BUSINESS MATHEMATICS

B.B.A., First Year

Director

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B.B.M., FIRST YEAR

BUSINESS MATHEMATICS

Syllabus

The objectives of this course are to familiarise the students with mathematical tools useful for decision marking.

(Proofs and derivations are excluded)

1. Introduction - Concept and nature of decision making-decision making process – need for information, Computation and analysis role of mathematical models- role of computers.
2. Linear and quadratic equations- progressions- permutation and combinations –Binomial theorem.
3. Set theory- sets and operation on sets- Functions- limits
4. Vectors-type- Geometric interpretation and linear dependence.
5. Matrix algebra- addition, subtraction and multiplication of matrices- ad joint matrix – Inverse of matrix-rank of matrix-rank of matrix – solutions of simultaneous equations.
6. Differentiation – basic laws of derivatives- higher order derivatives- partial differentiation- maxima and minima of functions.
7. Integration – concept- methods- definite integrals- integration by parts.
8. Elements of OR- concept of modeling- modeling procedures –OR techniques – an overview.
9. Linear programming – graphic solution – simplex method.

BOOKS RECOMMENDED

1. SIVAYYA K. V., Satya Rao, B.H. Satyanarayana and B. Satyanarayana **Business Mathematics** Technical Publishers, Guntur.
2. Sancheti and Kapoor **V.K.,Business Mathematics** Sultan Chand & Sons, New Delhi.

(DBBM 11)

B.B.A. DEGREE EXAMINATION, DECEMBER 2011.

First Year

Part II — Business Management

Paper II — BUSINESS MATHEMATICS

Time : Three hours **Maximum : 100 marks**

Answer any FIVE of the following.

All questions carry equal marks.

1. (a) What is Decision making? Explain the process. What are its challenges?

   (b) In a Geometric Progression, if the sum, of three consecutive terms is 14 and their product is 64, find them.

2. (a) (i) Describe the Mathematical model.

   (ii) Describe the role of computers.

   (b) A town has a population of 50,000. Out of it 28,000 read economic times and 23,000 read times of India while 4000 read both the papers. Indicate how many of them read neither of the papers.

3. (a) Explain de Morgan's Laws of sets.

   (b) Determine whether the vectors.

   $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k}$

   $\bar{b} = \bar{i} - 4\bar{k}$

   $\bar{c} = 4\bar{i} + 3\bar{j} - \bar{k}$

   are Linearly dependent or independent.

4. (a) What do you mean by Geometrical Interpretation? Elucidate its properties.

   (b) In how many ways can a committee of 5 be formed from 6 commerce, 5 English and

3 Hindi students so that each branch is represented.

5. (a) Define Matrix. Mention any three kinds of matrices.

   (b) Find the minimum average cost if

   (i) $T_a = 20 - 2x + x^2$

   (ii) $T_a = 4x + 12 - \frac{4}{x}$

6. (a) Integrate the following

(i) $\int \frac{x^2}{\sqrt{x^3 - 1}} dx$

(ii) $\int \frac{(2x + 3)}{(x^2 + 3x + 5)^8} dx$

(b) What do you mean by Rank of a Matrix? Find the rank of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 8 \end{bmatrix}$$

7. (a) Define Linear Programming. What are its various applications?

(b) Evaluate the following using integration by parts.

(i) $\int \log x \, dx$

(ii) $\int xe^x \, dx$

8. (a) What is Operations Research? Describe the various models used.

(b) Explain the methodology of operations Research.

9. (a) A trader deals in Rice and Wheat. He has room to store only 100 bags and his capacity to invest is Rs. 18,000/-. The cost of one bag rice is Rs. 200 and cost of one bag of wheat is Rs. 150. He makes a profit of Rs. 15 per bag of rice and Rs. 12 per bag of Wheat, find out how he should invest money in order to maximise the profit.

(b) The demand and cost function of a firm are given by $P = 12 - 3x$ and $T = x^2 + 2x$

Find the following :

(i) Average cost

(ii) Average Revenue

(iii) Marginal Cost

(iv) Maximum profit.

(DBBM 11)

B.B.A. DEGREE EXAMINATION, DECEMBER 2010.

(Examination at the end of First Year)

Part II — Business management

Paper II — BUSINESS MATHEMATICS

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Explain the process of decision making. Discuss the role of mathematical models in decision making process. State the advantages of these models.

   (b) Three numbers are in geometric progression. Their sum is 21 and their product is 216. Find those numbers.

2. (a) Find the first four terms in the binomial expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$. Obtain the middle term of this expansion and the term independent of x.

   (b) Out of 6 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways this can be done if the committee consists of

   (i) exactly 2 ladies

   (ii) at least 2 ladies

   (iii) there is no restriction in its formation.

3. (a) If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = 3x - 2$ and $g(x) = x^2 + 1$, then find

   (i) $(f \circ g)(2)$

   (ii) $(f \circ f)(x)$

   (iii) $(g \circ f)(x+1)$.

   (b) Evaluate :

   (i) $\underset{x \to 2}{lt} \frac{7x^2 - 11x - 6}{3x^2 - x - 10}$

   (ii) $\underset{x \to \frac{\pi}{2}}{lt} \frac{1 - \sin x}{x \cos x}$.

4. (a) When do we say that a finite set of vectors are linearly independent? Determine whether the following vectors : $\overline{a} + 2\overline{b} - \overline{c}$, $2\overline{a} - 3\overline{b} + 2\overline{c}$ and $4\overline{a} + \overline{b} + 3\overline{c}$ are linearly independent?

   (b) If $\overline{a}, \overline{b}, \overline{c}$ are three non zero non-coplanar vectors, examine whether the vectors $2\overline{a} - \overline{b} + 3\overline{c}$, $\overline{a} + \overline{b} - 2\overline{c}$ and $\overline{a} + \overline{b} - 3\overline{c}$ are coplamar. Justify your claim.

5. (a) If $A = \begin{bmatrix} 0 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 1 \end{bmatrix}$ them show that $B\,A' = (AB')'$.

(b) Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \\ -3 & -4 & 1 \end{bmatrix}$.

6. (a) Solve the system of equations :

$x + y + z = 6$, $x - y + z = 2$ and $2x + y - z = 1$ using matrix inversion method.

(b) Define the rank of a matrix. Find the rank of the matrix : $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$.

7. (a) Differentiate the following functions w.r.t

(i)

(ii)

(iii) .

(b) A manufacturing firm assumes the following relationships for the revenue (R) and cost function (C) :

$R(Q) = 1000\,Q - 2Q^2$

$C(Q) = Q^3 - 59\,Q^2 + 1315\,Q + 5000$

If Q is measured in tons per week, find out at what level of output Q, the profit is maximum?

8. (a) Evaluate the following integrals :

(i)

(ii) .

(b) If find .

9. (a) Discuss the importance, scope and limitations of operations research in the modern management.

(b) Solve the following linear programming problem using simplex method :

Maximize

Subject to :

and .

(DBBM 11)

B.B.A. DEGREE EXAMINATION, MAY 2011.

(Examination at the end of First Year)

Part II – Business Management

Paper II – BUSINESS MATHEMATICS

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) What is the function of a model in decision making name the type of models? What are the advantages of models? What are the pitfalls of models?

   (b) The length of a rectangular plot of ground is greater than its breadth by 36 m. If the area of the plot is 460 m^2. Find its dimensions.

2. (a) The first and last terms of an AP are – 4 and 146 respectively and the sum of AP is 7171. Find the number of terms in AP and the common difference.

   (b) A family of four brothers and 3 sister are to be arranged in a row for a photography. In how many ways can they be seated if all the sisters sit together.

3. (a) If f and g are real valued functions defined by $f(x) = 4x - 2$ and $g(x) = x^2$ find $(fog)(x)$.

   (b) Find $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a}$.

4. (a) If $\bar{a}, \bar{b}, \bar{c}$ are linearly independent, show that $\bar{a} - 2\bar{b} + 3\bar{c}$, $-2\bar{a} + 3\bar{b} - 4\bar{c}$ and $-\bar{b} + 2\bar{c}$ are linearly dependent.

   (b) If the position vectors of the vertices of a triangle are $2i + j + 3k$, $3i + 2j + k$, $i + 3j + 2k$, show that the triangle is an equilateral one.

5. (a) Find the adjoint of the square matrix

$$A = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 1 & 2 \\ 4 & 3 & 1 \end{bmatrix}.$$

   (b) Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

6. (a) Find the rank of the following matrix.

$$A = \begin{bmatrix} 4 & 8 & 2 & 1 \\ 5 & 7 & 1 & 0 \\ 5 & 3 & 9 & 2 \end{bmatrix}.$$

(b) Solve the following system of linear equations by Cramer's rule.

$x - 2y + 3z = 1$

$3x - y + 4z = 3$

$2x + y - 2z = 1$

7. (a) Find the differential coefficients of $(3x^3 - 5x^2 + 8)^3$ w.r. to .x

(b) Find the maximum and minimum values of functions.

(i) $y = x^3 - 3x^2 + 20$

(ii) $y = x^4 - 2x^2 + 2$

8. Evaluate

(a) $\int \frac{x^3 + 5x^2 + 4x + 1}{x^2}$

(b) $\int (2x+7)^5 dx$

(c) $\int (\log x)^2 \, dx$

9. (a) Explain the concept, scope and tools of operations research as applicable to business and industry.

(b) Use Simplex procedure to solve

Maximise $z = 3x_1 + 5x_2$

Subject to $2x_1 + 3x_2 \le 18$

$x_1 + 2x_2 \le 12$

$3x_1 + x_2 \le 15$

$x_1, x_2 \ge 0.$

CONTENTS

Lesson - 1

INTRODUCTION

OBJECTIVES:

The main objective of this lesson is to explain the importance of Business Mathematics in commerce. It also aims at explaining the relationship between the various subjects in commerce & Mathematics.

STRUCTURE:

1.1 Introduction

1.2 Summary

1.3 Model Questions

1.1 INTRODUCTION:

Mathematics is the base for Commerce. A combination of both is a better option. The course develops the methods of differentiation and integration to functions of one variable. Our aim is to supplement this course with suitable lessons from the existing Business Mathematics course, in order to provide an intoduction to, and illustrations of, techniques relevant to financial management. Mathematics is used by commercial enterprises to record and manage business operations. Mathematics typically used in commerce includes elementary arithmetic, such as fractions, decimals percentages, elementary algebra, statistics and probability. Business management can be made more effective in some cases by use of more advanced mathematics such as calculus, matrix algebra and linear programming. Commercial organizations use mathematics in accounting, inventory management, marketing, sales forecasting, and financial analysis.

In academia, " Business Mathematics" includes mathematics courses taken at an undergraduate level by Commerce students. These courses are slightly less difficult and do not always go into the same depth as other mathematics courses for people majoring in mathematics or science fields. The most common math courses taken in this form are Business Calculus, Algebra and Business Statistics. Examples used for problems in these courses are usually real-life problems from the business world.

An example of the differences in coursework from a business mathematics course and a regular mathematics course would be calculus. The accentuation in these courses is on computational skills and their practical application, with practical application being predominant. Many practical problems with solutions are also presented and emphasized. Derivatives are financial instruments whose values change in response to the changes in underlying variables. The main types of derivatives are futures, forwards, options, and swaps.

The main use of derivatives is to reduce risk for one party. The diverse range of potential underlying assets and pay-off alternatives leads to a wide range of derivatives contracts available to be traded in the market. Derivatives can be based on different types of assets such as commodities, equities (stocks), bonds, interest rates, exchange rates, or indexes (such as a stock market index, consumer price index (CPI) - see inflation derivatives - or even an index of weather conditions, or other derivatives). Their performance can determine both the amount and the timing of the pay-offs.

Widely hailed for its focus on student's needs, this classic algebra- based introduction to Business Mathematics takes care to present each topic in a clear and logical manner-with detailed explanations of all steps and concise discussions describing the business applications of each topic. This dual approach sharpens the mathematical skills of students preparing to enter business employment while also providing an introduction to accounting, finance, insurance, statistics, taxation, and other math-related subjects. Consumer math applications, such as bank reconciliation, discounting, mark-ups and markdowns, installment purchases, and simple and compound interest are also covered in depth.

" Mathematics provides a system of logic which is helpful in analysing many practical problems in science, social sciences and humanities. A comprehensive knowledge of mathematics is necessary for a meaningful study of business problems and in decision - making.

Modern Risk Management including Insurance, Stock Trading and Investment depend on Mathematics and it is a fact that one can use Mathematics advantageously to predict the future with more precision ! Not with 100% accuracy, of course. But well enough so that one can make a wise decision as to where to invest money. The idea of using Mathematics to predict the future goes back to two 17th Century French Mathematicians Pascal and Fermat. They worked out probabilities of the various outcomes in a game where two dice are thrown a fixed number of times.

Linear programming finds the least expensive way to meet given needs with available resources. Its results are used in every area of engineering and commerce :

Linear programming, sometimes known as linear optimization, is the problem of maximizing or minimizing a linear function over a convex polyhedron specified by linear and non- negativity constraints. Simplistically, linear programming is the optimization of an outcome based on some set of constraints using a linear mathematical model.

Linear programming theory falls within convex optimization theory and is also considered to be an important part of operations research. Linear programming is extensively used in business and economics.

Keeping all the above under consideration Business Mathematics curriculum is designed in appropriate way by including the relevant Mathematics subjects.

1.2. SUMMARY :

Business Mathematics is very useful in commerce and management. It will be used in almost all the aspects of commerce and management, particularly in the process of Decision making, maximisation of profits, minimisations of costs (linear programming) etc.

1.3. MODEL QUESTIONS :

1. What is the improtance Business Mathematics in commerce and management ?
2. Explain the relationship between commerce and Business Mathematics.
3. What is the importance of Business Mathematics in the process of Decision making?

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Lesson - 2

MATHEMATICAL MODELS

Objectives:

After studying this lesson, you should be able to understand -

→ The decision makers pay-off table, decision making under certainty.

→ Decision making under risk (with probability), dewsion tree analysis.

Structure:

This lesson has the following components:

2.1 Introduction

2.2 The decision maker

2.3 Pay-off

2.4 Regret (or Opportunity Loss)

2.5 Decision making under uncertainty (without probability)

2.6 Decision making under risk (with probability)

2.7 Decision Tree Analysis

2.8 Exercise

2.9 Answers

2.1 Introduction:

Decision theory is primarily concerned with helping people and organizations in making decisions. It provides a meaningful conceptual frame work for important decision making. The decision making refers to the selection of an act from amongst various alternatives, the one which is judged to be the best under given circumstances.

The management has to consider phases like planning, organization, direction, command and control. While performing so many activities, the management has to face many situations from which the best choice is to be taken. This choice making is technically termed as "decision making" or decision taking. A decision is simply a selection from two or more courses of action. Decision making may be defined as "a process of best selection from a set of alternative courses of action, that course of action which is supposed to meet objectives upto satisfaction of the decision maker.

The knowledge of statistical techniques helps to select the best action. the statistical decision theory refers to an optimal choice under condition of uncertainty. In this case probability theory has a vital role, as such, this probability thory will be used more frequently in the decision making theory under uncertainty and risk.

The statistical decision theory tries to reveal the logical structure of the problem into alternative action, states of nature, possible outcomes and likely pay-offs from each such outcome. Let us explain the concepts associated with the decision theory approach to problem solving.

2.2 The Decision Maker:

The decision maker refers to individual or a group of individual responsible for making the choice of an appropriate course of action amongst the avilable courses of action.

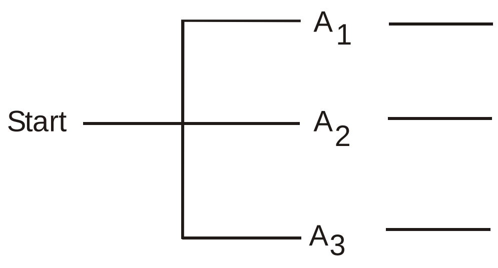
Decision making problems deals with the selection of a single act from a set of alternative acts. If two or more alternative courses of action occur ina problem, then decision making is necessary to select only one course of action.

Let the acts or action be $a_1, a_2, a_3, \cdots\cdots\cdots$ then the totality of all these actions is known as action space denoted by A. For three actions a_1, a_2, a_3 ; A = action space $= (a_1, a_2, a_3)$ or $A = (A_1, A_2, A_3)$. Acts may be also represented in the following matrix form i.e., either in row or column was

Acts
A_1
A_2
.   .
A_n

Acts	A_1	A_2	...	A_n

In a tree diagram the acts or actions are shown as



Events or States of Nature:

The events identify the occurrences, which are outside of the decision maker's control and which determine the level of success for a given act. These events are often called 'States of Nature' or outcomes. An example of an event or states of nature is the level of market demand for a particular item during a stipulated time period.

A set of states of nature may be represented in any one of the following ways:

$$S = \{S_1, S_2, \cdots\cdots\cdots, S_n\}$$

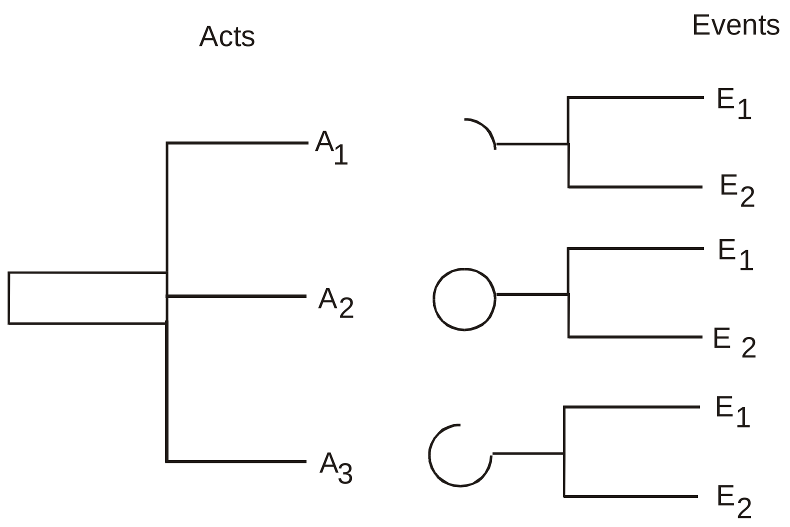
$$\text{or } E = \{E_1, E_2, \cdots\cdots\cdots, E_n\}$$

$$\text{or } \Omega = \{\theta_1, \theta_2, \theta_3\}$$

For example, if a washing powder is marketed, it may be highly liked by outcomes (outcome θ_1) or it may not appeal at all (outcome θ_2) or it may satisfy only a small fraction, say 25% (outcome θ_3).

$$\therefore \Omega = \{\theta_1, \theta_2, \theta_3\}$$

In a tree diagram the places are next to acts. We may also get another act on the happening of events as follows:



The matrix form, they may be represented as either of the two ways:

States of Nature → ↓ Acts	S_1	S_2
A_1 A_2		

or

States of Nature ↓	A_1 A_2 ...... A_N
S_1 S_2	

2.3 Pay-Off:

The result of combinations of an act with each of the states of nature is the outcome and momentary gain or loss of each such outcome is the pay-off. This means that the expression pay-off should be in quantitative form.

Pay-off may be also interms of cost saving or time saving. In general, if there are k alternatives and n states of nture, there will be k x n outcomes or pay-offs. These k x n pay-offs can be very conveniently represented in the form of a k x n pay-off table.

States of Nature	Decision Alternative			
	A_1	A_2	------	A_K
E_1	a_{11}	a_{12}	------	a_{1k}
E_2	a_{21}	a_{22}	------	a_{2k}
.	.	.	------	.
.	.	.	------	.
.	.	.	------	.
E_n	a_{n1}	a_{n2}	------	a_{nk}

where a_{ij} = conditional outcome (pay-off) of the ith event when jth alternative is chosen. The above pay-off table is called pay-off matrix.

For example,

A farmer can raise any one of three crops on his field. The yields of each crop depend on weather conditions. We have to show pay-off in each case, if prices of the three products are as indicated in the last column of yield matrix.

	Weather				
		Dry (E_1)	**Moderate (E_2)**	**Damp (E_3)**	**Price Rs.Per-Kg.**
Yield in Kg per hectare	**Paddy (A_1)**	500	1700	4500	1.25
	Groundnut (A_2)	800	1200	1000	4.00
	Tobacco (A_3)	100	300	200	15.00

Solution:

Pay-off Table

	E_1	E_2	E_3
A_1	500 x 1.25 = 625	1700 x 1.25 = 2125	4500 x 1.25 = 5625
A_2	800 x 4 = 3200	1200 x 4 = 4800	1000 x 4 = 4000
A_3	100 x 15 = 1500	300 x 15 = 4500	200 x 15 = 3000

2.4 Regret or Opportunity Loss:

The difference between the highest possible profit for a state of nature and the actual profit obtained for the particular actiontaken is known as opportunity loss. That is an opportunity loss is the loss incurred due to failure of not adopting the best possible course of action. Opportunity losses are calculated separately for each state of nature. For a given state of nature the opportunity loss of possible course of action is the difference between the pay-off value for that ourse of action and the pay-off for the best possibel course of action that could have been selected.

Let the pay-off of the outcomes in the 1^{st} row be P_{11} , P_{12} , ..., P_{1n} and similiarly for the other rows.

Pay of Table

Acts	States of Nature S_1	S_2	......	S_n
A_1	P_{11}	P_{12}	......	P_{1n}
A_2	P_{21}	P_{22}	......	P_{2n}
. . .	. . .	. . .		. . .
A_m	P_{m1}	P_{m2}	......	P_{mn}

Consider a fixed state of nature S_i. The pay-off corresponding to the n strategies are given by $P_{i1}, P_{i2}, \ldots\ldots, P_{in}$. Suppose M_i is the maximum of these quantities. The P_{i1} if A_1 is used by the decision maker there is loss of opportunity of $M_1 - P_{i1}$ and so on.

Then a table showing opportunity loss can be computed as follows:

Regret or Opportunity loss table

Acts	States of Nature S_1	S_2	......	S_n
A_1	$M_1 - P_{11}$	$M_2 - P_{12}$	......	$M_n - P_{1n}$
A_2	$M_1 - P_{21}$	$M_2 - P_{22}$	......	$M_n - P_{1n}$
. . .	. . .	. . .		. . .
A_m	$M_1 - P_{m1}$	$M_2 - P_{m2}$	......	$M_n - P_{mn}$

2.4.1 Types of Decision Making: Decisions are made based upon the information data available about the occurrence of events as well as the decision situation. There are three types of decisionmaking situations: Certainty, uncertainty and risk.

2.4.2 Decision Making Under Certainty: In this case the decision maker has the complete knowledge of consequence of every decision choice with certainty. In this decision model, assumed certainty means that only one possible state of nature exists.

Example 1:

A canteen prepares a food at a total average cost of Rs. 4 per plate and sells it at a price of Rs. 6. The food is prepared in the morning and is sold during the same day. Unsold food during the same day is spoiled and is to be thrown away. According to the pst sale, number of plates prepared past is not less than 50 or greater than 53. You are to formulate the (i) action space (ii) states of nature space (iii) pay-off table (iv) loss table.

Solution:

(i) The canteen will not prepare less than 50 plates or more than 53 plates.

Thus the acts or curses of action open to him are

a_1 = prepare 50 plates

a_2 = prepare 51 plates

a_3 = prepare 52 plates

a_4 = prepare 53 plates

Thus the action space is

$$A = \{a_1, a_2, a_3, a_4\}$$

(ii) The state of nature is daily demand for food plates.

Then are four possible state of nature i.e.

S_1 = demand is 50 plates

S_2 = demand is 51 plates

S_3 = demand is 52 plates

S_4 = demand is 53 plates

Hence the state of nature space, $S = \{S_1, S_2, S_3. S_4\}$

(iii) The uncertainty element in the given problem is the daily demand.

The profit of the canteen is subject to the daily demand.

Let n = quantity demanded

m = quantity produced

For $n \geq m$, profit = (cost price - selling price) x m

= (6 - 4) x m = 2m

For $m > n$,

Pay-Off Table

Supply (m)	Demand (n)			
	(S_1) 50	(S_2) 51	(S_3) 52	(S_4) 53
(a_1) 50	100	100	100	100
(a_2) 51	96	102	102	102
(a_3) 52	92	98	104	104
(a_4) 53	88	94	100	106

(iv) To calculate the opportunity loss we first determine the maximum pay-off in each state of nature. In this state

First maximum pay-off = 100

Second maximum pay-off = 102

Third maximum pay-off = 104

Fourth maximum pay-off = 106

Loss Table Corresponding to the Above Pay-Off Table

Supply (m)	Demand (n)			
	(S_1) 50	(S_2) 51	(S_3) 52	(S_4) 53
(a_1) 50	100 - 100 = 0	102 - 100 = 2	104 - 100 = 4	106 - 100 = 4
(a_2) 51	100 - 96 = 4	102 - 102 = 0	104 - 102 = 2	106 - 102 = 4
(a_3) 52	100 - 92 = 8	102 - 98 = 4	104 - 104 = 0	106 - 104 = 2
(a_4) 53	100 - 88 = 12	102 - 94 = 8	104 - 100 = 4	106 - 106 = 0

2.5 Decision Making Under Uncertainty (Without Probability):

Under conditions of uncertainty, only pay-offs are known and nothing is known about the likelihood of each state of nature. Such situations arise when a new product is introduced in the market or a new plant is set up. The number of different decision criteria available the condition of uncertainty is given below:

2.5.1 Certain of Optimism (Maximax): The maximax criterion finds the course of action or alternative strategy that maximizes the maximum pay-off. Since this decision criterion locates the alternative with the highest possible gain, it has also been called an optimistic decision criterion. The working method is

(i) Determine the best outcome for each alternative.

(ii) Select the alternative associated with the best of these.

2.5.2 Expected Monetary Value (EMV): The expected monetary value is widely used to evaluate the alternative course of action (or act). The EMV for given course of action is just sum of possible pay-off of the alternative each weighted by the probability of that pay-off occurring.

2.5.3 The Criteria of Pessimism or Maximin: This criterion is the decision to take the course of action which maximizes the minimum possible pay-off. since this decision criterion locates the alternative strategy that has the least possible loss, it is also known as a pessimistic decision criterion. The working method is:

1) Determine the lowest outcome for each alternative.

2) Choose the alternative associated with the best of these.

2.5.4 Minimax Regret Criterion (Savage Criterion): This criterion is also known as opportunity loss decision criterion because decision maker feels regret after adopting a wrong course of action (or alternative) resulting in an opportunity loss of pay-off. Thus he always intends to minimize this regret. The working method is

(a) Form the given pay-off matrix, develop an opportunity loss (or regret) matrix.

(i) Find the best pay-off corresponding to each state of nature and

(ii) Subtract all other entries (pay-off values) in that row from this value alternatives.

(iii) Select the alternative associated with the lowest of these.

2.5.5 Equally likely decision (Baye's or Laplace) Criterion: Since the probabilities of states of nature are not known, it is assumed that all states of nature will occur with equal probability i.e., each state of nature is assigned an equal probability. As states of nature are mutually exclusive and collectively exhaustive so the probability of each. These must be 1/(number of states of nature). The working method is

(a) Assign equal probability value to each state of nature by using the formula:

1/(number of states of nature)

(b) Compute the expected (or average) value for each alternative by multiplying each outcome by its probability and then summing.

(c) Select the best expected pay-off value (maximum for profit and minimum for loss)

This criterion is also known as the criterion of insufficient reason because, expect in a few cases, some information of the likelihood of occurrence of states of nature is available.

2.5.6 Criterion of Realism (Hurwicz Criterion): This criterion is a compromise between an optimistic and pessimistic decision criterion. To start with a co-efficient of optimism $\alpha \left(0 \le \alpha \le 1 \right)$ is selected.

When α is close to one, the decision maker is optimistic about the future and when α is close to zero, the decision maker is pessimistic about the future.

According to Hurwicz, select strategy which maximizes $H = \alpha$ (maximum pay-off in row) + (1 - α) minimum pay-off in row.

Example 2:

Consider the following pay-off (profit) matrix

Action	States			
	(S_1)	(S_2)	(S_3)	(S_4)
A_1	5	10	18	25
A_2	8	7	8	23
A_3	21	18	12	21
A_4	30	22	19	15

No probabilities are known for the occurrence of the nature states. Compare the solutions obtained by each of the following criteria:

(i) Maximin (ii) Lapalace (iii) Hurwicz (assume that $\alpha = 0.5$)

Solution:

(i) Maximin Cirterion:

					Minimum	
A_1 :	5	10	18	25	**5**	
A_2 :	8	7	8	23	7	
A_3 :	21	18	12	21	12	
A_4 :	30	22	19	15	**15**	**Maximum**

Best action is A_4

(ii) Laplace Criterion

$E(A_1) = 1/4\,[5 + 10 + 18 + 25] = 14 \cdot 5$

$E(A_2) = 1/4\,[8 + 7 + 8 + 23] = 11.5$

$E(A_3) = 1/4\,[21 + 18 + 12 + 21] = 18 \cdot 0$

$E(A_4) = 1/4\,[30 + 22 + 19 + 15] = 21 \cdot 5$ **maximum**

$E(A_4)$ is maximum. So the best action is A_4

(iii) Hurwicz Criterion (with $\alpha = 0 \cdot 5$)

	Minimum	**Maximum**	α (max) + $(1-\alpha)$ min
A_1	5	25	0.5 (25) + 0.5(5) = 15
A_2	7	23	0.5(7) + 0.5 (23) = 15
A_3	12	21	0.5 (12) + 0.5 (21) = 16.5
A_4	15	30	0.5 (15) + 0.5 (30) = **22.5 maximum**

Best action is A_4

Example 3:

Suppose that a decision maker faced with three decision alternatives and two states of nature. Apply (i) Maximin and (ii) Minimax regret approach to the following pay-off table to recommend the decisions.

States of Nature / **Act**	S_1	S_2
A_1	10	15
A_2	20	12
A_3	30	11

Solution:

(i) Maximin:

Act	Minimum	
A_1	10	
A_2	**12**	**Maximum**
A_3	11	

Act A_2 is recommended

(ii) Minimax Regret:

States of Nature Act	S_1	S_2	Maximum Regret
A_1	30 - 10 = 20	15 - 15 = 0	20
A_2	30 - 20 = 10	15 - 12 = 3	10
A_3	30 - 30 = 0	15 - 11 = 4	**4**

Mimum of the maximum regrets is 4 which coresponds to the act A_3 is recommended.

Example 4:

A business man has to select three alternatives open to him each of which can be followed by any of the four possible events. The conditional pay-off (in Rs.) for each action event combination are given below:

Alternative	Pay-offs Conditional Events			
	A	**B**	**C**	**D**
X	8	0	- 10	6
Y	- 4	12	18	- 2
Z	14	6	0	8

Determine which alternative should the businessman choose, if he adopts the

a) Maximin Criterion
b) Maximax Criterion
c) Hurwicz Criterion with degree of optimism is 0.7
d) Minimax Regret Criterion
e) Laplace Criterion

Solution:

For the given pay-off matrix, the maximum assured and minimum possible pay-off for each alternative are as given below:

Alternative	Maximum Pay-off (Rs.)	Minimum Pay-off (Rs.)	$(\alpha = 0\cdot7)$ $H = \alpha\,(\text{maximum pay-off}) + (1-\alpha)\,(\text{mimum pay-off})$
X	8	- 10	2.6
Y	18	- 4	**11.4**
Z	14	0	9.8

(a) Since z yields the maximum of the minimum pay-off, under maximin criterion, alternative z would be chosen.

(b) Under maximax criterion, the businessman would choose the alternative Y.

(c) It will be optimal to choose Y under Hurwicz Criterion.

(d) For the given pay-off matrix, we determine the regrets as shown below, when the regret pay-offs amounts when event A occurs, are computed by the relation.

Regret pay-off maximum pay-offs from A - pay-off.

Similarly for the other events.

Alternative	**Pay-off Amount**				**Regret Pay-off Amount**				**Maximum Regret**
	A	**B**	**C**	**D**	**A**	**B**	**C**	**D**	
X	8	0	- 10	6	6	12	28	2	28
Y	- 4	12	18	- 2	18	0	0	10	18
Z	14	6	0	8	0	6	18	0	18
Maximum Pay-off	14	12	18	8					

Since alternative Y and Z both corresponding to the minimal of the maximum possible regrets the decision maker would choose either of these two.

(e) **Laplace Criterion:** In this method assigning equal probabilities to the pay-off of each strategy results in the following expected pay-off.

Alternative	**Pay-off**				**Expected Pay-off Value**
	A **P = 1/4**	**B** **P = 1/4**	**C** **P = 1/4**	**D** **P = 1/4**	
X	8	0	- 10	6	$\frac{1}{4}[8+0-10+6]=1$
Y	- 4	12	18	- 2	$\frac{1}{4}[-4+12+18-2]=6$
Z	14	6	0	8	$\frac{1}{4}[14+6+0+8]=7$

Since the expected pay-off value for Z is the maximum the businessman would choose alternative Z.

2.6 Decision Making Under Risk (with probability):

Here the decision maker faces many states of nature. As such, he is supposed to believe authentic information, knowledge, past experience or happenings to enable him to assign probability values to the likelihood of coccurrence of each state of nature. Sometimes with reference to past records, experience or information, probailities to future events could be allotted. On the basis of probability distribution of the states of nature, one may select the best course of action having the highest expected pay-off value.

Example 5:

The pay-off table for three courses of action (A) with three states of nature (E) (or events) with their respective probabilities (p) is given. Find the best course of action.

Events	E_1	E_2	E_3
Probability → Acts ↓	0.2	0.5	0.3
A_1	2	1	- 1
A_2	3	2	0
A_3	4	2	1

The expected value for each act is

$$A_1 : 2(0\cdot 2) + 1(0\cdot 5) - 1(0\cdot 3) = 0\cdot 6$$

$$A_2 : 3(0\cdot 2) + 2(0\cdot 5) + 0(0\cdot 3) = 1\cdot 6$$

$$A_3 : 4(0\cdot 2) + 2(0\cdot 5) + 1(0\cdot 3) = 2\cdot 1$$

The expected monetary value for the act 3 is maximum. Therefore the best course of action is A_3.

Example 6:

Given the following pay-off of 3 acts: A_1, A_2, A_3 and their events E_1, E_2, E_3.

Act / **States of Nature**	A_1	A_2	A_3
E_1	35	- 10	- 150
E_2	200	240	200
E_3	550	640	750

The probabilities of the states of nature are respectively 0.3, 0.4 and 0.3. Calculate and tabulate EMV and conclude which of the acts can be chosen as the best.

Events	Prob.	A_1	A_2	A_3
E_1	0.3	$35 \times 0 \cdot 3 = 10 \cdot 5$	$-10 \times 0 \cdot 3 = -3$	$-150 \times 0 \cdot 3 = -45$
E_2	0.4	$200 \times 0 \cdot 4 = 80.0$	$240 \times 0 \cdot 4 = 96$	$200 \times 0 \cdot 4 = 80$
E_3	0.3	$550 \times 0 \cdot 3 = 165 \cdot 0$	$640 \times 0 \cdot 3 = 192$	$750 \times 0 \cdot 3 = 225$
EMV		255.5	285	260

The EMV of A_2 is maximum, therefore to choose A_2.

Example 7:

A shop keeper has the facility to store a large number of perishable items. He buys them at a rate of Rs. 3 per item and sells a the rate of Rs. 5 per item. If an item is not sold at the end of the day then there is a loss of Rs. 3 per item. The daily demand has the following probability distribution.

Number of Items demanded	3	4	5	6
Probability	0.2	0.3	0.3	0.2

How many items should he stored so that his daily expected profit is maximum?

Solution:

Let m = number of items stocked daily

n = number of items demanded daily

Now, for $n \geq m$ profit = 2 m

And for $m > n$ profit = 2n - 3 (m - n)

= 2n - 3m + 3n = 5n - 3m

Pay-off Table

Stock (m)	**Demand (n)**			
	3	**4**	**5**	**6**
3	6	6	6	6
4	3	8	8	8
5	0	5	10	10
6	- 3	2	7	12
Probability	0.2	0.3	0.3	0.2

Stock (m)	Expected gain
3	$6 \times 0\cdot2 + 6 \times 0\cdot3 + 6 \times 0\cdot3 + 6 \times 0\cdot2 =$ Rs. $6\cdot00$
4	$3 \times 0\cdot2 + 8 \times 0\cdot3 + 8 \times 0\cdot3 + 8 \times 0\cdot2 =$ Rs. $7\cdot00$
5	$0 \times 0\cdot2 + 5 \times 0\cdot3 + 10 \times 0\cdot3 + 10 \times 0\cdot2 =$ Rs. $6\cdot50$
6	$-3 \times 0\cdot2 + 2 \times 0\cdot3 + 7 \times 0\cdot3 + 12 \times 0\cdot2 =$ Rs. $4\cdot50$

Thus the highest expected gain is Rs. 7.00 when 4 units stocked. So, he can store 4 items to get maximum expected profit daily.

Example 8:

A magazine distributor assigns probabilities to the demand for a magazine as follows:

Copies demanded:	2	3	4	5
Probability :	0.4	0.3	0.2	0.1

A copy of magazine which he sells at Rs. 8 costs Rs. 6 How many should he stock to get the maximum possible expected profit if the distributor can return back unsold copies for Rs. 5 each?

Solution:

Let m = no of magazines stocked daily

n = no of magazines demanded

Now,

for $n \geq m$ profit = Rs. 2m

and for m > n, profit $= 8n - 6m + 5(m - n)$

$= 8n - 6m + 5m - 5n$

$= 3n - m$

Pay-off Table

Stock (m)	**Demand (n)**			
	2	**3**	**4**	**5**
2	4	4	4	4
3	3	6	6	6
4	2	5	8	8
5	1	4	7	10
Probability	**0.4**	**0.3**	**0.2**	**0.1**

Stock	Expected Profit (in Rs.)
2	$4 \times 0 \cdot 4 + 4 \times 0 \cdot 3 + 4 \times 0 \cdot 2 + 4 \times 0 \cdot 1 = 4 \cdot 0$
3	$3 \times 0 \cdot 4 + 6 \times 0 \cdot 3 + 6 \times 0 \cdot 2 + 6 \times 0 \cdot 1 = 4 \cdot 8$
4	$2 \times 0 \cdot 4 + 5 \times 0 \cdot 3 + 8 \times 0 \cdot 2 + 8 \times 0 \cdot 1 = 4 \cdot 7$
5	$1 \times 0 \cdot 4 + 4 \times 0 \cdot 3 + 7 \times 0 \cdot 2 + 10 \times 0 \cdot 1 = 4 \cdot 0$

Thus the highest expected profit is Rs. 4.8, when 3 magazines stocked. So, the distributor can stock 3 magazines to get the maximum possible expected profit.

2.7 Decision Tree Analysis:

A decision problem may also be represented with the help of a diagram. It shows all the possible courses of action, states of nature and the probabilities associated with the states of nature. The 'decision diagram' looks very much like a drawing of a tree, therefore also called 'decision tree'.

A decision tree consists of nodes, branches, probability estimates and pay-offs. Nodes are of two types, decision node (designated as a square) and chance node (designated as a circle). Alternative courses of action originate from decision node as the main branches (decision branches). Now at the terminal point of decision node, chance node exists from where chance nodes, emanate as sub-branches. The repsective pay-offs and the probabilities associated with alternative courses and the chance events are shown alongside the chance branches. At the terminal of the chance branches are shown the expected pay-off values of the outcome.

There are basically two types of decision trees-deterministic and probabilistic. These can further be divided into single stage and multistage trees. A single stage deterministic decision tree involves making only one decision under conditions of certainty (no chance events). In a multistage deterministic tree a sequence or chain of decisions are to be made. The optimal path (strategy) is one that corresponds to the maximum EMV.

In drawing a decision tree, one must follow certain basic rules and conventions as stated below:

1. Identify all decisions (and their alternatives) to be made and the order in which they must be made.

2. Identify the chance events or state of nature that might occur as a result of each decision alternative.

3. Develop a tree diagram showing the sequence of decisions and chance events. The tree is constructed starting from left and moving towards right. The square box denotes a decision point at which the available courses of action are considered. The circle O represents the chance node or event, the various states of nature or outcomes emanate from this chance event.

4. Estimate probbailities that possible events or states of nture will occur as a result of the decision alternatives.

5. Obtain outcomes (usually expressed in economic terms) of the possible interactions among decision alterntives and events.

6. Calculate the expected value of all possible decision alternatives.

7. Select the decision alternative (or course of action) offering the most attractive expected value.

Advantages of Decision Tree:

1. By drawing of decision tree, the decision maker will be in a position to visualise the entire complex of the problem.

2. Enable the decision maker to see the various elements of his problem in content and in a systematic way.

3. Multi-dimensional decision sequences can be strung on a decision tree without conceptual difficulties.

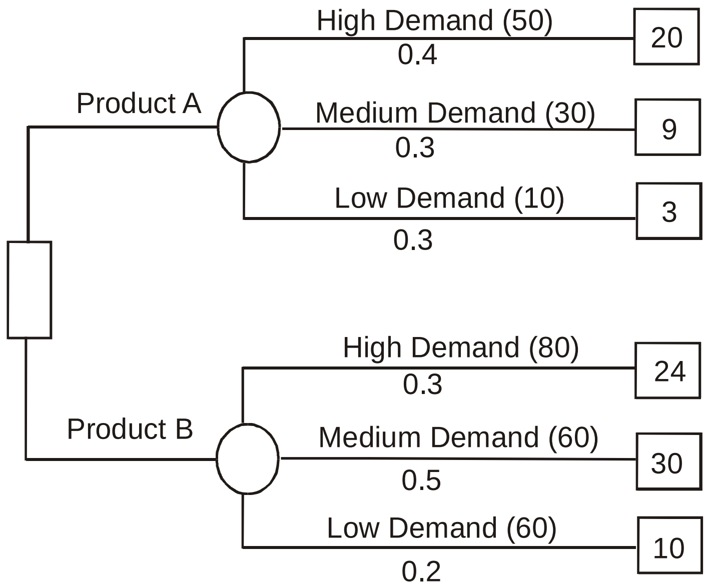
4. Decision tree model can be applied in various fields such as introduction of a new product, marketing strategy etc...

Example 9:

A manufacturing company has to select one of the two products A or B for manufacturing. Product A requires investment of Rs. 20,000 and product B of Rs. 40,000. Market research survey shows high, medium and low demands with corresponding probabilities and returns from sales in Rs. Thousand for the two produts in the following table.

Market Demand	Probbility		Return from Sales	
	A	B	A	B
High	0.4	0.3	50	80
Medium	0.3	0.5	30	60
Low	0.3	0.2	10	50

Construct an appropriate decision tree. What decision the company should take?



Market Demand	A			B		
	X('000)	P	PX	X('000)	P	PX
High	50	0.4	20	80	0.3	24
Medium	30	0.3	9	60	0.5	30
Low	10	0.3	3	50	0.2	10
Total			**32**			**64**

Product	Return (Rs.)	Investment (Rs.)	Profit (Rs.)
A	32,000	20,000	12,000
B	64,000	40,000	24,000

Since the profit is high in cse of product B, so the company's decision is in favour of B.

Example 10:

A farm owner is considering drilling a farm well. In the past, only 70% of wells drilled were successful at 20 metres of depth in that area. Moreover, on finding no water at 20 metres, some person drilled it further up to 25 metres but only 20% struck water at 25 metres. The prevailing cost of drilling is Rs. 500 per metres. The farm owner has estimated that in case he does not get his own well, he will have to pay Rs. 15,000 over the next 10 years to buy water from the neighbour.

Draw an appropriate decision tree and determined the farm owner's strategy under EMV approach.

Solution:

The given data is represented by the following decision tree diagram.

Decision	**Event**	**Probability**	**Cashout flows**	**Expected Cash out flow**
	Decision at point D_2			
1. Drill upto 25 metres	Water Struck	0.2	Rs. 12,500	Rs. 2,500
	No water struck	0.8	Rs. 27,500	Rs. 22,000
			EMV (Out flows)	Rs. 24,500
2. Do not Drill	EMV (out flow) = Rs. 25,000			

The decision at D_2 is : Drill upto 25 metres

Decision at point D_1

1. Drill Upto 20 metres	Water struck	0.7	Rs. 10,000	Rs. 7,000
	No water Struck	0.3	Rs. 24,500	Rs. 7,350
			EMV (out flows)	Rs. 14,350
2. Do not Drill	EMV (out flow) = Rs. 15,000			

The decision at D_1 is : Drill upto 20 metres.

Thus the optimal strategy for he farm-owner is to drill the well upto 20 metres.

2.8 Exercise:

1. Choose the correct answers:

1. Decision theory is concerned with
   (a) The amount of information that is available
   (b) Criteria for measuring the 'goodness' of a decision
   (c) Selecting optimal decisions in sequential problems
   (d) All of the above

2. Which of the following criteria does not apply to decision making under uncertainly
   (a) Maximin return
   (b) Maximax return
   (c) Minimax return
   (d) Maximize expected return

3. Maximin return, maximax return and minimax regret are criteria that
   (a) Lead to the same optimal decision
   (b) Cannot be used with probabilities
   (c) Both a and b
   (d) None of the above

4. Which of the following does not apply to a decision tree?
   (a) A square node is a point at which a decision must be made
   (b) A circular node represents an encounter with uncertainty
   (c) One chooses a sequence of decisions which have the greatest tprobability of success.
   (d) One attempts to maximize expected return

5. The criterion which selects the action for which maximum pay-off is lowest is known as
   (a) Max-min criterion (b) Min-max criterion
   (c) Max-max criterion (d) None of these

II. Fill in the blanks:

6. Decision trees involve _________________ of decisions and random outcomes.
7. One way to deal with decision making in the 'uncertainity' context is to treat all states of nature as _______________ and maximize expected return.
8. Maximizing expected net rupee return always yields the same optimal policy as ________ expected regret.
9. The different criteria for making decisions under risk always yields the same __________ choice.
10. In decision under uncertainty, the Laplace criterion is the least conservative while the _______________ criterion is the most conservative.

III. Answer the following:

11. Explain the meaning of 'statistical decision theory'.
12. What techniques are used to solve decision making problems under uncertainty?
13. Write a note on decision tree.
14. What is a pay-off matrix?
15. Desribe how you would determine the best decision using the EMV criterion with a decision tree.

IV. Problems:

16. The pay-off table for three courses of action (A) with three states of nature (E) (or events) with their respective probabilities (P) are given. Find the best course of action.

Events Acts	E_1	E_2	E_3
A_1	2.5	2.0	- 1
A_2	4.0	2.6	0
A_3	3.0	1.8	1
Probability	0.2	0.6	0.2

17. Calculate EMV and thus select the best act for the following pay-off table:

States of Nature	Probability	Pay-off (Rs.) by the player		
		A	B	C
X	0.3	- 2	- 5	20
Y	0.4	20	- 10	- 5
Z	0.3	40	60	30

18. Consider the pay-off matrix:

States of Nature	**Probability**	**Act A_1 do not Expand**	**Act A_2 Expand 200 units**	**Act A_3 Expand 400 units**
High Demand	0.4	2500	3500	5000
Medium Demand	0.4	2500	3500	2500
Low Demand	0.2	2500	1500	1000

Using EMV criterion decide the best act.

19. Apply (i) maximin (ii) minimax regret to the following pay-off matrix to recommended the decisions without any knowledge of probability.

States of nature

Act	S_1	S_2	S_3
a_1	14	8	10
a_2	11	10	7
a_3	9	12	13

20. A shop keeper of some highly perishable type of fruits sees that the daily demand X of this fruit in his area the following probability distribution.

Daily Demand (in Dozen) :	6	7	8	9
Probability :	0.1	0.3	0.4	0.2

He sells for Rs. 10.00 a dozen while he buys each dozen at Rs. 4.00. Unsold fruits in a day are traded on the next day at Rs. 2.00 per dozen, assuming that the stocks the fruits in dozen, how many should he stock so that his expected profit will be maximum?

[Hint: Profit $= 6m$ for $n \geq m$

$$= 10n - 4m + 2(m - n)$$

$$= 8n - 2m \text{ for } n < m]$$

21. A florist, in order to satisfy the needs of a number of regular and sophisticated customers, stocks a highly perishable flowers. A dozen flowers cost Rs. 3 and sell at Rs. 10.00 any flower not sold on the day are worthless. Demand distribution in dozen of flowers is as follows:

Demand	1	2	3	4
Probability	0.2	0.3	0.3	0.2

How many flowers should he stock daily in order to maximize his expected net profit?

22. A florist stock highly perishable flower. A dozen of flower costs Rs. 3.00 and sells for Rs. 10.00 Any flower not sold the day are worthless. Demand in dozen of flowers is as follows:

Demand in Dozen	0	1	2	3	4
Probability	0.1	0.2	0.4	0.2	0.1

Assuming that failure to satisfy any one customer's request will result in future lost profit amounting to Rs. 5.00, in addition to the lost profit on the immediate sale, how many flowers should the florist stock to expect maximum profit?

23. A newspaper agent's experience shows that the daily demand 'x' of newspaper in his area has the following probability distribution.

Daily Demand (x)	300	400	500	600	700
Probability	0.1	0.3	0.4	0.1	0.1

He sales the newspapers for Rs. 2.00 each while he buys each at Rs. 1.00 Unsold copies are treated as scrap and each such copy fetches 10 paisa. Assuming that he stock the newspapers in multiple of 100 only. How many should he stock that his expected profit is maximum?

24. Suppsoe that a decision maker faced with three decision alternatives and four states of nature. Given the following profit pay-off table.

	States of nature			
Acts	S_1	S_2	S_3	S_4
a_1	16	10	12	7
a_2	13	12	9	9
a_3	11	14	15	14

Assuming that he has no knowledge of the probabilities of occurrence of the states of nature, find the decisions to be recommended under each of the following criteria.

(i) Maximin

(ii) Maximax

(iii) Minimax Regret

25. Pay-off of three acts A, B and C and states of nature X, Y and Z are given below:

States of Nature	Pay-off (in Rs.) Acts		
	A	B	C
X	- 20	- 50	2000
Y	200	- 100	- 50
Z	400	600	300

The probabilities of the states of nature are 0.3, 0.4 and 0.3. Calculate the EMV for the above and select the best art.

2.9 Answers:

I.

1. (d) 2. (d) 3. (b) 4. (c) 5. (a)

II.

6. Sequence 7. Equally likely 8. Minimizing 9. Optimal 10. Minimax

IV.

16. A_2 is the best

17. Select A with the highest EMV Rs. 194

18. EMV: 3200, decide Act A_3, expand 400 units

19. (i) maximin : Act a_3

(ii) minimax regret Act a_1

20. So the shop keeper should stock 8 dozen of fruits to get maximum expected profit.

21. He should stock 3 dozen of flowers to get maximum expected net profit.

22. He stocks 3 dozen of flowers to expect maximum profit Rs. 9.50

23. To stock 405 copies so that his expected profit is maximum

24. (i) Act a_3 is recommended

(ii) Act a_1 is recommended

(iii) Act a_3 is recommended

25. EMA for A is highest. So the best act A is selected.

Lesson Writer

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Lesson - 3 Linear and Quadratic Equations

3.1 Objective of the lesson:

After studying this lesson, you should be able to understand -

→ Linear equation, identities.

→ To solve quadratic equations.

→ Nature of the roots.

→ To form an equation.

3.2 Structure:

This lesson has the following components:

3.3 Introduction:

Any two algebraic expressions connected by the sign of equality (=) form an equation.

Thus $3x + 1 = x + 5$; $x - 1 = 2x + 3$ are all equations.

A number which satisfies an equation is called the solution of the equation. A solution is also called root.

3.4 Linear Equation:

Def: An equation of the form $ax + b = 0$, $(a \neq 0)$ is called a linear equation or simple equation or first degree equation in x.

Here, the highest power of x (the unknown) is one, a and b are constant numbers, also a is known as a co-efficient of x.

Thus $2x + 5 = 0$, $4x = 0$, $\frac{2}{3}x + 7 = 3$ are the examples of linear equations.

3.4.1 Solution of the linear equations:

The solution of the linear equation $ax + b = 0$, $a \neq 0$ is $\frac{-b}{a}$

Note: A linear equation has one root only.

3.4.2 Rules for solving linear equations:

In solving the linear equation, the following steps are usually used.

Step 1: Clear the fractions by multiplying both sides of the equation, by the L.C.M. of denominators.

Step 2: Transpose all terms containing the unknown to the left side and remaining terms to the right side. Combine the terms of both sides.

Step 3: Divide both sides of the equation by the co-efficient of the unknown.

This gives the required root.

3.4.3 Example 1: Solve $\frac{2(x-4)}{3} = 5 - \frac{x+1}{6}$

Solution: To clear the fractions multiply both sides of the given equation by L.C.M of denominators 3, 6 ie 6 (L.C.M. of 3, 6 = 6) we get

$$\overset{2}{\cancel{6}} \times \frac{2(x-4)}{\cancel{3}} = 6 \times 5 - \cancel{6} \times \frac{(x+1)}{\cancel{6}}$$

$\Rightarrow 4(x-4) = 30 - (x+1)$

$\Rightarrow 4x - 16 = 30 - x - 1$

$\Rightarrow 4x + x = 30 - 1 + 16$ (transposing)

$\Rightarrow 5x = 45$

$\Rightarrow x = \frac{45}{5} = 9$ Which is the requined solution

3.4.4 **Example:** Solve $\frac{2x-3}{2x-1} = \frac{3x-1}{3x+5}$

Solution: Cross multiplying we find

$(2x-3)(3x+5) = (3x-1)(2x-1)$

$\Rightarrow 6x^2 + 10x - 9x - 15 = 6x^2 - 3x - 2x + 1$

$\Rightarrow x - 15 = -5x + 1$

$\Rightarrow x + 5x = 1 + 15$

$\Rightarrow 6x = 16$

$\Rightarrow x = \frac{16}{6} = \frac{8}{3}$

$x = \frac{8}{3}$ which is the required solution

3.4.5 **Example:** Solve $\frac{3}{x-2} + \frac{2}{x-3} = \frac{5}{x}$

Solution: $\frac{3}{x-2} + \frac{2}{x-3} = \frac{5}{x} \Rightarrow \frac{3(x-3)+2(x-2)}{(x-2)(x-3)} = \frac{5}{x}$

$\Rightarrow \frac{3x-9+2x-4}{x^2-5x+6} = \frac{5}{x}$

$$\Rightarrow \frac{5x - 13}{x^2 - 5x + 6} = \frac{5}{x} \Rightarrow x\,(5x - 13) = 5\left(x^2 - 5x + 6\right)$$

$$\Rightarrow 5x^2 - 13x = 5x^2 - 25x + 30 \Rightarrow 25x - 13x = 30$$

$$\Rightarrow 12x = 30 \Rightarrow x = \frac{30}{12} \Rightarrow x = \frac{5}{2} \Rightarrow x = 2\frac{1}{2}$$

Which is the required solution

3.4.6 Example: Solve $\frac{x-6}{5} + \frac{x-4}{3} = 8 - \frac{x-2}{7}$, find p if $x - p = 1$

Solution: $\frac{x-6}{5} + \frac{x-4}{3} = 8 - \frac{x-2}{7}$

$$\Rightarrow \frac{3(x-6) + 5(x-4)}{15} = \frac{7 \times 8 - (x-2)}{7}$$

$$\Rightarrow \frac{3x - 18 + 5x - 20}{15} = \frac{56 - x + 2}{7}$$

$$\Rightarrow \frac{8x - 38}{15} = \frac{58 - x}{7}$$

$$\Rightarrow 7\,(8x - 38) = 15\,(58 - x)$$

$$\Rightarrow 56x - 266 = 870 - 15x$$

$$\Rightarrow 56x + 15x = 870 + 266$$

$$\Rightarrow 71x = 1136$$

$$\Rightarrow x = \frac{1136}{71}$$

$$\Rightarrow x = 16$$

putting x = 16 is x - p = 1 we get

$$16 - p = 1 \Rightarrow -p = 1 - 16$$

$\Rightarrow -p = -15 \Rightarrow p = 15$ which is the required solution.

3.5 Problems based on equation:

According to given statement, equation is to be framed and hence to solve. For this one variable is to be used.

Steps: (i) Read the given problem carefully.

(ii) Represent the quantity to be calculated by the variable x (or y)

(iii) Form an equation in terms of x according to given problem.

(iv) Solve the equation.

3.6 Problems regarding members:

3.6.1 Example: Three times a number diminished by three is equal to 30. Find the number.

Solution: Let the number be x. According to problem we have

$$3x - 3 = 30 \Rightarrow 3x = 30 + 3$$

$$\Rightarrow 3x = 33$$

$$\Rightarrow x = \frac{33}{3}$$

$$\Rightarrow x = 11$$

$\therefore$ required number = 11

3.6.2 Example: A number consists of two digits, the digit in the unit's place is thrice that in the ten's place and if 1 is added to the sum of the digits, the addition is equal to one third of the number. Find the number.

Let x = the digit in the ten's place.

Then 3 x = The digit in the units place

Clearly the number is 10 x + 3 x

By condition of the problem, $(x + 3x + 1) = \frac{1}{3}(10x + 3x)$

$$\Rightarrow 4x + 1 = \frac{1}{3}(13x)$$

$$\Rightarrow 3(4x + 1) = 13x$$

$$\Rightarrow 12x + 3 = 13x$$

$\Rightarrow 12x - 13x = -3$

$\Rightarrow -x = -3$

$\Rightarrow x = 3$

The required number is $10x + 3x = 10(3) + 3(3) = 39$

3.7 Problems regarding ages:

3.7.1 Example: The age of a father and his son together are 90. IF the age of the son is doubled, it will exceed the father's age by 15 years. Find age of each.

Solution: Let father's age = x years then son's age = 90 - x

By condiiton, $2(90 - x) = x + 15$

$\Rightarrow 180 - 2x = x + 15$

$\Rightarrow -2x - x = 15 - 180$

$\Rightarrow -3x = -165$

$\Rightarrow 3x = 165 \Rightarrow x = \frac{165}{3} = 55$

$\Rightarrow x = 55$

$\therefore$ Father's age = 55 years

Son's age = 90 - x = 90 - 55 = 35 years

3.7.2 Example: A father's age is eight times that of his son. After 10 years the father's age will be three times that of his son. Find their present ages.

Solution: Let son's age = x years, then fathers age = 8 x years.

After 10 years, Son's age = (x + 10) years

Father's age = (8 x + 10) years

By condition, 8 x + 10 = 3 (x + 10)

$\Rightarrow 8x + 10 = 3x + 30$

$\Rightarrow 8x - 3x = 30 - 10$

$\Rightarrow 5x = 20$

$\Rightarrow x = 4$

$\therefore$ Son's present age = 4 years

Father's present age = 8 x = 8(4) = 32 years.

3.8 Problems regarding geometry:

3.8.1 Example: The length of rectangle is 4 cm more than the breadh. The perimeter is 11 cm more than the breadth. Find the length and breadth of the rectangle.

Solution: Let length = x cm, The breadth = (x - 4) cm

$\therefore$ Perimeter = 2 (length + breadth)

$$= 2\ (x + x - 4) = 2(2x - 4) = (4x - 8)\ \text{cm}$$

It is given that perimeter = breadth + 11 cm

$$\therefore\ 4x - 8 = (x - 4) + 11$$

$$\Rightarrow\ 4x - 8 = x - 4 + 11 \Rightarrow\ 4x - x = -4 + 11 + 8$$

$$\Rightarrow 3x = 15 \Rightarrow\ x = \frac{15}{3} = 5$$

Hence, length of the rectangle = 5 cm and breadth = 5 - 4 = 1 cm.

3.8.2 Example: Equal sides of an isoceless triangle are (3 x + 1) and 4 x - 3. IFthe third side is 2x. Find x and also the perimeter of the triangle.

Solution: In isoceless triangle two sides are equal

By condition we have, 3 x + 1 = 4 x - 3

$$\Rightarrow 3x - 4x = -3 - 1$$

$$\Rightarrow\ -x = -4$$

$$\Rightarrow x = 4$$

Perimeter $= 3x + 1 + 4x - 3 + 2x = 9x - 2$

For x = 4, Perimeter = 9(4) - 2 = 34 units.

3.9 Problems regarding costs:

3.9.1 Example: By selling a car for Rs. 72,000 a person made a profit of 20%. What was the cost price of the car?

Solution: Let the cost price of the car = Rs. x

Profit = 20% of Rs $x = \text{Rs}\ \frac{20}{100} \times x = \text{Rs}\ \frac{x}{5}$

$\therefore$ S.P. of the car $= \text{Rs}\left(x + \frac{x}{5}\right) = \frac{6x}{5}$

According to the given problem, we have

S.P of car = Rs. 72,000

$$\Rightarrow \frac{6x}{5} = 72,000$$

$$\Rightarrow 6x = 72,000 \times 5$$

$$\Rightarrow x = \frac{72,000 \times 5}{6} = 12,000 \times 5 = 60,000$$

Hence, the C.P. of the car = Rs. 60,000

3.9.2 Example: On the occasion of Diwali, Khadi Bhandar allowed discount of 20% on all textiles and 25% on readymade garments. Hari paid Rs. 180 for a gown., What was the marked price of the gown?

Solution: Let the marked price of the gown = Rs. x

Discount on all textiles @ 20 % $= \frac{20}{100} \times x = \frac{x}{5}$

$\therefore$ Cost Price after first discount $= \text{Rs.}\left(x - \frac{x}{5}\right) = \text{Rs}\ \frac{4x}{5}$

Discount on ready made garments @ 25% of $\frac{4x}{5}$

$$= \text{Rs.}\ \frac{25}{100} \times \frac{4x}{5} = \text{Rs.}\ \frac{x}{5}$$

$\therefore$ C.P. of the gown after second discount = Rs. $\left(\frac{4x}{5} - \frac{x}{5}\right)$

$$= \text{Rs}\ \frac{3x}{5}$$

According the given problem we have

$$\frac{3x}{5} = 180 \Rightarrow 3x = 180 \times 5$$

$$\Rightarrow x \frac{180 \times 5}{3} = 60 \times 5 = 300$$

Hence, marked price of gown = Rs. 300

3.10 Quadratic Equations (Q.E.):

Def: An equation with one variable, in which the highest power of the variable is two, is known as a quadratc equation.

The standard form of a quadratic equation is $a x^2 + b x + c = 0$ where a, b and c are all real numbers and $a \neq 0$.

For Example:

$4x^2 + 5x - 6 = 0$ is a quadratic equation in standard form.

3.10.1 Def: Every quadrtic equation gives two values of the unknown variable and these values are called roots of the equation.

3.10.2 Zero Product Rule: Whenever the product of two expression is zero ; atleast one of the expression is zero.

If $(x + 3)(x - 2) = 0$ then $x + 3 = 0$ or $x - 2 = 0$

$$\Rightarrow x = -3 \text{ or } x = 2$$

3.11 Solving (Q.E.) by factorization:

Steps: (i) Clear all fractions and brackets if necessary.

(ii) Transpose all the terms to the left hand side to get an equation in the form $a x^2 + b x + c = 0$

(iii) Factorise the expression on the left hand side.

(iv) Put each factor equal to zero and solve

3.11.1 Example: Solve $2x^2 - 7x = 39$

Solution: $2x^2 - 7x = 39 \Rightarrow 2x^2 - 7x - 39 = 0$

$\Rightarrow 2x^2 - 13x + 6x - 39 = 0$ (Factorising the lefthand side)

$\Rightarrow x(2x - 13) + 3(2x - 13) = 0$

$\Rightarrow (2x - 13)(x + 3) = 0$

$\Rightarrow 2x - 13 = 0$ or $x + 3 = 0$ (Zero product rule)

$\Rightarrow 2x = 13$ or $x = -3$

$\Rightarrow x = \frac{13}{2}$ or $x = -3$

3.11.2 Example: Solve $x^2 = 5x$

Solution: $x^2 = 5x \Rightarrow x^2 - 5x = 0$

$\Rightarrow x(x - 5) = 0$

$\Rightarrow x = 0$ or $x - 5 = 0$

$\Rightarrow x = 0$ or $x = 5$

3.11.3 Example: Solve $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$

Solution: $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$

$\Rightarrow \frac{x^2 + (x-1)^2}{x(x-1)} = \frac{5}{2}$

$\Rightarrow 2\left[x^2 + x^2 - 2x + 1\right] = 5x(x-1)$

$\Rightarrow 2(2x^2 - 2x + 1) = 5x^2 - 5x$

$\Rightarrow 4x^2 - 4x + 2 = 5x^2 - 5x$

$\Rightarrow -x^2 + x + 2 = 0$

$\Rightarrow x^2 - x - 2 = 0$ (Changing the sign of each term)

$\Rightarrow x^2 - 2x + x - 2 = 0$

$\Rightarrow x(x-2)+1(x-2)=0$

$\Rightarrow (x-2)\ (x+1)=0$ (on tactorising)

$\Rightarrow x-2=0$ or $x+1=0$

$\Rightarrow x=2$ or $x=-1$

3.11.4 Example: Find the quadratic equation whose solution set is $\{-2,\ 3\}$.

Solution: Since, solution set $=\{-2,\ 3\}$

$\Rightarrow x=-2$ or $x=3$

$\Rightarrow x+2=0$ or $x-3=0$

$\Rightarrow (x+2)(x-3)=0$

$\Rightarrow x^2-3x+2x-6=0$

$\Rightarrow x^2-x-6=0$

Which is the required quadratic equation.

3.12 Solving the quadratic equation using general method:

Let the quadratic equation be $ax^2+bx+c=0$, $a\neq 0$, $ax^2+bx=-c\cdot$

Multiplying both sides by 4a we get

$$4a^2x^2+4abx=-4ac$$

Adding b^2 on both sides, we get

$$4a^2x^2+4abx+b^2=b^2-4ac$$

$$\Rightarrow (2ax+b)^2=b^2-4ac$$

$$\Rightarrow 2ax+b=\pm\sqrt{b^2-4ac}$$

$$\Rightarrow 2ax = -b \pm \sqrt{b^2 - 4ac}$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.12.1 Formula: The roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

If α , β are the roots of the quadratic equation $ax^2 + bx + c = 0$ then

$$\alpha , \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum of the roots $= \alpha + \beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

$$\boxed{\text{sum of the roots} = \alpha + \beta = \frac{-b}{a} = \frac{-\text{coffi} \cdot \text{of } x}{\text{coffi} \cdot \text{of } x^2}}$$

Product of the roots $= \alpha\beta = \left(\dfrac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

$$\text{Product of the roots} = \alpha\beta = \frac{c}{a} = \frac{\text{constnat}}{\text{coffi of } x^2}$$

Formation of an equation:

If α, β are the roots of an equation are given. Then the equation can be written as

$$x^2 - (\text{sum of the roots})\, x + \text{Product of the roots} = 0$$

i.e. $$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

3.12.2 Example: Solve $x^2 - 7x + 12 = 0$

Solution: Comparing $x^2 - 7x + 12 = 0$ with $ax^2 + bx + c = 0$ we get

$a = 1$, $b = -7$, $c = 12 \cdot$

and so, $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \cdot 1 \cdot 12}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49 - 48}}{2} = \frac{7 \pm 1}{2} = \frac{7+1}{2}, \frac{7-1}{2}$$

$$= \frac{8}{2}, \frac{6}{2} = 4, 3 \cdot$$

3.12.3 Example: Solve $\sqrt{3}\, x^2 + 11x + 6\sqrt{3} = 0$

Solution: Hence $a = \sqrt{3}$, $b = 11$, $c = 6\sqrt{3}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-11 \pm \sqrt{11^2 - 4 \cdot \sqrt{3} \cdot 6\sqrt{3}}}{2 \cdot \sqrt{3}} = \frac{-11 \pm \sqrt{49}}{2\sqrt{3}}$$

$$= \frac{-11 \pm 7}{2\sqrt{3}} = \frac{-11+7}{2\sqrt{3}}, \frac{-11-7}{2\sqrt{3}} = \frac{-4}{2\sqrt{3}}, \frac{-18}{2\sqrt{3}}$$

$$= \frac{-2}{\sqrt{3}}, \frac{-9}{\sqrt{3}} = \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}, \frac{-9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \quad \text{(Rationalizing)}$$

$$= \frac{-2\sqrt{3}}{3}, -3\sqrt{3}$$

3.12.4 S.A.Q.: Form the quadratic equation whose roots are 2, 3.

3.13 Equations reducible to quadratics:

There are various types of equations, not quadratic in form, which can be reduced to quadratic forms by suitable transformation, as following examples.

3.13.1 Example: Solve $x^4 - 10x^2 + 9 = 0$

Solution: Taking $x^2 = u$, we find $u^2 - 10u + 9 = 0$

$$\Rightarrow u^2 - 9u - u + 9 = 0$$

$$\Rightarrow u(u-9) - 1(u-9) = 0$$

$$\Rightarrow (u-9)(u-1) = 0$$

$$\Rightarrow u - 9 = 0, u - 1 = 0$$

$$\Rightarrow u = 9 \text{ or } u = 1$$

when $u = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

when $u = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

3.13.2 Example: Solve $(1+x)^{1/3} + (1-x)^{1/3} = 2^{1/3}$

Solution: Given equation is

$$(1+x)^{1/3} + (1-x)^{1/3} = 2^{1/3}$$

$$\Rightarrow (1+x) + (1-x) + 3(1+x)^{1/3}(1-x)^{1/3}\left[(1+x)^{1/3} + (1-x)^{1/3}\right] = \left(2^{1/3}\right)^3$$

(cubing on both sides)

$\Rightarrow\ 2 + 3\left(1 - x^2\right)^{1/3} \cdot 2^{1/3} = 2$

$\Rightarrow 3\left(1 - x^2\right)^{1/3}\ 2^{1/3} = 0$

$\Rightarrow \left(1 - x^2\right)^{1/3} = 0$

$\Rightarrow \left(1 - x^2\right) = 0$

$\Rightarrow\ x^2 = 1$

$\Rightarrow x = \sqrt{1} = \pm 1$

3.13.3 Example: Solve $4^x - 3\ 2^{x+2} + 2^5 = 0$

Solition: Given equation is $4^x - 3\ 2^{x+2} + 2^5 = 0$

$\Rightarrow 2^{2x} - 3 \cdot 2^x \cdot 2^2 + 32 = 0$

$\Rightarrow \left(2^x\right)^2 - 12\ 2^x + 32 = 0$

putting $2^x = u$ then

$u^2 - 12u + 32 = 0 \Rightarrow u^2 - 8u - 4u + 32 = 0$

$\Rightarrow u\ (4 - 8) - 4(u - 8) = 0 \Rightarrow (u - 8)\ (u - 4) = 0$

$\Rightarrow u - 8 = 0$ or $u - 4 = 0 \Rightarrow u = 8\ ,\ 4$

when $u = 8 \Rightarrow 2^x = 2^3 \Rightarrow x = 3$

when $u = 4 \Rightarrow 2^x\ \ 2^2 \Rightarrow x = 2$

3.13.4 Example: Solve $2\left(x + \frac{1}{x}\right)^2 - 7\left(x + \frac{1}{x}\right) + 5 = 0$

Solution: Let $x + \frac{1}{x} = u$ then the equation is

$$2u^2 - 7u + 5 = 0 \Rightarrow 2u^2 - 5u - 2u + 5 = 0$$

$$\Rightarrow u(2u-5) - 1(2u-5) = 0$$

$$\Rightarrow (2u-5)(u-1) = 0$$

$$\Rightarrow 2u - 5 = 0 \quad \text{or} \quad u - 1 = 0$$

$$\Rightarrow u = \frac{5}{2} \quad \text{or} \quad u = 1$$

when $u = \frac{5}{2} \Rightarrow x + \frac{1}{x} = \frac{5}{2} \Rightarrow \frac{x^2+1}{x} = \frac{5}{2}$

$$\Rightarrow 2x^2 + 2 = 5x$$

$$\Rightarrow 2x^2 - 5x + 2 = 0$$

$$\Rightarrow 2x^2 - 4x - x + 2 = 0$$

$$\Rightarrow 2x(x-2) - 1(x-2) = 0$$

$$\Rightarrow (x-2)(2x-1) = 0$$

$$\Rightarrow x - 2 = 0 \quad \text{or} \quad 2x - 1 = 0$$

$$\Rightarrow x = 2 \text{ or } x = \frac{1}{2}$$

when $u = 1 \Rightarrow x + \frac{1}{x} = 1 \Rightarrow \frac{x^2+1}{x} = 1 \Rightarrow x^2 + 1 = x$

$$\Rightarrow x^2 - x + 1 = 0$$

$$\Rightarrow x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot1\cdot1}}{2(1)}$$ here a = 1 , b = - 1 , c = 1.

$= \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$, There values are rejected as they are not real

$$\therefore x = 2, \frac{1}{2}.$$

3.14 Discriminant and nature of the roots:

Def: For the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$; the expression $b^2 - 4ac$ is called discriminant and is, in general denoted by the letter Δ.

Thus, discriminant $\Delta = b^2 - 4ac$

3.14.1 To examine the nature of the roots: Examine the roots of a quadratic equation means to see the type of its roots i.e. whether they are real or imaginary, rational or irrational, equal or unequal.

The nature of the roots of a quadratic equation depends entirely on the value of its disriminant $b^2 - 4ac \cdot$

Case (1): When $b^2 - 4ac > 0$

$b^2 - 4ac > 0 \Rightarrow b^2 - 4ac$ is positive then the roots are real and un equal.

More over:

(i) If $b^2 - 4ac$ is a perfect square and b is irrational then the roots are irrational and unequal.

(ii) If $b^2 - 4ac$ is a perfect square and a, b and c are rational then, the roots are rational and un equal.

(iii) If $b^2 - 4ac$ is not a perfect square, then the roots are irrational and unequal.

Case (2): When $b^2 - 4ac = 0$

If $b^2 - 4ac = 0$, then the roots are real and equal and the root is $\frac{-b}{2a}$

Case (3): When $b^2 - 4ac < 0$

If $b^2 - 4ac < 0$ i.e. $b^2 - 4ac$ is negative, then the roots are not real i.e. the roots are imaginary.

3.14.2 Example: Without solving, examine the nature of the roots of the equations.

(i) $5x^2 - 6x + 7 = 0$ (ii) $x^2 + 6x + 9 = 0$ (iii) $2x^2 + 6x + 3 = 0$ (iv) $3x^2 - 5x + 2 = 0$

Solution: (i) For the equation $5x^2 - 6x + 7 = 0$

$a = 5$, $b = -6$, $c = 7$.

$\therefore$ Discriminant $\Delta = b^2 - 4ac = (-6)^2 - 4 \cdot (5) \cdot (7) = 36 - 140 = -104$

$\Rightarrow$ Δ is negative

$\Rightarrow$ the roots are not real. i.e. the roots are imaginary.

(ii) For the equation $x^2 + 6x + 9 = 0$, $a = 1$, $b = 6$, $c = 9$.

Discriminant $\Delta = b^2 - 4ac = b^2 - 4 \cdot 1 \cdot 9 = 36 - 36 = 0$

$\Rightarrow$ the roots are real and equal.

(iii) For the equation $2x^2 + 6x + 3 = 0$, $a = 2$, $b = 6$, $c = 3$.

Discriminant $\Delta = b^2 - 4ac = 6^2 - 4 \cdot 2 \cdot 3 = 36 - 24 = 12$

$\Delta = 12$ which is positive

$\Rightarrow$ the roots are irrational and un equal.

(iv) For the equation $3x^2 - 5x + 2 = 0$, $a = 3$, $b = -5$, $c = 2$.

Discriminant, $\Delta = b^2 - 4ac = (-5)^2 - 4 \cdot 3 \cdot 2 = 25 - 24 = 1$

$\Delta = 1$ which is a perfect square.

$\Rightarrow$ the roots are rational and un equal.

3.14.3 Example: For what value of m will the equation.

$(m+1)x^2 + 2(m+3)x + (2m+3) = 0$ have equal roots.

Solution: Since the discriminant for equal roots is zero. We have

$\Delta = b^2 - 4ac = 0$

$\Rightarrow 4(m+3)^2 - 4(m+1)(2m+3) = 0$

$\Rightarrow m^2 - m - 6 = 0$

$\Rightarrow m = 3, -2$.

3.14.4 Symmetrical Expressions: An expression in α and β is said to be symmetrical if it remains unchanged by interchange of α and β. Thus $\alpha+\beta$ becomes $\beta+\alpha$ by inter change of α and β. There fore $\alpha+\beta$ is symmetric in α and β.

Similarly $\alpha^2+\beta^2$, $\alpha^3+\beta^3$, $\alpha\beta$, $\frac{1}{\alpha}+\frac{1}{\beta}$, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$, $\cdots$ are symmetric in α and β.

Note: $\alpha-\beta$, $\alpha^3+\beta^2$, $\alpha^2-\alpha\beta+\beta^2$ are not symmetric in α and β.

3.14.5 Example: If α and β are the roots of the equation $ax^2+bx+c=0$ find the value of

(i) $\alpha-\beta$, (ii) $\alpha^2+\beta^2$, (iii) $\alpha^4\beta^7+\alpha^7\beta^4$, (iv) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$ and (v) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.

Solution: Since α,β are the roots of equation $ax^2+bx+c=0$,

We have $\alpha+\beta=\frac{-b}{a}$, $\alpha\beta=\frac{c}{a}$

(i) $$\alpha-\beta=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\sqrt{\left(\frac{-b}{a}\right)^2-4\cdot\frac{c}{a}}=\frac{\sqrt{b^2-4ac}}{a}$$

(ii) $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{-b}{a}\right)^2-2\frac{c}{a}=\frac{b^2-2ac}{a^2}$$

(iii) $$\alpha^4\beta^7+\alpha^7\beta^4=\alpha^4\beta^4\left(\alpha^3+\beta^3\right)$$

$$=(\alpha\beta)^4\cdot(\alpha+\beta)\left(\alpha^2+\beta^2-\alpha\beta\right)$$

$$=(\alpha\beta)^4\cdot(\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right]$$

$$=\left(\frac{c}{a}\right)^4\cdot\left(\frac{-b}{a}\right)\left[\left(\frac{-b}{a}\right)^2-3\cdot\frac{c}{a}\right]$$

$$= \frac{-b c^4}{a^5}\left[\frac{b^2 - 3ac}{a^2}\right]$$

$$= \frac{b c^4}{a^7}\left(3 a c - b^2\right)$$

(iv) $$\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2 = \left(\frac{\alpha^2 - \beta^2}{\alpha \beta}\right)^2 = \frac{(\alpha+\beta)^2 (\alpha-\beta)^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 \left[(\alpha+\beta)^2 - 4\alpha\beta\right]}{(\alpha\beta)^2}$$

$$= \frac{\left(\frac{-b}{a}\right)^2 \left[\left(\frac{-b}{a}\right)^2 - 4 \cdot \frac{c}{a}\right]}{\left(\frac{c}{a}\right)^2}$$

$$= \frac{b^2}{c^2}\left(\frac{b^2}{a^2} - \frac{4c}{a}\right)$$

$$= \frac{b^2}{c^2 a^2}\left(b^2 - 4ac\right)$$

(v) $$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\alpha^3 + \beta^3}{\alpha^3 \beta^3} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)}$$

$$= \frac{\left(\frac{-b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)}{\left(\frac{c}{a}\right)^3}$$

$$= \frac{\left(\frac{-b^3}{a^3} + \frac{3bc}{a^2}\right)}{c^3/a^3} = \left(\frac{-b^3 + 3abc}{\cancel{a^3}}\right) \cdot \frac{\cancel{a^3}}{c^3} = \frac{3abc - b^3}{c^3}$$

3.14.6 Example: Form the equation whose roots are (i) 6, 7 (ii) $5 \pm \sqrt{3}$

Solution: (i) The quadratic equation whose roots are 6, 7 is

$$x^2 - (6 + 7)\,x + 6 \cdot 7 = 0 \Rightarrow x^2 - 13\,x + 42 = 0$$

(ii) The quadratic equation whose roots are $5 + \sqrt{3}$, $5 - \sqrt{3}$ is

$$x^2 - \left(5 + \sqrt{3} + 5 - \sqrt{3}\right)x + \left(5 + \sqrt{3}\right)\left(5 - \sqrt{3}\right) = 0$$

i.e. $x^2 - 10\,x + 22 = 0$

3.14.7 Example: (a) If α , β are the roots of $x^2 + p\,x + q = 0$. Find the equation whose roots are $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$

(b) If α , β are the roots of the equation $x^2 - p\,x + q = 0$, find the equation whose roots are α^2 , β^2 .

Solution: (a) Since α , β are the roots of $x^2 + p\,x + q = 0$

$\therefore\ \alpha + \beta = -p,\ \alpha\,\beta = q$

sum of the roots $= \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\,\beta^2} = \frac{(\alpha + \beta)^2 - 2\,\alpha\,\beta}{(\alpha\,\beta)^2} = \frac{(-p)^2 - 2\,q}{q^2}$

$= \frac{p^2 - 2q}{q^2}$

Product of the roots $= \frac{1}{\alpha^2} \times \frac{1}{\beta^2} = \frac{1}{(\alpha\,\beta)^2} = \frac{1}{q^2}$

$\therefore$ The equation whose rotos are $\frac{1}{\alpha^2}$, $\frac{1}{\beta^2}$ is

$$x^2 - \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) x + \frac{1}{\alpha^2} \times \frac{1}{\beta^2} = 0$$

$$x^2 - \left(\frac{p^2 - 2q}{q^2}\right) x + \frac{1}{q^2} = 0$$

$$\Rightarrow q^2 x^2 - \left(p^2 - 2q\right) x + 1 = 0$$

(b) Since α , β are the roots of $x^2 - p x + q = 0$

$\therefore \alpha + \beta = p \; , \; \alpha \beta = q \cdot$

Sum of the roots $= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2 \alpha \beta = p^2 - 2 q$

Product of the roots $= \alpha^2 \beta^2 = (\alpha \beta)^2 = q^2$

$\therefore$ The equation whose roots are $\alpha^2 \; , \; \beta^2$ is $x^2 - \left(\alpha^2 + \beta^2\right) x + \alpha^2 \beta^2 = 0$

$\Rightarrow x^2 - \left(p^2 - 2q\right) x + q^2 = 0$

3.14.8 S.A.Q.: Find condition that one root of $a x^2 + b x + c = 0$ shall be n times the other.

3.14.9 S.A.Q.: Find K if the roots of $2 x^2 + 3x + k = 0$ are equal.

3.14.10 Example: If the roots of the equation $a x^2 + b x + c = 0$ may be in the ratio $m : n$, prove that $m n b^2 = a c (m + n)^2$.

Solution: Since the roots of the equation are in the ratio $m : n$. they can be taken as $m \alpha \; , n \alpha \cdot$

Then sum of the roots $= m \alpha + n \alpha = \frac{-b}{a} \Rightarrow (m + n) \alpha = \frac{-b}{a} \quad \cdots (1)$

Product of the roots $= m\alpha \cdot n \alpha = \frac{c}{a} \Rightarrow m n \alpha^2 = \frac{c}{a} \quad \cdots (2)$

The required condition can be obtained by eliminating α between (1) and (2).

From (1) $\alpha = \frac{-b}{a (m + n)}$

From (2) $\alpha^2 = \frac{c}{a\,m\,n}$

$$\Rightarrow \left[\frac{-b}{a\,(m+n)}\right]^2 = \frac{c}{a\,m\,n}$$

$$\Rightarrow \frac{b^2}{a^2\,(m+n)^2} = \frac{c}{a\,m\,n}$$

$$\Rightarrow m\,n\,b^2 = a\,c\,(m+n)^2$$

3.15 Answers to SAQ:

3.15.1 Solution of S.A.Q. 3.12.4:

The quadratic equation with roots 2, 3 is $x^2 - (2+3)x + 2 \times 3 = 0$

i.e. $x^2 - 5x + 6 = 0$

3.15.2 Solution of S.A.Q. 3.14.8:

Let one root of the equation be α. Then other will be $n\,\alpha$.

Sum of the roots: $\alpha + n\,\alpha = \frac{-b}{a} \Rightarrow \alpha\,(1+n) = \frac{-b}{a} \quad \cdots (1)$

Product of the roots: $(\alpha)(n\,\alpha) = \frac{c}{a} \Rightarrow n\,\alpha^2 = \frac{c}{a} \quad \cdots (2)$

Eliminating α from (1) & (2), the required condition is

$$\frac{b^2\,n}{a^2\,(1+n)^2} = \frac{c}{a} \Rightarrow b^2\,n = a\,c\,(1+n)^2$$

3.15.2 Solution of S.A.Q. 3.14.9:

Give equation is $2x^2 + 3x + k = 0$

Comparing to given equation with $a x^2 + b\,x + c = 0$ we get

$a = 2$, $b = 3$, $c = k \cdot$

Since the roots are equal $\therefore$ discriminant $\Delta = b^2 - 4ac = 0$

$$\Rightarrow 3^2 - 4(2)(k) = 0$$

$$\Rightarrow 9 - 8k = 0 \Rightarrow k = \frac{9}{8}\cdot$$

3.16 Summary:

In this lesson we discussed the various methods solving linear equations and quadratic equation.

Some of the important points are

1. The quadratic equation is of the form $ax^2 + bx + c = 0, a \neq 0, b, c \in \mathbb{Z}\cdot$

2. $ax^2 + bx + c = 0$ has two roots namely $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3. If α, β are the roots of the equation $ax^2 + bx + c = 0$ then $\alpha + \beta = \dfrac{-b}{a}, \alpha\beta = \dfrac{c}{a}$

4. $\Delta = b^2 - 4ac$ in the discriminant of $ax^2 + bx + c = 0\cdot$

5. Nature of the roots

   (i) If $\Delta > 0$ then $ax^2 + bx + c = 0$ has two real and distinct roots.

   (ii) If $\Delta = 0$ then $ax^2 + bx + c = 0$ has equal roots that are real.

   (iii) If $\Delta < 0$ then $ax^2 + bx + c = 0$ has imaginary roots.

6. The quadratic equation with roots α, β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0\cdot$

3.17 Technical Terms:

Linear Equation

Solution or Root

Quadratic Equation

Discriminant Δ

Symmetrical

3.18 Exercise:

Solve the following:

1. **(i)** $\dfrac{3}{x-2} + \dfrac{5}{x-6} = \dfrac{8}{x+3}$

(ii) $4x - \dfrac{x-1}{3} = x + \dfrac{2(x-1)}{5} + 3$

(iii) $6\dfrac{1}{3} - \dfrac{x-7}{3} = \dfrac{4x-2}{5}$

(iv) $\dfrac{x}{b} + \dfrac{b}{x} = \dfrac{a}{b} + \dfrac{b}{a}$

2. **(i)** $25x^2 = 16$ **(ii)** $\dfrac{x}{x+2} = \dfrac{x+3}{5(x+11)}$ **(iii)** $x^2 - 2\sqrt{3}\,x + 1 = 0$

3. **(i)** $\dfrac{x+2}{x-2} + \dfrac{x+3}{x-3} = \dfrac{x-2}{x+2} + \dfrac{x-3}{x+3}$ **(ii)** $\dfrac{x-p}{q} + \dfrac{x-q}{p} = \dfrac{q}{x-p} + \dfrac{p}{x-q}$

4. **(i)** $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$ **(ii)** $\sqrt{\dfrac{x}{1-x}} + \sqrt{\dfrac{1-x}{x}} = 2\dfrac{1}{6}$

5. Solve $\dfrac{x + \sqrt{12a - x}}{x - \sqrt{12a - x}} = \dfrac{\sqrt{a} + 1}{\sqrt{a} - 1}$

6. Find the value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}\ \infty$

7. If α, β are the roots of $2x^2 + 3x + 7\ \ 0$, find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\alpha^3 + \beta^3$ (iii) $\alpha^4 + \beta^4$ (iv) $\alpha\beta^{-1} + \beta\alpha^{-1}$

(v) $\left(\alpha^2 - \beta\right)^2 + \left(\beta^2 - \alpha\right)^2$ (vi) $\alpha^3 - \beta^3$

8. Form the quadratic equation whose roots are

(i) $4 + i\sqrt{2}\ ,\ 4 - i\sqrt{2}$ (ii) $p + \sqrt{q}\ ,\ p - \sqrt{q}$

9. If α, β are the roots of $x^2 - 2x + 3 = 0$ from the quadratic equations whose roots are

(i) $\alpha + 3, \ \beta + 3$ (ii) $2\alpha - 3\beta, \ 3\alpha - 2\beta$ (iii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (iv) $\frac{\alpha - 1}{\alpha + 1}, \frac{\beta - 1}{\beta + 1}$

10. If p, q are the roots of the equation $3x^2 + 6x + 2 = 0$ show that the equation whose roots are

$\frac{-p^2}{q}$ and $\frac{-q^2}{p}$ is $3x^2 - 18x + 2 = 0$

3.19 Answers to exercise:

1) (i) 3 (ii) 1 (iii) 8 (iv) $\frac{b^2}{a}$

2) (i) $\pm \frac{4}{5}$ (ii) $\frac{-25 \pm \sqrt{649}}{4}$ (iii) $\sqrt{3} \pm \sqrt{2}$

3) (i) $0, \pm\sqrt{6}$ (ii) $\frac{p^2 + q^2}{p + q}, \ 0, \ p + q$.

4) (i) $3 \pm 2\sqrt{3}, 5, 1$. (ii) $\frac{4}{13}, \frac{9}{13}$

5) $x = 3a$ or $-4a$

6) 3, - 2.

7) (i) $\frac{-19}{4}$ (ii) $\frac{99}{8}$ (iii) $\frac{-31}{16}$ (iv) $\frac{-19}{14}$ (v) $\frac{61}{16}$ (vi) $\frac{-5}{8}\sqrt{-47}$

8) (i) $x^2 - 8x + 18 = 0$ (ii) $x^2 - 2px + p^2 - q = 0$

9) (iii) $3x^2 + 2x + 3 = 0$ (iv) $3x^2 - 2x + 1 = 0$

3.20 Model questions:

1. If the roots of the equation $a x^2 + b x + c = 0$ be in the ratio $p : q$ prove that $a c (p + q)^2 = b^2 p q \cdot$

2. If the roots of $x^2 - p x + q = 0$ are α and β, prove that

(i) $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{p^3}{q^3} - 3 \frac{p}{q^2}$ (ii) $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - 4 \frac{p^2}{q} + 2$

3. Find K is

(i) The roots of $2x^2 + 3x + K = 0$ are equal.

(ii) One of the roots of the euation $x^2 - 6x + K = 0$ is $3 + i\sqrt{2}$

4. Solve $\sqrt{1 - 5x} + \sqrt{1 - 3x} = 2$

5. Solve $\left(x - \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 12 = 0$

3.21 References:

1. D.C. Sancheti, V.K. Kapoor : Business Mathematics, Sultan Chand & Sons, New Delhi.
2. S. SAHA : Business Mathematics, New Central Book Agency, Calcutta.

Lesson Writer

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Lesson-4 Arithmetic - Geometric and Harmonic Progressionss

4.1 Objective of the lesson:

After studying this lesson, you should be able to understand -

→ Arthmatic progession, its sum, arithmetic mean and its application in solving problems.

→ Gemetric mean and its applications in solving problems.

→ Harmonic mean and its application in solving problems.

→ Relation between A.M, G.M. and H.M.

4.2 Structure:

This lesson has the following components:

4.3 Introduction:

In this lesson we shall discuss two special types of series with sequences increasing or decreasing by an absolute quantity or a certain ratio designated as arthemetic, geometric and harmonic progressions respectively.

4.4 Arithmetic Progression (A.P.):

A.P. is a sequence of quantities, when the algebraic difference between any term and the preceding one is constant. This algebraic difference is known as Common Difference (C.D.). Thus the series

(i) 1, 3, 5, 7, ..., (ii) 2, 0, - 2, - 4, ... are in A.P.

Here C.D. n (i) is 2 (as 3 - 1 = 5 - 3 = 7 - 5 = ...) and C.D. in (ii) is

- 2 (as 0 - 2 = - 2 - 0 = - 4 - (- 2) = ...)

It is clear that in A.P. each term is formed from the preceding term by adding to it a constant qunatity (i.e. C.D.)

$$\boxed{C \cdot D \cdot = \text{Any term} - \text{its preceding term}}$$

4.4.1 The general form A.P. and Arithmetic Seris:

The arithemetic progression $a, a + d, a + 2d, a + 3d, \cdots$

Whose first term is a and the common difference is d, is designated as the standard form of an arithmetic progression.

The corresponding arithimetic series

$$a + (a + d) + (a + 2d) + (a + 3d) + \cdots$$

is designated as the standard form of Arithmetic Series.

4.4.2 The nth term of an A.P.:

Let a be the first term and d be the common difference. Then

First Term $t_1 = a$

Second Term $t_2 = a + d$

Third Term $t_3 = a + 2d$

----

n^{th} Term $\boxed{t_n = a + (n-1)d}$

4.4.3 Example: Which term of the series 12 + 9 + 6 + ...

is equal to (i) - 30 (ii) - 100 ?

Solution: The series is in A.P.

First Term a = 12

Common difference d = 9 - 12 = - 3

n^{th} term $t_n = a + (n-1)d = 12 + (n-1)(-3)$

$= 12 - 3n + 3$

$= 15 - 3n$

(i) Suppose n^{th} term is - 30. Then

$15 - 3n = -30 \Rightarrow n = 15$

(ii) Suppose n^{th} term is - 100. Then

$15 - 3n = -100 \Rightarrow n = \frac{115}{3}$

Which is impossible, because n must be a whole number. Hence there exists no term in the series which is equal to - 100.

4.4.4 S.A.Q.: Find the n^{th} term of the series. 3, 8, 13, 18, ...

4.4.5 Example: If a, b, c are the p^{th}, q^{th} and r^{th} terms of A.P.

Show that $a(q-r) + b(r-p) + c(p-q) = 0$

Solution: Let the A.P. A, A + D, A + 2 D + ... then

$a = A + (p-1)D$

$b = A + (q-1)D$

$c = A + (r-1)D$

$L \cdot H \cdot S \cdot = a(q-r) + b(r-p) + c(p-q)$

$= [A + (p-1)D](q-r) + [A + (q-1)D](r-p) + [A + (r-1)D](p-q)$

$= A[q - r + r - p + p - q] + D[(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)]$

$= A \times 0 + D \times 0 = 0 = R \cdot H \cdot S \cdot$

4.4.6 Sum of n terms of an A.P.:

Let a be the first term, d the common difference and ℓ be the n^{th} term (last term) of an A.P.

To find the sum S_n of the first n terms of the A.P. we have

$$S_n = a + (a + d) + (a + 2d) + \cdots + (\ell - d) + \ell$$

$$S_n = \ell + (\ell - d) + (\ell - 2d) + \cdots + (a + d) + a \quad \text{(reverse order)}$$

Adding $2 S_n = \underbrace{(a + \ell) + (a + \ell) + (a + \ell) + \cdots + (a + \ell) + (a + \ell)}_{\text{n terms}}$

$$2 S_n = n (a + \ell)$$

$$\Rightarrow \boxed{S_n = \frac{n}{2}(a + \alpha)} \quad \text{i.e. } S_n = \frac{n}{2}[\text{first term} + \text{last term}]$$

But $\ell = \text{last term} = n^{th} \text{ term} = a + (n-1) d$

$$\therefore S_n = \frac{n}{2}(a + a + (n-1) d)$$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1) d]}$$

4.4.7 Properties of an A.P.:

If a, b, c, d, ... are in A.P. and x is a constant quantity, then the following series

(i) $a + x, b + x, c + x, d + x, \cdots$ are in A.P.

(ii) $a - x, b - x, c - x, d - x, \cdots$ are in A.P.

(iii) $ax, bx, cx, dx, \cdots$ are in A.P.

(iv) $\frac{a}{x}, \frac{b}{x}, \frac{c}{x}, \frac{d}{x}, \cdots$ are in A.P.

4.4.8 Arithmetic Mean:

When three quantities are in A.P. the middle one is called the arthmetic mean (A.M.) of the other two. If a, m, b are in A.P. Then m is the A.M. of a and b., and we have

$$m - a = b - m \Rightarrow 2m = a + b \Rightarrow m = \frac{a + b}{2}$$

Hence, A.M. of a and b is equal to $\frac{a + b}{2}$

4.4.9 Def: If $a, m_1, m_2, \cdots, m_n, b$ be in A.P., then the intermediate numbers $m_1, m_2, \cdots, m_n$ are callen the n arithmetic means between a and b.

4.4.10 To insert n arithmetic means between a and b:

If $a, m_1, m_2, \cdots, m_n, b$ be the n arithmetic means to be inserted between a and b, then we have an A.P. consisting of (n + 2) terms of which a is the first term and b is the $(n+2)^{th}$ term, then

$$b = a + (n + 2 - 1)\, d$$

$$b = a + (n+1)\, d \Rightarrow d = \frac{b-a}{n+1}$$

Hence the required n arithmetic means are

$$a + d, a + 2d, a + 3d, \cdots, a + nd \cdot$$

i.e., $a + \frac{b-a}{n+1}, a + \frac{2(b-a)}{n+1}, a + \frac{3(b-a)}{n+1}, \cdots, a + \frac{n(b-a)}{n+1}$

4.4.11 Example: Show that

(i) $1 + 2 + 3 + \cdots$ to n terms $= \frac{n(n+1)}{2}$

(ii) $1 + 3 + 5 + \cdots$ to n terms $= n^2$

(iii) $2 + 4 + 6 + \cdots$ to n terms $= n(n+1)$

Solution: (i) For sum to n terms of an A.P.

$$S_n = \frac{n}{2}\left[2a + (n-1)\, d\right]$$

Here, $a = 1, d = 1, n = n$

$$\therefore S_n = \frac{n}{2}\left[2 \cdot 1 + (n-1) 1\right] = \frac{n}{2}(2 + n - 1) \quad \frac{n(n+1)}{2}$$

(ii) Here $a = 1, b = 2, n = n$

$$S_n = \frac{n}{2}\left[2 \cdot 1 + (n-1) \cdot 2\right]$$

$= \frac{n}{2}[2 + 2n - 2] = n^2$

(iii) Here $a = 2$, $d = 2$, $n = n$

$\therefore S_n = \frac{n}{2}[2 \cdot 2 + (n-1)2]$

$= \frac{n}{2}[4 + 2n - 2] = n(n+1)$

4.4.12Example: Insert 4 arithmetic means between 5 and 20.

Solution: Let m_1, m_2, m_3, m_4 be the 4 arithmetic means between 5 and 20.

Then $5, m_1, m_2, m_3, m_4, 20$ are in A.P.

Here $a = 5$ and 6^{th} term $= 20$.

Let d be the common difference. Then

$20 = 6^{th}$ term $= a + (6-1)d = a + 5d$

$\Rightarrow 20 = 5 + 5d$

$\Rightarrow 20 - 5 = 5d \Rightarrow d = \frac{15}{5} = 3$

The required 4 arithmetic means are $a + d$, $a + 2d$, $a + 3d$, $a + 4d$.

i.e. $m_1 = a + d = 5 + 3 = 8$

$m_2 = a + 2d = 5 + 2(3) = 11$

$m_3 = a + 3d = 5 + 3(3) = 14$

$m_4 = a + 4d = 5 + 4(3) = 17$

Hence, the required 4 arithmetic means between 5 and 20 are 8, 11, 14, 17.

4.4.13 Example: Find the sum of the following series.

(i) $1 + 4 + 7 + 10 + \cdots$ to 40 terms.

(ii) $2 + 7 + 12 + 17 + \cdots + 102$.

Solution: (i) We have $S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]$

Here a = 1, d = 3, n = 40.

$\therefore\; S_{40} = \frac{40}{2}\left[2(1) + (40-1)\,3\right] = 20(2+117)\;\; 2380$

(ii) Let n^{th} term = 102 then

$a + (n-1)\,d = 102$

$\Rightarrow 2 + (n-1)\,5 = 102 \qquad (\because \text{ Here } a = 2,\ d = 5\cdot)$

$\Rightarrow 2 + 5n - 5 = 102$

$\Rightarrow 5n = 105 \Rightarrow n = 21$

$\therefore S_n = S_{21} = \frac{21}{2}\left[2(2) + (21-1)\,5\right] \qquad (\because a = 2,\ d = 5,\ n = 21\cdot)$

$$= \frac{21}{2}(4+100) = \frac{21}{2}(104) = 21 \times 52 = 1092$$

4.4.14 **Example:** Find the sum of all natural numbers between 200 and 400. Which are divisible by 7.

Solution: The natural numbers between 200 and 400 which are divisible by 7 are 203, 210, 217, ..., 399. They form an A.P.

Here a = 203, d = 210 - 203 = 7.

n^{th} term = 399

$\Rightarrow a + (n-1)\,d = 399 \Rightarrow 203 + (n-1)\,7 = 399$

$$\Rightarrow (n-1) = \frac{399 - 203}{7} = 28$$

$$\therefore\; n = 28 + 1 = 29\cdot$$

Now sum of all the numbers between 200 and 400 which are divisible by 7 is

$S_n = \frac{n}{2}\left[2a + (n-1)\,d\right]$

Here a = 203, d = 7 , n = 29.

$$S_{29} = \frac{29}{2}\left[2(203) + (29-1)7\right]$$

$$\Rightarrow \frac{29}{2}\left(406 + (28)7\right) = 8729 \cdot$$

4.4.15 Example: Find the sum of natural numbers from 1 to 200 excluding those divisible by 5.

Solution: The required sum

$$= (1 + 2 + 3 + \cdots + 200) - (5 + 10 + 15 + \cdots + 200) = S_1 - S_2$$

Where $S_1 = 1 + 2 + 3 + \cdots + 200 = \frac{200}{2}(1 + 200) = 20100$

$S_2 = 5 + 10 + \cdots + 200$

Since, the last term = 200, here a = 5, d = 5, $t_n = 200 \cdot$

$$\Rightarrow a + (n-1)d = 200$$

$$\Rightarrow 5 + (n-1)5 = 200 \Rightarrow 5n = 200$$

$$\Rightarrow n = 40$$

Hence $S_2 = 5 + 10 + \cdots$ to 40 terms

$$= \frac{40}{2}\left[2(5) + (40-1)5\right] = 20(10 + 195) = 4100$$

$\therefore$ The required sum $= S_1 - S_2 = 20{,}100 - 4{,}100 = 16{,}000 \cdot$

4.4.16 Example: If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. prove that a^2, b^2, c^2 are also in A.P.

Solution: Since, $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\therefore (c+a)(a+b), (b+c)(a+b), (b+c)(c+a)$ are also in A.P.

(by multiplying each term by (a + b) (b + c) (c + a))

$$\Rightarrow a^2 + (ab + bc + ca), b^2 + (ab + bc + ca), c^2 + (ab + bc + ca)$$

$\Rightarrow a^2, b^2, c^2$ are A.P. (by subtracting (ab + bc + ca) from each term)

4.4.17 Example: A man saved Rs. 16,500 in ten years. In each year after the first he saved Rs. 100 more than he did in the preceding year. How much did he saved in the first year?

Solution: Here a = savings in the first year = ?

n = number of years = 10, d = 100, Sn = 16,500.

Now, $Sn = \frac{n}{2}\left[2a + (n-1)\,d\right]$

$\therefore\ 16,500 = \frac{10}{2}\left[\,2a + (10-1)\,100\right]$

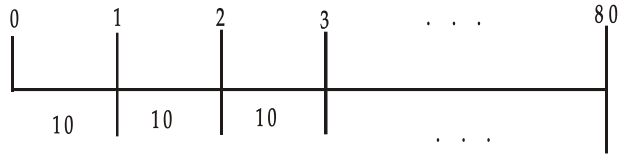
$16,500 = 5(2s + 900) \Rightarrow 2a + 900 = \frac{16500}{5} = 3300$

$\Rightarrow\ 2a = 3300 - 900 = 2400$

$\Rightarrow a = \frac{2400}{2} = 1200$

4.4.18 **Example:** 80 coins are placed in staright line on the ground. The distance between any two consecutive coins is 10 meters. How far must a person travel to bring them one by one to a basket placed 10 meters behind the first coin?

Solution:0



Let 1, 2, 3, ... , 80 represent the positions ofthe coins and 0 that of the basket.

The distance covered in bringing the first coin = 10 + 10 = 20

The distance covered in bringing the second coind = 20 + 20 = 40

The distance covered in bringing the third coin = 30 + 30 = 60

and so on.

$\therefore$ 20, 40, 60, ... are in A.P. Here a = 20, d = 20, n = 80.

$\therefore$ The total distance covered $= \frac{n}{2}\left[2a + (n-1)\,d\right]$

$= \frac{80}{2}\left[2\,(20) + (80-1)\,20\right]$

$= 40\,(40 + 1580) = 64,800$ meters

4.5 Geometric Progression (G.P.):

A geometric progression (G.P.) is a sequence of numbers such that the first term is non - zero, and the ratio of any term to the preceeding one is constant. This ratio is known as common ratio (C.R.) of G.P.

From the above definition of G.P., it follows that no term of a G.P. can be zero.

The following sequence of numbers

1, 3, 9, 27, . . . are in G.P. here, $\frac{3}{1}=\frac{9}{3}=\frac{27}{9}= \cdots$ and C.R. = 3.

4.5.1 General Expresentation of G.P.: The general representation of G.P. is

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}, ar^n, \cdots$$

where a = first term, r = common ratio.

4.5.2 n^{th} term of a G.P.:

Fromt he general representation of a G.P. $a, ar, ar^2, ar^3, \cdots$

The first term = a , The second term = a r

The third term = ar^2 , The fourth term = ar^3

The n^{th} term $(t_n) = ar^{n-1}$

i.e. $\boxed{t_n = ar^{n-1}}$ $\cdots\cdots\cdots\cdots\cdots\cdots$ (1)

The succesive terms $t_1, t_2, t_3, \cdots$ can be obtained by putting n = 1, 2, 3, ... is (1)

4.5.3 S.A.Q.: Find the 12^{th} term of the series 1, 2, 4, 8, 16, ...

4.5.4 Sum of series in G.P.: The sum of first n terms in G.P. is denoted by S_n. Then

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad \cdots\cdots\cdots\cdots (1)$$

$$r\,S_n = ar + ar^2 + ar^3 + \cdots + ar^n \quad \cdots\cdots\cdots\cdots\cdots\cdots (2)$$ (multiplying (1) by r on both sides)

subtracting (2) from (1) i.e. (1) - (2) $\Rightarrow$

$$S_n - r S_n = a - a r^n$$

$$\Rightarrow S_n (1 - r) = a (1 - r^n)$$

$$\Rightarrow S_n = \frac{a(1 - r^n)}{1 - r} \qquad \text{if } r < 1$$

$$= \frac{a(r^n - 1)}{r - 1} \qquad \text{if } r > 1$$

Note: 1) If $r = 1$, then $S = a + a + a + \ldots\ldots$ to n terms $= n a$

2) If ℓ denotes the last term of G.P. then $\ell = a r^{n-1}$ and $S = \frac{a - r\ell}{1 - r} = \frac{r\ell - a}{r - 1}$

4.5.5 S.A.Q.: Find sum of the first 10 terms of the series $1 + 3 + 9 + 27 + \ldots\ldots\ldots$

Gemetric mean (G.M.): If three quantities are in G.P. then the middle term is known as the Geometric Mean (G.M.) of the other.

Thus if a, G, b are in G.P.

$$\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \pm\sqrt{ab}$$

$$\boxed{\therefore \text{ G·M· of } a \text{ and } b = \pm\sqrt{ab}}$$

Note: A·M· $\geq$ G·M·

4.5.6 Properties of G.P.:

(1) If every term of a G.P. is multiplied by a fixed real number, then the resulting series is also a G.P.

(2) If every term of a G.P. is raised to the same power, then the resulting series is also a G.P.

(3) The reciprocals of the terms of a G.P. are also in G.P.

(4) If every term of a G.P. is positive, then the logarithms of terms of the G.P. are in A.P.

4.5.7 To insert n geometric means between a and b:

If $m_1, m_2, \cdots, m_n$ be the n G.M.'s which are to be inserted between a and b, then $a, m_1, m_2, \cdots, m_n, b$ are in G.P.

Here a is the first term and b is the (n + 2)th term.

If r is the common ratio, then $b = a\, r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Hence, required n geometric means are $ar,\ ar^2, ar^3, \cdots, ar^n$.

i.e. $m_i = ar^i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}}$ for i = 1, 2, ..., n.

4.5.8 Example: Three numbers whose sum is 15 are in A.P. If 1, 4 and 19 are added to them respectively, the results are in G.P. find the numbers.

Solution:

Let the three numbers in A.P. be a - d, a , a + d so that $(a - d) + a + (a + d) = 15$

$a = 5$

Also we are given $(a - d + 1), (a + 4), (a + d + 19)$ are in G..P.

$\therefore\ (a - d + 1)(a + d + 19) = (a + 4)^2$

$\Rightarrow (6 - d)(24 + d) = 81 \qquad (\because a = 5)$

$\Rightarrow d^2 + 18d - 63 = 0$

$\Rightarrow (d - 3)(d + 21) = 0$

$\therefore d = 3 \text{ or } -21$

Hence the numbers 2, 5, 8 (or) 26, 5, - 16.

4.5.9 Example: If a, b, c are the p^{th}, q^{th} and r^{th} terms of a G.P., Prove that $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$

Solution: Let A be the first term and R the common ratio of the G.P.

Then $a = A \cdot R^{p-1} \longrightarrow (1)$

$b = A \cdot R^{q-1} \longrightarrow (2)$

$c = A \cdot R^{r-1} \longrightarrow (3)$

Rasing (1) to power q- r, (2) to power r - p (3) to power p - q and multiplying them together, we get

$$a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = A^{q-r} \cdot A^{r-p} \cdot A^{p-q} \cdot R^{(p-1)(q-r)} \cdot R^{(q-1)(r-p)} \cdot R^{(r-1)(p-q)}$$

$$= A^{(qr + r-p + p-q)} \; R^{0}$$

$$= A^{0} \; R^{0} = 1$$

4.5.10 Example:

If a, b, c are in A.P. and x, y, z are in G.P. Prove that $x^b y^c z^a = x^c y^a z^b$

Solution:

Let d be the common difference and r the common ratio of the given A.P. and G.P. respectively.

We thus have $b = a + d$, $c = a + 2d$ and $y = xr$, $z = xr^2$.

$$L \cdot H \cdot S = x^b y^c z^a = x^{a+d} (xr)^{a+2d} \left(xr^2\right)^a$$

$$= x^{3a + 3d} \cdot r^{3a + 2d}$$

$$R \cdot H \cdot S = x^c y^a z^b = x^{a + 2d} (x\,r)^a \left(xr^2\right)^{a + d}$$

$$= x^{3a + 3d} \cdot r^{3a + 2d}$$

$\therefore$ $L \cdot H \cdot S = R \cdot H \cdot S$ hence the result

4.5.11. Example: Find the sum of the series.

1 + 3 + 9 + 27 + ...+ to 10 terms.

Solution: Using the formula

$$S_n = \frac{a\left(r^n - 1\right)}{(r - 1)} \qquad \text{We have } S_{10} = \frac{3^{10} - 1}{3 - 1}$$

$$= \frac{59,049 - 1}{2} = 29,524$$

4.5.12 Example: Sum of the series $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots +$ to 10 terms

Solution: Using the formula $S_n = \frac{a\left(1 - r^n\right)}{(1 - r)}$ $\left(\because \; r = \frac{1}{2} < 1\right)$

$$S_{10} = \frac{4\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{\left(1 - \frac{1}{2}\right)}$$

$$= \frac{4\left(1 - \frac{1}{1024}\right)}{\frac{1}{2}} = \frac{8 \times 1023}{1024} = 8 \text{ (approx)}$$

4.5.12 Example: Find the sum of the series 243 + 324 + 432 + to n terms

Solution: The above series is in G.P. with $r = \frac{4}{3} > 1$, $a = 243$

$$\therefore S_n = \frac{a\left(r^n - 1\right)}{(r - 1)} = \frac{243\left(\left(\frac{4}{3}\right)^n - 1\right)}{\frac{4}{3} - 1}$$

$$= 243 \times \frac{3\left(4^n - 3^n\right)}{3^n}$$

$$= \frac{3^6\left(4^n - 3^n\right)}{3^n} = 3^{6-n}\left(4^n - 3^n\right)$$

4.5.13 Example: Prove that the sum to n terms of the series

$$11 + 103 + 1005 + \cdots \text{ is } \frac{10}{9}\left(10^n - 1\right) + n^2$$

Solution: Let $S_n = 11 + 103 + 1005 + \cdots$

$$= (10 + 1) + \left(10^2 + 3\right) + \left(10^3 + 5\right) + \cdots$$

$$= \left(10 + 10^2 + 10^3 + \cdots + 10^n\right) + \left(1 + 3 + 5 + \cdots + (2n - 1)\right)$$

In the above series the first bracket is in G.P and the second bracket is in A.P.

$$\therefore S_n = \frac{10\left(10^n - 1\right)}{10 - 1} + \frac{n}{2}\left\{2 + (n - 1)\, 2\right\}$$

$$= \frac{10}{9}\left(10^n - 1\right) + \frac{n}{2}\left\{ \not{2} + 2n - \not{2} \right\}$$

$$= \frac{10}{9}\left(10^n - 1\right) + n^2$$

4.5.14 Example: Find the sum to n terms of the series

(a) $x\,(x + y) + x^2\left(x^2 + y^2\right) + x^3\left(x^3 + y^3\right) + \cdots$

(b) $(x + y) + \left(x^2 + xy + y^2\right) + \left(x^3 + x^2 y + xy^2 + y^3\right) + \cdots$

Solution:

Series (a) can be split into two parts, each of which is a G.P. to n terms

$$S_n = \left(x^2 + xy\right) + \left(x^4 + x^2 y^2\right) + \left(x^6 + x^3 y^3\right) + \cdots \text{ to n terms}$$

$$= \left(x^2 + x^4 + x^6 + \cdots \text{to n terms}\right) + \left(xy + x^2y^2 + x^3y^3 + \cdots \text{to n terms}\right)$$

$$= \frac{x^2\left(1 - \left(x^2\right)^n\right)}{1 - x^2} + \frac{xy\left(1 - (xy)^n\right)}{1 - xy}$$

$$= \frac{x^2\left(1 - x^{2n}\right)}{1 - x^2} + \frac{xy\left(1 - x^n y^n\right)}{1 - xy}$$

(b) Let $S_n = (x + y) + \left(x^2 + xy + y^2\right) + \left(x^3 + x^2 y + xy^2 + y^3\right) + \cdots \text{to n terms}$

Multiplying both sides by (x - y). We get

$$(x - y)\,S_n = \left(x^2 - y^2\right) + \left(x^3 - y^3\right) + \left(x^4 - y^4\right) + \cdots \text{ to 'n' terms}$$

$$= \left(x^2 + x^3 + x^4 + \cdots + \text{to n terms}\right) - \left(y^2 + y^3 + y^4 + \cdots \text{to n terms}\right)$$

$$= \frac{x^2\left(1 - x^n\right)}{1 - x} - \frac{y^2\left(1 - y^n\right)}{1 - y}$$

$$S_n = \frac{1}{x - y}\left\{\frac{x^2\left(1 - x^n\right)}{1 - x} - \frac{y^2\left(1 - y^n\right)}{1 - y}\right\}$$

4.5.15 **Example:** Sum to n terms the series

(a) 7 + 77 + 777 + ... (b) 0.7 + 0.77 + 0.777 + ...

Solution: We have $S_n = 7 + 77 + 777 + \cdots \text{ to n terms}$

$$= 7\left(1 + 11 + 111 + \cdots \text{ to n terms}\right)$$

$$= \frac{7}{9}\left(9 + 99 + 999 + \cdots \text{ to n terms}\right)$$

$$= \frac{7}{9}\left((10-1) + (100-1) + (1000-1) + \cdots\right)$$

$$= \frac{7}{9}\left[\left(10 + 10^2 + 10^3 + \cdots \text{ to n terms}\right) - (1 + 1 + \cdots \text{ to n terms})\right]$$

$$= \frac{7}{9}\left[\frac{10\left(10^n - 1\right)}{10 - 1} - n\right]$$

$$= \frac{7}{9}\left[\frac{10^{n+1} - 10}{9} - n\right] = \frac{7\left(10^{n+1} - 10\right)}{81} \qquad \frac{7}{9}n$$

(b) $S_n = 0\cdot7 + 0\cdot77 + 0\cdot777 + \cdots \text{ to n terms}$

$$= 7\left[0\cdot1 + 0\cdot11 + 0\cdot111 + \cdots \text{ to n terms}\right]$$

$$= \frac{7}{9}\left[0\cdot9 + 0\cdot99 + 0\cdot999 + \cdots \text{ to n terms}\right]$$

$$= \frac{7}{9}\left[(1-0\cdot1) + (1-0\cdot01) + (1-0\cdot001) + \cdots \text{ to n terms}\right]$$

$$= \frac{7}{9}\left[n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots \text{ to n terms}\right)\right]$$

$$= \frac{7}{9}\left[n - \frac{1}{10}\,\frac{\left(1 - \left(\frac{1}{10}\right)^n\right)}{\left(1 - \frac{1}{10}\right)}\right]$$

$$= \frac{7}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$$

4.5.16 Example: Find the least value of n for which the sum $1 + 3 + 3^2 + \cdots$ to n terms is greater than 7000.

Solution: Sum to n terms $= 1 + 3 + 3^2 + \cdots + 3^{n-1} = \dfrac{3^n - 1}{3 - 1} = \dfrac{3^n - 1}{2}$

This sum to n terms will be greater than 7000 . If $\dfrac{3^n - 1}{2} > 7000$

If $3^n - 1 > 14000$

If $3^n > 14001$

If $n \log 3 > \log 14001$

If $n > \dfrac{\log 14001}{\log 3}$

If $n > \dfrac{4 \cdot 1461}{0 \cdot 4771} = 8 \cdot 69\ (\text{APP})$

Hence the least value of n is 9.

4.5.17 **Example:** The sum of 3 numbers in G.P. is 35 and their product is 1000 find the numbers.

Solution: Let $\dfrac{a}{r}, a, ar$ be the three numbers in G.P.

The product of these numbers $= \dfrac{a}{r} \cdot a \cdot ar = 1000$

$\Rightarrow a^3 = 1000$

$\Rightarrow a = 10$

The sum of the three numbers is

$\dfrac{a}{r} + a + ar = 35$

$a\left(\dfrac{1}{r} + 1 + r\right) = 35$

$$10\left(1 + r + r^2\right) = 35\,r$$

$$2 + 2r + 2r^2 = 7r$$

$$2r^2 - 5r + 2 = 0$$

$$(2r - 1)(r - 2) = 0$$

$$r = \frac{1}{2} \text{ or } 2$$

Thus we have $a = 10$ and $r = \frac{1}{2}$ or 2

for $r = \frac{1}{2}$ the numbers will be 10 x 2, 10 , $10 \times \frac{1}{2}$ i.e. 20, 10, 5.

for $r = 2$ the numbers will be $\frac{10}{2}$, 10 , 10×2 i.e, 5, 10, 20.

Hence the three numbers 5, 10, 20.

4.5.18 **Example:** Insert 5 geometric means between 350 and 5.

Solution: We have in all 7 terms of which the first term is 320 and the 7th term is 5

Therefore using the formula $r^{n+1} = \frac{\ell}{a}$

we have $r^6 = \frac{1}{64}$

$$r = \frac{1}{6\sqrt{64}} \;[= \frac{1}{\left(2^6\right)^{1/6}} = \frac{1}{2}$$

$r = \frac{1}{2}$ which is the common ratio

Therefore the series is 320, 160, 80, 40, 20, 10 and 5 and the geometric means are 160, 80, 40, 20 and 10.

4.5.19 Example: At 10% per annum compound interest, a sum of money accumulates to Rs. 8750 in 4 years. Find the sum invested initially.

Solution: Let P be the principle then amount of P after 1 year $= P\left(1+\frac{10}{100}\right) = P \times 1{\cdot}1$

Let P be the principal then amount of P after 2 years = $P \times (1 \cdot 1)^2$

Let P be the principal then amount of P after 3 years = $P \times (1 \cdot 1)^3$

Let P be the principal then amount of P after 4 years = $P \times (1 \cdot 1)^4$

$$\therefore \; P \times (1 \cdot 1)^4 = 8750$$

$$P = \frac{8750}{(1{\cdot}1)^4} = \frac{8750}{1{\cdot}4641} = 5976 \cdot 37$$

which is required principal.

4.5.20 **Example:** If the value of Fiat Car depreciated by 25 percent annually. What will be its estimated value at the end of 8 years. If its present value is Rs. 2048.

Solution:

Present value of Car = Rs. 2048

Value of car depreciated = 25% annually

If present value is 100, then value after one year = Rs. 75

If present value is 1, then value after one year $= \frac{75}{100}$

If present value is 2048, then value after one year $= \frac{75}{100} \times 2048 \Rightarrow 1536$ Rs·

$\therefore$ a = 1536

We also note that value at the end of 2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 6^{th}, 7^{th} and 8^{th} years form a G.P. with common ratio $r = \frac{75}{100} = \frac{3}{4}$

$\therefore$ Value at the end of eight years $= a r^{8-1}$

$$= a r^7 = 1536 \times \left(\frac{3}{4}\right)^7 \Rightarrow 205 \cdot 03$$

4.5.21 Example: For three consecultine months, a person deposits some amount of money on the first day of each month in small savings fund. These three successive amounts in the deposit, the total values of which is Rs. 65, form a G.P. If the two extneme amounts be multiplied each by 3 and the mean by 5, the products form an A.P. Find the amounts in the first and second deposits.

Solution: Let the three successive deposits be Rs. a, Rs. ar and Rs. ar^2

Thus $a + ar + ar^2 = 65 \longrightarrow (1)$

Also $3a$, $5ar$ and $3ar^2$ form an A.P.

Thus $3a - 5ar = 5ar - 3ar^2$

$3ar^2 - 10ar + 3a = 0$

$\Rightarrow (r-3)(3r-1) = 0$

$\Rightarrow r = 3, \frac{1}{3}$

when r = 3, from (1) we get $a + 3a + 9a = 65 \Rightarrow a = 5$

Thus the amount are Rs. 5 , Rs. 15 and Rs. 45.

Again if $r = \frac{1}{3}$ then from (1) we get $a + \frac{a}{3} + \frac{a}{9} = 65 \Rightarrow a = 5$

Thus the successive deposits are Rs. 45, Rs. 15, Rs. 5.

Hence the amounts in the first and second deposits are either Rs. 5, Rs. 15 or Rs. 45, Rs. 15.

4.6 Harmonic Progression (H.P.):

Harmonic progression (H.P.) is a series of quantities called terms, such that their reciprocals are in A.P. Thus every harmonic series has a cooresponding A.P. series.

The following series are in G.P.

(1) $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots$ (as $2, 4, 6, \cdots$ are in $A \cdot P \cdot$)

(2) $\frac{-1}{7}, \frac{-1}{4}, -1, 2, \cdots$ (as $-7, -4, -1, 2, \cdots$ are in $A \cdot P \cdot$)

4.6.1 General Representation of H.P. Series:

A series of the form

$$\frac{1}{a} + \frac{1}{a+d} + \frac{1}{a+2d} + \cdots + \frac{1}{a+(n-1)d} + \cdots \text{ is a H.P.}$$

Here

(i) n^{th} term of H.P. $t_n = \dfrac{1}{a+(n-1)d}$

(ii) There is no formula to find the sum of the first n terms of the H.P.

4.6.2 Harmonic Mean (H.M.):

If n harmonic means (H.M.) are inserted between two non zero numbers a and b then

$a, t_1, t_2, \cdots, t_n, b$ are in H.P.

and k^{th} $H \cdot M = t_k = \dfrac{ab(n+1)}{b(n+1)+k(a-b)}$ for k = 1, 2, ..., n.

4.6.3 Remark: If A, G, H are A.M., G.M. and H.M. between two positive numbers a and b, then

(i) $A = \dfrac{a+b}{2}$, $G = \sqrt{ab}$, $H = \dfrac{2ab}{a+b}$.

(ii) A, G, H are in G.P. i.e. $G^2 = A \cdot H$

i.e. G is the mean proportional between A and H.

(iii) $A \geq G \geq H$

4.6.4 S.A.Q.: Prove that for any two real quantities.

$A \cdot M \cdot \geq G \cdot M \cdot \geq H \cdot M \cdot$

4.6.5 Example: Find the 5^{th} and n^{th} terms of the H.P. $4, 2, \frac{4}{3}, \cdots$

Solution: The reciprocals of the given series is

$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \cdots$ they are in A.P. so that

$d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, $a = \frac{1}{4}$

$$\therefore\ t_5 = a + (5-1)\,d = \frac{1}{4} + 4\cdot\frac{1}{4} = \frac{1}{4} + 1 = \frac{5}{4}$$

and $$t_n = a + (n-1)\,d = \frac{1}{4} + (n-1)\,\frac{1}{4} = \frac{n}{4}$$

So in H.P. 5^{th} term $= \frac{1}{t_5} = \frac{4}{5}$ and

$$n^{th} \text{ term} = \frac{1}{n/4} = \frac{4}{n}$$

4.6.6 Example: If the 7^{th} and the 10^{th} terms of an H.P. are respectively $\frac{2}{5}$ and $\frac{2}{7}$, find the 1^{st} and the n^{th} term of H.P.

Solution: Let the first term and the common difference of the corresponding A.P. be respectively a and d.

Clearly the 7^{th} and the 10^{th} terms of the A.P. are $\frac{5}{2}$ and $\frac{7}{2}$

$$\therefore\ a + 6d = \frac{5}{2} \longrightarrow (1)$$

$$a + 9d = \frac{7}{2} \longrightarrow (2)$$

(1) - (2) $\Rightarrow -3\,d = -1 \Rightarrow d = \frac{1}{3}$

from (1) $a = \frac{5}{2} - 6d = \frac{5}{2} - 6\left(\frac{1}{3}\right) = \frac{1}{2}$

n^{th} terms of the A.P. $= a + (n-1)d = \frac{1}{2} + (n-1)\,\frac{1}{3} = \frac{2n+1}{6}$

Hence the first term of the H.P. = 2 and the n^{th} term $= \frac{6}{2n+1}$

4.6.7 Example: Insert 4 harmonic means between 22 and 2.

Solution: Let h_1, h_2, h_3, h_4 be 4 harmonic means between 22 and 2.

Then 22, h_1, h_2, h_3, h_4, 2 are in H.P.

$\therefore \frac{1}{22}, \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \frac{1}{h_4}, \frac{1}{2}$ are in A.P.

Let d be the common difference of the A.P.

$\frac{1}{2} = 6^{th}$ term of the A.P. $= \frac{1}{22} + (6-1)d = \frac{1}{22} + 5d$ or $\frac{1}{2} - \frac{1}{22} = 5d$

$\Rightarrow 5d = \frac{10}{22} \Rightarrow d = \frac{10}{22} \cdot \frac{1}{5} = \frac{1}{11}$

$\therefore \frac{1}{h_1} = \frac{1}{22} + \frac{1}{11} = \frac{3}{22}$, $\frac{1}{h_2} = \frac{3}{22} + \frac{1}{11} = \frac{5}{22}$

$\frac{1}{h_3} = \frac{5}{22} + \frac{1}{11} = \frac{7}{22}$, $\frac{1}{h_4} = \frac{7}{22} + \frac{1}{11} = \frac{9}{22}$

Hence $h_1 = \frac{22}{3}$, $h_2 = \frac{22}{5}$, $h_3 = \frac{22}{7}$, $h_4 = \frac{22}{9}$.

4.7 Answers to S.A.Q.:

4.7.1 Solution of S.A.Q. 4.4.4:

The given seris is 3, 8, 13, 18, ...

It is a A.P. Hence, a = 3, d = 5.

n^{th} term of the series $A \cdot P \cdot = a + (n-1)d$

$$= 3 + (n-1)5$$

$$= 3 + 5n - 5$$

$$= 5n - 2$$

$$\therefore \boxed{t_n = 5n - 2}$$

4.7.2 Solution of S.A.Q. 4.5.3:

Given series is 1, 2, 4, 8, 16, ...

It is a G.P. Hence a = 1, r = 2.

n^{th} term $t_n = a r^{n-1} = 1 \cdot 2^{n-1} = 2^{n-1}$

$$t_n = 2^{n-1}$$

12^{th} term = $t_{12} = 2^{12-1} = 2^{11} = 2048$

4.7.3 Solution of S.A.Q. 4.5.5:

Given Series is 1 + 3 + 9 + 27 + ...

It is a G.P. Here a = 1, r = 3.

Sum to n term = $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$

Sum to 10 terms $= S_{10} = \dfrac{a\left(r^{10} - 1\right)}{r-1} = \dfrac{1\left(3^{10} - 1\right)}{3-1}$

$$= \frac{1}{2}\left(3^{10} - 1\right)$$

4.7.4 Solution of S.A.Q. 4.6.4:

Let A, G, H be the A.M., G.M. and H.M. between a and b then

$$A = \frac{a+b}{2}, \; G = \sqrt{ab}, \; H = \frac{2ab}{a+b} \cdot$$

$$A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab = G^2$$

Again, $A - G = \dfrac{a+b}{2} - \sqrt{ab} = \dfrac{a+b-2\sqrt{ab}}{2} = \dfrac{\left(\sqrt{a} + \sqrt{b}\right)^2}{2} \geq 0$

$\therefore$ $A \geq G$, equality occurs only when a = b.

Again, since $AH = G^2$ and $A \geq G$ $\therefore H \leq G \cdot$

Combining $A \geq G$ and $G \geq H$, we get $A \geq G \geq H$ $\cdot$

4.8 Summary:

In this lesson we discussed the A.P., G.P. and H.P. and their properties.

1) The arthemetic progression whose first term is a and common differene d is of the form

$a,\ a + d,\ a + 2d,\ a + 3d,\ ....$

2) The n^{th} term of an A.P. = $a + (n - 1) d$

3) Sum of first n terms in an A.P., $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

4) A.M. of a and b is equal to $\frac{a + b}{2}$

5) If $a, m_1, m_2, m_3, \cdots, m_n, b$ be the n arithemetic mean's to be inserted between a and b then

$m_r = a + r\left(\frac{b - a}{n + 1}\right)$ for $r = 1, 2, ..., n.$

6) The general representation of G.P. is $a, ar, ar^2, ar^3, \cdots, ar^{n-1}, ar^n, \cdots$

where a is the first term and r is the common ratio.

7) n^{th} term of $G \cdot P \cdot = ar^{n-1}$

8) Sum of first n terms of the G.P., $S_n = \frac{a(1 - r^n)}{1 - r}$ if $r < 1$

$$= \frac{a(r^n - 1)}{r - 1} \text{ if } r > 1$$

9) The G.M. of a and b is equal to $\sqrt{ab}$

10) A series of the form $\frac{1}{a} + \frac{1}{a + d}, \frac{1}{a + 2d} + \cdots + \frac{1}{a + (n-1)d} + \cdots$ is a H.P.

11) n^{th} term of $H \cdot P \cdot = \frac{1}{a+(n-1)d}$

12) $a_1, a_2, \cdots, a_n$ are in H.P. $\Leftrightarrow \frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n}$ are in A.P.

13) If A, G, H are A.M., G.M., H.M. between two positive numbers a and b then

(i) $A = \frac{a+b}{2}$, $G = \sqrt{a\ b}$, $H = \frac{2ab}{a+b}$

(ii) $G^2 = AH$ (iii) $A \geq G \geq H$

4.9 Technical Terms:

Arithmetic progression, Common difference, A.M.

Geometric progression , G.M.

Harmonic Progression, H.M.

4.10 Exercise:

Arithmetic Progression:

1. Find the 10^{th} and 15^{th} terms of the series 2, 6, 10, 14, ...

2. Find 120^{th} term of the progression 56, 53, 50, 47, ...

3. If the 5^{th} and the 12^{th} terms of an A.P. are 14 and 35 respectively, find the first term and the common difference.

4. Find the sum of the following series

   i) 2 + 4 + 6 + 8 + ... to n terms.

   ii) 1 + 4 + 7 + 10 + ... to 20 terms

5. If you save 1 P. to day, 2P the next day, 3p the succeeding day and so on, what will be your total savings in 365 days?

6. Insert 4 arithmetic means between 52 and 77.

7. The sum of three integers in A.P. is 15 and their product is 80, find them.

8. If the 3rd and the 6^{th} terms of A.P are 7 and 13 respectively, find the sum of the first 20 terms of the series.

9. Sum the first n terms of the series

i) 1.3 + 2.5 + 3.7 + ...

ii) 1.2.4 + 2.3.7 + 3.4.10 + ...

10. The sum of the three numbers in A.P. is 12 and the sum of their squares is 66. Find the numbers.

Geometric Mean:

11. Find the 10^{th} term of the series $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$

12. Find the sum of the following series:

(i) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ to 11 terms

(ii) $1 - 3 + 9 - 27 + \cdots$ to 9 terms

13. Insert 4 geometric means between 9 and 288

14. If the 4^{th} and the 8^{th} terms of a G.P. are 24 and 384 respectively, find the first term and the common ratio.

15. Find the three terms in G.P. whose sum is 14 and product is 64.

16. The sum of the firt eight terms of a G.P. is five times the sum of the first fourterms. Find the common ratio.

17. If a, b, c are in G.P. prove that $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are in A.P.

18. Find sum to n terms of the series 4 + 44 + 444 + ...

19. Find sum to n terms 0.3 + 0.33 + 0.333 + ...

20. Find the least value of n for which the sum $1 + 3 + 3^2 + 3^3 + \cdots$ to n terms is greater than 7000.

Harmonic Progression:

21. Find the 20^{th} term of the H.P. $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \cdots$

22. Insert 4 harmonic means between 20 and 5.

23. If a^2, b^2, c^2 are in A.P., Prove that b + c, c + a, a + b are in H.P.

24. If a, b, c are in H.P. prove that $\frac{a}{b+c-a}$, $\frac{b}{c+a-b}$, $\frac{c}{a+b-c}$ are in H.P.

25. Find two positive numbers such that their A.M. is 10 and G.M. is 8. Also find their H.P.

4.11 Answers to Exercise:

1) 38, 58 2) - 301 3) a = 2, d = 3

4) (i) n(n+1), (ii) 590 5) Rs. 667.95 6) 57, 62, 67, 72

7) 2, 5, 8 8) 440

9) (i) $\frac{1}{6} n (n+1)(4n+5)$ (ii) $\frac{1}{12} n (n+1)\left(9n^2+25n+4\right)$

10) 1, 4, 7 11) $\frac{1}{19683}$ 12) (i) $\frac{2047}{1024}$, (ii) 4921

13) 18, 36, 72, 144 14) a = 3, r = 2. 15) 2, 4, 8.

16) $\pm\sqrt{2}$, -1. 18) $\frac{4}{9}\left\{\frac{10}{9}\left(10^n-1\right)-n\right\}$ 19) $\frac{1}{3}\left[n-\frac{1}{9}\left\{1-(0\cdot1)^n\right\}\right]$

20) n = 9 21) $\frac{1}{58}$ 22) $\frac{9}{25}, \frac{11}{100}, \frac{7}{50}, \frac{17}{100}$.

25) 4, 16 or 16, 4 ; 6.4

4.12 Model Questions:

1. Insert 4 arithmetic means between 5 and 20.
2. Find the sum of all natural numbers between 200 and 400 which are divisible by 7.
3. Find sum of the series 243 + 324 + 432 + ... to n terms.
4. At 10 % per annum compound interest, a sum of money accumulater to Rs. 8750 in four years. Find the sum invested initially.

4.13 References:

1. D.C. Sancheti, V.K. Kapoor , "Business Mathematics", Sultan Chand and Sons, New Delhi.
2. S. SHAH, " Business Mathematics", NEw Central Book Agency, Calcutta.

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Lesson - 5 Permutations and Combinations Binomial Theorem

5.1 Objective of the lesson:

After studying this lesson, you should be able to understand:

→ Permutaions, factorial notations and problems involving permutations.

→ Combinations and problems involving combinations.

→ Difference between permutation and combination.

→ Binomial theorem, position of terms, binomial coefficients and its application.

→ Binomial theorem with any index and calculation of square root, cube root etc. and simplification.

→ Summation of series using binomial theorem.

5.2 Structure:

This lesson has the following components:

5.3 Introduction:

Permutations refer to different arrangements of things from a given lot taken one or more at a time where as combinations refer to different sets or groups made out of a given lot, with out repeating an element, taking one or more of them at a time. The distinction will be clear from the following illustration of combinations and permutations made out of a set of three elements $\{a, b, c\}$.

	Combinations	Permutations
(1)	One at a time : $\{a\}, \{b\}, \{c\}$	$\{a\}, \{b\}, \{c\}$
(2)	Two at a time: $\{a, b\}, \{b, c\}, \{a, c\}$	$\{a, b\}, \{b, a\}, \{b, c\}, \{c, b\}$, $\{a, c\}, \{c, a\}$
(3)	Three at a time $\{a, b, c\}$	$\{a, b, c\}$ $\{a, c, b\}$ $\{b, c, a\}$ $\{b, a, c\}$ $\{c, a, b\}$ $\{c, b, a\}$

It may be noticed that on the left side above every set has different combination where as on the right side above there all sets with different arrangements where ever possible of the same group. However no element appears twice in any set e.g. $\{a, a\}, \{b, b\}, \{a, b, b\}, \{a, a, a\}$ etc $\cdot$

5.3.1 Fundamental Rule of counting:

If one operation can be performed in 'm' different ways and corresponding to any one of such operations of a second operation can be performed in 'n' different ways then the total number of performingthe two operations is $m \times n \cdot$

This principle can be explained clearly by the following examples:

5.3.2 Example: There are five routes for journey from station A to station B. In how many different ways can a man go from A to B and return, If for returning

(i) Any of the routes is taken

(ii) Some route is taken

(iii) The same route is not taken

Solution:

(i) The man can go from A to B in 5 different ways for he may take any one of the five routes. When he has done so in any of the 5 ways, he may return in 5 different ways.

$\therefore$ The total number of different ways are $5 \times 5 = 25$

(ii) In case there is only one way of returning, then the total number of different ways are $5 \times 1 = 5$.

(iii) If there are 4 different ways of returning, then the total number of different ways are $5 \times 4 = 20$.

5.3.3 Example: How many telephone connections can be allotted with 5 and 6 digits from the natural numbers 1 to 9 inclusive.

Solution:

From the fundamental Rules of counting the total number of telephone cannections can be

$$N^r = 9^5 = 59,049 \quad \text{and} \quad N^6 = 9^6 = 5,31,441 \cdot$$

5.3.4 Factorial Notation:

The product of the first 'n' natural numbers 1, 2, ..., n is called factorial n (or) n factorial and is written as $\lfloor n$ (or) $n!$

Thus $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$

Thus $\lfloor 5 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$$\lfloor 7 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6)\, 7 = 7\,(\lfloor 6) = 7 \cdot 6 \cdot \lfloor 5$$

$$= 7 \cdot 6 \cdot 120 = 5040$$

Note: 1) $\boxed{\lfloor n \,:= n \lfloor n-1}$

2) $0! = 1$

5.4 Permutation:

Permutation refers to different arrangements of certain number of objects say r at a time taken from 'n' different objects.

Eg: A Book seller has recieved. Three new books A, B, C. He can place them in his show case in any of the following 6 ways.

ABC, ACB, BAC, BCA, CAB, CBA.

Thus we have 6 ways of arranging. Three distinct objects. When each arrangement is of all 3 objects.

Mathematically we can say that three distinct objects can be arranged in 3.2.1 = 6 ways.

We can reason out this as follows: "There are three places to be filled the first can be filled in 3 ways, the second in 2 ways while for the third in only 1 way".

Hence there are 3.2.1 ways in all.

5.4.1 Permutations of 'n' different things:

Permutations of n different things taken r at a time, where $r \le n$ are $n(n-1)(n-2)\cdots(n-r+1)$. It is denoted by ${}^{n}P_{r}$

Thus the number of permutations (arrangements) of n differnt hings taking 'r' at atime is given by ${}^{n}P_{r} = n(n-1)(n-2)\cdots(n-r+1)$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)(\lfloor n-r)}{\lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor n-r} \quad \text{or} \quad \frac{(n!)}{(n-r)!}$$

$$\therefore \boxed{{}^{n}P_{r} = \frac{\lfloor n}{\lfloor (n-r)}}$$

5.4.2 Remark:

1) The number of permutations of n different things taken all at a time is

$$n! = 1\cdot 2\cdot 3 \cdots (n-1)\,n = {}^{n}P_{n}$$

2) ${}^{n}P_{n-1} = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = {}^{n}P_{n}$

3) ${}^{n}P_{r} = n \times {}^{(n-1)}P_{r-1}$

4) We have ${}^{n}P_{n} = n!$

Also ${}^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{o\,!}$

$$n! = \frac{n!}{o\,!}$$

$$o\,! = \frac{n!}{n!} = 1$$

$$\therefore \boxed{o\,! = 1}$$

5.4.3 Example:

Find how many four letters words can be formed out of the word LOGARITHMS (The words may not have any meaning)

Solution: There are 10 different letters, therfore, n is equal to 10 and since we have to find four - letter words, r is 4 .

Hence, the required number of words are

$$^{10}P_4 = \frac{10\,!}{(10-4)\,!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= 10 \times 9 \times 8 \times 7 = 5,040$$

5.4.4 Example: Indicate how many 4 digit numbers greater than 7,000 can be formed form the digits 3, 5, 7, 8, 9.

Solution:

If the digits are to be greater than 7,000 then the first digit can be any one of the 7, 8 and 9 .

Now the first digit can be choosen in 3 ways $\left(\because\ 3P_1 = 3\right)$ and the remaining three digits can be any of the four digits left, Which can be choosen in $^4P_3$ ways. Therefore the total number of ways

$= 3 \times {}^4P_3 \quad = 3 \times 4 \times 3 \times 3 \quad = 72$

5.4.5 Example:

In how many ways can 5 telugu, 3 English and 3 Tamil books be arranged. If the books of each different language are kept together.

Solution:

The each language book amongst themselves can be arranged in the following ways:

Telugu: 5 books in $^5P_5$ ways or 5 !

English: 3 books in $^3P_3$ ways or 3 !

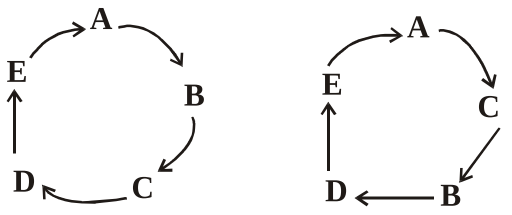
Tamil: 3 books in $^3P_3$ ways or 3 !

Also arrangement of these groups can be made in $^3P_3$ ways. Hence by fundamental theorem the required arrangements are

$5! \times 3! \times 3! \times 3! = 25,920$

5.5 Circular Permutations:

These are related with arrangement of objects as in the case of a sitting arrangement of members in a round table conference. Here the arrangement does not change unless the order changes. Let us consider the following two arrangements of 5 members.



It may be seen that the above two arrangementsare the same. But it is not so in the following cases where the order changes.

Therefore in the circular arrangement the relative position of the other objects depends on the position of the object placed first it is only the arrangement of the remaining objects is made. Therefore the circular arrangement of n objects will be in (n - 1) ! ways. Thus the circular arrangement of 5 persons will be in 4! ways.

5.5.1 Example: In how many ways can 5 boys and 5 girls can be seated around a table so that no 2 boys are adjacent,

Solution: Let the girls be seated first.

They can sit in 4 ! ways.

Now since the places for the boys in between girls are fixed. The option is there for the boys to occupy the remaining 5 places. There are 5! ways for the boys to fill up the 5 places in between 5 girls seated around a table already.

Thus the total number of ways in which both girls and boys can be seated sych that no 2 boys are adjacent are 4 ! x 5 ! = 2880 ways.

5.5.2 Example:

In how many ways can 4 Indians and 4 Pakistans be seated at a round table. So that two Indians may be together.

Solution:

Put one of the Pakistini in a fixed position and then arrange the remaining three pakistanis in all possible ways. Thus the number of ways in which the four pakistan is be seated ata round table is 3!. After they have taken their seats in any one way. There are four seats for the Indians each between two pakistan is therefore the Indians can be seated in 4! ways corresponding to one way of seating the Pakistani

$\therefore$ The total number of arrangements is $4! \times 3! = 144$

5.6 Permutations of things not all different:

The number of permutations of n things of which P things are of one kind, q things are of a second kind, r things are of third kind and all the rest are different is given by

$$x = \frac{n!}{p! \times q! \times r!}$$

5.6.1 Example: Find the number of permutations of the word ACCOUNTANT, ENGINERRING.

Solution:

(a) The word ACCOUNTANT has 10 letters, of which 2 are A's, 2 are C's , 2 are N's and 2 are T's, the rest are different

Therefore the number of permutations is

$$\frac{10!}{2! \quad 2! \quad 2! \quad 2!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdot 2} = 2,26,800 \cdot$$

(b) Since the word ENGINEERING consists of 11 letters, in which there are 3 Es, 3 Ns, 2 Gs, 2 Is and One R.

The total number of permutations is $\frac{11\,!}{3\,! \quad 3! \quad 2! \quad 2!}$

5.6.2 Example:

How many numbers greater than a million can be formed with the digits 4, 5, 5, 0, 4, 5, 3.

Solution:

Each number must consist of 7 or more digits. there are 7 digits in all of which there are 2 fours, 3 fives and therest different,

$\therefore$ The total numbers are $\frac{7!}{2\,! \quad 3!} = 420$

5.6.3 Example:

How many different words can be made out of the letters in the word ALLAHABAD. In how many of there will the vowels occupy the even places.

The word 'ALLAHABAD' consists of 9 letters of which A is repeated four times, L is repeated twice and the rest all different. Hence therequired number of words are $\frac{9!}{4! \quad 2!} = 7560$

(ii) Since the word ALLAHABAD consists of 9 letters, there are 4 even places which can be filled up by the 4 vowels in 1 way only, since all the vowels are similar. Further the remaining 5 places can be filled up by the 5 consonants of which two are similar which can be filled in $\frac{5!}{2!}$ ways. Hence the required number of arrangements are $1 \times \frac{5!}{2!} = 60$

5.6.4 Example:

In how many ways can the letters of word 'ARRANGE' be arranged? How many of these arrangements are there in which

(i) The two Rs come together

(ii) The two Rs do not come together

(iii) The two Rs and the two As come together.

Solution:

The word ARRANGE consists of 7 letters of which two are As two are Rs and the rest all different.

Hence they can be arranged among themselves in $\frac{7!}{2!\ \ 2!} = 1260$ ways

(i) The number of arramgements in which the two Rs come together can be obtained by treating the two R's as one letter. Thus there are 6 letters of which two A's are similar and so the total number of arrangements $= \frac{6!}{2!} = 360$

(ii) The number of arrangements in which the two R's do not come together can be obtained by subtracting from the total number of arrangements the arrangements in which the two R's come together. Thus the required number if 1260 - 360 = 900.

(iii) The number of arrangements in which the two Rs and the Two As come together can be obtained by treating the two Rs and the two A's as a single letter. Thus there are 5 letters which are all different and so the number of arrangements is 5 ! = 120.

5.7 Restricted Permutations:

(i) The number of permutations of n different things taken r at a time in which P particular things donot occur is $(n-p)\ P_r$.

(ii) The number of permutations of n different things taken r ata time is which P particular things are present is $(n-P)\ P_{n-p} \times {}^{r}P_p$.

5.7.1 Example:

Find the value of n if four times the number of permutations of n things taken 3 together is equal to 5 times the number of permutaions of (n - 1) things taken 3 together.

Solution:

We are given that

$$4 \times n P_3 = 5 \times (n-1) P_3$$

$$\Rightarrow 4n \ (n-1)(n-2) = 5(n-1)(n-2)(n-3)$$

$$\Rightarrow 4n = 5(n-3) \Rightarrow n = 15$$

5.7.2 Example:

How many numbers of six digits can be formed from the digit 4, 5, 6, 7, 8, 9 no digit being repeated. How many of them are not divisible by 5.

Solution:

The six digits being different, they can be arranged among themselves in $\lfloor 6$ ways all the digits being taken at a time.

Let us find the digits divisible by 5, such digits can be obtained when 5 occurs in te unit place. The position of 5 being fined, the remaining 5 digits can be arranged among themselves 5 $\lfloor 5$ ways. So the numbers divisible by 5 are $\lfloor 5$ or 5!.

Hence the numbers not divisible by 5 are 6 ! - 5 ! = 600.

5.7.3 Example:

In how many ways can 5 boys and 6 girls be arranged in a row so that all the 5 boys are together.

Solution:

Consider the 5 boys as one unit. Now there are 7 units and they can be arranged among themselves in 7 ! ways. In each of such arrangements, he 5 boys can be arranged among themselves in 5 ! ways.

$\therefore$ Total number of arrangements in which the boys are together = 7 ! x 5 !

5.7.4 Example:

If the letters of the word PRISON are permuted is all possible ways and the words thys formed are arranged in dictionary order, then find the ranks of the words

(i) PRISON, (ii) SIPRON.

Solution:

The letters of the given word in dictionary order are I N O P R S

(i) In the dictionary order, first all the words that being with I come. If I occupies the first place, then the remaining 5 plus can be fileld with the remaining 5 letters in 5 ! ways. Thus, there are 5 ! number of words that beign with I. On proceeding like this we get

I ------------------------------ $\rightarrow$ 5 ! ways

N ------------------------------ $\rightarrow$ 5 ! ways

O ------------------------------ $\rightarrow$ 5 ! ways

P I ------------------------------ $\rightarrow$ 4 ! ways

P N ------------------------------ $\rightarrow$ 4 ! ways

P O ------------------------------ $\rightarrow$ 4 ! ways

P R I N ------------------------- $\rightarrow$ 2 ! ways

P R I O ------------------------ $\rightarrow$ 2 ! ways

P R I S N ----------------------- $\rightarrow$ 1 ! ways

P R I S O N ------------------- $\rightarrow$ 1 ! ways

Hence, the rank of the PRISON is $3 \times 5! + 3 \times 4! + 2 \times 2! + 1 + 1 = 438$

(ii) Now we find the rank of the word SIPRON as above

I ------------------------------ $\rightarrow$ 5 ! ways

N ------------------------------ $\rightarrow$ 5 ! ways

O ------------------------------ $\rightarrow$ 5 ! ways

P -------------------------------- $\rightarrow$ 5 ! ways

R --------------------------------- $\rightarrow$ 5 ! ways

S I N --------------------------- $\rightarrow$ 3 ! ways

S I O ---------------------------- $\rightarrow$ 3 ! ways

S I P N ------------------------- $\rightarrow$ 2 ! ways

S I P O -------------------------- $\rightarrow$ 2 ! ways

S I P R N ---------------------- $\rightarrow$ 1 ! ways

S I P R O N -------------------- $\rightarrow$ 1 ! ways

Hence therank of SIPRON is $5 \times 5! + 2 \times 3! + 2 \times 2! + 1 + 1$

$= 600 + 12 + 4 + 2$

$= 618$

5.7.5 Example: Six papers are set in an examination, of which two are 'statistics'. In howmany different orders can the papers be arranged so that the two statistics papers are not together?

Solution: Number of ways in which six papers can be arranged = $6\,!$

If two statistics papers are to be kept togehter then the six papers can be arranged in $5\,! \times 2\,!$ ways. Hence, the number of arrangements in which six papers can be arranged so that the two statistics papers are not together

$$= 6! - 5! \times 2! = 5! \times 4! = 480$$

5.8 Combinations:

In permutations objects are based on the order of the arrangements. But in combinations order does not matter, it is simply the identity of items in the constitute a new combination. For example, the number of different ways in which 6 people can be arranged in a queue is a question of permutations where order matters. The number of different ways in which 6 people can sit in a committe of 3 in a question of combinations where order does not matter, but the constitute of each selection do matter. However, repetion in each combination is not allowed except otherwise stated.

5.8.1 The mathematical formula for finding out combinations:

The number of combinations of 'n' different things taken 'r' at a teme are given by

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}} \quad \text{where } (r \leq n)$$

i.e. $${}^{n}C_{r} = \frac{n(n-1)(n-2)\ \cdots(n-(r-1))}{r\cdot(r-1)\ (r-2)\ \cdots 1}$$

5.8.2 Formulas:

1) $${}^{n}C_{r} = {}^{n}C_{n-r}$$

2) $${}^{n}C_{0} = {}^{n}C_{n} = 1$$

3) $${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{(n+1)}C_{r}$$

4) $${}^{n}C_{r} = {}^{n}C_{s} \Rightarrow r + s = n \quad \text{or } r = s$$

5.8.3 In how many ways can 4 white and 3 black balls be selected from a box containing 20 white and 15 black balls.

Sol: This problem involves merely selection and hence is a problem of combinations. 4 out of 20 white balls can be selected in $20\,c_4$ i·e·, $\frac{20\times19\times18\times17}{4\times3\times2\times1} = 4845$ ways. call this processa as the first process. 3 out of 15 black balls can be selected in ${}^{15}C_3$ i·e·, $\frac{15\times14\times13}{3\times2\times1} = 455$ ways. Call this process as second process. The two processes can be carried out together in $4845\times455 = 2,204,475$ ways.

5.8.4 From 6 boys and 4 girls 5 are to be selected for admission for a particular course. In how many ways can this be done if there must be exactly 2 girls?

Sol: Since there has to exactly 2 girls, there should be 3 boys the possible combinations would therefore be $4C_2 \times 6C_3 = \frac{\overset{2}{\not{4}}\times3}{\not{2}\times1}\times\frac{\not{6}\times5\times4}{\not{3}\times\not{2}\times1} = 120$ ways

5.8.5 In a mercantile form 4 posts fall vacant and 35 condidates apply for the posts. In how many way can be selection be made

(i) if one particular condidate is always included?

(ii) if one particular condiate is always excluded?

(i) Since a particular person is always to be seleced, we must select the remaining 3 condidates out of the remaining 34.

$$\Rightarrow 34\,C_3 = \frac{34\times33\times32}{3\times2\times1} = 5984$$

(ii) Since a particular person is always to be excluded, the choice in restricted to 4 condidates out of the remaining 34

$$\Rightarrow 34C_4 = \frac{34\times33\times32\times31}{4\times2\times3\times1} = 46,376$$

5.8.6 A father takes 8 children, three at a time to the zoo, as often as he can without taking the same three together more than once (i) how often will each child go?

(ii) how often will he go?

Sol: (i) A particular child goes as often as that child can be included in the combinations of 8 childrent aken 3 at a time.

Let us select one child first of all, then we have to select only 2 from the remaining 7. This can be done $7C_2$ ways.

$\therefore$ The number of times the particular child will go is

$$7\,C_2 = \frac{7 \times \not{6}^{\,3}}{\not{2} \times 1} = 21$$

(ii) The father goes as often as the combinations of 8 children taken 3 at a time.

$\therefore$ The number of times the father will go is

$$8\,C_3 = \frac{8 \times 7 \times \not{6}}{\not{3} \times \not{2} \times 1} = 56$$

5.8.7 At an election there are five candidates out of whom three are to be elected and a votes is entitled to vote for any number of candidates not greater than the number to be elected.In how many ways may a voter choose to vote.

Sol: The voter may vote for one, two or three candidates out out of the 5 candidates .He can choose to vote for one candidate in $5C_1$ ways,two candidates in $5C_2$ ways and 3 candidates in $5C_3$ ways. Hence the total number of ways the votes can choose to vote is

$$5C_1 + 5C_2 + 5C_3 + 5C_4 = 5 + 10 + 10 = 25$$

5.8.8 for an examination , candidate has to select 7 subjects from three different groups A, B and c. The three groups A ,B,C contain 4,5,6 subjects respectively .In how many different ways can a candidates make his selection if he has to select at least 2 subjects from each group.

Sol: The different ways in which a condidate can make a selection of 7 subjects so as to have at least 2 from each group can be as follows:

(i) 2 from A, 2 from B and 3 from C

(ii) 2 from A, 3 from B and 2 from C

(iii) 3 from A, 2 from B and 2 from C

Now the selection of (I) can be done in ${}^4C_2 \times {}^5C_2 \times {}^6C_3$

$$= 6 \times 10 \times 20 = 1200 \text{ ways}$$

Again the selection of (II) can be done in ${}^4C_2 \times {}^5C_3 \times {}^6C_2$

$$= 6 \times 10 \times 15 = 900 \text{ ways}$$

Again the selection of (III) can be done in ${}^4C_3 \times {}^5C_2 \times {}^6C_2$

$$= 4 \times 10 \times 15 = 600 \text{ ways}$$

$\therefore$ The required number of selections are 1200 + 900 + 600 = 2700

5.8.9 Out of 10 consonants and 4 vowels, how many words can be formed each containing 6 consonants and 3 voweld s?

Sol: 6 consonants can be chosen out of 10 in $^{10}C_5$ ways and 3 vowels can be chosen out of 4 in $^{4}C_3$ ways.

Combining each way of selecting the consonants with each way of selecting the vowels, the number of selections having 6 consonants and 3 vowels $= {}^{10}C_6 \times {}^{4}C_3$. Each of these selections contains 9 letters can be arranged among themselves in 9! ways.

$$\therefore \text{ The total number of words } = {}^{10}C_6 \times {}^{4}C_3 \times 9!$$

$$= {}^{10}C_4 \times {}^{4}C_1 \times 9!$$

$$= \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} \times 4 \times 362880$$

$$= 304, 819, 200$$

5.8.10 A boat's crew consist of 8 men, 3 of whom can only row on one side and 2 only one the other. Find the number of ways in which the crew be arranged.

Sol: 4 men must row on each side. But on one side there are already three men and on the other side two. So one must be placed on the side of three men and two on the other side. This can be done in 3 C, or 3 ways.

Again 4 men on each side can be arranged among them seleves in 4! ways. Hence the required number of ways

$$= {}^{3}C_1 \times 4! \times 4!$$

$$= 3 \times 24 \times 24$$

$$= 1728$$

5.8.11 A party of 6 is to be formed from 10 boys and 7 girls so as to include 3 boys and 3 girls. In how many different ways can the party be formed if two paticular girls refuse to join the same party?

Sol: If the two particular girls do not refuse to join the same party, then we can select 3 girls from 7 in $^{7}C_3$ ways and 3 boys from 10 in $^{10}C_3$ ways. Hence a party of 6 including 3 boys and 3 girls can be formed in $^{7}C_3 \times {}^{10}C_3$

$$= 35 \times 120 = 4200 \text{ ways} \quad \cdots\cdots\cdots\cdots(1)$$

Now let us find the number of ways such that 2 girls whose refuse join the same party are included in the same party. For such arrangements we have to select only 1 girl from theremaining 5 and 3 boys from the total of 10.

This can be done in $^{5}C_{1} \times {}^{10}C_{3} = 5 \times 120 = 600$ ways ....(2)

We notice that in arrangement (2) those two particular girls who refus to join are included hence the required number of arrangements can be obtained by subtracting (1)

i.e. 4200 - 600 = 3600

5.8.12 A party of 3 ladies and 4 gentle men is to be formed from 8 ladies and 7 gentle men. In how many differentways can the party be formed if Mrs. X and Mr. 'Y' refuse to join the same party?

Sol: 3 ladies can be selected out of 8 ladies in $^{8}C_{3}$ ways and 4 gentle men can be selected out of 7 gentlemen in $^{7}C_{4}$ ways.

$\therefore$ The number of ways of choosing the committee

$$\Rightarrow {}^{8}C_{3} \times {}^{7}C_{4} = \frac{8!}{3! \times 5!} \times \frac{7!}{4! \times 3!} = 1960$$

If Mrs. X and Mr. Y are members, there remain to be selected 2 ladies from 7 ladies and 3 gentlemen from 6 gentlemen. This can be done in

$$^{7}C_{2} \times {}^{6}C_{3} = \frac{7!}{2!\ 4!} \times \frac{6!}{3! \times 3!} = 420 \text{ ways}$$

The number of ways of forming the party in which Mrs. X and Mr. Y refuse to join

= 1960 - 420 = 1540

5.8.13 A cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket keepers. In how many different ways can a team be formed so that the team contains (a) exactly 3 bowlers and 1 wicket - keeper (b) at least 3 bowlers and at least one wicket - keeper.

Sol: **(a)** Here a cricket team of 11 is exactly to contain 3 bowlers and a wicket keepr. 3 bowlers can be selected out of 4 in $^{4}C_{3}$ i.e., 4 ways. 1 wicket keeper canbe selected out of 2 in $^{2}C_{1}$ i.e. 2 ways. Now the remaining 10 players in $^{10}C_{3}$, i.e. 120 ways. Hence by the fundamental principle. The total number of ways in which the team can be formed.

$$= 4 \times 2 \times 120 = 960$$

(b) In the case the cricket team of 11 can be formed in the following ways:

(I) 3 bowlers, 1 wicket keeper and 7 other players.

(II) 3 bowlers, 2 wicket keepers and 6 other players.

(III) 4 bowlers, 1 wicket keepr and 6 other players.

(IV) 4 bowlers, 2 wicket keeprs and 5 other players.

(i) 3 bowlers, 1 wicket keeper and 7 other players can be selected in

$$^{4}C_{3} \times {}^{2}C_{1} \times {}^{10}C_{7} = 4 \times 2 \times 120 = 960 \text{ ways}$$

(ii) 3 bowlers, 2 wicket keepers and 6 other player can be selected in

$$^{4}C_{3} \times {}^{2}C_{2} \times {}^{10}C_{6} = 4 \times 1 \times 210 = 840 \text{ ways}$$

(iii) 4 bowlers, 1 wicket keepers and 6 other players can be selected in

$$^{4}C_{4} \times {}^{2}C_{1} \times {}^{10}C_{6} = 1 \times 2 \times 210 = 420 \text{ ways}$$

(iv) 4 bowlers, 2 wicket keepers and 5 other players can be selected in

$$^{4}C_{4} \times {}^{2}C_{2} \times {}^{10}C_{5} = 1 \times 1 \times 252 = 252 \text{ ways}$$

Hence the total number of different ways

$$= 960 + 840 + 420 + 252 = 2472$$

5.9 Restricted Combinations:

(i) The number of combinations of 'n' things taken 'r' at a time in which 'p' particular things always occur is $^{n-p}C_{r-p}$.

If the 'p' particular things are set aside, there remain (n - p) things out of which (r - p) things may be chosen in $^{n-p}C_{r-p}$ ways with each of these groups we combine the p particular things so that we get all the combinations in each of which the 'p' particular things will always occur.

$\therefore$ The required number of combinations $= {}^{n-p}C_{r-p}$

(ii) The number of combinations of 'n' things taken 'r' at a time in which 'p' particular things never occur is $^{n-p}C_{r}$

Let the 'p' particular things be set a side then there will remain (n - p) things out of which 'r' things may be selected in ${}^{n-p}C_r$ ways in none of these groups p particular things will occur. Hence the required number of combinations $= {}^{n-p}C_r$

5.9.1 Prove that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

Sol: We know that

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$${}^nC_r + {}^nC_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = {}^{(n+1)}C_r$$

5.9.2 Find the value of 'r' if ${}^{18}C_r = {}^{18}C_{r+2}$

Sol: Since ${}^nC_r = {}^nC_{n-r}$, we have ${}^{18}C_r = {}^{18}C_{18-r}$

But, we are given ${}^{18}C_{18-r} = {}^{18}C_{r+2}$

$\Rightarrow 18 - r = r + 2$

$\Rightarrow 18 - 2 = r + r$

$\Rightarrow 16 = 2r$

$\Rightarrow r = 8$

5.10 Combinations of Things not all different:

We will here show that the total number of combinations of 'n' different things taken some or all at a time is $= 2^n - 1$.

The first thing can be dealt with in 2 ways, for it may either be left way of dealing with the second,t he total number of ways of dealing with the two things.

$$= 2 \times 2 = 2^2$$

Proceeding similary, the total number of ways $= 2^n$

But this number includes one case in which none of the things are taken.

$\therefore$ The required number of combinations $= 2^n - 1$

Now, we consider the number of combinations of 'n' things not all different. The total number of combinations of (p + q + r + ...) things, where p are of one kind, q of the second kind and r of the third kind and so on, taken any number at a time are

$$= (p+1)(q+1)(r+1) \cdots\cdots\cdots\cdots\cdots\cdots - 1$$

Consider the p like things. The 'p' things can be dealt with in (p + 1) ways, for we may take 1 or 2 or 3, ... or p or none in any selection similarly the q like things can be dealt with in (q + 1) ways, r things in (r + 1) ways etc. Associating each group of selections with the others the other numbers of dealing with them is

$$(p+1)\ (q+1)\ (r+1) \cdots\cdots\cdots\cdot$$

But this number includes one case when all things are left. Therefore, the total number of ways.

$$= (p+1)\ (q+1)(r+1) \cdots\cdots\cdots\cdots - 1$$

5.10.1 In order to pass C.A. (Intermediate) examination minimum - marks have to be secured in each of the 7 subjects. In how many cases can a student fail?

Sol: Each subject can be dealt in two ways. The student may pass or fail in it. So the 7 subjects can be dealt in 2^7 ways but this includes the case in which the student passes in all the 7 subjects. Excluding, the number of ways in which the student can fail is $2^7 - 1 = 127$.

5.10.2 A question paper contains 6 questions, each having an alternative. In how many ways can an examinee answer one or more questions?

Sol: The first question can be dealt with in 3 ways, for the question itself may be answered, or its alternative may be asuered or none of them may be answered.

Similarly, the second question also can be dealt with in 3 ways. Hence the first two questions can be dealt with in 3×3 or 3^2 ways. Proceeding thus, all the 6 questions may be dealt with in 3^6 ways.

But this number includes one case in which none of the questions is answered.

5.10.3 There are 'n' points in a plane, no three of which are collinear (lying on the same stratight line) with the exception of p points which are collinear, find the number of

(i) Different straight lines and

(ii) Different triangles formed by joining them

Sol: (i) Any two points when joined give a straight line

The number of possible straight lines formed by joining 'n' points in pairs $= {}^nC_2$.

But 'p' of the points lie in the same straight line.

$\therefore$ PC_2 straight lines are lost and instead we get only 1 straight line in which they lie.

$\therefore$ The required number of straight lines in

$$ {}^nC_2 - {}^PC_2 + 1 = \frac{n(n-1)}{2} - \frac{p(p-1)}{2} + 1 $$

(ii) Any 3 non - collinear points give a triangle.

$\therefore$ The number of triangles formed by joining 'n' points taken three at a time $= {}^nC_3$

Since 'p' of the points are collinear. $P\,C_3$ triangles are lost.

$\therefore$ The required number of triangles is

$$ {}^nC_3 - PC_3 = \frac{n(n-1)\,(n-2)}{6} - \frac{p\,(p-1)\,(p-2)}{6} $$

5.11 Binomial Theorem:

A binomial expression in mathematics in one which has two terms, e.g., (a + b), (2 x + 3 y), (x + b) etc. These terms are at time complementary when the expression is used for objects which are of dichotomous charactor, i.e. success or failure, true or false, male or female. In business mathematics and statistics, there are various problems based on such classification where the theorem is found to be very useful.

From elementary algebra, we know

$$ (a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2 $$

$$ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 $$

These are quite simple but if the expression is to a higher order or with negative and fractional indices the problem becomes quite complicated.

5.11.1 Theorem (Binomial Theorem):

If 'n' is a positive integer. Then

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots\cdots + {}^nC_r x^{n-r} a^r + \cdots\cdots + {}^nC_n a^n$$

(The prrof is beyond the scope of the syllabus)

5.11.2 Note: (1) $(x-a)^n = (x+(-a))^n = {}^nC_0 x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots +$

$(-1)^r \; {}^nC_r \, x^{n-r} a^r + \cdots\cdots + (-1)^n \; {}^nC_n \, a^n$

(2) The number of terms in $(x+a)^n$ is n + 1.

(3) The general term of $(x+a)^n$ is $(r+1)^{th}$ term $= T_{r+1} = {}^nC_r \, x^{n-r} a^r$

(4) The simplest form of binomial expansion in the general form is given below:

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n$$

$$(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 + \cdots\cdots + (-1)^r \cdot {}^nC_r x^r + \cdots\cdots + {}^nC_n (-x)^n$$

5.11.3 Expand $\left(x - \frac{1}{x}\right)^5$:

Sol: $\left(x - \frac{1}{x}\right)^5 = {}^5C_0 x^5 - {}^5C_1 x^4 \cdot \frac{1}{x} + {}^5C_2 x^3 \cdot \frac{1}{x^2} + {}^5C_3 x^2 \cdot \frac{1}{x^3} + {}^5C_4 x \cdot \frac{1}{x^4} + {}^5C_5 \cdot \frac{1}{x^5}$

$= {}^5C_0 x^5 - {}^5C_1 x^3 + {}^5C_2 x + {}^5C_3 \cdot \frac{1}{x} + {}^5C_4 \cdot \frac{1}{x^3} + {}^5C_5 \cdot \frac{1}{x^5}$

5.11.4 Evaluate $\left\{x + \sqrt{x^2+1}\right\}^6 + \left\{x - \sqrt{x^2+1}\right\}^6$

Sol: Let $\left\{x + \sqrt{x^2+1}\right\}^6 = (x+y)^6$, where $y = \sqrt{x^2+1}$

$$\left\{x+\sqrt{x^2+1}\right\}^6 = 6C_0\,x^6 + 6C_1\,x^5\,y + 6C_2\,x^4\,y^2$$
$$+\,6C_3\,x^3\,y^3 + 6C_4\,x^2\,y^4 + 6C_5\,x\,y^5 + 6C_6\,y^6 \cdots\cdots\cdots(1)$$

Also $\left\{x-\sqrt{x^2+1}\right\}^6 = (x-y)^6$

$$= 6C_0\,x^6 - 6C_1\,x^5\,y + 6C_2\,x^4\,y^2 - 6C_3\,x^3\,y^3 + 6C_4\,x^2\,y^4 - 6C_5\,x\,y^5 + 6C_6\,y^6 \cdots\cdots(2)$$

Adding (1) and (2) we get

$$\left\{x+\sqrt{x^2+1}\right\}^6 + \left\{x-\sqrt{x^2+1}\right\}^6$$

$$= 2\left\{6C_0\,x^6 + 6C_2\,x^4\,y^2 + 6C_4\,x^2\,y^4 + 6C_6\,y^6\right\} \qquad \boxed{\because\ y=\sqrt{x^2+1}}$$

$$= 2\left\{6C_0\,x^6 + 6C_2\,x^4\left(x^2+1\right) + 6C_4\,x^2\left(x^2+1\right)^2 + 6C_6\left(x^2+1\right)^3\right\}$$

$$= 2\left\{x^6 + \frac{\overset{3}{\not{6}}\cdot 5}{1\cdot \not{2}}\,x^4\left(x^2+1\right) + \frac{6\cdot 5}{1\cdot 2}x^2\left(x^4+2x^2+1\right) + \left(x^6+3x^4+3x^2+1\right)\right\}$$

$$= 2\left\{32\,x^6 + 48x^4 + 18x^2 + 1\right\}$$

5.11.5 Write down the 7th term in the expansion of $\left(\dfrac{4x}{5} - \dfrac{5}{2x}\right)^9$

Sol: In the expansion of $(y+a)^n$ the general term is

$$t_{r+1} = {}^nC_r\; y^{n-r}\, a^r$$

Putting $r = 6$, $y = \dfrac{4x}{5}$, $a = \left(\dfrac{-5}{2x}\right)$ and $n = 9$ we get

$$t_{6+1} = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(\frac{-5}{2x}\right)^6$$

$$= \frac{9!}{6!\ 3!} \cdot \frac{(4x)^3}{5^3} \cdot \frac{(-5)^6}{(2x)^6}$$

$$= \frac{\overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}} \times 7}{\cancel{3} \times \cancel{2} \times 1} \cdot \frac{\cancel{64}\ x^3}{\cancel{125}} \cdot \frac{\cancel{125} \times 125}{\cancel{64}\ x^6}$$

$$= \frac{84 \times 125}{x^3} = \frac{10500}{x^3}$$

5.11.6 (i) Find the coefficient of x^4 in the expansion of $\left(1 + x - 2x^2\right)^6$. If the complete expansion of the expression is given by

$$1 + a_1 x + a_2 x^2 + \cdots\cdots\cdots\cdots + a_{12} x^{12}$$

Prove that $a_2 + a_4 + \cdots\cdots\cdots\cdots + a_{12} = 31$

Sol: (i) $\left(1 + x - 2x^2\right)^6 = \left[1 + x(1 - 2x)\right]^6$

$$= 1 + 6C_1 x(1-2x) + 6C_2 x^2 (1-2x)^2 + 6C_3 x^3 (1-2x)^3 + \cdots\cdots + 6C_6 x^6 (1-2x)^6$$

$$= 1 + 6C_1 x(1-2x) + 6C_2 x^2 \left(1-4x+4x^2\right) + 6C_3 x^3 (1-6x+\cdots)$$
$$+ 6C_4 x^4 (1-2x)^4 + 6C_5 x^5 (1-2x)^5 + 6C_6 x^6 (1-2x)^6$$

Coefficient of $x^4 = 4 \cdot 6C_2 - 6 \cdot 6C_3 + 6C_4$

$= 60 - 120 + 15$

$= -45$

(ii) Now $\left(1 + x - 2x^2\right)^6 = 1 + a_1 x + a_2 x^2 + \cdots\cdots + a_{12} x^{12}$

Putting x = 1, we have

$$1 + a_1 + a_2 + \cdots\cdots\cdots + a_{12} = 0 \cdots\cdots\cdots(1)$$

Putting x = - 1, we have

$$1 - a_1 + a_2 - a_3 + \cdots\cdots\cdots + a_{12} = 64 \cdots\cdots\cdots\cdots(2)$$

Adding (1) and (2) we get

$$2\left(1 + a_2 + a_4 + \cdots\cdots\cdots + a_{12}\right) = 64$$

$$\Rightarrow a_2 + a_4 + \cdots\cdots\cdots + a_{12} = 31$$

5.11.7 Using the binomial theorem, calculate $(1\cdot1)^{10}$ correct to 6 decimal places.

Sol: We have

$$(1\cdot1)^{10} = (1 + 0\cdot1)^{10}$$

$$= 1 + 10(0\cdot1) + 45(0\cdot01) + 120(0\cdot001) + 210(0\cdot0001) + 252(0\cdot00001) +$$

$$+ 210(0\cdot000001) + 120(0\cdot0000001) + 45(0\cdot00000001) + \cdots\cdots\cdots$$

$$= 1 + 10(0\cdot1) + 45(0\cdot01) + 120(0\cdot001) + 210(0\cdot0001) + 252(0\cdot00001)$$

$$+ 210(0\cdot000001)\, 120(0\cdot0000001) + 45(0\cdot00000001) + \cdots\cdots\cdots$$

$$= 1 + 1\cdot0 + 0\cdot45 + 0\cdot120 + 0\cdot0210 + 0\cdot00252 + 0\cdot000210$$
$$+ 0\cdot0000120 + 0\cdot00000045$$

$$= 2.593742$$

5.12 Positions of Terms:

We have already explained that in the expansion of $(1 + x)^n$, the coefficients of terms equidistant from the beginning and the end are equal. We also know that the coefficient of $(r + 1)^{th}$ term from the beginning is ${}^nC_r$. The $(r + 1)^{th}$ terms from the end has $\{(n + 1) - (r + 1)\}$ or n - r terms before it, therefore, f rom the beginning it is $(n - r + 1)^{th}$ term and its coefficient is ${}^nC_{n-r}$, which has been shown to be equal to ${}^nC_r$.

Therefore, when 'n' is even the greatest coefficient is ${}^{n}C_{n/2}$ and when 'n' is odd it is ${}^{n}C_{\frac{n-1}{2}}$ or ${}^{n}C_{\frac{n+1}{2}}$ these two coefficients are how ever equal.

5.13 Binomial Coefficients:

The coefficients of the expansion of the binomial are the prefixes of each term, the elements are with variable powers of the binomials. The coefficients are ${}^{n}C_0$, ${}^{n}C_1$, ………, ${}^{n}C_n$ indicated briefly as C_0 , C_1 , ………, C_n . The properties of these coefficientsa re given below:

5.13.1 If C_0 , C_1 , ………, C_n denote the coefficients of the expansion of $(1+x)^n$, prove that

(a) $C_1 + 2C_2 + 3C_3 + \cdots\cdots + n\,C_n = n \cdot 2^{n-1}$

(b) $C_0 + 2C_1 + 3C_2 + \cdots + (n+1)\,C_n = (n+2)\,2^{n-1}$

(c) $C_0 + 3C_1 + 5C_2 + \cdots\cdots + (2n+1)\,C_n = (n+1)\;2^{n}$

Sol: (a) $L \cdot H \cdot S = C_1 + 2C_2 + 3C_3 + \cdots\cdots + nC_n$

$$= n + 2 \cdot \frac{n(n-1)}{2} + 3 \cdot \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} + \cdots\cdots + n$$

$$= n\left[1 + (n-1) + \frac{(n-1)(n-2)}{2} + \cdots\cdots + 1\right]$$

$$= n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \cdots\cdots + {}^{n-1}C_{n-1}\right]$$

$$= n\left[[1+1]^{n-1}\right]$$

$$= n \cdot 2^{n-1} = R \cdot H \cdot S$$

(b) $L \cdot H \cdot S = C_0 + 2C_1 + 3C_2 + \cdots\cdots + (n+1)\,C_n = (n+2)\,2^{n-1}$

$$= \left[(C_0 + C_1 + C_2 + \cdots\cdots + C_n) + (C_1 + 2C_2 + \cdots\cdots + nC_n)\right]$$

$$= 2^n + n \cdot 2^{n-1}$$

$$= 2^{n-1}(n+2) = R \cdot H \cdot S$$

(c) $L \cdot H \cdot S = C_0 + 3C_1 + 5C_2 + \cdots\cdots\cdots + (2n+1)\,C_n$

$$= (C_0 + C_1 + C_2 + \cdots\cdots\cdots + C_n) + 2\,(C_1 + 2C_2 + \cdots\cdots + {}^nC_n)$$

$$= 2^n + 2(n\,2^{n-1})$$

$$= 2^n + n\,2^n$$

$$= (n+1)\,2^n$$

$= $ R.H.S.

5.13.2 If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots\cdots\cdots\cdots + C_n\,x^n$ Prove that

(a) $\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \cdots\cdots\cdots\cdots + \dfrac{nC_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$

(b) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \cdots\cdots\cdots\cdots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1} - 1}{n+1}$

Sol: (a) $L \cdot H \cdot S = \dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \cdots\cdots\cdots + \dfrac{nC_n}{C_{n-1}}$

$$= \frac{n}{1} + \frac{\dfrac{2 \cdot n(n-1)}{2!}}{n} + \frac{\dfrac{3 \cdot n(n-1)(n-2)}{3!}}{\dfrac{n\,(n-1)}{2!}} + \cdots\cdots\cdots + \frac{n \cdot 1}{n}$$

$$= n + (n-1) + (n-2) + \cdots\cdots\cdots + 1$$

$$= \frac{n\,(n+1)}{2} = R \cdot H \cdot S$$

(b) $L \cdot H \cdot S = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots\cdots + \frac{C_n}{n+1}$

$$= 1 + \frac{n}{2} + \frac{n(n-1)}{1 \cdot 2 \cdot 3} + \cdots\cdots\cdots + \frac{1}{n+1}$$

$$= \frac{1}{n+1}\left[n + 1 + \frac{(n+1)\,n}{1 \cdot 2} + \frac{(n+1)\,n\,(n-1)}{1 \cdot 2 \cdot 3} + \cdots\cdots + 1\right]$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \cdots\cdots\cdots + {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1}\left[(1+1)^{n+1} - 1\right]$$

$$= \frac{2^{n+1} - 1}{n+1} = R \cdot H \cdot S$$

5.13.3 Find the coefficient of x^4 in the expansion of $\left[x^4 + \frac{1}{x^3}\right]^{15}$

Sol: T_{r+1} in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$

$$T_{r+1} = {}^{15}C_r \left(x^4\right)^{15-r} \left(\frac{1}{x^3}\right)^r$$

$$= {}^{15}C_r \; x^{60-4r} \; x^{-3r}$$

$$= {}^{15}C_r \; x^{60-7r}$$

This term will contain x^4 if $60 - 7r = 4$

$$7r = 56 \Rightarrow r = 8$$

$\therefore$ The required coefficeint of $x^4 = {}^{15}C_8$

$$= {}^{15}C_8 \Rightarrow \frac{15!}{8!\,7!}$$

5.13.4 Find the term independent of n in the expansion of $\left(\frac{3}{5}x^2 - \frac{1}{2x}\right)^9$

Sol: T_{r+1} in the expansion of $\left(\frac{3}{5}x^2 - \frac{1}{2x}\right)^9$

$$T_{r+1} = {}^9C_r \left(\frac{3}{5}x^2\right)^{9-r} \left(\frac{-1}{2x}\right)^r$$

$$T_{r+1} = {}^9C_r \left(\frac{3}{5}\right)^{9-r} x^{18-2r} (-1)^r (2x)^{-r}$$

$$T_{r+1} = {}^9C_r \left(\frac{3}{5}\right)^{9-r} (-1)^r x^{18-2r}\, 2^{-r}\, x^{-r}$$

$$T_{r+1} = {}^9C_r \left(\frac{3}{5}\right)^{9-r} (-1)^r\, 2^{-r}\, x^{18-3r}$$

This term will be independent of x if

$18 - 3r = 0$ i.e. if $r = 6$

$\therefore$ The required term independent of x

$$= (-1)^6 \cdot {}^9C_6 \left(\frac{3}{5}\right)^{9-6} 2^{-6}$$

$$= \frac{9!}{6!\,3!} \cdot \frac{3^3}{5^3} \cdot \frac{1}{2^6}$$

$$= \frac{2268}{8000}$$

5.13.5 Find the term independent of x in the expansion of $\left(x - \frac{1}{x^2}\right)^{3n}$

Sol: $T_{r+1} = {}^{3n}C_r \; x^{3n-r} \left(\frac{-1}{x^2}\right)^r$

$= {}^{3n}C_r \; x^{3n-r} \; \frac{(-1)^r}{x^{2r}}$

$= (-1)^r \cdot {}^{3n}C_r \; x^{3n-3r}$

This term will be independent of x, if 3n - 3r = 0, i.e., r = n

$\therefore$ required term $= (-1)^n \cdot {}^{3n}C_n = (-1)^n \cdot \frac{(3n)!}{n!(2n)!}$

5.13.6 (a) If the 21st and 22nd terms in the expansion of $(1+x)^{44}$ are equal, find the value of x.

(b) In the expansion of $(1+x)^{11}$ the fifth term is 24 times the thired term. Find the value of x.

Sol: (a) $(r+1)^{th}$ term in the expansion of $(1+x)^{44}$

$T_{r+1} = {}^{44}C_r \; x^r$

$T_{21} = {}^{44}C_{20} \; x^{20}$

and $T_{22} = {}^{44}C_{21} \; x^{21}$

Now $T_{21} = T_{22}$ (is given)

${}^{44}C_{20} \; x^{20} = {}^{44}C_{21} \; x^{21}$

$x = \frac{{}^{44}C_{20}}{{}^{44}C_{21}}$

$$x = \frac{\cancel{44!}}{24!\ 20!} \times \frac{23!\ 21!}{\cancel{44!}}$$

$$x = \frac{\cancel{23!} \times 21 \times \cancel{20!}}{\cancel{23!} \times 24 \times \cancel{20!}} = \frac{21}{24} = \frac{7}{8}$$

(b) $T_{r+1} = {}^{11}C_r\ x^r$

$T_5 = {}^{11}C_4\ \ x^4$ and $T_3 = {}^{11}C_2\ \ x^2$

We are given $T_5 = 24\ t_3$

$\Rightarrow {}^{11}C_4\ x^4 = 24 \cdot {}^{11}C_2\ \ x^2$

$$\Rightarrow x^2 = 24 \cdot \frac{{}^{11}C_2}{{}^{11}C_4}$$

$$\Rightarrow x^2 = 24 \cdot \frac{\cancel{11!}}{9!\ 2!} \times \frac{7!\ 4!}{\cancel{11!}}$$

$$\Rightarrow x^2 = 24 \cdot \frac{\cancel{7!} \times 12 \times \cancel{2!}}{\cancel{7!} \times 72 \times \cancel{2!}}$$

$$x^2 = \overset{4}{\cancel{24}} \cdot \frac{\cancel{12}}{\underset{6}{\cancel{72}}}$$

$x^2 = 4$

$x = \pm 2$

5.13.7 (a) If the coefficient of x in the expansion of $\left(x^2 + \frac{k}{x}\right)^5$ is 270 find k.

(b) If the absolute term in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405 find the value of k.

Sol: **(a)** T_{r+1} in the expansion of $\left(x^2 + \frac{k}{r}\right)^5$

$$T_{r+1} = {}^5C_r \left(x^2\right)^{5-r} \left(\frac{k}{x}\right)^r$$

$$= {}^5C_r\ x^{10-2r}\ k^r\ x^{-r}$$

$$= {}^5C_r\ k^r\ x^{10-3r}$$

This term will contain x if 10 - 3 r = 1, i.e., if r = 3.

Now Coefficient of x is given to be 270

$$\therefore\ {}^5C_3\ k^3 = 270\ ,\ \text{i.e.,}\ \ k^3 = \frac{270}{{}^5C_3} = \frac{270}{10} = 27$$

$$\Rightarrow\ \therefore\ k = 3$$

(b) T_{r+1} term in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$

$$T_{r+1} = {}^{10}C_r \left(\sqrt{x}\right)^{10-r} \left(\frac{-k}{x^2}\right)^r$$

$$T_{r+1} = {}^{10}C_r\ (-1)^r \cdot k^r\ x^{5-r/2}\ \ x^{-2r}$$

$$T_{r+1} = (-1)^r\ {}^{10}C_r\ k^r \cdot x^{(10-5r)/2}$$

This term will be independent of x , i.e. the power of x will be zero of

$$\frac{10-5r}{2} = 0\ \ \text{i.e., if}\ r = 2$$

Thus the third term is independent of x, and its value is

$$T_3 = (-1)^2 \; {}^{10}C_2 \; k^2$$

and we are given

$$(-1)^2 \cdot {}^{10}C_2 \; k^2 = 405$$

i.e., $k^2 = \dfrac{405}{{}^{10}C_2}$

$$k^2 = \frac{405}{45} = 9$$

$$k = \pm 3$$

5.13.8 Find the coefficient of x^{32} in the expansion of $\left(x^4 - \dfrac{1}{x^3}\right)^{15}$

Sol: Let us assume that x^{32} occurs inthe $(r+1)^{th}$ term of the expansion of $\left(x^4 - \dfrac{1}{x^3}\right)^{15}$

$$\therefore T_{r+1} = {}^{15}C_r \left(x^4\right)^{15-r} \left(\frac{-1}{x^3}\right)^r \quad \cdots\cdots\cdots\cdots(1)$$

$$= {}^{15}C_r \cdot x^{60-4r} \cdot (-1)^r \cdot x^{-3r}$$

$$= {}^{15}C_r \cdot (-1)^r \cdot x^{60-7r}$$

Since x^{32} is the $(r+1)^{th}$ term, we must have

$$60 - 7r = 32$$

$$7r = 28$$

$$r = 4$$

$$\therefore T_{4+1} = {}^{15}C_4 \left(x^4\right)^{15-4} \left(\frac{-1}{x^3}\right)^4$$

$$T_5 = {}^{15}C_4 \; x^{32} \; (-1)^4$$

$$T_5 = \frac{15!}{11!\ 4!} x^{32}$$

$$T_5 = 1365\ x^{32}$$

$\therefore$ Coefficient of x^{32} is 1365.

5.13.9 Find the coefficient of x^{11} in the expansion of $\left(1 - 2x + 3x^2\right)(1+x)^{11}$

Sol: Using binomial theorem, we have

$$\left(1 - 2x + 3x^2\right)(1+x)^{11}$$

$$= \left(1 - 2x + 3x^2\right)\left(1 + {}^{11}C_1\ x + {}^{11}C_2\ x^2 + \cdots + {}^{11}C_9\ x^9 + {}^{11}C_{10}\ x^{10} + {}^{11}C_{11}\ x^{11}\right)$$

$\therefore$ coefficient of x^{11} in the expansion of $\left(1 - 2x + 3x^2\right)(1+x)^{11}$

$$= 1 \times {}^{11}C_{11} - 2\ {}^{11}C_{10} + 3 \times {}^{11}C_9$$

$$= 1 - 2 \times {}^{11}C_1 + 3 \times {}^{11}C_2 \qquad \left(\because {}^nC_r = {}^nC_{n-r}\right)$$

$$= 1 - 22 + 165$$

$$= 144$$

5.13.10 If the 2nd, 3rd and 4th term in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, find x, a and n.

Sol: $(x+a)^n = {}^nC_0\ x^n + {}^nC_1\ x^{n-1}\ a + {}^nC_2\ x^{n-2}\ a^2 + \cdots\cdots\cdots\cdots$

$$= x^n + n\ x^{n-1}\ a + \frac{n(n-1)}{2} x^{n-2}\ a^2 + \frac{n(n-1)(n-\ )}{6} x^{n-3}\ a^3 + \cdots\cdots\cdots$$

Equating 2nd, 3rd and 4th terms, we get

$$n\ x^{n-1}\ a = 240 \cdots\cdots\cdots\cdots\cdots(1)$$

$$\frac{n(n-1)}{2} x^{n-2}\ a^2 = 720 \cdots\cdots\cdots\cdots(2)$$

$$\frac{n(n-1)(n-2)}{6} x^{n-3} a^3 = 1080 \cdots\cdots\cdots\cdots(3)$$

Multiplying (1) and (3) we get

$$\frac{n^2 (n-1)(n-2)}{6} x^{2n-4} a^4 = 240 \times 1080 \cdots\cdots(4)$$

Squaring (2) we get

$$\frac{n^2 (n-1)^2}{4} x^{2n-4} a^4 = 720 \times 720 \cdots\cdots\cdots(5)$$

Dividing (4) by (5) we get

$$\frac{n^2 (n-1)(n-2) x^{2n-4} a^4}{6} \times \frac{4}{x^2 (n-1)^2 x^{2n-4} a^4} = \frac{240 \times 1080}{720 \times 720}$$

$$\Rightarrow \frac{4(n-2)}{6(n-1)} = \frac{1}{2}$$

$$\Rightarrow 8n - 16 = 6n - 6 \Rightarrow n = 5$$

Substituting the value of 'n' in (1) and (2) we get

$$5x^4 a = 240$$

$$10 x^3 a^2 = 720$$

Divide square of (6) and (7) we get

$$\frac{25 x^8 a^2}{10 x^3 a^2} = \frac{240 \times 240}{720} \Rightarrow x^5 = 32, \text{ i} \cdot \text{e} \cdot x = 2$$

Substituting values of x and 'n' in (1) we get

$$5 \times 2^{5-1} \times a = 240 \Rightarrow a = \frac{240}{5 \times 16} = 3$$

Hence $x = 2,\ a = 3$ and $n = 5 \cdot$

5.14 Binomial Theorem With Any Index:

We illustrate these with refrence to the simplest form of expressions $(1+x)^{1/n}$ and $(1-x)^{-n}$ from which other binomials can be reduced. The two such expansions are (i) with a fractional index and (ii) with a negative index.

Eg: $$(1+x)^{1/2} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{19}x^3 \cdots\cdots\cdots\cdots$$

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots\cdots\cdots\cdots$$

In each of these series the number of terms are infinite. The general form of expansion which can be used in the first case is

$$1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots$$

If we put $n = \frac{1}{2}$ in that case the above expansion will take the following form:

$$1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots\cdots$$

Now, if the index is negative, the general form of the expansion is

$$1 + (-n)x + \frac{(-n)(-n-1)}{1\cdot 2}x^2 + \frac{(-n)(-n-1)(-n-2)}{1\cdot 2\cdot 3}x^3 + \cdots\cdots\cdots$$

5.14.1 Expand $(1-x)^{3/2}$ upto four terms

Sol: $$(1-x)^{3/2} = 1 + \frac{3}{2}(-x) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1\cdot 2}(-x)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1\cdot 2\cdot 3}(-x)^3 + \cdots\cdots\cdots$$

$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \cdots\cdots\cdots\cdots$$

Eg: Expand $(2+3x)^{-4}$ upto four terms

Sol: $$(2+3x)^{-4} = 2^{-4}\left(1+\frac{3x}{2}\right)^{-4}$$

$$= \frac{1}{2^4}\left[1+(-4)\left(\frac{3x}{2}\right)+\frac{(-4)(-4-1)}{1\cdot 2}\left(\frac{3x}{2}\right)^2+\frac{(-4)(-4-1)(-4-2)}{1\cdot 2\cdot 3}\left(\frac{3x}{2}\right)^3+\cdots\cdots\right]$$

$$= \frac{1}{16}\left[1-6x+\frac{45}{2}x^2-\frac{135}{2}x^3+\cdots\cdots\cdots\cdots\right]$$

5.14.2 Find the general term of expansion of $(1-x)^{-2}$

Sol: The $(r+1)^{th}$ term $= \dfrac{n(n-1)(n-2)\cdots\cdots\cdots\cdots(n-r+1)}{r!}\, x^r$

The $(r+1)^{th}$ term $= \dfrac{(-3)(-4)(-5)\cdots\cdots\cdots(-3-r+1)}{r\,!}\,(-x)^r$

$$= (-1)^r\ \frac{3\cdot 4\cdot 5\cdots\cdots\cdots\cdots(r+2)}{r\,!}\,(-1)^r\ x^r$$

$$(-1)^{2r}\ \frac{3\cdot 4\cdot 5\cdots\cdots\cdots\cdots(r+2)}{1\cdot 2\cdot 3\cdots\cdots\cdots\cdots r}\, x^r$$

$$= \frac{(r+1)(r+2)}{1\cdot 2}\, x^r$$

5.14.3 Using Binomial Theorem, find the value of $\sqrt[3]{126}$ to four decimal places.

Sol: $\sqrt[3]{126} = (126)^{1/3} = (125+1)^{1/3}$

$$= (125)^{1/3}\left[1+\frac{1}{125}\right]^{1/3}$$

$$= \left(5^3\right)^{1/3}\ [1+0\cdot 008]^{1/3}$$

$$= 5\ [1+0\cdot 008]^{1/3}$$

$$= 5\left[1+\frac{1}{3}\times 0\cdot 008+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2\cdot 1}\times(0\cdot 008)^2\right]$$ (neglecting the other terms)

$\therefore \sqrt[3]{125} = 5\left[1 + 0 \cdot 002666 - 0 \cdot 00000711\right]$

$= 5\left[1 \cdot 002659\right]$

$= 5.013295$

$= 5.0133$ (correct upto four places of decimal)

5.14.4 Extract of fifth root of 244 correct to the three places of decimal.

Sol: $(244)^{1/5} = (244)^{1/5} = (243 + 1)^{1/5}$

$$= \left[243\left(1 + \frac{1}{243}\right)\right]^{1/5}$$

$$= 3\left[1 + \frac{1}{3^5}\right]^{1/5}$$

$$= 3\left[1 + \frac{1}{5} \cdot \frac{1}{3^5} + \frac{\frac{1}{5}\left(-\frac{4}{5}\right)}{2!} \cdot \left(\frac{1}{3^5}\right)^2 + \cdots\cdots\cdots\right]$$

$$= 3\left[1 + \frac{1}{5} \cdot \frac{1}{35} - \frac{2}{25} \cdot \frac{1}{3^{10}} + \cdots\cdots\cdots\right]$$

$$= 3 + \frac{1}{5} \cdot \frac{1}{3^4} - \frac{8}{100} \cdot \frac{1}{3^9} + \cdots\cdots\cdots\cdot$$

$$= 3 + \frac{1}{5}(0 \cdot 01234) - \frac{8}{100}(0 \cdot 00005)$$

$$= 3 + 0 \cdot 00247 - 0 \cdot 000004$$

$$= 3.002466$$

5.14.5 Find the coefficient of x^{10} int he expansion of $\frac{1 + 2x}{(1 - 2x)^2}$

Sol: $\frac{1 + 2x}{(1 - 2x)^2} = (1 + 2x)(1 - 2x)^{-2}$

$$= (1+2x)\left[1+2(2x)+3(2x)^2+4(2x)^3+\cdots\cdots+10(2x)^9+11(2x)^{10}+\cdots\cdots\right]$$

$$= (1+2x)\left[1+4x+12x^2+\cdots\cdots+10\cdot 2^9\cdot x^9+11\cdot 2^{10}\cdot x^{10}+\cdots\cdots\right]$$

$\therefore$ required coefficient $= 11\cdot 2^{10} + 2\cdot 10\cdot 2^9 = 21504\cdot$

5.14.6 Find the coefficient of x^n in the expansion of $\left(1+x+x^2+x^3+x^4+\cdots\cdots\cdots\right)^{2/3}$ where $|x|<1$.

Sol: We have

$$\left(1+x+x^2+x^3+x^4+\cdots\cdots\cdots\cdots\right)^{2/3}$$

$$= \left(\frac{1}{1-x}\right)^{2/3} = \left\{(1-x)^{-1}\right\}^{2/3} = (1-x)^{-2/3}$$

$$= (1-x)^{-2/3}$$

$$= \left(1+(-x)\right)^{-2/3}$$

$$= 1+\left(\frac{-2}{3}\right)(-x)+\frac{\left(\frac{-2}{3}\right)\left(\frac{-2}{3}-1\right)}{2\,!}(-x)^2+\frac{\left(\frac{-2}{3}\right)\left(\frac{-2}{3}-1\right)\left(\frac{-2}{3}-2\right)}{3!}(-x)^3$$

$$+\frac{\left(\frac{-2}{3}\right)\left(\frac{-2}{3}-1\right)\left(\frac{-2}{3}-2\right)\left(\frac{-2}{3}-3\right)}{4\,!}(-x)^4+\cdots\cdots\cdots$$

$$= 1+\frac{\left(\frac{2}{3}\right)}{1!}x+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)}{2!}x^2+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)}{3\,!}x^3+\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)}{4\,!}x^4+\cdots\cdots\cdots$$

$\therefore$ Coefficient of x^n in the expansion of

$\left(1+x+x^2+x^3+\cdots\cdots\cdots\cdots\right)^{2/3}$ is given by

$$\frac{\left(\frac{2}{3}\right)\left(\frac{5}{3}\right)\left(\frac{8}{3}\right)\left(\frac{11}{3}\right)\cdots\cdots\cdots\cdots\text{..to n factors}}{n\,!}$$

$$= \frac{2\cdot 5\cdot 11\cdots\cdots\cdots\cdots(3n-1)}{3^n\cdot n!}$$

5.14.7 If x is very small compared to 1, prove that $\dfrac{\sqrt{1+x}+\sqrt[3]{(1-x)^2}}{(1+x)+\sqrt{1+x}} = 1-\dfrac{5x}{6}$ nearly.

Sol: The given expansion

$$= \frac{(1+x)^{1/2}+(1-x)^{2/3}}{(1+x)+(1+x)^{1/2}} = \frac{\left(1+\frac{1}{2}x\right)+\left(1-\frac{2}{3}x\right)}{(1+x)+\left(1+\frac{1}{2}x\right)} \quad \text{(neglecting the other terms)}$$

$$= \frac{2-\frac{1}{6}x}{2+\frac{3}{2}x} = \frac{1-\frac{1}{12}x}{1+\frac{3}{4}x}$$

$$= \left(1-\frac{1}{12}x\right)\left(1+\frac{3}{4}x\right)^{-1}$$

$$= \left(1-\frac{1}{12}x\right)\left(1-\frac{3}{4}x\right)$$

$$= 1-\frac{1}{12}x-\frac{3}{4}x$$

$$= 1-\frac{5}{6}x$$

5.15 Summation of a series:

We can find approximate value of a series if it is on lines of the expansion of the binomial.

We know $(1+x)^n = 1+nx+\dfrac{n(n-1)}{1\cdot 2}x^2+\cdots\cdots$

If x is small compared to 1, we find that term which get smaller and smaller and we take 1 as the first approximation of the value of $(1+x)^n$ and $(1+nx)$ as a second approximation etc. The following examples will illustrate.

5.15.1 Find the sum of infinite series:

$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}+\frac{2\cdot5\cdot8}{3\cdot6\cdot9}\cdot\frac{1}{2^3}+\cdots\cdots\cdots\cdots$$

Sol: Comparing the given series with the expansion

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1\cdot2}x^2+\cdots\cdots\cdots \text{ we have}$$

$$nx=\frac{2}{3}\cdot\frac{1}{2}\cdots\cdots\cdots\cdots\cdots(1)$$

$$\frac{n(n-1)}{1\cdot2}x^2=\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^3}\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

From (1), we get $x=\frac{1}{3n}$

Substituting the value of 'x' in (2) we have

$$\frac{n(n-1)}{2}\cdot\frac{1}{9n^2}=\frac{2\cdot5}{3\cdot6}\cdot\frac{1}{2^2}$$

$$\Rightarrow\frac{(n-1)}{n}=\frac{5}{2}\text{ , i·e, } 2n-2=5n$$

$$n=\frac{-2}{3}$$

and from (1), we get $x=\frac{-1}{2}$

Hence the given series $=(1+x)^n$

$$=\left(1-\frac{1}{2}\right)^{-2/3}$$

$$= \left(\frac{1}{2}\right)^{-2/3}$$

$$= 2^{2/3}$$

5.15.2 Sum of the following series:

$$2 + \frac{5}{2!\,3} + \frac{5 \cdot 7}{3!\ 3^2} + \frac{5 \cdot 7 \cdot 9}{4!\,3^3} + \cdots\cdots\cdots\cdot$$

Sol: The given series is

$$\left[1 + 1 + \frac{5}{2!} \cdot \frac{1}{3} + \frac{5 \cdot 7}{3!} \cdot \left(\frac{1}{3}\right)^2 + \cdots\cdots\cdot\right]$$

Comparing the series in the braket with the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \cdots\cdots\cdot \quad \text{we have}$$

$$nx = 1 \cdots\cdots\cdots\cdot (1)$$

$$\frac{n(n-1)}{2} x^2 = \frac{5}{2!} \cdot \frac{1}{3} \cdots\cdots\cdots\cdot (2)$$

From (1) we get $x = \frac{1}{n}$

Substituting in (2), we get $\frac{n(n-1)}{2} \cdot \frac{1}{n^2} = \frac{5}{6}$

$$\Rightarrow \frac{n-1}{2n} = \frac{5}{6}, \text{ i·e·, } 6n - 6 = 10n$$

$$\Rightarrow n = \frac{-3}{2}$$

From (1) we get $x = \frac{-2}{3}$

Hence the given series

$$= \left(1 - \frac{2}{3}\right)^{-3/2}$$

$$= \left(\frac{1}{3}\right)^{-3/2}$$

$$= 3^{3/2}$$

$$= 3\sqrt{3}$$

5.15.3 If $y = \frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12} + \cdots\cdots\cdots$

Prove that $y^2 + 2y - 2 = 0$

Sol: The given series may be written as

$$y = -1 + \left[1 + \frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \cdots\cdots\cdots\right]$$

Comparing the series in the bracket with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \cdots\cdots\cdots \text{ we have}$$

$$nx = \frac{1}{3} \cdots\cdots\cdots\cdots(1)$$

$$\text{and } \frac{n(n-1)}{1 \cdot 2}x^2 = \frac{1 \cdot 3}{3 \cdot 6} \cdots\cdots\cdots\cdots(2)$$

Solging for n and x, we get

$$n = \frac{-1}{2} \text{ and } x = \frac{-2}{3}$$

$$\therefore y = (1+x)^n - 1 = \left(1 - \frac{2}{3}\right)^{-1/2} - 1$$

$$y = \sqrt{3} - 1$$

$$\Rightarrow y + 1 = \sqrt{3}$$

$$\Rightarrow (y+1)^2 = 3$$

$$\Rightarrow y^2 + 2y + 1 = 3$$

$$\Rightarrow y^2 + 2y - 2 = 0$$

5.15.4 Sum the series:

$$1 + \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \cdots\cdots\cdots$$

Sol: Let the given series be iolentical with

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \cdots\cdots\cdots(1)$$

Equating the second and third terms in the two series, we get

$$ny = \frac{1}{3}x \cdots\cdots\cdots\cdots(2)$$

and $$\frac{n(n-1)}{2!}y^2 = \frac{1\cdot 4}{3\cdot 6}x^2 \cdots\cdots\cdots(3)$$

From (2) we get $y = \frac{x}{3n}$. Substituting this value of y in (3), we get

$$\frac{n(n-1)}{2}\cdot\frac{x^2}{9x^2} = \frac{4}{18}x^2$$

$$\Rightarrow \frac{n-1}{n} = 4$$

$$\Rightarrow n - 1 = 4n$$

$$\Rightarrow 3n = -1$$

$$\Rightarrow n = \frac{-1}{3}$$

$$\therefore\ y = \frac{3}{3n} = \frac{x}{3 \cdot \left(\frac{-1}{3}\right)} = -x$$

Hence the sum ofthe given series $= (1+y)^n = (1-x)^{-1/3}$

5.15.5 Sum of the series to infinity

$$1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1 \cdot 4}{3 \cdot 6} \cdot \frac{1}{4^2} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} \cdot \frac{1}{4^3} + \cdots\cdots\cdots$$

Sol: Comparing the given series with the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \cdots\cdots\cdots \text{ we have}$$

$$nx = \frac{1}{3} \cdot \frac{1}{4}$$

$$nx = \frac{1}{12} \cdots\cdots\cdots\cdots\cdots (1)$$

$$\frac{n(n-1)}{1 \cdot 2} x^2 = \frac{1 \cdot 4}{3 \cdot 6} \cdot \frac{1}{4^2} \cdots\cdots\cdots\cdots (2)$$

From (1) we get $x = \frac{1}{12n}$

Substituting the value of x in (2) we have

$$\frac{\not{n}(n-1)}{1 \cdot 2} \times \frac{1}{\underset{2}{\not{12}}\, n} \times \frac{1}{\underset{3}{\not{12}}\, \not{n}} = \frac{1 \cdot \not{4}}{\not{3} \cdot \not{6}} \cdot \frac{1}{\not{4}^{\not{2}}}$$

$$\frac{(n-1)}{2} \times \frac{1}{2n} = 1$$

$$n - 1 = 4n$$

$$3n = -1$$

$$n = \frac{-1}{3}$$

and from (1) we get $x = \frac{-1}{4}$

Hence the given series $= (1+x)^n = \left(1 - \frac{1}{4}\right)^{-1/3}$

$$= \left(\frac{3}{4}\right)^{-1/3}$$

$$= \left(\frac{4}{3}\right)^{1/3}$$

5.15.6 Sum of the series to infinity

$$1 + \frac{1}{3^2} + \frac{1 \cdot 4}{1 \cdot 2} \cdot \frac{1}{3^4} + \frac{1 \cdot 4 \cdot 7}{1 \cdot 2 \cdot 3} \cdot \frac{1}{3^6} + \cdots\cdots\cdots$$

Sol: Comparing the given series with the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \cdots\cdots\cdots \text{ we have}$$

$$nx = \frac{1}{9} \cdots\cdots\cdots (1)$$

$$\frac{n(n-1)}{2} x^2 = \frac{1 \cdot 4}{1 \cdot 2} \cdot \frac{1}{3^4} \cdots\cdots\cdots\cdots (2)$$

From (1), we get $x = \frac{1}{9n}$

Substituting the value of x in (2) we have

$$\frac{\not{n}(n-1)}{\not{2}} \frac{1}{\not{81} n} \times \frac{1}{\not{n}} = \frac{4}{\not{2}} \cdot \frac{1}{\not{81}}$$

$$n - 1 = 4n$$

$$n = -1/3$$

and form (1), we get $x = \frac{-1}{3}$

Hence the given series $= (1 + x)^n$

$$= \left(1 - \frac{1}{3}\right)^{-1/3}$$

$$= \left(\frac{2}{3}\right)^{-1/3}$$

5.16 Answers to SAQ:

5.16.1 Answer to S.A.Q. of 5.12.3:

$$\left(x - \frac{1}{x}\right)^5 = {}^5C_0\, x^5 + {}^5C_1\, x^4 \left(\frac{-1}{x}\right) + {}^5C_2\, x^3 \left(\frac{-1}{x}\right)^2$$

$$+ {}^5C_3\, x^2 \left(\frac{-1}{x}\right)^3 + {}^5C_4\, x \left(\frac{-1}{x}\right)^4 + {}^5C_5 \left(\frac{-1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - 10 \cdot \frac{1}{x} + 5 \cdot \frac{1}{x^3} - \frac{1}{x^5}$$

5.17 Summary:

In this lesson we `discussed permutations and combinations, circular permutations and their related problems. Also we discussed Binomial theorem , summation of series and its applications.

5.18 Technical Terms:

Permutations,cambinations factorials

Binomial theorem, summation of series

Binomial coefficients.

5.19 Exercise:

1) Evaluate (i) $15\ P_3$ (ii) $4\ P_3$

2) Find 'n' if (i) $(n-1)\,P_3 : (n+1)\,P_3 = 5 : 12$

(ii) $(n+3)\,P_4 : (n+2)\,P_4 = 14 : 1$

3) How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5 ? How many of these have 0 in ten's place?

4) How many different arrangements can be made by using all the letters of the word

(i) MONDAY (ii) ORIENTAL?

How many of these arrangements being with A and end with N ?

5) How many numbers greater than a million can be formed with the digits 1, 7, 10, 7, 3, 7 ?

6) In how many ways can 8 boys form a ring?

7) In how many ways can 10 letters be posted in 5 letter boxes?

8) Find the value of ${}^{8}C_{3}$, ${}^{21}C_{19}$, ${}^{n}C_{n-3}$.

9) There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further in how many of these committees?

(i) a particular professor is included?

(ii) a perticular student is included?

10) From 7 gentlemen and 4 ladies a committee of 5 is to be formed. In howmany ways can this be done so as to include at least one lady?

11) Expand (i) $(3x - 4)^4$ (ii) $\left(\frac{x}{3} + \frac{2}{4}\right)^4$

12) Write down and simplify

(i) The 11^{th} term in the expansion of $(y + 4x)^{30}$

(ii) The 5^{th} term in theexpansion of $\left(\frac{3x}{4} + \frac{4}{3x}\right)^{12}$

13) Find the coefficient of

(i) x^{18} in the expasnion of $(a x^4 - b x)^9$

(ii) x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$

(iii) x^{-2} in the expansion of $\left(2x - \frac{1}{x^2\sqrt{3}}\right)^{10}$

14) Find the term independent of x in

(i) $\left(2x + \frac{1}{3x^2}\right)^9$ (ii) $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ (iii) $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$

15) If the coefficients of x^7 and x^8 in the expansion of $\left(3 + \frac{x}{2}\right)^n$ are equal, find the value of n.

Sum of the following series:

16) $1+\frac{3}{4}+\frac{3\cdot 5}{4\cdot 8}+\frac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\cdots\cdots\cdots\infty$

17) $1+2\cdot\frac{1}{3}+3\cdot\frac{1}{3^2}+4\cdot\frac{1}{3^3}+\cdots\cdots\cdots\cdots$

18) $1+\frac{2}{9}+\frac{2\cdot 5}{9\cdot 18}+\frac{2\cdot 5\cdot 8}{9\cdot 18\cdot 27}+\cdots\cdots\cdots$

19) Prove that $\frac{1\cdot 3}{3\cdot 6}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}+\frac{1\cdot 3\cdot 5\cdot 7}{3\cdot 6\cdot 9\cdot 12}+\cdots\cdots = 0\cdot 04$

20) Prove that $\sqrt{2}=\frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1\cdot 3}{1\cdot 2}\cdot\frac{1}{10^4}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\;\frac{1}{10^6}+\cdots\cdots\right]$

15.21 Model Examination Questions:

1) Find the value of "r" if $^{18}C_r = {}^{18}C_{r+2}$

2) How many ways can 5 boys and 6 girls be arranged in a row so that all the 5 boys are together.

3) A question paper contains 6 questions, each having an alternative. In how many ways can an examine one or more questions?

4) Sum the following series:

$$1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1\cdot 4}{3\cdot 6}\cdot\frac{1}{4^2}+\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}\cdot\frac{1}{4^3}+\cdots\cdots$$

5) Prove that $\sqrt{2}=\frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1\cdot 3}{1\cdot 2}\cdot\frac{1}{10^4}+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\cdot\frac{1}{10^4}+\cdots\cdots\right]$

5.22 References:

(1) D.C. SAncheti, V.K. Kapoor, " Business Mathematics", Sultan Chand and Sons, New Delhi.

(2) S. SAHA, " Business Mathematics, New Central Book Agency, Calcutta".

Lesson Writer

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Lesson - 6

SET THEORY

Objective of the lesson:

After studying this lesson the studentw will be in a position to know about methods of decribing a set, types of sets, operations on sets, demorgan laws, cartision products.

Structure:

This lesson has the following compenets.

6.1 Introduction:

In sets we deal with a group of objects which can be defined in terms of their distinctive charactorstics, magnitudes etc. In set theory the oeprations are called as intersection $\cap$, union $\cup$ and complementation $\{\ \}$ respectively.

Set theory play an important role in modern mathematics. In almost whole of the business mathematics the set theory is applied in one form or the other.

6.2 A Set:

A set is a collection of well defined and well-distinguished objects. From a set it is possible to tell whether a given object belongs to a set or not.

Examples: (1) The vowels in English alphabets.

(2) The integers from 1 to 50

Note: The set of all beautiful girls in the class is not a set. It is not well defined set.

6.3 Methods of Decribing a Set:

There are two methods to describing a set.

1. Tabular, Roster or Enumeration method.
2. Selector, property builder or rule method.

1. Tabular Method: Under this method we enumerate or list all the elements of the set within braces.

Example: (1) A set of vowels : $A = \{a, e, i, o, u\}$

(2) A set of even numbers $= \{2, , 4, 6, \cdots\cdots\cdots\}$

(3) A set of prime ministers $P = \{\text{Nehru , Shastri , Indira Gandhi}\}$

2. Selector Method: Under this method the elements are not listed but are indicated by description of their characterstics.

Example: (1) $A = \{x/x \text{ is a vowel in English alphabet}\}$

$B = \{x/x \text{ is an even natural number}\}$

$C = \{x/x \text{ is a prime minister of India}\}$

The vertical line "/" after x to be read as 'suchthat'. Some times we use ":" to denote 'such that'.

6.4 Types of sets:

Finite Set: The number of elements in a set is finite than the set is called finite set.

Eg: (i) $A = \{1, 2, 3, 4, 5, 6\}$

(ii) $B = \{1, 2, \cdots\cdots\cdots\cdots\cdots\cdots, 100\}$

(iii) $C = \{x/x \text{ is a natural number} \leq 20\}$

6.4.1 Infinite Set: The number of elements ina set is infinite than the set is called an infinite set.

Eg: (i) $A = \{1, 2, 3, \cdots\cdots\cdots\}$

(ii) $B = \{ x/x \text{ is an even integer} \}$

(iii) $C = \{ x/x \text{ is a prime number} \}$

6.4.2 Singleton Set: A set containing only one element is called a singleton set.

Eg: (i) $A = \{ a \}$

(ii) $B = \{ 0 \}$

6.4.3 Empty Set or Null Set or Void Set: Any set which has no element in it is called an empty set. It is denoted by ϕ (radius phi).

Eg: (i) $A = \{ x/x \text{ is a perfect square of an integer}, 31 \leq x \leq 35 \}$

(ii) B = The set of integers between 1 and 2.

6.4.4 Equal Sets: Two sets A and B are said to be equal if for every element of A is also an element of B and every element of B is also an element of A. i.e.

$A = B$ if and only if $\{ x \in A \Leftrightarrow x \in B \}$

Eg: (i) If $A = \{3, 4, 4, 7, 9\}$, $B = \{3, 4, 7, 9\}$ Then A = B

6.4.5 Equivalent Sets: FI the elemens of one set can be put into one to one correspondence with the elements of another set then the two sets are called equivalent sets.

The symbol $\equiv$ isused to denote equivalent sets.

Eg: $A = \{a, b, c, d, e\}$ & $B = \{2, 3, 4, 5, 6\}$

Here the elements of A can be put into one-to-one correspondence with those of B thus

a	b	c	d	e
2	3	4	5	6

Hence $A \equiv B$

6.4.6 Sub Sets: If every element of a set A is also an element of a set B. Then the set A is called subset of B. It is denoted as $A \subseteq B$ and read as 'A is a subset of B' or 'A is contained in B'. Some times this relationship is written as $B \supseteq A$ and is read as B is a super set of A or B contains A.

i.e. $A \subseteq B$ if $x \in A \Rightarrow x \in B$

If A is not contained in B then we write it as $A \not\subseteq B$

Examples of subsets. (i) $A = \{2, 3, 4, 7\}, B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$\therefore A \subseteq B$

(ii) If A is a set of books on 9^{th} class mathematics in library and B is a set of books on mathematics in library then $A \subseteq B$

(iii) Let $A = \{x/x \text{ is a } +\text{ve power of } 3\} = \{3, 3^2, 3^3, \cdots\cdots\} = \{3, 9, 27, \cdots\cdots\cdots\}$

$B = \{x/x \text{ is an odd } +\text{ve integers}\} = \{1, 3, 5, 7, 9, 11, \cdots\cdots\}$

Then $A \subseteq B$

6.4.7 Proper Sub Sets: Set A is called proper subset of super set B if each and every element of set A is the element of the set B and atleast one element of super set B is not an element of A.

It is denoted as $A \subset B$

Example: $A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5, 6\}, C = \{1, 2, 3, 4, 5, 6, 8\}$

Here, $A \subset B$ & $B \subset C$

6.4.8 Family of Sets: If all the elements of a set are sets them solves then it is called a set of sets or is a 'family of sets'.

For example, if $A = \{a, b\}$ then the set $A = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ is a family of sets whose elements are subsets of the set A.

6.4.9 Power Set: From a set containing n elements 2^n subsets can be formed. The set consisting of these 2^n subsets is called a power set. In otherwords, if A be a given set thent he family of sets each of whose number is a subset of the given set A is called the power set of A and is denoted as P(A).

For example (i) If $A = \{a\}$ then its subsets are $\phi, \{a\}$

$\therefore P(A) = \{\phi, \{a\}\}$

(ii) If $A = \{a, b\}$ then $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

(iii) If $A = \{a, b, c\}$ then

$$P(A) = \{\phi \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$$

6.4.10 Universal Set: When analysing some particular situation we are never required to go beyond some particular well defined limits. This particular well-defined set may be called the universal set for that particular situation. The universal set generally denoted by the symbol $\cup$.

Examples: (i) A set of integers may be considered as a universal set for a set of odd or even integers.

(ii) A set of chartered accountants in India may be considered as a universal set for a set of fellows or associates of C.A.'s

(iii) A deck of cards may be a universal set for a set of spade.

6.5 Venn Diagrams:

The venn diagrams are named after english logician John Venn (1834 - 1923) to present pictorial representation. The universal set, say U is denoted by a regin enclosed by a rectangle and one or moe sets say A, B, C are shown through circles o closed curver within these rectangles. these circles intersect each other if there are any common elements amongst them, if there are no common elements then they are shown separately as disjoints.

6.6 Operations on sets:

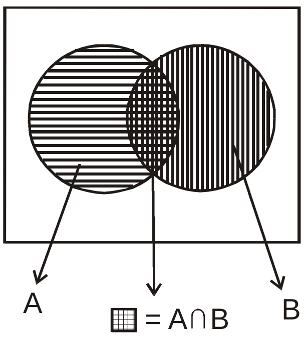
Operations on ssets are intersection $\cap$, union $\cup$ and complementation $(\sim)$

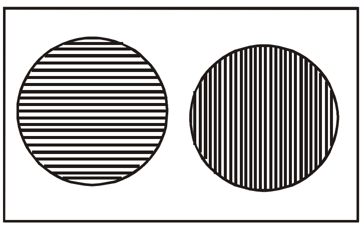
6.6.1 Intersection of Sets: The intersection of two sests A and B is the set consisting of all elements which belong to both A and B. The interection of A and B is denoted as $A \cap B$ which is read as "A intersection B".

i.e. $A \cap B = \{x/x \in A \text{ and } x \in B\}$

In other words $x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$

The following diagram show the intersection of sets can be expresed by venn diagrams.





$A \cap B = \phi$

Examples: (1) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 9\}$ and $C = \{3, 4, 6, 8\}$

Then

$A \cap B = \{4,5\}$

$(A \cap B) \cap C = \{4\}$

$B \cap C = \{4, 6\}$

$A \cap (B \cap C) = \{4\}$

Note that

$A \cap (B \cap C) = (A \cap B) \cap C$

(2) If $A = \{x / x \text{ is an integer}, 1 \le x \le 40\}$ and

$B = \{x / x \text{ is an integer}, 21 \le x \le 100\}$, then

$A \cap B = \{x / x \text{ is an integer}, 21 \le x \le 40\}$

6.6.2 Properties of intersection of sets:

(1) $A \cap B$ is the subset of both the set A and the set B.

i.e. $(A \cap B) \le A$ and $(A \cap B) \le B$

(2) Intersection of any set with an empty set is the null set. i.e.

$A \cap \phi = \phi$ fo every set A.

(3) Intersection of a set with itself is the set itself. i.e.

$A \cap A = A$ for every set A.

(4) Intersection has commutative property. i.e.

$A \cap B = B \cap A$ for every sets A, B.

(5) Intersection as associative property. For any three sets A, B and C.

$(A \cap B) \cap C = A \cap (B \cap C)$

6.6.3 Union of Sets: The union of two sets A and B is the set consisting of all elements which belong to either A or B or both. The union of A and B is denoted as $A \cup B$ read as A union B i.e.

$A \cup B = \{x / x \in A \text{ or } x \in B \text{ or } x \in \text{both } A \text{ and } B\}$

Inother words

$x \in A \cup B \Rightarrow x \in A$ or $x \in B$

Example: (1) Let $A = \{1, 2, 3, 4\}, B = \{2, 4, 5, 6\}$ and $C = \{3, 4, 6, 8\}$.

Then

$A \cup B = \{1, 2, 3, 4, 5, 6\}$, $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\}$

$B \cup C = \{2, 3, 4, 5, 6, 8\}$, $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$

Note that $(A \cup B) \cup C = A \cup (B \cup C)$

(2) $A = \{1, 2, 3, 4\}, B = \{0\}, C = \phi$ then

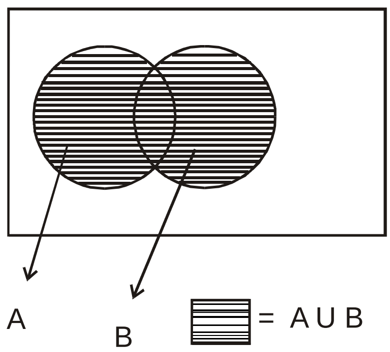
$A \cup B = \{0, 1, 2, 3, 4\}$, $A \cup C = \{1, 2, 3, 4\} = A$ & $B \cup C = \{0\} = B$

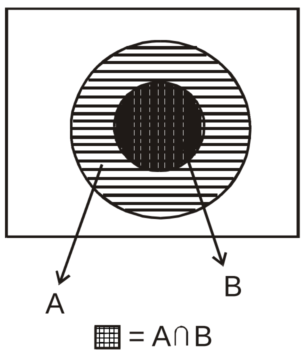
(3) If $A = \{x/x \text{ is an intger}, 1 \le x \le 40\}$ and

$B = \{x/x \text{ is an integer}, 21 \le x \le 100\}$ then

$A \cup B = \{x/x \text{ is an integer}, 1 \le x \le 100\}$

The union of two sets can be illutrated more clearly by ven diagrams as shown below:





Total shaded = $A \cup B$

$A \cup B = A$, $B \subseteq A$

6.6.4 Properties of union of sets:

1. The individual sets compoing a union are member of the union. In other words

$A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$

2. It has an identity property in an empty set. Therefore

$A \cup \phi = A$, for every set A.

3. Union of a set with itself is the set itself. i.e.

$A \cup A = A$, for every set A.

4. It has a commutative property. i.e.

$A \cup B = B \cup A$ for every sets

5. It has an associated property is for any three sets A, B and C.

$(A \cup B) \cup C = A \cup (B \cup C)$

6.6.5 Distribution laws of union and interection: Let A, B and C be any three sets then

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

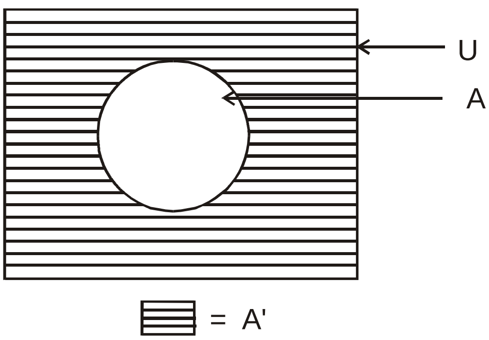
6.6.6 Complement of a set: The complement of a set is the set of all those elements which do not belong to that set. In other words If U be the universal set and A be any set then the complement of set A i the set U - A and s dentoed as A' or A^c, $\overline{A}$ or $\sim A$ i.e.

$A' = U - A = \{x / x \in U, x \notin A\} = \{x/x \notin A\}$

Example: If $U = \{1, 3, 5, 9, 10, 18\}$ and $A = \{3, 5, 10\}$ then

$A' = U - A = \{1, 9, 18\}$

Venn diagram showing thecomplement of the set A int heset U is given below:



6.6.7 Propertes of Complementation:

1. The intersection of a set A and its complement A' is a null set.

i.e. $A \cap A' = \phi$

2. The union of a set A and ts complement A' is the universal set.

   i.e. $A \cup A' = U$

3. The complement of the universal set is the empty set and the complement of the empty set is the universal set i.e.

   $U^1 = \phi$ and $\phi^1 = U$

4. The complement of the complement of a sete is the set itself.

   i.e. $(A')' = A$

5. If $A \subset B$ then $B' \subset A'$

6. Expansion or contration of sets is possible by taking into account the complements of a set.

   (i) $(A \cap B) \cup (A \cap B') = A$ (ii) $(A \cup B) \cap (A \cup B') = A$

6.7 De. Morgan Laws:

1. Complement of a union is the intersecton of complements. i.e.

   $(A \cup B)' = A' \cap B'$ for any sets A and B

2. Complement of an intersections is the union of the complements.

   $(A \cap B)' = A' \cup B'$

3. If A, B, C are any three sets, then

   (i) $A - (B \cup C) = (A - B) \cap (A - C)$

   (ii) $A - (B \cap C) = (A - B) \cup (A - C)$

6.7.1 Some useful results on difgference, union and intersection:

1. $A \cup B = (A - B) \cup B$ for any sets A and B

2. $A \cap (B - A) = \phi$

3. $A - (A - B) = A \cap B$

4. $A - B = A \Rightarrow A \cap B = \phi$

5. $A \cap (B - A) = \phi$

6. $A - B = A \cap (-B)$

7. $A' - B' = B - A$

8. $A \cap (B - C) = (A \cap B) - (A \cap C)$

9. $A \cap (B - C) = (A \cap B) - C$

10. $A \cup (B - C) \neq (A \cup B) - (A \cup C)$

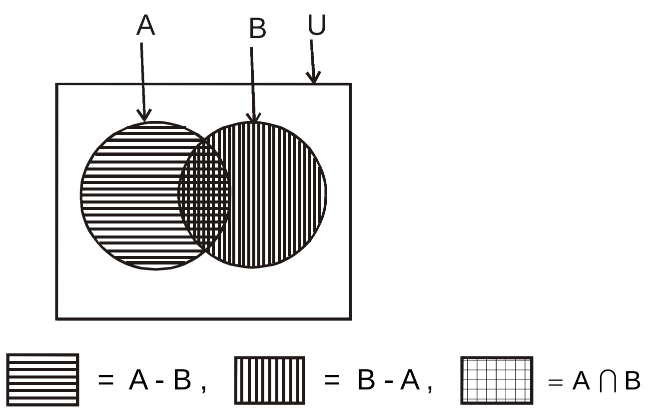
6.8 Number of elemens in a finite set:

From operations on abstract sets we now switch over to the numbers attached to a set which is of great practical utility in finding out the values of new sets formed through operations on some basic sets. Therefore, we find it convenient to introduce a symbol "$n(A)$" to denote the number of elements in a set A. We derive a formula for $n(A \cup B)$ interms of $n(A)$, $n(B)$ and $n(A \cap B)$.

First we observe that it A and B are disjoit i.e. $A \cap B = \phi$ then $n(A \cup B) = n(A) + n(B)$.

If A and B are not disjoint then they have some elements commn to them.

Consider the following venn diagram.



Formula: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

6.9.1 Example: A company studies the product preferences of 20,000 consumers. It was found that each of the products A, B, C was liked by 7020 , 6230 and 5980 respectively and all the products were liked by 1500; Products A and B were liked by 2580, products A and C were liked by 1200 and products B and C were liked by 1950. Prove that the study results are not correct.

Solution: Let A, B, C denote the set of people who like products A, B, C respectively.

The given data means

$$n(A) = 7420,\ n(A \cap B) = 2580,\ n(A \cap B \cap C) = 1500 \cdot$$

$$n(B) = 6230,\ n(A \cap C) = 1200,\ n(A \cup B \cup C) = 20{,}000 \cdot$$

$$n(C) = 5980,\ n(B \cap C) = 1950 \cdot$$

We also know that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 7020 + 6230 + 5980 - 2580 - 1200 - 1950 + 1500$$

$$= 15{,}000 \neq 20{,}000$$

6.9.2 Example: In a class of 25 students, 12 students have taken economics. 8 have taken economics but not polities. Find the number of students who have taken economics and politics and those who have taken politics but not economics.

Solution: Total number of students = 25

n (A) = Number of students taking economics = 12

n (B) = Number of students taking politics.

We have to find $n(A \cap B)$ and $n(B \cap A')$

Now $n(A) - n(A \cap B') + n(A \cap B)$ $\quad [A = (A \cap B') \cup (A \cap B)]$

$\Rightarrow$ $12 = 8 + n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = 12 - 8 = 4$

Also $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow$ $25 = 12 + n(B) - 4$

$n(B) = 17$

Again $n(B) = n(A \cap B) + n(A' \cap B)$

$17 = 4 + n(A' \cap B)$

$n(A' \cup B) = 17 - 4 = 13$

6.9.3 Example: Out of 880 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball of the total 64 played both basket ball and hockey 80 played cricket and basketball and 40 played cricket and hockey 24 boys played all the three games. How many boys did not play any game and how many played only one game?

Solution: Let the C, H and B denote the sets of player playing Cricket, Hockey and Basket ball respectively. Now we are given

$$n(C) = 224 \qquad n(H) = 240 \qquad n(B) = 336$$

$$n(H \cap B) = 64 \qquad n(C \cap B) = 80 \qquad n(C \cap H) = 40$$

and $n(C \cap H \cap B) = 24$ and $n(S) = 880$

(i) Number of players who played at least one game is given by

$$n(C \cup H \cup B) = n(C) + n(H) + n(B) - n(C \cap H) - n(H \cap B) - n(C \cap B) + n(C \cap H \cap C)$$

$$= 224 + 240 + 336 - 40 - 64 - 80 + 24 = 824 - 184 = 640$$

$\therefore$ Number of boys who did not play any game

$$= n(S) - n(C \cup H \cup B) = 880 - 640 = 240$$

(ii) Now $n(C \cap H) = n(C \cap H \cap B) + n(C \cap H \cap B')$

$$\Rightarrow \qquad 40 = 24 + n(C \cap H \cap B')$$

$$\Rightarrow \qquad n(C \cap H \cap B') = 40 - 24 = 16$$

Similarly

$$n(C \cap B \cap H') = 80 - 24 = 56 \text{ and } n(B \cap H \cap C') = 64 - 24 = 40$$

Number of boys who played only basket ball can be obtained from

$$n(B) = n(B \cap H' \cap C') + n(B \cap H \cap C) + n(B \cap H \cap C') + n(B \cap H \cap C)$$

$$336 = n(B \cap H' \cap C') + 56 + 40 + 24$$

$$n(B \cap H' \cap C') = 216$$

Hence number of boys who played only basket ball are 216.

Also $n(H) = n(H \cap B' \cap C') + n(H \cap B \cap C')$

$$+ n(H \cap B' \cap C) + n(H \cap B \cap C)$$

$$\Rightarrow \qquad 240 = n(H \cap B' \cap C') + 40 + 16 + 24$$

$\Rightarrow \quad n(H \cap B' \cap C') = 160$

Similarly $n(C) = n(C \cap B' \cap H') + n(C \cap B \cap H') + n(C \cap B' \cap H) + n(C \cap B \cap H)$

$\Rightarrow \quad 224 = n(C \cap B' \cap H') + 56 + 16 + 24$

$\Rightarrow \quad n(C \cap B' \cap H') = 128$

Hence the number of students who play only hockey and cricket are 160 and 128 respectively.

$\therefore$ Number of boys who played only one game.

$= n(B \cap H' \cap C') + n(H \cap B' \cap C') + n(C \cap B' \cap H') = 216 + 160 + 128$

$= 504$

6.9.4 Example: An inquiry into 1,000 candidates who failed at ICWA final examination revealed the following data:

658 failed in the aggregate	166 failed in the aggregate and in group I
372 in group I	434 failed in the aggregate and in group II
590 in group II	126 failed in both groups

You have to find out how many condidates failed in:

(a) all the three

(b) in aggregate but not in group II

(c) group I but not in the aggregate

(d) group II but not in group I

(e) aggregate or group II but not in group I

(f) aggregate but not in group I and II

Solution: Let n(4) denote the students who fail in the aggregate n(B), those who fail in the group I and n(C) those who fail in the group II.

Therefore those who fail in all three will be represented by $n(A \cap B \cap C)$ and those who fail in A or B or C by $n(A \cup B \cup C)$. WE are using + for union, - for not and we are not using sign of intersection (in the diagram). The number of elements in a set are shown here by putting n before the braces indicating a given set. We know that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Now by substituting values in the above, we get

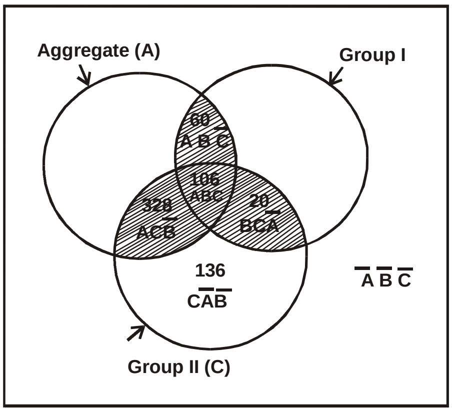
(a) $1{,}000 = 658 + 372 + 590 - 166 - 434 - 126 + n(A \cap B \cap C)$

$$n(A \cap B \cap C) = 106$$

(b) Now we have to find out the value of $n(A \cap C)$ viz,

$$n(A \cap C) = n(A) - n(A \cap C) = 658 - 434 = 224$$

This we can very easily be verfied by use of the following figure also:



(c) In this case we have to find the value of $n(B \cap \bar{A})$ which can the obtained as follows:

$$n(B \cap \bar{A}) = n(B) - n(B \cap A) = 372 - 166 = 206$$

(d) Here we have to find the value of $n(C \cap \bar{B})$, which we can obtain as follows:

$$n(C \cap \bar{B}) = n(C \cap \bar{B} \cap A) + n(C \cap \bar{B} \cap \bar{A}) = 328 + 136 = 464$$

(e) Here $n[(A \cup C) - B] = n[(A - B) \cup (C - B)]$

$$= n(A \cap \bar{B}) + n(C \cap \bar{B}) - n(A \cap \bar{B} \cap C)$$

$$= n(A \cap \bar{B} \cap C) + n(A \cap \bar{B} \cap \bar{C}) + n(C \cap \bar{B} \cap A)$$

$$+ n(C \cap \bar{B} \cap \bar{A}) - n(A \cap \bar{B} \cap C)$$

$$= n(A \cap \bar{B} \cap \bar{C}) + n(A \cap \bar{B} \cap \bar{C}) + n(C \cap \bar{B} \cap \bar{A})$$

$$= 164 + 328 + 136 = 628$$

(f) We have the value of $n\left(A \cap \bar{B} \cap \bar{C}\right)$ equal to 164.

6.10 Ordered Pair:

An ordered pair of objects consists of two elements a and b written in parentheses (a, b) such that one of them, say a is designated as the first member and b as the second member.

Illustration:

(i) The natural numbers and their squares can be represented by ordered pairs in the following manner:

$$(1,1)\ ;\ (2,4)\ ;(3,9);(4,16)\ ;\cdots\cdots\cdots\cdots$$

(ii) The points in plane can be represented by an ordered pair (x, y) where x is the first coordinate called abscissa and y the second coordinate called ordinate. Thus a point represents an ordered pair.

(iii) The order of occurrence of members is of prime importance in an ordered pair. For example an ordered pair (3, 5) is not the same as an ordered pair (5, 3).

(iv) Two odered pais (a, b) and (c, d) will be equal if and only if a = c and b = d.

In other words

$$(a, b) = (c, d) \Rightarrow a = c, b = d \cdot$$

6.11 Cartesion Products:

Definition:

If A and B be any two sets then the set of all ordered pairs whose first membe belongs to set A and second member belongs to set B is called is cartesian product of A and B in that order and is denoted by $A \times B$, to be read as 'A cross B'.

In other words, if A, B are two sets then the set of all ordered pairs of the form (x, y) where $x \in A$ and $y \in B$ is called the cartesian product of the sets A and B. Symbolically

$$A \times B = \left\{(x, y) : x \in A \text{ and } y \in B\right\}$$

Illustration:

Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\} \cdot$

$$A \times B = \{(1, a), (1, b), (1, c), (1, d), (2, a), (2, b), (2, c), (2, d), (3, a), (3, b)$$

$$(3, c), (3, d), (4, a), (4, b), (4, c), (4, d), (5, a), (5, b)\ (5, c), (5, d)\}$$

Similarly we can write down $B \times A$

Example:

If $A = \{1, 2, 3\}$ and $B = \{2, 3\}$ prove that

$$A \times B \neq B \times A$$

Solution:

The cartesian product of sets are:

$$A \times B = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

and $B \times A = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

we notice that $(1, 2)$ and $(1, 3)$ which are the elements of $A \times B$ are not elements of $B \times A$

$$\therefore \quad A \times B \neq B \times A$$

Example:

$A = \{1, 4\}, B = \{2, 3\}, C = \{3, 5\}$ prove that $A \times B \neq B \times A$

Also find $(A \times B) \cap (A \times C)$

Solution:

$A \times B = \{(1, 2), (1, 3), (4, 2), (4, 3)\}$ and

$B \times A = \{(2, 1), (2, 4), (3, 1), (3, 4)\}$

Clearly $A \times B \neq B \times A$

Moreover $A \times C = \{(1, 3), (1, 5), (4, 3), (4, 5)\}$

Thus $(A \times B) \cap (A \times C) = \{(1, 3), (4, 3)\}$

If be a set of real numbers then $R \times R = R^2$ denotes the cartesian plane as

$$R \times R = \{(x, y) : x \in R, y \in R\}$$

each element of this set represents cartesian coordinates of a point in plane.

Example:

Let $A = \{a,\ b\}, B = \{\ p, q\}$ and $C = \{\ q, r\ \}$

Find (a) $A \times (B \cup C)$

(b) $(A \times B) \cup (A \times C)$

(c) $A \times (B \cap C)$

(d) $(A \times B) \cap (A \times C)$

Solution:

(a) If $B \cup C = (p, q, r)$

Then $A \times (B \cup C) = \{a, b\} \times \{p,q,r\}$

$= \{(a, p), (a, q), (a, r), (b, p), (b, q), (b, r)\}$

(b) Since $A \times B = \{(a, p), (a, q), (a, r), (b, p), (b, q), (b, r)\}$

$A \times C = \{(a, q), (a, r), (b, q), (b, r)\}$

Then

$(A \times B) \cup (A \times C) = \{(a, p), (a, q), (b, p), (b, q), (a, r), (b, r)\}$

From (a) and (b) we get

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

(c) $B \cap C = \{q\}$

$A \times (B \cap C) = \{a, b\} \times \{q\} = \{(a, q), (b, q)\}$

(d) $(A \times B) \cap (A \times C) = \{(a, q), (b, q)\}$

From (c) and (d) we find that

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Example:

If $A = \{1, 4\}; B = \{4, 5\}; C = \{5, 7\}$ verify that

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution:

We have

$B \cap C = \{5\}$

$A \times (B \cap C) = \{(1,5), (4,5)\}$ (1)

$A \times B = \{(1,4), (1,5), (4,4), (4,5)\}$

$A \times C = \{(1,5), (1,7), (1,5), (4,7)\}$

$(A \times B) \cap (A \times C) = \{(1,4), (4,5)\}$ (2)

$\therefore$ From (1) and (2) we have

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

6.12 Exercise:

1. Define the following and give an example of each

   (i) Subset of a set (ii) Complement of a set (iii) Union of two sets

   (iv) Intersection of two sets (v) Disjoint of sets.

2. Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $X = \{0, 2, 4, 6, 8\}$, $Y = \{3, 5, 7\}$ and $Z = \{3, 7\}$

   Find (i) $Y \cup Z$ (ii) $V \cup Y \cap X$ (iii) $(X \cup Z) \cup V$

   (iv) $(X \cup Y) \cap Z$ (v) $(\phi \cup V) \cap \phi$

3. If $A = \{5, 6, 7, 8, 9\}$, $B = \{2, 4, 6, 8, 10, 12\}$, $C = \{3, 6, 9, 12\}$

   Verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. Let $P = \{1, 2, x\}$, $Q = \{a, x, y\}$, $R = \{x, y, z\}$.

   Find (i) $P \times Q$ (ii) $P \times R$ (iii) $Q \times R$ (iv) $(P \times Q) \cap (P \times R)$

   (v) $(P \times Q) \cap (R \times P)$ (iv) $(P \times Q) \cup (R \times P)$

Lesson Writer

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Lesson - 7

FUNCTIONS - LIMITS AND CONTINUITY

Objectives:

After studying this lesson, the student should be able to understand -

→ Functions, mappings, notations for functions and types of functions.

→ The concept of limit of a function

→ The continuous function at a point and in an interval.

Structure:

This lesson has the following components:

7.1 Introduction:

A function is a technical term used to symbolise relationship between variables. When two variables are so related that, for any arbitrarily assigned value to one of them, there corresponds a definite value (or a set of definite values) for the other, the second variable is said to be the function of the first.

For example, the distance covered is a function of time and speed, the railway freight charged is a function fo weight or volume, the quantity demanded or supplied is a function of price, etc. The area of a circle depends upon the length of its radius and so the area is said to be the function of radius.

The idea of function is sometimes expressed as:

The relationship between two real variables say x and y which are so related that corresponding to every value 'a' of x defined as the domain we get a finite value 'b' of y defined as the range then y is said to be the function of x.

The domain of variation of x in a function is called the domain of the definition.

The fact that y is a function of a variable x is expressed symbolically by equations:

$$y = f(x),\ y = F(x),\ y = \varphi(x) \text{ etc...}$$

The set of the values of x belonging to the domain or a function generate another set which consists of values of $y\{\text{or } f(x)\}$. This generated set is called the range of the function $f(x)$.

If a is any particular value of x, the value of the function $f(x)$ for $x = a$ is denoted by $f(a)$.

We shall now clarify the above concept of function by following examples:

(i) If y is always equal to x^2 then y is a function of x and we write $y = x^2$

$$\left[\text{here } f(x) = x^2\right]$$

Similarly $\cos x$, e^x, $\log x$, $(x+a)^n$ etc., are all functions of x

(ii) If y is defined by saying that

$$\begin{cases} y = x^2 & \text{when} \quad x > 2 \\ y = x - 1 & \text{when} \quad x \le 2 \end{cases}$$

Here y is defined as a function of x but two formulae have been used to defined the function, one holds for one part of the domain and the other for the remaining part of the domain.

(iii) Consider the two numbers x and y with their relationship defined by the equations:

$$\begin{cases} y = x^2 & \text{when} \quad x < 0 \\ y = x & \text{when} \quad 0 \le x \le 1 \\ y = \dfrac{1}{x} & \text{when} \quad x > 1 \end{cases}$$

The domain of definition of this function which is expressed by three formulae is the whole set of real numbers. The first formula is used for the domain of all real numbers less than 0. The second formula is used for the domain of all real numbers lying between 0 and 1. The third formula isused for the domain of all those real numbers which are greater than 1.

(iv) Let $y = x!$

Here y is a function of x defined for aggregate of positive integers only.

(v) Let $y = |x|$

Here y is a function of x defined for the entire field of real numbers. The same function can also be defined as follows:

$y = x$ when $x \geq 0$

$y = -x$, when $x < 0$

7.2 Mapping:

In modern mathematics the equivalent expression for a function is mapping.

Definition:

If f is a rule which associates every element of set X with one and only one element of set Y, then the rule f is said to be the function or mapping from the set X to the set Y. This we write symbolically as

$$f : X \rightarrow Y$$

If y is the element of Y corresponding to an element x of X, given by rule f, we write this as $y f x$ or $y = f(x)$ and read as 'y' is the value of f at x^2

7.3 Notations for functions:

The mere fact that a quantity is a function of a single variable, say x, is indicated by writing the function in one of the forms $f(x), F(x), \phi(x), g(x), \cdots\cdots\cdots, f_1(x), f_2(x), \cdots\cdots$ If one of these occurs alone, it is read "a function of x" or "some function of x" if some of these are together, they are read "the f - function of x", "the F function of x", "the phi-function of x", ... The latter y is often used to denote a function of x.

Sometimes the exact relation between the function and the dependent variable (or variables) is stated as for example

$$f(x) = x^2 + 3x - 7 \text{ or } y = x^2 + 3x - 7, F(x, y) = 2e^x + 7e^y + xy - 1$$

In such cases f - function of any other number is obtained by substituting that number for x in f(x) and the F - function of any two numbers is obtianed by substituting those for x and y respectively in F(x , y). Thus

$$f(z) = z^2 + 3z - 7 \text{ , } f(4) = 4^2 + 3 \times 4 - 7 = 21$$

(i) Constants: The symbols which retain the same value throughout a set of mathematical operations are called constants.

It has become customary to useinitial alphabets a, b, c, ... as symbols for constants. These are of two types:

(a) Absolute Constants: Those which have the same value in all operations and discussions. For example $\pi, \sqrt{2}$ etc., e are absolute constants.

(b) Arbitrary Constants: Those which may have any assigned value throught a set of mathematical operations.

For example, theradius ofa circle or the sides of a right angled trianlge informing the trigonometric ratios are he arbitrary constants.

(ii) Variables: If a symbol x denotes any element of a given set of numbers, then x is said to be a variable.

The last few alphabets x, y, z or u, v, w, ... are generally used to denote variables.

The variables which can take arbitrarily assigned vlues are usually termed as independent variables. The other variables whose values must be determined in order that they may correspond to these assigned values are usually termed asdependent variables. It will be seen later that a "function" and "dependent variable" are synonymous terms.

(iii) A continuous Real Variable: If x assumes successively every numerical value of an aggregate of all real numbers from a given number 'a' to another given number 'b' then x is called acontinuous real variable.

(iv) A Domain Interval: If a variable x which can take only those numerical values which lie between two given numbres a and b then all the numerical values between a and b taken collectively is called domain or interval ofthe variable x and is usually denoted by (a, b).

If the set of values say x is such that $a \le x \le b$ then the domain or interval (a, b) is called a closed domain or interval. In this case the number a and b are also included in the domain.

If the set of x is such that $a < x < b$ then it is called an open domain interval which is denoted by $[a, b]$ to distinguish it from (a, b). Here the numbers a and b do not belong to the domain.

We may also have semi-closed intervals like.

$(a, b]$ or $a \le x < b$; $[a, b)$ or $a < x \le b$

The first interval is closed on theleft and the second one is closed on the right.

We may have domains of variation extending without bound in one or the other direction, which we write

$(-\infty, b)$ or $x \le b$; (a, ∞) or $x \ge a$; $(-\infty, \infty)$ or any x.

7.4 Types of functions:

We shall now introduce somedifferent types of functions which are particularly useful in calculus.

I. One Valued Functions: When a function has only one value corresponding to each value of the independent variable, the function is called a one valued function. If it has two values corresponding to each value it is called a two valued function. In case a function has several values corresponding to each value of the independent variable, it is called a multiple valued function or a many valued function e.g.,

(i) If $y = x^2$, y is a single valued function of x

(ii) If $y = \sqrt{x}$, y is a two valued function of $x\left(+\sqrt{x} \text{ and } -\sqrt{x}\right)$

II. Explicit Functions: A function expressed directly in terms of the dependent variable is said to be an explicit function, e.g. $y = x^2 + 2x - 5$.

In it one of the variables is dependent on the other and the relationship is not mutual so that the other could be expressed as a dependent variable.

The function which is not expressed directly in terms of the dependent varaible there is a mutual relationship between two variables and either variable determines the other is said to be an implicit function e.g.,

(i) $2x - 3y = 0$ then $x = \frac{3}{2}y$ and $y = \frac{2}{3}x$

(ii) $x^2 + y^2 = 16$ sothat $x = \pm\sqrt{16 - y^2}$ and $y = \pm\sqrt{16 - x^2}$

(iii) $x^2 - y^2 - 6x - 8y - 7 = 0$ so that $y = \pm(y + 4) + 3$

III. Algebraic and Transcendeatal Functions: Functions may also be classified according to the operations involved in the relation connecting a function and its dependent variable (or variables). When the relation which involoves only a finite number of terms and the variables are affected only by the operations of addition, subtraction, multiplication, division, powers and roots the function is said to be an algebraic function.

Thus $2x^2 + 3x^2 - 9, \sqrt{x} - \frac{1}{x^3}$ are algebraic functions of x.

All the functions of x which are not algebraic are called transcendetal functions. We have the following sub-classes of transecendental functions.

(i) Exponential Functions

(ii) Logarithmic Functions

(iii) Trigonometric Functions

(iv) Inverse Trigonometric Functions.

Functions, e.g., $\cos x$, $\tan(x)$, $\sin^{-1} x e^{2x}$ $\log x$ and $\log(4x+5)$ are transcendental functions of x.

IV. Rational and Irrational Functions: Expressions involving x which consist of a finite number of terms of the form ax^n in which 'a' is a constant and n a positive integer e.g.,

$$4x^4 + 3x^3 - 2x^2 + 9x - 7$$

is called a rational integral function of x.

When an expression having more than two terms but only one variable it is called polynomial in x. For example in

$$a_0 x^m + a_1 x^{m-1} + \cdots\cdots\cdots + a_m$$

where $a_0, a_1, a_2, \cdots\cdots\cdots, a_m$ are constants and m is the degree of the polynomial.

If an expression in x, in which x has positive integral exponenets only and a finite number of terms including the division by a rational integral function of x, it is a called a rational function of x, e.g.

$$\frac{x^2+6}{3x^2+9}, \frac{7x^2-4x+7}{(x+5)(2x-3)}, \frac{\frac{x-2}{x+7}}{4x^2+5} + 8x - 9$$

Rational integral functions and rational functions are included in rational functions.

An expression involving x which involves root extraction of terms is called an irrational function, e.g.,

$$\sqrt{x}\ , \sqrt{x^2 + 4x + 5 + 9x - 7}$$

We can say a rational function is an algebraic expression which involves no variable in an irreducible radical form (or under a fractional exponent) a function which can be written as a quotient of polynomials. The expressions $2x^2 + 1$ and $2 + \frac{1}{x}$ are rational but $\sqrt{x+1}$ and $x^{3/2} + 1$ are not.

V. **Monotone Functions:** When the dependent variable increases with an increase in the independent variable, the function is called a monotonically increasing function. For example, the function of supply is a monotonically increasing function of price. As against this a demand functions is a monotonically decreasing function of price because the quantity demanded decreases with every increase in price. A function $y = f(x)$ is called a monotone increasing function in an interval if a larger value of x gives a larger value of y i.e., an increase in x causes an increase in y in an interval. Similarly a function is called a monotone decreasing function in an interval if an increase in the value of x always brings out a decrease in the value the function in the interval. Thus if x_1 and x_2 are only two numbers in the interval such that $x_2 > x_1$ then

$f(x)$ is monotonically increasing if $f(x_2) > f(x_1)$

and $f(x)$ is monotonically decreasing if $f(x_2) < f(x_1)$

VI. **Even and Odd Functions:** If a function $f(x)$ is such that $f(-x) = f(x)$ then it is said to be an even function of x, e.g., x^4, $5x^2$, $7x^2 + \cos x$ are all even functions of x.

Now if a function $f(x)$ is such that

$$f(-x) = -f(x)$$

then it is said to be an odd function of x, e.g., x^3, $5x + 6x^3$, $\sin x$ are all odd functions of x.

VII. **Periodic Functions:** A function such that the range of the independent variable can be separated into equal sub-intervals such that the graph of the function is the same in each part interval. The length of the smallest such part is the period. Technically if $f(x + p) = f(x)$ for all x or $f(x)$ and $f(x+p)$ are both undefined, then p is the period of f. For example, the trigonometric frunction of sine has period 2π radians since

$$\sin(x + 2\pi) = \sin x \text{ of all } x$$

VIII. **Composite Functions:** If $y = g(u)$ and $u = f(x)$ then $y = g\{f(x)\}$ is called a function of function or a composite function. For example the volume of a cylindrical water tank is the function of area and depth. While, area itself is a function of the radius $\left(\because A = \pi r^2 \right)$, so the function of the volume is the function of the area.

IX. Inverse Functions: If $y = f(x)$, defined in an interval (a, b) is a function such that we can express x as a function of y, say $x = \phi(y)$, then $\phi(y)$ is called the inverse of $f(x)$ e.g.,

(i) If $y = \frac{5x + 3}{2x + 9}$, then $x = \frac{3 - 9y}{2y - 5}$ is the inverse of the first function

(ii) $y = \sin^{-1} x$ is the inverse function of $x = \sin y$

(iii) $x = \sqrt[3]{y}$ is inverse function of $y = x^3$

X. Continuous and Discontinuous Functions: A discussion on an exceedingly important classification of functions is given in the next section on limits of a function.

7.5 Limit of a Function:

The limit of a function is that fixed value to which a function approaches as the variable approaches a given value. The function approaches this fixed constant in such a way that the absolute value of the differnce between the function and the constant may be made smaller and smaller than any positive number, however small. This difference continues to remain les than this assigned number say ε when the variable approaches still nearer to the particular value chosen for it.

The limit of a function say ℓ is then that value to which a function $f(x)$ approaches, as x approaches a given value say a. In other words, as x reaches closer and closer to a, the function $f(x)$ reaches closer and closer to ℓ so that given a positive number ε (epsilon), however small we can find a number $\delta = |f(x) - \ell|$ such that $\delta < \varepsilon$ as x approaches closer and closer to a.

Def:

If corresponding to a positive number ε, however small we are able to find a number δ such that

$|f(x) - \ell| < \varepsilon$ for all values of x satisfying $|x - a| < \delta$ then we say that $f(x) \to \ell$ as $x \to a$ and write this symbolically as

$$\lim_{x \to a} f(x) = \ell$$

It should be remembered that thefunction may not actually reach the limit ℓ but it may get closer and closer to ℓ as x approaches a so that $|f(x) - \ell|$ is less than any given value. For example, let us have a function $f(x) = x^2 - 2$. The function approahes the limit 7 as x approaches

3, we can express it is $\lim_{x \to 3} \left(x^2 - 2\right) = 7$. This can be shown below first with x approaching closer and closer to 3 from the lower side:

when $x = 2.99$ $f(x) = 6.99401$

when $x = 2.999$ $f(x) = 6 \cdot 994001$

when $x = 2.9999$ $f(x) = 6 \cdot 99940001$

Now when x approaches 3 from the higher side, we have

when $x = 3.01$ $f(x) = 7 \cdot 0601$

when $x = 3.001$ $f(x) = 7 \cdot 006001$

when $x = 3.0001$ $f(x) = 7 \cdot 00060001$

It is evident from the above that as x is taken closer and closer to 3, $f(x)$ moves closer and closer to 7.

7.6 Methods of Evaluating Limit of a Function:

In this section we shall give the various methods of finding the limits. The following are some theorems on the limits which are often used for evaluating the limits of a function. The proofs are however, beyond the scope of the book.

If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} \phi(x) = B$ then

I. $\lim_{x \to a} \left[f(x) \pm \phi(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} \phi(x) = A \pm B$

This can be extended to any finite number of functions.

II. $\lim_{x \to a} \left[f(x) \cdot \phi(x) \right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \phi(x) = AB$

This can also be extended to any finite number of functions.

III. It obviously follows that

$$\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) = k \cdot A$$

IV. $\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)} = \frac{A}{B}$ where $\lim_{x \to a} \phi(x) \neq 0$

V. $\lim\limits_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim\limits_{x\to a} \cdot f(x)} = \frac{1}{A}$ provided $\lim\limits_{x\to a} f(x) \neq 0$

VI. $\lim\limits_{x\to a} \log f(x) = \log \lim\limits_{x\to a} f(x) = \log A$

7.7 Some Important Limits:

I. $\lim\limits_{x\to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$

II. $\lim\limits_{n\to 0} (1+n)^{1/n} = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

III. $\lim\limits_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

IV. $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

7.8 Left Hand Side and Right Hand Side:

$$\lim_{x\to a-} f(x) = \lim_{h\to 0} f(a-h) = \text{limit of } f(x)$$

when x approaches 'a' from the L.H.S.

also $\lim\limits_{x\to a+} f(x) = \lim\limits_{h\to 0} f(a+h) = \text{limit of } f(x)$

when x approaches 'a' from the R.H.S.

Therefore to find the left hand side limit, we write a - h for x in $f(x)$ and take the limit as $h \to 0$. Similalry to find the right hand side limit, we write a + h for x in $f(x)$ and take the limit as $h \to 0$, where h is always positive.

7.9 Solved Problems:

Problem 1:

Find the behaviour of $\frac{1}{x}$ as $x \to 0$ from the left hand side as well as fromt he right hand side.

Solution:

For L.H.S. we have

$$\lim_{x \to 0-} \frac{1}{x} = \lim_{h \to 0} \frac{1}{0-h} = \lim_{h \to 0} \left(\frac{-1}{h} \right) = -\infty$$

and $$\lim_{x \to 0+} \frac{1}{x} = \lim_{h \to 0} \frac{1}{0+h} = \lim_{h \to 0} \left(\frac{1}{h} \right) = +\infty$$

Problme 2:

Find $\lim_{x \to a} e^{\frac{1}{x-a}}$ when $x \to a$ from the left hand as well as from the right hand side.

Solution:

For L.H.S. limit, we have

$$\lim_{x \to a-} e^{\frac{1}{x-a}} = \lim_{h \to 0} e^{\frac{1}{a-h-a}} = \lim_{h \to 0} e^{-\frac{1}{h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{\frac{1}{h}}} = 0 \quad \text{for } \frac{1}{h} \to \infty \text{ as } h \to 0$$

For the R.H.S. limit, we have

$$\lim_{x \to a+} e^{\frac{1}{x-a}} = \lim_{h \to 0} e^{\left(\frac{1}{a+h-a}\right)} = \lim_{h \to 0} e^{\frac{1}{h}} = \infty$$

Problem 3:

Find the limit of $\frac{4x^4 + 3x^2 - 1}{x^3 + 7}$ when $x \to 1$

Solution:

We have to find $\lim_{x \to 1} \frac{4x^4 + 3x^2 - 1}{x^3 + 7}$

Substituting $x = 1$ in the expression we find that it comes out to be a definite number $\frac{6}{8}$.

Hene the required limit is $\frac{3}{4}$.

Problem 4:

Evaluate $\lim_{x \to 3} \frac{x^2 + 2x - 15}{x^2 - 9}$

Solution:

Replacing x by 3 in the expression, we get $\frac{0}{0}$ which is indeterminate x - 3 must therefore be a factor of the numerator as well as of the denominator.

Factorising we get

$$\lim_{x \to 3} \frac{x^2 + 2x - 15}{x^2 - 9} = \lim_{x \to 3} \frac{(x - 3)(x + 5)}{(x - 3)(x + 3)}$$

$$= \lim_{x \to 3} \frac{x + 5}{x + 3} = \frac{8}{6} = \frac{4}{3} \text{ by putting } x = 3$$

Problem 5:

Prove that $\lim_{x \to 0} \frac{\sqrt{a + x^2} - \sqrt{a - x^2}}{x^2} = \frac{1}{\sqrt{a}}$

Solution:

We find that if we put $x = 0$, we get $\frac{0}{0}$. In such cases retaionalising the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{a+x^2}-\sqrt{a-x^2}}{x^2}=\lim_{x\to 0}\left\{\frac{a+x^2-a+x^2}{x^2\left[\sqrt{\left(a+x^2\right)}+\sqrt{\left(a-x^2\right)}\right]}\right\}$$

$$=\frac{2}{\sqrt{a+0}+\sqrt{a-0}}=\frac{1}{\sqrt{a}}$$

Problem 6:

Evaluate: $\lim_{x\to 0}\frac{\sqrt{1+x}}{x}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\left[\frac{\sqrt{1+x}-1}{x}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right]$$

$$=\lim_{x\to 0}\left[\frac{(1+x)-1}{x\left(\sqrt{1+x}+1\right)}\right]=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}=\frac{1}{2}$$

Problem 7:

Evaluate: $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

where $F(x)=\sin^2 x$

Solution:

$$\lim_{h\to 0}\frac{\sin^2(x+h)-\sin^2 x}{h}=\lim_{h\to 0}\frac{\sin(2x+h)\sin h}{h}$$

$$=\lim_{h\to 0}\left[\sin(2x+h)\frac{\sin h}{h}\right]=\sin 2x$$

Problem 8:

Find $\lim_{x\to 1}\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

Solution:

Substituting x = 1, we get $\frac{0}{0}$, hence by rationalising.

$$\lim_{x\to 1}\frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$$

$$=\lim_{x\to 1}\frac{\left[\sqrt{3+x}-\sqrt{5-x}\right]\left[\sqrt{3+x}+\sqrt{5-x}\right]}{\left(x^2-1\right)\left[\sqrt{3+x}+\sqrt{5-x}\right]}$$

$$=\lim_{x\to 1}\frac{2}{(x+1)\left[\sqrt{3+x}+\sqrt{5-x}\right]}$$

$$=\frac{2}{(1+1)\left[\sqrt{3+1}+\sqrt{5-1}\right]}=\frac{1}{4}$$

Problem 9:

Show that $\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{1}{x^2-3x+2}\right]=1$

Solution:

$$\lim_{x\to 1}\left[\frac{1}{x-2}-\frac{1}{x^2-3x+2}\right]$$

$$=\lim_{x\to 2}\left[\frac{x^2-3x+2-(x-2)}{(x-2)\left(x^2-3x+2\right)}\right]$$

$$=\lim_{x\to 2}\frac{x^2-4x+4}{(x-2)(x-2)(x-1)}$$

$$=\lim_{x\to 2}\frac{1}{x-1}=\frac{1}{2-1}=1$$

Problem 10:

Show that $\lim_{n\to\infty}\frac{2^{-n}\left(n^2+5n+6\right)}{(n+4)(n+5)}=0$

Solution:

$$\text{The given limit} = \lim_{n\to\infty} 2^{-n} \lim_{n\to\infty} \frac{n^2 + 5n + 6}{(x+4)(n+5)}$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \cdot \lim_{n\to\infty} \frac{n^2 + 5n + 6}{n^2 + 9n + 20}$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \cdot \lim_{n\to\infty} \frac{1 + \frac{5}{n} + \frac{6}{n^2}}{1 + \frac{9}{n} + \frac{20}{n^2}}$$

$$= 0 \times 1 = 0$$

Problem 11:

Show that $\lim_{x\to 1} f(x)$ exists and is equal to $f(1)$ where $f(x) = x + 1$ for $x \le 1$

$= 3 - x^2$ for $x > 1$

Solution:

We have

$$f(1) = 1 + 1 = 2$$

$$L \cdot H \cdot L = \lim_{x\to 1-} f(x) = \lim_{h\to 0} f(1-h), h > 0$$

$$= \lim_{h\to 0} \left[(1-h) + 1 \right] = 2$$

$$R \cdot H \cdot L \cdot = \lim_{x\to 1+} f(x) = \lim_{h\to 0} f(1-h), h > 0$$

$$= \lim_{h\to 0} \left[3 - (1+h)^2 = 2 \right]$$

Since L.H.L. = R.H.L., $\lim_{x\to 1} f(x)$ exists and is equal to $f(1)$

Example 12:

Discus theexistence of $\lim_{x \to \frac{3}{2}} f(x)$, if

$$f(x) = 3 + 2x \text{ for } -\frac{3}{2} \le x < 0$$

$$= 3 - 2x \text{ for } 0 \le x < \frac{3}{2}$$

$$= -3 - 2x \text{ for } x \ge \frac{3}{2}$$

Solution:

$$R \cdot H \cdot L \cdot = \lim_{x \to \frac{3}{2}+0} f(x) = \lim_{h \to 0} f\left(\frac{3}{2} + h\right), h > 0$$

$$= \lim_{h \to 0} \left[-3 - 2\left(\frac{3}{2} + h\right)\right] = -3 - 3 = -6$$

$$L \cdot H \cdot L \cdot = \lim_{x \to \frac{3}{2}-} f(x) = \lim_{h \to 0} f\left(\frac{3}{2} - h\right), h > 0$$

$$= \lim_{h \to 0} \left[3 - 2\left(\frac{3}{2} - h\right)\right] = 3 - 3 = 0$$

Since $L \cdot H \cdot L \cdot \ne R \cdot H \cdot L \cdot$, $\lim_{x \to \frac{3}{2}} f(x)$ does not exist.

7.10 Continuity of a Functions:

A function $f(x)$ is said to be continuous at $x = a$ if corresponding to any arbitrarily assigned positive number ε, however small (but not zero) there exists a positive number δ such that

$$|f(x) - f(a)| < \varepsilon \text{ for all } |x - a| \le \delta$$

We note that in order to obtain the above we have to replace ℓ by $f(a)$ in the definition of $f(x)$ tending to a limit ℓ as $x \to a$. Hence the above definition becomes:

A function $f(x)$ is said to be continuous at a point $x = a$. If $f(x)$ possesses a finite and definite limit as x tends to the value 'a' from either side and each ofthese limits is equal to $f(a)$ so that

$$\lim_{x \to a-} f(x) = f(a) = \lim_{x \to a+} f(x)$$

Thus the continuity of a function at point $x = a$ boils down to the determination of three numbers:

$$\text{(i) } f(a), \text{ (ii) } \lim_{x \to a-} f(x), \text{ (iii) } \lim_{x \to a+} f(x)$$

which involves only the simple process of

(i) replacing x by 'a' in $f(x)$ and then finding if $f(a)$ is finite and definite.

(ii) evaluating the left hand limit
(iii) evaluating the right hand limit
} by methods already explained

If all the three number so obtained are equal, then $f(x)$ is continuous at $x = a$ otherwise it is discontinuous.

Example 1:

Show that $f(x) = 3x^2 + 2x - 1$ is continuous at x = 2. Hence prove that $f(x)$ is continuous for all values of x.

Solution:

The conditions to be satisfied by a function before we can say that it is continuous at a particular point say $x = a$ are

$$f(a), \lim_{x \to a-} f(x) \text{ and } \lim_{x \to a+} f(x)$$

should have definite and finite values and that

$$\lim_{x \to a-} f(x) = f(a) = \lim_{x \to a+} f(x)$$

Let us examine whether these conditions are satisfied by $f(x) = 3x^2 + 2x - 1$ for $x = 2\cdot$ Here $a = 2$ therefore, we have

(i) $f(2) = 3 \cdot 2^2 + 2 \cdot 2 - 1 = 15$

Again by the method of finding the left hand and right hand side limits, we have

(ii) $$\lim_{x \to 2-} \left(3x^2 + 2x - 1\right) = \lim_{h \to 0} \left\{3(2-h)^2 + 2(2-h) - 1\right\} = 15$$

$\therefore$ Left hand side limit = 15

Also (iii) $$\lim_{x \to 2+} \left(3x^2 + 2x - 1\right) = \lim_{h \to 0} \left\{3(2+h)^2 + 2(2+h) - 1\right\} = 15$$

$\therefore$ Right hand side limit = 15

We findt hat the value of the function at $x = 2$, the left hand and the right hand limits all exist, and are finite and equal.

We shall show further that $f(x) = 3x^2 + 2x - 1$ is continuous for values of x. The method followed is quite general and the students are required to note it carefully.

Let $x = k$ be any value of x arbitrarily selected and find out whether the given function is continuous at $x = k$.

Here $a = k$ therefore $f(k) = 3k^2 + 2k - 1$ (finite number)

Also $$\lim_{x \to k-} \left(3x^2 + 2x - 1\right) = \lim_{h \to 0} \left(3(k-h)^2 + 2(k-h) - 1\right) \cdots\cdots\cdots\cdots (1)$$

$$= \lim_{h \to 0} \left(3k^2 - 6kh + 3h^2 + 2k - 2h - 1\right)$$

$$= 3k^2 + 2k - 1 \cdots\cdots\cdots\cdots (2)$$

Similarly we find that

$$\lim_{x \to k+} \left(3x^2 + 2x - 1\right) = 3k^2 + 2k - 1 \cdots\cdots\cdots\cdots (3)$$

From (1), (2) and (3) we deduce that the given function is continuous at $x = k$. since k is any arbitrary value of x, therefore $f(x)$ is continuous for all values of x.

Example 2:

Discuss the continuity of

$$f(x) = \frac{|x|}{x}, x \neq 0$$

$f(x) = 0 \, , x = 0$

Solution:

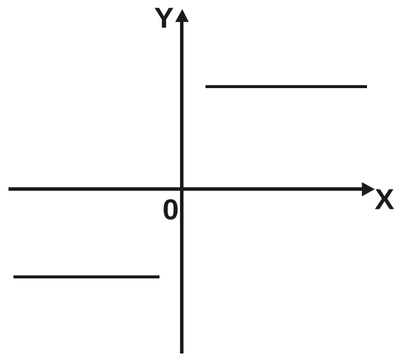
$f(x)$ can be written as

$f(x) = 1, \; x > 0$

$f(x) = 0 \, , \; x = 0$

$f(x) = -1, x < 0$

Graphically the function is discontinuous at $x = 0$. Analytically we have $f(0) = 0$



But $\lim_{x \to 0} f(x) = -1$ which is not equal to $f(0)$.

Hence the function is discontinuous at this point.

Example 3:

Show that

$$f(x) = \frac{e^{-1/x}}{1 + e^{1/x'}} \quad \text{when} \quad x \neq 0$$

= 0, when $x = 0$ is not continuous at $x = 0$

Solution:

Here $f(0) = 0$ given

$$L \cdot H \cdot L \cdot = \lim_{x \to 0-} \frac{e^{-\frac{1}{x}}}{1 + e^{\frac{1}{x}}} = \lim_{h \to 0} \frac{e^{-\frac{1}{0-h}}}{1 + e^{\frac{1}{0-h}}} \qquad (1)$$

$$= \lim_{h\to 0} \frac{e^{\frac{1}{h}}}{1+e^{-\frac{1}{h}}} = \infty \qquad (2)$$

$$\left[\because \frac{1}{h} \to \infty \text{ as } h \to 0, \text{ therefore } \frac{1}{e^{1/h}} \to 0 \text{ as } h \to 0 \text{ and } e^{1/h} \to \infty \text{ as } h \to 0\right] \qquad (3)$$

$$R\cdot H\cdot L\cdot = \lim_{x\to 0+} \frac{e^{-1/x}}{1+e^{1/x}} = \lim_{h\to 0} \frac{e^{-1/h}}{1+e^{1/h}} = 0$$

(2) and (3) $\Rightarrow R\cdot H\cdot L\cdot \neq L\cdot H\cdot L\cdot$

$\lim_{x\to 0} f(x)$ does not exist

$\Rightarrow f(x)$ is discontinuous at x = 0

Example 4:

Consider the functions defined as follows:

(a) $f(x) = \frac{x^2-4}{x-2}$, for $x < 2$

$f(x) = 4$, for $x = 2$

$f(x) = 2$, for $x > 2$

(b) $f(x) = \frac{x^2-4}{x-2}$, for $x < 2$

$f(x) = 2$, for $x \geq 2$

(c) $f(x) = \frac{x^2-4}{x-2}$, when $0 \leq x < 2$

$f(x) = 2$, when $x = 2$

$f(x) = x+1$, when $x > 2$

(d) $f(x) = \frac{x^2-4}{x-2}$ when $0 \leq x < 2$

$f(x) = 2$, when $x = 2$

$f(x) = \frac{3x+2}{x}$, when $x > 2$

Discuss the continuity at $x = 2$

Solution:

(a) Here $f(2) = 4$ a finite and definite given number. The first condition is, therefore, satisfied.

The function is defined for values of $x < 2$ by $f(x) = \frac{x^2-4}{x-2}$

$\therefore$ For the left hand limit we have to take up the first part of function. We then have

$$\lim_{x \to 2-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2-} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2-} (x+2) = 4$$

Again the function is defined for values of $x > 2$ by $f(x) = 2$, therefore the right hand limit is to be calculated from $f(x) = 2$.

Here we have $\lim_{x \to 2+} 2 = 2$

Also we see that $f(2) = \lim_{x \to 2-} f(x) \neq \lim_{x \to 2+} f(x)$

Hence the function is discontinuous at x = 2.

(b) Here $f(2) = 2$ and $\lim_{x \to 2+} f(x) = \lim_{x \to 2+} 2 = 2$

But $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2-} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2-} (x+2) = 4$

Hence $f(2) = 2 \neq \lim_{x \to 2-} \frac{x^2 - 4}{x - 2}$

$\therefore$ Hence $f(x)$ is discontinuous at $x = 2$

(c) Here $f(2) = 2$ (given)

$$\lim_{x \to 2-} \frac{x^2 - 4}{x - 2} = 4 \text{ and } \lim_{x \to 2} (x+1) = 3$$

Which show that

$$\lim_{x \to 2-} \frac{x^2 - 4}{x - 2} \neq \lim_{x \to 2+} (x+1) \neq f(2)$$

Hence the function is discontinuous at $x = 2$

(d) Here $\lim_{x \to 2-} \frac{x^2 - 4}{x - 2} = 4$, $\lim_{x \to 2+} \frac{3x + 2}{x} = 4$

but $f(2) = 2$ which means that

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) \neq f(2)$$

Hence the function is discontinuous at $x = 2$

Example 5: Show that the function $f(x)$ as defined below, is discontinuous at $x = \frac{1}{2}$

$f(x) = x$, when $0 \leq x < \frac{1}{2}$

$f(x) = 1$, when $x = \frac{1}{2}$

$f(x) = 1 - x$, when $\frac{1}{2} < x < 1$

Solution:

We are given that

$f(x) = 1$ when $x = \frac{1}{2}$

which means that $f\left(\frac{1}{2}\right) = 1$, i.e. the value of the function at $x = \frac{1}{2}$ is 1.

$\therefore$ The first condition is satisfied

Now let us find $\lim_{x \to \frac{1}{2}-} f(x)$ and $\lim_{x \to \frac{1}{2}+} f(x)$

In $\lim_{x \to \frac{1}{2}-} f(x)$, we have to find the limit when $x \to \frac{1}{2}$ through those values of x which are less than $\frac{1}{2}$.

Now $f(x) = x$ is the only part of the function which is defined for values of x such that $0 \leq x < \frac{1}{2}$

Therefore, we should find $\lim_{x \to \frac{1}{2}-} x$. By the method of finding the left hand limit, we have

$$\lim_{x \to \frac{1}{2}-} x = \lim_{h \to 0} \left(\frac{1}{2} - h \right) = \frac{1}{2}$$

at this stage we stop and say that $f(x)$ is discontinuous at $x = \frac{1}{2}$ because

$$f\left(\frac{1}{2}\right) \neq \lim_{x \to \frac{1}{2}-} x$$

There is no need of finding the right hand limit. In case we want to find the right hand limit then we must select $f(x) = 1 - x$ as our function because this is the only part of the given function which is defined for values of x greater than $\frac{1}{2}$.

7.11 Continuity in an Interval:

A function $f(x)$ is said to be continuous in the closed interval (a, b) if it is continuous for every value of x in $a < x < b$, and if $f(x)$ is continuous from the right at 'a' and from the left at 'b' i.e., if $\lim_{x \to a+} f(x)$ exists and is equal to $f(a)$, and $\lim_{x \to b-} f(x)$ exists and is equal to $f(b)$.

It is easily deduced from the theorems on limits that the sum, product, difference or quotient of two functions which are continuous at a certain point are themselves continuous at that point, except that in the case of quotient in which the denominator must not vanish at the point in question. Further it is true that the function of a continuous function is a continuous function.

We now take up a few examples to illustrate the method of application of the set of conditions arrived at in the previous sections to prove the continuity of a function at a point as well as in an interval.

Example 1:

A function $f(x)$ is defined as follows:

$$f(x) = \frac{9x}{x+2}, \text{ for } x < 1$$

$$f(1) = 3$$

and $$f(x) = \frac{x+3}{x}, \text{ for } x > 1$$

Examine the continuity of $f(x)$ in the interval $(-3, 3)$

Solution:

$f(x) = \frac{9x}{x+2}$ is to be considered for values of x lying between - 3 and 1 because this is he part of the function which is defined for value of x < 1.

The denominator of $\frac{9x}{x+2}$ becomes 0 when $x = -2$. But - 2 is a point between -3 and 1. Hence $f(x)$ is discontinuous at x = -2 as the function is not defined at x = -2. Again $f(1) = 3$.

and $\lim_{x \to 1-} \frac{9x}{x+2} = \lim_{h \to 0} \frac{9(1-h)}{(1-h)+2} = 3$

also $\lim_{x \to 1+} \frac{x+3}{x} = \lim_{h \to 0} \frac{1+h+3}{1+h} = 4$

Since the right hand limit is not equal to the left hand limit and is not equal to the value of the function at $x = 1$, therefore the given function is discontinuous at $x = 1$.

Hence $f(x)$ is discontinuous at $x = -2$ and 1, for all other values of x it is continuous.

Example 2:

Show that the function $f(x) = x \sin(1/x)$, $x \neq 0$ is continuous at $x = 0$, where $f(0) = 0$

Solution:

Here $f(0) = 0$

and $\lim_{x \to 0-} x \sin \frac{1}{x} = \lim_{h \to 0} (-h) \sin(1/-h) = \lim_{h \to 0} h \sin(1/h) = 0$

Also $\lim_{x \to 0+} x \sin \frac{1}{x} = \lim_{h \to 0} h \sin \frac{1}{h} = 0$

Hence the function is continuous at $x = 0$

Example 3:

Show that the function defined as under is ontinuous at $x = 0$

$f(x) = x^2 \cos e^{1/x}$ for all x excepting $x = 0$

Render it continuous by changing its definition.

Solution:

Here $f(0)=1$

$$\lim_{x\to 0-} x^2 \cos^{1/x} = \lim_{x\to 0} h^2 \cos\frac{1}{e^{1/h}} = 0$$

Also $$\lim_{x\to 0+} x^2 \cos e^{1/x} = \lim_{h\to 0} h^2 \cos e^{1/h} = 0$$

$$\therefore \lim_{x\to 0} x^2 \cos e^{1/x} = 0$$

Since $\lim_{x\to 0} f(x) \neq f(0)$, the function is nto continuous at $x=0$.

The function becomes continuous at $x=0$ if we define the function as follows:

$$f(x) = x^2 \cos e^{1/x},\ x \neq 0$$

$$f(x) = 0, \text{ when } x = 0$$

7.12 Exercise:

Evaluate the following limits:

1. (a) $\lim_{x\to 2} \frac{2x^2 - 7x + 6}{5x^2 - 11x + 2}$ (b) $\lim_{x\to 1} \frac{x^3 - 5x^2 + 2x + 2}{x^3 + 2x^2 - 6x + 3}$

   (c) $\lim_{x\to 0} \frac{4x^4 + 5x^3 + 7x^2 + 6x}{5x^5 + 7x^2 + x}$

2. (a) $\lim_{x\to 2} \frac{(x^2 - 5x + 6)(x^2 - 3x + 2)}{x^3 - 3x^2 + 4}$ (b) $\lim_{x\to \infty} \frac{\sqrt{3x^4 - 5x^3 + 7x + 5}}{4x^2}$

3. (a) $\lim_{x\to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x}$ (b) $\lim_{x\to a} \frac{\sqrt{x+a} - \sqrt{2a}}{x-a}$

   (c) $\lim_{x\to 2} \frac{\sqrt{x^2 + x - 3} - \sqrt{x+1}}{x-2}$

4. (a) $\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{2}{x(x-1)(x-2)}\right]$ (b) $\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{3}{x\left(x^2-5x+6\right)}\right]$

(c) $\lim_{x\to a}\frac{x^{-3}-a^{-3}}{x^{-2}-a^{-2}}$

5. Prove that the function x^2+4x-2 iscontinuous at $x=1$

6. Prove that the function $\frac{x^2-9}{x-3}$ is discontinuous at $x=3$

7. Discuss the ontinuity of the following functions:

(i) $f(x)=\frac{1}{x}$ at $x=0$ (ii) $f(x)=\frac{1}{x^2}$ at $x=0$

8. Show that the function defined as

$f(x)=1$ for $x=0$

$f(x)=x$ for $x>0$

is discontinuousat the end point x = 0

9. (a) A function $f(x)$ is defined as follows:

$f(x)=\frac{1}{2}-x$, when $0<x<\frac{1}{2}$

$f(x)=0$, when $x=\frac{1}{2}$

$f(x)=\frac{3}{2}-3x$, when $\frac{1}{2}<x<1$

Show that $f(x)$ is continuous at $x=\frac{1}{2}$

Lesson Writer

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Lesson - 8

MATRICES

Objectives:

- → After studying this lesson, you should be able to understand -
- → Types of matrices, scalar multiplication of matrix, equality of matrices, additions, subtraction, multiplication of matrices.

Structure:

This lesson has the following components:

8.1 Introduction:

A matrix consists of a rectangular presentation of symbols or numerical elements aranged systematically in rows and columns describing various aspects of a phenomenon inter-related in some manner.

It is a powerful tool in modern mathematics having wide applications. Scociologists use matrices to study the dominance within a group. Demographers use matrices in the study of births and survivals, marriage and descent, class structure and mobility etc. Matrices are all the more useful for practical business purposes and therefore occupay an important place in Business Mathematics. Obviously, because business problems can be presented more easily in distinct finite number of gradations than ion infinite gradations as we hve in calculus. The matrix form therefore suits very well for games theory, allocation of expenses budgeting for by-products etc. Economists now use matrices very extensively in 'social accounting', 'input-output tables' and in thestudy of 'inner-industry economics'.

A matrix to put in simple language is a rectangular array of numbers. Now what is a rectangular array? For this, we consider the following illustrations:

1. In an elocution contest, a participant can speak either of the five languages: Hindi, English, Punjabi, Gujarati and Tamil. A college (say No. 1) sent 30 students of which 10 offered to speak in Hindi, 9 in English, 6 in Punjabi, 3 in Gujarati and the rest in Tamil. Another college (say No. 2) sent 25 students of which 7 spoke in Hindi, 8 in English and 10 in Punjabi. Out of 22 students from the third college (say No. 3), 12 offered to speak in Hindi, 5 in English and 5 in Gujarati.

The information furnished int he above manner is somewhat cumbersome. It can be written in a more compact manner if we consider the following tabular form:

	Hindi	**English**	**Punjabi**	**Gujarati**	**Tamil**
College 1	10	9	6	3	2
College 2	7	8	10	0	0
College 3	12	5	0	5	0

The number in the above data are said to form a rectangular array. In any such array, lines across the page are called rows and lines down the page are called columns. Any one number within the arrangement is called an entry or an element. Thus int he above data there are 3 rows and 5 columns and hence $3 \times 5 = 15$ elements. If it is enclosed by a pair of square brackets then

$$\begin{bmatrix} 10 & 9 & 6 & 3 & 2 \\ 7 & 8 & 10 & 0 & 0 \\ 12 & 5 & 0 & 5 & 0 \end{bmatrix}$$

is called a matrix.

Since it has 3 rows and 5 columns it is said to be a matrix of order 3×5 or simply a 3×5 (read as '3 by 5') matrix. It may be noted that a matrix can have any number of rows and any number of columns. Thus in the above illustration if there are entries from 12 colleges and if the competition is held in 8 languages then we can construct a 12×8 matrix.

2. Consider a system of two linear equations in the unknown, viz.,

$$2x - 3y + z = 7$$

$$4x + 5y - 3z = 5$$

The co-efficients of x, y, z in thefirst equation are 2, -3, 1 and those in the second are 4, 5, -3 respectively. They form the matrix (called the co-efficient matrix).

$$\begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{pmatrix}$$

which is a 2×3 matrix.

Remark: The reason for enclosing a rectangular array by a pair of brackets is that hereafter we shall treat a rectangular array (and hence a matrix) as a single entity. In fact we shall develop a new algebra which may be called 'Algebra of Matrices' where operatiosn are performed on the whole array of numbers and not on a single number. It will be seen that this algebra bears a close resemblance to the Algebra of Sets.

8.2 Def. of a Matrix:

A matrix is a rectangular array of numbers arranged in rows and columns enclosed by a pair of brackets and subject to certain rules of presentation. The numbers can be substituted by symbols, with appropriate suffixes indicating the row and column numbers. It will be possible to identify the exact location of a number or a symbol in the whole arrangement of a matrix. We will find that through a matrix form of presentation, the complex phenomeana with various characteristics or relations would be presented in a very concise manner.

Sometimes a pair of brackets [ ], or a pair of double bars || || are used instead of a pair of parentheses e.g., the matrix

$$\begin{pmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{pmatrix}$$

may also be written as

$$\begin{bmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{bmatrix} \text{ or } \begin{Vmatrix} 2 & -3 & 1 \\ 4 & 5 & -3 \end{Vmatrix}$$

Notations:

A matrix is usually denoted by a capital letter and its elements by corresponding small letters followed by two suffixes, the first one indicating the row and the second one the column in which the element appears.

For example, in the first illustration just as the colleges were numbered from 1 to 3, let the languages be numbered from 1 to 5. Then the matrix can eb written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$$

where $a_{11} =$ number of students from college no 1 who offered language No. 1 (i.e., Hindi) = 10, a_{12} those from College No. 1 offering language No. 2 (i.e., English) = 9 and so on.

It should be noted that all the elements in the 1st row hve 1 as the first suffix, those in the 2nd and 3rd rows have respectively 2 and 3 as the first suffix. Also all the elements in the 1st column have 1 as the second suffix those in the 2nd, 3rd, 4th and 5th columns have respectively 2, 3, 4 and 5 as the second suffix.

A general form of a Matrix:

A matrix of order $m \times n$ (i.e., one having m rows and n colums) can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

where $a_{11}, a_{12}, \cdots$ stand for real numbers. The above matrix can also be written in a more concise form as:

$$A = \left[a_{ij} \right]_{m \times n}$$

where $i = 1, 2, \cdots\cdots, m$; $j = 1, 2, \cdots\cdots, n$ and where a_{ij} is the element in the i^{th} row and j^{th} column and is referred as $(i, j)^{th}$ element.

Illustration:

Read the elements $a_{24}, a_{41}, a_{13}, a_{22}$ and the corresponding 'b' elements in the following matrices.

$$A = \begin{pmatrix} 3 & 4 & 5 & 9 \\ 2 & 0 & -6 & 2 \\ 1 & 3 & 7 & 8 \\ 3 & -6 & -2 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 7 & 8 \\ -1 & 2 & 6 \end{pmatrix}$$

Solution:

(i) Let $A = \begin{pmatrix} 3 & 4 & 5 & 9 \\ 2 & 0 & -6 & 2 \\ 1 & 3 & 7 & 8 \\ 3 & -6 & -2 & -4 \end{pmatrix}_{4 \times 4}$

Now a_{24} indicates the element which appears in the second row and fourth column.

$$\therefore\ a_{24} = 2$$

Again a_{41} indicates the element which appears in teh fourth row and first column.

$\therefore\ a_{41} = 3$

Similarly $a_{13} = 5$, $a_{22} = 0$ ·

(ii) Here $B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 7 & 8 \\ -1 & 2 & 6 \end{pmatrix}_{4 \times 3}$

b_{24} = not possible, $b_{41} = -1, b_{13} = 1$ and $b_{22} = 2$ ·

8.3 Types of Matrices:

8.3.1 Square Matrix: A matrix in which the number of rows is equal to the number of columns is called a square matrix. Thus a $m \times n$ matrix will be a square matrix if $m = n$ and it will be referred as a square matrix of order n or n-rowed matrix. Thus are square matrices.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2 \times 2} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{3 \times 3} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$$

Remark: In a square matrix all those elements a_{1j} for which $i = j$, i.e., those which occur in the same row and same column namely $a_{11}, a_{22}, \cdots\cdots, a_{nn}$ are called the diagonal elements. A square matrix has of course two diagonals. Diagonal extending from the upper left to the lower right is more important than the other diagonal. This is known as the principal diagonal or the main diagonal and its elements are called the diagonal elements.

Illustration:

$$\begin{bmatrix} 1 & 2 & -3 \\ 6 & 8 & 5 \\ 2 & -1 & 6 \end{bmatrix}$$

3×3 (square) Matrix

Principal Diagonal is $(1, 8, 6)$

8.3.2 Row and Column Matrices: A row matrix is defined as a matrix having a single row and a column matrix is one having a single column, e.g.,

$\left[a_{11}, a_{12}, \cdots, a_{1n} \right]_{1 \times n}$ is a row matrix

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{mi} \end{bmatrix}_{m \times 1} \text{ is a column matrix}$$

Remark: Row and Column matrices are sometimes called the row and column vectors. The latter names are also used to designate any row or column of a $m \times n$ matrix.

8.3.3 Diagonal Matrix: A square matrix all of whose elements, except those in the leadingdiagonal are zero is called a diagonal matrix. Thus the matrix is a diagonal matrix and may be written as $A = \text{diag}\left(a_{11} \quad a_{22} \quad \cdots \quad a_{nn}\right)$

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}_{n \times n}$$

Remarks:

1. The square matrix A will be a diagonal matrix if all elements a_{ij} for which $i \neq j$ are zero.

2. A diagonal matrix whose all the diagonal elements are equal is called a scalar matrix e.g.,

$$A = \begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} = \text{diag}(\ a, a, \cdots\cdots\cdots, a\ )$$

8.3.4 Unit Matrix: A scalar matrix each of whose diagonal element is unity (or one) is called a unit matrix or an identity matrix. A unit matrix of order n is written as I_n. Thus

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ are unit matrices of order 2 and 3 respectively.

Remark: In general for a unit matrix.

$$\begin{cases} a_{ij} = 0, & i \neq j \\ a_{ij} = 1, & i = j \end{cases}$$

8.3.5 Zero Matrix or Null Matrix: A matrix, rectangular or square, each of whose elements are zero is called a zero matrix or a null matrix and is denoted by 0.

$$0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a zero matrix of order } 4 \times 4$$

8.3.6 Triangular Matrices: A square matrix $A = \left(a_{ij}\right)_{n \times n}$ is called upper triangular matrix if $a_{ij} = 0$ for i > j and is called lower triangular matrix if $a_{ij} = 0$ for i < j.

Thus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

are upper and lower triangular matrices.

8.3.7 Sub Matrix: A matrix obtained by deleting some rows or columns or both of a given matrix is called a sub matrix of a given matrix.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{4 \times 4}$$

If we delete the first row and first column, the sub matrix of A is

$$\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}_{3 \times 3}$$

8.3.8 Scalar Matrix: A square matrix when given in the form of a scalar multiplication to an identity matrix is called a scalar matrix. For example

(i) $$aI = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) $aI = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

are scalar matrices.

8.3.9 Symmetric Matrices: A symmetric matrix is a special kind of a square matrix $A = (a_{ij})$ for which

$a_{ij} = a_{ji}$ for all i and j

i.e., the $(i, j)^{th}$ element $= (j, i)^{th}$ element. i.e. $A^T = A$ For example the matrices.

$$\begin{pmatrix} 5 & 2 & 1 \\ 2 & 6 & -1 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

are symmetric matrices.

8.3.10 Complex Conjugate of a Matrix: It is a matrix obtained by replacing all its elements by their respective complex conjugates.

For example

If $A = \begin{bmatrix} 2+3i & 4 \\ 5-3i & 7 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2-3i & 4 \\ 5+3i & 7 \end{bmatrix}$

8.3.11 Skew-Symmetric Matrix: It is square matrix A if

$$A^t = -A$$

i.e., the transpose of a square matrix is equal to the negative of that matrix. For example the following matrix is skew symmetric.

$$A = \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}$$

8.4 Scalar Multiplication of a Matrix:

A real number is referred to as a scalar when it occurs in operations involving matrices. The scalar multiple kA of a matrix A by scalar k, is a matrix obtained by multiplying every element of A by the scalar k, i.e., the scalar multiple of the matrix $A = [a_{ij}]_{m \times n}$ by scalar k is the matrix $X = [c_{ij}]_{m \times n}$ where $c_{ij} = k a_{ij}$, $i = 1, 2, \cdots\cdots, m$; $j = 1, 2, \cdots\cdots, n$. Thus if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ then } kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Illustrations:

1. $$3\begin{pmatrix} 4 & -3 \\ 8 & -2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 12 & -9 \\ 24 & -6 \\ -3 & 0 \end{pmatrix}$$

2. $$5\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$$

3. If $$A = \begin{pmatrix} 3 & 7 & 6 & -5 \\ 2 & -6 & 0 & 4 \\ 5 & 2 & 8 & 8 \\ -1 & 6 & 5 & -3 \end{pmatrix}$$

then $$-A = \begin{pmatrix} -3 & -7 & -6 & 5 \\ -2 & 6 & 0 & -4 \\ -5 & -2 & -8 & -8 \\ 1 & -6 & -5 & 3 \end{pmatrix} \quad 4A = \begin{pmatrix} 12 & 28 & 24 & -20 \\ 8 & -24 & 0 & 16 \\ 20 & 8 & 32 & 32 \\ -4 & 24 & 20 & -12 \end{pmatrix}$$

8.5 Equality of Matrices:

Two matrices are said to equal if and only if

(i) They are comparable, i.e., they are of the same order if one is 3×2, the other one is also 3×2 and not 2×3.

(ii) Each element of one is equal to the corresponding element of the other, i.e., if

$A = \left[a_{ij} \right]_{m \times n}$ and $B = \left[b_{ij} \right]_{m \times n}$ then

$$A = B \text{ iff } a_{ij} = b_{ij} \quad \forall \quad \left. \begin{matrix} i = 1, 2, \cdots\cdots, m \\ j = 1, 2, \cdots\cdots, n \end{matrix} \right\}$$

Illustrations:

1. If

$$A = \begin{pmatrix} 4 & 7 & 0 \\ 7 & -2 & 5 \end{pmatrix}, B = \begin{pmatrix} 4 & 7 & 0 \\ 7 & -2 & 5 \end{pmatrix}$$

then A = B.

2. If

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$$

then $A \neq B$ $(\text{since } a_{22} = -2 \text{ and } b_{22} = 1)$

3. If

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 8 & 6 \end{pmatrix}, B = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 8 & 6 \\ 1 & 2 & 5 \end{pmatrix}$$

$A \neq B$ because first they are not comparable, matrix A being 2×3 and B being 3×3. Second, the elements are not the same in respective columns and rows.

4. The following is a statement of matrix equality given the values of the components.

$$\begin{pmatrix} x+y & 2z+w \\ x-y & z-w \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$$

if $x = 2, y = 1, z = 3$ and $w = -1$.

8.6 Matrix Operations:

In matrix algebra the elements are ordered numbers and therefore operations on them have to be done in a manner an army sergeant gives drill to the platoon. Every cadet has to maintain his position vis-a-vis his fellow cadets. Again the main operations are addition and multiplication while the subtraction and division is derived out of these oeprations.

8.6.1 Addition and Subtraction:

(i) Matrices can be added or subtracted if and only if they are of the same order.

(ii) The sum or difference of two $(m \times n)$ matrices is another matrix $(m \times n)$ whose elements are the sum or differences of the corresponding elements in the component matrices.

Symbolically let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of order $m \times n$ each then their sum (difference) $A \pm B$ is the matrix $C = [c_{ij}]_{m \times n}$ where

$c_{ij} = a_{ij} \pm b_{ij}$; $\left\{ \begin{matrix} i = 1, 2, \cdots\cdots, m \\ j = 1, 2, \cdots\cdots, n \end{matrix} \right\}$ is the matrix each element of which is the sum (difference) of the corresponding element of A and B. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}_{m \times n}$$

8.6.2 Properties:

Commutative:

I. If A and B are any two matrices of order $m \times n$ each, then

$$A + B = B + A$$

II. If A, B and C are any three comparable matrices of the same type $m \times n$, then

$$(A + B) + C = A + (B + C)$$

III. Distributive with respect to scalar.

$$k(A + B) = kA + kB$$

IV. Existence of An additive Identity,

$$A + O = A = O + A$$

where O is the null matrix of the same type.

V. Existence of an Inverse: If A be any given matrix then the matrix - A which must exist, is the additive inverse of A.

$$A + (-A) = O = (-A) + A$$

VI. Cancellation Law: If A, B, C are matrices of the same order, then

$$A + C = B + C \Rightarrow A = B$$

Example 1: If $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{pmatrix}$ find the value of $2A + 3B$.

Solution:

$$2A = 2\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 0+21 & 4+18 & 6+9 \\ 4+3 & 2+12 & 8+15 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$$

8.6.3 Multiplication: Earlier we considered scalar product of a matrix. To recollect if

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} \text{ then } 3A = \begin{pmatrix} 3\times 2 & 3\times 0 \\ 3\times 1 & 3\times 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 3 & 12 \end{pmatrix}$$

Now, a step ahead we take a vector product of a matrix. If

vector $A = (1, \quad 2, \quad 3)$ and matrix $B = \begin{pmatrix} 4 & 9 \\ 6 & 3 \\ 8 & 0 \end{pmatrix}$

then $AB = (1 \quad 2 \quad 3) \times B = \begin{pmatrix} 4 & 9 \\ 6 & 3 \\ 8 & 0 \end{pmatrix}$

$$= [1\cdot 4 + 2\cdot 6 + 3\cdot 8 \quad 1\cdot 9 + 2\cdot 3 + 3\cdot 0]$$

$$= [4 + 12 + 24 \quad 9 + 6 + 0]$$

$$= [40 \quad 15]$$

It was a pre-multiplication of a matrix by a vector. A post-multiplication in the following form is not possible.

$$\begin{pmatrix} 4 & 9 \\ 6 & 3 \\ 8 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

The reason being where as in the earlier case the columns in the vector were 3 which were equal tot he number of rows of the matrix which were also 3. But, in the latter situation the matrix has 2 columns but the vector had only one row. For matrix multiplication, the number of columns in the first matrix or vector must be equal to the number of rows in the second matrix or the vector.

The rule is to multiply the first element in the first row of the first matrix with the first element in the first column of the second matrix, the second element in the first column of the second matrix, the n^{th} element in the first column of the second matrix. This further proves the need of the number of columns in the first matrix to be equal to the number of rows in the second matrix. Now, these products are added together to give the first element of the first row and the first column of the product matrix. Next we multiply the elements of first row of the first matrix with the elements of the second column fo the second matrix and obtain the second element of the first row of the product matrix and so on.

Thus the two matrices are conformable for multiplication if the number of columns of first matrix is equal to the number of rows of the second matrix. If the matrix A is of type $m \times n$ i.e., has m rown and n columns, then B must be of the type $n \times p$ where n is the number of rows which are the same as number of columns in A and p is any number not necessarily m. Then the product AB is another matrix $C = A \times B$ of the type $m \times p$ (number of rows of A and number of columns of B).

Let $A = \left[a_{ij} \right]_{m \times n}$ and $B = \left[b_{jk} \right]_{n \times p}$ be two matrices then the product AB is the matrix.

$$C = \left[c_{ik} \right]_{m \times p}$$

where c_{ik} is obtained by multiplying the corresponding entries of the i^{th} row of A and those of k^{th} column of C and then adding the results. Thus

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & 22 & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{i1}} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2k} & \cdots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{ji} & b_{j2} & \cdots & b_{jk} & \cdots & b_{jp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2k} & \cdots & c_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ik} & \cdots & c_{ip} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} & \cdots & c_{mp} \end{bmatrix}_{m \times p}$$

where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \cdots + a_{in}b_{nk}$

Remarks:

1. The rule of multiplication fo two matrices is Row-Columnwise $(\rightarrow\downarrow \text{wise})$, i.e., row of one matrix is multiplied with column of the second matrix to get the corresponding elements of the product. In short first row of AB is obtained by multiplying the first row of A with 1st, 2nd, 3rd column of B respectively. Similarly the second row of AB is obtained by multiplying the second row A with 1st, 2nd, 3rd columns of B respectively and so on.

2. The rule of multiplication (viz, $\rightarrow\downarrow$ wise) is the same for matrices of any order provided the matrices are conformable for multiplication.

3. If

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{bmatrix}_{m \times n} \qquad B = \begin{bmatrix} C_1 & C_2 & \cdots & C_j & \cdots & C_p \end{bmatrix}_{n \times p}$$

where R_i denotes the i^{th} row of matrix A and can be regarded as $m \times n$ matrix.

where C_j denotes the j^{th} column of matrix B and can be regarded as an $n \times p$ matrix. Then

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_P \\ R_2C_1 & R_2C_2 & \cdots & R_2C_P \\ \vdots & \vdots & \vdots & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_p \end{bmatrix}_{m \times p}$$

4. In the product AB, A is said to have been post-multiplied by B and B is said to have been premultiplied by A, i.e., AB is called the sub-multiplication of A by B or premultiplication of B by A.

5. Matrix multiplication in general is not commutative: If AB is defined, it is not necessary that BA is also defined, e.g., if A is of the type $m \times n$ and B of the type $n \times p$ then AB is defined. Even if AB and BA are both defined, it is not necessary that they are equal e.g., if A is $m \times n$ and B is $n \times m$ then AB is $m \times m$ and BA is $n \times n$ so that $AB \neq BA$ because they arenot of the same order.

8.6.4 Properties:

I. Multiplication is distributive w.r.t. addition:

If A, B, C are $m \times n$, $n \times p$ and $n \times p$ matrices respectively, then

$$A(B + C) = AB + AC$$

II. Multiplication is associative if conformability is assured:

If A, B, C are $m \times n$, $n \times p$ and $p \times q$ matrices respectively, then

$$(AB)C = A(BC)$$

III. If A is $n \times m$, O is $m \times n$ then

$$AO = O = OA$$

IV. Multiplication of a Matrix by Unit Matrix:

If A is a square matrix of order $m \times n$ and I is the unit matrix of the same order then

$$AI = A = IA$$

V. AB = 0 (null matrix) does not necessarily imply that A = 0 or B = 0 or both = 0 e.g.,

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \neq 0 \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq 0$$

But

$$AB = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

VI. Multiplication of Matrix by Itself:

The product A.A is defind if the number of columns of A is equal tot he number of rows of A., i.e, if A is a swuare matrix and in that case A.A will also be a square matrix of the same order.

$A^2A = (AA)A = A(AA)$ (By Associative Law)

$A^2A = AAA = A^3$

Similarly $A \cdot A \cdot A \cdots\cdots\cdots n \text{ times} = A^n$

Remark:

If I is a unit matrix, then

$I = I^2 = I^3 = \cdots\cdots\cdots = I^n$

Example 2:

Write down the product AB of the two matrices A and B where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Solution:

Sine A is 1×4 matrix, B is 4×1 matrix, AB will be 1×1 matrix.

$$\therefore AB = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}_{1 \times 4} \times \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}_{4 \times 1} = [1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 4] = [30]_{1 \times 1}$$

Example 3:

If $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}$, find AB and BA. Is AB = BA?

Solution:

Here $AB = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 5 \times (-3) & 2 \times (-1) + 5 \times 2 \\ 1 \times 1 + 3 \times (-3) & 1 \times (-1) + 3 \times 2 \end{pmatrix}$

$$= \begin{pmatrix} -13 & 8 \\ -8 & 5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1\times 2 + (-1)\times 1 & 1\times 5 + (-1)\times 3 \\ (-3)\times 2 + 2\times 1 & (-3)\times 5 + 2\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -4 & -9 \end{pmatrix}$$

Thus $AB \neq BA$

Example 4:

Obtain the product

$$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

Solution:

Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}_{3\times 3}$ $\quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}_{3\times 4}$

Since A is 3×3 and B is 3×4, product AB is valied and AB is 4×4.

$$AB = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+2+0 & 4+0+0 & 6+1+0 & 8+2+0 \\ 3+4+3 & 6+0+1 & 9+2+0 & 12+4+5 \\ 1+0+3 & 2+0+1 & 3+0+0 & 4+0+5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{pmatrix}$$

Example 5:

Final (a): $(x, y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (b): $\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Solution:

(a) $$(x\ ,\ y)_{1\times 2}\left\{\begin{pmatrix} a & h \\ h & b \end{pmatrix}_{2\times 2}\begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}\right\}$$

$$=(x\ ,\ y)_{1\times 2}\begin{pmatrix} ax+hy \\ hx+by \end{pmatrix}_{2\times 1}$$

$$=\left[\,x(ax+hy)\,+y(hx+by)\right]_{1\times 1}$$

$$=ax^2+hxy+hxy+by^2=ax^2+2hxy+by^2$$

(b) $$(x\quad y\quad z)_{1\times 3}\left\{\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}_{3\times 3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{3\times 1}\right\}$$

$$=(x\quad y\quad z)_{1\times 3}\begin{pmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{pmatrix}_{3\times 1}$$

$$=\left[\,x(ax+hy+gz)+y(hx+by+fz)+z(gx+fy+cz)\right]_{1\times 1}$$

$$=ax^2+hxy+gxz+hxy+by^2+fyz+gzx+fyz+cz^2$$

$$=ax^2+by^2+cz^2+2(hxy+fyz+gzx)$$

Example 6:

If $A=\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}$ find the matrix X such that $3A+5B+2X=0$

Solution:

$$3A+5B+2X=0\Rightarrow X=-\frac{1}{2}[3A+5B]$$

$$X=-\frac{1}{2}\left\{3\begin{pmatrix} 9 & 1 \\ 4 & 3 \end{pmatrix}+5\begin{pmatrix} 1 & 5 \\ 7 & 12 \end{pmatrix}\right\}$$

$$= -\frac{1}{2}\left\{\begin{pmatrix} 27 & 3 \\ 12 & 9 \end{pmatrix} + \begin{pmatrix} 5 & 25 \\ 35 & 60 \end{pmatrix}\right\}$$

$$= -\frac{1}{2}\left\{\begin{pmatrix} 27+5 & 3+25 \\ 12+5 & 9+60 \end{pmatrix}\right\}$$

$$= \begin{pmatrix} -\frac{32}{2} & -\frac{28}{2} \\ -\frac{47}{2} & -\frac{69}{2} \end{pmatrix} \begin{pmatrix} -16 & -14 \\ -\frac{47}{2} & -\frac{69}{2} \end{pmatrix}$$

Example 7:

If $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}$

(a) Find a 2×4 matrix X such that $A - X = 3B$

(b) Find a 2×4 matrix Y such that $A + 2Y = 4B$

Solution:

(a) $A - X = 3B$

$X = A - 3B$

$$\Rightarrow X = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix} - 3\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix} + \begin{pmatrix} -6 & -3 & 0 & -9 \\ -3 & +3 & -6 & -9 \end{pmatrix}$$

$$= \begin{pmatrix} 1-6 & 2-3 & 0 & 4-9 \\ 2-3 & 4+3 & -1-6 & 3-9 \end{pmatrix} = \begin{pmatrix} -5 & -1 & 0 & -5 \\ -1 & 7 & -7 & -6 \end{pmatrix}$$

(b) $A + 2Y = 4B \rightarrow Y = 2B - \frac{1}{2}A$

$$\Rightarrow Y = 2\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 & 0 & 6 \\ 2 & -2 & 4 & 6 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{2}{2} & 0 & \frac{1}{2} \\ \frac{2}{2} & \frac{4}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 4-\frac{1}{2} & 2-1 & 0 & 6-2 \\ 2-1 & -2-2 & 4+\frac{1}{2} & 6-\frac{3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 1 & 0 & 4 \\ 1 & -4 & \frac{9}{2} & \frac{9}{2} \end{pmatrix}$$

Example 8:

When $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix}$

and $i = \sqrt{-1}$ determine AB. Compute also BA.

Solution:

$$AB = \begin{pmatrix} 1 \times i - i \times 1 & (+1) \times 1 + (i)(-i) \\ i \times (-i) - 1 \times 1 & (-i)(-1) + 1(-i) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $BA = \begin{pmatrix} 2i & -2 \\ -2 & -2i \end{pmatrix}$

Example 9:

Given $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

prove the following relations:

$A^2 = B^2 = C^2 = I$ (unit matrix)

$AB = -BA$, $AC - -CA$, $BC = -CB$.

Solution:

$$A^2 = A \cdot A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$B^2 = B \cdot B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0-i^2 & 0+0 \\ 0+0 & -i^2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Similarly $C^2 = I$

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0+i & 0+0 \\ 0+0 & -i+0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$-BA = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0-i & 0+0 \\ 0+0 & i+0 \end{pmatrix}$$

$$= -\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i & o \\ 0 & -i \end{pmatrix}$$

$\therefore$ $AB = -BA$, similarly we can prove the other relations.

Example 10:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, show that $A^2 - (a+d)A = (bc - ad)I$

Solution:

We have

$$A^2 = A \cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix}$$

$$\therefore A^2 - (a+d)A = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

or $$A^2 - (a+d)A = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix} + \begin{pmatrix} -a(a+d) & -b(a+d) \\ -c(a+d) & -d(a+d) \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc-a(a+d) & b(a+d)-b(a+d) \\ c(a+d)-c(a+d) & bc+d^2-d(a+d) \end{pmatrix}$$

$$= \begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix} = (bc-ad)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (bc-ad)I$$

Example 11:

If $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}$

and $(A+B)^2 = A^2 + B^2$ find a and b.

Solution:

$$A + B = \begin{pmatrix} (1+a) & 0 \\ (2+b) & -2 \end{pmatrix}$$

$$(A+B)^2 = \begin{pmatrix} (1+a) & 0 \\ (2+b) & -2 \end{pmatrix}\begin{pmatrix} (1+a) & 0 \\ (2+b) & -2 \end{pmatrix} = \begin{pmatrix} (1+a)^2 & 0 \\ 2a+ab-b-2 & 4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix}\begin{pmatrix} a & 1 \\ b & -1 \end{pmatrix} = \begin{pmatrix} a^2+b & a+1 \\ ab-b & b+1 \end{pmatrix}$$

$$A^2 + B^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{pmatrix} = \begin{pmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{pmatrix}$$

Now $(A+B)^2 \Rightarrow A^2 + B^2$

$$\Rightarrow \begin{pmatrix} (1+a)^2 & 0 \\ 2a-b+ab-2 & 4 \end{pmatrix} = \begin{pmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{pmatrix}$$

$\Rightarrow a-1=0$ or $a=1$ and $b=4$

Example 12:

Given the matrices A, B, C.

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ C = \begin{pmatrix} 1 & -2 \end{pmatrix}$$

verify that $(AB)C = A(BC)$

Solution:

Clearly AB is defined and will be 2×1 matrix and hence $A_{2\times3}\,(BC)_{3\times3}$ is also defined and will be 2×2 matrix.

$$(AB) = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{pmatrix}_{2\times2} \times \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_{3\times1} = \begin{pmatrix} 2\cdot1+3\cdot1-1\cdot2 \\ 3\cdot1+0\cdot1+2\cdot2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 7 \end{pmatrix}_{2\times1}$$

$$(AB)C = \begin{pmatrix} 3 \\ 7 \end{pmatrix}_{2\times1} (1\ -2)_{1\times2} = \begin{pmatrix} 3\cdot1 & 3\cdot(-2) \\ 7\cdot1 & 7\cdot(-2) \end{pmatrix}_{2\times2}$$

$$= \begin{pmatrix} 3 & -6 \\ 7 & -14 \end{pmatrix}_{2\times2} \qquad \text{...........(I)}$$

Again

$$BC = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}_{3\times1} (1-2)_{1\times2} = \begin{pmatrix} 1\cdot1 & 1\cdot(-2) \\ 1\cdot1 & 1\cdot(-2) \\ 2\cdot1 & 2\cdot(-2) \end{pmatrix}_{3\times2} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \\ 2 & -4 \end{pmatrix}_{3\times2}$$

$$\therefore A(BC) = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \end{pmatrix}_{2\times 3} \begin{pmatrix} 1 & -2 \\ 1 & -2 \\ 2 & -4 \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} 2\cdot 1 + 3\cdot 1 + (-1)\cdot 2 & 2\cdot(-2) + 3(-2) + (-1)(-4) \\ 3\cdot 1 + 0\cdot 1 + 2\cdot 2 & 3\cdot(-2) + 0\cdot(-2) + 2(-4) \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} 3 & -6 \\ 7 & -14 \end{pmatrix}_{2\times 2} \qquad ....(II)$$

Thus we observe that

$$(AB)C = A(BC)$$

Example 13:

If $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$

Show that $A^3 - 3A^2 - A + 9I = 0$

Solution:

$$A^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{pmatrix}$$

Now $A^3 - 3A^2 - A + 9I = \begin{pmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{pmatrix} - \begin{pmatrix} 12 & 9 & 0 \\ -9 & 6 & -6 \\ 18 & 12 & 15 \end{pmatrix}$

$$- \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Example 14:

A finance company has offices located in every division, every district and every taluka in a certain state in India. Assume that there are five divisions, 30 districts and 200 talukas in the State. Each office has 1 head clerk, 1 cashier, 1 clerk and 1 peon. A divisional office has, in addition an office superintendent, 2 clerks, 1 typist and 1 peon. A district office has, in addition, 1 clerk and 1 peon. The basic monthly salaries are as follows:

Office superintendent Rs. 500, Head clerk Rs. 200, Cashier Rs. 175, Clerks and Typist Rs. 150 and peons Rs. 100. Using matrix notation find

(i) The total number of posts of each kind in all the offices taken together.

(ii) The total basic monthly salary bill of each kind of office and

(iii) The total basic monthly salary bill of all the offices taken together

solution:

The number of offices can be arranged as elements of a row vector, say

$$A = \begin{matrix} \text{Division} & \text{District} & \text{Taluka} \\ (\;5 & 30 & 200) \end{matrix}$$

Staff composition can be arranged in 3×6 matrix B,

$$B = \begin{pmatrix} O & H & C & T & CL & P \\ 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

where O = Office superintendent, H = Head Clerk, C = Cashier, T = Typist, CL = Clerk, P = Peon.

Column vector D will have the elements that correspond to basic monthly salaries.

$$D = \begin{matrix} O \\ H \\ C \\ T \\ CL \\ P \end{matrix} \begin{bmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{bmatrix}$$

(i) Total number of posts of each kind in all the offices are the elements of the matrix

$$AB\ ,\ i \cdot e., \begin{pmatrix} O & H & C & T & CL & P \\ 5 & 235 & 235 & 5 & 275 & 270 \end{pmatrix}$$

(ii) Total basic monthly salary bill of each kind of offices is the elements of matrix

$$BD = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \begin{vmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{vmatrix} = \begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix}$$

(iii) Total bill of all these offices is the element of the matrix

$$\begin{pmatrix} 5 & 30 & 200 \end{pmatrix} \begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix} \quad 1{,}59{,}625$$

8.7 Tranpose of a Matrix:

The matrix obtained by interchanging rows and columns of the matrix A is called the transpose of A and is denoted by A' or A' (read as A transpose), e.g., if

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 7 & -5 \end{pmatrix}_{3\times 2} \quad \text{then} \quad A' = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 1 & -5 \end{pmatrix}_{2\times 3}$$

$$A = \left(a_{ij}\right)_{m \times n} \quad \text{then} \quad A' = \left(a_{ji}\right)_{n \times m}$$

i.e., the $(i, j)^{th}$ element of $A = (j, i)^{th}$ element of A'. In other words, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \quad A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{i1} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{i2} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1i} & a_{2i} & \cdots & a_{ji} & \cdots & a_{mj} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{jn} & \cdots & a_{mn} \end{bmatrix}_{n\times m}$$

Remark:

1. If A is $m \times n$ matrix, then A' will be a $n \times m$ matrix.
2. The transpose of a row (column) matrix is a column (row) matrix.

3. $(A')' = A$

4. The transpose of the sum of two matrices is the sum of their transposes, i.e.,

$(A+B)' = A' + B'$

5. The transpose of the product AB is equal to the product of the transposes taken in the reverse order, i.e., $(AB)' = B'A'$

Example 15:

Let $A = \begin{pmatrix} 2 & -3 & 1 \\ 4 & 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 4 \\ 1 & 3 & -5 \end{pmatrix}$

Show that $(A+B)' = A' + B'$

Solution:

$$A+B = \begin{pmatrix} 2+3 & (-3)+(-2) & 1+4 \\ 4+1 & 2+3 & 3+(-5) \end{pmatrix} = \begin{pmatrix} 5 & -5 & 5 \\ 5 & 5 & -2 \end{pmatrix}$$

$$\therefore (A+B)' = \begin{pmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{pmatrix} \qquad \text{..........(I)}$$

Now $A' = \begin{pmatrix} 2 & 4 \\ -3 & 2 \\ 1 & 3 \end{pmatrix}$ and $B' = \begin{pmatrix} 3 & 1 \\ -2 & 3 \\ 4 & -5 \end{pmatrix}$

$$\therefore A' + B' = \begin{pmatrix} 2+3 & 4+1 \\ (-3)+(-2) & 2+3 \\ 1+4 & 3+(-5) \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -5 & 5 \\ 5 & -2 \end{pmatrix} \qquad \text{............(II)}$$

Hence $(A+B)' = A' + B'$

8.7.1 Symmetric Matrix: A square matrix is said to be symmetric if the transpose of a matrix is equal to the matrix itself, e.g.,

$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \begin{pmatrix} 1 & 4 & 9 \\ 4 & 7 & 5 \\ 9 & 5 & 8 \end{pmatrix}$ are symmetric matrices.

Symbolically $A = (a_{ij})_{n\times m}$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j.

8.7.2 Skew Symmetric Matrix: A square matrix $A = (a_{ij})_{n\times n}$ is said to be skew matrix if

$a_{ij} = -a_{ji}$ for all i and j

e.g., $\begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & f & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 6 & 8 \\ -6 & 0 & 9 \\ -8 & -9 & 0 \end{pmatrix}$

are skew symmetric matrices.

orthogonal matrix $A'A = I = AA'$

8.8 Determinant of a Square Matrix:

Let $A = [a_{ij}]$ be a square matrix. We can associate with the square matrix A a determinant which is formed by exactly the same array of elements of the matrix A. A determinant formed by the same array of elements of the matrix A. A determinant formed by the same array of elements of the square matrtix A is called the determinant of the square matrix A and is denoted by the symbol det. A or $|A|$. It should be remembered that the determinant of a square matrix will be a scalar quantity, i.e., with a determinant we associate some value whereas a matrix is essentially an arrangement of numbers and as such has no value.

For example, let a matrix $A = \begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix}$ so that $|A| = \begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix}$

$$= 6 \times 2 - 5 \times 3 = -3$$

$$= -3$$

Here $|A| = -3$ where as A is a matrix giving only an arrangement of the four numbers 6, 5, 3, 2 in two rows and two columns. It should be noted that the positions occupied by the elements of a matrix are important. A change in the positions of the elements of a matrix gives rise to a different matrix.

For example $\begin{pmatrix} 6 & 5 \\ 3 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 5 \\ 3 & 6 \end{pmatrix}$ are different matrices. Although formed by the same elements of a number 6, 5, 3 and 2. However, the determinants of these two square matrices are

$\begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix}$ and $\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$

and have the same value namely - 3.

We will now takeup determinants of various orders, viz, two three and higher order.

8.8.1 Determinats of Order Two: The determinant of a 2×2 matrix is denoted by any of the following ways.

(i) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb \text{ or } ad - bc$

(ii) $\begin{vmatrix} a_{11} & a_{22} \\ a_{21} & a_{12} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

(iii) $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$ or $a_1b_2 - a_2b_1$

It should be remembered that the numbers enclosed by straight lines do not constitue a matrix they are the coefficients or the numbers assigned to a square matrix. We will now illustrate its use in soluton of simultaneous equations.

8.8.2 Determinant of Order Three: In a 3 by 3 matrix, the determinants are defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 \left(b_2c_2 - b_3c_2 \right) - b_1\left(a_2c_3 - a_3c_2 \right) + c_1 \left(a_2b_3 - a_3b_2 \right)$$

It may be noticed that in each case a 2 by 2 determinant has been taken by omiting the row and column of a particular row element in order a_1, b_1 and c_1. Another thing to note is the alternating signs for this row elements.

Example 17: Compute the determinant of the following matrix:

$$A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$$

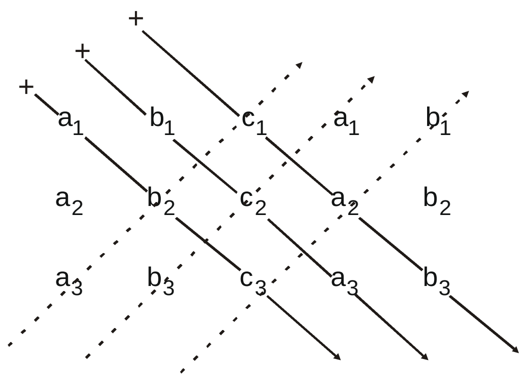
Solution:

$$|A| = \det(A) = 2\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} + (-4)\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix}$$

$$= 2(-20 + 2) - 30\ (0 - 2) - 4(0 + 4)$$

$$= -36 + 6 - 16 = -46$$

8.8.3 Sarrus Diagram: We can find out determinants ofa given marix very conveniently if we extend the matrix by adding the first two columns and connect the elements by arrows downwards preceded by a plus sign and upwards by a minus sign as illustrated below:



The product of elements joined by download arrows preceded by plus signs are

$$a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1$$

Example 18:

Find the value of the determinants

$$\begin{vmatrix} 2x & 4y \\ x & 3y \end{vmatrix}, \quad \begin{vmatrix} x & x+1 \\ x+2 & x+3 \end{vmatrix}$$

Solution:

(i) $\begin{vmatrix} 2x & 4y \\ x & 3y \end{vmatrix} = 2x \cdot 3y - x \cdot 4y = 2xy$

(ii) $\begin{vmatrix} x & x+1 \\ x+2 & x+3 \end{vmatrix} \; x(x+3) - (x+1)(x+2) = -2$

Example 19:

Find the value of

$$\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & -7 \\ 0 & 3 & 4 \end{vmatrix}$$

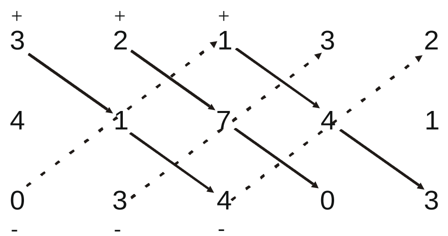
Solution:

Since there is zero in the first column, we expand by the elements of the first column,

$$\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & -7 \\ 0 & 3 & 4 \end{vmatrix} = 3\begin{vmatrix} 1 & -7 \\ 3 & 4 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & -7 \end{vmatrix}$$

$$= 3(4 + 21) - 4(8 - 3) + 0 = 55$$

By sarrus diagram



$$= 12 + 0 + 12 - 32 + 63 + 0$$
$$= 55$$

8.9 Properties of Determinants:

I. If the rows of a determinant are chaged into columns and vice versa the value of the determinant remains unchanged. i.e., $\det A = \det A'$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

For example

$$\begin{vmatrix} 1 & 5 & 6 \\ 2 & 8 & 7 \\ 3 & -9 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 8 & -9 \\ 6 & 7 & 0 \end{vmatrix}$$

II. If any two rows (or columns) are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

III. If any two rows or any two columns of a determinant are identical the value of the determinant is zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

IV. If the elements of a row (column) of determinant are added (subtracted) k -times the corresponding elements of another row (column), the value of the determinant remains unchanged.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} + k a_{11} - m a_{12} \\ a_{21} & a_{22} & a_{23} + k a_{21} - m a_{22} \\ a_{31} & a_{32} & a_{33} + k a_{31} - m a_{32} \end{vmatrix}$$

V. If the elements of a row (column) of amatrix multiplied by the same number, k say the determinant of the matrix thus obtained is k times the determinant of the original matrix.

$$\begin{vmatrix} k a_{11} & a_{12} & a_{13} \\ k a_{21} & a_{22} & a_{23} \\ k a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

VI. If the elements of any row or any column of a determinant is sum (difference) of two or more elements then the determinant can be expressed as sum (difference) of two or more determinatns.

$$\begin{vmatrix} a_{11}+\alpha_{11} & a_{12} & a_{13} \\ a_{21}+\alpha_{21} & a_{22} & a_{23} \\ a_{31}+\alpha_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix}$$

Explain 20:

Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} \quad \text{Apply } C_1 - C_2,\ C_2 - C_3$$

$$= (a-b)(b-c)\left\{\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}\right\}$$

$$= (a-b)\ (b-c)(b+c-a-b)$$

$$= (a-b)(b-c)(c-a)$$

Example 21:

Prove taht

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2 \end{vmatrix} \quad \text{Apply } c_1 + c_2$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \quad \text{Apply } R_2 - R_1,\ R_3 - R_1$$

$$= 2(a+b+c)\left\{\begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}\right\}$$

$$= 2\,(a+b+c)^3$$

Example 22:

Evaluate $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = abc \begin{vmatrix} 0 & b^2 & c^2 \\ a^2 & 0 & c^2 \\ a^2 & b^2 & 0 \end{vmatrix}$$

$$abc\left(a^2b^2c^2\right)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= a^3b^3c^3\left[-1(0-1)+1(1-0)\right] = 2a^3b^3c^3$$

8.9 Exercise:

1. Given $A = \begin{bmatrix} 2 & 0 & 6 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 & -2 \\ 0 & -2 & 0 \\ 2 & 2 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 2 & 0 \\ 0 & 2 & -6 \\ -8 & 4 & -10 \end{bmatrix}$

   Compute the following

   (i) A + B (ii) A - B (iii) A + (B + C) (iv) (A + B) + C

   (v) (A - B) + C (vi) A - B -C (vii) 2 (A + B) (viii) 2 A + 2 B

2. Find the matrix B if

   (i) $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ and $A + 2B = A^2$

   (ii) $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ and $A^2 + 3A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

3. $A = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$ find the matrix C if

   (i) 2 C = A + B (ii) $C + A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (iii) 5 C + 2 B = A

4. If $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ show that $A^2 = 2A$ and $A^3 = 4A$

5. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$ compute show that $A^2 = 0$

6. For each of the following matrices verify that $\left(A^t\right)^t = A$

(i) $\begin{bmatrix} 2 & 8 & 4 \\ 8 & 6 & -1 \\ 4 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 7 & 9 & 2 & -7 \\ 7 & 8 & 5 & 6 & 0 \end{bmatrix}$

7. If $A = \begin{bmatrix} 3 & -3 & 0 \\ 6 & 3 & 9 \\ 12 & 3 & 24 \end{bmatrix}$, $B = \begin{bmatrix} 12 & 3 & 0 \\ 6 & -9 & 3 \\ 3 & 3 & -3 \end{bmatrix}$ Verify that $(A + B)^t = A^t + B^t$ and $(AB)^t = B^t A^t$

8. If $A = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ Find AB, BA and verify that $(AB)^t = B^t A^t$

9. Show that

$$\begin{vmatrix} 3 & -7 \\ 8 & 6 \end{vmatrix} = 74 \quad , \quad \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y \\ -1 & +1 \end{vmatrix} = x + y \,, \begin{vmatrix} -4 & 2 \\ -3 & -4 \end{vmatrix} = 22$$

10. (a) Show that

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \quad \begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 4$$

(b) Show that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} b & q \\ p & c \end{vmatrix} + \begin{vmatrix} p & d \\ a & q \end{vmatrix} = 0$$

11. Show that

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{vmatrix} = -48, \begin{vmatrix} 3 & 4 & 8 \\ 2 & 1 & 3 \\ 7 & -2 & 0 \end{vmatrix} = 14$$

12. Show that

$$\begin{vmatrix} 3 & 4 & 7 \\ 2 & 1 & 3 \\ -5 & -1 & 2 \end{vmatrix} = -40$$

13. Show that

$$\begin{vmatrix} 1 & 2 & 3 \\ a & -a & b \\ -a & 0 & -b \end{vmatrix} = ab - 3a^2$$

14. Show that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc - af^2 - bg^2 - ch^2 + 2fgh$$

15. Evaluate the following:

$$\begin{vmatrix} x & 1 & 2 \\ 2 & x & 2 \\ 3 & 1 & x \end{vmatrix}, \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 1 & b^1 & b^2 \end{vmatrix}, \begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$$

16. Show that

$$\begin{vmatrix} 2 & 45 & 55 \\ 1 & 92 & 32 \\ 3 & 68 & 87 \end{vmatrix} = 54$$

(Hint: Apply $R_1 - 2R_2$, $R_3 - 3R_2$ and expand)

Lesson Writer

B. Rami Reddy

Lesson - 9

RANK OF A MATRIX

Objective of the Lesson:

After studying this lesson, the student will be in a position to know about Rank of a Matrix, solving of simulatancous equations by matrix inversion, Gramer's rule.

Structure:

This lesson has the following components:

9.1 Introduction:

Sub Martix of a Matrix: Suppose A is any matrix of the type $m \times n$. Then a matrix obtained by leaving some rows and columns from A is called a **sub matrix** of A.

Minors of a Matrix: We know that every square matrix posses a determinant. If A be an $m \times n$ matrix, then the determinant of every square sub-matrix of A is called a **minor** of the matrix A. For example, let

$$A = \begin{bmatrix} 2 & 4 & 1 & 9 & 1 \\ 0 & 5 & 2 & 5 & 2 \\ 1 & 9 & 7 & 3 & 4 \\ 3 & -2 & 8 & 1 & 8 \end{bmatrix}_{4\times5}$$

If we leave any column from A, we shall get a square sub matrix of A of order 4. Thus

$$\begin{bmatrix} 2 & 4 & 1 & 9 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 7 & 3 \\ 3 & -2 & 8 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 9 & 1 \\ 5 & 2 & 5 & 2 \\ 9 & 7 & 3 & 4 \\ -2 & 8 & 1 & 8 \end{bmatrix}$$, etc. are 4 rowed minors of A.

9.2 Rank of a Matrix:

A number r is said to be the rank of a matrix A if it posses the following two properties.

(i) There is at least one square sub matrix of A of order r whose determinant is not equal to zero.

(ii) If the matrix A contains any sequare sub matrix of order $r+1$, then thedeterminant of every square submatrix of A of order $r+1$ should be zero.

We shall denote the rank of a matrix A by the symbol $\rho(A)$

Note:

1. The rank of every non singular matrix of order n is n.
2. Since the rank fo every non zero matrix is ≥ 1, the rank of null matrix is zero.
3. The rank of a matrix is $\leq r$, if there is at least one r.rowed minor of the matrix which is not equal to zero.

Example:

1. Let $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a unit matrix of order 3. We have $|A| = 1 \therefore A$ is a non singular matrix. Hene the rank A = 3. In particular the rank of a unit matrix of order n is n.

2. Let $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ be a null matarix. $\therefore$ the rank A = 0.

3. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$, $|A| = 1(24-25) - 2(18-20) + 3(15-16) = 0$

$\therefore$ The rank of A is lessthan 3, now there is at least one minor of A of order 2, namely $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = 2 \neq 0$. Hence the rank A = 2.

9.3 Echelon Form of a Matrix:

A matrix A is said to be in Echelon form if

(i) Every row of A which has all its entries '0' occures below every row which has a non zero entry.

(ii) The first non zero entry in each non zero row is equal to 1.

(iii) The number of zero's before the first non-zero element in a row is lessthan the number of such zeros in the rest row.

Important Result:

The rank of a matrix in Echelon Form is equal to the number of non zero rows of the matrix.

Example:

Find the rank of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix A has one zero row. We see that it occurs below every non zero row.

Further the number of zero's before the first non zero element in the first row is one. The number of zeros before the first non zero element in the third row is three.

$\therefore$ The given matrix is in Echelon Form.

Rank A = The number of non zero rows of A ine Echelor Form = 2

9.4 Elementary Row Transformations of a Matrix:

An elementary transformation is an operation of any one of the following types:

(i) The interchange of any two rows. The interchange of i^{th} and j^{th} rows will be denoted by $R_i \leftrightarrow R_j$.

(ii) The multiplication of the elements of any row by any non-zero number. i.e. the multiplication of the i^{th} row by a non zero number k will be denoted by $R_i \rightarrow k\,R_i$

(iii) The addition to the elements of any other row the corresponding elements of any other row multiplied by any number. i.e., The addition of k times the j^{th} row to the i^{th} row will be denoted by $R_i \rightarrow R_i + k\,R_j$.

Similarly, Elementary column transformation are defined.

Statements of Important Theorems:

1. The rank of the transpose of a matrix is the same as that of the original matrix.
2. Elementary row transformation donot change the rank of a matrix.

9.5 Solved Examples of Rank of a Matrix:

Example:

(i) $A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

By elementary row transformations

$R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$ we get

$$A \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_4 \to R_4 - 2R_2$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 12 & 16 & 4 \\ 0 & 12 & 12 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_2 \to 4R_2, R_3 \to 3R_3$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 12 & 16 & 4 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

Which is in Echelon Form. The number of non zero rows inthis matrix is 3.

Therefore Rank $A = 3$.

Let us denote the given matrix by A

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_2$$

$$\sim\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 3 & 3 & -3\\ 0 & -2 & -2 & 2\\ 0 & 1 & 1 & -1\end{bmatrix} \text{ by } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - R_1$$

$$\sim\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & 1 & 1 & -1\\ 0 & 1 & 1 & -1\end{bmatrix} \text{ by } R_2 \to \frac{1}{3}R_2,\ R_3 \to -\frac{1}{2}R_3$$

$$\sim\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \text{ by } R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2$$

Which is in Echelon Form. The number of non zero rows is 2.

$\therefore$ Rank A = 2.

9.6 Inverse of a Matrix:

9.6.1 Minors and Cofactors of the Elements of a Determinant:

The minor of an element a_{ij} of a determinant A is denoted by M_{ij} and is the determinant obtained form A by deleting the row and the column where a_{ij} occurs.

The cofactor of an element a_{ij} with minor M_{ij} is denoted by C_{ij} and is defined as

$$C_{ij} = \begin{cases} M_{ij}, & \text{if } i + j \text{ is even} \\ -M_{ij}, & \text{if } i + j \text{ is odd} \end{cases}$$

Thus cofactors are signed minors.

In the case of $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, we have

Also $M_{11} = a_{22},\quad M_{12} = a_{21},\quad M_{21} = a_{12},\quad M_{22} = a_{11}$

$C_{11} = a_{22},\quad C_{12} = -a_{21},\quad C_{21} = -a_{12},\quad C_{22} = a_{11}$

In the case of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, we have

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$ and so on.

9.6.2 Adjoint of a Square Matrix:

The transpose of the matrix got by replacing all the elements of a square matrix A by their corresponding cofactors in $|A|$ is called the Adjoint of A or Adjugate of A and is denoted by Adj A.

Thus, $\text{Adj } A = A_c^t$

Note:

(i) Let $A = \begin{pmatrix} a & b \\ -c & d \end{pmatrix}$ then $A_c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

$$\therefore \text{Adj } A = A_c^t = \begin{pmatrix} a & -b \\ c & a \end{pmatrix}$$

Thus the Adjoint of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

can be written instantly as $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(ii) $\text{Adj } I = I$, where I is the unit matrix.

(iii) $A\,(\text{Adj } A) = (\text{Adj } A)\,A = |A|\,I$

(iv) $\text{Adj}(AB) = (\text{Adj } B)(\text{Adj } A)$

(v) If A is a square matrix of order 2, then $|\text{Adj } A| = |A|$

If A is a square matrix of order 3, then $|\text{Adj } A| = |A|^2$

Example 1:

Write the adjoint of the matrix $A = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix}$

Solution:

$$\text{Adj } A = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$$

Example 2:

Find the Adjoint of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}, \ \text{Adj } A = A_c^t$$

Now,

$$C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8, C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6, C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2, C_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$\therefore A_c = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Hence

$$\text{Adj } A = \begin{pmatrix} -1 & 8 & - \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{pmatrix}^t = \begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix}$$

9.6.3 Inverse of a Non Singular Matrix:

The inverse of a non singular matrix A is the matrix B such that $AB = BA = I$. B is then called the inverse of A and dentoed by A^{-1}.

Note:

(i) A non square matrix has no inverse.

(ii) The inverse of a square matrix A exists only when $|A| \neq 0$ that is, if A is a singular matrix then A^{-1} does not exist.

(iii) If B is the inverse of A then A is the inverse of B. That is $B = A^{-1} \Rightarrow A = B^{-1}$

(iv) $A\,A^{-1} = I = A^{-1}\,A$

(v) The inverse of a matrix, if it exists is unique. That is, no matrix can have more than one ivnerse.

(vi) The order of the matrix A^{-1} will be the same as that of A.

(vii) $I^{-1} = I$

(viii) $(AB)^{-1} = B^{-1}\,A^{-1}$, provided the inverses exist.

(ix) $A^2 = I$ implies $A^{-1} = A$

(x) If $AB = C$ then

(a) $A = CB^{-1}$ (b) $B = A^{-1}C$ provided the inverses exist.

(xi) We have seen that

$$A(\text{Adj } A) = (\text{Adj } A)\,A = |A|\; I$$

$$\therefore\; A\,\frac{1}{|A|}(\text{Asj } A) = \frac{1}{|A|}(\text{Adj } A)A = I \qquad (|A| \neq 0)$$

This suggests that

$$A^{-1} = \frac{1}{|A|}(\text{Adj } A)$$

That is $A^{-1} = \frac{1}{|A|} A_c^t$

(xii) $\left(A^{-1}\right)^{-1} = A$ provided the inverse exists

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|A| = ad - bc \neq 0$

Now $A_c = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, $A_c^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\sim A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus the inverse of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written

instantly as $\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided $ad - bc \neq 0$

Example 3:

Find the inverse of $A = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$ if it exists.

Solution:

$|A| = \begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} = -2 \neq 0$ $\therefore A^{-1}$ exists

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$$

Example 4:

Show that the inverses of the following do not exist:

(i) $A=\begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix}$ (ii) $A=\begin{pmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 6 & 2 & -4 \end{pmatrix}$

Solution:

(i) $|A|=\begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = 0$ $\therefore A^{-1}$ does not exist

(ii) $|A|=\begin{vmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 6 & 2 & -4 \end{vmatrix} = 0$ $\therefore A^{-1}$ does not exist

Example 5:

Find the inverse of $A=\begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, if it exists.

Solution:

$$|A|=\begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 15 \neq 0 \quad \therefore A^{-1} \text{ exists}$$

We have, $A^{-1}=\frac{1}{|A|} A_c^t$

Now, the cofactors are

$$C_{11}=\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -5,\ C_{12}=-\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = 7,\ C_{13}=\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1 \cdot$$

$$C_{21}=-\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = 10,\ C_{22}=\begin{vmatrix} 2 & 4 \\ 1 & -2 \end{vmatrix} = -8,\ C_{23}=-\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1 \cdot$$

$$C_{31}=\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5,\ C_{32}=-\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix} = 10,\ C_{33}=\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -5 \cdot$$

Hence

$$A_c = \begin{pmatrix} -5 & 7 & 1 \\ 10 & -8 & 1 \\ -5 & 10 & -5 \end{pmatrix}, \; A_c^t = \begin{pmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{pmatrix} \qquad \therefore \; A^{-1} = \frac{1}{15}\begin{pmatrix} -5 & 10 & -5 \\ 7 & -8 & 10 \\ 1 & 1 & -5 \end{pmatrix}$$

Example 6:

Show that $A = \begin{pmatrix} 3 & - & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} \frac{1}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{8}{17} & \frac{6}{17} & -\frac{9}{17} \\ \frac{10}{17} & -\frac{1}{17} & -\frac{7}{17} \end{pmatrix}$ are inverse of each other.

Solution:

$$AB = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{17} & \frac{5}{17} & \frac{1}{17} \\ \frac{8}{17} & \frac{6}{17} & -\frac{9}{17} \\ \frac{10}{17} & -\frac{1}{17} & -\frac{7}{17} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix} \frac{1}{17}\begin{pmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 17 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 17 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Since A and B are square matrices and AB = I, A and B are inverse of each other.

9.7 Exercise:

1) Find the Adjoint of the matrix $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$

2) Find the Adjoint of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

3) Show that the Adjoint of the matrix $A = \begin{pmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{pmatrix}$ is A itself.

4) If $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, verify that $A\,(\text{Adj } A)\,A = |A|\,I$

5) Given $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$

verify that $\text{Adj}\,(AB) = (\text{Adj}\,B)\,(\text{Adj}\,A)$

6) In the second order matrix $A = (a_{ij})$ given that $a_{ij} = i + j$ write out the matrix A and verify that $|\text{Adj } A| = |A|$.

7) Given $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$ verify that $|\text{Adj } A| = |A^2|$

8) Write the inverse of $A = \begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}$

9) Find the inverse of $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

10) Find the inverse of $A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and verify that $A\,A^{-1} = I$

11) Find the Rank of the Matrix $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

12) Find the Rank of the Matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \end{bmatrix}$

13) Find the Rank of the Matrix $A = \begin{bmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{bmatrix}$

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Lesson - 10 VECTORS

10.1 Objective of the lesson:

After studying this lesson, you should be able to understand -

→ Difference between vectors and scalars

→ Addition of vectors and their properties and its applications in solving problems

→ Linearly independent and dependent vectors.

10.2 Structure:

This lesson has the following components:

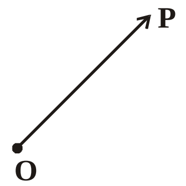
10.3 Introduction:

In mathematics we generally deal with two kinds of physical quantities. Those which are specified by a single real number called the magnitude, in otherwords those which masure quantities but not related to any direction in space. Such quantities are called scalars.

The examples of scalars are mass, length, density, volume etc...

The other types of physical qunatities are those which have got magnitute as well as a definite direction in space. Such quantities are called vector quantities or simply vectors. the examples of vectors are velocity, acceleration, fource, displacement etc...

10.3.1 Representation of vectors: We shall represent vectors by directed line segments. Let O be any arbitrary fixed point in the space and p be any other point. Then the straight line OP has magnitude as well as direction. Therefore the directed line segment OP is capable of representing a vector quantity. We denote this vector by $\overrightarrow{OP}$ and read it as vector $\overrightarrow{OP}$.



The length OP represents the magnitude of the vector $\overrightarrow{OP}$. The point 'O' is called the origin or the initial point of the vector $\overrightarrow{OP}$ while P is called the terminal point.

10.3.2 Notation of Vectors: Vectors are generally represented by clarendon letters (leeters in bold faced type) and their magnitudes by the corresponding italic letters. Thus we may denote $\overrightarrow{OP}$ by **a** and its magnitude by a. However it is more convenient to represent vectors by $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$ etc and their magnitudes by a, b, c, d etc...

10.3.3 Modulus of a Vector: The non - negative number which isthe magnitude of a vector is called its modulus. Thus the length of the line segment OP is called the modulus of $\overrightarrow{OP}$. The modulus a of a vector $\bar{a}$ is some times written as $|\bar{a}|$

10.4 Different Kinds of Vectors:

10.4.1 Null Vector or Zero Vector: If the origin and terminal points of a vector coincide it is said to be a zero or null vector. Evidently its length is zero and its direction is indeterminate. A null vector is denoted by the bold faced type **0**. All zero vectors are equal and they can be expressed as $\overrightarrow{AA}$, $\overrightarrow{BB}$ etc.

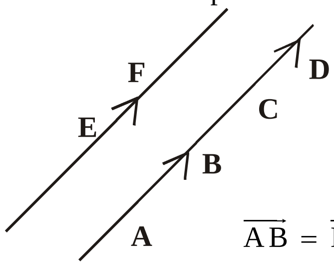
10.4.2 Unit Vector: A vector whose modulus is unity is called a unit vector. If there be any vector a whose modulus is a, then the corresponding unit vector in that direction is denoted by $\hat{a}$ which has its parallel supports.

$$a = a\,\hat{a} \quad \text{or} \quad \hat{a} = \frac{a}{a}$$

10.4.3 Reciprocal Vector: A vector whose direction is the same as that of a given vector a but whose magnitude is the reciprocal of the magnitude of thegiven vector, it is called thereciprocal of a and is written as a^{-1}

Thus if $a = a\,\hat{a}$, then $a^{-1} = \frac{1}{a}\,\hat{a} = \frac{a}{a^2}$

10.4.4 Equal Vectors: Two vectors are said to be equal if they have (i) the same length (magnitude), (ii) the same sense and (iii) the same or parallel supports. The equality is symbolically denoted by a = b. Thus equal vectors may be represented by parallel lines of equal length drawn in the same sense or direction irrespective of the origin.



$\overrightarrow{AB} = \overrightarrow{EF} = \overrightarrow{CD} \neq \overleftarrow{CD}$

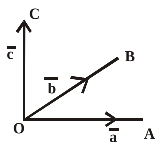
10.4.5 Collinear Vectors: Any number of vectors are said to be collinear when they are parallel to the same line what ever their magnitudes may be.

10.4.6 Negative Vector: The vector which has the same modules as the vector $\overline{a}$ but opposite direction, is called the negative of $\overline{a}$.

The negative of $\overline{a}$ is represented by - $\overline{a}$. Thus if $\overline{AB} = \overline{a}$ then $\overline{BA} = -\overline{a}$.

10.4.7 Co-Initial Vectors: The vectors which have the same initial point are called co-initial vectors.

Eg: Thevectors $\overline{a}$, $\overline{b}$, $\overline{c}$ are co-initial vectors.

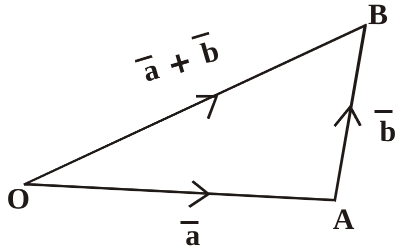


10.4.8 Coplanar Vectors: The vectors which are parallel to the same plane or which lie in the same plane are said to be coplanar.

10.4.9 Localised and Free Vectors: A vector which is drown parallel to a gien vector through a specified point in space is called a localised vector. There can be one and only one such vector. But if the origin of vectors is not specified, thevectos are said to be free vectors.

10.5 Addition of Vectors:

Let $\overline{a}$ and $\overline{b}$ be any two given vectors. If three points O, A,B are taken such that $\overrightarrow{OA} = \overline{a}$, $\overline{AB} = \overline{b}$. Then the vector $\overline{OB}$ (= say $\overline{c}$) is called the vector sum or resultent of the given vectors $\overline{a}$ and $\overline{b}$ and we write $\overline{OB} = \overline{OA} + \overline{AB}$ or $\overline{c} = \overline{a} + \overline{b}$.



It should be noted that the terminal point of vector $\bar{a}$ is the initial point of vector $\bar{b}$ and the resultent vector $\bar{c}$ is the join of the initial point of $\bar{a}$ and the terminal point of $\bar{b}$.

The above law of addition is known as the triangle law of addition.

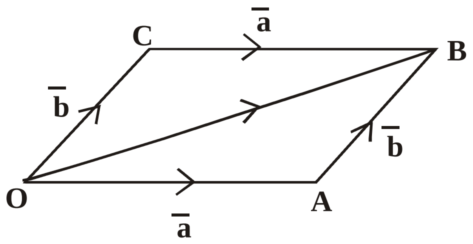
Properties of Vector Addition:

10.5.1 Theorem: Vector addiiton is commutative. i.e. $\bar{a}+\bar{b}=\bar{b}+\bar{a}$, where $\bar{a}$ and $\bar{b}$ are any two vectors.

Proof: Let $\bar{a}$ and $\bar{b}$ be two vectors represented by $\overline{OA}$ and $\overline{AB}$.

Then by definition of addition of two vectors, we have

$$\overline{OB}=\overline{OA}+\overline{AB}=\bar{a}+\bar{b}\;\cdots\cdots\cdots\cdots(1)$$



Compute the parallelogram OABC

We have by definition of equality of two vectors,

$$\overline{OC}=\overline{AB}=\bar{b} \quad \text{and} \quad \overline{CB}=\overline{OA}=\bar{a}$$

Now, by definition of addition of two vectors, we have

$$\overline{OB}=\overline{OC}+\overline{CB}=\bar{b}+\bar{a}\;\cdots\cdots\cdots(2)$$

from (1) and (2) we have

$$\bar{a}+\bar{b}=\bar{b}+\bar{a}$$

10.5.2 Theorem: Vector addiiton is associative

i.e., $(\bar{a}+\bar{b})+\bar{c}=\bar{a}+(\bar{b}+\bar{c})$

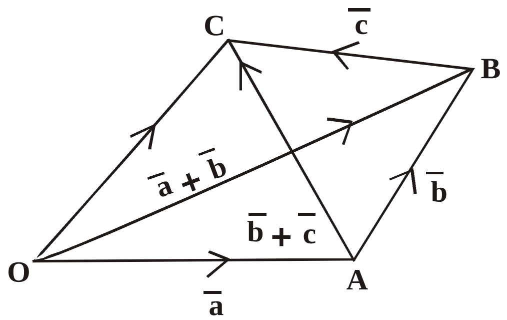
where $\bar{a}$, $\bar{b}$, $\bar{c}$ by any three vectors

Proof: Let $\overline{OA}=\bar{a}$, $\overline{AB}=\bar{b}$ and $\overline{BC}=\bar{c}$

Compute the quadrilateral OABC

By definition of addition of two vectors

$$\overline{OC}=\overline{OA}+\overline{AC}$$



But by definition of addition of two vectors, we have

$$\overline{AC}=\overline{AB}+\overline{BC}=\bar{b}+\bar{c}$$

$$\therefore \overline{OC}=\overline{OA}+\overline{AC}$$

$= \overline{OA} + \left(\overline{AB} + \overline{BC}\right)$

$= \bar{a} + \left(\bar{b} + \bar{c}\right) \cdots\cdots\cdots\cdots(1)$

Again we have by definition of addition of two vectors

$\overline{OC} = \overline{OB} + \overline{BC}$

But $\overline{OB} = \overline{OA} + \overline{AB} = \bar{a} + \bar{b}$

$\therefore \overline{OC} = \left(\bar{a} + \bar{b}\right) + \bar{c} \cdots\cdots\cdots\cdots(2)$

from (1) and (2) we have $\bar{a} + \left(\bar{b} + \bar{c}\right) = \left(\bar{a} + \bar{b}\right) + \bar{c}$

Note: From the above property we notice that the sum of three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is independent of the order in which they are added. Hence this sum can be written as $\bar{a} + \bar{b} + \bar{c}$ without any doubt.

10.5.3 Theorem: For every vector $\bar{a}$, $\bar{a} + \bar{0} = \bar{a}$ where $\bar{0}$ is the zero vector.

Proof: Let $\overline{OA} = \bar{a}$, $\overline{AA} = \bar{0}$

We have by definition of addition of two vectors

$\overline{OA} = \overline{OA} + \overline{AA} = \bar{a} + \bar{0}$

$\therefore \bar{a} = \bar{a} + \bar{0}$

10.5.4 Theorem: To every vector $\bar{a}$, there corresponds the vector - $\bar{a}$ such that $\bar{a} + \left(-\bar{a}\right) = \bar{0}$ where $\bar{0}$ is the zero vector.

Proof: Let $\overline{OA} = \bar{a}$, then $\overline{A0} = -\bar{a}$

We have by definition of addition of two vectors

$\overline{OA} + \overline{AO} = \overline{00}$

$\therefore \bar{a} + \left(-\bar{a}\right) = \bar{0}$

10.5.5 Subtraction of Vectors: If $\bar{a}$, $\bar{b}$ be any two vectors, then we write $\bar{a} + \left(-\bar{b}\right) = \bar{a} - \bar{b}$ and call the operation subtraction.

10.6 Multiplication of a vector by a scalar:

Let m be a scalar and $\bar{a}$ be a vector, then m $\bar{a}$ is defined as a vector whose modulus in $|m|$ times the modulus of the vector $\bar{a}$ or opposite to the vector $\bar{a}$ according as m is positive or negative.

10.6.1 Remark: From the above definition, it is obvious that two non - zero vectors $\bar{a}$ and $\bar{b}$ are collinear iff there exists a non zero scalar m such that $\bar{a} = m\,\bar{b}$.

10.6.2 Remark: If a denotes the modulus of a non - zero vector $\bar{a}$, then the unit vector $\hat{a}$ in the direction of $\bar{a}$ is given by $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$ or $\hat{a} = \dfrac{\bar{a}}{a}$

10.6.3 Remark:

(i) The scalar multiple of a vector satisfies associative law, i.e.,

$$m\left(n\,\bar{a}\right) = (m\,n)\,\bar{a} = n\left(m\,\bar{a}\right)$$

(ii) The scalar multiple of a vector satisfies the distribution laws:

i.e. $(m+n)\,\bar{a} = m\,\bar{a} + n\,\bar{a}$

$$m\left(\bar{a} + \bar{b}\right) = m\,\bar{a} + m\,\bar{b}$$

where m, n are scalars and $\bar{a}$ and $\bar{b}$ are vectors.

10.6.4 Linear Combination: A vector $\bar{r}$ is said to be a linear combination of the vectors $\overline{a_1}, \overline{a_2}, \cdots\cdots, \overline{a_n}$, if there exist scalars $x_1, x_2, x_3, \cdots\cdots, x_n$ such that

$$\bar{r} = x_1\,\overline{a_1} + x_2\,\overline{a_2} + \cdots\cdots + x_n\,\overline{a_n}$$

10.7 Linearly Independent and Dependent System of Vectors:

A system of vectors $\overline{a_1}, \overline{a_2}, \cdots\cdots, \overline{a_n}$ is said to be linearly dependent if there exist a system of scalars $x_1, x_2, x_3, \cdots\cdots, x_n$ not all zero such that

$$x_1\overline{a_1} + x_2\,\overline{a_2} + \cdots\cdots + x_n\,\overline{a_n} = \bar{0}$$

A system of vectors which is not linearly dependent is said to be linearly independent. Thus a system of vectors $\overline{a_1}, \overline{a_2}, \cdots\cdots, \overline{a_n}$ is said to be linearly indepenedent if every relation of the form.

$$x_1\overline{a_1} + x_2\,\overline{a_2} + \cdots\cdots + x_n\,\overline{a_n} = \bar{0} \Rightarrow x_1 = 0, x_2 = 0, \cdots, x_n = 0\,\cdot$$

10.7.1 Theorem: If $\bar{a}$, $\bar{b}$ are two non - zero non - collinear vectors and x, y are scalars such that

$$x\,\bar{a} + y\,\bar{b} = \bar{0} \quad \text{Then} \quad x = 0,\ y = 0.$$

Proof: It is given that $x\,\bar{a} + y\bar{b} = \bar{0}$ ....(1)

Suppose $x \neq 0$ then (1) can be written as

$$\bar{a} = \frac{-y}{x}\,\bar{b} \quad \text{............}(2)$$

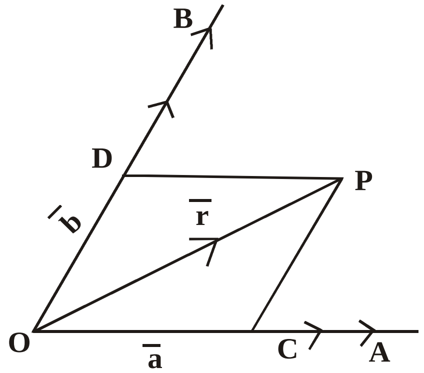
Since $\frac{y}{x}$ is a scalar, therefore (2) implies that the vectors $\bar{a}$ and $\bar{b}$ are collinear. This is a countrary to the hypothesis. Hence x = 0.

Similarly it can be shown that y = 0.

Resolution of a vector in terms of coplanar vectors.

10.7.2 Theorem: If $\bar{a}$, $\bar{b}$ be two given non - collinear vectors, then every vector, $\bar{r}$ coplanar with $\bar{a}$ and $\bar{b}$ can be uniqualy represented as a linear combination $x\,\bar{a} + y\,\bar{b}$; x y being scalars.

Proof: Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ and $\overline{OP} = \bar{r}$. The lines OA, OB, OP are coplanar. Through the point P draw lines parallel to OB and $\overline{OA}$ meeting OA and OB at C and D respectively.



We have $\overline{OP} = \overline{OC} + \overline{CP}$

$$= \overline{OC} + \overline{OD}$$

But $\overline{OC}$ and $\overline{OD}$ are collinear with $\overline{OA}$ and $\overline{OB}$ respectively.

There fore we can find suitable scalars x and y such that

$\overline{OC} = x\,\overline{OA} = x\,\bar{a}\,,\ \overline{OD} = y\,\overline{OB} = y\,\bar{b}$

$\therefore\ \overline{OP} = x\,\bar{a} + y\,\bar{b}$

$\therefore\ \bar{r} = x\,\bar{a} + y\,\bar{b}$

Uniqueness: If possible let $\bar{r} = x_1\,\bar{a} + y_1\,\bar{b}$

Then we have $x\,\bar{a} + y\,\bar{b} = x_1\,\bar{a} + y_1\,\bar{b}$

$$\Rightarrow (x - x_1)\,\bar{a} + (y - y_1)\,\bar{b} = 0 \cdots\cdots\cdots\cdots(1)$$

Since $\bar{a}$ and $\bar{b}$ are non - collinear vectors. Thirefore from (1)

$x - x_1 = 0,\ \ y - y_1 = 0$

$\Rightarrow\ x = x_1\ ,\ y = y_1\ .$

10.7.3 Theorem: If $\bar{a}\,,\bar{b}\,,\bar{c}$ are non coplanar vectors and x, y, z are scalars such that $x\,\bar{a} + y\bar{b} + z\,\bar{c} = \bar{0}$ when $x = y = z = 0$.

Proof: Given that $x\,\bar{a} + y\bar{b} + z\,\bar{c} = \bar{0} \cdots\cdots\cdots\cdots\cdots\cdots(1)$

Suppose $x \neq 0$. Then from (1)

$$\bar{a} = \frac{-y}{x}\,\bar{b} - \frac{z}{x}\,\bar{c} \cdots\cdots\cdots\cdots(2)$$

Since $\frac{y}{x}, \frac{z}{x}$ are scalars, therfore (2) implies that the vector $\bar{a}$ is coplanar with the vectors $\bar{b}$ and $\bar{c}$. This is contrary to the hypothesis. Hence x = 0. Similalry y = 0 and z = 0.

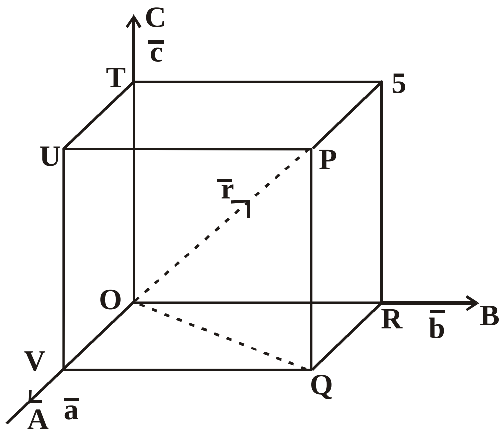
$\therefore\ x = y = z = 0$

10.7.4 Theorem: If $\bar{a}\,,\bar{b}\,,\bar{c}$ are three non - coplanar vectors and $\bar{r}$ is any vector. Then there exist unique scalars x, y, z such that $\bar{r} = x\,\bar{a} + y\,\bar{b} + z\,\bar{c}$.

Proof: Given that $\bar{a}\,,\bar{b}\,,\bar{c}$ are three non - coplanar vectors. Let 0 be any point in space.

Let $\overline{OA} = \bar{a}\,,\ \overline{OB} = \bar{b}\,,\ \overline{OC} = \bar{c}$

Let $\overline{OP} = \bar{r}$



With OP as diagonal, constract a parallelopiped whose three coterminous edges OV, OR and OT are along OA, OB and OC respectively.

Since $\overline{OV}$, $\overline{OR}$, $\overline{OT}$ are collinear with $\overline{OA}$, $\overline{OB}$ and $\overline{OC}$ respectively. Therefore we can find scalars x, y, z such that $\overline{OV} = x\,\overline{OA}$, $\overline{OR} = y\,\overline{OB}$ and $\overline{OT} = z\,\overline{OC}$

or $\overline{OV} = x\,\bar{a}$, $\overline{OR} = y\,\bar{b}$ and $\overline{OT} = z\,\bar{c}$

Now, $\overline{OP} = \overline{OQ} + \overline{QP} = \left(\overline{OV} + \overline{VQ}\right) + \overline{QP}$

$= \overline{OV} + \overline{OR} + \overline{OT}$ $\qquad \left(\because\ \overline{VQ} = \overline{OR}\ ,\ \overline{QP} = \overline{OT}\right)$

$= x\,\bar{a} + y\,\bar{b} + z\,\bar{c}$

$\therefore\ \bar{r} = x\,\bar{a} + y\,\bar{b} + z\,\bar{c}$

Uniqueness: If possible let $\bar{r} = x_1\,\bar{a} + y_1\,\bar{b} + z_1\,\bar{c}$

Then we have $x\,\bar{a} + y\,\bar{b} + z\,\bar{c} = x_1\,\bar{a} + y_1\,\bar{b} + z_1\,\bar{c}$

$\Rightarrow \left(x - x_1\right)\bar{a} + \left(y - y_1\right)\bar{b} + \left(z - z_1\right)\bar{c} = \bar{0}$

Since $\bar{a}$, $\bar{b}$, $\bar{c}$ are on coplanar, we should have

$x - x_1 = 0$, $y = y_1 = 0$ and $z - z_1 = 0$

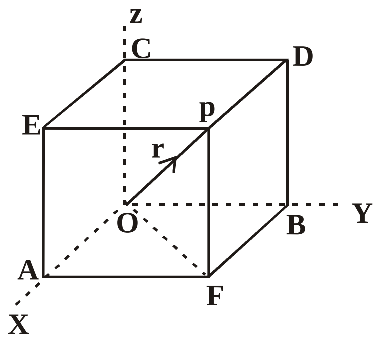
$\therefore\ x = x_1$, $y = y_1$, $z = z_1$.

Note: In the relation $\bar{r} = x\,\bar{a} + y\,\bar{b} + z\,\bar{c}$, we call the vector $\bar{r}$ as the resultent of three vectrs $x\,\bar{a}$, $y\,\bar{b}$, $z\,\bar{c}$ which are called the components of the vector $\bar{r}$.

10.8 The Unit Vectors i, j, k. (Orthonormal System of Unit Vectors):

Let us consider three mutually perpendicular straight lines OX, OY and OZ in the right handed rotaion. By right handed rotation we mean that if one rotates from OX to OY, then OZ lies in the direction in which a right handed screw will move. These three mutually perpendicular lines can determine uniquely the position of a point. Hence these lines can be taken as the coordinate axes, with 0 as origin. The planes XOY, YOZ and ZOX are called coordinate planes.

Let us now consider a vector r. Through the point 0, draw a vector $\overrightarrow{OP}$ equal to the vector r.



With OP as diagonal, construct a rectangular parallelopiped whose three conterminous edges OA, OB, OC are along OX, OY, OZ respectively. Let i, j, k denote the unit vectors along OX, OY, OZ respectively.

Let OA = x, OB = y, and OC = z.

Then we have $\overrightarrow{OA} = xi$, $\overrightarrow{OB} = yj$ and $\overrightarrow{OC} = zk$ ·

Now we have $\overrightarrow{OP} = \overrightarrow{OF} + \overrightarrow{FP}$

$$= \overrightarrow{OA} + \overrightarrow{AF} + \overrightarrow{FP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

i.e., $r = xi + yj + zk$ ·

Here x, y, z are called the coordinates of the point P referred to the axes OX, OY and OZ. Also xi, yj, zk are called rsolved parts of the vector r in the directions of i, j, k respectively.

If α, β, γ be the angles which O P makes with the coordinate axes OX, OY and OZ respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines (d. c. 's) of the line OP. These are usually denoted by ℓ, m, n ·

If OP = r clearly

$$x = r\cos\alpha = \ell r,\ y = r\cos\beta = mr,\ z = r\cos\gamma = nr \quad ....(2)$$

$\therefore$ (1) gives $\bar{r} = \ell r\, i + m r\, j + n r\, k$

$$\therefore\ \hat{r} = \frac{\bar{r}}{r} = \ell i + m\, j + n k \ \ ....(3)$$

Thus the direction cosines of vector $\bar{r}$ are the coefficients of $i,\ j,\ k$ in the rectangular resolution of the unit vector $\hat{r}$.

Also $OP^2 = OF^2 + FP^2$

or $OP^2 = OA^2 + AF^2 + FP^2 = OA^2 + OB^2 + OC^2$

i.e. $r^2 = x^2 + y^2 + z^2$

i.e. $\boxed{|r|^2 = x^2 + y^2 + z^2}$ $\cdots\cdots\cdots\cdots\cdots\cdots (4)$

Thus the square of the module of the vector $\bar{r}$ is the sum of the squares of the coefficients of i, j, k when $\bar{r}$ is expressed in terms of i, j and k.

Also from $r^2 = x^2 + y^2 + z^2$, on dividing both sides by r^2 we get

$$1 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \ell^2 + m^2 + n^2$$

which shows that the sum of the squares of direction cosines is equal to unity.

Again if $r_1 = x_1 i + y_1 j + z_1 k$; $r_2 = x_2 i + y_2 j + z_2 k$;

$r_3 = x_3 i + y_3 j + z_3 k$ etc.

then $\sum r_n = r_1 + r_2 + r_3 + \cdots + r_n$

$$= \left(\sum x_n\right) i + \left(\sum y_n\right) j + \left(\sum z_n\right) k \cdots\cdots\cdots\cdots\cdots (5)$$

where $\left(\sum x_n\right) i$ is the resolved part of $\sum r_n$ along i.

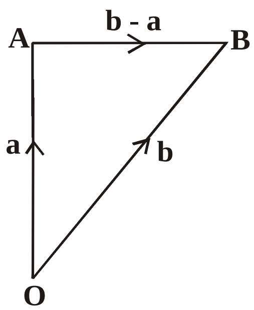
10.8.1 Important Note: If $\bar{r}$ is any vector, then $\bar{r}$ can be uniquely represented as a linear combination of the vectors i, j, k. Thus there exist three unique scalars x, y and z such that $\bar{r} = x i + y j + z k$. Instead of writing $\bar{r} = x i + y j + z k$, we often find it convenient to writer $\bar{r}$ (x, y, z). Thus in this notation if a is the vector $(3, -4, 7)$ then we have $\bar{a} = 3i - 4j + 7k \cdot$

10.8.2 Position Vector: If the vector $\overrightarrow{OP}$ represents the position of the point P in space relative to the point 0, then $\overrightarrow{OP}$ is called the position vector of P referred to 0 as origin.

If we say that A is the point r, then we mean that the position vector of A is r with respect to some given origin 0.

10.8. Important: To express a vector in terms of the position vectors of its and points.

Let 0 be the origin and let a and b be the position vectors of the points A and B with respect to 0 as origin.



Then $\overrightarrow{OA} = a,\ \overrightarrow{OB} = b$

We have $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$$\therefore\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= b - a$$

Thus $\overrightarrow{AB} =$ Position vector of B - posotion vector of A.

Similarly, $\overrightarrow{BA} =$ Position Vector of A - Position Vector of B.

10.8.4 Note: If the coordinates of point P are (x, y, z) referred to OX, OY, OZ as rectangular coordinate axes, then the position vector of the point P with respect to 0 as origin is $xi + yj + zk$. Some times by saying the point P as (x, y, z) we mean the point whose position vector is $xi + yj + zk$. Thus according to this notation if B is the point (- 3, 4, 8) then the position vector of B is $-3i + 4j + 8k$.

10.8.5 To find the position vector of a point which divides the joint of two given points whose position vectors are a and b in given ratio m : n.

Let a, b be the position vectors of two given points A and B referred to 0 as origin.

Let r be the position vector of the point P which divides AB internally in the ratio m : n,

i.e., $$\frac{AP}{PB} = \frac{m}{n}$$

$\therefore \; nAP = mPB$

Hence $n\overrightarrow{AP} = m\overrightarrow{PB}$

But $\overrightarrow{AP} = r - a$ and $\overrightarrow{PB} = b - r$

$\therefore \; n(r - a) = m(b - r)$

$\therefore \; nr - na = mb - mr$

$\therefore \; (m + n)\, r = mb + na$

$$\therefore \; r = \frac{mb + na}{m + n}$$

In particular if m = n, then P is the middle point of AB and its position vector is $\frac{a + b}{2}$.

10.8.6 Collinearity of three points: The necessary and sufficient condition for three points with position vectors a, b, c to be collinear isthat there exist three scalars x, y, z not all zero, such that $xa + yb + zc \;\; 0$, where $x + y + z = 0$.

The condition is sufficient. Let x, y, z be three scalars, not all zero, such that

$$xa + yb + zc = 0, \; x + y + z = 0$$

Let $z \neq 0$, then $x + y = -z \neq 0 \cdot$

We have $xa + yb + zc = 0$

$$\therefore \; \frac{xa + yb}{-z} = c$$

$$\therefore \; c = \frac{xa + yb}{x + y}$$

Thus c is the position vector of the point dividing the join of a and b in the ratio y : x.

$\therefore$ The points a, b, c are collinear.

The condition is necessary. Let the three points be collinear. Suppose c divides the join of a and b in the ratio m : n.

Then $c = \dfrac{na + mb}{n + m}$

$\therefore (n+m)c = na + mb$

$\therefore na + m - (n+m)c = 0$

If we take $n = x, m = y, -(n+m) = z\cdot$

We get $xa + yb + zc = 0,$

where $x + y + z = n + m - (n+m) = 0$

Hence the condition is necessary.

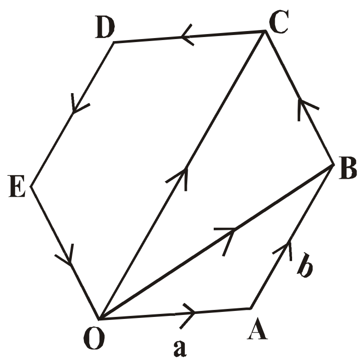
10.8.7 **S.A.Q.:** What is the modulus of the vector $3i - 2j + 6k$? Also write a unit vector in the direction of this vector.

10.8.8 **S.A.Q.:** Find the value or values of k for which k (i + j + k) is a unit vector.

10.9 Solved Examples:

10.9.1 **Example:** If a, b are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order?

Sol: OABCDE is a regular hexagon.



Let $\overrightarrow{OA} = a$ and $\overrightarrow{AB} = b$

Join OB and OC

We have $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = a + b$

Since OC is parallel to AB and double of AB,

$\therefore \overrightarrow{OC} = 2\,\overrightarrow{AB} = 2b$

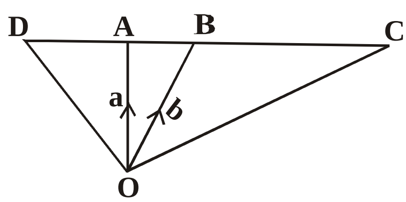
Now $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2b - (a+b) = b - a,$

$\overrightarrow{CD} = -\overrightarrow{OA} = -a$ and $\overrightarrow{DE} = -\overrightarrow{AB} = -b$

Also $\overrightarrow{EO} = -\overrightarrow{BC} = -(b-a) = a - b\cdot$

10.9.2 **Example:** If a, b are the position vectors of A, B respectively, find that of a point C in AB produced such that AC = 3 AB; and that of a point D in BA produced such that BD = 2 BA.

Sol: Let a and b be the position vectors of A and B



with repect to 0 as origin.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = b - a$

Since AC = 3 AB.

$\therefore \overrightarrow{AC} = 3\,\overrightarrow{AB} = 3\,(b - a)$

Now $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= a + 3\,(b - a) = 3b - 2a$

Again $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = a - b$

Since BD = 2 BA

$\therefore \overrightarrow{BD} = 2\,\overrightarrow{BA} = 2\,(a - b)$

Now $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = b + 2\,(a - b) = 2a - b$

10.9.3 Example: Show that $\overrightarrow{DA} + \overrightarrow{CD} = \overrightarrow{CB} - \overrightarrow{AB}$

Sol: We have $\overrightarrow{DA} + \overrightarrow{CD} - \overrightarrow{DA} + \left(-\overrightarrow{DC}\right)$ $\left[\because \overrightarrow{CD} = -\overrightarrow{DC}\right]$

$= \overrightarrow{DA} - \overrightarrow{DC} = \overrightarrow{CA} \cdots\cdots\cdots\cdots(1)$

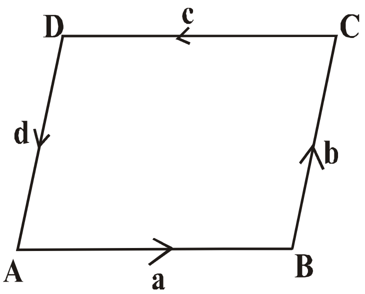
Also $\overrightarrow{CB} - \overrightarrow{AB} = -\overrightarrow{BC} - \left(-\overrightarrow{BA}\right) = -\overrightarrow{BC} + \overrightarrow{BA} = \overrightarrow{BA} - \overrightarrow{BC}$

$= \overrightarrow{CA} \cdots\cdots\cdots\cdots(2)$

From (1) and (2) we see that $\overrightarrow{DA} + \overrightarrow{CD} = \overrightarrow{CB} - \overrightarrow{AB}$

10.9.4 Example: a, b, c, d are the vectors forming the consecutive sides of a quadrilateral. Show that a necessary and sufficient condition that the figure be a parallelogram is that a + c = 0 and that this implies b + d = 0.

Sol: ABCD is a quadrilateral.



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = a + b$

Also $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\therefore a + b + c = -d$

$\therefore a + b + c + d = 0 \cdots\cdots\cdots\cdots(1)$

Now if ABCD is parallelogram, then AB And DC are parallel and equal.

$\therefore \overrightarrow{AB} = -\overrightarrow{CD}$ · $\therefore a = -c$ or $a + c = 0$ ·

Hence the condition is necessary. Also with the help of (1), We get in this case b + d = 0.

Sufficient. Since a + c = 0, $\therefore$ a = – c

$$\therefore \overrightarrow{AB} = -\overrightarrow{CD}$$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$. Thus AB And DC are parallel and equal.

When a + c = 0. We have from (1) b + d = 0.

$\therefore$ b = – d $\cdot$ $\therefore \overrightarrow{BC} = -\overrightarrow{DA}$ $\therefore \overrightarrow{BC} = \overrightarrow{AD}$

$\therefore$ BC and AD are parallel and equal.

Hence ABCD is a parallelogram.

10.9.5 Example: The position vectors of the four points A, B, C and D are a, b, 2a + 3b, a - 2b respectively.

Express the vectors $\overrightarrow{AC}$, $\overrightarrow{DB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ in terms of a and b.

Sol: We have

$\overrightarrow{AC}$ = position vector of C - position vector of A

$$= (2a + 3b) - a = a + 3b$$

$\overrightarrow{DB}$ = position vector of B - position vector of D

$$= b - (a - 2b) = 3b - a$$

$\overrightarrow{BC}$ = position vector of C - position vector of B

$$= (2a + 3b) - b = 2a + 2b$$

and $\overrightarrow{CA}$ = position vector of A - position vector of C

$$= a - (2a + 3b) = -a - 3b = -(a + 3b)$$

10.9.6 Example: D, E, F are the middle points of the sides BC, CA, AB respectively of a triangle ABC. Show that

(i) FE is parallel to BC and half of its length.

(ii) The sum of the vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$, $\overrightarrow{CF}$ is zero.

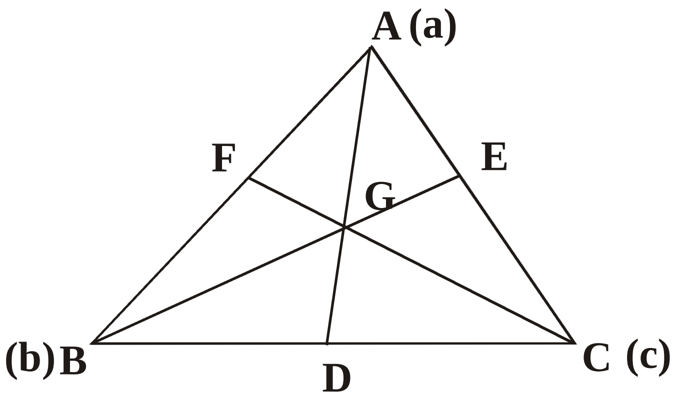
(iii) The medians have a common point of trisection.

Sol: Let a, b, c be the position vectors of A, B and C with respect to any point as origin.

(i) Position vector of D is $\frac{b+c}{2}$

Position vector of E is $\frac{c+a}{2}$ and

Position vector of F is $\frac{a+b}{2}$



$\overrightarrow{BC} =$ position vector of C - position vector of B = c - b,

$\overrightarrow{FE} =$ Position vector of E - position vector of F

$$= \frac{c+a}{2} - \frac{a+b}{2} = \frac{c-b}{2}$$

$\therefore \overrightarrow{FE} \ \frac{1}{2}\overrightarrow{BC}$. Hence FE is parallel to BC and half of its length.

(ii) $\overrightarrow{AD} =$ position vector of D - position vector of A

$$= \frac{b+c}{2} - a$$

Similarly $\overrightarrow{BE} = \frac{c+a}{2} - b$ and $\overrightarrow{CF} = \frac{a+b}{2} - c$

$$\therefore \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \left(\frac{b+c}{2} - a\right) + \left(\frac{c+a}{2} - b\right) + \left(\frac{a+b}{2} - c\right)$$

$$= \frac{b+c-2a+c+a-2b+a+b-2c}{2} = 0$$

(iii) The position vector of the point G dividing AD in the ratio 2 : 1 is

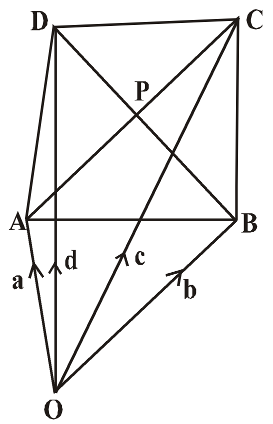
$$= \frac{1(\text{Position Vector of } A) + 2(\text{Position Vector of } D)}{1+2}$$

$$= \frac{a + 2\left(\frac{b+c}{2}\right)}{1+2} = \frac{a+b+c}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The symmetry of the result in (1) shows that G also lies in the other two medians BE and CG at their point of trisection. Hence the medians are concurrent and the point of concurrency G divides each median in the ratio 2 : 1.

10.9.7 Example: Prove by vector methods that the diagonals of a parallelogram bisect each other; conversely, if the diagonals ofa quadrilateral bisect each other it is a parallelogram.

Sol: Let ABCD be a parallelogram. Reffered to 0 as the origin of vectors let a, b, c, d be the position vectors of A, B, C, D respectively.



Since ABCD is a parallelogram.

$\therefore$ AB and DC are parallel and equal.

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

$\therefore \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$

$\therefore b - a = c - d$

$\therefore b + d = a + c$

The position vector of the middle point of AC is $\frac{a+c}{2}$ while the position vector of the middle point of BD is $\frac{b+d}{2}$

These two points coincide by virtue of (1). HEnce the diagonals of a parallelogram bisect each other.

Converse. AC and BD have the same point as their mid-point.

$\therefore \frac{a+c}{2} = \frac{b+d}{2}$ $\quad \therefore a + c = b + d$ $\quad \therefore b - a = c - d$

$\therefore \overrightarrow{AB} = \overrightarrow{DC}$ $\quad \therefore$ AB and DC are parallel and equal.

Similarly AD and BC are parallel and equal. HEnce ABCD is a parallelogram.

10.9.8 Example: ABCD is a parallelogram and P the point of intersection of its diagonals. Show that for any origin 0 (not necessarly in the plane of hte figure) the sum of the position vectors of the vertices is equal to fourtimes that of P.

Sol: Refer fig of Ex. 7 . The diagonals of a parallelogram bisect each other. Therefore P is the middle point of AC and BD both.

$$\therefore \overrightarrow{OA} + \overrightarrow{OC} = 20\vec{P} \cdots\cdots\cdots\cdots\cdots(1)$$

and $$\overrightarrow{OB} + \overrightarrow{OD} = 2\overrightarrow{OP} \cdots\cdots\cdots\cdots\cdots(2)$$

Adding (1) and (2) we get,

$$\left(\overrightarrow{OA} + \overrightarrow{OC}\right) + \left(\overrightarrow{OB} + \overrightarrow{OD}\right) = 40\,\vec{P}$$

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 40\,\vec{P}$$

10.9.9 **Example:** If the position vectors of P and Q are $2i + 3j - 7k$ and $4i - 3j + 4k$ respectively, find $\overrightarrow{PQ}$ and determine its direction cosines.

Sol: We have $\overrightarrow{PQ} =$ position vector of Q - Position vector of P

$$= 4i - 3j + 4k - (2i + 3j - 7k) = 2i - 6j + 11k$$

Modulus of $\overrightarrow{PQ} = \sqrt{\left\{(2)^2 + (-6)^2 + (11)^2\right\}} = \sqrt{(4 + 36 + 121)}$

$$= \sqrt{(161)}$$

$\therefore$ Unit vector in the direction of $\overrightarrow{PQ} = \dfrac{2i - 6j + 11k}{\sqrt{(161)}}$

$$= \frac{2}{\sqrt{(161)}}\,i - \frac{6}{\sqrt{(161)}}\,j + \frac{11}{\sqrt{(161)}}\,k$$

$\therefore$ Direction cosines of $\overrightarrow{PQ}$ are $\dfrac{2}{\sqrt{(161)}}, \dfrac{-6}{\sqrt{(161)}}, \dfrac{11}{\sqrt{(161)}}$

10.9.10 **Example:** Find the fum of the vectors $4i + 2j + 3k$, $i - 7j - 2k$, $5i + 4j - 2k$ and $2i + j + k$. Also calculate the module and direction cosines of each.

Sol: The sum of the given vectors

$$= (4i + 2j + 3k) + (i - 7j - 2k) + (5i + 4j - 2k) + (2i + j + k)$$

$= 12i + 0j + 0k = 12i$

Modulus of vector $4i + 2j + 3k$

$$= \sqrt{\left\{(4)^2 + (2)^2 + (3)^2\right\}} = \sqrt{(16 + 4 + 9)} = \sqrt{(29)}$$

Unit vector in the direction of this vector $= \dfrac{4i + 2j + 3k}{\sqrt{(29)}}$

$$= \frac{4}{\sqrt{(29)}} i + \frac{2}{\sqrt{(29)}} j + \frac{3}{\sqrt{(29)}} k \cdot$$

$\therefore$ Direction cosines of this vector are

$$= \frac{4}{\sqrt{(29)}}, \frac{2}{\sqrt{(29)}}, \frac{3}{\sqrt{(29)}}$$

Similarly unit vector in the direction of the vector $i - 7j - 2k$

$$= \frac{i - 7j - 2k}{\sqrt{\left(1^2 + 7^2 + 2^2\right)}} = \frac{i - 7j - 2k}{\sqrt{(1 + 49 + 4)}}$$

$$= \frac{1}{\sqrt{(54)}} i - \frac{7}{\sqrt{(54)}} j - \frac{2}{\sqrt{(54)}} k \cdot$$

$\therefore$ Direction cosines of this vector are

$$\frac{1}{\sqrt{(54)}}, \frac{-7}{\sqrt{(54)}}, \frac{-2}{\sqrt{(54)}}$$

Dimilstly for other vectors.

10.9.11 Example: If $a = 2i + j - k$, $b = i - j, c = 5i - j + k$ find the unit vector parallel to $a + b - c$ but in the opposite direction.

Solution: We have $a + b - c = -2i + j - 2k$

$$\therefore |a + b - c| = \sqrt{\left\{(-2)^2 + (1)^2 + (-2)^2\right\}} = \sqrt{(9)} = 3$$

$\therefore$ The unit vector in the direction of $a + b - c = \dfrac{a + b - c}{|a + b - c|}$

$$= \frac{1}{3}(-2i + j - 2k)$$

$\therefore$ The unit vector parallel to $a + b - c$ but in the opposite direction

$$= -\frac{1}{3}(-2i + j - 2k) = \frac{1}{3}(2i - j + 2k)$$

10.9.12 Example: If the vertices of a triangle are the points $x_1 i + y_1 j + z_1 k$, $x_2 i + y_2 j + z_2 k$ and $x_3 i + y_3 j + z_3 k$ what are the vectors determined by its sides? Find the lengths of these vectors.

Solution: Let ABC be a triangle whose vertices A, B, C are the points $x_1 i + y_1 j + z_1 k$, $x_2 i + y_2 j + z_2 k$ and $x_3 i + y_3 j + z_3 k$ respectively. Thus the position vectors of A, B and C are $x_1 i + y_1 j + z_1 k$, $x_2 i + y_2 j + z_2 k$ and $x_3 i + y_3 j + z_3 k$ respectively.

We have $\overrightarrow{AB} =$ position vector of B - position vector of A

$$= (x_2 i + y_2 j + z_2 k) - (x_1 i + y_1 j + z_1 k)$$

$$= (x_2 - x_1) i + (y_2 - y_1) j + (z_2 - z_1) k \cdot$$

Length of AB = modulus of $\overrightarrow{AB}$

$$= \sqrt{\left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right\}}$$

Similarly $\overrightarrow{BC} =$ position vector of C - position vector of B

$$= (x_3 - x_2) i + (y_3 - y_2) j + (z_3 - z_2) k$$

Length of $BC = \sqrt{\left\{ (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 \right\}}$

Finally $\overrightarrow{CA} =$ position vector of A - position vector of C

$$= (x_1 - x_3) i + (y_1 - y_3) j + (z_1 - z_3) k$$

Length of $CA = \sqrt{\left\{ (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 \right\}}$

10.9.13 Example: If the position vectors of P, Q, R, S be

$2i + 4k,\ 5i + 3\sqrt{(3)}\,j + 4k,\ -2\sqrt{(3)}\,j + k,\ 2i + k$ prove that RS is parallel to PQ and is two third of PQ.

Sol: Let 0 be the origin of vectors. Then we have

$$\overrightarrow{OP} = 2i + 4k,\ \overrightarrow{OQ} = 5i + 3\sqrt{3}j + 4k,\ \overrightarrow{OR} = -2\sqrt{3}j + k,\ \overrightarrow{OS} = 2i + k$$

Now $\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = 2i + k - \left(-2\sqrt{3}j + k\right) = 2i + 2\sqrt{3}j$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 5i + 3\sqrt{3}j + 4k - \left(2i + 4k\right) = 3i + 3\sqrt{3}j$$

$$\therefore \overrightarrow{RS} = \frac{2}{3}\ \overrightarrow{PQ}$$

$\therefore \overrightarrow{PQ}$ and $\overrightarrow{RS}$ are collinear.

Also modulus of $\overrightarrow{RS} = \frac{2}{3}$ modulus of $\overrightarrow{PQ}$

Hence PQ is parallel to RS and $RS = \frac{2}{3}\ PQ$

10.9.14 Example:

The position vectors of four points P, Q, R, S are a, b, $2a + 3b,\ 2a - 3b$ respectively. Express the vectors $\overrightarrow{PR}$, $\overrightarrow{RS}$ and $\overrightarrow{PQ}$ in terms of a and b.

Sol: Let 0 be the origin

$$\therefore \overrightarrow{OP} = a,\ \overrightarrow{OQ} = b,\ \overrightarrow{OR} = 2a + 3b,\ \overrightarrow{OS} = 2a - 3b$$

Now $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 2a + 3b - a = a + 3b$

$$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = 2a - 3b - 2a - 3b = -6b$$

and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = b - a$

10.9.15 Example: Show that the points $a - 2b + 3c,\ \ 2a + 3b - 4c$ and $-7b + 10c$ are collinear.

Solution: Let the given points be denoted by A, B and C. Let '0' be the origin reference, then

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2a + 3b - 4c - a + 2b - 3c = a + 5b - 7c$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -7b + 10c - a + 2b - 3c$$

$$= -a - 5b + 7c$$

$$= -(a + 5b - 7c)$$

$$\therefore \overrightarrow{AB} = -\overrightarrow{AC}$$

Thus the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are either parallel or collinear.

Further because these vectors are coterminus, hence the points A, B, C are collinear.

10.9.16 Example:

Show that the points $-6a + 3b + 2c$, $3a - 2b + 4c$, $5a + 7b + 3c$, $-13a + 17b - c$ are coplanar a, b, c being three coplanar vectors.

Sol: Let the given points be A, B, C and D.

Let O be the origin of reference, then

$$\overrightarrow{OA} = -6a + 3b + 2c, \quad \overrightarrow{OB} = 3a - 2b + 4c$$

$$\overrightarrow{OC} = 5a + 7b + 3c, \overrightarrow{OD} = -13a + 17b - c$$

Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3a - 2b + 4c + 6a - 3b - 2c$

$$= 9a - 5b + 2c$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 5a + 7b + 3c + 6a - 3b - 2c$$

$$= 11a + 4b + c$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -13a + 17b - c + 6a - 3b - 2c$$

$$= -7a + 14b - 3c$$

Let us first prove that the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are linearly connected.

Let $\ell \cdot \overrightarrow{AB} + m\,\overrightarrow{AC} - \overrightarrow{AD}$, then

$$\ell[9a - 5b + 2c] + m[11a + 4b + c] = -7a + 14b - 3c$$

$$\Rightarrow (9\ell + 11m)a + [-5\ell + 4m]b + (2\ell + m)c = -7a + 14b - 3c$$

$$\therefore 9\ell + 11m = -7$$

$-5\ell + 4m = 14$

$2\ell + m = -3$

Sovling (1) and (2), we get $\ell = -2,\ m = 1\cdot$

These values of 'ℓ' and 'm' satisfy the equation (3) also. Hence vectos $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar.

Because these vectors are coterminus, Hence the four points A, B, C and D are coplanar.

10.10 Answers to S.A.Q:

10.10.1 Answer to S.A.Q. of 10.8.7

Let $\bar{a} = 3i - 2j + 6k$. Then

$$\left|\bar{a}\right| = \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7\cdot$$

The unit vector in the direction of $\bar{a}$ is equal to $\dfrac{\bar{a}}{\left|\bar{a}\right|}$

$$= \frac{3i - 2j + 6k}{7}$$

10.10.2 Answer to S.A.Q. of 10.8.8

Let $\bar{a} = k\,(i + j + k)$

Since $\bar{a}$ is a unit vector $\therefore \left|\bar{a}\right| = 1$

$$\Rightarrow \left|k\,(i + j + k)\right| = 1$$

$$\Rightarrow \pm\sqrt{k^2 + k^2 + k^2} = 1$$

$$\Rightarrow \pm\sqrt{3k^2} = 1$$

$$\Rightarrow \pm\sqrt{3}\,k = 1$$

$$\Rightarrow k = \pm\frac{1}{\sqrt{3}}$$

10.10.3 Answer to S.A.Q. of 10.9.16

Let $2i - j + k = x\left(i - 3j - 5k\right) + y\left(3i - 4j - 4k\right)$

Then

$$2i - j + k = \left(x + 3y\right)i + \left(-3x - 4y\right)j + \left(-5x - 4y\right)k$$

$$\Rightarrow x + 3 = 2 \cdots\cdots\cdots\cdots(1)$$

$$-3x - 4 = -1 \cdots\cdots\cdots\cdots(2)$$

$$-5x - 4y = 1 \cdots\cdots\cdots\cdots(3)$$

from (1) and (2) we get x = - 1, y = 1.

These values of x and y satisfy the equation (3)

Hence the given vectors are collinear.

10.11 Summary:

In thsi lesson we discussed, scalars vectors, addition of vectors, properties of vector addition collinear vectors, coplanar vectors and related problems.

10.12 Technical Terms:

Scalars

Vectors

Modulus of a Vector

Unit Vector

Receprocal Vector

Collinear Vectors

Co - initial Vectors

Coplanar Vectors

Localised and Free Vectors

10.13 Exercise:

1) Define the terms scalars and vectors giving one example of each.

2) Show that the vectors $4i - 6j + 9k$ and $-6i + 9j - \frac{27}{2}k$ are collinear.

3) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$ if A and B are the point $(2, 3m-5)$ and $(-4, 6, 7)$ respectively.

4) Prove by vector method that the diagonals of a parallelogram bisect each other.

5) If the position vectors of two points A and B are $2\bar{a}+\bar{b}-\bar{c}$ and $-\bar{a}+2\bar{c}$ respectively, then find the position vector of the middle point of AB.

6) If $\bar{a}=3i-j-4k, \bar{b}=-2i+4j-3k$ and $\bar{c}=i+2j-k,$ find a unit vector parallel to $3\bar{a}-2\bar{b}+4\bar{c}$.

7) If ABCD is a parallelogram and L, M are the mid points of BC and CD respectively, Show that $\overline{AL}+\overline{AM}=\frac{3}{2}\overline{AC}$

8) Show that the points $A(1,1,1)$, $B(1, 2, 3)$, $C(2, -1, 1)$ are vertices of an isoseeles triangle.

9) If P is a poitn and ABCD is a quadrilateral, and $\overline{AP}+\overline{PB}\ \overline{PD}=\overline{PC}$, show that ABCD is a parallelogram.

10) Show that the points $\bar{a}-2\bar{b}+3\bar{c}$, $2\bar{a}+3\bar{b}-4\bar{c}$, $-7\bar{b}+10\bar{c}$ are collinear.

10.14 Answers to Exercise:

3) $\overrightarrow{AB}=(-6, 3, 12)$, $\overrightarrow{BA}=(6, -3, -12)$

5) $\frac{1}{2}(\bar{a}+\bar{b}+\bar{c})$

6) $\pm\frac{1}{\sqrt{398}}(17i-3j-10k)$

10.15 Model Questions:

1) If $\bar{a}=2i+j-k$, $\bar{b}=i-j$, $\bar{c}=5i-j+k$ find the unit vector parallel to $\bar{a}+\bar{b}-\bar{c}$.

2) Show that the points $\bar{a}-2\bar{b}+3\bar{c}$, $2\bar{a}+3\bar{b}-4\bar{c}$ and $-7\bar{b}+10\bar{c}$ are collinear.

10.16 References:

1) D.C. Sancheti, V.K. Kapoor, "Business Mathematics".
Sultan Chand and Sons, New - Delhi - 110002.

2) A.R. Vasistha, "Vector Algebra", Krishna - Prakasan Mandir, Meerut.

Lesson Writer

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Lesson - 11

Product of Two Vectors

11.1 Objective of the lesson:

After studying this lesson, you should be able to understand -

→ Scalar product or dot product of two vectors and their geometrical interpretations.

→ Vector product or cross product of two vectors and their geiometrical inter pretation.

→ Application of scalar and vector product of vectors and study its properties.

11.2 Structure:

This lesson has the following components:

11.3 Introduction
11.4 The scalar or dot product of two vectors
11.5 Properties of scalar or dot products
11.6 Solved examples of scalar product of vectors
11.7 Vector product or cross product
11.8 Properties of vector product
11.9 Geometrical interpretation of vector product
11.10 Solved exaples of cross product'
11.11 Answers to S.A.Q.
11.12 Summary
11.13 Technical Terms
11.14 Exercise
11.15 Answers to exercise
11.16 Model questions
11.17 References

11.3 Introduction:

In this lesson we shall discuss the operations of multiplication of a vector by another vector. These operations are called products of vectors. There are two distinct kinds of products oftwo vectors. One being a pure number is called the scalar product while the other being a vector quantity is called the scalar product while the other being a vector quantity is called the vector product. Each of these products is jointly proportional to the moduli of two vectors; and each follows the distributive law.

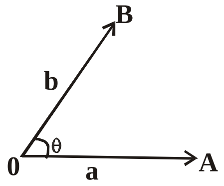
11.4 The Scalar or Dot Product of Two Vectors:

11.4.1 Definition: The scalar product of two vectors $\bar{a}$ and $\bar{b}$ whose moduli are a and b respectively, is defined as the real number

$$\bar{a}\ \bar{b} = |\bar{a}|\ |\bar{b}|\ \cos\theta$$

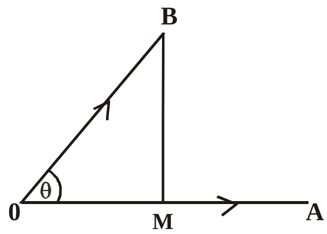
where θ is the angle between the two vectors when drawn from a common origin.

In this the angle θ is restricted to the interval $0 \le \theta \le \pi$. It makes no difference whether we choose θ or $-\theta$ since $\cos\theta = \cos(-\theta)$.



The scalar product of two vectors '$\bar{a}$' and '$\bar{b}$' is written as $\bar{a}\cdot\bar{b} = (\bar{a}, \bar{b})\cdot$ The first is read as $\bar{a}$ dot $\bar{b}$. It is on account of this notation that scalar product is sometimes called dot product. Thus $\bar{a}\cdot\bar{b} = ab\ \cos\theta$. This product will be positive, negative or zero according as the angle θ is less thab $\pi/2$, greater than $\pi/2$ or right angle.

11.4.2 Physical Illustration of scalar product: An example of scalar product is the work done by a force during a displacement of the particle acted upon.



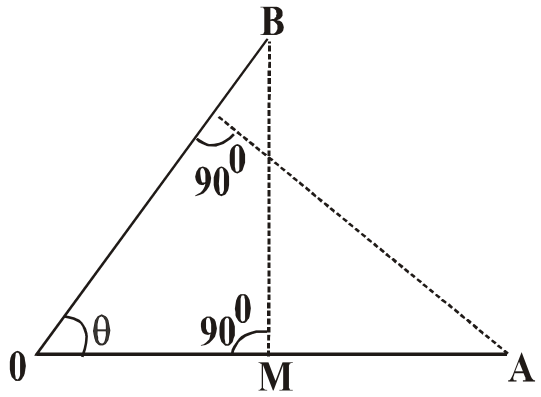
Suppose $\overrightarrow{OA} = \bar{a}$ represents a force in magnitude and idrection acting upon a particle at 0. Suppose $\overrightarrow{OB} = \bar{b}$ represents the displacement of the particle from 0 to B. Draw BM perpendicular to OA. Then the displacement of the particle in the direction of the force.

$$= OM = OB\ \cos\theta,\ \text{where } \angle BOA = \theta$$

$$= b\ \cos\theta$$

$\therefore$ Work done by the force a in moving its point of application from 0 to $B = ab\ \cos\theta = \bar{a}\cdot\bar{b}$

11.4.3 Geometrical Interpretation of Dot Product: The scalar product of two vector is the product of the modulus of either vector and the scalar component ofthe other in its direction.



Let $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ and $\angle BOA = \theta$.

Then $\bar{a} \cdot \bar{b} = ab \cos\theta$, where $|a| = a$ and $|b| = b = a[0B \cos\theta] = a[0L] = |a| \times$ projection of $\bar{b}$ in the direction of $\bar{a}$.

The quantity b cos θ is called the scalar component of $\bar{b}$ along $\bar{a}$.

Similarly $\bar{a} \cdot \bar{b} = b[0A \cos\theta] = b[0M]$

$$= |b| \times \text{ projection of } \bar{a} \text{ ain the direction of } \bar{b}.$$

The quantity a cos θ is called the scalar component of $\bar{a}$ along $\bar{b}$.

Remember: If '$\bar{a}$' and '$\bar{b}$' are two vectors and $\bar{a} \neq 0$, then the projection of $\bar{b}$ on the direction of $\bar{a} = \dfrac{\bar{a} \cdot \bar{b}}{|\bar{a}|}$.

11.5 Properties of Scalar or Dot Products:

11.5.1 The Scalar Product of Two Vectors is Commutative, i.e.,

$$\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$$

where '$\bar{a}$' and '$\bar{b}$' are any two vectors.

11.5.2 It can easily be shown that

$$\bar{a}\cdot(-\bar{b}) = -(\bar{a}\cdot\bar{b})\ ;\ (-\bar{a})\cdot(-\bar{b}) = \bar{a}\cdot\bar{b}$$

for every pair of vectors $\bar{a}$, $\bar{b}$.

11.5.3 If m is any scalar and '$\bar{a}$' and '$\bar{b}$' are any vectors, then $(m\bar{a})\cdot\bar{b} = m(\bar{a}\cdot\bar{b}) = \bar{a}\cdot(m\bar{b})$

11.5.4 If two vectors '$\bar{a}$' and '$\bar{b}$' have the same direction, then $\theta = 0$, i.e., $\cos\theta = 1$ and $\bar{a}\cdot\bar{b} = |\bar{a}|\ |\bar{b}|\cos 0 = a\ b$.

11.5.5 If two vectors '$\bar{a}$' and '$\bar{b}$' have opposite directions then $\theta = \pi$, $\cos\pi = -1$ and $\bar{a}\cdot\bar{b} = |\bar{a}|\ |\bar{b}|\cos\pi = -\ a\ b$.

11.5.6 Angle between two vectors int erms of scalar product.

Let θ be the angle between two non - zero vectors '$\bar{a}$' and '$\bar{b}$', then $\bar{a}\cdot\bar{b} = |\bar{a}|\ |\bar{b}|\cos\theta$.

$$\therefore \cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} \text{ or } \theta = \cos^{-1}\left(\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right)$$

In particular if each of the vectors '$\bar{a}$' and '$\bar{b}$' is a unit vector, then $\bar{a}\cdot\bar{b} = \cos\theta$. Thus the scalar product of two unit vectors is equal to the cosine of the angle between their directions.

11.5.7 **Condition for perpendicularity of two vectors:**

Theorem: If '$\bar{a}$' and '$\bar{b}$' are perpendicular vectors, then

$$\bar{a}\cdot\bar{b} = 0$$

Conversely if $\bar{a}\cdot\bar{b} = 0$, then at least one of the following must be true : $\bar{a} = 0$, $\bar{b} = 0$, '$\bar{a}$' and '$\bar{b}$' are perpendicular to each other.

Hence the necessary and sufficient condition that two non - zero vectors should be perpendicular is that their scalar product should be zero.

11.5.8 **Length of a vector as a scalar product:** If '$\bar{a}$' be any vector, then the scalar product

$$\bar{a}\cdot\bar{a} = |\bar{a}|\ |\bar{a}|\cos\theta = |\bar{a}|^2 = a^2$$

or $|\bar{a}| = +\sqrt{(\bar{a}\cdot\bar{a})}$ i.e. the length $|\bar{a}|$ of any vector '$\bar{a}$' is the non negative square root of the scalar product $\bar{a}\cdot\bar{a}$

Note: As a convention $\bar{a} \cdot \bar{a}$ will be denoted by Thus the square of a vector will be equal to the square of its modulus. In particular, the square of a unit vector will be equal to unity.

11.5.9 Orthonomral Vector Triads i, j, k:

We have the following important results:

$$\bar{i} \cdot \bar{i} = \bar{i}^2 = 1, \ \bar{j} . \bar{j} = \bar{j}^2 = 1 ; \ \bar{k} \cdot \bar{k} = \bar{k}^2 = 1 \qquad [\because \ \bar{i}, \bar{j}, \bar{k} \text{ are unit vectors}]$$

$$\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{i} = 0 \ ; \ \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{j} = 0 ; \ \bar{k} \cdot \bar{i} = \bar{i} \cdot \bar{k} = 0 \qquad [\because \ \bar{i}, \bar{j}, \bar{k} \text{ are mutually perpendicular}]$$

11.5.10 Since the scalar product is a mere number, therefore it may occur as the numerical coefficient of a vector. Thus $(\bar{a} \cdot \bar{b})\,\bar{c}$ is a vector, obtained on multiplying c by the number $\bar{a} \cdot \bar{b}$. Similarly the combination $(\bar{a} \cdot \bar{b})(\bar{c} \cdot \bar{d})$ of four vectors is merely the product of two numbers $\bar{a} \cdot \bar{b}$ and $\bar{c} \cdot \bar{d}$.

11.5.11 Distributive Law:

To prove that $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$ i.e.

to rpove that the dot product of two vectors is distributive with respect to vector addition.

11.5.12 Some simple identities based on distributive law.

(i) $(\bar{a} + \bar{b}) \cdot (\bar{a} - \bar{b}) = \bar{a}^2 - \bar{b}^2$

We have $(\bar{a} + \bar{b}) \cdot (\bar{a} - \bar{b}) = \bar{a} \cdot \bar{a} + \bar{b} \cdot \bar{a} - \bar{b} \cdot \bar{b}$

$$= \bar{a}^2 - \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{b} - \bar{b}^2 = \bar{a}^2 - \bar{b}^2$$

(ii) $(\bar{a} + \bar{b})^2 = \bar{a}^2 + 2\bar{a} \cdot \bar{b} + \bar{b}^2$

We have $(\bar{a} + \bar{b})^2 = (\bar{a} + \bar{b}) \cdot (\bar{a} + \bar{b}) = \bar{a} \cdot \bar{a} + \bar{a} \cdot \bar{b} + \bar{b} \cdot \bar{a} + \bar{b} \cdot \bar{b}$

$$= \bar{a}^2 + \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{b} + \bar{b}^2 = \bar{a}^2 \ 2\,\bar{a} \cdot \bar{b} + \bar{b}^2$$

(iii) $(\bar{a} - \bar{b})^2 = \bar{a}^2 - 2\,\bar{a} \cdot \bar{b} + \bar{b}^2$

We have $(\bar{a} - \bar{b})^2 = (\bar{a} - \bar{b}) \cdot (\bar{a} - \bar{b})$

$$= \bar{a} \cdot \bar{a} - \bar{a} \cdot \bar{b} - \bar{b} \cdot \bar{a} + \bar{b} \cdot \bar{b}$$

$$= \bar{a} \cdot \bar{a} - \bar{a} \cdot \bar{b} - \bar{a} \cdot \bar{b} + \bar{b} \cdot \bar{b} = \bar{a}^2 - 2\,\bar{a} \cdot \bar{b} + \bar{b}^2$$

11.5.13 Scalar Product of two vectors in terms of their rectangular components.

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

Then $\bar{a} \cdot \bar{b} = \left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right) \cdot \left(b_1\bar{i} + b_2\bar{j} + b_3\bar{k}\right)$

$$= a_1b_1\bar{i}^2 + a_1b_2\bar{i} \cdot \bar{j} + a_1b_3\bar{i} \cdot \bar{k} + a_2b_1\bar{j} \cdot \bar{i} + \cdots \qquad \text{(By Distributive Law)}$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

$$\left[\because \bar{i}^2 = \bar{j}^2 = \bar{k}^2 = 1 \text{ and } \bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{i} = 0\right]$$

Thus the scalar product of two vectors is equal to the sum of the products of their corresponding rectangular components.

In particular $a^2 = \left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right) \cdot \left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right)$

$$= a_1^2 + a_2^2 + a_3^2 \cdot$$

Thus the square of 'a' vector is equal to sum of the squares of its rectangular components.

Imp: We know that the square of a vector is equal to the square of its module. Therefore if 'a' be the module of the vector

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, we have $a^2 = \bar{a}^2 = a_1^2 + a_2^2 + a_3^2$

$$\therefore a = +\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)}$$

11.5.14 To find the distance between the points $A\left(a_1, a_2, a_3\right)$ and $B\left(b_1, b_2, b_3\right)$.

Let O be the origin of reference. Then $\overrightarrow{OA} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\overrightarrow{OB} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}\cdot$

We have $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(b_1\bar{i} + b_2\bar{j} + b_3\bar{k}\right) - \left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right)$

$$= \left(b_1 - a_1\right)\bar{i} + \left(b_2 - a_2\right)\bar{j} + \left(b_3 - a_3\right)\bar{k}$$

Now $AB^2 = (\overrightarrow{AB})^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$

$\therefore AB = \sqrt{\{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2\}}$.

11.5.15 To find the direction cosines of the vector

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ where $|\bar{r}| = r$

Suppose the vector $\bar{r}$ makes angles α, β, γ with OX, OY and OZ respectively, where $\bar{i}$, $\bar{j}$ and $\bar{k}$ are unit vectors along OX, OY and OZ respectively.

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ .... (1)

Multiplying (1) scalarly with the vectors $\bar{i}, \bar{j}, \bar{k}$ successively

We get

$\bar{r} \cdot \bar{i} = x(\bar{i} \cdot \bar{i}) + y(\bar{j} \cdot \bar{i}) + z(\bar{k} \cdot \bar{i}) = x$

$\bar{r} \cdot \bar{j} = x(\bar{i} \cdot \bar{j}) + y(\bar{j} \cdot \bar{j}) + z(\bar{k} \cdot \bar{j}) = y$

and $\bar{r} \cdot \bar{k} = x(\bar{i} \cdot \bar{k}) + y(\bar{j} \cdot \bar{k}) + z(\bar{k} \cdot \bar{k}) = z$

Now $\bar{r} \cdot \bar{i} = r(1) \cos\alpha = r\cos\alpha$

$\therefore r\cos\alpha \Rightarrow x$ or $\cos\alpha = x/r$

Again $\bar{r} \cdot \bar{j} = r(1) \cos\beta = r\cos\beta$

$\therefore r\cos\beta = y$ or $\cos\beta = y/r$

Also $\bar{r} \cdot \bar{k} = r(1) \cos\gamma = r\cos\gamma$

$\therefore r\cos\gamma = z$ or $\cos\gamma = z/r$

But $r = |\bar{r}| = \sqrt{(x^2 + y^2 + z^2)}$

$\therefore$ If l, m, n be the direction cosines of the vector $\bar{r}$, we have

$$\ell = \cos\alpha = \frac{x}{\sqrt{(x^2+y^2+z^2)}}\ ,\quad m = \cos\beta = \frac{y}{\sqrt{(x^2+y^2+z^2)}},$$

$$n = \cos\gamma = \frac{z}{\sqrt{(x^2+y^2+z^2)}}.$$

Moreover unit vector in the direction of $\bar{r} = \hat{r}$

$$= \frac{\bar{r}}{r} = \frac{x\bar{i} + y\bar{j} + z\bar{k}}{\sqrt{(x^2+y^2+z^2)}} = \ell\bar{i} + m\bar{j} + n\bar{k}$$

Note: The quantities which are proportional to direction cosines are called direction rations.

11.5.16 To find the angle between two vectors in terms of the rectangular components ofthe given vectors.

Let $\overrightarrow{OA} = a = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

and $\overrightarrow{OB} = b = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

Let $|\bar{a}| = a$ and $|\bar{b}| = b$ and let θ be the angle between the two vectors $\bar{a}$ and $\bar{b}$.

We have $\bar{a}\cdot\bar{b} = ab\cos\theta$

$$\therefore \cos\theta = \frac{\bar{a}\cdot\bar{b}}{ab}$$

Now $\bar{a}\cdot\bar{b} = \left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right)\cdot\left(b_1\bar{i} + b_2\bar{j} + b_3\bar{k}\right)$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

$$a = \sqrt{\bar{a}^2} = \left(a_1^2 + a_2^2 + a_3^2\right)$$

and $b = \sqrt{\bar{b}^2} = \left(b_1^2 + b_2^2 + b_3^2\right)$

$$\therefore \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\left(a_1^2+a_2^2+a_3^2\right)}\sqrt{\left(b_1^2+b_2^2+b_3^2\right)}}$$

$$= \frac{a_1}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)}} \cdot \frac{b_1}{\sqrt{\left(b_1^2 + b_2^2 + b_3^2\right)}} + \frac{a_2}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)}} \cdot \frac{b_2}{\sqrt{\left(b_1^2 + b_2^2 + b_3^2\right)}}$$

$$+ \frac{a_3}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)}} \cdot \frac{b_3}{\sqrt{\left(b_1^2 + b_2^2 + b_3^2\right)}}$$

$= \ell_1\ell_2 + m_1m_2 + n_1n_2$ where $\left(\ell_1, m_1, n_1\right)$ and $\left(\ell_2, m_2, n_2\right)$ are the direction cosines of the lines along the vectors $\bar{a}$ and $\bar{b}$.

11.5.17 To find an explicit expression of r as a linear combination of the orthonormal traid of vectors $\bar{i}$, $\bar{j}$ and $\bar{k}$.

Any vector $\bar{r}$ can be expressed uniquely as linear combination of any three non - coplanar vectors. Therefore

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ ....(1)

Multiplying (1) scalarly with the vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ successively,

We get $\bar{r} \cdot \bar{i} = x\,\bar{i} \cdot \bar{i} + y\,\bar{j} \cdot \bar{i} + z\bar{k} \cdot \bar{i} = x$,

$\bar{r} \cdot \bar{j} = x\bar{i} \cdot \bar{j} + y\bar{j} \cdot \bar{j} + z\bar{k} \cdot \bar{j} = y$,

and $\bar{r} \cdot \bar{k} = x\bar{i} \cdot \bar{k} + y\bar{j} \cdot \bar{k} + z\bar{k} \cdot \bar{k} = z\cdot$

Putting the values of x, y and z in (1), we get

$\bar{r} = \left(\bar{r} \cdot \bar{i}\right)\bar{i} + \left(\bar{r} \cdot \bar{j}\right)\bar{j} + \left(\bar{r} \cdot \bar{k}\right)\bar{k}$ which is the required expression.

11.6 Solved Examples of Scalar Product of Vectors:

11.6.1 **Example:** Show that the vectors $2\bar{i} - 3\bar{j} + 5\bar{k}$, $-2\,\bar{i} + 2\bar{j} + 2k$ are perpendicular to each other.

Sol: We have $\left(2\bar{i} - 3\bar{j} + 5\bar{k}\right) \cdot \left(-2\,\bar{i} + 2\bar{j} + 2k\right) = -4 - 6 + 10 = 0$

Moreover $\left|2\bar{i} - 3\bar{j} + 5\bar{k}\right| = \sqrt{(4 + 9 + 25)} = \sqrt{38}$ and $\left|-2\bar{i} + 2\bar{j} + 2\bar{k}\right| = \sqrt{12}$

Thus the dot product of the given vectors is zero and neither of the vectors is a zero vector. Hence the two vectors are perpendicular.

11.6.2 Example: Prove that the points $2\bar{i} - \bar{j} + \bar{k}$, $\bar{i} - 3\bar{j} - 5\bar{k}$ and $3\bar{i} - 4\bar{j} - 4\bar{k}$ are the vertices of a right angled triangle. Also find the remaining angles of the triangle.

Sol: Let the given points be A, B, C respectively. We have

$$\overrightarrow{AB} = (\bar{i} - 3\bar{j} - 5\bar{k}) - (2\bar{i} - \bar{j} + \bar{k}) = -\bar{i} - 2\bar{j} - 6\bar{k} ;$$

$$\overrightarrow{BC} = (3\bar{i} - 4\bar{j} - 4\bar{k}) - (\bar{i} - 3\bar{j} - 5\bar{k}) = 2\bar{i} - \bar{j} + \bar{k}$$

and $$\overrightarrow{CA} = (2\bar{i} - \bar{j} + \bar{k}) - (3\bar{i} - 4\bar{j} - 4\bar{k}) = -\bar{i} + 3\bar{j} + 5\bar{k}$$

Now $\overrightarrow{BC} \cdot \overrightarrow{CA} = (2\bar{i} - \bar{j} + \bar{k}) \cdot (-\bar{i} + 3\bar{j} + 5\bar{k}) = -2 - 3 + 5 = 0$

Thus BC is perpendicular to CA. Hence $\angle C$ is a right angle.

Now in order to find the angle 'B', we have $\overrightarrow{BC} = 2\bar{i} - \bar{j} + \bar{k}$ and $\overrightarrow{BA} = -\overrightarrow{AB} = \bar{i} + 2\bar{j} + 6\bar{k}$

So $\overrightarrow{BC} \cdot \overrightarrow{BA} = |\overrightarrow{BC}|\,|\overrightarrow{BA}| \cos B$

$$\therefore \cos B = \frac{\overrightarrow{BC} \cdot \overrightarrow{BA}}{|\overrightarrow{BC}|\,|\overrightarrow{BA}|} = \frac{(2\bar{i} - \bar{j} + \bar{k}) \cdot (\bar{i} + 2\bar{j} + 6\bar{k})}{|2\bar{i} - \bar{j} + \bar{k}|\,|\bar{i} + 2\bar{j} + 6\bar{k}|}$$

$$= \frac{2 - 2 + 6}{\sqrt{(4+1+1)}\,\sqrt{(1+4+36)}} = \frac{6}{\sqrt{6}\sqrt{41}}$$

$$\therefore B = \cos^{-1} \sqrt{\left(\frac{6}{41}\right)}$$

Similarly $$\cos A = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{|\overrightarrow{AC}|\,|\overrightarrow{AB}|} = \frac{(\bar{i} - 3\bar{j} - 5\bar{k}) \cdot (-\bar{i} - 2\bar{j} - 6\bar{k})}{|\bar{i} - 3\bar{j} - 5\bar{k}|\,|-\bar{i} - 2\bar{j} - 6\bar{k}|}$$

$$= \frac{-1 + 6 + 30}{\sqrt{(1+9+25)}\,\sqrt{(1+4+36)}}$$

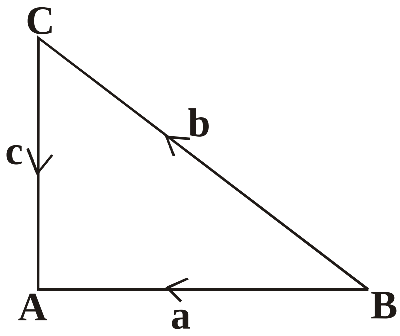
$$= \frac{35}{\sqrt{35}\sqrt{41}} = \sqrt{\left(\frac{35}{41}\right)}$$

$$\therefore A = \cos^{-1} \sqrt{\left(\frac{35}{41}\right)}$$

11.6.3 Example: Show that the vectors $a = 3\bar{i} - 2\bar{j} + \bar{k}$, $b = \bar{i} - 3\bar{j} + 5\bar{k}$, $c = 2\bar{i} + \bar{j} - 4\bar{k}$ from a right angled triangle.

Sol: We have $\bar{b} + \bar{c} = (\bar{i} - 3\bar{j} + 5\bar{k}) + (2\bar{i} + \bar{j} - 4\bar{k})$

$$= 3\bar{i} - 2\bar{j} + \bar{k} = \bar{a}$$



Thus if $\overrightarrow{BC} = \bar{b}$ and $\overrightarrow{CA} = \bar{c}$ and $\overrightarrow{BA} = \bar{a}$,

then $\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$ i.e. $\bar{a} = \bar{b} + \bar{c}$

Hence $\bar{a}, \bar{b}, \bar{c}$ form the sides of a triangle.

Also $\bar{a} \cdot \bar{c} = (3\bar{i} - 2\bar{j} + \bar{k}) \cdot (2\bar{i} + \bar{j} - 4\bar{k}) = 6 - 2 - 4 = 0$

Showing that $\overrightarrow{BA}$ and $\overrightarrow{CA}$ are perpendicular to each other i.e. ΔABC is right angled at A.

11.6.4 Example: If $a = \bar{i} + 2\bar{j} - 3\bar{k}$ and $b = 3\bar{i} - j + 2\bar{k}$

(i) Show that $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ are orthogonal,

(ii) Calculate the angle between $2\bar{a} + \bar{b}$ and $\bar{a} + 2\bar{b}$

Sol: (i) We have $\bar{a} + \bar{b} = 4\bar{i} + \bar{j} - \bar{k}$ and $\bar{a} - \bar{b} = -2\bar{i} + 3\bar{j} - 5\bar{k}$

Now $(\bar{a} + \bar{b}) \cdot (\bar{a} - \bar{b}) = (4\bar{i} + \bar{j} - \bar{k}) \cdot (-2\bar{i} + 3\bar{j} - 5\bar{k})$

$$= 4(-2) + 1(3) - 1(-5) = -8 + 3 + 5 = 0$$

$\therefore$ the vectors $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ are orthogonal.

(ii) We have $2\bar{a} + \bar{b} = 2(\bar{i} + 2\bar{j} - 3\bar{k}) + (3\bar{i} - \bar{j} + 2\bar{k}) = (2\bar{i} + 4\bar{j} - 6\bar{k})$

$+ (3\bar{i} - \bar{j} + 2\bar{k}) = 5\bar{i} + 3\bar{j} - 4\bar{k}$ and $\bar{a} + 2\bar{b} = (\bar{i} + 2\bar{j} - 3\bar{k}) + 2(3\bar{i} - \bar{j} + 2\bar{k})$

$$= (\bar{i} + 2\bar{j} - 3\bar{k}) + (6\bar{i} - 2\bar{j} + 4\bar{k}) = 7\bar{i} + \bar{k}$$

Let θ be the angle between the vectors $2\bar{a} + \bar{b}$ and $\bar{a} + 2\bar{b}$. Then

$$\cos\theta = \frac{(2\bar{a}+\bar{b})\cdot(\bar{a}+2\bar{b})}{|2\bar{a}+\bar{b}|\ |\bar{a}+2\bar{b}|} = \frac{(5\bar{i}+3\bar{j}-4\bar{k})\cdot(7\bar{i}+\bar{k})}{|5\bar{i}+3\bar{j}-4\bar{k}|\ |7\bar{i}+\bar{k}|}$$

$$= \frac{5(7)+3(0)-4(1)}{\sqrt{\{5^2+3^2+(-4)^2\}}\cdot\sqrt{(7^2+0^2+1^2)}} = \frac{35-4}{\sqrt{50}\ \sqrt{50}} = \frac{31}{50}$$

$$\therefore\ \theta = \cos^{-1}(31/50)$$

11.6.5 Prove that in a right angled triangle the square of the hypoienuse is equal to the sum of the squares of the other two sides.

Sol: Let ABC be a rt. angled triangle. Since AB and AC are perpendicular,

$$\therefore\ \overrightarrow{AC}\cdot\overrightarrow{AB} = 0 \quad ....(1)$$

Now $\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$$\therefore |\overrightarrow{BC}| = |\overrightarrow{AC} - \overrightarrow{AB}| \qquad \therefore |\overrightarrow{BC}|^2 = |\overrightarrow{AC} - \overrightarrow{AB}|^2$$

$$\therefore (\overrightarrow{BC})^2 = (\overrightarrow{AC} - \overrightarrow{AB})^2$$

[Since the square of a vector is equal to the square of its module].

$$\therefore (\overrightarrow{BC})^2 = (\overrightarrow{AC})^2 + (\overrightarrow{AB})^2 - 2\,\overrightarrow{AC}\cdot\overrightarrow{AB}$$

$$\therefore\ BC^2 = AC^2 + AB^2 - 2(0) \qquad \text{[From (1)]}$$

$$\therefore\ BC^2 = AC^2 + AB^2$$

$\therefore$ Square of the hypotenuse BC = sum of the squares of the other two sides AC and AB.

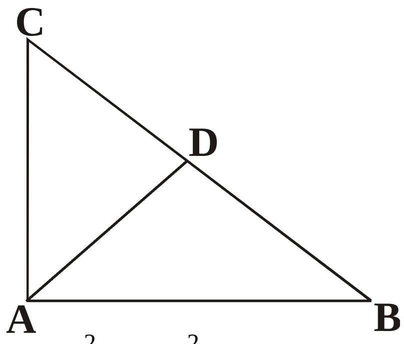
11.6.6 **Example:** Prove that the mid point of the hypotenuse of a right angled triangle is equidistant from its vertices.

Sol: ABC is a triangle, right angled at A. D is the middle point of the hypotenuse BC. We have

$$\overrightarrow{BD} = \overrightarrow{DC}$$

Now $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$ and

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{DB}$$



But AB and AC are perpendicular to each other

$$\therefore \overrightarrow{AB} \cdot \overrightarrow{AC} = 0$$

Hence $\left(\overrightarrow{AD} + \overrightarrow{DB}\right) \cdot \left(\overrightarrow{AD} - \overrightarrow{DB}\right) = 0$

$\therefore \left(\overrightarrow{AD}\right)^2 - \left(\overrightarrow{DB}\right)^2 = 0$ $\therefore AD^2 - DB^2 = 0$ $\therefore AD^2 = DB^2$

$\therefore AD = DB = DC$

11.6.7 Example: The necessary and sufficient condition that two non - null vectors $a = \left(a_1 i + a_2 j + a_3 k\right)$ and $b = \left(b_1 i + b_2 j + b_3 k\right)$ are parallel is $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

The necessary and sufficient condition that these two vectors are erpendicular is $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$.

Sol: We know that the necessary and sufficient condition that two non - null vectors $\bar{a}$ and $\bar{b}$ are parallel is that there should exist a scalar t such that

$$\bar{a} = t\,\bar{b}$$

i.e. $a_1 i + a_2 j + a_3 k = t\left(b_1 i + b_2 j \;\; b_3 k\right)$

This equation is consistent if and only if

$$a_1 = t b_1 \;,\; a_2 = t b_2 \;,\; a_3 = t b_3 \cdot$$

or $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = t$

For the second part, we know that the necessary and sufficient condition that two non - null vectors 'a' and 'b' are perpendicular is that $a \cdot b = 0$

i.e. $\left(a_1 i + a_2 j + a_3 k\right) \cdot \left(b_1 i + b_2 j + b_3 k\right) = 0$

i.e. $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

11.6.8 Example:

If a, b are unit vectors and θ is the angle between them show that

$$\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} - \hat{b}\right|$$

Sol: We have $\left(\hat{a} - \hat{b}\right)^2 = (\hat{a})^2 + \left(\hat{b}\right)^2 - 2\left(\hat{a} \cdot \hat{b}\right)$

$$= 1 + 1 - 2(1)(1)\cos\theta = 2\left(1 - \cos\theta\right) = 4\sin^2\frac{\theta}{2}$$

Now square of a vector is equal to square of its module.

$$\therefore \left|\hat{a} - \hat{b}\right|^2 = 4\sin^2\frac{\theta}{2} \text{ or } \left|\hat{a} - \hat{b}\right| = 2\sin\frac{\theta}{2}$$

or $$\sin\frac{\theta}{2} = \frac{1}{2}\left|\hat{a} - \hat{b}\right|$$

11.6.9 Example: Show that if $\left|\bar{a} + \bar{b}\right| = \left|\bar{a} - \bar{b}\right|$ then $\bar{a}$ and $\bar{b}$ are perpendicular.

Sol: We have $\left|\bar{a} + \bar{b}\right| = \left|\bar{a} - \bar{b}\right|$

$$\therefore \left|\bar{a} + \bar{b}\right|^2 = \left|\bar{a} - \bar{b}\right|^2$$

But the square of a vector is equal to the square of its module.

$$\therefore \left(\bar{a} + \bar{b}\right)^2 = \left(\bar{a} - \bar{b}\right)^2$$

or $\bar{a}^2 + \bar{b}^2 + 2\bar{a} \cdot \bar{b} = \bar{a}^2 + \bar{b}^2 - 2\bar{a} \cdot \bar{b}$ or $4\,\bar{a} \cdot \bar{b} = 0$ or $\bar{a} \cdot \bar{b} = 0$

Hence $\bar{a}$ and $\bar{b}$ are perpendicular.

11.6.10 Example: Prove that perpendiculars from the vertices of a triangle of the opposite sides are concurrent.

11.6.11 Example: A, B, C, D are the points $i - k, -i + 2j, 2i - 3k, 3i - 2j - k$ respectively. Show that the projection of AB on CD is equal to that of CD on AB. Also find the cosine of their inclinations.

Sol: $\overrightarrow{AB} = \left(-i + 2j\right) - \left(i - k\right) = -2i + 2j + k$

$\therefore \left|\overrightarrow{AB}\right| = \sqrt{(4+4+1)}$

and $\overrightarrow{CD} = (3i - 2j - k) - (2i - 3k) = i - 2j + 2k$

$\therefore \left|\overrightarrow{CD}\right| = \sqrt{(1+4+4)} = 3$

Now $\overrightarrow{AB} \cdot \overrightarrow{CD} = (-2i + 2j + k) \cdot (i - 2j + 2k)$

$= -2 - 4 + 2 = -4$

But $\overrightarrow{AB} \cdot \overrightarrow{CD} = \left|\overrightarrow{CD}\right|$ (projection of AB on CD)

$\therefore$ Projection of AB on CD $= \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\left|\overrightarrow{CD}\right|} = \dfrac{-4}{3}$

Similarly projection of CD on AB $= \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\left|\overrightarrow{AB}\right|} = \dfrac{-4}{3}$

Hence projection of AB on CD = projection of CD on AB.

Also if θ be the angle between AB and CD, we have

$\overrightarrow{AB} \cdot \overrightarrow{CD} = \left|\overrightarrow{AB}\right| \left|\overrightarrow{CD}\right| \cos\theta$

$\therefore \cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\left|\overrightarrow{AB}\right| \left|\overrightarrow{CD}\right|} = \dfrac{-4}{3 \times 3} = \dfrac{-4}{9}$

11.6.12 Example: Prove that the angle in a semi - circle is a right angle.

Sol: Let O be the centre of the semi circle and AB be the diameter.

Let P be any point on the circumference of the semi - circle.

Let $\overrightarrow{OA} = a$. Then $\overrightarrow{OB} = -a$

Also let $\overrightarrow{OP} = r$

Now $OP = OA \quad \therefore OP^2 = OA^2 \quad \therefore \overrightarrow{OP}^2 = \overrightarrow{OA}^2$

$\therefore r^2 = a^2 \qquad \therefore r^2 - a^2 = 0$

$\therefore (r - a) \cdot (r + a) = 0 \cdots\cdots\cdots (1)$

Now $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = r - a$

and $\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = r - (-a) = r + a$

$\therefore$ From (1) we get $\overrightarrow{AP} \cdot \overrightarrow{BP} = 0$. Hence AP and BP are mutually perpendicular. Therefore $\angle APB = \pi/2$.

11.6.13 Example:

Find the angle between the vectors

(i) $\bar{a} = i + 2j + 2k$, $\bar{b} = i - 2j + 2k$

(ii) $\bar{a} = 2i + j + k$, $\bar{b} = -2i + 2j + 2k$

(iii) $\bar{p} = a_1 i + b_1 j + c_1 k$, $\bar{q} = a_2 i + b_2 j + c_2 k$.

and find the condition that they are perpendicular to each other.

Sol: (i) $\bar{a} \cdot \bar{b} = (i + 2j + 2k) \cdot (i - 2j + 2k)$

$= (1)(1) + (2)(-2) + (2)(2)$

$= 1 - 4 + 4 = 1$

$a = \sqrt{1^2 + 2^2 + 2^2}$, $b = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

$a = 3$

$$\cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|\ |\bar{b}|} = \frac{1}{9} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{9}\right)$$

(ii) $\bar{a} \cdot \bar{b} = 2(-2) + 1 \cdot 2 + 1 \cdot 2 = 0$

$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

i.e. vectors $\bar{a}$ and $\bar{b}$ are perpendicular to each other.

(iii) $\bar{p} \cdot \bar{q} = a_1a_2 + b_1b_2 + c_1c_2$

$p = \sqrt{a_1^2 + b_1^2 + c_1^2}$, $q = \sqrt{a_2^2 + b_2^2 + c_2^2}$

$$\cos\theta = \frac{\bar{p} \cdot \bar{q}}{|\bar{p}|\,|\bar{q}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)\left(a_2^2 + b_2^2 + c_2^2\right)}}$$

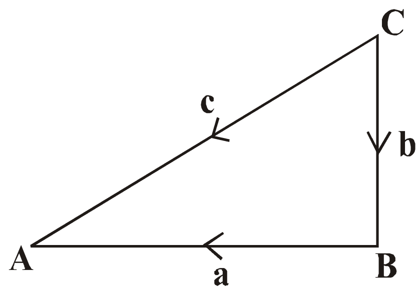
The condition that $\theta = \frac{\pi}{2}$ provides $a_1a_2 + b_1b_2 + c_1c_2 = 0$

11.6.14 Example: Show that the vectors $2i - j + k,\ i - 3j - 5k$ and $3i - 4j - 4k$ from the sides of a right angled triangle. Also find the remaining angles of the triangle.

Sol: Let us suppose that $\bar{a} = 2i - j + k$, $\bar{b} = i - 3j - 5k$

and $\bar{c} = 3i - 4j - 4k$

Let $\bar{a}$ and $\bar{b}$ represent the vectors BA and CB respectively.



Then $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$

$= (i - 3j - 5k) + (2i - j + k)$

$\because \bar{a}$, $\bar{b}$ and $\bar{c}$ form the sides of triangle

Again $\bar{a} \cdot \bar{b} = (2i - j + k) \cdot (i - 3j - 5k)$

$= (2)(1) + (-1)(-3) + (1)(-5)$

$= 2 + 3 - 5$

$= 0$

$\therefore \bar{a}$ and $\bar{b}$ i.e., $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are perpendicular to each other.

Also $\cos A = \frac{\bar{a} \cdot \bar{c}}{|\bar{a}|\ |\bar{c}|}$

$$= \frac{(2i - j + k) \cdot (3i - 4j - 4k)}{\sqrt{2^2 + 1^2 + 1^2}\ \sqrt{3^2 + 4^2 + 4^2}}$$

$$= \frac{(2)\ (3) + (-1)\ (-4) + (1)\ (-4)}{\sqrt{6}\ \sqrt{41}} = \frac{6 + 4 - 4}{\sqrt{6}\ \sqrt{41}}$$

or $\quad A = \cos^{-1}\sqrt{\frac{6}{41}}$

and $\quad \cos c = \frac{\bar{b} \cdot \bar{c}}{|\bar{b}|\ |\bar{c}|}$

$$= \frac{(i - 3j - 5k) \cdot (3i - 4j - 4k)}{\sqrt{1^2 + 3^2 + 5^2} \cdot \sqrt{3^2 + 4^2 + 4^2}}$$

$$= \frac{(1)\ (3) + (-3)(-4) + (-5)(-4)}{\sqrt{35}\ \sqrt{41}}$$

$$= \sqrt{\frac{35}{41}}$$

or $\quad c = \cos^{-1}\sqrt{\frac{35}{41}}$

11.6.15 Example:

Given three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ such that

$$\overrightarrow{7a} = \overrightarrow{2i} + \overrightarrow{3j} + \overrightarrow{6k}$$

$$\overrightarrow{7b} = \overrightarrow{3i} - \overrightarrow{6j} + \overrightarrow{2k}$$

$$\overrightarrow{7c} = \overrightarrow{6i} + \overrightarrow{2j} - \overrightarrow{3k}$$

Show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are each of unit length and are mutually perpendicular.

Sol: We are given the vectors

$$\vec{a} = \frac{2}{7}\vec{i} + \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$$

$$\vec{b} = \frac{3}{7}\vec{i} - \frac{6}{7}\vec{j} + \frac{2}{7}\vec{k}$$

$$\vec{c} = \frac{6}{7}\vec{i} + \frac{2}{7}\vec{j} - \frac{3}{7}\vec{k}$$

Magnitude of $\vec{a} = \left|\vec{a}\right| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{49}{49}} = 1$

Magnitude of $\vec{b} = \left|\vec{b}\right| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{49}{49}} = 1$

Magnitude of $\vec{c} = \left|\vec{c}\right| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2} = \sqrt{\frac{49}{49}} = 1$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are all of maginitude equal to 1, i.e., they are unit vectors.

Now to prove that they are mutually perpendiuclar

Let us take their dot products

$$\vec{a} \cdot \vec{b} = \left(\frac{2}{7}\vec{i} + \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}\right) \cdot \left(\frac{3}{7}\vec{i} - \frac{6}{7}\vec{j} + \frac{2}{7}\vec{k}\right)$$

$$= \frac{2 \cdot 3}{7 \cdot 7} - \frac{3 \cdot 6}{7 \cdot 7} + \frac{2 \cdot 6}{7 \cdot 7} = \frac{6 - 18 + 12}{49} = 0$$

$\Rightarrow$ vectors $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\vec{b} \cdot \vec{c} = \left(\frac{3}{7}\vec{i} - \frac{6}{7}\vec{j} + \frac{2}{7}\vec{k}\right) \cdot \left(\frac{6}{7}\vec{i} + \frac{2}{7}\vec{j} - \frac{3}{7}\vec{k}\right)$$

$$= \frac{3 \cdot 6}{7 \cdot 7} - \frac{6 \cdot 2}{7 \cdot 7} - \frac{2 \cdot 3}{7 \cdot 7} = \frac{18 - 12 - 6}{49} = 0$$

$\Rightarrow$ vectors $\vec{b}$ and $\vec{c}$ are perpendicular.

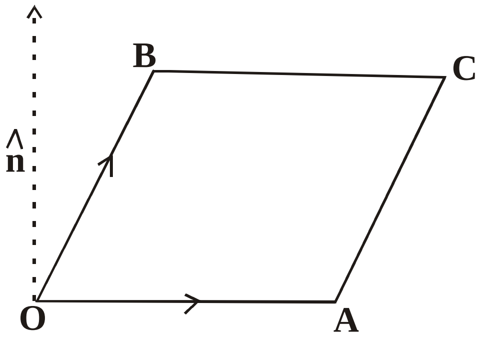
$$\vec{a} \cdot \vec{c} = \frac{2 \cdot 6}{7 \cdot 7} + \frac{3 \cdot 2}{7 \cdot 7} - \frac{3 \cdot 6}{7 \cdot 7} = \frac{12 + 6 - 18}{49} = 0$$

Hence vector $\vec{a}$, $\bar{b}$ and $\vec{c}$ are mutually perpendicular.

11.7 Vector Product or Cross Product:

The vector (or cross) product of two vectors $\bar{a}$ and $\bar{b}$ written as $\bar{a} \times \bar{b}$ is a vector c, where

(i) Modulus $|\bar{c}| = |\bar{a}| \, |\bar{b}| \sin\theta$ where θ is the angle between vectors $\bar{a}$, $\bar{b}$ and $0 \le \theta \le 180^0$.



(ii) The support of the vector $\bar{c}$ is perpendicular to that of $\bar{a}$ as well as of $\bar{b}$.

(iii) The sense of the vector $\bar{c}$ is such that $\bar{a}$, $\bar{b}$, $\bar{c}$ is $\bar{a}$ right handed system.

Thus the vector product of two vectors $'\bar{a}'$ and $'\bar{b}'$ whose directions are inclined at an angle θ is defined as $\bar{a} \times \bar{b} = ab \sin\theta \, \hat{n}$

Where $a = |\bar{a}|$, $b = |\bar{b}|$ and $\hat{n}$ is a unit vector perpendicular to both $'\bar{a}'$ and $'\bar{b}'$ and the sense of $\hat{n}$ is such that $\bar{a}$, $\bar{b}$ $\hat{n}$ from a right - handed triad of vectors.

Also the modulus $ab \sin\theta$ of $\bar{a} \times \bar{b}$ is the area of parallelogram whose adjacent sides are $\bar{a}$ and $\bar{b}$.

11.8 Properties of Vector Product:

11.8.1 The vector product is not commutative. In fact

$$\bar{a} \times \bar{b} = -\left(\bar{b} \times \bar{a}\right)$$

This follows fron the fact that the magnit vole and support of $b \times a$ are the same as those of $a \times b$ but the sense are different.

11.8.2 $-\bar{a} \times \bar{b} = -(\bar{a} \times \bar{b})$, $\bar{a} \times (-\bar{b}) = -(\bar{a} \times \bar{b})$

$$(-\bar{a}) \times (-\bar{b}) = \bar{a} \times \bar{b}$$

Generally $m\bar{a} \times n\bar{b} = mn(\bar{a} \times \bar{b}) = \bar{a} \times mn\bar{b}$

Where m and n are any scalars, positive or negative.

11.8.3 The vector product of two parallel or equal vectors is the zero vector, for this case $\theta = 0$ or 180^0 so that $\sin\theta = 0$ and as such $\bar{a} \times \bar{b} = 0$.

From here it also follows that $\bar{a} \times \bar{a} = 0$

Conversely, If $\bar{a} \times \bar{b} = 0$, $ab \sin\theta\, \hat{n} = 0$ then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or $\sin\theta = 0$ i.e., either of the vectors is a zero or null vector and in case neither of the vectors is a zero vector, then $\sin\theta$ being zero shows that they are parallel.

11.8.4 In case the vectors are perpendicular i.e., $\theta = 90^0$ then $\sin\theta = 1$ so that $\bar{a} \times \bar{b} = ab \cdot \hat{n}$

Thus the cross product of two perepndicular vectors is a vector whose modulus is equal to the product of the modulus of the given vectors and whose direction is such that $\bar{a}$, $\bar{b}$ and $\hat{n}$ from a right handed system of mutually perpendicular vectors.

11.8.5 Vector product of unit vector i, j, k.

We have $i \times i = j \times j = k \times k = 0$

$$i \times j = k = -j \times i$$

$$j \times k = i = -k \times j$$

$$k \times i = j = -i \times k$$

11.8.6 To express the vector product as determinant.

Let a and b be the two vectors. Let us express them in terms of orthonormal unit vector i, j, k i.e.,

$$\bar{a} = a_1 i + a_2 j + a_3 k \quad \text{and} \quad \bar{b} = b_1 i + b_2 j + b_3 k$$

$$\bar{a} \times \bar{b} = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$$

$$= \{a_1 b_1 (i \times j) + a_1 b_2 (i \times j) + a_1 b_3 (i \times k)\}$$

$$+ \{a_2 b_1 (j \times i) + a_2 b_2 (j \times j) + a_2 b_3 (j \times k)\}$$

$$+ \left\{a_3b_1 \; (k \times i) + a_3b_2 \; (k \times j) + a_3b_3 \; (\; k \times k)\right\}$$

$$\bar{a} \times \bar{b} \; = \left(a_2b_3 - a_3b_2\right) \, i \; + \left(a_3b_1 - a_1b_3\right) j \, + \left(a_1b_2 \; - a_2b_1\right) k$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

11.8.7 $\sin \theta = \dfrac{|\bar{a} \times \bar{b}|}{|\bar{a}| \cdot |\bar{b}|}$

$$= \frac{\sqrt{\left(a_2b_3 \; - a_3b_2\right)^2 + \left(a_3b_1 - a_1b_3\right)^2 + \left(a_1b_2 \; - a_2b_1\right)^2}}{\sqrt{\left(a_1^2 + a_2^2 + a_3^2\right) \; \left(b_1^2 + b_2^2 + b_3^2\right)}}$$

11.8.8 The magnitude of $a \times b$ can be expressed in terms of scalar products i.e.,

$$\left(\bar{a} \times \bar{b}\right)^2 = \bar{a}^2 \cdot \bar{b}^2 - \left(\bar{a} \cdot \bar{b}\right)^2$$

Proof: $\left(\bar{a} \times \bar{b}\right) \cdot \left(\bar{a} \times \bar{b}\right) = \left|\bar{a} \times \bar{b}\right|^2$

$$= \left|\bar{a}\right|^2 \; \left|\bar{b}\right|^2 \; \sin^2 \theta$$

$$= \left|\bar{a}\right|^2 \; \left|\bar{b}\right|^2 \; \left(1 - \cos^2 \theta\right)$$

$$= \bar{a}^2 \cdot \bar{b}^2 \; - \left|\bar{a}\right|^2 \; \left|\bar{b}\right|^2 \; \cos^2 \theta$$

$$= \bar{a}^2 \cdot \bar{b}^2 \; - \left(\bar{a} \cdot \bar{b}\right)^2$$

If a, b, c be three vectors then, $\bar{a} \times \left(\bar{b} + \bar{c}\right) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

11.9 Geometrical Interpretation of Vector Product:

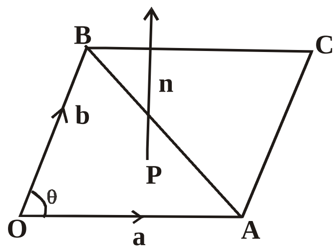
11.9.1 **Vector Area:** Definition - The vector area of a plane closed curve C is a vector of magnitude equal to the area enclosed by the curve and of direction normal to the plane of the curve in the sense of right handed rotation. By right handed rotation we mean that the direction of the normal vector will be the same as the direction of the motion of translation of a right handed screw when rotated in the sense in which the boundary of the area has been described.

11.9.2 **Theorem:** The magnitude of the vector product of two vectors is the area of a parallelogram whose adjacent sides are represented by these vectors.

Let OACB be a paralleolgram

Let $\overrightarrow{OA} = \bar{a}$, $\overrightarrow{OB} = \bar{b}$ and $\angle BOA = \theta$

Then $\bar{a} \times \bar{b} = |\bar{a}|\ |\bar{b}| \sin\theta\ n$ where 'n' is a unit vector normal to the plane of the parallelogram pointing in the direction in which a right handed screw would move if rotated in the sense OACB.



Now $|\bar{a} \times \bar{b}| = |\bar{a}|\ |\bar{b}|\ \sin\theta = (OA)\ (OB) \sin\theta = OA\ (OB \sin\ \theta)$

$=$ area of the parallelogram OACB

Thus $\bar{a} \times \bar{b}$ is a vector whose modulus is equal to the area of the parallelogram OACB and whose direction is normal to the plane of the parallelogram OACB in the sense of right handed rotation. Hence $\bar{a} \times \bar{b}$ represents the vector area of the parallelogram OACB, whose boundary is described in this sense.

Note 1: The vector area of the parallelogram OBCA will be represented by $\bar{b} \times \bar{a}$

Note 2: The area of triangle OAB, two of whose adjacent sides are represented by the vectors $\bar{a}$ and $\bar{b}$ is $\frac{1}{2}\ |\bar{a} \times \bar{b}|$. The vector area of triangle OAB is $\frac{1}{2}\ \bar{a} \times \bar{b}$.

If we are concerned with plane areas and if we take care to see that their boundaries are described in the same sense, then their sum and difference are respectively given by the magnitudes of the sum and difference of the corresponding vector areas.

11.9.3 **S.A.Q:** If $\bar{a} = 3i \cdot j + 2k,\ \bar{b} = 2i + j - k\ ,\ \bar{c} = i - 2j + 2k$ find $(\bar{a} \times \bar{b}) \times \bar{c}$ and $\bar{a} \times (\bar{b} \times \bar{c})$ and hence show that $(\bar{a} \times \bar{b}) \times \bar{c} \neq \bar{a} \times (\bar{b} \times \bar{c})$

11.10 Solved Examples of Cross Product:

11.10.1 Example: If $\bar{a} = 2i - j + k$ and $\bar{b} = 3i + 4j - k$ prove that $\bar{a} \times \bar{b}$ represents a vector which is perpendicular to both $\bar{a}$ and $\bar{b}$.

Sol: $$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3i + 5j + 11k$$

$$\therefore \left(\bar{a} \times \bar{b}\right) \cdot \bar{a} = \left(-3i + 5j + 11k\right) \cdot \left(2i - j + k\right)$$

$$= -6 - 5 + 11$$

$$= 0$$

Here $\left(\bar{a} \times \bar{b}\right)$ is perpendicular to $\bar{a}$ similalry we can prove that $\left(\bar{a} \times \bar{b}\right)$ is [er[endicular to $\bar{b}$.

11.10.2 Two vectors $\vec{a}$ and $\vec{b}$ are expressed interms of unit vect as follows:

$$\vec{a} = \overrightarrow{2i} - \overrightarrow{6j} - \overrightarrow{3k} \quad \text{and} \quad \vec{b} = \overrightarrow{4i} + \overrightarrow{3j} - \vec{k}$$

What is the unit vector perpendicular to each of the vectors. Also determine the sine of the angle between the gien vectors.

Sol: $$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15i - 10j + 30k$$

$$\left|\bar{a} \times \bar{b}\right| = \sqrt{15^2 + 10^2 + 30^2} = \sqrt{1225} = 35$$

$\therefore$ Unit vector perpendicular to each of the vectors a and b

$$= \frac{\bar{a} \times \bar{b}}{\left|\bar{a} \times \bar{b}\right|} = \frac{15i - 10j + 30k}{35}$$

$$= \frac{3}{7}i - \frac{2}{7}j + \frac{6}{7}k$$

$$\sin\theta = \frac{|\bar{a}\times\bar{b}|}{|\bar{a}|\,|\bar{b}|} = \frac{35}{\sqrt{2^2 +^2 + 3^2}\ \sqrt{4^2 + 3^2 + 1}}$$

$$= \frac{35}{\sqrt{49}\ \sqrt{26}} = \frac{5}{\sqrt{26}} = \sqrt{\frac{25}{26}}$$

11.10.3 Example: Prove that $\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = 0$

Sol: L.H.S. $= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$

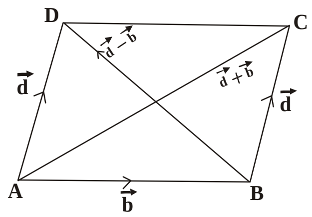
$$= (\vec{a}\times\vec{c}) - (\vec{c}\times\vec{a}) + (\vec{b}\times\vec{c}) - (\vec{a}\times\vec{b}) + (\vec{c}\times\vec{a}) - (\vec{b}\times\vec{c})$$

$$= \bar{0} = R\cdot H\cdot S\cdot$$

11.10.4 Example: Show that the diagonals of a rhombus are at right angles.

Sol: With A as origin Let $\vec{b}, \vec{d}$ be the position vectors of B, D then

$\vec{b}+\vec{d}$ is the position vector of c.



Now $AC = \vec{b}+\vec{d}$

$\overrightarrow{BD} = \vec{d}-\vec{b}$

$\therefore\ \overrightarrow{AC}\cdot\overrightarrow{BD} = (\vec{b}+\vec{d})\cdot(\vec{d}-\vec{b}) = d^2 - b^2 = 0$

$[\because\ AB = AD\ \text{i·e·},\ b = d]$

Since the scalar product of $\overrightarrow{AC}$ and $\overrightarrow{BD}$ iszero it follows that AC and BD are at right angles.

11.10.5 Example: D is the mid point of the side BC of a triangle ABC show that

$$AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$$

Sol: With A as origin Let $\vec{b}, \vec{c}$ be the position vectors of B and C

so that the postion vector of B is $\frac{1}{2}(\vec{b}+\vec{c})$.

Now $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

$$= \left(\vec{b} + \vec{c}\right) - \vec{b} = \frac{1}{2}\left(\vec{c} - \vec{b}\right)$$

Again $AB^2 + AC^2 = b^2 + c^2$ $\qquad (\because AB = b, \; AC = c)$

Thus $AD^2 + BD^2 + \frac{1}{4}\left(\vec{b} + \vec{c}\right)^2 = \frac{1}{4}\left(\vec{c} - \vec{b}\right)^2$

$$= \frac{1}{4}\left[\left(b^2 + c^2\right) + 2\left(\vec{b} \cdot \vec{c}\right) + c^2 + b^2 - 2\left(\vec{b} \cdot \vec{c}\right)\right]$$

$$= \frac{1}{2}\left(b^2 + c^2\right)$$

$$\therefore 2\left(AD^2 + BD^2\right) = AB^2 + AC^2$$

11.10.6 Example: In any triangle ABC, show that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Sol: Let $\vec{a}, \vec{b}, \vec{c}$ represent the sides of the Δ ABC

$$\therefore \vec{a} + \vec{b} + \vec{c} = 0$$

$$\therefore \vec{a} \times \left(\vec{a} + \vec{b} + \vec{c}\right) = 0$$

i.e., $\left(\vec{a} \times \vec{a}\right) + \left(\vec{a} \times \vec{b}\right) + \left(\vec{a} \times \vec{c}\right) = 0$

But $\left(\vec{a} \times \vec{a}\right) = 0$

$\therefore \left(\vec{a} \times \vec{b}\right) = -\left(\vec{a} \times \vec{c}\right) = \left(\vec{c} + \vec{a}\right)$

$\therefore \left(\vec{a} \times \vec{b}\right) = \left|\vec{c} \times \vec{a}\right|$

Similarly $\left|\vec{c} \times \vec{a}\right| = \left|\vec{b} \times \vec{c}\right|$

Now $\left|\vec{a} \times \vec{b}\right| = \left[ab \sin\left(\pi - c\right)\right] = ab \sin c$ etc.

Now $ab \sin C = ca \sin B = bc \sin A$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

11.10.7 Example: What is the unit vector perpendicular to each of the vectors $2i - j + k$ and $3i + 4j - k$? Calculate the sine of the angle between these vectors.

Sol: Let $a = 2i - j + k$ and $b = 3i + 4j - k$

We know that $a \times b$ is a vector perpendicular to both 'a' and 'b'.

$$\text{Now } a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -3i + 5j + 11k$$

$$\therefore |a \times b| = |-3i + 5j + 11k| = \sqrt{\left(3^2 + 5^2 + 11^2\right)} = \sqrt{155}$$

Now unit vector perpendicular to the plane of 'a' and 'b'

$$= \frac{a \times b}{|a \times b|} = \frac{-3i + 5j + 11k}{\sqrt{155}}$$

Further if θ be the angle between the vectors $\bar{a}$ and $\bar{b}$, we have $\bar{a} \times \bar{b} = |\bar{a}|\ |\bar{b}| \sin \theta n$, where 'n' is a unit vector.

$$\therefore \left(\bar{a} \times \bar{b}\right)^2 = \left[\,|\bar{a}|\ |\bar{b}| \sin \theta\,\right]^2 n^2 = |\bar{a}|^2\ |\bar{b}|^2 \sin^2 \theta \qquad \left[\because\ n^2 = 1\right]$$

$$\therefore \sin^2 \theta = \frac{\left(\bar{a} \times \bar{b}\right)^2}{|\bar{a}|^2\ |\bar{b}|^2} = \frac{(-3i + 5j + 11k)^2}{\left(2^2 + 1^2 + 1^2\right)\left(3^2 + 4^2 + 1^2\right)}$$

$$= \frac{9 + 25 + 121}{(6)\ (26)} = \frac{155}{156}$$

$$\therefore \sin \theta = \sqrt{\frac{155}{156}}$$

11.10.8 Example: Find the cosine and sine of the angle between the two vectors

$a = 3i + j + 2k,\ b = 2i - 2j + 4k \cdot$

Sol: Let θ be the angle between the vectors 'a' and 'b'

We have $a \cdot b = |a|\ |b| \cos\theta$

$$\therefore \cos\theta = \frac{a \cdot b}{|a||b|} = \frac{a \cdot b}{\sqrt{a^2}\ \sqrt{+b^2}}$$

Now $a \cdot b = (3i + j + 2k) \cdot (2i - 2j + 4k) = 6 - 2 + 8 = 12$

$a^2 = 9 + 1 + 4 = 14$ and $b^2 = 4 + 4 + 16 = 24$

$$\therefore \cos\theta = \frac{12}{\sqrt{14}\ \sqrt{24}} = \frac{12}{4\sqrt{7}\sqrt{3}} = \sqrt{\frac{3}{7}}$$

Now $a \times b$ is a vector perpendicular to the plane of 'a' and 'b'.

We have $a \times b = (3i + j + 2k) \times (2i - 2j + 4k)$

$$= \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 3i - 8j - 8k$$

We have $a \times b = |a||b| \sin\theta\, n = \sqrt{a^2}\ \sqrt{b^2}\ \sin\theta\, n$

$$\therefore (a \times b)^2 = (a^2)(b^2)\sin^2\theta\, n^2 = (a^2)(b^2)\sin^2\theta \qquad [\because n^2 = 1]$$

Now $(a \times b)^2 = (8i - 8j - 8k)^2 = 64 + 64 + 64 = 64(3) = 192$

$a^2 = (3i + j + 2k)^2 = 9 + 1 + 4 = 14$ and

$b^2 = (2i - 2j + 4k)^2 = 4 + 4 + 16 = 24$

$$\therefore \sin^2\theta = \frac{192}{14 \times 24} = \frac{192}{336} \qquad \therefore \sin\theta = \sqrt{\frac{192}{336}} = \frac{2}{\sqrt{7}}$$

11.10.9 Example: Solve the simultaneous equations $r \times b = a \times b$ and $r \cdot c = 0$, where 'b' and 'c' are not orthogonal to each other.

Sol: We have $r \times b = a \times b \Rightarrow r \times b - a \times b = 0$

$\Rightarrow (r - a) \times b = 0 \Rightarrow r - a$ is parallel to 'b'

$\Rightarrow r - a = \lambda b$ where λ is some scalar.

$\therefore r \times b = a \times b \Rightarrow r = a + \lambda b$

Putting $r = a + \lambda b$ in the second equation $r \cdot c = 0$ we get

$(a + \lambda b) \cdot c = 0$ or $a \cdot c + \lambda (b \cdot c) = 0$

or $\lambda = -\frac{a \cdot c}{b \cdot c}$ [Note that 'b' and 'c' are not orthogonal implies that $b \cdot c \neq 0$]

$$\therefore r = a - \frac{a \cdot c}{b \cdot c} b$$

11.10.10 Example: Find the area of the parallelogram determined by the vectors

$\bar{i} = 2\bar{j} + 3\bar{k}$ and $-3\bar{i} - 2\bar{j} + \bar{k}$

Sol: Let $\bar{a} = \bar{i} = 2\bar{j} + 3\bar{k}$ and $\bar{b} = -3\bar{i} - 2\bar{j} + \bar{k}$

The vector area of the parallelogram whose adjacent sides are represented by vectors $\bar{a}$ and $\bar{b} = \bar{a} \times \bar{b}$ ·

Now $\bar{a} \times \bar{b} = (\bar{i} = 2\bar{j} + 3\bar{k}) \times (-3\bar{i} - 2\bar{j} + \bar{k})$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= 8i - 10j + 4k$$

$\therefore$ The area of the parallelogram

$$= |\bar{a} \times \bar{b}| = |8\bar{i} - 10\bar{j} + 4\bar{k}|$$

$$= \sqrt{(8^2 + 10^2 + 4^2)} = \sqrt{(64 + 100 + 16)} = \sqrt{180} \text{ square units} \cdot$$

11.10.11 Example: Show that the area of the triangle whose adjacent sides are determined by the vectors $\bar{a} = 3\bar{i} + 4\bar{j}$ and $\bar{b} = -5\bar{i} + 7\bar{j}$ is $20\frac{1}{2}$ square units.

Sol: The vector area of the triangle whose adjacenet sides are represented by the vectors $\bar{a}$ and $\bar{b}$ is

$$= \frac{1}{2}\,\bar{a} \times \bar{b}$$

Now $\bar{a} \times \bar{b} = \left(3\bar{i} + 4\bar{j} + 0\bar{k}\right) \times \left(-5\bar{i} + 7\bar{j} + 0\bar{k}\right)$

$$= \begin{vmatrix} i & j & k \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix}$$

$$= 41k$$

$\therefore$ The area of the triangle $= \frac{1}{2}\left|\bar{a} \times \bar{b}\right| = \frac{1}{2}\left|41\,\bar{k}\right|$

$$= 41/2 \text{ sq} \cdot \text{units} \cdot$$

11.10.12 Example: Show that $\frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{BD}$ represents the vector area of the plane quadrilateral ABCD.

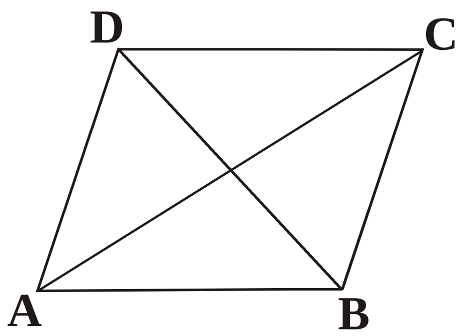
Sol: Vector area of the quadrilateral ABCD = vector area of Δ ABC $\times$ vector area of ΔACD.

$$= \frac{1}{2}\,\overrightarrow{AB} \times \overrightarrow{AC} + \frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{AD}$$

$$= -\frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{AB} + \frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{AD}$$

$$= \frac{1}{2}\,\overrightarrow{AC} \times \left(\overrightarrow{AD} - \overrightarrow{AB}\right)$$

$$= \frac{1}{2}\,\overrightarrow{AC} \times \overrightarrow{BD}$$



11.10.13 Example: ABCD is a quadrilateral such that

$$\overrightarrow{AB} = \bar{b},\ \overrightarrow{AD} = \bar{d} \text{ and } \overrightarrow{AC} = m\bar{b} + n\bar{d}$$

Show that the area of the quadrilateral ABCD is

$$\frac{1}{2}\left|(m+n)\right|\ \left|\bar{b} \times \bar{d}\right|$$

Sol: For fig. refer exercise 7.

Vector area of the quadrilateral ABCD = Vector area of ΔABC + Vector area of ΔACD

$$= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD} = \frac{1}{2}\bar{b}\times\left(m\bar{b}+n\bar{d}\right) + \frac{1}{2}\left(m\bar{b}+n\bar{d}\right)\times\bar{d}$$

$$= \frac{1}{2}m\bar{b}\times\bar{b} + \frac{1}{2}n\,\bar{b}\times\bar{d} + \frac{1}{2}m\bar{b}\times\bar{d} + \frac{1}{2}n\bar{d}\times\bar{d}$$

$$= \frac{1}{2}(m+n)\left(\bar{b}\times\bar{d}\right) \qquad \left[\because\ \bar{b}\times\bar{b}=0 \text{ and } \bar{d}\times\bar{d}=0\right]$$

$\therefore$ Area of quadrilateral ABCD $= \left|\frac{1}{2}(m+n)\ \bar{b}\times\bar{d}\right|$

$$= \frac{1}{2}\,|m+n|\,\left|\bar{b}\times\bar{d}\right|$$

11.10.14 Example: Show that the area of a parallelogram having diagonals $3i + j - 2k$ and $i - 3j + 4k$ is $5\sqrt{3}$.

Sol: Let ABCD be a parallelogram such that

$$\overrightarrow{AC} = 3i + j - 2k \text{ and } \overrightarrow{BD} = i - 3j + 4k$$

Vector area of the parallelogram ABCD = Vector area of Δ ABC + Vector area of ΔACD

$$= \frac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC} + \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD} = -\frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{AD}$$

$$= \frac{1}{2}\overrightarrow{AC}\times\left(\overrightarrow{AD} - \overrightarrow{AB}\right) = \frac{1}{2}\overrightarrow{AC}\times\overrightarrow{BD}$$

$$= \frac{1}{2}(3i + j - 2k)\times(i - 3j + 4k)$$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= \frac{1}{2}(-2i - 14j - 10k) = -i - 7j - 5k$$

$\therefore$ area of the parallelogram ABCD $= |-i - 7j - 5k|$

$$= \sqrt{(1 + 49 + 25)} = \sqrt{75} = 5\sqrt{3}$$

11.11 Answers to S.A.Q.:

11.11.1 S.A.Q. of 11.9.3:

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = -i + 7j + 5k$$

$$\therefore (\bar{a} \times \bar{b}) \times \bar{c} = (-i + 7j + 5k) \times (i - 2j + 2k)$$

$$= \begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 24i + 7j - 5k \cdots\cdots\cdots\cdots(1)$$

Similarly we can prove that

$$\bar{b} \times \bar{c} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 2 & -2 & 2 \end{vmatrix} = -5j - 5k$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = (3i - j + 2k) \times (-5j - 5k)$$

$$= \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= 15i + 15j - 15k \cdots\cdots\cdots\cdots\cdots(2)$$

from (1) & (2) we carclude that

$$(\bar{a} \times \bar{b}) \times \bar{c} \neq \bar{a} \times (\bar{b} \times \bar{c})$$

11.12 Summar:

In this lesson we discussed the operations of multiplication of a vector by another vector. We also discussed the dot and cross product of two vectors and their properties some problems also discussed.

11.13 Technical Terms:

Scalar or dot Product

Cross Product

Vector Area

11.14 Exercise:

1) Define dot product of two vectors $\bar{a}$ and $\bar{b}$ and give its geometrical interpretation.

2) If $\bar{a}$ and $\bar{b}$ are any two vectors, then what do you conclude if $\bar{a} \cdot \bar{b} = 0$

3) If $\bar{a} = (2, 3, -1)$, $\bar{b} = (0, 4, 2)$ find $\bar{a} \cdot \bar{b}$

4) If $\bar{a} = i - j$, $\bar{b} = -j + 2$ show that $(\bar{a} + \bar{b}) \cdot (\bar{a} - 2\bar{b}) = -9$

5) Find the direction cosines of the vector $\bar{a} = 5i - j + k$

6) If $\bar{a} = 2i + xj - k$ and $\bar{b} = 4i - 2j + 2k$ find x such that $\bar{a}$ and $\bar{b}$ are othogonal.

7) If $\bar{a} = 2i + j + 2k$ and $\bar{b} = 5i - 3j + k$ find the projection of $\bar{b}$ on the direction of $\bar{a}$.

8) Define cross product of two vectors $\bar{a}$ and $\bar{b}$ and give its geometrical interpretation.

9) Show that if $\bar{a}$, $\bar{b}$ are non - sero vectors, such that $\bar{a} \times \bar{b} = \bar{0}$ then $\bar{a}$ and $\bar{b}$ are parallel vectors.

10) Given that $\bar{a} = 2i + 3j + 6k$, $\bar{b} = 3i - 6j + 2k$, $\bar{c} = 6i + 2j - 3k$ show that $\bar{a} \times \bar{b} = 7\bar{c}$.

11) Find the unit vector perpendicular to each of the vectors $4i - 2j + 3k$ and $5i + j - 4k$ ·

12) Show that the sine of the angle between the vectors $i + 3j + 2k$ and $2i - 4j + 4k$ is $\sqrt{\frac{115}{147}}$

13) Prove that $|\bar{a} \times \bar{b}|^2 + (\bar{a} \cdot \bar{b})^2 = \overline{a^2}\ \overline{b^2}$

14) If $\bar{a} + \bar{b} + \bar{c} = \bar{0}$, Show that $\bar{b} \times \bar{c} = \bar{c} \times \bar{a} = \bar{a} \times \bar{b}$.

15) Find the area of the parallelogram determined by the vectors $i + 2j + 3k$ and $3i - 2j + k$ ·

16) Find the area of the triangle whose vertices are P, Q, R given by

$P = (1, 2, 3),\ Q = (2, -1, 1),\ R = (1, 2, -4)\cdot$

11.15 Answers to Exercise:

2) $\bar{a} = 0$ or $\bar{b} = 0$ or $\bar{a} + \bar{b}\cdot$ If $\bar{a} \neq 0,\ \bar{b} \neq 0\cdot$ Then $\bar{a} + \bar{b}\cdot$

3) 10

5) $\frac{5}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}$

6) $x = 3$

7) 3

11) $\frac{1}{\sqrt{1182}}(5i + 31j + 14k)$

15) $8\sqrt{3}$ square units

16) $\frac{1}{2}\sqrt{490}$

11.16 Model Questions:

1) If $\bar{a}, \bar{b}, \bar{c}$ are non - coplanar vectors show that $\bar{b} + \bar{c},\ \bar{c} + \bar{a},\ \bar{a} + \bar{b}$ are also non coplanar.

2) Show that the diagonals of rhombus are at right angles.

3) Find the angle between the vectors $\bar{a} - i + 2j + 2k,\ \bar{b} = i - 2j + 2k\cdot$

11.17 References:

1) D.C. Sancheti, V.K. Kapoor, " Business Mathematics".
Sultan Chand and Sons, New Delho - 110002.

2) A.R. Vasistha, "VEctor Algebra", Krishna Prakasan Mandir, Meerut.

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Lesson - 12

CALCULAS - DIFFERENTIATION - I

OBJECTIVES:

After studying this chapter, you will be able to understand.

- the derivative and write the derivatives of standard functions.
- differentiate functions using standard derivatives and rules of differentiation.
- higher order derivatives of function.
- derivatives as a rate measure.

STRUCTURE:

12.1 Introduction

12.2 Differentiation - Derivatives

12.3 Derivative of composite function

12.4 Exercises.

12.1 INTRODUCTION:

The word calculus stands for the method of computation. There may be an arithmetic calculus or a probability calculus. The most common use of calculus is in regard to the computation of the rate of change in one variable with reference to an infinitesimal variation in the other variable. For example we know that given the speed, the distance covered is a function of time or given the distance, the time taken is a function of speed. Then there is a dependent variable which gets an impulse of change by a change in the independent variable. Calculus gives us the technique for measuring these changes in the dependent variable with reference to a very small change, approaching almost zero, in the independent variable or variables. The techniques concerning the calculation of the average rate of change are studied under differentiation or the Differential calculus and the calculation of the total amount of change in the given range of values is studied under integration or integral calculus.

The usefulness of both these is very great in business. Given certain functional relations, we can find out the average rate of change in the dependent variable with reference to a change in one or more independent variables. For example with a given demand function, it would be possible to find the degree of change in demand with reference to a small change in price or incene or both as the case may be and also the maximum and minimum values of the function.

12.2 DIFFERENTIATION - DERIVATIVES :

1. Differentiation :

To express the rate of change in any function we have the concept of derivative which involves infinitesimally small changes in the dependent variable with reference to a small change in independent variable.

Differentiation we can say is the process of finding out the derivative of a continuous function. A derivative is the limit of the ratio of the increment in the function corresponding to a small increment in the argument as the latter tends to zero.

Let y = f(x) be a continuous function. The rate of change in y is δy when the rate of change in x is δx.

$$\therefore \delta y = f(x+\delta x)$$

$$\therefore y+\delta y = f(x+\delta x)$$

$$\delta y = f(x+\delta x) - y$$

$$\delta y = f(x+\delta x) - f(x)$$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\overset{Lt}{\delta x \to 0} \frac{\delta y}{\delta x} = \overset{Lt}{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$\overset{Lt}{\delta x \to 0} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \overset{Lt}{h \to 0} \frac{f(x+h) - f(x)}{h}$ $f^1(x)$ when h = δx

$\frac{dy}{dx}$ means the rate of change in y w.r.t 'x'

If y = f(x) be a function then $\frac{dy}{dx} = \overset{Lt}{h \to 0} \frac{f(x+h) - f(x)}{h}$

Derivative of x^n :

$$\because f(x) = x^n$$

$$\because f^1(x) = \overset{Lt}{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{(x+h)^n - x^n}{h}$$

$$= \underset{h \to 0}{Lt} \frac{(x^n + {}^nc_1 x^{n-1}.h + {}^nc_2 x^{n-2}.h^2 + ... + h^n) - x^n}{h}$$

$$= \underset{h \to 0}{Lt} ({}^nc_1 x^{n-1} + {}^nc_2 x^{n-2} h + ... + h^{n-1})$$

$= n\ x^{n-1}$

$\frac{d}{dx}$ **(x^n) = n. x^{n+1}**

Derivative of e^x :

$$let\ f(x) = e^n$$

$$\therefore f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{e^{x+h} - e^x}{h}$$

$$= \underset{h \to 0}{Lt} e^x \frac{(e^h - 1)}{h}$$

= e^x **(1)**

= e^x $\left(\because \underset{h \to 0}{Lt} \frac{(e^h - 1)}{h} = 1 \right)$

$\frac{d}{dx}$ **(e^x) = e^x**

Derivative of const 'c' :

$$Let\ f(x) = c$$

$$\therefore f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{(c) - c}{h}$$

$= 0$

$\frac{d}{dx}$ **(c) = 0**

Derivative of $\sqrt{x}$:

$$let\ f(x)=\sqrt{x}$$

$$\because\ f^1(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\underset{h\to 0}{Lt}\ \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\ \frac{x}{\sqrt{x+o}+\sqrt{x}}$$

$$=\ \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\ (\sqrt{x})=\ \frac{1}{2\sqrt{x}}$$

Derivative of log x :

$$let\ f(x)=\log x$$

$$\because\ f^1(x)=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{\log\ (x+h)-\log x}{h}$$

$$=\underset{h\to 0}{Lt}\ \frac{1}{h}\log\ \left(\frac{x+h}{x}\right)$$

$$=\underset{h\to 0}{Lt}\ \log\ \left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$=\frac{1}{x}.\ \underset{h\to 0}{Lt}\ \log\ \left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} \log \lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} \log e$$ log x means $\log_e x$

$$= \frac{1}{x} (1)$$

$$= \frac{1}{x}$$

$$\frac{d}{dx} (\log x) = \frac{1}{x}$$

Derivative of Sin x:

$$let\ f(x) = Sinx$$

$$\therefore f^1(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin (x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2 \text{Cos} \left(\frac{(x+h)+x}{2}\right) \text{Sin}\left(\frac{(x+h)-x}{2}\right)}{h}$$

$$= \lim_{h \to 0} 2 \text{Cos} \left(x + \frac{h}{2}\right) \frac{\text{Sin} \frac{h}{2}}{\frac{h}{2}.2}$$

= Cos (x + 0) . 1

= Cos x

$$\frac{d}{dx}$$ **(Sin x)** = Cos x

Derivative of UV :

$$let\ f(x) = u(x).\ v(x)$$

$$\therefore\ f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{u\ (x+h)\ v\ (x+h) - u\ (x).\ v\ (x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{u\ (x+h)\ v\ (x+h) - u\ (x+h)\ u\ (x) + v\ (x+h)\ v\ (x) - u\ (x).\ v\ (x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{u\ (x+h)\ \left[v\ (x+h) - v\ (x)\right] + v\ (x)\left[u\ (x+h) - u\ (x)\right]}{h}$$

$$= \underset{h \to 0}{Lt}\ u\ (x+h).\ \underset{h \to 0}{Lt} \frac{v\ (x+h) - v\ (x)}{h} \underset{h \to 0}{Lt}\ v\ (x).\ \underset{h \to 0}{Lt} \frac{u\ (x+h) - u(x)}{h}$$

$$= u\ (x).v^1(x) - v(x)\ u^1(x)$$

$$\therefore \frac{d}{dx} \textbf{(uv)} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Derivative of $\frac{u}{v}$:

$$\text{let } f(x) = \frac{u(x)}{v(x)}$$

$$\therefore\ f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\frac{u\ (x+h)}{v\ (x+h)} - \frac{u\ (x)}{v\ (x)}}{h}$$

$$= \underset{h \to 0}{Lt} \frac{u\ (x+h)\ v\ (x) - v\ (x+h).\ u\ (x)}{h\ v\ (x+h).\ v\ (x)}$$

$$= \underset{h \to 0}{Lt} \frac{u\ (x+h)\ v\ (x) - u\ (x)\ v\ (x) + u\ (x)\ v\ (x) - v\ (x+h).\ u\ (x)}{h\ v\ (x+h).\ v\ (x)}$$

$$= \underset{h \to 0}{Lt} \frac{v\ (x)\ \left[u\ (x+h) - u\ (x)\right] - u\ (x)\left[v\ (x+h) - v\ (x)\right]}{h\ v(x+h)\ v\ (x)}$$

$$= \left[v(x).\underset{h\to 0}{Lt} \left(\frac{u(x+h) - u(x)}{h} \right) - u(x).\left(\underset{h\to 0}{Lt} \frac{v(x+h) - v(x)}{h} \right) \right].$$

$$\underset{h\to 0}{Lt} \frac{1}{v(x+h) \,.\, v(x)}$$

$$= \frac{v(x).\, u^1(x) - u(x)\, v^1(x)}{[v(x)]^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Formula :

1. $\frac{d}{dx}(x^n) = n.\, x^{n-1}$

2. $\frac{d}{dx}(e^x) = e^x$

3. $\frac{d}{dx}(a^x) = a^x \cdot \log a$; $\frac{d}{dx}(ax) = a$

4. $\frac{d}{dx}(\log x) = \frac{1}{x}$

5. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

6. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$, $\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$

7. $\frac{d}{dx}(c) = 0$

8. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

9. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

10. $\frac{d}{dx}$ (uvw) = uv$\frac{dv}{dx}$w+ uw$\frac{du}{dx}$(v) + vw$\frac{du}{dx}$(u)

11. $\frac{d}{dx}$ (Sin x) = Cos x

12. $\frac{d}{dx}$ (Cosx) = -Sin x

13. $\frac{d}{dx}$ (tanx) = Sec^2 x

14. $\frac{d}{dx}$ (Cot x) = - $Cosec^2$ x

15. $\frac{d}{dx}$ (Sec x) = Sec x. tan x

16. $\frac{d}{dx}$ (Cosec x) = - Cosec x. Cot x

SOLVED EXAMPLES :

i) Find the derivative of x^5

let y = x^5

$\frac{dy}{dx} = \frac{d}{dx}(x^5)$

= 5 x^{5-1}

=5x^u

ii) Find the derivative of $x^{2/3} + 2x^3 + 7x + 5$

let y = $x^{2/3} + 2x^3 + 7x + 5$

$\frac{dy}{dx} = \frac{d}{dx}(\ x^{2/3} + 2x^3 + 7x + 5\ )$

$= \frac{d}{dx}(\ x^{2/3}) + 2\frac{d}{dx}(\ x^3) + \frac{d}{dx}(7x) + \frac{d}{dx}(\ 5\ )$

$= {}^2/_3\ x^{2/3\ -1} + 2.3\ x^{3-1} + 7 + 0$

$= {}^2/_3\ x^{-1/3} + 6x^2 + 7$

iii) Find the derivative of $\frac{1}{x^2}$

let $y = \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= \frac{-2}{x^{2+1}}$$

$$= \frac{-2}{x^3}$$

iv) Find the derivative of $x^3 + \text{Sin } x + e^x$

let $y = x^3 + \text{Sin } x + e^x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + \text{Sin } x + e^x \right)$$

$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)$$

$$= 3x^2 + \text{Cos } x + e^x$$

v) Find the derivative of $\frac{x^3 + 2x + 7}{x^2}$

let $y = \frac{x^3 + 2x + 7}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3 + 2x + 7}{x^2} \right)$$

$$= \frac{d}{dx}\left(x + \frac{2}{x} + \frac{7}{x^2}\right)$$

$$= \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) + 7\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$= 1+2\left(\frac{-1}{x^2}\right)+\left(\frac{-2}{x^{2+1}}\right)$

$1-\frac{2}{x^2}-\frac{2}{x^3}$

vi) Find the derivative of $\left(x-\frac{1}{x}\right)^3$

Let y = $\left(x-\frac{1}{x}\right)^3$

$\frac{dy}{dx}=\frac{d}{dx}\left(x-\frac{1}{x}\right)^3$

$= \frac{d}{dx}\left(x^3-\frac{1}{x^3}-3.x^2.\frac{1}{x}+3x.\frac{1}{x^2}\right)$

$= \frac{d}{dx}\left(x^3-\frac{1}{x^3}-3x+\frac{3}{x}\right)$

$= \frac{d}{dx}(x^3)-\frac{d}{dx}\left(\frac{1}{x^3}\right) - 3\frac{d}{dx}(x) + 3\frac{d}{dx}\left(\frac{1}{x}\right)$

$= 3x^2-\left(\frac{-3}{x^4}\right) - 3(1) + 3\left(\frac{-1}{x^2}\right)$

$= 3x^2 - \frac{3}{x^4} - 3 - \frac{3}{x^2}$

vii) Find the derivative of x^2. Sinx

Let y = x^2. Sinx

$\frac{dy}{dx}=\frac{d}{dx}(x^2.\,\text{Sinx})$

$= x^2\frac{d}{dx}(\text{Sinx}) + \text{Sin x}\,\frac{d}{dx}(x^2)$

$= x^2 \text{Cos}\, x + \text{Sin}x .2x$

$= x^2 \text{Cos}\, x + 2x \text{ Sin } x$

vii) Find the derivative of $(x^2 + x + 1) \log x$

Let $y = (x^2 + x + 1) \log x$

$$\frac{dy}{dx} = \frac{d}{dx} [(x^2 + x + 1) \log x]$$

$$= (x^2 + x + 1) \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (x^2 + x + 1)$$

$$= (x^2 + x + 1) \left(\frac{1}{x}\right) + \log x (2x + 1 + 0)$$

$$= (x + 1 + \frac{1}{x}) + (2x + 1) \log x$$

viii) Find the derivative of $[(x^2 -3) (4x^3+1)]$

Let $y = (x^2 -3) (4x^3+1)$

$$\frac{dy}{dx} = \frac{d}{dx} [(x^3 -3) (4x^3+1)]$$

$$= (x^3 -3) \frac{d}{dx} (4x^3+1) + (4x^3+1) \frac{d}{dx} (x^3 -3)$$

$$= (x^3 -3) [4. 3x^2 + 0] + [4x^3 + 1] (3x^2 + 0)$$

ix) Find the derivative of $[(\sqrt{x} -3x) (x + \frac{1}{x})]$

Let $y = [(\sqrt{x} -3x) (x + \frac{1}{x})]$

$$\frac{dy}{dx} = \frac{d}{dx} [(\sqrt{x} -3x) (x + \frac{1}{x})]$$

$$= (\sqrt{x} -3x) \frac{d}{dx} (x + \frac{1}{x}) + (x + \frac{1}{x}) \frac{d}{dx} (\sqrt{x} -3x)$$

$= (\sqrt{x}$ $-3x)$ $(x + \frac{-1}{x^2}) + (x + \frac{1}{x}) [\frac{1}{2\sqrt{x}} - 3 (1)]$

$= (\sqrt{x}$ $-3x)$ $(x - \frac{1}{x^2}) + (x + \frac{1}{x}) [\frac{1}{2\sqrt{x}} - 3]$

x) Find the derivative of $e^x \log x - \sin x$

Let $y = e^x \log x - \sin x$

$\frac{dy}{dx} = \frac{d}{dx}$ $e^x \log x - \sin x$

$= e^x \frac{d}{dx} (\log x) + \log x \frac{d}{dx} e^x - \cos x$

$= e^x . \left(\frac{1}{x} \right) + \log x . e^x - \cos x$

$= \frac{e^x}{x} + e^x . \log x - \cos x$

xi) Find the derivative of $x e^x \sin x$

Let $y = x e^x \sin x$

$\frac{dy}{dx} = \frac{d}{dx} (x e^x \sin x)$

$= x e^x \frac{d}{dx} (\sin x) + x \sin x \frac{d}{dx} (e^x) + e^x \sin x \frac{d}{dx} (x)$

$= x e^x \cos x + x .e^x . \sin + e^x \sin x$

xii) Find the derivative of $\frac{x^2+2}{x^2-x-2}$ with respected to ' x '

Sol : $\frac{d}{dx} \left[\frac{x^2+2}{x^2-x-2} \right]$

$$= \frac{\left(x^2 - x - 2\right)\frac{d}{dx}\left(x^2 + 2\right) - \left(x^2 + 2\right)\frac{d}{dx}\left(x^2 - x - 2\right)}{\left(x^2 - x - 2\right)^2}$$

$$= \frac{\left(x^2 - x - 2\right)(2x) - \left(x^2 + 2\right)(2x - 1)}{\left(x^2 - x - 2\right)^2}$$

$$= \frac{2x^3 - 2x^2 - 4x - 2x^3 + x^2 - 4x + 2}{\left(x^2 - x - 2\right)^2}$$

$$= \frac{-x^2 - 8x + 2}{\left(x^2 - x - 2\right)^2}$$

xiii) Find the derivative of $\frac{a - b\ Cos\ x}{a + b\ Cos\ x}$ with respected to ' x '

Sol : $\frac{d}{dx}\left[\frac{a - b\ Cos\ x}{a + b\ Cos\ x}\right]$

$$= \frac{\left(a + b\ Cos\ x\right)\frac{d}{dx}\left(a - b\ Cos\ x\right) - \left(a - b\ Cos\ x\right)\frac{d}{dx}\left(a + b\ Cos\ x\right)}{\left(a + b\ Cos\ x\right)^2}$$

$$= \frac{\left(a + b\ Cos\ x\right)\left(b\ \text{Sin}\ x\right) - \left(a - b\ Cos\ x\right)\left(-b\ \text{Sin}\ x\right)}{\left(a + b\ Cos\ x\right)^2}$$

$$= \frac{ab\ Sin\ x + b^2\ \text{Sin}\ x\ Cos\ x + ab\ Sin\ x - b^2\ \text{Sin}\ x\ Cos\ x}{\left(a + b\ Cos\ x\right)^2}$$

$$= \frac{2ab\ Sin\ x\ Cos\ x}{\left(a + b\ Cos\ x\right)^2}$$

xiv) Find the derivative of $\frac{x^2}{\text{Sin } x} - \frac{a^x}{2x-3}$ with respected to ' x '

Sol : $\frac{d}{dx}\left[\frac{x^2}{\text{Sin } x} - \frac{a^x}{2x-3}\right]$

$$= \frac{d}{dx}\left[\frac{x^2}{\text{Sin } x}\right] - \frac{d}{dx}\left[\frac{a^x}{2x-3}\right]$$

$$= \frac{Sin\ x\ \frac{d}{dx}\left(x^2\right) - x^2\frac{d}{dx}(\text{Sin } x)}{(\text{Sin } x)^2} - \frac{(2x-3)\ \frac{d}{dx}\left(\text{a}^x\right) - a^x\frac{d}{dx}(2x-3)}{(2x-3)^2}$$

$$= \left[\frac{Sin\ x\ .\ 2x - x^2 Cos\ x}{\text{Sin}^2\ x}\right] - \left[\frac{(2x-3)\ a^x \log a - a^x .2}{(2x-3)^2}\right]$$

$$= \frac{2x\ Sin\ x\ \text{-}\ x^2 Cos\ x}{\text{Sin}^2\ x} - \frac{(2x-3)\ a^x \log a - 2a^x}{(2x-3)^2}$$

Exercise :

Find the derivative of the following with respect to ' x '

1. $3x^2+5x-1$
2. 2 Cos x - 4 e^x + 5 log x
3. a^x + log x + $\sqrt{x}$
4. x^7- 7 e^x + tan x
5. $\sqrt{2}$ Sin x + 7 x^6 + $\frac{3}{x^2}$
6. log a^x
7. x^3 log x
8. (x^2 + 1) Cot x
9. $4x^3$ - 3 Sin x log x + e^x log x
10. a^x tan x log x

11. $\frac{x(x+2)}{2x+5}$

12. $\frac{Cos\ x}{Sin\ x + Cos\ x}$

13. $\frac{Cos\ x}{x^2} - \frac{e^x}{5x + 4}$

12.3 DERIVATIVE OF COMPOSITE FUNCTION :

If ' f' is a differentiable function at 'x' and g is differentiable function at x then gof is differentiable at x and $(gof)^1(x) = g^1[f(x)].\ f^1(x)$.

Solved examples :

1) Find the derivative of $(5x - 6)^3$ with respected to ' x '

Sol : let $y = (5x - 6)^3$

$$\frac{dy}{dx} = \frac{d}{dx}(5x - 6)^3$$

$$= 3.\ (5x - 6)^{2}.\ \frac{d}{dx}(5x - 6)$$

$$= 3.\ (5x - 6)^{2}.\ 5$$

$$= 15.\ (5x - 6)^2$$

2) Find the derivative of a^{2x+3} with respected to ' x '

Sol : let $y = a^{2x+3}$

$$\frac{dy}{dx} = \frac{d}{dx}(a^{2x+3})$$

$$= a^{2x+3}.\ \log a.\ \frac{d}{dx}(2x + 3)$$

$$= a^{2x+3} \log a.\ 2$$

$$= 2.\ a^{2x+3} \log a.$$

3) Find the derivative of $Cosec^4 x$ with respected to ' x '

Sol : let y = $Cosec^4 x$

$$\frac{dy}{dx} = \frac{d}{dx}(Cosec^4 x)$$

$$= \frac{d}{dx}(Cosec\ x)^4$$

$$= 4\ Cosec^3 x . \frac{d}{dx}(Cosec\ x)$$

$$= 4\ Cosec^3 x . (-Cosec\ x. Cot\ x)$$

$$= -4\ Cosec^4 x . \ Cot\ x$$

4) Find the derivative of $20^{\log \tan x}$ with respected to ' x '

Sol : let y = $20^{\log \tan x}$

$$\frac{dy}{dx} = \frac{d}{dx}(20^{\log \tan x})$$

$$= 20^{\log \tan x} . \log 20. \frac{d}{dx}[\log(\tan x)]$$

$$= 20^{\log \tan x} . \log 20. \frac{1}{\tan x} . \frac{d}{dx}(\tan x)$$

$$= 20^{\log \tan x} . \log 20. \frac{1}{\tan x} . Sec^2 x$$

5) Find the derivative of log [x + $\sqrt{x^2+1}$] with respected to ' x '

Sol : let y = log [x + $\sqrt{x^2+1}$]

$$\frac{dy}{dx} = \frac{d}{dx}[x + \sqrt{x^2+1}]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \frac{d}{dx}[x + \sqrt{x^2+1}]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left[1 + \frac{1}{2\sqrt{x^2+1}} \frac{d}{dx} (x^2+1) \right]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left[1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x \right]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}} \right]$$

$$= \frac{1}{\sqrt{x^2+1}}$$

6) Find the derivative of $\sqrt{Cos\sqrt{x}}$ with respected to ' x '

Sol : let y = $\sqrt{Cos\sqrt{x}}$

$$\frac{dy}{dx} = \frac{d}{dx} \sqrt{Cos\sqrt{x}}$$

$$= \frac{1}{2\sqrt{Cos\sqrt{x}}} \frac{d}{dx} (Cos\sqrt{x})$$

$$= \frac{1}{2\sqrt{Cos\sqrt{x}}} (-\text{Sin } \sqrt{x}) \cdot \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{2\sqrt{Cos\sqrt{x}}} (-\text{Sin } \sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{-Sin\sqrt{x}}{4\sqrt{x}\ \sqrt{Cos\sqrt{x}}}$$

7) Find the derivative of x$\sqrt{Sin\ x}$ with respected to ' x '

Sol : let y = x. $\sqrt{Sin\ x}$

$$\frac{dy}{dx} = \frac{d}{dx}\ [x.\ \sqrt{Sin\ x}\ ]$$

$$= x.\ \frac{d}{dx}\sqrt{Sin\ x} + \sqrt{Sin\ x}\ .\ \frac{d}{dx}(x)$$

$$= x.\ \frac{1}{2\sqrt{Sin\ x}}\frac{d}{dx}(Sin\ X)\ .\ \sqrt{Sin\ x}\ .\ 1$$

$$= x.\ \frac{1}{2\sqrt{Sin\ x}}\ .\ Cos\ X + \sqrt{Sin\ x}$$

$$= \frac{x\ Cos\ x + 2\ Sin\ x}{2\sqrt{Sin\ x}}$$

7) Find the derivative of $\tan^2\left(\frac{1+x}{1-x}\right)$ with respected to ' x '

Sol : let y = $\tan^2\left(\frac{1+x}{1-x}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\ \tan^2\left(\frac{1+x}{1-x}\right)$$

$$= \frac{d}{dx}\left[\tan\left(\frac{1+x}{1-x}\right)\right]^2$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ \frac{d}{dx}\tan\left(\frac{1+x}{1-x}\right)$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right)\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right).\ \frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right).\ \frac{(1-x).1-(1+x)(-1)}{(1-x)^2}$$

$$= \frac{4}{(1-x)^2}.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right)$$

Exercise :

I

(i) Cos (4x +7) (ii) e^{ax+b} (iii) $(x^2-3x+5)^{7/3}$ (iv) log (2x - 1) (v) a^{4+5x} (vi) $\sqrt{2+3x}$

(vii) $Cos^n x$ (viii) log (tan 5x) (ix) $\log\left(Sin\frac{x}{2}\right)$

II

(i) log (x^2 +1) (ii) $(x+\sqrt{x^2+1})^n$ (iii) Sin $(3\sqrt{x^2+1})$ (iv) $\sqrt{x+\frac{1}{x}}$ (v) $\frac{1}{\sqrt{2+3x}}$

(vi) $\log\left[Cot(1-x^2)\right]$

III

(i) Sin mx . Cos nx (ii) x^2. $Cos^3 2x$ (iii) $(2x^2+1)e^x$ (iv) e^{5x}. log 6x (v) (x^2+1) log (log x)

(vi) e^{2x}. Sin 3x .Cos 4x (vii) Cos $\left[\frac{1-x^2}{1+x^2}\right]$

SUCCESSIVE DIFFERENATION :

If the function in 'x' is differentiated with respect to ' x' , the result is called first derivative at the function. If the 1st derivative is differentiative, the result is called second derivative. Similarly we proced n times, then result is called n^{th} derivative of function.

Let y = f (x) is a function, then first derivative $\frac{dy}{dx} = f^1(x)$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f^{II}(x)$$

Similarly n^{th} derivative

$$\frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^{n}(x)$$

Solved examples :

1) Find the second derivative of $x^3 + 2x^2 + 3x + 4$ with respected to ' x '

Sol : let $y = x^3 + 2x^2 + 3x + 4$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 2x^2 + 3x + 4)$$

$$= 3x^2 + 4x + 3$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(3x^2 + 4x + 3)$$

$$= (6x + 4)$$

2) Find the second derivative of e^{2x} with respected to ' x '

Sol : let $y = e^{2x}$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = e^{2x}\frac{d}{dx}(2x)$$

$$= e^{2x} . 2$$

$$= 2. e^{2x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2e^{2x})$$

$$= 2\frac{d}{dx}(e^{2x})$$

$= 2.2\ e^{2x}$

$= 4.\ e^{2x}$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\ \frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}(\ 4\ e^{2x}\ )$$

$$= 4\ \frac{d}{dx}(\ e^{2x}\ )$$

$= 4.2\ e^{2x}$

$= 8.\ e^{2x}$

12.4 EXERCISE :

I Find second derivative of

(i) x^4 (ii) Sin 3x (iii) tan x (iv) log x (v) $\frac{1}{x^2}$ (vi) x. log x (vii) e^x . sin x (viii) $x^2\ e^x$

II Find third derivative of

(i) $x^4 + 2x^3 + 7x + 5$ (ii) Cos x (iii) e^{5x} (iv) $\frac{1}{x}$ (v) $e^x\ x^3$

PARTIAL DIFFERENTIATION :

Partial Derivatives :

Let Z = f (x, y) Then $\underset{h\to 0}{Lt}\ \frac{f(a+h,b)-f(a,b)}{h}$ If it exist, is said to be the partial derivative of f (x,y) w.r. to 'x' at (a,b) and is denoted by $\left(\frac{\partial z}{\partial x}\right)$ or f_x (a,b).

If f (x,y) possesses a partial derivative with respect to x at every point of its domain, then the partial derivative with respect to x at every point of its domain, then the partial derivative of f (x,y) w.r. to x is $\underset{h\to 0}{Lt}\ \frac{f(x+h,y)-f(x,y)}{h}$ and is denoted by f_x (x, y) or $\left(\frac{\partial f}{\partial x}\right)$

Note 1 : Partial derivative of z w.r. to 'x' is the ordinary derivative of z w.r. to 'x' treating y as constant . Also partial derivative of z w.r to ' y' is the ordinary derivative of z w.r. to 'y' treating x as constant.

Note 2 : $\left(\frac{\partial z}{\partial x}\right)$ is called first order partial derivative of z w.r to 'x' and are some times denoted by z_x

Examples :

1) Find $\frac{\partial z}{\partial x}$ when $z = x^y + y^x$

Sol : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^y + y^x)$

$\frac{dy}{dx} = \frac{\partial}{\partial x}(x^y) + \frac{\partial}{\partial x}(y^x)$

$= y\,x^{y-1} + y^x \log y$

2) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ when $z = e^{ax} \sin by$

Sol : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^{ax} \sin by)$

$= \text{Sin } by \frac{\partial}{\partial x}(e^{ax})$

$= \text{Sin } by \cdot e^{ax} \frac{\partial}{\partial x}(ax)$

$= \text{Sin } by \cdot e^{ax} \cdot a$

$= a \cdot e^{ax} \cdot \text{Sin } by$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(e^{ax} \sin by)$

$= e^{ax} \cdot \frac{\partial}{\partial y}(\text{Sin } by)$

$= e^{ax} \text{Cos by} . \frac{\partial}{\partial x} (\text{by})$

$= e^{ax} \text{Cos by} . \ b$

$= b. \ e^{ax} \text{Cos by}$

3) If $u = f(x^2 + y^2)$ prove that $y \frac{\partial u}{\partial x} = x \frac{\partial u}{\partial y}$

Sol : $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(x^2 + y^2)$

$= f^{I}(x^2 + y^2) \frac{\partial}{\partial x} (x^2 + y^2)$

$= f^{I}(x^2 + y^2). \ 2x$

$y \frac{\partial u}{\partial y} = 2xy \ f^{I}(x^2 + y^2)$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x} f(x^2 + y^2)$

$= f^{I}(x^2 + y^2) \frac{\partial}{\partial y} (x^2 + y^2)$

$= f^{I}(x^2 + y^2). \ 2y$

$x \frac{\partial u}{\partial y} = 2xy \ f^{I}(x^2 + y^2)$

$\therefore y \frac{\partial u}{\partial x} = x \frac{\partial u}{\partial y}$

4) If $Z = e^{ax+by} f(ax - by)$ prove that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2 \ abz$

Sol : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [e^{ax+by} f(ax-by)]$

$= e^{ax+by} \frac{\partial}{\partial x} f(ax-by) + f(ax-by) \frac{\partial}{\partial x} e^{ax+by}$

$= e^{ax+by} f^{l}(ax - by) \frac{\partial}{\partial x}(ax - by) + f(ax - by) e^{ax+by} \frac{\partial}{\partial x}(ax + by)$

$= e^{ax+by} f^{l}(ax - by). a + f(ax - by) e^{ax+by}. a$

$b \frac{\partial z}{\partial x} = ab\, e^{ax+by} f^{l}(ax - by) + ab\, e^{ax+by} f(ax - by)$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [e^{ax+by} f(ax - by)]$

$= e^{ax+by} \frac{\partial}{\partial x} f(ax-by) + f(ax-by) \frac{\partial}{\partial x} e^{ax+by}$

$= e^{ax+by} f^{l}(ax - by) \frac{\partial}{\partial x}(ax - by) + f(ax - by) e^{ax+by} \frac{\partial}{\partial x}(ax + by)$

$= e^{ax+by} f^{l}(ax - by). (-b) + f(ax - by) e^{ax+by}. (b)$

$a \frac{\partial z}{\partial x} = - ab\, e^{ax+by} f^{l}(ax - by) + ab .e^{ax+by} . f(ax - by)$

$\therefore\ b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2\, ab .e^{ax+by} . f(ax - by)$

$= 2\, abz$

5) If $z = e^{x+y} + f(x) + g(y)$ find $\frac{\partial^2 z}{\partial x\, \partial y}$

Sol : $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ e^{x+y} + f(x) + g(y)$

$= \frac{\partial}{\partial x} [e^{x+y}] + \frac{\partial}{\partial x} [f(x)] + \frac{\partial}{\partial x} [g(y)]$

$= e^{x+y} + \frac{\partial}{\partial x}(x + y) + 0 + g^{l}(y)$

$= e^{x+y}(1) + g^{l}(y)$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[e^{x+y} + g^{I}(y) \right]$$

$$= \frac{\partial}{\partial x} e^{x+y} + \frac{\partial}{\partial x}\left[g^{I}(y) \right]$$

$$= e^{x+y} \frac{\partial}{\partial x}(x + y) + 0$$

$$= e^{x+y} (1)$$

$$= e^{x+y}$$

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Lesson - 13

CALCULAS - PARTIAL DIFFERENTIATION - MAXIMA - MINIMA

OBJECTIVES:

By the study of this lesson you will be able to understand the meaning, first order partial derivatives, second order partial derivatives in detail.

STRUCTURE:

13.1 Introduction

13.2 First order partial derivatives

13.3 Second order partial derivatives

13.4 Solved examples

13.5 Exercises

13.6 Homogeneous function

13.7 Examples

13.8 Statement of Euler's Theorem.

13.9 Examples

13.10 Exercises

13.11 Maxima - Minima - Stationary value

13.12 Critical points

13.13 Relative Maximum value

13.14 Relative Minimum Value

13.15 Extreme values

13.16 Working rule for finding Maximum and Minimum values of a function

13.17 Examples

13.18 Exercises

13.1 INTODUCTION :

Partial Derivation :

Let Z = f(x,y) be a function of two variables in a domain DCR let (a,b) $\in$ D

If $\lim_{h\to o} \frac{f(a+b,b)-f(a,b)}{h}$ exists, then it is said to be the partial derivative of f (x,y) w.r.t x at the point (a,b). It is denoted by $\left[\frac{\partial f}{\partial x}\right]_{(a,b)} (or) \left[\frac{\partial z}{\partial x}\right]_{(a,b)}$ (or) fx (a,b)

If $\lim_{k\to o} \frac{f(a+b,k)-f(a,b)}{k}$ exists, then it is said to be the partial derivative of f(x,y) w.r.t y at the point (a,b). It is denoted by $\left[\frac{\partial f}{\partial y}\right]_{(a,b)} (or) \left[\frac{\partial z}{\partial y}\right]_{(a,b)}$ (or) fy (a,b)

Usually, the partial derivates of Z = f(x,y) at a point (x,y) are denoted by (or) fx and (or)fy

Note :

Partial derivative of z = f(x,y) w.r.t x is the ordinary derivative of z w.r.t x treating y as a constant Partial derivative of z = f (x,y) w.r.t y is the ordinary derivative of z w.r.t y treating x as a constant.

13.2 FRIST ORDER PARTIAL DERIVATIVES :

$\frac{\partial f}{\partial x}$ or fx $\frac{\partial f}{\partial y}$ or fy are called frist order Partial derivatives of z = f(x,y)

13.3 SECOND ORDER PARTIAL DERIVATIVES :

If z = f (x,y) then the frist order partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ may again be functions of x,y.

The partial derivatives of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ w.r.t x and y are called Second order partial derivatives of z = f(x,y)

They are , $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or f x *x*

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x}$ or f y x

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y}$ or f x y

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ or f y y

Note : We consider the functions for which $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$

13.4 SOLVED EXAMPLES :

Examples 1 :

If $z = xe^y + ye^x$ then find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

Solution : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xe^y + ye^x)$

$= \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial x}(ye^x)$

$= e^y \frac{\partial}{\partial x}(x) + y \frac{\partial}{\partial x}(e^x)$

$= e^y(1) + y(e^x) = e^y + y\,e^x$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xe^y + ye^x)$

$= x \frac{\partial}{\partial x}(e^y) + e^x \frac{\partial}{\partial x}(y)$

$= x\,e^y + e^x\ (1) = xe^y + e^x$

Examples 2 :

If $z = x^2+y^2 + 2xy$ then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Solution : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2 + 2xy)$

$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(2xy)$

$= 2x + 0 + 2y \frac{\partial}{\partial x}(x)$

$= 2x + 2y (1)$

$= 2x + 2y$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2 + 2xy)$

$= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(2xy)$

$= 0+ 2y + 2x\frac{\partial}{\partial x}(y)$

$= 2y + 2x (1)$

$= 2x + 2y$

$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

Examples 3 :

If $f = x^y$ find $f\,y_y$

Solution : $fy = \frac{\partial f}{\partial y}$

$= \frac{\partial}{\partial y}(x^y) = x^y \log x$

$$fy_y = \frac{\partial f}{\partial y} \ (fy)$$

$$= \frac{\partial}{\partial y} \ (x^y \log x)$$

$$= \log x \ \frac{\partial}{\partial y} \ (x^y)$$

$$= \log x \,.\, x^y \,.\, \log x$$

$$= (\log x)^2 \,.\, x^y$$

Examples 4 :

If $f = x^2yz - 2xz^3 + xz^2$ then find $\frac{\partial^2 f}{\partial x\, \partial y}$ at (1,0,2)

Solution : $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2yz - 2xz^3 + xz^2)$

$$= \frac{\partial}{\partial y}(x^2 yz) - \frac{\partial}{\partial y} (2xz^3) + \frac{\partial}{\partial y} (xz^2)$$

$$= x^2 \frac{\partial}{\partial y} (y) - 0 + 0$$

$$= x^2 z (1)$$

$$= x^2z$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x} (x^2z)$$

$$= z\frac{\partial}{\partial x} (x^2)$$

$$= z (2x) = 2xz$$

$$\therefore \frac{\partial^2 f}{\partial x\, \partial y} \text{ at } (1,0,2) = 2 (1) (2) = 4$$

Examples 5 :

If $u = \log(x^2 + y^2 + z^2)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution : $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\log(x^2 + y^2 + z^2)\right)$

$$= \frac{1}{x^2 + y^2 + z^2}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$= \frac{1}{x^2 + y^2 + z^2}(2x)$$

$$x\frac{\partial u}{\partial x} = x.\frac{1}{x^2 + y^2 + z^2}(2x)$$

$$= \frac{2x^2}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^2 + y^2 + z^2)\right)$$

$$= \frac{1}{x^2 + y^2 + z^2}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)$$

$$= \frac{1}{x^2 + y^2 + z^2}(2y)$$

$$= \frac{2y}{x^2 + y^2 + z^2}$$

$$y\frac{\partial u}{\partial y} = y.\frac{2y}{x^2 + y^2 + z^2}$$

$$= \frac{2y^2}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\log (x^2 + y^2 + z^2) \right)$$

$$= \frac{1}{x^2 + y^2 + z^2} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$= \frac{1}{x^2 + y^2 + z^2} (2z)$$

$$z \frac{\partial u}{\partial z} = z. \frac{1}{x^2 + y^2 + z^2} (2z)$$

$$= \frac{2z^2}{x^2 + y^2 + z^2}$$

$$\therefore\ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

$$= \frac{2x^2}{x^2 + y^2 + z^2} + \frac{2y^2}{x^2 + y^2 + z^2} + \frac{2z^2}{x^2 + y^2 + z^2}$$

$$= \frac{2x^2 + 2y^2 + 2z^2}{x^2 + y^2 + z^2}$$

$$= \frac{2(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

$$= 2$$

Examples 6 :

If $z = x^3 + y^3 - 3\,axy$ show that $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$

Solution : $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 + y^3 - 3\,axy)$

$$= 0 + 3y^2 - 3ax$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(3y^2 - 3ax)$$

$$= 0 - 3a$$

$$= -3a$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 + y^3 - 3\,axy)$$

$$= 3x^2 + 0 - 3ay$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$$

$$= \frac{\partial}{\partial y}(3x^2 - 3ay)$$

$$= 0 - 3a$$

$$= -3a$$

$$\therefore \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$$

13.5 EXERCISE :

1. If $z = x^2 + y^2$, find $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$
2. If $u = e^{x-y}$, find $\frac{\partial^2 u}{\partial x^2}\ \frac{\partial^2 u}{\partial x\, \partial y}$
3. If $f = x^2 (y-z) + y^2 (z-x) + z^2 (x-y)$ show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$
4. If $u = e^{xy}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$
5. If $z = xy\, f\left(\frac{y}{x}\right)$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

13.6 HOMOGENEOUS FUNCTION:

Let $A \in R^2$, $f : A \to R$, $n \in R$. Then the function f (x,y) is said to be a homogeneous function of degree or order n in variables x, y if $f(Kx, Ky) = K^n f(x,y)$.

13.7 EXAMPLES :

Examples 1 :

If $f = \dfrac{x^3+y^3}{x^2+y^2}$, show that it is homogeneous function of order 1.

Solution : Let $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

$$\therefore f(Kx, Ky) = \frac{\left(kx^3\right)+\left(ky^3\right)}{\left(kx^2\right)+\left(ky^2\right)}$$

$$= \frac{k^3x^3+k^3y^3}{k^2x^2+k^2y^2}$$

$$= \frac{k^3\left(x^3+y^3\right)}{k^2\left(x^2+y^2\right)}$$

$$= k. \frac{x^3+y^3}{x^2+y^2}$$

$$= k^1 . f(x,y)$$

$\therefore$ f is homogeneous function of order 1 .

Examples 2 :

If $f(x,y) = ax^2 + 2hxy + by^2$ show that f is homogeneous function of order 2.

Solution : Let $f(x,y) = ax^2 + 2hxy + by^2$

$$\therefore f(Kx, Ky) = a(kx^2) + 2h(kx)(ky) + b(ky)^2$$

$$= a\,k^2x^2 + 2h\,kx\,ky + b\,k^2y^2$$

$$= k^2(ax^2 + 2hxy + by^2)$$

$$= k^2 . f(x,y)$$

$\therefore$ f is homogeneous function of order 2 .

13.8 STATEMENT OF EULER'S THEOREM :

If z = f (x,y) is a homogeneous function of degree n, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \; \forall$ x,y in the domain of the function.

13.9 EXAMPLES :

Examples 1 :

If $f = \frac{x^3 + y^3}{x - y}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$ by Euler's thorem.

Solution : Let $f(x,y) = \frac{x^3 + y^3}{x - y}$

$$\therefore f(kx, ky) = \frac{(kx^3) + (ky^3)}{(kx) - (ky)}$$

$$= \frac{k^3x^3 + k^3y^3}{kx - ky}$$

$$= \frac{k^3(x^3 + y^3)}{k(x - y)}$$

$$= k^2 . \frac{x^3 + y^3}{x - y}$$

$$= k^2 . f(x,y)$$

$\therefore$ f (x,y) is homogeneous function of order 2 .

By Euler's thorem.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

Examples 2 :

If f = (x,y), = x Cos $\left(\frac{y}{x}\right)$ + y Cos$\left(\frac{x}{y}\right)$ show that x$\frac{\partial f}{\partial x}$ + y$\frac{\partial f}{\partial y}$ = f by Euler's thorem.

Solution : Let f (x,y) = x Cos $\left(\frac{y}{x}\right)$ + y Cos$\left(\frac{x}{y}\right)$

$$\therefore f(kx, ky) = kx.\text{Cos}\left(\frac{ky}{kx}\right) + ky.\text{Cos}\left(\frac{kx}{ky}\right)$$

$$= kx\,\text{Cos}\left(\frac{y}{x}\right) + ky\,\text{Cos}\left(\frac{x}{y}\right)$$

$$= k\left[x\,\text{Cos}\left(\frac{y}{x}\right) + y\,\text{Cos}\left(\frac{x}{y}\right)\right]$$

$$= k^1.\ f(x,y)$$

$\therefore$ f (x,y) is homogeneous function of order 1 .

By Euler's thorem.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1f$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

13.10 EXERCISE :

1. If z = $\frac{x^{1/3} + y^{1/3}}{x^{1/3} + y^{1/3}}$ find x$\frac{\partial z}{\partial x}$ + y$\frac{\partial z}{\partial y}$ = $\frac{z}{6}$ by Euler's theorem

2. If z = $\frac{x^2 y}{x^3 + y^3}$ find x$\frac{\partial z}{\partial x}$ + y$\frac{\partial z}{\partial y}$ = 0 by Euler's theorem

3. If u = $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ find x$\frac{\partial z}{\partial x}$ + y$\frac{\partial z}{\partial y}$ = 0 by Euler's theorem.

4. If u = $x^3 + 3x^2y - 2xy^2 + y^3$ show that x $\frac{\partial u}{\partial x}$ + y $\frac{\partial u}{\partial y}$ = 3u by Euler's theorem.

13.11 MAXIMA AND MINIMA :

Stationary Value : Let f be a real function which is differentiable at ' a'. If $f^1(a) = 0$ then we say that f(x) is sationary at x =a . (a, f(a)) is called sationary point and f(a) is called sationary value.

13.12 CRITICAL POINTS :

The points at which f^1 (x) = 0 or f^1 (x) does not exists are called critical points of the function f(x).

13.13 RELATIVE MAXIMUM VALUE :

f (x) has relative maximum at x = a there exist $\delta > 0$ such that f(x) $\leq$ f(a) for $a-\delta < x < a + \delta$. f(a) is called relative maximum value.

13.14 RELATIVE MINIMUM VALUE :

f (x) has relative maximum at x = a if there exist $\delta > 0$ such that f(x) $\geq$ f(a) for $a-\delta < x < a + \delta$. f(a) is called relative minimum value.

13.15 EXTREME VALUE :

The points at which a function attains either maximum or minimum are called extreme points or turnins points of the function maximum or minimum values of a function are called extreme values or turning values of the function

Necessary condition for extreme value of function :

If a function f(x) has extreme value f(a) then $f^1(a)$ =0 if it exists.

Sufficient condition for extreme values :

Let f(x) be derivable at x = a and f^{11} (a) exists and is non - zero.

a) $f^1(a) = 0$ and $f^{11}(a) < 0 \Rightarrow$ x = a is a point of relatively maximum.

b) $f^1(a) = 0$ and $f^{11}(a) > 0 \Rightarrow$ x = a is a point of relatively minimum.

13.16 WORKING RULE FOR FINDING MAXIMUM AND MINIMUM VALUES OF A FUNCTION :

Step I : Find $\frac{dy}{dx}$ for the given function y = f(x)

Step II : Find the values at x when $\frac{dy}{dx}$ = 0 let these values a,b,c,......

Step III : Find $\frac{d^2y}{dx^2}$

Step IV : Find x = a in $\frac{d^2y}{dx^2}$

i) if the result is -Ve, the function is maximum at x = a and that maximum value is f(a)

ii) If the result is +ve, the function is minimum at x = a that minimum value is f(a).

Step V : When $\frac{d^2y}{dx^2}$ = 0 for a particular value x = a, then the either employ the first method or find $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ and put x = a successively in these derivatives.

13.17 EXAMPLES :

Examples 1 :

Find the stationary points and stationary values of the function $3x^4 - 4x^3 + 1$.

Solution : Let $y = 3x^4 - 4x^3 + 1$.

$$\frac{dy}{dx} = 12x^3 - 12x^2$$

$$\frac{d^2y}{dx^2} = 36x^2 - 24x$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow 12x^3 - 12x^2 = 0$

$\Rightarrow 12x^2(x-1) = 0$

$\Rightarrow x^2 = 0,\ x-1 = 0$

$\Rightarrow x = 0,\ x = 0$

Stationary points 0,1

Stationary values = Y x= 0, Y x=1

$= 3(0)^4 - 4(0)^3 + 1,\ 3(1)^4 - 4(1)^3 + 1$

= 1,0

Examples 2 :

Find the maximum and minimum values of the function $x^3 - 9x^2 + 24x - 12$

Solution : Let $y = x^3 - 9x^2 + 24x - 12$

$$\frac{dy}{dx} = 3x^2 - 18x + 24$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow x^2 - 2x - 4x + 8 = 0$

$\Rightarrow x(x-2) = 4(x-2) = 0$

$\Rightarrow (x-2)(x-4) = 0$

Stationary Values 2,4

i) When x = 2

$$\frac{d^2y}{dx^2} = 6(2) - 18 = -6 < 0$$

$y = 2^3 - 9.2^2 + 24.2 - 12$

$= 8 - 36 + 48 - 12$

$= 8$

When x = 2 the given function has maximum value and that maximum value is 8.

ii) When x = 4

$\frac{d^2y}{dx^2}$ = 6(4) - 18 = 6>0

y = $4^3 - 9.4^2 + 24.4 - 12$

= 64 - 144 + 96 -12

= 160 - 156

= 4

When x = 4 the given function has minimum value and that minimum value is 4.

Examples 3 :

Show that the function f (x) = $\frac{x}{\log x}$ has a minimum value at x = e.

Solution : f (x) = $\frac{x}{\log x}$

$$f^1(x) = \frac{\log x \frac{d}{dx}(x) - x\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{\log x.1}{(\log x)^2}$$

$$= \frac{\log x - 1}{(\log x)^2}$$

$$f^{11}(x) = \frac{\left(\log x\right)^2 \frac{d}{dx}(\log x - 1) - (\log x - 1)\ \frac{d}{dx}(\log x)^2}{(\log x)^4}$$

$$= \frac{\left(\log x\right)^2 . \frac{1}{x} - (\log x - 1)\ .2\log x.\frac{1}{x}}{(\log x)^4}$$

$$= \frac{\frac{1}{x}\left(\log x\right)^2 . \frac{2}{x}\log x\ .(\log x - 1)}{(\log x)^4}$$

Put $f^1(x) = 0$

$$\Rightarrow \frac{\log x - 1}{(\log x)^2} = 0$$

$\Rightarrow \log x - 1 = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log_e x = \log_e e$

$\Rightarrow x = e$

Stationary value $= e$

When $x = e$

$$f^{11}(e) = \frac{\frac{1}{e}(\log e)^2 . \frac{2}{e}\log e \ .(\log e - 1)}{(\log e)^4}$$

$$= \frac{\frac{1}{e}1^2 . \frac{2}{e}.1 \ (1-1)}{(1)^4}$$

$$= \frac{1}{e} > 0$$

$$f(e) = \frac{e}{\log e} = \frac{e}{1} = e$$

$\therefore$ f (x) has minimum value at x = e and that minimum value is e

Examples 4 :

Investigate the maxima and minima of the function $2x^3 + 3x^2 - 36x + 10$

Solution : Let $y = 2x^3 + 3x^2 - 36x + 10$

$$\frac{dy}{dx} = 6x^2 + 6x - 36$$

$$\frac{d^2y}{dx^2} = 12x + 6$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow$ (x+3) (x-2) = 0

Stationary Values -3,2

When x = -3

$\frac{d^2y}{dx^2}$ = 12 (-3) + 6 = -30 < 0

x = - 3

y = 2 $(-3)^3$ + 3 $(-3)^2$ - 36 (-3) +10

= - 54 + 27 + 108 +10

= 91

$\therefore$ The function has maximum value at x = -3 and that maximum value is 91

When x = 2

$\frac{d^2y}{dx^2}$ = 12 (2) + 6 = 30 > 0

y = 2 $(2)^3$ + 3 $(2)^2$ - 36 (2) +10

= 16 +12 -72 +10

= -34

$\therefore$ The function has maximum value at x = 2 and that maximum value is -34

Examples 5 :

A company has examined its cost structure and revenue structure and has determined that c the total cost , R total revenue, and x the number of units produced are related as

C = 100 + 0.015 x^2 and R = 3x

Find the production rate x that will maximise points of the company. Find the profit. Find also the profit when x = 120.

Solution : Let p denote the profit of the company, then

p = R - c

$\Rightarrow$ p = 3x (100 + 0.015 x^2)

= 3x - 100 - $\frac{15}{1000}x^2$

$$\frac{dp}{dx} = 3 - \frac{30x}{1000}$$

$$\frac{d^2p}{dx^2} = \frac{-30}{1000} < 0$$

Put $\frac{dp}{dx} = 0$

$\Rightarrow 3 - \frac{30x}{1000} = 0$

$\Rightarrow$ x = 100 units

Stationary value x = 100.

When x = 100

$$\frac{d^2p}{dx^2} = \frac{-30}{1000} < 0$$

Put = 3(100) -100 - $\frac{15}{1000}$ $(100)^2$

= 3 -100 - $\frac{15 \text{ x } 100 \text{ x } 100}{1000}$

= 3 -100 - 150

= 50 rupees

When x = 100 the profit is maximum and that maximum profit is 50 rupees .

The profit when x = 120 is

p = 3 x120 - 100 - 0.015 x $(120)^2$

= 360 - 100 - 216

= 44 rupees.

Examples 6 :

The cost C of manu facturing a certain article is given by the formula

$$C = 5 + \frac{48}{x} + 3x^2$$

Where x is the number of articles manufactured. Find minimum value of c.

Solution : $C = 5 + \frac{48}{x} + 3x^2$

$$\frac{dc}{dx} = 0 + 48\left(\frac{-1}{x^2}\right) + 6x$$

$$\frac{d^2c}{dx^2} = 48\left(\frac{2}{x^3}\right) + 6$$

Put $\frac{dc}{dx} = 0$

$$\Rightarrow \frac{-48}{x^2} + 6x = 0$$

$$\Rightarrow 6x = \frac{48}{x^2}$$

$$\Rightarrow 6x^3 = 48$$

$$\Rightarrow x^3 = 8$$

$$\Rightarrow x = 2$$

Stationary value 2

When x = 2

$$\frac{d^2c}{dx^2} = \frac{96}{2^3} + 6$$

$$= 12 + 6$$

$$= 18 > 0$$

$$C = 5 + \frac{48}{2} + 3(2^2)$$

$$= 5 + 24 + 24$$

= 53 rupees

When x = 2 the cost is minimum and that minimum cost is 53 rupees .

13.18 EXERCISES :

Find the maximum (or) minimum values of the function.

1. $y = 27 - 6x + x^2$
2. $y = 8 + 4x - x^2$

Find the maximum and minimum values of the function.

3. $y = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$
4. $y = x^3 - 3x + 15$
5. $y = x^3 - 2x^2 + x + 6$
6. $y = 2x^3 + 3x^2 - 36x + 10$
7. By an ' Economic order Quantity; We mean a quantity Q,Which when purchased in each order, minimizes the total cost T incurred in obtaining and staring material for a certain time period to 10 fulfil a given rate of demand for the material during the time period.

   The material demanded is 10,000 units per year, the cost price of material Re.1 per unit, the cost of replenishing the stock of material per order regardless of the size Q of the order is Rs. 25, and the cost of storing the material is $12\frac{1}{2}$ percent per year on the rupee value of average quantity Q/2 on hand.

   (i) show that $T = 10,000 + \frac{2,50,000}{Q} + \frac{Q}{16}$

   (ii) Find the Economic order quantity and the cost T corresponding to that.

   (iii) Find the total cost when each order is placed for 2500 units.
8. The demand function for a particular commodity is $y = 15\,e^{-x/3}$ for $0 \leq x \leq 8$, where y is the price per unit and 'x' is the number of units demanded. Determine the price and the quantity for which the revenue is maximum.

   (Hint : Revenue ; R = y , x)
9. A firm has to produce 144,000 units of an item per year. It cost Rs. 60 to make the factory ready for a product run of the item regardless of units x produced in a run. The cost of material is Rs. 5 per unit and the cost of storing the material is 50 paiseper item per year on the average inventory $\left(\frac{x}{2}\right)$ in hand. Show that the total cost cis given by

$$C = 720{,}000 + \frac{23{,}040{,}000}{x} + \frac{x}{4}$$

Find also the economic lost size, i.e. value of x for which ' C ' is minimum.

10. A company notices that higher sales, of a particular item which it produces are achieved by lowering the price charged. As a result the total revenue from the sales at first rises as the number of units sold increases, reaches the highest point and then falls off. This pattern of total revenue is described. by the relation $y = 40{,}00{,}000 - (x - 2000)^2$ Where y is the total revenue and x the number of units sold.

    i) Find what number of units sold maximizes total revenue ?

    ii) What is the amount of this maximum revenue ?

    iii) What would be the total revenue if 2500 units were sold ?

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Lesson - 14

CALCULAS - INTEGRATION

OBJECTIVES:

By the study of this lesson you will be able to understand the meaning & important of the indefinite integrals with examples. You will also be thorough with integration by substitution, Definite integrats in detail.

STRUCTURE:

14.1 Introduction

14.2 Theorems

14.3 Solved examples

14.4 Exercises

14.5 Integration by substitution

14.6 Solved Examples

14.7 Exercises

14.1 INTODUCTION :

Antiderivation :

If f(x) and g(x) are two functions such that $f^{I}(x) = g(x)$ then f(x) is called antiderivative or primitive of g(x) with respect to x

Ex : $\frac{d}{dx}$ $(x^2 +2x+5) = 2x+2$

$x^2 +2x+5$ is an antiderivative of 2x + 2.

If f(x) is Primitive of f(x) then F(x)+c is called indefinite integral of f(x) with respect to x. It is denoted by $\int f(x)dx$.

$\int f(x)dx$ = F(x)+c where c is constant.

Ex : 1 - $\int\left(x^4 + 2x\right)dx = \frac{x^5}{5} + x^2 + c$

Ex : 2 - $\int Sinx\ dx =$ - Cos x + c

14.2 THEOREMS :

Theorem 1 :

If n $\neq$ - 1 then $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

Proof :

$$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\therefore \int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

Ex : $\int x^7 \, dx = \frac{x^{7+1}}{7+1} + c$

$$= \frac{x^8}{8} + c$$

Ex : $\int x^{-6} \, dx = \frac{x^{-6+1}}{-6+1} + c$

$$= \frac{x^{-5}}{-5} + c$$

Ex : $\int x^{2/3} \, dx = \frac{x^{2/3+1}}{2/3+1} + c$

$$= \frac{3}{5} x^{5/3} + c$$

Theorem 2 :

$$\int a\,dx = ax + c$$

Proof :

$$\because \frac{d}{dx}(ax) = a$$

$$\therefore \int a\,dx = ax + c$$

Ex : $\int 2\,dx = 2 + c$

Ex : $\int dx = x + c$

Theorem 3 :

$$\int \frac{1}{x}dx = \log|x| + c$$

Proof :

$$\because \frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x}\,dx = \log|x| + c$$

Ex : $\int \frac{1}{ax}dx = \frac{1}{a}\int \frac{1}{x}dx = \frac{1}{a}\log|x| + c$

Ex : $\int \frac{1}{5x}dx = \frac{1}{5}\int \frac{1}{x}dx = \frac{1}{5}\log|x| + c$

Theorem 4 : $\int e^x dx = e^x + c$

Proof :

$$\because \frac{d}{dx}\left(e^x\right) = e^x$$

$$\therefore \int e^x\,dx = e^x + c$$

Theorem 5 : $\int \frac{1}{x^n}\,dx = \frac{1}{(n-1)x^{n-1}} + c$

Proof :

$$\because \frac{d}{dx}\left[\frac{-1}{(n-1)x^{n-1}}\right]$$

$$= \frac{d}{dx}\left[\frac{-1}{(n-1)}.x^{n-1}\right]$$

$$= \frac{-1}{(n-1)}\left[-(n-1)\right]x^{-(n-1)-1}$$

$$= x^{-n+1-1}$$

$$= \frac{1}{x^n}$$

$$\therefore \int \frac{1}{x^n}\,dx = \frac{1}{(n-1)x^{n-1}} + c$$

Ex : $\int \frac{1}{x^5}\,dx = \frac{-1}{(5-1)x^{5-1}} + c$

$$= \frac{-1}{4x^4} + c$$

Ex : $\int \frac{1}{x^{5/3}}\,dx = \frac{-1}{\left(\frac{5}{3}-1\right)x^{(5/3-1)}} + c$

$$= \frac{-1}{\frac{2}{3}x^{2/3}} + c$$

Theorem 6 : $\int a^x dx = \frac{a^x}{\log a} + c$

Proof :

$$\because \frac{d}{dx}\left[\frac{a^x}{\log a}\right] = \frac{1}{\log a}\frac{d}{dx}(a^x)$$

$$= \frac{1}{\log a}.\ a^x.\ \log a$$

$$\therefore \int a^x dx = \frac{a^x}{\log a} + c$$

Ex : $\int 2^x dx = \frac{2^x}{\log 2} + c$

Ex : $\int \left(\frac{5}{2}\right)^x dx = \frac{\left(\frac{5}{2}\right)^x}{\log\left(\frac{5}{2}\right)} + c$

Theorem 7 : $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

Proof :

$$\because \frac{d}{dx}\left[2\sqrt{x}\right] = 2\frac{d}{dx}(\sqrt{x})$$

$$= 2.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}$$

$$\therefore \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$$

Ex : $\int \frac{1}{\sqrt{5x}} dx = \int \frac{1}{\sqrt{5}\sqrt{x}} dx$

$= \frac{1}{\sqrt{5}} \int \frac{1}{\sqrt{x}} dx$

$= \frac{1}{\sqrt{5}} 2\sqrt{x} + c$

$= \frac{2}{\sqrt{5}} \sqrt{x} + c$

Ex : $\int \frac{1}{\sqrt{\frac{x}{7}}} dx = \int \sqrt{\frac{7}{x}} dx$

$= \int \frac{\sqrt{7}}{\sqrt{x}} dx$

$= \sqrt{7} \int \frac{1}{\sqrt{x}} dx$

$= \sqrt{7}.2\sqrt{x} + c$

$= 2\sqrt{7}\sqrt{x} + c$

Theorem 8 : $\int 0.dx = c$

Proof :

$\because \frac{d}{dx}(c) = 0$

$\therefore \int 0.dx = 0 + c = c$

Theorem 9 : $\int \text{Cos x dx} = \text{Sin x} + \text{c}$

Proof :

$$\because \frac{d}{dx}(Sin\ x) = Cos\ x$$

$$\therefore \int \text{Cos x dx} = \text{Sin x} + \text{c}$$

Theorem 10 : $\int \text{Sin x dx} = -\text{ Cos x} + \text{c}$

Proof :

$$\because \frac{d}{dx}(-Cos\ x) = -\left[\frac{d}{dx}(Cos\ x)\right]$$

$= -(-\text{Sin x}) = \text{Sin x}$

$$\therefore \int \text{Sin x dx} = -\text{ Cos x} + \text{c}$$

Theorem 11 : $\int \text{Sec}^2\text{ x dx} = \tan \text{x} + \text{c}$

Proof :

$$\because \frac{d}{dx}(\tan\ x) = Sec^2 x$$

$$\therefore \int \text{Sec}^2\text{ x dx} = \tan \text{x} + \text{c}$$

Theorem 12 : $\int \text{Cosec}^2\text{ x dx} = -\text{ Cot x} + \text{c}$

Proof :

$$\because \frac{d}{dx}(-Cot\ x) = -\left[\frac{d}{dx}(Cot\ x)\right]$$

$= -[-\text{Cosec}^2\text{x}]$

$= \text{Cosec}^2\text{x}$

$$\therefore \int \text{Cosec}^2\text{ x dx} = -\text{ Cot x} + \text{c}$$

Theorem 13 : $\int \text{Sec} \tan x \, dx = \text{Sec } x + c$

Proof :

$$\because \frac{d}{dx}\left(Sec \text{ x }\right) = Sec \text{ x tan x}$$

$$\therefore \int \text{Sec tan x dx} = \text{Sec x + c}$$

Theorem 14 : $\int \text{CoSec x Cot x dx} = \text{- Cosec x + c}$

Proof :

$$\because \frac{d}{dx}\left(-Cosec \text{ x }\right) = -\left[\frac{d}{dx}\left(Cosec \text{ x }\right)\right]$$

$$= -[\,-\text{Cosec x Cot x}\,]$$

$$= \text{Cosec x Cot x}$$

$$\therefore \int \text{CoSec x Cot x dx} = \text{- Cosec x + c}$$

14.3 SOLVED EXAMPLES :

Examples 1 : Evaluate $\int (3x^2 - 5x + 4)\, dx$

Solution : $\int (3x^2 - 5x + 4)\, dx$

$$= 3\int x^2\, dx - 5\int x\, dx + \int u\, dx$$

$$= 3\frac{x^{2+1}}{2+1} - 5.\frac{x^{1+1}}{1+1} + 4x + c$$

$$= x^3 - \frac{5}{2}x^2 + 4x + c$$

Examples 2 : Evaluate $\int \left(\frac{3x+4}{2\sqrt{x}}\right)^2 dx$

Solution : $\int \left(\frac{3x+4}{2\sqrt{x}}\right)^2 dx$

$$= \int \frac{9x^2+24x+16}{4x} dx$$

$$= \int \left(\frac{9}{4}x^2+12x+\frac{4}{x}\right) dx$$

$$= \frac{9}{4}\int x^2\, dx + 12\int x\, dx + 4\int \frac{1}{x} dx$$

$$= \frac{9}{4}\frac{x^{2+1}}{2+1} + 12\frac{x^{1+1}}{1+1} + 4\log x + c$$

$$= \frac{3}{4}x^3 + 6x^2 + 4\log x + c$$

Examples 3 : Evaluate $\int (1-x)(4-3x)dx$

Solution : $\int (1-x)(4-3x)dx$

$$= \int \left(4-7x+3x^2\right)dx$$

$$= \int 4dx - 7\int x\, dx + 3\int x^2 dx$$

$$= 4x.-7\frac{x^{1+1}}{1+1} + 3.\frac{x^{2+1}}{2+1} + c$$

$$= 4x - \frac{7}{2}x^2 + x^3 + c$$

Examples 4 : Evaluate $\int\left(\frac{3x^2+4x+5}{\sqrt{x}}\right)dx$

Solution : $\int\left(\frac{3x^2+4x+5}{\sqrt{x}}\right)dx$

$$= \int\left(\frac{3x^2}{\sqrt{x}}+\frac{4x}{\sqrt{x}}+\frac{5}{\sqrt{x}}\right)dx$$

$$= \int\left(3x^{3/2}+4\sqrt{x}+\frac{5}{\sqrt{x}}\right)dx$$

$$= 3\int x^{3/2}\,dx+4\int\sqrt{x}\,dx+5\int\frac{1}{x}\,dx$$

$$= 3\frac{x^{3/2}+1}{\frac{3}{2}+1}+4\frac{x^{1/2}+1}{\frac{1}{2}+1}+5\,.2\sqrt{x}+c$$

$$= 3.\ \frac{2}{5}x^{\frac{5}{2}}+4\ \frac{2}{3}x^{\frac{3}{2}}+10\sqrt{x}+c$$

$$= \frac{6}{5}x^{\frac{5}{2}}+\frac{8}{3}x^{\frac{3}{2}}+10\sqrt{x}+c$$

Examples 5 : Evaluate $\int\left(\frac{a^x - b^x}{a^x.b^x}\right)^2 dx$

Solution : $\int\left(\frac{a^x - b^x}{a^x.b^x}\right)^2 dx$

$$= \int\frac{a^{2x}-2a^xb^x+b^{2x}}{a^x.\,b^x}dx$$

$$= \int \left(\frac{a^{2x}}{a^x . b^x} - \frac{2a^x b^x}{a^x . b^x} + \frac{b^{2x}}{a^x . b^x} \right) dx$$

$$= \int \left(\frac{a^{2x}}{b^x} - 2 + \frac{b^x}{a^x} \right) dx$$

$$= \int \left(\frac{a}{b} \right)^x dx - \int 2\, dx + \int \left(\frac{b}{a} \right)^x dx$$

$$= \frac{\left(\frac{a}{b} \right)^x}{\log \left(\frac{a}{b} \right)} - 2x + \frac{\left(\frac{b}{a} \right)^x}{\log \left(\frac{b}{a} \right)} + c$$

Examples 6 : Evaluate $\int (e^x + \sin x + \sqrt{x}\ ) dx$

Solution : $\int (e^x + \sin x + \sqrt{x}\ ) dx$

$$= \int e^x dx + \int \sin x\, dx + \int \sqrt{x}\, dx$$

$$= \int e^x - \text{Cos } x + \frac{x^{1/2} + 1}{\frac{1}{2} + 1} + c$$

$$= e^x - \text{Cos } x + \frac{2}{3} x^{3/2} + c$$

Examples 6 : Evaluate $\int (\text{Cosec}^2 x - \text{Cos } x + \frac{1}{x}) dx$

Solution : $\int (\text{Cosec}^2 x - \text{Cos } x + \frac{1}{x}) dx$

$$= \int \text{Cosec}^2 x\, dx - \int \text{Cos } x\, dx + \int \frac{1}{x} dx$$

$$= \int Cosec^{2} x\, dx - \int Cos\, x\, dx + \int \frac{1}{x} dx$$

$$= \int -Cot\, x - Sin\, x + \log x + c$$

Examples 7 : Evaluate $\int (8e^{x} - 4a^{x} + 3\cos x + \sqrt[4]{x})\, dx$

Solution : $\int (8e^{x} - 4a^{x} + 3\cos x + \sqrt[4]{x})\, dx$

$$= \int 8e^{x} dx - \int 4a^{x} dx + \int 3\cos x\, dx + \int x^{1/4}\, dx$$

$$= 8\int e^{x} dx - 4\int a^{x} dx + 3\int \cos x\, dx + \int x^{1/4}\, dx$$

$$= 8e^{x} - 4\frac{a^{x}}{\log a} + 3\sin x + \frac{x^{1/4 + 1}}{\frac{1}{4} + 1} + c$$

$$= 8e^{x} - \frac{4a^{x}}{\log a} + 3\sin x + \frac{4}{5} x^{\frac{5}{4}} + c$$

Examples 8 : Evaluate $\int \left(\frac{a + b\sin x}{\cos^{2} x} \right) dx$

Solution : $\int \left(\frac{a + b\sin x}{\cos^{2} x} \right) dx$

$$= \int \left(\frac{a}{\cos^{2} x} + \frac{b\sin x}{\cos^{2} x} \right) dx$$

$$= \int (a\sec^{2} x + b\sec x \tan x)\, dx$$

$$= a\int \sec^{2} x + b\int \sec x \tan x\, dx$$

$$= a\tan x + b\sec x + c$$

Examples 9 : Evaluate $\int \frac{\sin x}{1+\sin x} dx$

Solution : $\int \frac{\sin x}{1+\sin x} dx$

$$= \int \frac{\sin x\,(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= \int \frac{\sin x - \sin^2 x}{1-\sin^2 x} dx$$

$$= \int \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx$$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \int (\sec x \tan x - \tan^2 x) dx$$

$$= \int \sec x \tan x\, dx - \int (\sec^2 x - 1)\, dx$$

$$(\because \sec^2 x \tan x = 1)$$

$$= \int \sec x \tan x\, dx - \int \sec^2 x\ dx + \int 1 dx$$

$$= \text{Sec } x - \tan x + x + c$$

14.4 EXERCISE - 1 :

Evaluate

1. $\int 3x^4\, dx$

2. $\int 7x^{3/2}\, dx$

3. $\int \left(2x^{3/2} + 4x + 5\right) dx$

4. $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

5. $\int\left(1+\frac{2}{x}+\frac{3}{x^2}\right)dx$

6. $\int\left(\frac{x^3+2x^2-4x+5}{x^2}\right)dx$

7. $\int\frac{(2x+3)^2}{3x}dx$

8. $\int(2x+3)(4x-1)\,dx$

9. $\int\left(1-x^2\right)^2 dx$

10. $\int\left(x-\frac{1}{x}\right)^3 dx$

11. $\int\sqrt[5]{x^3}\,dx$

12. $\int\left(\frac{3}{\sqrt{x}}-\frac{2}{x}+\frac{1}{3x^2}\right)dx$

13. $\int\left(\frac{1-\sqrt{x}}{x}\right)dx$

14. $\int\left(\frac{1}{\sqrt{x}}+\frac{7}{x^{3/2}}+\frac{3}{2x^2}\right)dx$

15. $\int\left(\frac{2\cos x}{5\sin^2 x}+\frac{1}{5\cos^2 x}\right)dx$

16. $\int\sqrt{1+\sin 2x}\,dx$

17. $\int\left(2x+\frac{1}{2}e^{-x}+\frac{4}{x}-\frac{1}{\sqrt[3]{x}}\right)dx$

14.5 INTEGRATION BY SUBSTITUTION :

Theorem : If $\int f(x)\,dx = g(x)$ and $a \neq 0$ then $\int f(ax+b)\,dx = \frac{1}{a}$

Proof :

Put ax + b = t

$\Rightarrow \frac{d}{dx}(ax+b) = \frac{d}{dx}(t)$

$\Rightarrow a+0 = \frac{dt}{dx}$

$\Rightarrow a.dx = dt$

$\Rightarrow dx = \frac{1}{a}dt$

$\therefore \int f\,(ax+b)\,dx$

$= \int f\,(t)\,.\,\frac{1}{a}dt$

$= \frac{1}{a}\int f\,(t)\,dt$

$= \frac{1}{a}\int g\,(t) + c$

$= \frac{1}{a}g\,(ax+b) + c$

Theorem : $\int \tan dx = \log|\sec x| + c$

Proof : $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Put cos x = t Differentiate

- sin x dx = dt

sin dx = - dt

$= \int \frac{-dt}{t}$

$= - \log(t) + c$

$= \log(\frac{1}{t}) + c$

$= \log\left(\frac{1}{\text{Cos } x}\right) + c$

$= \log|\sec x| + c$

Theorem : $\int \cot dx = \log|\sin x| + c$

Proof : $\int \cot dx = \int \frac{\cos x}{\sin x} dx$

Put sin x = t Differentiate

cos x dx = dt

$= \int \frac{dt}{t}$

$= \log(t) + c$

$= \log|\sin x| + c$

Theorem : $\int \sec x \, dx = \log|\sec x - \tan x|$

Proof : $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

Put sec x - tan x = t Differentiate

$(\sec x \tan x + \sec^2 x) \, dx = dt$

$\sec x(\tan x + \sec x) \, dx = dt$

$= \int \frac{dt}{t}$

$= \log(t) + c$

$= \log|\sec x + \tan x| + c$

Theorem : $\int \text{cosec } x \, dx = \log\left|\text{cosec } x - \cot x\right| + c$

Proof : $\int \text{cosec } x \, dx = \int \frac{\text{cosec } x\,(\text{cosec } x - \cot x)}{\text{cosec } x - \cot x} dx$

Put cosec x - cot x = t Differentiate

$(-\text{cosec } x \cot x + \text{cosec}^2 x)\ dx = dt$

$\text{cosec } x\,(\text{cosec } x - \cot x)\, dx = dt$

$= \int \frac{dt}{t}$

$= \log(t)$

$= \log\left|\text{cosec } x - \cot x\right| + c$

14.6 SOLVED EXAMPLES :

Examples 1 : Evaluate $\int \text{Sin}\,(2x+3)\,dx$

Sol : Put 2x + 3 = t Differentiate

$(2 + 0)\ dx = dt$

$dx = \frac{dt}{2}$

$\therefore \int \text{Sin}\,(2x+3)\,dx$

$= \int \text{Sin } t \ \frac{dt}{2}$

$= \frac{1}{2}\int \text{Sin } t \ dt$

$= \frac{1}{2}(-\cos t) + c$

$= -\frac{1}{2}\cos(2x+3) + c$

Examples 2 : Evaluate $\int \frac{dx}{\sqrt{1+5x}}$

Sol : Put 1 + 5x = t Differentiate

$$(0+5)\ dx = dt$$

$$dx = \frac{dt}{5}$$

$$\therefore \int \frac{dx}{\sqrt{1+5x}}$$

$$= \int \frac{\frac{dt}{5}}{\sqrt{t}}$$

$$= \frac{1}{5}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{5} 2.\sqrt{t} + c$$

$$= \frac{2}{5}\sqrt{1+5x} + c$$

Examples 3 : Evaluate $\int \frac{e \log x}{x} dx$

Sol : Put log x = t Differentiate

$$\frac{1}{x}\ dx = dt$$

$$\therefore \int \frac{e \log x}{x} dx$$

$$= \int e^{t}\ dt$$

$$= \int e^{t} + c$$

$$= \int e^{\log x} + c$$

Examples 4 : Evaluate $\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx$

Sol : Put log $\sqrt{x}$ = t Differentiate

$$\frac{1}{2\sqrt{x}}\, dx = dt$$

$$\frac{1}{\sqrt{x}}\, dx = 2.\, dt$$

$$\therefore \int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx$$

$$= \int \cot t.\, 2\, dt$$

$$= 2\int \cot t\, dt$$

$$= 2 \sin t + c$$

$$= 2 \sin \sqrt{x} + c$$

Examples 5 : Evaluate $\int \sec (\tan x) \sec^2 x\, dx$

Sol : Put log tan x = t Differentiate

$$\sec^2 dx = dt$$

$$\therefore \int \sec (\tan x) \sec^2 x\, dx$$

$$= \int \sec t.\ dt$$

$$= \log \left| \sec t + \tan t \right| + c$$

$$= \log \left| \sec (\tan x) + \tan (\tan x) \right| + c$$

Examples 6 : Evaluate $\int \frac{\operatorname{cosec}^2 x}{a+b\cot x}\,dx$

Sol : Put a+b cot x = t Differentiate

$\Rightarrow [0 + b(-\operatorname{cosec}^2 x)]\ dx = dt$

$\Rightarrow -b\ \operatorname{cosec}^2 x\ \ dx = dt$

$\Rightarrow \operatorname{cosec}^2 x\ \ dx = \frac{dt}{-b}$

$\therefore \int \frac{\operatorname{cosec}^2 x}{a+b\cot x}\,dx$

$= \int \frac{\frac{dt}{-b}}{t}$

$= \frac{-1}{b}\int \frac{dt}{t}$

$= \frac{-1}{b}\log|t| + c$

$= \frac{-1}{b}\log|a+b\cot x| + c$

Examples 7 : Evaluate $\int \frac{2x+3}{\sqrt{x^2+3x-4}}\,dx$

Sol : Put $x^2 + 3x - 4 = t$ Differentiate

$(2x+3)\,dx = dt$

$\therefore \int \frac{2x+3}{\sqrt{x^2+3x-4}}\,dx$

$= \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + c$

$= 2\sqrt{x^2+3x-4} + c$

Examples 8 : Evaluate $\int \left[\dfrac{\sec x}{(1-\tan x)^2}\right]^2 dx$

Sol : Put $1-\tan x = t$ Differentiate

$(0 - \sec^2 x)\,dx = dt$

$\sec^2 x \; dx = -\,dt$

$$\therefore \int \left[\frac{\sec x}{(1-\tan x)^2}\right]^2 dx$$

$$= \int \frac{\sec x}{(1-\tan x)^4} dx$$

$$= \int \frac{-dt}{t^4}$$

$$= \int \frac{dt}{t^4}$$

$$= - \left[\frac{-1}{3t^3}\right] + c$$

$$= \frac{1}{3t^3} + c$$

$$= \frac{1}{3(1-\tan x)^3} + c$$

14.7 EXERCISE - 2 :

1. $\int \dfrac{1}{5x-6} dx$
2. $\int (7x-4)^{3/4} dx$
3. $\int \dfrac{dx}{\sqrt{11-5x}}$
4. $\int \dfrac{6x}{3x^2-2} dx$

5. $\int \frac{2x - 3}{x^2 - 3x + 4}\, dx$

6. $\int \frac{1}{x \log x}\, dx$

7. $\int x^3 . \sin^4\, dx$

8. $\int 2x . e^{x^2}\, dx$

9. $\int 2x . \cos(1 + x^2)\, dx$

10. $\int e^x . \sin(e^x)\, dx$

11. $\int \left(3x^2 - 4\right) x\, dx$

12. $\int \frac{1}{1 + (2x + 1)}\, dx$

13. $\int \frac{\log x}{x}\, dx$

14. $\int \frac{\sin\left(\log x\right)}{x}\, dx$

15. $\int \frac{\cos\left(\log x\right)}{x}$

16. $\int \frac{\log\left(1 + x\right)}{1 + x}\, dx$

17. $\int \frac{\left(1 + \log x\right)^n}{x}\, dx$

18. $\int \cos^3 x \sin x\, dx$

19. $\int \sqrt[3]{\sin x} . \cos x\, dx$

20. $\int \tan^5 x . \sec^2 x\, dx$

20. $\int \tan^5 x.\sec^2 x\, dx$

21. $\int \operatorname{cosec}^5 x.\sqrt{\cot x}\, dx$

22. $\int \frac{\cos x}{(1+\sin x)^2} dx$

23. $\int \frac{\sec^2 x}{(1+\tan x)^4} dx$

24. $\int \frac{\operatorname{cosec}^2 x}{(a+b\cot x)^6} dx$

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Lesson - 15

DEFINITE INTEGRALS

OBJECTIVES:

By the study of this lesson you will be able to understand the meaning of definite integrals, upper and lower bounds of a function, definite integral as the limit of a sun with examples.

STRUCTURE:

15.1 DEFINITE INTEGRALS :

Upper and Lower Bounds of a function :

Let s be a subset of the domain of f. If there exists a number M such that $f(x) \leq M$ for every x in s, then we say that f is bounded above and M is called an upper bound. If there exists a number m such that $m \leq f(x)$ for every x in S, then we say that f is bounded below and m is called a lower bound. If a function is both bounded above and below then we say that f is bounded.

Definit integral as the limit of a sun :

Definition :

Let f be a bounded function in an interval [a,b] contained in the domain of f. Let $x_0, x_1, x_2, ..., x_{r-1}, x_r, ...x_{n-1}, x_n$ be numbers such that

$a = x_0 < x_1 < x_2 < ......, x_{r-1} < x_r < .....< x_{n-1} < x_n = b$

Let δ_r be the length of the r^{th} interval. $[x_{r-1}, x_r]$ $\delta_r = x_r - x_{r-1}$ m_r and M_r are the lower and upper bounds in the r^{th} interval.

$$\Rightarrow m_r \leq M_r \Rightarrow m_r \delta_r \leq M_r \delta_r \Rightarrow \sum_{r=1}^{n} m_r \delta_r \leq \sum_{r=1}^{n} M_r \delta_r .$$

Now as $n \to \infty$ every $\delta_r \to 0$ and if $\sum_{r=1}^{n} m_r \delta_r$ and $\sum_{r=1}^{n} M_r \delta_r$. approach the same limit, then f is said to be Rienann integrable or simply integrable. If f(x) is integrable in [a,b], then the limit is called the definite integral of f from a to be and its denoted by $\int_a^b f(x)dx.$

Definition :

Let f(x) be a function defined on [a,b]. If $\int f(x)dx$ = F(x), then F(b) - F(a) is called the definite integral of f(x) over [a,b]. It is denoted by $\int_a^b f(x)\, dx$. Where a is called lower limit and b is called upper limit.

Ex : $\int_a^b \sin\ x\ dx = [-\cos x]_a^b$

= (- cos b) - (- cos a)

= cos a - cos b

Ex : $\int_2^4 x^4\ dx > [\frac{x^5}{5}]_2^4$

$= \frac{4^5}{5} - \frac{2^5}{5}$

$= \frac{102}{5} - \frac{32}{5}$

$= \frac{992}{5}$

15.2 SOLVED EXAMPLES :

Examples 1 :

Evaluate $\int_0^1 (2x^6 + 4x^3 + 3)\ dx$

Solution : $\int_0^1 (2x^6 + 4x^3 + 3)\ dx$

$$= \left[\frac{2x^{6+1}}{6+1} + \frac{4x^{3+1}}{3+1} + 3x\right]_0^1$$

$$= \left[\frac{2x^7}{7} + \frac{4x^4}{4} + 3x\right]_0^1$$

$$= \left[\frac{2(1)^7}{7} + 1^4 + 3(1)\right] - \left[\frac{2(0)^7}{7} + 0^4 + 3(0)\right]$$

$$= \frac{2}{7} + 1 + 3$$

$$= \frac{2}{7} + 4$$

$$= \frac{30}{7}$$

Examples 2 :

Evaluate $\int_0^{\pi/2} \sin x \, dx$

Solution : $\int_0^{\pi/2} \sin x \, dx$

$$= [- \cos x]_0^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{2}\right] - [-\cos 0]$$

= (-0) – (-1)

= 1

Examples 3 :

Evaluate $\int_1^2 e^{5x-4} \, dx$

Solution : $\int_1^2 e^{5x-4} \, dx$

$$= \left[\frac{e^{5x-4}}{5+1}\right]_1^2$$

$$= \left[\frac{e^{5x-4}}{6}\right]_0^1$$

$$= \frac{e^{5(1)-(4)}}{6} - \frac{e^{5(0-4)}}{6}$$

$$= \frac{e^{5-4}}{6} - e^0$$

$$= \frac{e}{6} - 0$$

Examples 4 :

Evaluate $\int_1^2 \left(\frac{x^2+2x+1}{x}\right) dx$

Solution : $\int_1^2 \left(\frac{x^2+2x+1}{x}\right) dx$

$$= \int_1^2 \left(x+2+\frac{1}{x}\right) dx$$

$$= \left[\frac{x^2}{2} + 2x + \log x\right]_1^2$$

$$= \left[\frac{2^2}{2} + 2.2 + \log 2\right] - \left[\frac{1^2}{2} + 2.1 + \log 1\right]$$

$$= [2+4+\log 2] - \left[\frac{1}{2} + 2 + 0\right] \qquad (\because \log 1 = 0)$$

$$= \frac{7}{2} + \log 2$$

Examples 5 : Evaluate $\int_2^3 \frac{2x}{1+x^2}\,dx$

Solution : Put $1 + x^2 = t$

Differentiate

$(0+2x)\,dx = dt$

$$\therefore \int \frac{2x\,dx}{1+x^2} = \left[\int \frac{dt}{t}\right]$$

$= [\log t]$

$= \log(1+x^2)$

$$\therefore \int_2^3 \frac{2x\,dx}{1+x^2} = \left[\log\left(1+x^2\right)\right]_2^3$$

$= \log(1+3^2) - \log(1+2^2)$

$= \log 10 - \log 5$

$= \log\left(\frac{10}{5}\right)$

$= \log 2$

15.3 EXERCISE :

Evaluate

1. $\int_0^1 2x^8\,dx$
2. $\int_1^2 \left(2-3x+x^2\right)dx$
3. $\int_1^2 \frac{(5x+2)^2}{3x}\,dx$
4. $\int_0^{\pi/2} \sec^2 x\,dx$
5. $\int_0^5 e^{2x+3}\,dx$
6. $\int_3^4 \frac{1}{x^3}\,dx$

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Lesson - 16

SYSTEM OF LINEAR EQUATIONS

Objective:

After studying this lesson the student will be in a position to know about system of linear equations, consisting of equations by rank method.

Structure:

This lesson has the following components:

16.1 Introduction:

Suppose we are given data on prices in (Rs. per kg.) of wheat and rice in the months of August and September.

	Wheat	Rice
August	3	2
Sept.	4	3

The family can spend Rs. 80 and Rs. 90 in Auguest and September respectively on wheat and rice. Now if the family wants to purchase the same combination of wheat and rice in August and Sept. the question is "How much wheat and how much rice it can buy in each month?"

Assuming they spent x kg. of wheat and y kg. of rice in each month. Then the amount spent are

$3x + 2y$ in August

and $4x + 3y$ in September

Since the family can spent Rs. 80 in Auguest and Rs. 90 in Sept., we must have

$$\left.\begin{array}{l} 3x + 2y = 80 \\ 4x + 3y = 90 \end{array}\right\} \qquad \text{..........(*)}$$

Solving these equations for x and y we get the required combination. The given data on prices can be written in the matrix form as

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}, \text{ the price matrix}$$

The purchase of the family may be expressed as

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ the required matrix.}$$

Then $AX = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 \\ 4x_1 + 3x_2 \end{pmatrix}$

Writing $B = \begin{pmatrix} 80 \\ 90 \end{pmatrix}$ the equations (*) can now be written as

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 80 \\ 90 \end{pmatrix}$$

$$AX = B$$

In general, the two simultaneous equations in the two variables x_1 and x_2 are

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

can be written in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$AX = B \qquad \text{...........(*)}$$

Similarly the three simultaneous equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}$$

can be written in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\Rightarrow \quad A\,X = B$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Since $|A| \neq 0$, A^{-1} exists

Multiply $(*)$ by A^{-1} we get

$$A^{-1} A X = A^{-1} B, \text{ i.e.,}$$

$$\Rightarrow \quad IX = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B$$

Remarks:

By elementary algebra, we can conveniently express $x_1, x_2, \cdots\cdots, x_n$ interms of $b_1, b_2, \cdots\cdots$ then the co-efficient matrix of this latter system is the inverse A^{-1} of A.

Illustration 1:

$$\begin{aligned} x + 2y - z &= 5 \\ 3x - y + 2z &= 9 \\ 5x + 3y + 4z &= 15 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 5 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 15 \end{pmatrix}$$

2. $$\begin{pmatrix} 3 & -1 & 5 \\ 5 & 3 & -1 \\ -1 & 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$

From this we get the simultaneous equations as

$$3x - y + 5z = 4$$
$$5x + 3y - z = -3$$
$$-x + 5y + 3z = 2$$

16.2 System of Linear Equations:

A system of (simultaneous) equations in which the variables (i.e. the unknowns) occur only in the first degree is said to be linear.

A system of linear equations can be represented in the form $AX = B$. For example, the equations $x - 3y + z = -1$, $2x + y - 4z = -1$, $6x - 7y + 8z = 7$ can be written in the matrix form as

$$\begin{pmatrix} 1 & -3 & 1 \\ 2 & 1 & -4 \\ 6 & -7 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}$$

A X = B

A is called the coefficient matrix. If the matrix A is augmented with thecolumn matrix B, at the end we get the augmented matrix.

$$\begin{pmatrix} 1 & -3 & 1 & \vdots & -1 \\ 2 & 1 & -4 & \vdots & -1 \\ 6 & -7 & 8 & \vdots & 7 \end{pmatrix}$$ denoted by (A, B)

A system of (simultaneous) linear equations is said to be homogeneous if the constant term in each of the equations is zero. A system of linear homogeneous equations can be represented in the form $AX = O$. For example, the equations $3x + 4y - 2z = 0$, $5x + 2y = 0$, $3x - y + z = 0$ can be written in the matrix form as

$$\begin{pmatrix} 3 & 4 & -2 \\ 5 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A X = O

16.3 Consistency of Equations:

A system of equations is said to be consistent if it has at least one set of solution. Otherwise it is said to be inconsistent.

Consistent equations may have

(i) unique solution (that is, only one set of solution) or

(ii) infinite sets of solution

By way of illustration, consider first the case of linear equations in two variables.

The equations $4x - y = 8$, $2x + y = 10$ represent two straight lines intersecting at $(3, 4)$. They are consistent and have the unique solution $x = 3, y = 4$.

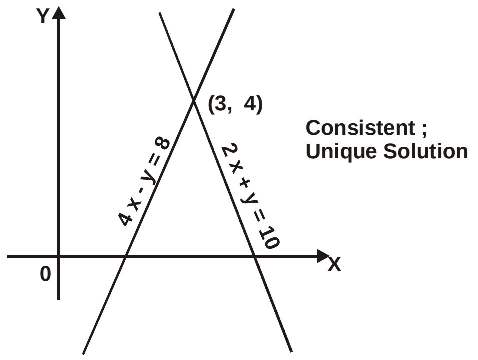

Fig 16.1

The equations $5x - y = 15, 10x - 2y = 30$ represent two coincident lines. We find that any point on the line is a solution. The equations are consistent and have infinite sets of solution such as $x = 1, y = -10$; $x = 3, y = 0$, $x = 4$, $y = 5$ and so on (Fig 16.2). Such equations are called dependent equations.

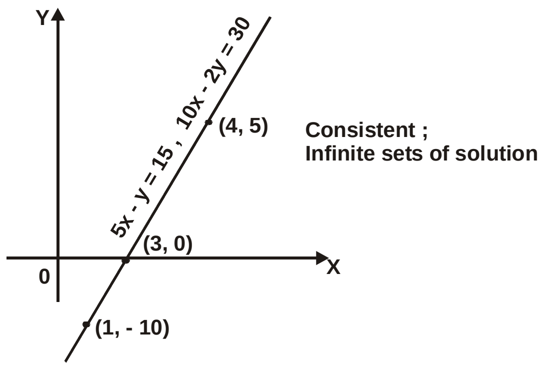

Fig 16.2

The equations $4x - y = 4, 8x - 2y = 5$ represent two parallel straightr lines. The equations are inconsistent and have no solution (Fig 1.3)

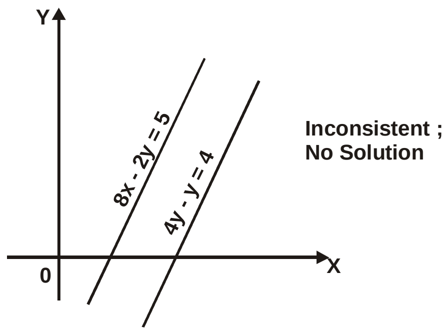

Fig 1.3

Now consider the case of linear euations in three variables. The equations $2x + 4y + z = 5$, $x + y + z = 6, 2x + 3y + z = 6$ are consistent and have only one set of unique solution viz. $x = 2, y = -1, z = 5$. On the other hand, the equations $x + y + z = 1, x + 2y + 4z = 1, x + 4y + 10z = 1$ are consistent and have infinite sets of solution such as $x = 1, y = 0, z = 0;\ x = 3, y = -3, z = 1$ and so on. All these solutions are included in $x = 1 + 2k\ , y = -3k, z = k$ where k is a parameter.

The equations $x + y + z = -3, 3x + y - 2z = -2, 2x + 4y + 7z = 7$ do not have even a single set of solution. They are inconsistent.

All homogeneous equations do have the trivial solution $x = 0, y = 0, z = 0$. Hence the homogeneous equations are all consistent and the question of their being consistent or otherwise does not arise at all.

The homogeneous equations may or may not have solutions other than the trivial solution. For example, the equations $x + 2y + 2z = 0, x - 3y - 3z = 0, 2x + y - z = 0$ have only the trivial solution viz., $x = 0, y = 0, z = 0$. On the other hand the equations $x + y - z = 0, x - 2y + z = 0$, $3x + 6y - 5z = 0$ have infinite sets of solution such as $x = 1, y = 2, z = 3\ ; x = 3, y = 6, z = 9$ and so on. All these non trival solutions are included in $x = t, y = 2, z = 3 ; x = 3, y = 6, z = 9$ and so on. All these non trival solutions are included in $x = t, y = 2t, z = 3t$ where t is a parameter.

16.4 Testing the consistency of equations by rank method:

Consider the equations $AX = B$ in 'n' unknowns:

1. If $\rho(A, B) = \rho(A)$ then the equations are consistent.

2. If $\rho(A, B) \neq \rho(A)$ then the equations are inconsistent.

3. If $\rho(A, B) = \rho(A) = n$ then the equations are consistent and have unique solution.

4. If $\rho(A, B) = \rho(A) < n$ then the equations are consistent and have infinite sets of solution.

Consider the equations AX = 0 in 'n' unknowns:

1) If $\rho(A) = n$ then equations have the trivial solution only.

2) If $\rho(A) < n$ then equations have the non trivial solutions also.

16.5 Solved Problems:

Problem 1:

Show that the equations $2x - y + z = 7, 3x + y - 5z = 13$, $x + y + z = 5$ are consistent and have unique solution.

Solution:

The equations take the matrix form as

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \\ 5 \end{pmatrix}$$

$$A \qquad X = B$$

Now $(A, B) = \begin{pmatrix} 2 & -1 & 1 & \vdots & 7 \\ 3 & 1 & -5 & \vdots & 13 \\ 1 & 1 & 1 & \vdots & 5 \end{pmatrix}$

Applying $R_1 \leftrightarrow R_3$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 3 & 1 & -5 & \vdots & 13 \\ 2 & -1 & 1 & \vdots & 7 \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & -2 & -8 & \vdots & -2 \\ 0 & -3 & -1 & \vdots & -3 \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{3}{2} R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & -2 & -8 & \vdots & -2 \\ 0 & 0 & 11 & \vdots & 0 \end{pmatrix}$$

Obviously

$\rho(A, B) = 3, \rho(A) = 3$

The number of unknowns is 3.

Hence $\rho(A, B) = \rho(A) =$ the number of unknowns.

$\therefore$ The euations are consistent and have unique solution.

Problem 2:

Show that theequations $x + 2y = 3,\ y - z = 2, x + y + z = 1$ are consistent and have infinite sets of solution.

Solution:

The equations take the matrix form as

$$\underset{A}{\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}$$

Now, $$(A, B) = \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 1 & 1 & 1 & \vdots & 1 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -2 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 2 & 0 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

obviously

$$\rho(A, B) = 2, \rho(A) = 2$$

The number of unknowns is 3.

Hence $\rho(A, B) = \rho(A) <$ the number of unknowns.

$\therefore$ The equations are consistent and have infinite sets of solution.

Problem 3:

Show that the equations $x - 3y + 4z = 3$, $2x - 5y + 7z = 6$, $3x - 8y + 11z = 1$ are inconsistenet.

Solution:

The equations take the matrix form as

$$\begin{pmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}$$

$$A \quad X = B$$

Now,

$$(A, B) = \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 2 & -5 & 7 & \vdots & 6 \\ 3 & -8 & 11 & \vdots & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$(A, B) \sim \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & -8 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$(A, B) \sim \begin{pmatrix} 1 & -3 & 4 & \vdots & 3 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & -8 \end{pmatrix}$$

Obviously

$$\rho(A, B) = 3, \ \rho(A) = 2$$

Hence $\rho(A, B) \neq \rho(A)$

$\therefore$ The equations are inconsistent.

Problem 4:

Show that the equations $x + y + z = 0,\ 2x + y - z = 0,\ x - 2y + z = 0$ have only the trivial solution.

Solution:

The matrix form of the equations is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \quad X = O$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & -3 & 0 \end{pmatrix}$$

Applying $R_3 \to R_3 - 3R_2$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 9 \end{pmatrix}$$

Obviously

$\rho(A) = 3$

The number of unknowns is 3

Hence $\rho(A) =$ the number of unknowns.

$\therefore$ The equations have only the trival solution.

Problem 5:

Show that the equations $3x + y + 9z = 0, 3x + 2y + 12z = 0, 2x + y + 7z = 0$ have non trivial solutions also.

Solution:

The matrix form of the equations is

$$\begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \quad X = O$$

$$A = \begin{pmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{vmatrix} = 0 \ , \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \neq 0$$

$$\therefore \rho(A) = 2$$

The number of unknowns is 3.

Hence $\rho(A) <$ the number of unknowns.

$\therefore$ Theequations have non trivial solutions also.

Problem 6:

Find k if the equations $2x + 3y - z = 5, \ 3x - y + 4z = 2, \ x + 7y - 6z = k$ are consistent.

Solution:

$$(A, B) = \begin{pmatrix} 2 & 3 & -1 & \vdots & 5 \\ 3 & -1 & 4 & \vdots & 2 \\ 1 & 7 & -6 & \vdots & k \end{pmatrix}, \ A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & -1 & 4 \\ 1 & 7 & -6 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -1 & 4 \\ 1 & 7 & -6 \end{vmatrix} = 0 \ , \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -11 \neq 0$$

Obviously $\rho(A) = 2$

For the equations to be consistent, $\rho(A, B)$ should also be 2.

Hence every minor of (A, B) of order 3 should be zero.

$$\therefore \begin{vmatrix} 3 & -1 & 5 \\ -1 & 4 & 2 \\ 7 & -6 & k \end{vmatrix} = 0$$

Expanding and simplyfying we get k = 8.

Problem 7:

Find k if the equations $x + y + z = 3, x + 3y + 2z = 6, x + 5y + 3z\ k$ are consistent.

Solution:

$$(A,B) = \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 3 & 2 & \vdots & 6 \\ 1 & 5 & 3 & \vdots & k \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 5 & 3 \end{pmatrix}$$

We find,

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$$

Obviously $\rho(A) = 2$

For the equations to be inconsistent $\rho(A, B)$ should not be 2.

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 3 & 2 & \vdots & 6 \\ 1 & 5 & 3 & \vdots & k \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 3 \\ 0 & 4 & 2 & \vdots & k-3 \end{pmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$(A, B) \sim \begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 2 & 1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & k-9 \end{pmatrix}$$

$\rho(A, B) \neq 2$ only when $k \neq 9$

$\therefore$ The equations are inconsistent when k assumes any real value other than 9.

Problem 8:

Find the value of k for the equations $kx + 3y + z = 0$, $3x - 4y + 4z = 0$, $kx - 2y + 3z = 0$ to ahve non trivial solution.

Solution:

$$A = \begin{pmatrix} k & 3 & 1 \\ 3 & -4 & 4 \\ k & -2 & 3 \end{pmatrix}$$

For the homogeous equations to have non trivial solution, $\rho(A)$ should be less than the number of unknowns viz., 3.

$$\therefore \rho(A) \neq 3$$

Hence $\begin{vmatrix} k & 3 & 1 \\ 3 & -4 & 4 \\ k & -2 & 3 \end{vmatrix} = 0$

Expanding and simplifying we get $k = \frac{11}{4}$

Problem 9:

Find k if the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y + kz = 0$ have only the trivial solution.

Solution:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{pmatrix}$$

For the homogeneous equations to have only the trivial solution, $\rho(A)$ should be equal to the number of unknowns viz., 3.

$$\therefore \begin{vmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{vmatrix} \neq 0 \ , \quad k \neq 1$$

The equations have only the trivial solution when k assumes any real value other than 1.

16.6 Exercise:

1. Find the rank of each of the following matrices.

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 4 & 2 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \\ 3 & 6 & 6 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

(iv) $\begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -2 & -4 \end{pmatrix}$ (vi) $\begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ -1 & -3 & -4 & -7 \end{pmatrix}$

(vii) $\begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 3 \end{pmatrix}$ (viii) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (ix) $\begin{pmatrix} 9 & 6 \\ -6 & 4 \end{pmatrix}$

2. Find the ranks of A + B and AB where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$$

3. Prove that the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear

If the rank of the matrix $\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$ is less than 3.

4. Show that theequations $2x + 8y + 5z = 5$, $x + y + z = -2$, $x + 2y - z = 2$ are consistent and have unique solution.

Lesson Writer

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