# ANALYSIS

## (DM02)

## (MSC MATHEMATICS)


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Lesson : 1

# THE REAL NUMBER SYSTEM

## 1.0 Introduction :

We know that a rational number is defined as $\frac{m}{n}$ where $m, n$ are integers and $n \neq 0$. We also know that the set $Q$ of all rational numbers is an ordered field (in the sense of Definition 1.2.5).

We now observe that the equation

$$p^2 = 2$$

is not satisfied by any rational $p$. Suppose there is a rational $p = \frac{m}{n}$ satisfying $p^2 = 2$. Without loss of generality, we can assume that both $m$ and $n$ are not even. So, $m^2 = 2n^2$. This shows that $m^2$ is even. Hence $m$ is even (otherwise $m^2$ is odd). So, $m^2$ is divisible by 4 so that $n^2$ is even and hence $n$ is even. So, both $m$ and $n$ are even, a contradiction to the choice of $m$ and $n$. Hence there is no rational number $p$ satisfying $p^2 = 2$.

Now consider the sets

$$A = \{p \in Q / p > 0,\ p^2 < 2\} \text{ and}$$

$$B = \{p \in Q / p > 0,\ p^2 > 2\}$$

we, now, show that neither $A$ contains greatest element nor $B$ contains least element. Let $p \in Q$ be such that $p > 0$.

Let $q = p - \frac{p^2 - 2}{p+2} = \frac{2p+2}{p+2}$.

Then $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$

Now,

$$p \in A \Rightarrow p \in Q,\ p > 0,\ p^2 < 2$$

$$\Rightarrow q \in Q,\ q > 0,\ q^2 < 2,\ q > p$$

$$\Rightarrow q \in A,\ q > p$$

$$p \in B \Rightarrow p \in Q,\ p > 0,\ p^2 > 2$$

$$\Rightarrow p \in Q,\ p > 0,\ q^2 > 2,\ q < p$$

$$\Rightarrow q \in B,\ q < p.$$

Thus, $A$ contains no greatest element and $B$ contains no least element.

The purpose of the above discussion to exhibit certain gaps in the rational number system inspite of the fact that between any two rationals, there is another rational. The real number system fill these gaps.

We also study the extended real number system, the field of complex numbers and Schwarz inequality.

Further, we study "sets and functions" which play an important role in the study of Modern Mathematics, in particular, this study is useful in the study of countable sets.

## 1.1 ORDERED SETS

**1.1.1 Definition :** Let $S$ be a set. An order on $S$ is a binary relation $<$ on $S$ satisfying the following properties.

(i) If $x \in S$ and $y \in S$ then one and only one of the statements.

$x < y,\ x = y,\ y < x$ is true.

(ii) If $x, y, z \in S$ and if $x < y$ and $y < z$ then $x < z$.

**1.1.1.1 Note :** The condition 1.1.1(i) is called the law of Trichotomy.

**1.1.2 Definition :** Any pair $(S, <)$ where $S$ is a set and $<$ is an order on $S$ is called an ordered set.

**1.1.3 Definition :** Let $(S, <)$ be an ordered set. Let $E \subseteq S$.

(i) An element $\beta \in S$ is called an upper bound of $E$ if $x \le \beta$ for all $x \in E$.

(ii) An element $\beta \in S$ is called a least upper bound of $E$ (briefly lub $E$) or Supremum of $E$ (briefly $\sup E$) if

(a) $\beta$ is an upper bound of $E$ and

(b) $\alpha \in S,\ \alpha < \beta$ implies $\alpha$ is not an upper bound of $E$.

In this case, we write $\beta = \text{lub } E$ or $\beta = Sup\ E$.

(iii) An element $r \in S$ is called a lower bound of $E$ if $r \le x$ for all $x \in E$.

(iv) An element $r \in S$ is called a lower bound of $E$ (briefly glb $E$) or Infimum of $E$ (briefly $Inf\ E$) if

(a) $r$ is a lower bound of $E$ and

(b) $\delta \in S,\ r < S$ implies $\delta$ is not a lower bound of $E$.

In this case, we write $r = \text{glb } E$ or $r = Inf\ E$.

(v) $E$ is said to be bounded above if $E$ has an upper bound in $S$

(vi) $E$ is said to be bounded below if $E$ has a lower bound in $S$.

(vii) An element $x \in E$ is called least element $E$ if $x \le s$ for all $s \in E$.

(viii) An element $y \in E$ is called greatest (largest) element of $E$ if $s \le y$ for all $s \in E$.

**1.1.4 Lemma :**

(a) The second condition ----- " $\alpha \in S,\ \alpha < \beta \Rightarrow \alpha$ is not an upper bound of $E$", in the definition of lub $E = \beta$ is equivalent to" $\alpha \in S,\ \alpha$ is an upper bound of $E \Rightarrow \beta \le \alpha$ ".

(b) The second condition ------" $\delta \in S,\ r < \delta \Rightarrow \delta$ is not a lower bound of $E$", in the definition of glb $E = r$ is equivalent to "$\delta \in S,\ \delta$ is a lower bound of $E \Rightarrow \delta \le r$ ".

**Proof :** Clear (We leave this as an exercise).

**1.1.5 Definition :** An ordered set $(S,<)$ is said to have the least upperbound property (briefly lub property) if the following is true :

If $E \subseteq S,\ E \ne \phi$ and $E$ is bounded above, then $\sup E\ (= \text{lub } E)$ exists in $S$.

Similarly, we define glb property.

**1.1.6 Definition :** An ordered set $(S,<)$ is said to have the greatest lower bound property (briefly glb property) if the following true :

If $E \subseteq S,\ E \ne \phi$ and $E$ is bounded below, then $Inf\ E\ (= \text{glb } E)$ exists in $S$.

The following theorem relates the lub property and glb property.

**1.1.7 Theorem :** An ordered set $(S,<)$ has the lub property if and only if it has the glb property.

**Proof :** Let $(S,<)$ be an ordered set. Assume that $(S,<)$ has lub property.

Let $E \subseteq S$, $E \neq \phi$ and $E$ be bounded below by $x$ in $S$. Let $L$ be the set of all lower bounds of $E$ in $S$. Clearly, $L \neq \phi$ (since $x \in F$), $L \subseteq S$ and $L$ is bounded above by every element of $E$. Since $(S,<)$ has lub property, $\sup L = s$ exists in $S$.

Now, we show that $s = \text{glb}\, E$. Since every element of $E$ is an upper bound of $L$, we have that $s$ is a lower bound of $E$. Clearly, every lower bound of $E$ is less than or equal to $(\leq)$ $s$. So, $s = \text{glb}\, E$. Hence, $(S,<)$ has the glb property.

We leave the converse part as an exercise.

## 1.2 FIELDS

**1.2.1 Definition :** A field is a set $F$ with two operations, called addition and multiplication (respectively denoted by $+$ and $\cdot$), which satisfy the following so-called "field axioms" $(A)$, $(M)$ and $(D)$.

**(A) Axioms for addition :**

(A1) : If $x \in F$ and $y \in F$ then $x+y \in F$

(A2) : Addition is commutative : $x+y=y+x$ for all $x, y \in F$.

(A3) : Addition is associative : $(x+y)+z=x+(y+z)$ for all $x, y, z \in F$

(A4) : $F$ contains an element 0 such that $0+x=x$ for each $x \in F$.

(A5) : To each $x \in F$, there exists an element $-x \in F$ such that $x+(-x)=0$

**(M) Axioms for multiplication :**

(M1) : If $x \in F$ and $y \in F$ then $xy \in F$

(M2) : Multiplication is commutative : $xy = yx$ for all $x, y \in F$

(M3) : Multiplication is associative : $(xy)z = x(yz)$ for all $x, y, z \in F$

(M4) : $F$ contains an element $1 \neq 0$ such that $x1 = x$ for every $x \in F$

(M5) : To each $x \in F - \{0\}$, there exists an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.

**(D) The distributive law :**

$$x(y+z) = xy + xz \text{ for all } x, y, z \in F.$$

**1.2.1.1 Note :** In any field, we write

$x - y,\ \frac{x}{y},\ x+y+z,\ xyz,\ x^2, x^3, 2x, 3x$........ in place of

$x + (-y),\ x.\frac{1}{y},\ (x+y)+z,\ (xy)z,\ x \cdot x,\ x \cdot x \cdot x,\ x+x,\ x+x+x,$........ respectively.

**1.2.2 Theorem :** In any field $F$, the following hold.

(a) $x + y = x + z \Rightarrow y = z$ (Cancellation law)

(b) $x + y = x \Rightarrow y = 0$

(c) $x + y = 0 \Rightarrow y = -x$

(d) $-(-x) = x$

**Proof :** Exercise.

**1.2.3 Theorem :** In any field $F$, the following hold

(a) $x \neq 0,\ xy = xz \Rightarrow y = z$

(b) $x \neq 0,\ xy = x \Rightarrow y = 1$

(c) $x \neq 0,\ xy = 1 \Rightarrow y = \frac{1}{x}$

(d) $x \neq 0 \Rightarrow {}^{1}\!/\!\left({}^{1}\!/\!_{x}\right) = x$ (${}^{1}\!/\!_{x}$ means $\frac{1}{x}$)

**Proof :** Exercise.

**1.2.4 Theorem :** In any field, the following hold

(a) $0x = 0$

(b) $x \neq 0, y \neq 0 \Rightarrow xy \neq 0$

(c) $(-x)y = -(xy) = x(-y)$

(d) $(-x)(-y) = xy$

**Proof :** (a) $0x + 0x = (0+0)x = 0x \Rightarrow 0x = 0$

(b) Suppose $x \neq 0,\ y \neq 0$ and $xy = 0$. Since $x \neq 0$ there exist elements $\frac{1}{x}$ such that $x \cdot \frac{1}{x} = 1$.

$$xy = 0 \Rightarrow \frac{1}{x}(x\ y) = \frac{1}{x}\ 0$$

$$\Rightarrow \left(\frac{1}{x}\ x\right) y = 0$$

$\Rightarrow 1\ y = 0$ i.e. $y = 0$, a contradiction to $y \neq 0$

Hence $xy \neq 0$

(c) $xy + (-x)y = (x + (-x))y = 0\,y = 0$

$$\Rightarrow (-x)y = -(xy)$$

Similarly, we can prove that $x(-y) = -(xy)$

(d) $(-x)(-y) = -(x(-y)) = -[-(xy)] = xy$

**1.2.5 Definition :** An ordered field is a field $F$ which is also an ordered set, such that

(i) $x, y, z \in F,\ y < z \quad \Rightarrow x + y < x + z$

(ii) $x, y \in F, x > 0, y > 0 \Rightarrow xy > 0$

**1.2.6 Definition :** Let $F$ be an ordered field. Let $x \in F$. We say that $x$ is positive if $x > 0$; $x$ is negative if $x < 0$.

**1.2.7 Theorem :** In any ordered field, the following hold

(a) $x > 0 \Leftrightarrow -x < 0$

(b) $x > 0, y < z \Rightarrow xy < xz$

(c) $x < 0, y < z \Rightarrow xy > xz$

(d) $x \neq 0 \Rightarrow x^2 > 0$. In particular, $1 > 0$.

(e) $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$.

**Proof :** Let $F$ be an ordered field

(a) Let $x > 0$. By definition 1.2.5. (i), $x + (-x) > 0 + (-x)$

i.e., $0 > -x$ i.e., $-x < 0$.

$-x < 0 \Rightarrow x + (-x) < x + 0$ (by Definition 1.2.5.(i))

i.e., $0 < x$

(b) Let $x > 0, y < z$

$y < z \Rightarrow y + (-y) < z + (-y)$

$\Rightarrow 0 < z + (-y)$

$\Rightarrow x0 < x(z + (-y))$ (By Definition 1.2.5. (ii))

$\Rightarrow 0 < xz + x(-y)$

$= xz + [-(xy)]$

$\Rightarrow 0 + xy < (xz + [-(xy)]) + xy$ (by Definition 1.2.5. (i))

$\Rightarrow xy < xz + ([-(xy)] + xy)$

$\Rightarrow xz + 0 = xz$

(c) Let $x < 0, y < z$. So, $-x > 0$ and $z + (-y) > 0$

By Definition 1.2.5.(ii), $(-x)(z + (-y)) > 0$

i.e., $(-x)z+(-x)(-y)>0$

i.e., $-(xz)+xy>0$

i.e., $xy>xz$

(d) Let $x\neq 0$. So, either $x>0$ or $x<0$ (since $F$ is an ordered field).

$x<0$ : So, $-x>0$. By Definition 1.2.5 (ii),

$$(-x)(-x)>0 \text{ i.e., } x^2>0$$

$x>0$ : By Definition 1.2.5 (ii), $x^2>0$.

**1.2.8 Definition :** Let $(F,+,\cdot)$ (where + and $\cdot$ denote the operations of addition and multiplication respectively) be a field. A subset $K$ of $F$ is called a subfield of $F$ if the following conditions are satisfied.

(a) $0,1\in K$

(b) $x,y\in K \Rightarrow x-y\in K$

(c) $x,y\in K-\{0\}\Rightarrow x/y\in K$

## 1.3 THE REAL FIELD

Now, we state the existence theorem which is the core of this chapter (with out proof).

**1.3.1 Theorem :** There exists an ordered field $\mathbb{R}$, which has the least upper bound property.

More over, $\mathbb{R}$ contains $Q$, the field of rationals as a subfield.

**1.3.1.1 Note :** The members of $\mathbb{R}$ are called ~~real~~ numbers.

**1.3.2 Theorem :**

(a) If $x\in\mathbb{R}$, $y\in\mathbb{R}$, and $x>0$ then there exists a positive integer $n$ such that $nx>y$.

(b) If $x\in\mathbb{R}$, $y\in\mathbb{R}$ and $x<y$ then there exists a $p\in Q$ such that $x<p<y$.

**Proof :**

(a) Let $x\in\mathbb{R}$, $y\in\mathbb{R}$ and $x>0$. Let

$$A=\{nx/n \text{ is a positive integer}\}.$$

Suppose the conclusion of (a) is false i.e., for every positive integer $n$, $nx \leq y$. So, $y$ is an upper bound of $A$. Thus, $A$ is a non empty subset of real numbers which is bounded above. Since $\mathbb{R}$ has lub property, lub $A = \alpha$ (say) exists in $\mathbb{R}$. Since $x > 0, \alpha - x < \alpha$. So, $\alpha - x$ is not an upper bound of $A$. Hence $\alpha - x < mx$ for some positive integer $m$. Then $\alpha < mx + x = (m+1)x$, $(m+1)x \in A$, a contradiction to that $\alpha$ is an upper bound of $A$. Hence, there exists a positive integer $n$ such that $nx > y$.

(b) Let $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$.

$x < y \Rightarrow y - x > 0$ ( by definition 1.2.5 (i))

By (a), there exists a positive integer $n$ such that

$n(y-x) > 1$ -------- (1)

Since $1 > 0$; by (a), there exist positive integers $m_1$ and $m_2$ such that $m_1 > nx$, $m_2 > -nx$.

Then

$$-m_2 < nx < m_1$$

Hence there is an integer $m$ such that

$-m_2 \leq m \leq m_1$ and $m - 1 \leq nx < m$ ------ (2)

From (1) and (2), we have

$$nx < m \leq nx + 1 < ny.$$

Since $n > 0$,

$$x < \frac{m}{n} < y.$$

By Definition, $\frac{m}{n}$ is a rational number.

**1.3.2.1 Note :** The property (a) of the above theorem is called Archimedian property. The property (b) of the above theorem is nothing but $Q$ is dense in $\mathbb{R}$ (in the sense of Definition 4.1.3 ($\ell$)).

**1.3.2.2. Note :** Any real number which is not rational is called an irrational number i.e., each $x \in \mathbb{R} - Q$ is called an irrational number.

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**13.4 Theorem :** For every real $x>0$ and every integer $n>0$, there is one and only one positive real $y$ such that $y^n = x$.

(This number $y$ is denoted by $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$).

**Proof :** Let $x$ be a real number such that $x>0$. Let $n$ be a positive integer. Let

$$E=\left\{t \in \mathbb{R} / t>0,\ t^n < x\right\}$$

$E \neq \phi$ : Put $t=\dfrac{x}{1+x}$ then $0<t<1$. Hence $t^n < t < x$. So, $t \in E$. Hence $E \neq \phi$.

$E$ is bonded above : Now,

$$t \in \mathbb{R},\ t>1+x \Rightarrow t^n > t > 1+x \Rightarrow t \notin E.$$

So, $t \leq 1+x$ for any $t \in E$ i.e. $1+x$ is an upper bound of $E$.

Thus, $E$ is a non empty subset of $\mathbb{R}$ which is bounded above. Since $\mathbb{R}$ has lub property, lub $E$ exists in $\mathbb{R}$. Let $y = \text{lub } E$. Now, we prove that $y^n = x$.

Clearly,

$$0<a<b \Rightarrow b^n - a^n = (b-a)\left(b^{n-1} + b^{n-2} \cdot a + \dots\dots + a^{n-1}\right) < (b-a)\, n b^{n-1}$$

Case (i) : Assume $y^n < x$ : Choose $h$ such that $0<h<1$ and $h < \dfrac{x-y^n}{n(y+1)^{n-1}}$

Put $a=y,\ b=y+h$. Then

$$(y+h)^n - y^n < hn(y+1)^{n-1} < x - y^n.$$

So, $(y+h)^n < x$ i.e., $y+h \in E$. Since $y = \text{lub } E,\ y+h \leq y$ and hence $h \leq 0$, a contradiction to $h>0$. So, $y^n \nless x$.

**Case (ii) :** Assume $y^n > x$ : Put

$$R = \frac{y^n - x}{n y^{n-1}}$$

Then $0 < R < y$. If $t \geq y - R$,

$$y^n - t^n \leq y^n - (y-R)^n < R\, n\, y^{n-1} = y^n - x \Rightarrow t^n > x \Rightarrow t \notin E.$$

So, $y - R$ is an upper bound of $E$, a contradiction to $y = \text{lub}\, E$

So, $y^n \not> x$

Hence $y^n = x$.

**Uniqueness :** Suppose $y_1$ and $y_2$ are two positive reals such that $y_1^n = x,\ y_2^n = x$.

$0 < y_1 < y_2 \Rightarrow y_1^n < y_2^n$ *i.e.* $x < x$, a Contradiction.

$0 < y_2 < y_1 \Rightarrow y_2^n < y_1^n$ i.e., $x < x$, a Contradiction.

Hence $y_1 = y_2$.

**1.3.4.1 Corollary :** If $a$ and $b$ are positive real numbers and $n$ is a positive integer, then

$$(a \cdot b)^{1/n} = a^{1/n} \cdot b^{1/n}$$

**Proof :** Put $\alpha = a^{1/n},\ \beta = b^{1/n}$. So, $\alpha > 0,\ \beta > 0$ and $\alpha^n = a,\ \beta^n = b$.

Now, $ab = \alpha^n \beta^n = (\alpha\beta)^n$ (Since multiplication is commutative). By the uniqueness assertion of the theorem 1.3.4,

$$(a\, b)^{1/n} = \alpha\beta = a^{1/n}\, b^{1/n}$$

**1.3.5 Decimals :** Let $x > 0$ be real. Let $n_0$ be the largest integer such that $n_0 \leq x$ (such integer exists by Archimedian property of $\mathbb{R}$). Having choosen $n_0, n_1, \ldots\ldots, n_{R-1}$, let $n_R$ be the largest integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots\dots + \frac{n_{R-1}}{10^{R-1}} + \frac{n_R}{10^R} \le x.$$

Let $E$ be the set of these numbers

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots\dots + \frac{n_R}{10^R} \quad (R = 0, 1, 2, \dots..)$$

Then $x = Sup\ E$. The decimal expansion of $x$ is

$$n_0 . n_1 n_2 \dots\dots$$

Conversely, if

$$x = n_0 \cdot n_1 n_2 \dots\dots$$

is an infinite decimal expansion of $x$, then the set of numbers

$n_0 + \frac{n_1}{10} + \dots\dots + \frac{n_R}{10^R} (R = 0, 1, 2, \dots..)$ in bonded above and $x = Sup\ E$.

## 1.4 THE EXTENDED REAL NUMBER SYSTEM

**1.4.1 Definition :** The extended real number system consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the order in $\mathbb{R}$ and define

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$. We write

**1.4.1.1 Note :** If $E$ is any subset of the extended real number system, then $-\infty$ and $+\infty$ are lower and upper bounds of $E$ respectively. Clearly, every nonempty subset of the extended real number system has both lub and glb.

**1.4.1.2 Note :** If $E$ is a nonempty subset of $\mathbb{R}$ which is not bounded above in $\mathbb{R}$, then $Sup\ E = +\infty$ in the extended real number system. Similarly, if $E$ is a nonempty subset of reals which is not bounded below in $\mathbb{R}$. Then $Inf\ E = -\infty$ in the extended real number system. If $E$ is the empty set then $Sup\ E = -\infty$ and $Inf\ E = +\infty$ in the extended real number system. Actually, the empty set is neither bounded above nor bounded below $\mathbb{R}$.

**1.4.1.3 Note :** The extended real number system does not form a field, but it is customary to make the following convertions :

(a) If $x$ is real, then

$$x+\infty=+\infty,\ x-\infty=-\infty,\ \frac{x}{+\infty}=\frac{x}{-\infty}=0.$$

(b) If $x>0$ then

$$x\cdot(+\infty)=+\infty,\ x\cdot(-\infty)=-\infty$$

(c) If $x<0$ then

$$x\cdot(+\infty)=-\infty,\ x\cdot(-\infty)=+\infty.$$

## 1.5 COMPLEX FIELD

**1.5.1 Definition :** A complex number is an ordered pair of real numbers.

**1.5.1.1 Note :** In the above definition ordered means that $(a,b)$ and $(b,a)$ are treated as distinct if $a \neq b$.

**1.5.2 Definition :** Let $x=(a,b)$ be a complex number. Then $a$ and $b$ are called real and imaginary parts of $x$. We write

$$a=\text{Re } x,\ b=\text{Im } x.$$

**1.5.3 Definition :** Let $x=(a,b)$ and $y=(c,d)$ be complex numbers.

(i) We say that $x$ and $y$ are equal and we write $x=y$ if and only if $a=c$ and $b=d$.

(ii) Define $+$ (the addition of complex numbers) and (the multiplication of complex numbers) as

$$x+y=(a+c,\ b+d),\ xy=(ac-bd,\ ad+bc)$$

**1.5.4 Theorem :** The set $\mathbb{C}$ of all complex numbers form a field with respect to the addition $+$ and the multiplication of complex numbers defined in the definition 1.5.3.

**Proof :**

**I. Axioms for addition :** Let $x=(a,b),\ y=(c,d),\ z=(e,f)$

$(A1)$: $x+y\in\mathbb{C}$ is clear.

$(A2)$: Addition is commutative. $x+y=(a+c,\ b+d)$

$$=(c+a,\ d+b)=(c,d)+(a,b)=y+x$$

$(A3)$: Addition is associative : $(x+y)+z=(a+c,\ b+d)+(e,f)$

$$=((a+c)+e,\ (b+d)+f)=(a+(c+e),b+(d+f))$$

$$=x+(c+e,\ d+f)=x+(y+z)$$

$(A4)$: Let $0=(0,0)$. Then $0\in\mathbb{C}$ and $x+0=(a,b)+(0,\ 0)$

$$=(a+0,\ b+0)=(a,\ b)=x$$

$(A5)$: Put $-x=(-a,-b)$. Then $-x\in\mathbb{C}$ and $x+(-x)=(0,0)=0$

**II. Axioms for multiplication :** Let $x=(a,b),\ y=(c,d),\ z=(e,f)$

$(M1)$ $x\cdot y\in\mathbb{C}$ (Clear).

$(M2)$ Multiplication is commutative : $xy=(ac-bd,\ ad+bc)$

$$=(ca-db,\ da+cb)=yx$$

$(M3)$ $(xy)z=(ac-bd,\ ad+bc)(e,f)$

$$=((ac-bd)e-(ad+bc)f,\ (ac-bd)f+e(ad+bc))$$

$$=(ace-bde-adf-bcf,\ acf-bdf+ead+ebc)$$

$$=(a(ce-df)-b(de+cf),\ a(de+cf)+b(ce-df))$$

$$=(a,b)(ce-df,\ de+cf)$$

$$=x(yz)$$

$(M4)$ Let $1=(1,0)$. So, $1\in\mathbb{C}$ and $x1=(a,b)(1,0)$

$$=(a\cdot 1-b\cdot 0,\ a\cdot 0+b\cdot 1)=(a,\ b)=x.$$

$(M5)$ Let $x=(a,b)\neq 0$. So at least one of $a$, $b$ is different from 0.

So, $a^2+b^2>0$. Define

$\dfrac{1}{x}=\left(\dfrac{a}{a^2+b^2},\dfrac{-b}{a^2+b^2}\right)$. Then

$$x\cdot\frac{1}{x}=(a,\ b)\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(1,0)=1.$$

**III. (D) Distributive Law :**

$$x(y+z)=(a,b)(c+e,\ d+f)$$

$$=\big(a(c+e)-b(d+f),\ \ a(d+f)+b(c+e)\big)$$

$$=(ac+ae-bd-bf,\ ad+af+bc+be)$$

$$=(ac-bd,\ ad+bc)+(ae-bf,\ af+be)$$

$$=xy+xz$$

Hence $\mathbb{C}$ is a field.

**1.5.5 Theorem :** For any real numbers $a$ and $b$, we have

$$(a,0)+(b,0)=(a+b,\ 0),$$

$$(a,0)\ (b,0)=(ab,0)\quad .$$

**Proof :** Excercise

**1.5.5.1 Note :** We can treat each real number $a$ as the complex number $(a,0)$. So, $\mathbb{R}$ is a subfield of the field of complex numbers.

**1.5.6 Definition :** $i=(0,\ 1)$.

**1.5.7 Theorem :** $i^2 = -1$

**Proof :** $i^2 = (0,1)(0,1) = (0.0 - 1.1,\ 0.1 + 1.0) = (-1,\ 0) = -1$

**1.5.8 Theorem :** If $a$ and $b$ are reals, then $(a,b) = a + ib$.

**Proof :** $a + ib = (a,0) + (0,1)(b,0)$

$$= (a,0) + (0 \cdot b - 1 \cdot 0,\ 0 \cdot 0 + 1 \cdot b)$$

$$= (a,\ 0) + (0,\ b)$$

$$= (a,\ b)$$

**1.5.8.1 Note :** Let $z = a + bi$, $w = c + di$. Then $z = (a,b)$, $w = (c,d)$. So, $z + w = (a+c,\ b+d)$ $= (a+c) + (b+d)i$ and $zw = (ac - bd,\ ad + bc) = (ac - bd) + (ad + bc)i$. Infact,

$$zw = (a+bi)(c+di) = (a+bi)c + (a+bi)di$$

$$= ac + bci + adi + bdi^2$$

$$= ac - bd + i(ad + bc) \quad \text{(since } i^2 = -1\text{)}$$

**1.5.9 Definition :** If $z = a + bi$ is the complex number then the complex number $\overline{z} = a - bi$ is called the conjugate of $z$.

**1.5.10 Theorem :** If $z$ and $w$ are complex numbers, then

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z}\ \overline{w}$

(c) $z + \overline{z} = 2\operatorname{Re} z,\quad z - \overline{z} = 2i \operatorname{Im} z$

(d) $z\overline{z}$ is real and positive (except when $z = 0$)

**Proof :** Let $z = a + bi$, $w = c + di$. So, $\overline{z} = a - bi$, $\overline{w} = c - di$.

(a) $\overline{z + w} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i$

$= a + c - bi - di = a - bi + c - di = \overline{z} + \overline{w}$

(b) $\overline{z}\ \overline{w} = (a, -b)(c, -d) = (ac - (-b)(-d),\ a(-d) + (-b)c)$

$= (ac - bd, -(ad + bc))$

$= \overline{(ac - bd, ad + bc)} = \overline{zw}$

(c) $z + \overline{z} = a + bi + a - bi = 2a = 2 \operatorname{Re} z$

$z - \overline{z} = a + bi - (a - bi)$

$= a + bi - a + bi$

$= 2bi = 2i \operatorname{Im} z$

(d) $z\,\overline{z} = (a, b)(a, -b) = (a \cdot a - b \cdot (-b),\ a(-b) + b \cdot a)$

$= (a^2 + b^2, 0) = a^2 + b^2.$

$z\,\overline{z} = 0 \Rightarrow a^2 + b^2 = 0 \Rightarrow a = b = 0 \Rightarrow z = 0;$

$z = 0 \Rightarrow a = 0 = b \Rightarrow z\,\overline{z} = a^2 + b^2 = 0$

**1.5.11 Definition :** If $z$ is a complex number, then the absolute value of $z$, denoted by $|z|$ is defined as $(z \cdot \overline{z})^{\frac{1}{2}}$.

**1.5.11.1 Note :** (i) If $z = a + bi$, then $|z| = (z \cdot \overline{z})^{\frac{1}{2}} = \sqrt{a^2 + b^2}$;

$|z|^2 = a^2 + b^2$

(ii) $z = a + bi$ is real $\Leftrightarrow b = 0$ and hence $z = a$

$\Leftrightarrow z = \overline{z}$

(iii) $z = a + ib$ is pure imaginary

$\Leftrightarrow a = 0$ and hence $z = ib$

**1.5.12 Theorem :** Let $z$ and $w$ be complex numbers. Then

(a) $|z| > 0$ unless $z=0$, $|0|=0$

(b) $|\overline{z}| = |z|$

(c) $|z\,w| = |z|\,|w|$

(d) $|\text{Re } z| \le |z|$, $|\text{Im } z| \le |z|$

(e) $|z+w| \le |z| + |w|$

**Proof :** Let $z=a+bi$, $w=c+id$

(a) $|z| = \sqrt{a^2+b^2}$. Now,

$z=0 \Rightarrow a=0,\ b=0 \Rightarrow |z|=0$.

$|z|=0 \Rightarrow |z|^2 = a^2+b^2 = 0 \Rightarrow a=b=0 \Rightarrow z=0$.

(b) $|\overline{z}| = |a-bi| = |a+(-b)i| = \sqrt{a^2+(-b)^2}$

$= \sqrt{a^2+b^2} = |z|$

(c) $|z\,w|^2 = zw\,\overline{zw} = zw\,\overline{z}\,\overline{w} = z\,\overline{z}\,w\,\overline{w} = |z|^2\,|w|^2$

$= (|z|\,|w|)^2$.

So, $|z\,w| = |z|\,|w|$

(d) $a^2 \le a^2+b^2 \Rightarrow |a| \le \sqrt{a^2+b^2}$ i.e., $|\text{Re } z| \le |z|$

and $b^2 \le a^2+b^2 \Rightarrow |b| \le \sqrt{a^2+b^2}$ i.e. $|\text{Im } z| \le |z|$

(e) $|z+w|^2 = (z+w)\,\overline{(z+w)}$

$$=(z+w)\left(\overline{z}+\overline{w}\right)$$

$$=(z+w)\overline{z}+(z+w)\overline{w}$$

$$=z\overline{z}+w\overline{z}+z\overline{w}+w\overline{w}$$

$$=|z|^2+\overline{z\overline{w}}+z\overline{w}+|w|^2$$

$$=|z|^2+2\operatorname{Re}\left|z\overline{w}\right|+|w|^2$$

$$\leq|z|^2+2\left|z\overline{w}\right|+|w|^2$$

$$=\left|z^2\right|+2|z|\left|\overline{w}\right|+|w|^2$$

$$=|z|^2+2|z||w|+|w|^2$$

$$=(|z|+|w|)^2$$

Hence, $|z+w| \leq |z|+|w|$

**1.5.13 Notation :** If $x_1, x_2, \ldots\ldots, x_n$ are complex numbers, we write

$$x_1+x_2+\ldots..+x_n=\sum_{i=1}^{n} x_i$$

**1.5.14 Theorem :** If $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n}\left|a_j\right|^2 \sum_{j=1}^{n}\left|b_j\right|^2.$$

(This theorem is called Schwarz inequality).

**Proof :** Put $A=\sum_{j=1}^{n}\left|a_j\right|^2$, $B=\sum_{j=1}^{n}\left|b_j\right|^2$, $C=\sum_{j=1}^{n} a_j \overline{b_j}$.

If $B=0$, then $b_1=b_2=\cdots\cdots\cdots=b_n=0$ and hence the conclusion is clear.

Assume that $B>0$. Now,

$$\sum|Ba_j - Cb_j|^2 = \sum(Ba_j - Cb_j)(B\overline{a_j} - \overline{C}\,\overline{b_j})$$

$$= B^2\sum|a_j|^2 - B\overline{C}\sum a_j\overline{b_j} - BC\sum\overline{a_j}\,b_j + |C|^2\sum|b_j|^2$$

$$= B^2A - B|C|^2$$

$$= B\left(AB - |C|^2\right).$$

So, $B\left(AB - |C|^2\right) \geq 0$.

Since $B > 0$, $AB - |C|^2 \geq 0$

i.e. $|C|^2 \leq AB$ and hence the result.

## 1.6 SETS AND FUNCTIONS

**1.6.1 Definition :** Let $A$ and $B$ be sets.

(i) The set

$A \times B = \{(a,b)/a \in A,\ b \in B\}$ is called the Cartesian product of $A$ and $B$.

(ii) Any subset of $A \times B$ is called a relation from $A$ to $B$.

If $R$ is a relation from $A$ to $B$ and if $(a,b) \in R$ then we write $aRb$.

(iii) Any relation from $A$ to $A$ (i.e. any subset of $A \times A$) is called a binary relation on $A$.

**1.6.2 Definition :** Let $A$ and $B$ be two sets. A relation $f$ from $A$ to $B$ is called a mapping (or a function) from $A$ to $B$ and we write $f : A \to B$ or $A \xrightarrow{f} B$ if the following conditions are satisfied.

(i) to each $a$ in $A$ there exists an element $b$ in $B$ such that $(a,b) \in f$;

(ii) $(a,b_1) \in f,\ (a,b_2) \in f \Rightarrow b_1 = b_2$

**1.6.2.1 Note :** Infact, a function $f$ from $A$ to $B$ can also be defined as a relation from $A$ to $B$ such that to each $a$ in $A$ there is a unique $b$ in $B$ such that $(a,b) \in f$.

**1.6.3 Definition :** Let $f: A \to B$

(i) If $(a,b) \in f$ then we write $b = f(a)$; we call $b$ as the image of $a$ and $a$ is called a preimage of $b$.

(ii) If $X \subseteq A$ and $Y \subseteq B$ then

$f(x) = \{f(x)/x \in X\}$ is called the image of $X$ with respect to $f$; and the set

$f^{-1}(Y) = \{x \in A/f(x) \in Y\}$ is called the inverse image of $Y$ with respect to $f$.

**1.6.3.1 Note :** In order to define a function $f$ from $A$ to $B$, it is enough to specify the image for each element of $A$.

**1.6.4 Definition :** Let $f: A \to B$. $f$ is called

(i) one - one (or one to one or injection) if distinct elements in $A$ have distinct images in $B$.

i.e. $x \neq y$ in $A \Rightarrow f(x) \neq f(y)$ in $B$.

(ii) onto (or surjection) if to each $b$ in $B$, there exists atleast one element $a$ in $A$ such that $f(a) = b$.

(iii) bijection if $f$ is both one - one and onto.

**1.6.5 Theorem :** Let $f: A \to B$. Then

(i) $f$ is one - one if and only if "$f(x) = f(y) \Rightarrow x = y$"

(ii) $f$ is onto if and only if $B = f(A)$.

**Proof :** (i) Assume that $f$ is one - one. Suppose $f(x) = f(y)$. If $x \neq y$ then $f(x) \neq f(y)$ (since $f$ is one - one), a contradiction. So, $x = y$.

Conversely assume that "$f(x) = f(y) \Rightarrow x = y$". Let $x, y$ be in $A$ such that $x \neq y$. If $f(x) = f(y)$ then $x = y$ (by our assumption), a contradiction. So, $f(x) \neq f(y)$. Hence $f$ is one-one.

(ii) Assume that $f$ is onto i.e. to each $b$ in $B$. There exists atleast one $b$ in $B$ such that $b = f(a)$. So, $B \subseteq f(A)$. Clearly $f(A) \subseteq B$. Hence $f(A) = B$.

Conversely assume that $B = f(A)$. Clearly, $f$ is onto.

**1.6.6 Definition :** The mapping $I_A : A \to A$ defined by $i_A(a)=a$ for all $a$ in $A$ is called the identity mapping.

**1.6.6.1 Note :** Clearly, any identity mapping is a bijection.

**1.6.7 Definition :** Let $f$ and $g$ be relations from $A$ to $B$ and from $B$ to $C$ respectively. The composite relation $g \circ f$ from $A$ to $C$ of $f$ and $g$ is defined by

$$g o f = \{(a,c) \in A \times C / \exists b \in B \ni (a, b) \in f \text{ and } (b, c) \in g\}$$

**1.6.8 Theorem :** If $f : A \to B$ and $g : B \to C$ then the composite relation $g o f$ of $f$ and $g$ is a mapping from $A$ to $C$.

**Proof :** Let $f : A \to B$ and $g : B \to C$.

(i) Let $a \in A$. Clearly $(a, f(a)) \in f$ and hence $(f(a), g(f(a))) \in g$. So, $(a, g(f(a))) \in g o f$.

(ii) Let $(a, c_1), (a, c_2) \in g \ o f$. So, there exist elements $b_1, b_2$ in $B$ such that $(a, b_1) \in f$, $(b_1, c_1) \in g$, $(a, b_2) \in f$ and $(b_2, c_2) \in g$. Now $b_1 = b_2$ (since $f$ is a function) and hence $c_1 = c_2$ (since $g$ is a function).

Hence $g \ o \ f : A \to C$.

**1.6.8.1 Note :** Suppose $f : A \to B$ and $g : B \to C$. Then $g o f : A \to C$. Let $a \in A$ and $(g o f)(a) = c$. So, $(a, c) \in g o f$. So, there exists $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$. i.e. $b = f(a)$, $c = g(b)$.

Hence $(g o f)(a) = c = g(b) = g(f(a))$. Thus, the mapping $g o f : A \to C$ is given by $(g \ o \ f)(a) = g(f(a))$.

**1.6.9 Theorem :** Composition of relations is associative.

**Proof :** Let $f, g, h$ be relations from $A$ to $B$; $B$ to $C$ and $C$ to $A$ respectively. Clearly, $g o f \subseteq A \times C$, $h o g \subseteq B \times D$ and hence $h \ o \ (g \ o \ f) \subseteq A \times D$; $(h \ o \ g) \ o \ f \subseteq A \times D$

$$(a, d) \in h \circ (g \circ f) \Leftrightarrow \exists c \in C \ni (a, c) \in g \circ f, (c, d) \in h$$

$$\Leftrightarrow \exists b \in B \ni (a,b) \in f, (b,c) \in g, (c,d) \in h$$

$$\Leftrightarrow (a,b) \in f, \ (b,d) \in h \circ g$$

$$\Leftrightarrow (a, d) \in (h \circ g) \circ f$$

Hence, $h \circ (g \circ f) = (h \circ g) \circ f$

**1.6.9.1 Note :** Suppose $f : A \to B$ and $g : C \to D$. Suppose $f$ and $g$ are equal as sets. Let $a \in A$. Then $(a, f(a)) \in f = g$. So, $a \in C$ and $f(a) = g(a)$. Hence, $A \subseteq C$ and $f(a) = g(a)$ for any $a$ in $A$. Similarly, $C \subseteq A$ and $g(c) = f(c)$ for any $c$ in $C$. Thus, $A = C$ and $f(a) = g(a)$ for any $a$ in $A$.

Now, we are in a position to define the equality of two function.

**1.6.10 Definition :** Two functions $f : A \to B$ and $g : C \to D$ are equal if (i) $A = C$ and (ii) $f(a) = g(a)$ for all $a$ in $A$.

**1.6.11 Theorem :** Compositions of functions is associative.

**Proof :** I : As a corollary of theorem 1.6.9

II : Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Clearly, $g \circ f : A \to C$ and $h \circ g : B \to D$. Hence, $h \circ (g \circ f) : A \to D$ and $(h \circ g) \circ f : A \to D$. For any $a$ in $A$,

$$[h \circ (g \circ f)](a) = h((g \circ f)(a))$$

$$= h(g(f(a)))$$

$$= (h \circ g)(f(a))$$

$$= [(h \circ g) \circ f](a)$$

Hence, $h \circ (g \circ f) = (h \circ g) \circ f$

**1.6.12 Definition :** Write here the definition of $f^{-1}$.

**1.6.13 : Theorem :** Let $f: A \to B$. Then, $f$ is a bijection if and only if $f^{-1}: B \to A$ is a mapping (infact, $f^{-1}$ is a bijection).

**Proof :** Assume that $f$ is a bijection.

**(1) $f^{-1}$ is a mapping from $B$ to $A$ :**

(i) Let $b \in B$. Since $f$ is onto, there exists atleast one $a$ in $A$, such that $b = f(a)$ i.e., $(a, b) \in f$ i.e. $(b, a) \in f^{-1}$.

(ii) Let $(b, a_1), (b, a_2) \in f^{-1}$ i.e. $(a_1, b) \in f$ and $(a_2, b) \in f$ i.e. $f(a_1) = b = f(a_2)$. Since $f$ is one-one, $a_1 = a_2$.

Hence $f^{-1}$ is a mapping.

**(2) $f^{-1}$ is one - one :** $f^{-1}(b_1) = f^{-1}(b_2)$ ( $= a$ say) $\Rightarrow b_1 = f(a) = b_2$. So, $f^{-1}$ is one - one.

**(3) $f^{-1}$ is onto :** Let $a \in A$. So, $(a, f(a)) \in f$ i.e. $(f(a), a) \in f^{-1}$, i.e. $f^{-1}(f(a)) = a$. So $f^{-1}$ is onto.

Hence $f^{-1}: B \to A$ is a bijection.

Conversely assume that $f^{-1}: B \to A$ is a function.

**(1) $f$ is one - one :** $f(a_1) = f(a_2)$ $(= b \text{ say}) \Rightarrow (a_1, b) \in f, (a_2, b) \in f$ $\Rightarrow (b, a_1), (b, a_2) \in f^{-1} \Rightarrow a_1 = a_2$ (since $f^{-1}: B \to A$ is a mapping).

**(2) $f$ is onto :** Let $b \in B$. Put $a = f^{-1}(b) (\in A)$ (since $f^{-1}$ is a mapping). So, $(b, a) \in f^{-1}$ i.e. $(a, b) \in f$ i.e. $b = f(a)$. Hence $f$ is onto.

Thus, $f$ is a bijection.

## Lesson 2

# COUNTABILITY

## 2.0 INTRODUCTION :

In this lesson, we study the concepts - finite sets, countable sets and infinite sets. In lesson 1, we have studied the equivalence of sets and this equivalence is an equivalence relation on any non-empty family of sets (see theorem 1.6.20). We prove some important theorems like - countable union of countable sets is countable (see theorem 2.1.8).

## 2.1 COUNTABILITY

**2.1.1 Definition :** We say that a set $A$ is

(i) finite if $A$ is empty $(i.e.\ A=\phi)$ or $A \sim J_n$ for some positive integer $n$.

(ii) infinite if $A$ is not finite.

(iii) countable or denumerable or enumerable if $A \sim J$.

(iv) atmost countable if $A$ is finite or countable.

(v) uncountable if $A$ is not atmost countable i.e. $A$ is neither finite nor countable.

**2.1.2 Theorem :** Any non-empty finite set can be written as $\{x_1, x_2, ......, x_n\}$ for some positive integer $n$, where $x_i \neq x_j$ whenever $i \neq j$.

**Proof :** Let $A$ be a non-empty finite set. So $A \sim J_n$ for some positive integer $n$. Hence, there is a bijection $f : J_n \to A$. So, $A = f(J_n) = \{f(i)/i \in J_n\} = \{x_i/i \in J_n\}$ (where $f(i) = x_i$) $= \{x_1, x_2, ........, x_n\}$. Since $f$ is one - one, $x_i \neq x_j$ whenever $i \neq j$.

**2.1.2.1 Note :** In view of theorem 2.1.2, if $A \sim J_n$, then we say that $A$ contains $n$ elements.

**2.1.3 Theorem :** Every countable set can be written as

$$\{x_1, x_2, ................\}$$

where $x_i \neq x_j$ whenever $i \neq j$

**Proof :** Let $A$ be a countable set i.e. $A \sim J$ i.e. there exists a bijection $f : J \to A$. So,

$A = f(J)$ (since $f$ is onto)

$= \{f(i)/i \in J\}$

$= \{x_i / i \in J\}$

$= \{x_1, x_2, ..............\}$

Since $f$ is one - one, $x_i \neq x_j$ whenever $i \neq j$.

**2.1.4 Example :** Let $\mathbb{Z}$ be the set of all integers. Define $f : J \to \mathbb{Z}$ as

$$f(n) = \begin{cases} \frac{n}{2} \text{if } n \text{ is even} \\ -\frac{n-1}{2} \text{if } n \text{ is odd} \end{cases}$$

Infact, $f$ is given by

$$\begin{array}{l} J : 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7, ......... \\ f \downarrow : \\ \mathbb{Z} : 0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3........ \end{array}$$

Clearly, $f$ is a bijection. So $J \sim \mathbb{Z}$. Hence $\mathbb{Z}$ is countable.

**2.1.5 Example :** The identity mapping $I_J$ of the set $J$ of all positive integers is a bijection. So, $J \sim J$ i.e. $J$ is countable.

**2.1.6 Lemma :** Let $A \sim B$

(i) If $A$ is finite then $B$ is finite and hence both $A$ and $B$ have the same number of elements (namely, $n$ elements)

(ii) If $A$ is countable, then $B$ is countable.

**Proof :** (i) Suppose $A$ is finite. So, $A \sim J_n$ for some positive integer '$n$'. Since $\sim$ is an equivalence relation and $A \sim B$, we have that $B \sim J_n$. So $B$ is finite. In view of Note 2.1.2.1, both $A$ and $B$ have the same number $n$ of elements.

(ii) Suppose $A$ is countable i.e. $A \sim J$. Since $\sim$ is an equivalence relation and since $A \sim B$, we have that $B \sim J$. So, $B$ is countable.

**2.1.7 Theorem :** Every subset of a countable set is atmost countable. Infact, every infinite subset of a countable set is countable.

**Proof :** Let $A$ be a countable set. By theorem 2.1.3, $A$ can be written as

$$A=\{x_1,\ x_2,\ ............\}$$

where $x_i \neq x_j$ whenever $i \neq j$. Let $B$ be an infinite subset of $A$.

Now, we show that $B$ is countable.

Put $S_1=\{n \in J / x_n \in B\}$. Since $B$ is infinite, $B \neq \phi$. So, $S_1 \neq \phi$. By the well ordering principle, $S_1$ contains least element $n_1$ say. Thus, $x_{n_1} \in B$ and $x_n \notin B$ for $n \not< n_1$.

Put $S_2 = \left\{n \in J \middle/ x_n \in B-\left\{x_{n_1}\right\}\right\}$. Since $B$ is infinite, $B-\left\{x_{n_1}\right\}$ is infinite and hence $S_2 = \phi$. By the well ordering principle, $S_2$ contains least element $n_2$ say. Thus, $x_{n_2} \in B,\ n_1 < n_2$ and $x_n \notin B$ for $n_1 < n < n_2$.

After choosing positive integers $n_1, n_2, ......, n_k$ such that $n_1 < n_2 < ......... < n_k, x_{n_i} \in B(i=1,2,........K)$ and $x_n \notin B$ for $n_i < n < n_{i+1}(i=1,2,.....,K-1)$, we choose $n_{K+1}$ as the least element of

$$S_{K+1} = \left\{n \in J \middle/ x_n \in B-\left\{x_{n_1},\ x_{n_2},\ .....x_{n_K}\right\}\right\}$$

(which exists by the well ordering principle).

Thus, we have that $B = \left\{x_{n_1},\ x_{n_2},\ .......\right\}$. Clearly $i \neq j$ implies $n_i \neq n_j$ (infact $n_i < n_j$ if $i < j$) which implies $x_{n_i} \neq x_{n_j}$. Define $g: J \to B$ by $g(i) = x_{n_i}$. Clearly, $g$ is a bijection. So, $J \sim B$. Hence, $B$ is countable.

**2.1.8 Theorem :** Countable union of countable sets is countable. i.e. If $\{E_n\}_{n=1,2,...}$ is a sequence of countable sets then $S = \bigcup_{n=1}^{n} E_n$ is countable.

**Proof :** Let $\{E_n\}_{n=1,2,.....}$ be a sequence of countable sets. By theorem 2.1.3 each $E_n$ can be written as

$$E_n = \left\{ x_{n_1},\ x_{n_2}, \ldots\ldots\ldots\ldots\ldots\ldots \right\}$$

where $i \neq j$ implies $x_{n_i} \neq x_{n_j}$ we, now write the elements of $E_1, E_2, \ldots\ldots.$ as follows.

2nd arrow

1st arrow     3rd arrow

$$\begin{array}{lllll}
E_1 : & x_{11}, & x_{12}, & x_{13}, & \ldots\ldots\ldots\ldots\ldots\ldots \\
E_2 : & x_{21}, & x_{22}, & x_{23}, & \ldots\ldots\ldots\ldots\ldots\ldots \\
E_3 : & x_{31}, & x_{32}, & x_{33}, & \ldots\ldots\ldots\ldots\ldots\ldots \\
 & & & & \\
E_i : & x_{i_1}, & x_{i_2}, & x_{i_3}, & \ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

Now, the elements in $S = \bigcup_{n=1}^{\infty} E_n$ can be arranged (numbered) as follows :

1st arrow : $x_{11}$

(1) 1

2nd arrow : $x_{21}, \quad x_{12}$

(1) 2 (2) 3

3rd arrow : $x_{31}, \quad x_{22}, \quad x_{13}$

(1) 4, (2) 5, (3) 6

4th arrow : $x_{41}, \quad x_{32}, \quad x_{23}, \quad x_{14}$

(1) 7 (2) 8 (3) 9 (4) 10

.....................................................

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$(i+j-1)$th arrow : $x_{i+j-11} \quad x_{i+j-22}$ ........................ $x_{ij}$ ............... $x_{1\,i+j-1}$

(1) $\ell+1$ (2) $\ell+2$ $(j)$ $\ell+j$ $(i+j)\ell+i+j$

.....................................................

.....................................................

.....................................................

Where $\ell$ = The number of all elements in the first $(i+j-2)$ rows (irrespective of repetetions of $x_{mn}$ s.

$$=1+2+.........+(i+j-2)$$

$$=\frac{(i+j-2)\,(i+j-1)}{2}$$

In this process, the element $x_{ij}$ will be given the positive integer $\frac{(i+j-2)\,(i+j-1)}{2}+j$

Clearly,

(1) $K^{\text{th}}$ arrow contains $K$ elements;

(2) $x_{ij}$ lies in the $(i+j-1)^{\text{th}}$ arrow - as $j^{\text{th}}$ element.

Clearly, each $E_n$ contains distinct elements. But, two distinct $E_n$ s may have a common element. On ommitting the repetetion $-s$ of $x_{ij}$ $s$ in $S$ along with the positive integers associated with them, we get a subset $T$ of positive integers such that $S \sim T$. By Theorem 2.1.7, $T$ is countable. By Lemma 2.1.6(ii), $S$ is countable.

**2.1.8.1 Corollary :** Suppose $A$ is atmost countable and for each $\alpha \in A$, $E_\alpha$ is atmost countable, put

$$S=\bigcup_{\alpha\in A} E_\alpha$$

Then, $S$ is atmost countable.

**Proof :** Since $A$ is atmost countable, either $A=J_n$, for some positive integer $n$ or $A=J$. When $A=J_n$, we can take $\{E_\alpha\}_{\alpha\in A}$ as

$$\{E_\alpha\}_{\alpha\in A}=\{E_1,\,E_2,.............,E_n\}$$

$$=\{E_1,\,E_2,.......E_n,\,E_n,\,E_n,.............\}.$$

When $A=J$, we can take $\{E_\alpha\}_{\alpha\in A}$ as

$$\{E_\alpha\} = \{E_1, E_2, ..........\}$$

By Theorem 2.1.8, there exists a subset $T$ of positive integers such that $S \sim T$. Since $T$ is atmost countable, $S$ is atmost countable (by Lemma 2.1.6.)

**2.1.9 Theorem :** If $A_1, A_2, ..........., A_n$ are countable sets then $A_1 \times A_2 \times ........... \times A_n$ is countable.

**Proof :** We prove this by induction on '$n$'. To each positive integer $n$, write

$p(n)$ : If $A_1, A_2, ........., A_n$ are countable sets then $A_1 \times A_2 \times ........... \times A_n$ is countable.

**Truth of** $P(1)$ **:** Clear

**Truth of** $P(2)$ **:** Suppose $A$ and $B$ are countable. So, we can write

$A = \{a_1, a_2, ............\}$ and

$B = \{b_1, b_2, ...............\}$.

Now $A \times B = \{(a_i, b_j) / i = 1, 2, ......... \& \ j = 1, 2, .........\}$.

Clearly, $A \times B = \bigcup_{i=1}^{\infty} E_i$

Where $E_i = \{a_i\} \times B \ (i = 1, 2, .........)$

Clearly, for each $i$, $E_i \sim B$ and hence $E_i$ is countable (by Lemma 2.1.6 (ii)). By theorem 2.1.8, $A \times B$ is countable.

**Truth of** $P(n)$ $\Rightarrow$ **Truth of** $P(n+1)$ **:** Assume that $P(n)$ is true. Let $A_1, A_2, ........ A_{n+1}$ be countable sets. Since $P(n)$ is true, $A_1 \times A_2 .......... \times A_n = B_n$ is countable. Clearly,

$$A_1 \times A_2 \times .......... \times A_{n+1} \sim B_n \times A_{n+1}.$$

Since $P(2)$ is true, $B_n \times A_{n+1}$ is countable. By Lemma 2.1.6(ii) $A_1 \times A_2 \times \cdots \times A_{n+1}$ is countable i.e. $P(n+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all positive integers $n$.

**2.1.9.1 Corollary :** Let $A$ be a countable set. For any positive integer $n$, $A^n = A \times A \times \ldots\ldots\ldots\ldots \times A$ ($n$ - times) = The set of all $n$-tuples of elements of $A$ is countable.

**Proof :** Exercise

**2.1.9.2 Corollary :** The set $Q$ of all rationals is countable.

**Proof :** From example 2.1.4, the set $\mathbb{Z}$ of all integers is countable. By theorem 2.1.7, $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ is countable. By Theorem 2.1.9, $\mathbb{Z} \times \mathbb{Z}^*$ is countable. Define $f : \mathbb{Z} \times \mathbb{Z}^* \to Q$ by $f(m, n) = m/n$. Clearly, $f$ is onto (by the definition of rational number). By theorem 1.6.16, there exists a $1-1$ function. $g : Q \to \mathbb{Z} \times \mathbb{Z}^*$. So, $Q \sim g(Q)$. Since $g(Q)$ is an infinite subset of the countable set $\mathbb{Z} \times \mathbb{Z}^*$, $g(Q)$ is countable (by Theorem 2.1.7) and hence $Q$ is countable.

Now, we recall the definition of a sequence (see Definition 1.6.22)

**2.1.10 Definition :** By a sequence in a set $A$, we mean any function $f : J \to A$

**2.1.10.1 Note :** Let $f$ be a sequence in $A$. So, $f : J \to A$. We know that $f$ is completely given by specifying the images of elements of $J$. Let $f(i) = x_i$. We can represent $f$ by

$$x_1, x_2, \ldots\ldots\ldots\ldots$$

or

simply $\{x_n\}_{n=1,2,\ldots\ldots\ldots}$

**2.1.11 Theorem :** Let $A$ be the set of all sequences of the digits 0s and 1s. Then $A$ is uncountable.

**Proof :** Suppose $A$ is countable. So, we can write

$$A = \{x_1, x_2, \ldots\ldots\ldots\}$$

where $x_i \neq x_j$ whenever $i \neq j$. Since each $x_i$ is a sequence of 0s and 1s we can write

$$x_1 = x_{11} \;\; x_{12} \;\; x_{13} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_2 = x_{21} \;\; x_{22} \;\; x_{23} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_3 = x_{31} \;\; x_{32} \;\; x_{33} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

..................................................

$x_i = x_{i1}\ \ x_{i2}\ \ x_{i3}$ ..........................

..................................................

..................................................

where each $x_{ij}$ is 0 or 1. Define a sequence

$a = a_1\ a_2\ a_3$ .............

of 0s and 1s such that $a_i \neq x_{ii}$ for all $i$. Clearly, $a \in A$ and $a \neq x_i$ for all $i$, a contradiction. So, $A$ is countable.

**2.1.12 Definition :** Let $a, b \in \mathrm{R}$ be such that $a < b$. The set

$$(a,\, b) = \{x \in \mathrm{R} / a < x < b\}$$

is called a segment (or finite open interval) in the real line.

**2.1.13 Theorem :** The segment (0, 1) is uncountable.

**Proof :** The proof of this theorem is almost on the similar lines of that of theorem 2.1.11.

Assume (0, 1) is uncountable. By theorem 2.1.3, we can write $(0,1) = \{x_1,\ x_2, ..................\}$, where $x_i \neq x_j$ whenever $i \neq j$. Now, we will write decimal representation for $x_1,\ x_2,$........ as given below.

$x_1 = 0 \cdot x_{11}\ \ x_{12}$ ...................

$x_2 = 0 \cdot x_{21}\ \ x_{22}$ ...................

..........................................

$x_i = 0 \cdot x_{i1}\ \ x_{i2}$ ...................

..........................................

..........................................

where each $x_{ij}$ is one of the digits 0, 1, 2, .........., 9.

Take $a = 0 \cdot a_1\ a_2\ a_3$ ................

where $a_1, a_2, .........$ are digits such that $a_i \neq x_{ii}$ for all $i$. Clearly, $a \in (0,1)$ and $a \neq x_i$ for $i$, a contradiction. Hence, $(0,1)$ is uncountable.

**2.1.13.1 Note :** The idea of the proofs of theorems 2.1.11 and 2.1.13 was first used by Cantor and is called Cantor's diagonal process.

## 2.2 SHORT ANSWER QUESTIONS :

**2.2.1 :** Define a countable set.

**2.2.2 :** Prove that the set of all even integers is countable.

**2.2.3 :** Prove that the set of all odd integers is countable.

**2.2.4 :** Define a sequence.

## 2.3 MODEL EXAMINATION QUESTIONS :

**2.3.1 :** Define countable set and prove that the set of all integers is countable.

**2.3.2 :** Prove that the countable union of countable sets is countable.

**2.3.3 :** Prove that every countable set is equal to a proper subset of itself.

**2.3.4 :** If $f : X \to Y$ is onto and $Y$ is countbale, prove that $X$ is countable.

**2.3.5 :** Prove that the set of all sequences of the digits 0 and 1 is uncountable.

**2.3.6 :** Prove that the set of all rational numbers is countable.

**2.3.7 :** If $A$ and $B$ are countable sets, prove that $A \times B$ is countable.

## 2.4 EXERCISES :

**2.4.1 :** Let $a, b \in \mathrm{R}$ be such that $a < b$. Prove that the interval $(a,b) = \{x \in \mathrm{R} / a < x < b\}$ is uncountable.

(Hint : The mapping $x \mapsto \dfrac{x-a}{b-a} : (a,\ b) \to (0,1)$ is a bijection)

**2.4.2 :** Let $f : A \to B$. (i) If $B$ is countable and $f$ is 1 - 1, prove that $A$ is atmost countable

(ii) If $A$ is countable and $f$ is onto, prove that $B$ is atmost countable.

(Hint : (i) $A \sim f(A)$, $f(A) \subseteq B$. Use theorem 2.1.7 and Lemma 2.1.6.

(ii) Use theorems 1.6.16, 2.1.7)

**2.4.3 :** (i) Prove that every superset of an uncountable set is uncountable

(ii) Show that the set R of all reals is uncountable.

(Hint : Use theorem 2.1.13 and (i))

(iii) Prove that the set of all irrational numbers is uncountable.

**2.4.4 :** Prove that countable union of countable sets is countable.

**2.4.5 :** Prove Corollary 2.1.9.1.

## 2.5 ANSWERS TO SHORT ANSWER QUESTIONS :

**2.2.1 :** See definition 2.1.1 (iii)

**2.2.2 :** Define $f : J \to E$, where $E$ is the set of all even integers by

$$\begin{array}{cccccccccl} \mathrm{J} & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots\ldots\ldots\ldots \\ f\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \mathrm{E} & : & 0 & 2 & -2 & 4 & -4 & 6 & -6 & \ldots\ldots\ldots\ldots \end{array}$$

i.e. $f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ -(n-1) & \text{if } n \text{ is odd} \end{cases}$

Clearly, $f$ is a bijection. So, $J \sim E$. Hence $E$ is countable.

**2.2.3 :** Define $f : J \to O$, where $O$ is the set of all odd integers by

$$\begin{array}{ccccccccl} \mathrm{J} & : & 1 & 2 & 3 & 4 & 5 & 6 & \ldots\ldots\ldots\ldots \\ f\downarrow & & & & & & & & \\ \mathrm{O} & : & 1 & -1 & 3 & -3 & 5 & -5 & \ldots\ldots\ldots\ldots \end{array}$$

i.e. $f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ -(n-1) & \text{if } n \text{ is even} \end{cases}$

Clearly, $f$ is a bijection. So, $J \sim O$. Hence $O$ is countable.

**2.2.4 :** See definition 2.1.10

## 2.6 REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin.

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Lesson - 3

# EUCLIDEAN SPACES

## 3.0 INTRODUCTION :

Our main interest of this lesson is to study the Euclidean space $\mathbb{R}^k$. We observe that $\mathbb{R}^k$ is an inner product space (see Example 3.1.3). Infact, $\mathbb{R}^k$ is a normal linear space with respect to the norm induced by the inner product and we study the properties of this norm.

## 3.1 INNER PRODUCT SPACES AND NORMED LINEAR SPACES :

**3.1.1 Definition :** Let $F$ be the field of real numbers or the field of complex numbers. Let $V$ be a vector space over $F$. A mapping $<,\ >: V \times V \Rightarrow F$ is called <u>inner product</u> on $V$ if the following conditions are satisfied.

$(\mathrm{I}_1): <\alpha+\beta, \gamma> = <\alpha, \gamma> + <\beta, \gamma>$

$(\mathrm{I}_2): <c\alpha,\ \beta> = c<\alpha,\ \beta>$

$(\mathrm{I}3): <\beta, \alpha> = \overline{<\alpha,\ \beta>}$

$(\mathrm{I}4): <\alpha, \alpha> >0 \text{ if } \alpha \neq 0$

for all $\alpha, \beta, \gamma$ in $V$ and $c$ in $F$.

**3.1.2 Definition :** An <u>inner product space</u> is a real or complex vector space together with a specified inner product on that space.

**3.1.2.1 Notation :** If $x_1, x_2, \ldots\ldots\ldots\ldots, x_n$ are elements in a field, we write

$$x_1, x_2, \ldots\ldots\ldots\ldots, x_n = \sum_{i=1}^{n} x_i$$

**3.1.3 Example :** Let $F$ be the field of real or complex numbers. Let $F^n$ be the set of all $n-$tuples of elements in $F$.

i.e. $F^n = \{x = (x_1,\ x_2,\ \ldots\ldots, x_n) / x_i \in F\,(1 \leq i \leq n)\}$.

we say that two elements $x=(x_1,x_2,\ldots\ldots\ldots,x_n)$, $y=(y_1,y_2,\ldots\ldots,y_n)$ in $F^n$ are equal and we write $x=y$ if $x_i=y_i\,(1\le i\le n)$.

(1) Define the binary operation + on $F^n$ and the scalar multiplication

$F\times F^n\to F^n$ as follows.

Let $x=(x_1,x_2,\ldots\ldots\ldots,x_n)$, $y=(y_1,y_2,\ldots\ldots\ldots,y_n)\in F^n$ and $a\in F$.

$x+y=(x_1+y_1,\,x_2+y_2,\ldots\ldots\ldots,x_n+y_n)$

$ax=(ax_1,\,ax_2,\ldots\ldots\ldots,ax_n)$.

Then $(F^n,+)$ is a vector space over $F$,

(2) The mapping $<,>:F^n\times F^n\to F$ defined by

$$<x,\,y> = \sum_{i=1}^{n} x_i\,\overline{y_i}$$

where $x=(x_1,x_2,\ldots\ldots\ldots,x_n)$, $y=(y_1,y_2,\ldots\ldots\ldots,y_n)$ in $F^n$ is an innerproduct on $F^n$. This inner product is called standard inner product.

**(1)** $\mathrm{I}:(F^n,+)$ **is an abelian group :**

(i) + **is commutative :** For any $x=(x_1,x_2,\ldots\ldots\ldots,x_n)$, $y=(y_1,y_2,\ldots\ldots\ldots,y_n)$ in $F^n$,

$x+y=(x_1+y_1,\,x_2+y_2,\ldots\ldots\ldots,x_n+y_n)$

$=(y_1+x_1,\,y_2+x_2,\ldots\ldots\ldots,y_n+x_n)$ (since addition of complex numbers is commutative)

$=y+x$

(ii) + **is associative :** For any $x=(x_1,x_2,\ldots\ldots\ldots,x_n)$, $y=(y_1,y_2,\ldots\ldots\ldots,y_n)$ & $z=(z_1,z_2,\ldots\ldots\ldots,z_n)$ in $F^n$,

$$x+(y+z)=x+(y_1+z_1,\ y_2+z_2,..........,y_n+z_n)$$

$$=(x_1+(y_1+z_1),\ x_2+(y_2+z_2),...........,x_n+(y_n+z_n))$$

$$=[(x_1+y_1)+z_1,\ (x_2+y_2)+z_2,............,(x_n+y_n)+z_n]$$

(since addition of complex numbers is associative)

$$=(x_1+y_1,\ x_2+y_2,........,x_n+y_n)+z$$

$$=(x+y)+z$$

(iii) **Existence of identity** : Clearly, $O=(0,\ 0,\ .........,0)\in F^n$ and for any $x=(x_1,\ x_2,\ ......,x_n)$ in $F^n$,

$$x+0=(x_1+0,\ x_2+0,.........,x_n+0)$$

$$=(x_1,\ x_2,.........,x_n)=x.$$

So, 0 is identity with respect to '+'.

(iv) **Existence of inverse :** Let $x=(x_1,x_2,...............,x_n)\in F^n$. Then

$$y=(-x_1,-x_2,...........,-x_n)\in F^n \text{ and}$$

$$x+y=(x_1+(-x_1),\ x_2+(-x_2),..............,x_n+(-x_n))$$

$$=(0,\ 0,...............,0)=0.$$

So, $-y$ is inverse of $x$ with respect to '+'.

(From the definition of scalar multiplication, $y=(-1)x$.

So, $-x=y=(-1)x$)

Hence, $(F^n,+)$ is an abelian group.

**II Properties of Scalar multiplication :** For any $x=(x_1,x_2,............,x_n)$, $y=(y_1,y_2,............,y_n)$ in $F^n$ and $a,b$ in $F$,

(i) $a(x+y)=a(x_1+y_1,\ x_2+y_2,..........,x_n+y_n)$

$=(a(x_1+y_1),\ a(x_2+y_2),................,a(x_n+y_n))$

$=(ax_1+ay_1,\ ax_1+ay_2,\ ...........,ax_n+ay_n)$

(since multiplication of complex numbers is distributive over addition)

$=(ax_1,\ ax_2,\ ...........,ax_n)+(ay_1,\ ay_2,..........,ay_n)$

$=ax+ay$;

(ii) $(a+b)x=((a+b)x_1,\ (a+b)x_2,............(a+b)x_n)$

$=(ax_1+bx_1,\ ax_2+bx_2,.............,ax_n+bx_n)$

(since distributive law holds for complex numbers)

$=(ax_1,\ ax_2,.........,ax_n)+(bx_1,bx_2,........,bx_n)$

$=ax+bx$;

(iii) $a(bx)=a(bx_1,bx_2,............,bx_n)$

$=(a(bx_1),a(bx_2),...........,a(bx_n))$

$=((ab)x_1,(ab)x_2,.........,(ab)x_n)$

(since multilplication of complex numbers is associative)

$=(ab)x$;

(iv) $|x=(|x_1,|x_2,.............,|x_n)=(x_1,x_2,.............,x_n)=x.$

Hence, $(F^n,+)$ is a vector space over $F$.

**(2) <, > defined is an inner product :**

Let $x=(x_1,x_2,............,x_n)$, $y=(y_1,y_2,............,y_n)$, $z=(z_1,z_2,........,z_n)\in F^n$ and $c\in F$.

$(I_1)$ : $<x+y,\ z>\ =\ <(x_1+y_1,\ x_2+y_2,............,x_n+y_n),\ z>$

$$= \sum_{i=1}^{n} (x_i + y_i)\overline{z_i} = \sum_{i=1}^{n} \left(x_i\, \overline{z_i} + y_i\, \overline{z_i}\right) \text{ (since distributive laws hold for complex numbers)}$$

$$= \sum_{i=1}^{n} x_i\, \overline{z_i} + \sum_{i=1}^{n} y_i\, \overline{z_i} = < x, z > + < y, z >$$

$$(\mathrm{I}_2) : < cx, y > = < (c\,x_1, c\,x_2, \, ......., c\,x_n), y >$$

$$= \sum_{i=1}^{n} (c\,x_i)\overline{y_i} = \sum_{i=1}^{n} c\left(x_i\, \overline{y_i}\right) = c\sum_{i=1}^{n} x_i\, \overline{y_i} = c < x, y >$$

$$(\mathrm{I}_3) : < y, x > = \sum_{i=1}^{n} y_i\, \overline{x_i} = \sum_{i=1}^{n} \overline{\left(\overline{y_i}\, x_i\right)} = \overline{\sum_{i=1}^{n} x_i\, \overline{y_i}}$$

$$= \overline{< x, y >}$$

$(\mathrm{I}_4)$: $< x, x > = \sum_{i=1}^{n} x_i\, \overline{x_i} = \sum_{i=1}^{n} |x_i|^2$. If $x \neq 0$ then $x_i \neq 0$ for atleast one $i$ so that $|x_i|^2 > 0$ and hence $< x, x > > 0$.

Hence, $F^n$ is an innerproduct space.

**3.1.4 Lemma :** Let V be an innerproduct space. For any $u, v, w$ in V and $c, d$ in $F$, the following hold

(i) $< u, cv + dw > = \overline{c} < u, v > + \overline{d} < u, w >$. Consequently $< u, cv > = \overline{c} < u, v >$

(ii) $u = 0$ or $v = 0$ implies $< u, v > = 0$

(iii) $< u, v >$ is real and $< u, u > \geq 0$.

(iv) $< u, u > = 0$ if and only if $u = 0$.

**Proof :**

(i) $< u, cv + dw > = \overline{< cv + dw, u >} = \overline{c < v, u > \;\; d < w, u >}$ (by $\mathrm{I}_1$)

$$= \overline{c}\; \overline{< v, u >} + \overline{d} < w, u >$$

$=\bar{c}<\overline{v, u}> + \bar{d}<u, w>$ (by $I_3$ )

The rest is clear by taking $d=0$.

(ii) $u=0$ : $<u, v> = <o \cdot O, v> = o<O, v>$ (by $I_2$)

$=0$

$v=0$ : $<u, v> = <u, o \cdot O> = \bar{o}<u, 0>$ (by (i))

$=0$

(iii) $<u, u> = \overline{<u, u>}$ (by $I_3$ ). So, $<u,u>$ is real.

If $u \neq 0$ then $<u,u> > 0$ (by $I_4$). If $u=0$ then $<u, u> = 0$ (by (ii)). Thus, $<u, u> \geq 0$.

(iv) $<u, u> = 0 \Rightarrow u=0$ (by $I_4$ and (iii))

$u=0 \Rightarrow <u, u> = 0$ (by (ii))

**3.1.5 Definition :** Let V be a real or complex vector space. A mapping $\| \ \| : V \to \mathbb{R}$ satisfying the following conditions is called a norm on V.

$(N1)$ : $\| x \| \geq 0$ for all $x \in V$

$(N2)$: $\| x \| = 0$ if and only if $x = 0$

$(N3)$ : $\| cx \| = |c| \|x\|$ for any scalar $c$ and $x \in V$.

$(N4)$: $\|x+y\| \leq \| x \| + \| y \|$ for any $x, y \in V$.

**3.1.6 Definition :** Any real or complex vector space V together with a norm defined on it is called a normed linear space.

**3.1.7 Definition :** Let V be an inner product space. If $v \in V$ then we define the length of $v$ or norm of $v$ denoted by $\| v \|$ as

$$\| v \| = \sqrt{<v, v>}$$

**3.1.7.1 Note :** For any vector $v$ in an innerproduct space V, $\| v \| \geq 0$ (in view of Lemma 3.1.4 (iii)).

Now, we prove that the function $\| \ \|$ defined on the inner product space V in the definition 3.1.7 is a norm on V in the sense of definition 3.1.5 and hence every inner product space is a normed linear space (with respect to the definition 3.1.6). In order to prove this, we prove a sequence of lemmas and theorems. Here after (throughout this section), V stands for the inner product space with inner product <, >.

**3.1.8 Lemma :** For any $u, v$ in V and scalars $c, d$,

$$< cu + dv, \ cu + dv > = 1$$

$$c\,\overline{c} \ < u,u > + \ c\,\overline{d} \ < u,v > + \ \overline{c}\,d \ < v,u > + \ d\,\overline{d} < v,v >$$

$$= |c|^2 \ \|u\|^2 + 2\,\mathrm{Re} \ c\overline{d}\,(u, \ v) + |d|^2 \ \|v\|^2$$

**Proof :** Follows directly.

**3.1.8.1 Corollary :** $\|c\ u\| = |c|\ \|u\|$

**Proof :** Obvious by taking $d = 0$. We prove this directly.

$$\|cu\|^2 = < cu, \ cu > = c \ < u, \ cu > \text{ (by I2) } = c\ \overline{c} < u,u >$$

by lemma (3.1.4(i)) = $|c|^2 \ \|u\|^2$. Hence $\|cu\| = |c|\ \|u\|$.

**3.1.9 Lemma :** If $a, b, c$ are reals such that $a > 0$ and $a\lambda^2 + 2b\lambda + c \geq 0$ for all real $\lambda$, then $b^2 \leq ac$.

**Proof :** Take reals $a, b, c$ such that the hypothesys holds.

$$a\lambda^2 + 2b\lambda + c = \frac{1}{a}\left[a^2\lambda^2 + 2ab\lambda\right] + c$$

$$= \frac{1}{a}(a\lambda + b)^2 + c - \frac{b^2}{a}.$$

Taking $\lambda = -\frac{b}{a}$, we have the conclusion.

**3.1.10 Theorem (Schwarz inequality) :** If $u, v \in V$ then

$$|< u, v >| \leq \| u \| \| v \|.$$

**Proof :** If $u = 0$ then $< u, v > = 0$ and $\| u \| = 0$ so that $|< u, v >| = 0 = \| u \| \| v \|$. Suppose $x \neq 0$.

**Case (i) :** Assume that $<u, v>$ is real. For any real number $\lambda$,

$$0 \leq < \lambda u + v, \lambda u + v > \text{ (by Lemma 3.1.4 (ii))}$$

$$= \lambda^2 \| u \|^2 + 2\langle u, v \rangle \lambda + \|v\|^2 \text{ (by Lemma 3.1.8)}$$

By Lemma 3.1.9

$$< u, v >^2 \leq \| u \|^2 \| v \|^2$$

i.e. $|< u, v >| \leq \| u \| \| v \|$.

**Case (ii) : Assume that $\langle u, v \rangle = \alpha$ is not real.** So,

$$< \frac{u}{\alpha}, v > = \frac{1}{\alpha} < u, v > = \frac{1}{\alpha} \cdot \alpha = 1$$

is real. By case (i),

$$\left| < \frac{u}{\alpha}, v > \right| \leq \left\| \frac{u}{\alpha} \right\| \| v \|$$

i.e. $\left| \frac{1}{\alpha} < u, v > \right| \leq \left| \frac{1}{\alpha} \right| \| u \| \| v \|$ (by corollary 3.1.8.1)

i.e., $|< u, v >| \leq \| u \| \| v \|$.

**3.1.11 Theorem :** For any $u$, $v$ in V,

$$\|u + v\| \leq \| u \| + \| v \|.$$

**Proof :** Let $u, v \in V$.

$$\|u + v\|^2 = < u + u, u + v >$$

$$= \| u \|^2 + 2\text{Re} < u, v > + \| v \|^2 \text{ (by}$$

$$\leq \| u \|^2 + 2 \left| \langle u, v \rangle \right| + \| v \| + \| v \|^2 \text{ (for any complex number } z \text{, we have } \text{Re}\, z \leq | z |)$$

$$\leq \|u\|^2 + 2\|u\| \, \|v\| + \|v\|^2 \text{ ( by Schwarz in equality)}$$

$$= \left(\|u\| + \|v\|\right)^2.$$

Hence, $\|u + v\| \leq \|u\| + \|v\|$.

**3.1.12 Theorem :** Any inner product space is a normed linear space with respect to the norm defined in the definition 3.1.7.

**Proof :** Let V be an inner product space together with the norm defined in 3.1.7.

$(N1)$: Holds in view of Lemma 3.1.4 (iii).

$(N2)$: Holds in view of Lemma 3.1.4 (ii).

$(N3)$: Holds in view of Corollary 3.1.8.1

$(N4)$: Holds in view of Theorem 3.1.11

Hence V is a normed linear space.

**3.1.13 Theorem :** Let V be a normed linear space with norm $\| \;\|$ defined on V. Define $d : V \times V \to \mathbb{R}$ by $d(u, v) = \|u - v\|$. Then

(i) $d(u,v) \geq 0$

(ii) $d(u,v) = 0$ if and only if $u = v$

(iii) $d(u,v) = d(v,u)$

(iv) $d(u,v) \leq d(u,w) + d(w,v)$ for all $u, v, w$ in V.

**Proof :** (i) $d(u, v) = \|u - v\| \geq 0$ (by $N1$)

(ii) $d(u,v)=\|u-v\|=0$ if and only if $u-v=0$ (by $N2$) i.e. $u=v$

(iii) $d(u,v)=\|u-v\| = \|-(v-u)\| = \|(-1)(v-u)\|$

$=|-1|\ \|v-u\|$ (by $N3$) = $d(v,u)$

(iv) $d(u,v)=\|u-v\|=\|u-w+w-v\|$

$\leq \|u-w\|+\|w-v\|$ (by $N4$)

$=d(u,w)+d(w,v)$

## 3.2 EUCLIDEAN SPACES

Our main interest is the Example 3.1.3

**3.2.1 Example :** Consider the Example 3.1.3 with $F = \mathbb{C}$, the field of complex numbers $F^n = \mathbb{C}^n$ is an inner product space with inner product defined by

$$< x,\ y > = \sum_{i=1}^{n} x_i,\ \overline{y_i}$$

Where $x=(x_1,\ x_2,.........,x_n),\ y=(y_1,y_2,........,y_n)$ in $\mathbb{C}^n$. This innerproduct is called standard inner product.

If $x=(x_1,x_2,.........,x_n)$ then,

$$< x,\ x > = \sum_{i=1}^{n} x_i\,\overline{x_i} = \sum_{i=1}^{n} |x_i|^2$$

and hence

$$\|x\| = \sqrt{< x,\ x >} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

**3.2.1.1 Note :** We sometimes denote $< x, y >$ by $x \cdot y$ and $|x|$ for $\|x\|$.

**3.2.2 Example :** Consider the Example 3.1.3. With $F = \mathbb{R}$, the field of real numbers. Now, $F^n = \mathbb{R}^n$ is an inner product space with inner product defined by

$$< x, y > = \sum_{i=1}^{n} x_i y_i$$

where $x = (x_1, x_2, \ldots\ldots, x_n)$, $y = (y_1, y_2, \ldots\ldots, y_n)$ in $\mathbb{R}^n$. If $x = (x_1, x_2, \ldots\ldots, x_n)$ in $\mathbb{R}^n$, then

$$< x, x > = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{i=1}^{n} x_i^2$$

and hence

$$\|x\| = \sqrt{< x, x >} = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$$

**3.2.2.1 Note :** $\mathbb{C}^n$ and $\mathbb{R}^n$ are normed linear spaces ( for any positive integer $n$ ).

**3.2.3 Theorem (Schwarz inequality ) :** If $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

**Proof :** Let $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ be complex numbers.

**Method I :** Put $x = (a_1, a_2, \ldots\ldots, a_n)$, $y = (b_1, b_2, \ldots\ldots, b_n)$. By theorem 3.1.10,

$$|< x, y >| \leq \| x \| \, \| y \|$$

i.e. $$|< x, y >|^2 \leq \| x \|^2 \, \| y \|^2$$

i.e. $$\left| \sum_{j=1}^{n} a_j \overline{b_j} \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

**Method II :** See Theorem 1.5.14

**3.2.4 Theorem :** Suppose $x, y, z \in \mathbb{R}^k$, and $\alpha$ is real. Then

$(a)$ $|x| \geq 0$

$(b)$ $|x| = 0$ if and only if $x = 0$

$(c)$ $|\alpha x| = |\alpha||x|$

$(d)$ $|x \cdot y| \leq |x|\,|y|$

$(e)$ $|x+y| \leq |x| + |y|$

$(f)$ $|x-z| \leq |x-y| + |y-z|$

**Proof :** Exercise.

## 3.3 SHORT ANSWER QUESTIONS :

**3.3.1 :** If $\mathbb{C}$ is the field of complex numbers, define the standard inner product.

**3.3.2 :** State Schwarz inequality.

**3.3.3 :** Define a normed linear space.

**3.3.4 :** Define the norm induced by the inner product.

**3.3.5 :** State Schwarz inequality in an inner product space.

## 3.4 MODEL EXAMINATION QUESTIONS :

**3.4.1 :** State and prove Schwarz inequality

**3.4.2 :** Define the standard inner product on $\mathbb{R}^k$ and the norm induced by this inner product. For any $x, y$ in $\mathbb{R}^k$ and real $\alpha$, prove the following

$(a)$ $|x| \geq 0$

$(b)$ $|x| = 0 \Leftrightarrow x = 0$

$(c)$ $|\alpha x| = |\alpha||x|$

$(d) \quad |x+y| \le |x|+|y|$

(Here $|x|$ and $\|x\|$ represent the same).

## 3.5 EXERCISES :

**3.5.1 :** Give proof of Theorem 3.2.4.

**3.5.2 :** Define normed linear space. Prove that $\mathbb{R}^k$ is a normed linear space with respect to the norm $\| \ \|$ on $\mathbb{R}^k$ defined by

$$\| x \| = \left\{ \sum_{i=1}^{k} x_i^2 \right\}^{\frac{1}{2}}$$

where $x = (x_1, x_2, \ldots\ldots\ldots, x_k)$

## 3.6 ANSWERS TO S.A.Q.s :

**3.3.1 :** See Definition given in Example 3.2.1

**3.3.2 :** See the statement of Theorem 3.2.3

**3.3.3 :** See Definition 3.1.5

**3.3.4 :** See Definition 3.1.7

**3.3.5 :** See the statement of the Theorm 3.1.10

## 3.7 REFERENCE BOOK :

Principles of Mathematical Analysis, Third edition, Mc Graw - Hill International Editions : Walter Rudin

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Lesson - 4

# METRIC SPACES

## 4.0 INTRODUCTION

In this lesson, we study the notion of distance (also called metric) on a set. Any set together with a metric is called a metric space. In a metric space, we study the concepts - neighborhood, openset, closed set, perfect set and dense sets.

## 4.1 METRIC SPACES

**4.1.1 Definition :** Let $X$ be a non-empty set. A mapping $d: X \times X \to \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers) is called a metric or distance function on $X$ if the following conditions are satisfied

$(D1)$: $d(p,q) \geq 0$

$(D2)$: $d(p,q)=0$ if and only if $p=q$

$(D3)$: $d(p,q)=d(q,p)$ (symmetry)

$(D4)$: $d(p,q) \leq d(p,r)+d(r,q)$ (triangle inequality) for all $p,q,r$ in $X$.

**4.1.2 Definition :** By a metric space we mean any pair $(X,d)$ where $X$ is a non-empty set and $d$ is a metric on $X$. Elements of $X$ are called points. If $p,q \in X$ then $d(p,q)$ is called the distance between the points $p$ and $q$.

**4.1.3 Example :** Let $\mathbb{R}$ be the set of real numbers. Define $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x,y)=|x-y|$. Then $d$ is a metric on $\mathbb{R}$ (called the usual metric).

We know that the absolute value of $|a|$ of a real number $a$ defined by $|a| = \max\{a, -a\}$ has the following properties.

(1) $|x| \geq 0$

(2) $|x|=0 \Leftrightarrow x=0$

(3) $|-x| = |x|$

(4) $|x+y| \leq |x|+|y|$ for all reals $x, y$

$(D1)$: Clearly, $d(x,y)=|x-y|\geq 0$ for all $x, y$ in $\mathbb{R}$.

$(D2)$: $d(x,y)=0 \Leftrightarrow |x-y|=0$

$$\Leftrightarrow x-y=0 \text{ i.e. } x=y$$

$(D3)$: $d(x,y)=|x-y|=|-(x-y)|=|y-x|=d(y,x)$

for any $x, y$ in $\mathbb{R}$

$(D4)$: $d(x,y)=|x-y|\leq|x-z+z-y|$

$$\leq|x-z|+|z-y|$$

$$=d(x,z)+d(z,y)$$

for any $x, y, z$ in $\mathbb{R}$.

Hence, $d$ is a metric on $\mathbb{R}$. Thus $(\mathbb{R}, d)$ is a metric space.

**4.14 Example :** Let $\mathbb{C}$ be the set of complex numbers. Define $d: \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ by

$$d(z_1, z_2)=|z_1 - z_2|$$

where $|z|$ denotes the absolute value of the complex number $z=a+ib$ given by $|z|=\sqrt{a^2+b^2}$.

We know that the absolute value (i.e. $|\ |$) satisfies

$(i)$ $|z|\geq 0$;

$(ii)$ $|z|=0 \Leftrightarrow z=0$;

$(iii)$ $|z_1 z_2|=|z_1||z_2|$;

$(iv)$ $|z_1+z_2| \leq |z_1|+|z_2|$ for any complex numbers $z, z_1, z_2$.

If $z_1=a_1+ib_1$, $z_2=a_2+ib_2$ then

$$|z_1 - z_2| = |(a_1 - a_2) + i(b_1 - b_2)|$$

$$= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

= distance between $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$.

Now, we prove that $d$ is a metric on $\mathbb{C}$.

$(D1)$: $d(z_1, z_2) = |z_1 - z_2| \geq 0$

$(D2)$: $d(z_1, z_2) = 0 \Leftrightarrow |z_1 - z_2| = 0$

$$\Leftrightarrow z_1 - z_2 = 0$$

$$\Leftrightarrow z_1 = z_2$$

$(D3)$: $d(z_1, z_2) = |z_1 - z_2|$

$$= |-1(z_2 - z_1)|$$

$$= |-1||z_2 - z_1|$$

$$= d(z_2, z_1)$$

$(D4)$: $d(z_1, z_2) = |z_1 - z_2| = |z_1 - z_3 + z_3 - z_2|$

$$\leq |z_1 - z_3| + |z_3 - z_2|$$

$$= d(z_1, z_3) + d(z_3, z_2)$$

Hence, $d$ is a metric on $\mathbb{C}$.

**4.1.5 Example :** Inview of Theorem 2.1.13, every normal linear space $V$ is a metric space with respect to the metric $d$ on $V$ defined by $d(x, y) = \|x - y\|$.

**4.1.6 Example :** Every inner product space is a normed linear space with respect to the norm induced by the inner product (i.e. $\|x\| = \sqrt{<x, x>}$ (by Theorem 2.1.12) and hence a metric space (byTheorem 2.1.13) Example 4.3.3 is infact, can be obtained by taking $F = \mathbb{R}$, $n = 1$ in Example 2.1.3 and Example 4.1.4 can be obtained by taking $F = \mathbb{C}$, $n = 1$ in Example 2.1.3.

**4.1.7 Example :** Let $X$ be any non-empty set. Define $d: X \times X \to \mathbb{R}$ by

$$d(x,y)=\begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{other wise} \end{cases}$$

Then $d$ is a metric on $X$.

$(D1)$ : $d(x,y) \geq 0$ for all $x, y$ in $X$ holds by definition.

$(D2)$: $d(x,y)=0$ if and only if $x=y$ (by definition)

$(D3)$ : $d(x,y)=d(y,x)$ for all $x, y$ in $X$ (by definition)

$(D4)$ : Let $x, y, z \in X$.

| Case | Sub-case | $d(x,y)$ | $d(x,z)$ | $d(z,y)$ | $d(x,z)+d(z,y)$ |
|---|---|---|---|---|---|
| $x=y$ | -- | 0 | $\geq 0$ | $\geq 0$ | $\geq 0$ |
| $x \neq y$ | $x=z$ | 1 | 0 | 1 | 1 |
| | $x \neq z$ | 1 | 1 | 0 or 1 | 1 or 2 |

Clearly, $d(x,y) \leq d(x,z)+d(z,y)$.

Hence, $d$ is a metric on $X$. This metric is called discrete metric and this metric space is called discrete metric space.

**4.1.7.1 Note :** Any non-empty set $X$ can be converted into a metric space by defining $d$ as in the Example 4.1.8.

**4.1.8 Example :** Let $X$ be a non-empty set. Let $r>0$ be real. Define $d: X \times X \to \mathbb{R}$ by

$$d(x,y)=\begin{cases} r \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$$

Then $d$ is a metric on $X$.

**4.1.9 Definition :** Let $\mathbb{R}$ be the set of real numbers.

(i) $(a,b)=\{x \in \mathbb{R} / a < x < b\}$ (where $a, b \in \mathbb{R}$, $a < b$) is called a segment.

(ii) $[a,b]=\{x\in\mathbb{R}/a\le x\le b\}$ (where $a,b\in\mathbb{R},\ a\le b$) is called an interval.

(iii) $[a,b)=\{x\in\mathbb{R}/a\le x<b\}$ (where $a,b\in\mathbb{R},\ a<b$) is called a left closed right open interval.

(iv) $(a,b]=\{x\in\mathbb{R}/a<x\le b\}$ (where $a,b\in\mathbb{R},\ a<b$) is called a left open right closed interval.

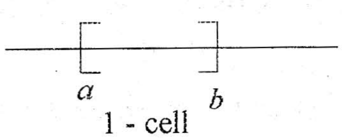
**4.1.9.1 Note :** Clearly, the segment $(a,b)$ contains all points between $a$ and $b$; except $a$ and $b$; the interval $[a,b]$ contains all points between $a$ and $b$ (inclusive of $a$ and $b$); $[a,b)$ contains all points between $a$ and $b$ inclusive of $a$ but not $b$; $(a,b]$ contains all points between $a$ and $b$ inclusive of $b$ but not $a$.

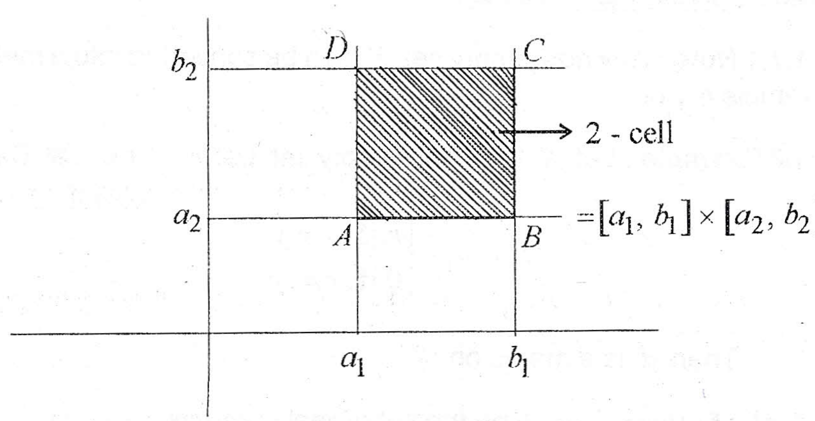
**4.1.10 Definition :** Let $a_i<b_i\ (i=1,2,......,k)$ hold in $\mathbb{R}$. Then the set

$$\left\{x=(x_1,x_2,.......,x_k)\in\mathbb{R}^k/a_i\le x_i\le b_i\ (i=1,2,......,k)\right\}$$ is called a $k$ - cell.

**4.1.10.1 Note :** (i) The $k$ - cell defined above is, infact, the Cartesian product of the intervals $[a_1,b_1],[a_2,b_2],..,[a_k,b_k]$ i.e. $\prod_{i=1}^{k}[a_i,b_i]$ (where $a_i,b_i$ s are in $\mathbb{R}$ and $a_i<b_i$)

(ii) I - cell is an interval and 2 - cell is a rectangle.



**4.1.11 : Definition :** A subset $E \subseteq \mathbb{R}^k$ is called convex if

$$\lambda x + (1-\lambda) y \in E$$

whenever $x \in E,\ y \in E$ and $0 \leq \lambda \leq 1$.

**4.1.12 Definition :** Let $x, y \in \mathbb{R}^k$. The line segment joining $x$ and $y$ is defined as the set

$$\{(1-\lambda)x + \lambda y / 0 \leq \lambda \leq 1\}$$

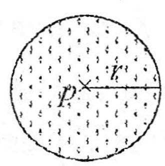
This set is denoted by $[x, y]$

**4.1.12.1 Note :** Inview of the definition 4.1.11, a subset $E \subseteq \mathbb{R}^k$ is called convex whenever $x, y \in E$, the line segment joining $x$ and $y$ lies entirely in $E$.

**4.1.13 Definition :** Let $(X, d)$ be a metric space. Let $E \subseteq X$.

(a) Let $p \in X$. For any real $r > 0$, the set

$N_r(p) = \{q \in X / d(p,q) < r\}$ is called a neighbourhood of $p$. $p$ is called the center and $r$ is called radius of $N_r(p)$. Some times, we call $N_r(p)$ as the open sphere or open ball centered at $p$ with radius $r$.




(b) Let $p \in X$. For any $r > 0$, the set

$N_r[p] = \{q \in X / d(p,q) \leq r\}$ is called a closed sphere or closed ball with centre $p$ and radius $r$.

(c) A point $p \in X$ is called a limit point of $E$ if every neighbourhood of $p$ contains at least one point of $E$ other than $p$.

i.e. $N \cap (E - \{p\}) \neq \phi$ for every neighbourhood of $N$ of $p$.

$\{\text{Here } N \cap (E - \{p\}) = (N \cap E) - \{p\}\}$

(d) A point $p \in E$ is called an isolated point of $E$ if $p$ is not a limit - point of $E$.

(e) $E$ is called closed if $E$ contains all of its limit points.

(f) A point $p \in E$ is called an interior point of $E$ if there exists a neighbourhood $N$ of $p$ such that $N \subseteq E$.

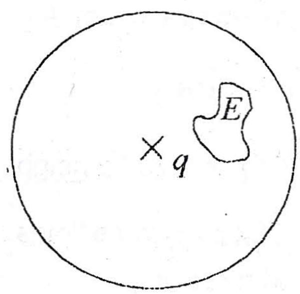
(g) $E$ is called open if every point of $E$ is an interior point of $E$.

(h) The complement of $E$ is defined as the set

$$E^c = \{x \in X / x \notin E\}$$

(i) $E$ is called perfect if $E$ is closed and every point of $E$ is a limit point of $E$.

(j) $E$ is bounded if there exists a real number $M$ and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$. Diagramatically




(k) The closure of $E$ denoted by $\overline{E}$ is defined as the set $\overline{E} = E \cup E'$.

where $E'$ is the set of all limit points of $E$.

(l) $E$ is called dense if every point of $X$ is either in $E$ or a limit point of $E$ i.e. $X = E \cup E'$

(m) $X$ is called separable if $X$ has a countable dense set i.e. there exists a countable set $E \subseteq X$ such that $X = \overline{E}$.

**4.1.14 Lemma :** $p \in E$ is an isolated point of $E$ iff there exists a neighborhood $N$ of $p$ such that

$$N \cap E - \{p\} = \phi$$

i.e. $N \cap E \subseteq \{p\}$

i.e. $N \cap E = \{p\}$ (since $p \in E$)

Proof is clear.

**4.1.15 Lemma :** If $E$ is bounded then to each $x \in X$, there exists a real number $M(x)$ such that $d(p,x) < M(x)$ for all $p \in E$.

**Proof :** Assume that $E$ is bounded, so, there exists a real number $M$ and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$. Let $x \in X$. For any $p \in E$, $d(x,p) \leq d(x,q) + d(q,p)$ $< d(x,q) + M = M(x)$ (say).

**4.1.16 Lemma :** Balls are convex in $\mathbb{R}^k$

**Proof :** We know that the balls in $\mathbb{R}^k$ are of the form $N_r(p)$ or $N_r[p]$.

$N_r(p)$ **is convex :** Let $x, y \in N_r(p)$ and let $0 \leq \lambda \leq 1$.

$$d((1-\lambda)x + \lambda y,\ p) = \|(1-\lambda)x + \lambda y - p\| = \|(1-\lambda)x + \lambda y - (1-\lambda)p - \lambda p\|$$

$$= \|(1-\lambda)(x-p) + \lambda(y-p)\|$$

$$\leq \|(1-\lambda)(x-p)\| + \|\lambda(y-p)\|$$

$$= |1-\lambda|\ \|x-p\| + |\lambda|\|y-p\| = (1-\lambda)d(x,p) + \lambda d(y,p)$$

$$< (1-\lambda)r + \lambda r = r$$

So, $(1-\lambda)x + \lambda y \in N_r(p)$. Hence, $N_r(p)$ is convex.

Similarly, we can prove that $N_r[p]$ is convex.

**4.1.17 Theorem :** Every neighborhood is an openset.

**Proof :** Let $N = N_r(p) = \{q \in X / d(p,q) < r\}$ be a neighborhood of $p$. Let $q \in N$. So, $d(p,q) < r$.

Choose $\delta$ such that $0 < \delta < r - d(p,q)$.

Now, we show that $N_\delta(q) \subseteq N_r(p)$.

$$x \in N_\delta(q) \Rightarrow d(x,q) < \delta$$

$$\Rightarrow d(p,x) \leq d(p,q) + d(q,x)$$

$$< d(p,q)+\delta$$

$$< r.$$

$$\Rightarrow x \in N_r(p).$$

So, $N_\delta(q) \subseteq N_r(p)$. Thus, every point of $N_r(p)$ is an interior point of $N_r(p)$. Hence, $N_r(p)$ is open.

**4.1.18 Theorem :** Every closed sphere is a closed set.

**Proof :** Consider the closed sphere $N_r[p]$. To prove that $N_r[p]$ is closed, it is enough if we prove that no point outside $N_r[p]$ is a limit point of $N_r[p]$.

Let $q \notin N_r[p]$. So, $d(p,q) > r$. Choose $\delta$ such that $0 < \delta < d(p,q) - r$. Now,

$$x \in N_\delta(q) \Rightarrow d(x,q) < \delta < d(p,q) - r$$

$$\Rightarrow d(p,q) \le d(p,x) + d(x,q)$$

$$\le d(p,x) + \delta$$

$$< d(p,x) + d(p,q) - r$$

$$\Rightarrow r < d(p,x)$$

$$\Rightarrow x \in N_r[p]^c.$$

So, $N_\delta(q) \subseteq N_r[p]^c$

i.e. $N_\delta(q) \cap N_r[p] = \phi$

Thus, $q$ is not a limit point of $N_r[p]$. Hence, $N_r[p]$ contains all its limit points i.e. $N_r[p]$ is closed.

**4.1.19 Theorem :** $p \in X$ is a limit point of $E$ if and only if every neighbourhood of $p$ contains infinitely many points of $E$.

**Proof :** Assume that $p \in X$ is a limit point of $E$. Let $N = N_r(p)$ be a neighbourhood of $p$. Suppose

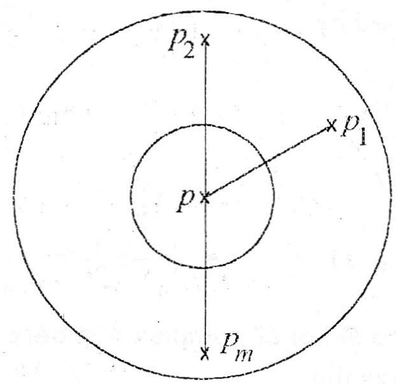
$N$ contains only finite number of points of $E$ and hence only a finite number of points $E-\{p\}$. Let

$$N \cap (E-\{p\}) = \{p_1, p_2 ................, p_m\}$$

Put $\in = \min_{1 \le i \le m} d(p, q_i)$

Clearly $\in > 0$. Let $0 < \delta < \in$. Since $\delta \le d(p, p_i)$ for $i = 1, 2, ......., m$, no $p_i$ is in $N_\delta(p)$. i.e.

$N_\delta(p) \cap (E-\{p\}) = \phi$, a contradiction to the fact that $p$ is a limit point of $E$.




Conversely assume that every neighbourhood of $p$ contains infinitely many points of $E$ and hence every neighbourhood of $p$ contains atleast one point of $E$ other than $p$.

i.e. $p$ is a limit point of $E$.

**4.1.19.1 Corollary :** Any finite subset of a metric space is closed.

**Proof :** Let $E$ be a finite subset of a metric space $X$. By Theorem 4.1.19, $E$ has no limit points. So, $E$ contains all of its limit points. Hence $E$ is closed.

Now, we examine the form of neighbourhood in $(\mathbb{R}^k, d)$ when $k = 1, 2$.

**4.1.20 Example :** Consider the example 4.1.3 whre the set $\mathbb{R}$ of all real numbers equipped with the usual metric $d$ defined by $d(x, y) = |x-y|$. For any $p \in \mathbb{R}$, $r(>0)$ in $\mathbb{R}$,

$$q \in N_r(p) \Leftrightarrow d(p, q) < r \text{ i.e. } |p-q| < r$$

$$\Leftrightarrow p-r < q < p+r$$

$$\Leftrightarrow q \in (p-r, p+r).$$

i.e. $N_r(p)=(p-r,\ p+r)$.

Thus, every neighbourhood in $\mathbb{R}$ is a bounded open interval i.e. a segment.

Conversely, if the segment $(a,b)$ is given then it is clear that

$$(a,b)=N_r(p)$$

where $p=\dfrac{a+b}{2}$, $r=\dfrac{b-a}{2}$.

**4.1.21 Example :** Consider the example 4.1.4 where the set $\mathbb{C}$ of all complex numbers - equipped with metric $d:\mathbb{C}\times\mathbb{C}\rightarrow\mathbb{R}$ defined by

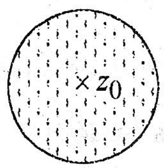
$$d(z_1,z_2)=|z_1-z_2|$$

Let $z_0\in\mathbb{C}$ and $r>0$. Now,

$$z\in N_r(z_0)\Leftrightarrow d(z_0,z)<r \quad \text{i.e. } |z-z_0|<r$$

$\Leftrightarrow$ The set of all complex numbers $z$ whose distance from $z_0$ is less than $r$.

So, $N_r(z_0)$ is shown below




**4.1.21.1 Note :** Consider the metric space $(\mathbb{R}', d)$ where $d(x,y)=|x-y|$ mentioned in example 4.1.3. Let $E\subseteq\mathbb{R}$. Let $p\in\mathbb{R}$. Plot the points of $E$ on $\mathbb{R}'$.

(i) To check whether $p$ is a limit point of $E$, we have to check whether every segment containing $p$ contains infinitely many points of $E$ or not.

(ii) To check whether $p\in E$ is an interior point of $E$, we have to search for a segment containing $p$ which is fully contained in $E$. If atleast one such segment is there then we conclude that $p$ as an interior point of $E$.

(iii) To check whether $E$ is bounded we have to try to find a segment (bounded open interval) containing $E$. If such a segment exists, then we conclude that $E$ is bounded.

**4.1.21.2 Note :** Consider the metric space $(\mathbb{R}^2, d)$ where $d(z_1, z_2) = |z_1 - z_2|$ mentioned in example 4.1.4. Let $E \subseteq \mathbb{R}^2$. Plot the points of $E$ on $\mathbb{R}^2$ (i.e. two dimensional plane).

(i) To check whether $p$ is a limit point of $E$, we have to check whether every circle centered at $p$ with positive radius contains infinitely many points or not.

(ii) To check whether $p \in E$ is an interior point of $E$, we have to search for a circle with center $p$ with some positive radius that is contained in $E$. If such a circle exists then we conclude that $p$ is an interior point of $E$.

(iii) To check whether $E$ is bounded, we have to try to draw a circle so that $E$ is fully contained in the interior of the circle. If such a circle exists, then we conclude that $E$ is bounded.

**4.1.22 Examples :** Consider the subsets of $\mathbb{R}^2$.

(a) The set of all complex numbers $z$ subset $|z| < 1$.

(b) The set of all complex numbers $z$ such that $|z| \leq 1$.

(c) The set of all integers.

(d) The set of all complex numbers $z = r e^{i\theta}$ with $0 \leq \theta \leq 60^0$ and $r \geq 0$.

(e) Let $a, b \in \mathbb{R}$ be such that $a < b$. Consider $(a, b)$ the set of all real numbers $x$ such that $a < x < b$.

(f) The set of all complex numbers i.e. $\mathbb{R}^2$.

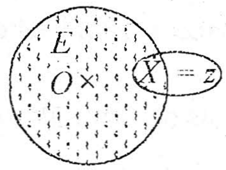
(g) $\left\{ \frac{1}{n} \middle/ n = 1, 2, \ldots\ldots\ldots\ldots \right\}$

(h) A finite set.

(i) The set of all complex numbers $z$ such that $|z| = 1$.

We now examine whether the sets are closed, open, perfect and bounded.

(a) : $E = \left\{ z \in \mathbb{R}^2 \middle/ |z| < 1 \right\}$. Clearly, $E = N_1(0)$. By theorem 4.1.17, $E$ is open.

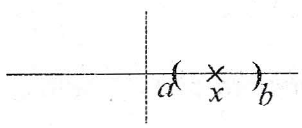



Take a point $z$ such that $|z|=1$.

If we consider any neighbourhood of $z$, it contains infinitely many points of $E$ and hence $z$ is a limit point of $E$. Thus, each complex number $z$ with $|z|=1$ is a limit point of $E$. Clearly no complex number $z$ with $|z|=1$ is in $E$. So, $E$ is not closed. Hence, $E$ is not perfect. $E$ is clearly bounded since $E$ is contained in $N_2(0)$.

(e) $E = \text{Segment } (a,b) = \{x \in \mathbb{R} / a < x < b\}$

(i) Let $x \in (a,b)$. Clearly, we cannot draw any circle with center $x$ so that it is contained in $(a,b)$. So, $x$ is not an interior point of $(a,b)$. Thus, no point of $E$ is an interior point of $(a,b)$.




(ii) Let $x \in (a,b)$. Clearly, every circle centered at $x$ contains infinitely many points of $(a,b)$. So, $x$ is a limit point of $(a,b)$. Clearly $a$ is a limit point of $(a,b)$ which is not in $(a,b)$. So, $(a,b)$ is not closed. Similarly, $b$ is a limit point of $(a,b)$ which is not in $(a,b)$. So, $(a,b)$ is not perfect.

(iii) Clearly, $(a,b) \subseteq N_r(0)$ = circle with center 0 and radius $r$, for some appropriate $r > 0$. So, $(a,b)$ is bounded.

We now give the answers for (a), (b), (c), (d), (e), (f), (g), (h) and (i). The student is advised to check.

| | Closed | Open | Perfect | Bounded |
|---|---|---|---|---|
| (a) | No | Yes | No | Yes |
| (b) | Yes | No | Yes | Yes |
| (c) | Yes | No | No | No |

| | | | | |
|---|---|---|---|---|
| (d) | Yes | No | Yes | No |
| (e) | No | No | No | Yes |
| *(f)* | *Yes* | *Yes* | *Yes* | *No* |
| (g) | No | No | No | Yes |
| (h) | Yes | No | No | Yes |
| (i) | Yes | No | Yes | Yes |

Now, we prove a set-theoretic result which is useful.

**4.1.23 Theorem :** Let $\{E_\alpha\}_\alpha$ be a (finite or infinite) collection of sets $E_\alpha$. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c \text{ and}$$

$$\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$$

**Proof :** (i) $x \in \left(\bigcup_\alpha E_\alpha\right)^c \Leftrightarrow x \notin \bigcup_\alpha E_\alpha$

$\Leftrightarrow x \notin E_\alpha$ for any $\alpha$

$\Leftrightarrow x \in E_\alpha^c$ for any $\alpha$

$\Leftrightarrow x \in \bigcap_\alpha E_\alpha^c$

Hence, $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$

(ii) $x \in \left(\bigcap_\alpha E_\alpha\right)^c \Leftrightarrow x \notin \bigcap_\alpha E_\alpha$

$\Leftrightarrow x \notin E_\alpha$ for some $\alpha$

i.e. $x \in E_\alpha^c$ for some $\alpha$

$$\Leftrightarrow x \in \bigcup_{\alpha} E_{\alpha}^{c}$$

Hence, $\left(\bigcap_{\alpha} E_{\alpha}\right)^{c} = \bigcup_{\alpha} E_{\alpha}^{c}$

**4.1.24 Theorem :** A set $E$ is open if and only if its complement is closed.

**Proof :** Assume that $E$ is open. Let $x$ be a limit point of $E^c$. Suppose $x \notin E^c$ i.e. $x \in E$. Since $E$ is open, $x$ is an interior point of $E$. So, there exists $r > 0$ such that

$$N_r(x) \subseteq E$$

i.e. $N_r(x) \cap E^c = \phi$

i.e. $N_r(x) \cap E^c - \{x\} = \phi$ (since $x \notin E^c$)

Thus, we have a neighbourhood of $x$ which does not contain any point of $E^c$ other than $x$. So, $x$ is not a limit point of $E^c$ other than $x$, a contradiction. Hence $x \in E^c$. Thus, $E^c$ contains all of its limit points. i.e. $E^c$ is closed.

Conversely, assume that $E^c$ is closed. Let $x \in E$ i.e. $x \notin E^c$. Since $E^c$ is closed, $x$ is not a limit point of $E^c$. So, there exists a neighbourhood $N$ of $x$ such that

$$N \cap E^c - \{x\} = \phi$$

i.e. $N \cap E^c = \phi$ ( since $x \in E$)

i.e. $N \subseteq E$.

So, $x$ is an interior point of $E$. Thus, every point of $E$ is an interior point of $E$ i.e. $E$ is open.

**4.1.24.1 Corollary :** A set $F$ is closed if and only if its complement is open.

**Proof :** Clear

**4.1.25 Theorem :**

(a) For any collection $\{G_{\alpha}\}$ of opensets, $\bigcup G_{\alpha}$ is open

(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_{\alpha} F_\alpha$ is closed.

(c) For any finite collection $G_1, G_2, \ldots\ldots\ldots, G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open

(d) For any finite collection of closed sets $F_1, F_2, \ldots., F_n$, $\bigcup_{i=1}^{n} F_i$ is closed.

**Proof :** (a) Let $\{G_\alpha\}_\alpha$ be a collection of open sets. Put $G = \bigcup G_\alpha$

$x \in G \Rightarrow x \in G_\alpha$ for some $\alpha$

$\Rightarrow \exists$ a neighbourhood $N$ of $x \ni N \subseteq G_\alpha$ (since $G_\alpha$ is open) $\subseteq G$.

$\Rightarrow x$ is an interior point of $G$.

Thus, every point of $G$ is an interior point of $G$. Hence $G$ is open.

(c) Let $G_1, G_2, \ldots.., G_n$ be open sets. Put $H = \bigcap_{i=1}^{n} G_i$.

$x \in H \Rightarrow x \in G_i$ for $i = 1, 2, \ldots\ldots, n$.

$\Rightarrow$ For $i = 1, 2, \ldots\ldots, n$, there exists positive reals $r_1, r_2, \ldots\ldots, r_n$ such that

$N_{r_i}(x) \subseteq G_i$

$N_r(x) \subseteq H$ (where $r = \min\{r_i / 1 \le i \le n\}$)

$\Rightarrow x$ is an interior point of $H$.

Thus, every point of $H$ is an interior point of $H$ i.e. $H$ is open.

(b) Let $\{F_\alpha\}$ be a collection of closed sets. By Corollary 4.1.24.1, each $F_\alpha^c$ is open. By (a),

$\bigcup_{\alpha} F_\alpha^c = \left(\bigcap_{\alpha} F_\alpha\right)^c$ is open. By Theorem 4.3.24, $\bigcap_{\alpha} F_\alpha$ is closed.

(d) Let $F_1, F_2, ..., F_n$ be closed sets. So, $F_i^c \ (1 \le i \le n)$ are open. By (c),

$$\bigcap_{i=1}^{n} F_i^c = \left( \bigcup_{i=1}^{n} F_i \right)^c$$ is open. By Theorem 4.3.24, $\bigcup_{i=1}^{n} F_i$ is closed.

The above Theorem 4.1.25 does not give answers for the following questions.

Q1 : Is the intersection of an arbitrary family of opensets open ?

Q2 : Is the union of an arbitrary family of closed sets closed ?

Consider the following

**4.1.26 Example :** Consider the metric space $(\mathbb{R}', d)$ where $d$ is the usual metric on $\mathbb{R}$.

(i) Put $G_n = \left( -\frac{1}{n}, \frac{1}{n} \right)$ for $n = 1, 2, ..........$ Clearly, each $G_n$ is open and

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

which is finite and hence not open (No non-empty finite set is open. For, let $F$ be a non-empty finite subset of $\mathbb{R}$. Suppose $F$ is open. Since $F \neq \phi$, we can choose $x \in F$. Since $F$ is open, there exists $r > 0$ such that $N_r(x) = (x - r, x + r) \subseteq F$. Since $(x - r, x + r)$ is uncountable, $F$ is uncountable, a contradiction to that $F$ is finite).

Thus, arbitrary intersection of open sets is not open.

(ii) Put $F_n = \left[ \frac{1}{n}, 1 - \frac{1}{n} \right]$ for $n = 3, 4, 5, ..........$ Clearly each $F_n$ is closed. Clearly,

$$\bigcup_{n=3}^{\infty} F_n = (0, 1)$$

which is not closed (as 0 is a limit point of (0, 1) which is not in (0, 1)). Thus, arbitrary union closed sets need not closed.

**4.1.27 Theorem :** Let $(X, d)$ be a metric space. Let $E \subseteq X$. Then

(a) $\bar{E}$ is a closed set containing $E$.

(b) $E = \bar{E}$ if and only if $E$ is closed.

(c) $\bar{E}$ is the smallest closed set containing $E$.

i.e. $\bar{E}$ is closed and $\bar{E} \subseteq F$ for every closed set $F$ containing $E$.

**Proof :**

(a) By definition $\bar{E} = E \cup E'$ where $E'$ is the set of all limit points of $E$. Clearly $\bar{E}$ contains $E$. Suppose $x \notin \bar{E}$.

Now,

$$x \notin \bar{E} \Rightarrow x \notin E \text{ and } x \notin E'$$

$$\Rightarrow \exists\, \delta > 0 \ni N_\delta(x) \cap E = \phi$$

$\Rightarrow$ no point of $N_\delta(x)$ is a limit point of $E$

i.e. $N_\delta(x) \cap E' = \phi$

$$\Rightarrow N_\delta(x) \cap (E \cup E') = \phi$$

i.e. $N_\delta(x) \cap \bar{E} = \phi$

$\Rightarrow x$ is not a limit point of $\bar{E}$

Thus, $\bar{E}$ contains all of its limit points. Hence $\bar{E}$ is closed.

(b) $E = \bar{E} \Rightarrow E$ is closed (by (a)).

Assume that $E$ is closed. So, $E' \subseteq E$ (by definition). Hence $\bar{E} = E \cup E' = E$.

(c) Let $F$ be a closed set containing $E$. $E \subseteq F \Rightarrow \bar{E} = E \cup E' \subseteq F \cup F' = \bar{F} = F$ (since $F$ is closed).

**4.1.27.2 Note :** Infact, $\bar{E}$ is the intersection of all closed sets containing $E$.

For, let $\mathfrak{F} = \{F \subseteq X / F \text{ is a closed set, } F \supseteq E\}$. By (a), $\bar{E} \in \mathfrak{F}$. Put $C = \bigcap_{F \in \mathfrak{F}} F$.

By Theorem 4.3.25(b), $C$ is a closed set containing $E$. By (c), $\bar{E} \subseteq C \subseteq \bar{E}$ i.e. $\bar{E} = C$.

**4.1.27.2 Note :** While proving (c), we have used the fact that $E \subseteq F \Rightarrow E' \subseteq F'$. This holds since $N \cap E - \{x\} \subseteq N \cap F - \{x\}$ holds whenever $E \subseteq F$.

**4.1.28 Definition :** Let $E$ be a subset of a metric space $X$. The set of all interior points of $E$ is the Interior of $E$ and is denoted by $E^0$ or $Int(E)$.

**4.1.29 Theorem :** Let $E$ be a subset of a metri space $(X,d)$. Then $Int(E)$ is the largest opensubset of $E$.

**Proof :** To prove the theorem we have to prove :

(i) $E^0$ is an open subset of $E$

(ii) $G$ is open in $X$, $G \subseteq E$ implies $G \subseteq \overset{\circ}{E^0}$.

(i) $E^0 = Int(E)$ is an open sub set of $E$ : Clearly, $E^0 \subseteq E$.

Let $x \in E^0$ i.e. $x$ is an interior point of $E$ i.e. there exists $\delta > 0$ such that $N_\delta(x) \subseteq E$. Now,

$y \in N_\delta(x) \Rightarrow y$ is an interior point of $N_\delta(x)$ (since $N_\delta(x)$ is open)

$\Rightarrow$ There exists $r > 0$ such that $N_r(y) \subseteq N_\delta(x) \subseteq E$

$\Rightarrow y$ is an interior point of $E$

i.e. $y \in E^0$

Thus, $N_\delta(x) \subseteq E^0$. So, $x$ is an interior point of $E^0$ Thus, every point of $E^0$ is an interior point of $E^0$. Hence $E^0$ is open.

(ii) Let $G$ be an open subset of $E$. So,

$x \in G \Rightarrow x$ is an interior point of $G$.

$\Rightarrow$ there exists $r > 0$ such that $N_r(x) \subseteq G \subseteq E$

$\Rightarrow x$ is an interior point of $E$.

$\Rightarrow x \in Int(E) = E^0$

Hence, $G \subseteq E^0$. So, $E^0$ is the largest open subset of $E$.

**4.1.30 Theorem :** Let $(X,d)$ be a metric space. A subset $E$ of $X$ is open if and only if $E = E^0$.

**Proof :** Assume that $E$ is open. So, the largest open subset of $E$ is $E$ itself i.e. $Int(E)=E$, i.e.

$E^0 = E$.

Converse is clear.

**4.1.31 Theorem :** Let $E$ be a closed set of real numbers which is bounded above. Let $y$ be the lub of $E$ Then $y \in E$.

**Proof :** Suppose $y \notin E$. Let $\in >0$. Now, $y-\in <y$. Since $y=\text{lub}\,E$, $y-\in$ is not an upper bound of $E$. So, there exists $x \in E$ such that $x \nleq y-\in$. i.e. $y-\in <x \leq y$ (since $y$ is an upper bound of $E$). Since $y \notin E$, $x \in E$, we have that $x \neq y$. So, $y-\in <x<y<y+\in$. So, $(y-\in, y+\in)$ contains atleast one point of $E$ other than $x$. This holds for all $\in >0$. So, $y$ is a limit point of $E$. Since $E$ is closed, $y \in E$, a Contradiction. Hence $y \in E$.

**4.1.32 Definition :** Let $(X,d)$ be a metric space. Let $Y$ be a nonempty subset of $X$. Let $E \subseteq Y$. We say that $E$ is open relative to $Y$ if $E$ is open in the metric space $(y,d)$.

**4.1.32.1 Note :** Let $(X,d)$ be a metric space. Let $Y(\neq \phi) \subseteq X$. So, $(Y,d)$ is also a metric space. Let $p \in Y$ and $r>0$ be real. Suppose $N_r(p)$ and $N_r^Y(p)$ denote the nighbourhood of $p$ in $X$ and $Y$ respectively. So,

$$N_r^Y(p) = \{q \in Y / d(p,q) < r\}$$

$$= Y \cap N_r(p)$$

**4.1.32.2 Note :** Let $Y(\neq \phi) \subseteq X$ where $X$ is a metric space. So, $Y$ is also a metric space with respect to the metric on $X$. We call $Y$ as a subspace of $X$. Suppose $E \subseteq Y$. Then, $E \subseteq X$ (since $Y \subseteq X$). So, we can talk of openness of $E$ in $Y$ as well as in $X$. What is the relation between the openness of $E$ in $X$ and that of $E$ in $Y$? There are open sets in $Y$ without being open in $X$. Consider the following example.

**4.1.33 Example :** Consider the metric space $(\mathbb{R}', d)$. Take $Y=(0,1]$. Take $E=Y$. Clearly $Y$ is not open in $(\mathbb{R}', d)$ (as 1 is not an interior point of $Y$). Thus, $E$ is open relative to $Y$ but $Y$ is not open relative to $(\mathbb{R}', d)$. Infact, let $Y$ be a nonempty subset of a metric space $(X,d)$ which is not open. Take $E=Y$. Then $E$ is clearly open relative to $Y$ but $E$ is not open relative to $X$.

The following theorem characteristics the open sets in a subspace of a metric space.

**4.1.34 Theorem :** Let $(X,d)$ be a metric space. Let $Y(\neq\phi)\subseteq X$. A subset $E$ of $Y$ is open relative to $Y$ if and only if $E=Y\cap G$ for some open set $G$ in $X$.

**Proof :** Let $E\subseteq Y$. Assume that $E$ is open relative to $Y$. So, $E$ is open in $(Y,d)$. Let $p\in E$. So, $p$ is an interior point of $E$ (in the metric space $(Y,d)$). So, there exists a positive real $r_p$ such that

$$N_{r_p}^{Y}(p)\subseteq E$$

i.e. $Y\cap N_{r_p}(p)\subseteq E$ --------- (1)

Thus, to each $p\in E$, there exists a positive real $r_p$ such that

(1) holds. Put

$$G=\bigcup_{p\in E} N_{r_p}(p)$$

Clearly, $G$ is open and

$$E=\bigcup_{p\in E} N_{r_p}^{Y}(p)=\bigcup_{p\in E}\left(Y\cap N_{r_p}(p)\right)=Y\cap\left(\bigcup_{p\in E} N_{r_p}(p)\right)$$

$$=Y\cap G.$$

Conversely assume that $E=Y\cap G$ for some openset $G$ in $X$. Let $p\in E$. So, $p\in E$ and $p\in G$. Since $G$ is open, there exists $r>0$ such that

$$N_r(p)\subseteq G$$

and hence

$$Y\cap N_r(p)\subseteq Y\cap G$$

i.e. $N_r^{Y}(p)\subseteq E$.

So, $p\in E$ is an interior point of $E$ (in the metric space $(Y,d)$). Thus, every point of $E$ is an interior point of $E$ in $Y$. So, $E$ is open relative to $Y$.

**4.1.35 Problem :** Let $(X,d)$ be a metric space. Let $x,y$ be in $X$ such that $x\neq y$. Prove that there exist disjoint neighbourhoods for $x$ and $y$ respectively. In fact there exists $\delta > 0$ such that $\delta < d(x,y)$ and

$$N_\delta(x) \cap N_\delta(Y) = \phi$$

**Solution :** Since $x \neq y$, $d(x,y) > 0$. Choose $\delta$ such that $0 < 2\delta < d(x,y)$. If

$z \in N_\delta(x) \cap N_\delta(y)$ then

$d(x,y) \leq d(x,z) + d(z,x)$ (by $D4$)

$< \delta + \delta$ (since $z \in N_\delta(x) \cap N_\delta(y)$)

$= 2\delta < d(x,y)$ (By the choice of $\delta$), a contradiction.

**4.1.36 Theorem :** Let $A$ be a subset of a metric space $X$. Then $x \in \bar{A}$ if and only if every neighbourhood of $x$ intersects $A$.

**Proof :** Assume that $x \in \bar{A} = A \cup A'$. If $x \in A$, then the conclusion is clear. If $x \notin A$ then $x \in A'$ i.e. $x$ is a limit point of $A$ and hence the conclusion follows from the definition of limit point.

Conversely assume that every neighbourhood of $x$ intersects $A$. If $x \in A$, it is well and good. Suppose $x \notin A$. For any neighbourhood $N$ of $x$,

$N \cap A \neq \phi$

i.e. $N \cap (A - \{x\}) \neq \phi$ (since $x \notin A$)

i.e. $(N \cap A) - \{x\} \neq \phi$

and hence $x$ is a limit point of $A$ i.e. $x \in A' \subseteq \bar{A}$. Hence the theorem.

**4.1.37 Theorem :** Let $(X,d)$ be a metric space. Let $A \subseteq X$. Then $A$ is dense in $X$ if and only if every non-empty open set intersects $A$.

**Proof :** Assume that $A$ is dense in $X$ i.e. $X = \bar{A}$. Let $G$ be a non-empty open set in $G$. Let $x \in G$. Since $G$ is open there exists $r > 0$ such that

$$N_r(x) \subseteq G$$

Now, $x \in X = \overline{A}$. By theorem 4.3.35,

$$N_r(x) \cap A \neq \phi$$

and hence

$$G \cap A \neq \phi.$$

Conversely assume that every non-empty open set intersects $A$. Now,

$x \in X \Rightarrow$ for any $r > 0$, $N_r(x) \cap A \neq \phi$ (since every $nbd$ is an open set by Theorem 4.3.17)

$\Rightarrow$ every $nbd$ of $x$ intersects $A$

$\Rightarrow x \in \overline{A}$ (by Theorem 4.3.35)

Thus, $X \subseteq \overline{A}$ i.e., $X = \overline{A}$ i.e., $A$ is dense in $X$.

**4.1.38 Definition** : Let $E$ be a subset of a metric space $(X, d)$.

(i) Let $x \in X$. The distance of $x$ from $E$ is defined as infimum or greatest lower bound of the set

$$\{d(x,a)/a \in E\}$$

i.e. $\inf\{d(x,a)/a \in E\}(= \text{glb}\{d(x,a)/a \in E\})$ and is denoted by $d(x, E)$.

(ii) The diameter of $E$ is defined as the least upper bound or Supremum of the set

$$\{d(x, y)/x \in E, y \in E\}$$

i.e., $\sup\{d(x, y)/x \in E, y \in E\}\ (= \text{lub}\{d(x,y)/x \in E, y \in E\})$ and is denoted by diam $E$.

**4.1.39 Problem :** Let $A$ be a subset of a metric space $(X, d)$. Let $x \in X$. Prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$.

**Solution :** Assume that $x \in \overline{A}$. So, every neighbourhood of $x$ intersects $A$ (by Theorem 4.1.36). Let $n$ be a positive integer. So,

$$A \cap N_{\frac{1}{n}}(x) \neq \phi$$

Take a point $a_n$ in this set. Now,

$$d(x,A) \le d(x,a) < \frac{1}{n} \text{ ----------- (1)}$$

Thus, (1) holds for each positive integer $n$. Hence $d(x,A)=0$. Otherwise, $d(x,A)>0$. By Archimedian property, there exists a positive integer $n$ such that

$$n \cdot d(x,A) > 1$$

i.e. $d(x,A) > \frac{1}{n}$, a contradiction to (1)

Conversely assume that $d(x,A)$ = 0. Let $\in > 0$ (i.e. we are considering the $\in$ - neighbourhood $N_\in(x)$ of $x$).

Since $0 = d(x,A)$

$$= \text{glb}\{d(x,a)/a \in A\}$$

$\in$ is not a lower bound of the set

$$\{d(x,a)/a \in A\}$$

So, there exists an element $a \in A$ such that

$$\in \not\le d(x,a) \text{ i.e. } d(x,a) < \in$$

i.e. $a \in A \cap N_\in(x)$

Thus, every neighbourhood of $x$ intersects $A$. By theorem 4.1.36, $x \in \overline{A}$.

**4.1.40 Problem :** Let $(X, d)$ be a metric space. A subset $G$ of $X$ is open if and only if $G$ is a union of open spheres.

**Solution :** Let $G \subseteq X$. Assume that $G$ is open. Therefore, each $x \in G$ is an interior point of $G$ and hence to each $x \in G$, there exists a $nbd$ $N_x$ of $x$ such that

$$N_x \subseteq G$$

So, $$G = \bigcup_{x \in G} N_x$$

Thus, $G$ is a union of neighbourhoods (i.e. open spheres)

Conversely assume that there exists a family $\{N_i\}_{i \in I}$ of open spheres such that

$$G = \bigcup_{i \in I} N_i .$$

We know that each neighbourhood is open (by theorm 4.1.17) and hence each $N_i$ is open. We know that arbitrary union of open sets is open (by theorem 4.1.25(a)). Hence $G$ is open.

**4.1.41 Problem :** Let $(X, d)$ be a metric space. Define $\mu : X \times X \to \mathbb{R}$ by

$$\mu(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Prove that $\mu$ is a metric on $X$.

**Solution :** (D 1) : Clearly $\mu(x, y) \geq 0$ for all $x, y$ in $X$.

(D 2) : $\mu(x, x) = \dfrac{d(x, x)}{1 + d(x, x)} = 0$ (since $d$ satisfies (D 2)) : Now,

$$\mu(x, y) = 0 \Rightarrow d(x, y) = 0$$

$$\Rightarrow x = y \text{ (since } d \text{ satisfies (D 3))}$$

Thus, $\mu$ satisfies (D 2).

(D 3) : $\mu(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$

$$= \frac{d(y, x)}{1 + d(y, x)} = \mu(y, x)$$

for any $x, y$ in $X$.

(D 4) : Let $x, y, z \in X$. Suppose

$$\mu(x, y) \not\leq \mu(x, z) + \mu(z, y)$$

i.e. $\frac{d(x,y)}{1+d(x,y)} > \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$

i.e. $\left[d(y,z)+d(x,z)-d(x,y)\right]+2d(x,z)d(y,z)$

$+d(x,y)d(y,z)d(z,x)<0$

i.e. $2d(x,z)d(y,z)+d(x,y)d(y,z)d(z,x)<0$,

a contradiction. So, $\mu$ satisfies (D 4). Hence $\mu$ is a metric on $X$.

**4.1.42 Problem :** Prove that the set of all limit points of any subset of a metric space is closed.

**Solution :** Let $(X,d)$ be a metric space. Let $E \subseteq X$. Let $E'$ be the set of all limit points of $E$. Now, we prove that $E'$ is closed i.e. $E'$ contains all its limit points. Let $x \in X$ be a limit point of $E'$. Let $\delta > 0$. Since $x$ is a limit point of $E'$.

$$N_\delta(x) \cap E' - \{x\} \neq \phi$$

Choose a point $y$ in this set. So, $0 < d(x,y) < \delta$ and $y \in E'$. Choose $\in > 0$ such that

$$\in < \text{Min}\{d(x,y),\ \delta - d(x,y)\}$$

Clearly, $N_\in(y) \subseteq N_\delta(x)$. Since $y \in E'$, $N_\in(y)$ contains atleast one point of $E$ other than $y$ and hence $N_\delta(x)$ contains atleast one point of $E$ other than $x$. Thus, each neighbourhood of $x$ contains atleast one point of $E$ other than $x$. So, $x$ is a limit point of $E$ i.e. $x \in E'$. Thus, $E'$ contains all of its limit points. Hence $E'$ is closed.

**4.1.43 Problem :** For any subset $E$ of a metric space $X$, prove that

$$E^0 = \overline{E^c}^{\,c} \text{ or } E^{0\,c} = \overline{E^c}$$

**Solution :** Let $E$ be a subset of a metric space $X$. Now,

$$E^c \subseteq \overline{E^c} \text{ (by Theorem 4.1.27(a))}$$

$$\Rightarrow \overline{E^c}^{\,c} \subseteq E^{c\,c} = E$$

Since $\overline{E^c}$ is closed, $\overline{\overline{E^c}}^{\,c}$ is open. Thus, $\overline{\overline{E^c}}^{\,c}$ is an open subset of $E$. Let $G$ be an open subset of $E$. Now,

$$G \subseteq E \Rightarrow E^c \subseteq G^c$$

$$\Rightarrow \overline{E^c} \subseteq \overline{G^c} = G^c \text{ (since } G \text{ is open, } G^c \text{, is closed by Theorem 4.1.24}$$

and hence by Theorem 4.1.27(b)) $\Rightarrow G \subseteq \overline{\overline{E^c}}^{\,c}$

Thus $\overline{\overline{E^c}}^{\,c}$ is the largest open subset of $E$. Hence

$$E^0 = \overline{\overline{E^c}}^{\,c} \text{ (by Theorem 4.1.29)}$$

## 4.2 SHORT ANSWER QUESTIONS :

**4.2.1 :** Is (5, 7) a neighbourhood of some point in $(\mathbb{R}', d)$ ?

**4.2.2 :** Is every segment in $\mathbb{R}'$ a neighbourhood of some point in $(\mathbb{R}', d)$ ?

**4.2.3 :** Let $(X, d)$ be a discrete metric space. Describe $N_{\frac{1}{2}}(x)$ in this space.

**4.2.4 :** Define a $k-$ cell.

**4.2.5 :** Prove that balls are convex (balls means open or closed balls in $\mathbb{R}^k$.

**4.2.6 :** Is 0 an interior point of the set

$$E = (-1, 1) = \{x \in \mathbb{R} / -1 < x < 1\} \text{ in the metric space } (\mathbb{C}, d).$$

**4.2.7 :** Let $E$ be a subset of a metric space $X$. If $p \in E$ is an isolated point of $E$ prove that there exists a neighbourhood $N$ of $p$ such that $N \cap E = \{p\}$.

**4.2.8 :** Is (2, 3) open in $(\mathbb{R}', d)$? Is (2, 3) open in $(\mathbb{C}, d)$ ?

**4.2.9 :** Is $[2, 3](\subseteq \mathbb{R}) \subseteq \mathbb{C}$ is perfect in $(\mathbb{C}, d)$ ?

**4.2.10 :** Let $E$ be a subset of a metric space $X$ such that $E$ is contained in a neighbourhood of some point. Then only one of the following is most appropriate.

(A) $E$ is open (B) $E$ is closed (C) $E$ is perfect (D) $E$ is bounded

**4.2.11 :** Consider the set $Q$ of all rationals in the metric space $(\mathbb{R}', d)$. Then $Q$ is

(A) open (B) Closed (C) Perfect (D) Dense

**4.2.12 :** Is the set $\mathbb{Z}$ of all integers closed in $(\mathbb{R}', d)$ ?

**4.2.13 :** Is every point of the set $\mathbb{Z}$, of all integers an isolated point of $\mathbb{Z}$ in the metric space $(\mathbb{R}', d)$? Justify.

**4.2.14 :** Is $Q$, the set of all rationals is dense in $(\mathbb{R}', d)$ ? Justify.

**4.2.15 :** Consider the metric space $(X, \mu)$ obtained from the metric space $(X, d)$. What is the relationship between the neighbourhoods in these two metric spaces ?

## 4.3 MODEL EXAMINATION QUESTIONS :

**4.3.1 :** Define metric space. Give two examples.

**4.3.2 :** Define neighbourhood of a point.

Prove that every neighbourhood is an open set.

Characterize the neighborhoods in the metric space $(\mathbb{R}, d)$ where $\mathbb{R}$ is the set of all real numbers and $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $d(x, y) = |x - y|$

**4.3.3 :** Define limit point.

Let $(X, d)$ be a metric space. Let $E \subseteq X$. Prove that a point $x \in X$ is a limit point $E$ if and only if every neighbourhood of $x$ contains infinitely many points of $E$.

**4.3.4 :** Define bounded set. If a subset $E$ of a metric space $X$ is bounded, prove that to each point $x$ in $X$, there exists a real number $M(x)$ such that $d(x, y) \leq M(x)$ for all $y$ in $E$.

**4.3.5 :** Define (i) Closed set (ii) Open set

Prove that a subset $E$ of a metric space $X$ is open if and only if the complement of $E$ is closed.

**4.3.6 :** Define open set. Prove that arbitrary union of opensets is open. Is arbitrary intersection of opensets open ? Justify your answer.

**4.3.7 :** Define the closure of a set. In any metric space, for any set $E$, prove that $\overline{E}$ is the smallest closed set containing $E$.

**4.3.8 :** Define interior of a set. For any subset $E$ of a metric space $X$, prove that $E^0$ = interior of $E$ is the largest open subset of $E$.

**4.3.9 :** Let $A$ be a subset of a metric space $X$. Prove that a point $x \in \overline{A}$ if and only if every neighbourhood intersects $A$.

Deduce that $A$ is dense if and only if every non-empty open set intersects $A$.

**4.3.10 :** Define distance of a point from a set in a metric space.

Let $A$ be a subset of a metric space $X$. Let $x \in X$. Prove that $x \in \overline{A} \Leftrightarrow d(x, A) = 0$.

**4.3.11 :** Let $(X, d)$ be a metric space. Define $\mu : X \times X \to \mathbb{R}$ by

$$\mu(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Prove that $(X, \mu)$ is a metric space. What is the connection between the neighbourhoods in $(X, d)$ and $(X, \mu)$.

**4.3.12 :** For any subset $E$ of a metric space $X$, prove that $E^{0c} = \overline{E^c}$

**4.3.13 :** Prove that the set of all limit points of a set is closed.

## 4.4 EXERCISES :

**4.4.1 :** Construct a bounded set of real numbers with exactly three limit points.

**4.4.2 :** Which of the following sets are (i) open ? (ii) closed ? (iii) bounded ? (iv) perfect ? in $(\mathbb{R}', d)$.

(a) $(2, 3)$ (b) $[2, 5)$ (c) $(1, \infty)$ (d) The set $Q$ of all rational numbers.

**4.4.3 :** Answer the problem 4.7.2 in ($\mathbb{C}$, $d$).

**4.4.4 :** In any metric space, prove that any two distinct points can be separated by open sets in the sense that if $x$ and $y$ are distinct points in a metric space, then there exist neighbourhoods of $M$ and $N$ of $x$ and $y$ respectively such that $M \cap N = \phi$ (observe that the radii of each $M$ and $N$ can be taken as less than $\frac{1}{2} d(x, y)$).

**4.4.5 :** Prove that any two disjoint closed sets in any metric space can be separated by open sets i.e. if $A$ and $B$ are disjoint closed sets in a metric space $(X,d)$ then there exist disjoint open sets $G$ and $H$ containing $A$ and $B$ respectively.

(Hint : Use the Exercise 4.4.4 or see the proof of Theorem 6.2.2).

**4.4.6 :** Let $E$ be a subset of a metric space $X$. Prove that the set $E'$ of all limit points of $E$ is closed.

**4.4.7 :** Define $\overline{E}$, the closure of a subset $E$ of a metric space $X$. Prove that $\overline{E}$ is the intersection of all closed sets in $X$ containing $E$.

**4.4.8 :** Let $(X,d)$ be a metric space.

(1) For any subsets $A$, $B$ of $X$, prove that

(i) $A \subseteq B \Rightarrow A^0 \subseteq B^0$

(ii) $A^0 \cup B^0 \subseteq (A \cup B)^0$

(iii) $A^0 \cap B^0 = (A \cap B)^0$

(2) : Is $A^0 \cup B^0 = (A \cup B)^0$ true ? (Hint : In $(\mathbb{R}', d)$, take $A = (0,1]$, $B = (1,3])$

**4.4.9 :** Let $(X,d)$ be a metric space.

For any subsets $A$, $B$ of $X$, Prove that

(i) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

(2) Is $\overline{A \cap B} = \overline{A} \cap \overline{B}$ true ? (Hint : $In (\mathbb{R}', d)$ take $A = [0,1]$, $B = [1,2])$

**4.4.10 :** For any subset $E$ of a metric space $X$, prove that diam $E$ = diam $\overline{E}$. (use Theorem 4.1.35 or see Theorem 8.1.8).

## 4.5 ANSWERS TO SHORT ANSWER QUESTIONS

**4.2.1 :** (In view of Example 4.1.20),

$$(5, 7) = N_1(6) = 1 \text{ - neighbourhood of } 6.$$

**4.2.2 :** Consider the segment $(a,b)$ in $\mathbb{R}'$ where $a, b \in \mathbb{R}$ with $a<b$. Taking $p=\dfrac{a+b}{2}$, $r=\dfrac{b-a}{2}$, we have

$$N_r(p)=(p-r,\ p+r)=(a,\ b)$$

Thus, every segment is a neighbourhood of some point (infact its mid point).

**4.2.3 :** $N_{\frac{1}{2}}(x)=\left\{y\in x \big/ d(x,y)<\frac{1}{2}\right\}$

$$=\{x\}$$

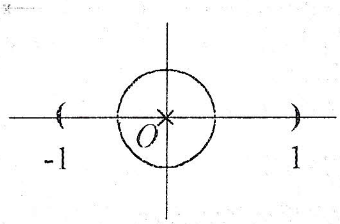
**4.2.4 :** See definition 4.1.10

**4.2.5 :** See Lemma 4.1.16

**4.2.6 :** Consider the $\in - nbd$ of 0 in ($\mathbb{C}$, $d$) i.e. $N_{\in}(0)=\{z\in\mathbb{C} \,/\, |z|<\in\}$.

Clearly, $N_{\in}(0)$ contains infinitely many points in $\mathbb{C}$ which are not in $(-1,1)$. Thus,

$$N_{\in}(0) \not\subset (-1,\ 1).$$



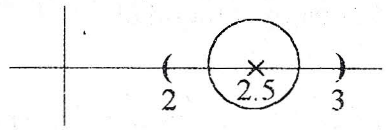

for any $\in > 0$. Hence 0 is not an interior point of $(-1,1)$.

**4.2.7 :** See Lemma 4.1.14.

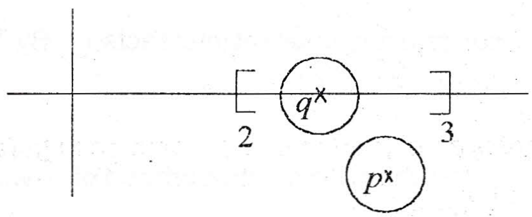
**4.2.8 :** (i) $(2,3)$ is open in $(\mathbb{R}', d)$ since

$(2,3)=N_{\frac{1}{2}}(2\cdot 5)$ and every neighbourhood is an openset.

(ii) $(2,3)$ is not open in ($\mathbb{C}$, $d$) since each neighbourhood of $2\cdot 5\in(2,3)$ contains infinitely many points which are not in $(2,3)$. See the following figure.




**4.2.9 :**




(i) Clearly no point outside [2, 3] is a limit point of [2, 3] : Take a point $p$ in $\mathbb{C}$ out side [2, 3]. We can draw a small circle centred at $p$ not muting the line segment joining 2 and 3. So, $p$ is not a limit point of [2, 3] i.e. [2, 3] contains all of its limit points. Hence [2, 3] is closed.

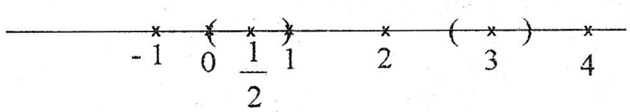
(ii) Let $q\in[2,3]$. Clearly, each $nbd$ of $q$ contains infinitely many points of [2, 3] i.e. $q$ is a limit point of [2, 3]. Thus, every point of [2, 3] is a limit point of [2, 3].

Hence [2, 3] is perfect in ($\mathbb{C}$, $d$).

**4.2.10 :** D

**4.2.11 :** D

**4.2.12 :** Yes (since it has no limit points)




Take a point $p$ in $\mathbb{R}'$.

(i) Suppose $p\in\mathbb{Z}$. $p=3$ say. If we consider $N_{\frac{1}{2}}(3)=(2.5,\ 3.5)$ in $(\mathbb{R}',\ d)$ then it contains no point of $\mathbb{Z}$ other than 3. So, 3 is not a limit point of $\mathbb{Z}$. Thus, no point of $\mathbb{Z}$ is a limit point of $\mathbb{Z}$.

(ii) Suppose $p\notin\mathbb{Z}$. $p=\frac{1}{2}$ say. Clearly $N_{\frac{1}{2}}\left(\frac{1}{2}\right)$ = (0, 1) in $(\mathbb{R}',\ d)$ contains no point of $\mathbb{Z}$ and hence $\frac{1}{2}$ is not a limit point of $\mathbb{Z}$. Thus, $p$ is not a limit point of $\mathbb{Z}$.

Hence, $\mathbb{Z}$ has no limit points in $(\mathbb{R}', d)$.

**4.2.13 :** If $p \in \mathbb{Z}$ then $N_{\frac{1}{2}}(p) \cap \mathbb{Z} = \left(p - \frac{1}{2}, p + \frac{1}{2}\right) \cap \mathbb{Z} = \{p\}$ and hence $p$ is an limited point of $\mathbb{Z}$. (by Lemma 4.1.14).

**4.2.14 :** Clearly, every segment in $(\mathbb{R}', d)$ contains infinitely many rationals i.e. every $nbd$ in $(\mathbb{R}', d)$ intersects $Q$. Hence every openset intersects $Q$. By Theorem 4.1.37, $Q$ is dense in $(\mathbb{R}', d)$.

**4.2.15 :** Let $x \in X$ and $\in > 0$. If $\in \geq 1$ then the neighbourhood of $x$ with radious $\in$ in $(X, \mu)$ is

$$\{y \in X / \mu(x, y) < \in\} = X$$

since $\mu(x, y) < 1$ holds for all $x, y$ in $X$.

Suppose $0 < \in < 1$. Then

$$\mu(x, y) < \in \Leftrightarrow \frac{d(x, y)}{1 + d(x, y)} < \in$$

$$\Leftrightarrow d(x, y) < \frac{\in}{1 - \in}$$

So, $$N_{\in}(x)(\text{in } (X, \mu)) = N_{\frac{\in}{1-\in}}(x) \ (\text{in}(X, d))$$

## 4.6 REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin

Lesson Writer :

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Lesson - 5

# COMPACT SETS

## 5.0 INTRODUCTION

In this lesson, we study the compactness of subsets of a metric space. We prove that every compact subset of a metric space is closed and bounded (see Theorem 5.1.6). Converse of this statement is not true (see Note 5.1.6.1). This converse is true in $\mathbb{R}^k$ (see Theorem 5.1.14); when $k=1$, this converse called Heine Borel Theorem (see note 5.1.12.1). We further study the properties of compact sets.

## 5.1 COMPACT SETS

**5.1.1 Definition :** Let $(X,d)$ be a metric space. Let $E \subseteq X$. By an <u>open cover</u> of $E$, we mean any collection $\{G_\alpha\}_{\alpha \in D}$ of open sets such that each point of $E$ is in atleast one $G_\alpha$ i.e.

$$E \subseteq \bigcup_{\alpha \in D} G_\alpha .$$

**5.1.2 Example :** Consider the metric space $(\mathbb{R}', d)$. Let $E=(0,1)$. Put $G_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ $(n=3, 4, 5, ...)$. Then $\{G_n\}$ is an open cover for $(0,1)$. Clearly, each $G_n$ is open and is contained in $(0,1)$. Furhter,

$$G_3 \subseteq G_4 \subseteq G_5 \subseteq ...............$$

So, $$\bigcup_{n=3}^{\infty} G_n \subseteq E$$

Let $x \in E=(0,1)$. So, $0<x<1$. Choose positive integer $n$ such that $\frac{1}{n} < \min\{x, 1-x\}$. Without loss of generality, we can assume that $n>3$ (or Take $N = \max\{n, 3\}$. Then $\frac{1}{N} \le \frac{1}{n} < \min\{x, 1-x\}$). Hence,

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$$\frac{1}{n} < x < 1 - \frac{1}{n}$$

i.e. $x \in G_n$

So, $$E \subseteq \bigcup_{n=3}^{\infty} G_n$$

Hence $$E = \bigcup_{n=3}^{\infty} G_n$$

i.e. $\{G_n\}_{n=3,4,\ldots\ldots\ldots}$ is an open cover for $E$.

**5.1.3 Definition :** A subset $K$ of a metric space $X$ is called compact if every open cover of $K$ has a finite sub cover.

**5.1.3.1 Explanation :** Let $K$ be a subset of a metric space $X$. The open cover $\{K_\alpha\}_{\alpha\in\Delta}$ has a finite sub cover means there exist finitely many indices $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots, \alpha_n$ in $\Delta$ such that

$$K \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots\ldots\ldots\ldots \cup G_{\alpha_n}$$

$$= \bigcup_{i=1}^{n} G_{\alpha_i}$$

Suppose $K \subseteq Y \subseteq X$ where $X$ is a metric space. Since $Y \subseteq X$, we have that $Y$ is also a metric space with respect to the metric in $X$. Now, we can talk of compactness of $K$ in $X$ as well as in $Y$. The compactness of $K$ in $X$ can be called as the compactness of $K$ relative to $X$ i.e. every open cover of $K$ relative to $X$ has a finite sub cover. Similarly, we have the compactness of $K$ in $Y$. We know that $K$ may be open in $Y$ without being open in $X$. The following theorem gives the relation between the compactness of $K$ in $X$ and the compactness in $Y$.

**5.1.4 Theorem :** Suppose $K \subseteq Y \subseteq X$, where $X$ is a metric space. Then $K$ is compact relative to $X$ if and only if $K$ is compact relative to $Y$.

**Proof :** Assume that $K$ is compact relative to $X$. Let $\{G_\alpha\}_{\alpha\in\Delta}$ be an open cover relative to $Y$

Thus, each $G_\alpha$ is open relative to $Y$ and

$$K \subseteq \bigcup_{\alpha \in \Delta} G_\alpha$$

By Theorem 4.3.33, to each $\alpha \in \Delta$, there exists an open set $H_\alpha$ (relative to $X$) such that

$$G_\alpha = Y \cap H_\alpha$$

So, $\{H_\alpha\}_{\alpha \in \Delta}$ is an open cover for $K$ relative to $X$. By our assumption (i.e. $K$ is compact relative to $X$), there exist $\alpha_1, \alpha_2, \ldots\ldots\ldots.\alpha_n$ in $\Delta$ such that

$$K \subseteq \bigcup_{i=1}^{n} H_{\alpha_i}$$

and hence $$K \subseteq Y \cap \left( \bigcup_{i=1}^{n} H_{\alpha_i} \right)$$

$$= \bigcup_{i=1}^{n} \left( Y \cap H_{\alpha_i} \right)$$

$$= \bigcup_{i=1}^{n} G_{\alpha_i}$$

Hence $K$ is compact relative to $Y$.

Conversely assume that $K$ is compact relative to $Y$.

Let $\{H_\alpha\}_{\alpha \in \Delta}$ be an open cover for $K$ relative to $X$. So, each $H_\alpha$ is open in $X$ and

$$K \subseteq \bigcup_{\alpha \in \Delta} H_\alpha$$

So, $$K \subseteq Y \cap \left( \bigcup_{\alpha \in \Delta} H_\alpha \right) = \bigcup_{\alpha \in \Delta} \left( Y \cap H_\alpha \right) = \bigcup_{\alpha \in \Delta} G_\alpha$$

Where $G_\alpha = Y \cap H_\alpha \, (\alpha \in \Delta)$. Since $H_\alpha$ is open in $X$, $G_\alpha$ is open relative to $Y$ (for each $\alpha \in \Delta$). Thus, $\{G_\alpha\}_{\alpha \in \Delta}$ is an open cover for $K$ relative to $Y$. By our assumption, there exist $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ in $\Delta$ such that

$$K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i} \subseteq \bigcup_{i=1}^{n} H_{\alpha_i}$$

Hence $K$ is compact relative to $X$.

**5.1.5 Theorem :** Closed subsets of compact sets are compact.

**Proof :** Let $K$ be a compact subset of a metric space $X$. Let $F$ be a closed subset of $X$ such that $F \subseteq K$. Let $\{V_\alpha\}_{\alpha \in \Delta}$ be an open cover for $F$ (relative to $X$). Since $F$ is closed, $F^c$ is open in $X$. Then open set $F^c$ together with $\{V_\alpha\}_{\alpha \in \Delta}$ form an open cover for $K$. Since $K$ is compact, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\Delta$ such that

$$K \subseteq F^c \cup \bigcup_{i=1}^{n} V_{\alpha_i} \text{ and hence}$$

$$K \subseteq F \cap \left\{F^c \cup \bigcup_{i=1}^{n} V_{\alpha_i}\right\} = \left(F \cap F^c\right) \cup \left(F \cap \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right)\right)$$

$$= F \cap \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right)$$

$$\subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$$

Hence $F$ is compact.

**5.1.6 Theorem :** Compact subsets of metric spaces are closed and bounded.

**Proof :** Let $K$ be a compact subset of a metric space $X$. $K$ is closed. Inview of theorem 5.1.5, to prove that $K$ is closed it is enough if we prove that the complement $K^c$ of $K$ is open.

Let $p \in K^c$. So, $p \notin K$. Let $q \in K$. So, $p \neq q$. By problem 4.1.34, there exist neighborhoods $V_q$ and $W_q$ of $p$ and $q$ respectively such that $V_q \cap W_q = \phi$. Without loss of generality, we can assume that the radius $\delta_q$ of each $V_q$ and $W_q$ satisfies

$$\delta_q < \frac{1}{2} d(p, q).$$

Thus, to each $q \in K$, we have an open set $W_q$ containing $q$. So, $\{W_q\}_{q \in K}$ is an open cover for $K$. Since $K$ is compact, there exist $q_1, q_2, ........, q_n$ in $K$ such that

$$K \subseteq \bigcup_{i=1}^{n} W_{q_i}$$

Put $\qquad V = N_r(p)$

Wherer $r = \min\{\delta_{q_i} / 1 \le i \le n\}$. Clearly, $V$ is a neighborhood of $p$ and

$$V \subseteq V_{q_i} \qquad (i = 1, 2, ........ n)$$

and hence $\quad V \cap W_{q_i} = \phi$ (Since $V_{q_i} \cap W_{q_i} = \phi$)

for $i = 1, 2, ......., n$. Now,

$$V \cap K \subseteq V \cap \left(\bigcup_{i=1}^{n} W_{q_i}\right)$$

$$= \bigcup_{i=1}^{n} \left(V \cap W_{q_i}\right) = \phi$$

i.e. $V \cap K = \phi$ i.e. $V \subseteq K^c$.

Thus, to each $p \in K^c$, there exists a neighborhood $V$ of $p$ such that $V$ is contained in $K^c$.

i.e. each point of $K^c$ is an interior point of $K^c$

i.e. $K^c$ is open. Hence $K$ is closed.

$K$ **is bounded :** Let $\in > 0$. Clearly, $\{S_\in(x)\}_{x \in K}$ is an open cover for $K$. Since $K$ is compact, there exist $x_1, x_2, ..., x_n$ in $K$ such that

$$K \subseteq \bigcup_{i=1}^{n} S_\in(x_i)$$

Put $M = \text{Max}\{d(x_1, x_i) / 1 \le i \le n\}$. For any $x \in K$. i.e. if $x \in S_\in(x_i) \cap K$, then

$$d(x_1,x) \le d(x_1,x_i)+d(x_i,x)$$
$$< M+\in$$

Thus, for any $x \in K$, $d(x_1,x) < M+\in$. So, $K$ is bounded.

**5.1.6.1 Note :** Converse of Theorem 5.1.6 is not true. Consider the infinite discrete metric space $(X,d)$. Consider the open cover $\{\{x\}/x \in X\}$ of $X$. This cover has no finite sub cover for $X$ as the union of members of every finite subcover of this cover provides only a finite subset of $X$. So $X$ is not compact. Clearly, $X$ is bounded.

**5.1.6.2 Corollary :** If $F$ and $K$ are closed and compact subsets of a metric space $X$ respectively, then $F \cap K$ is compact.

**Proof :** Let $F$ and $K$ be closed and compact subsets of a metric space $X$ respectively. By theorem 5.1.6, $K$ is closed. By theorem 4.1.25(b) $F \cap K$ is closed. Since $K$ is compact, $F \cap K$ is compact (By theorem 5.1.5).

**5.1.7 Definition :** A collection $\{K_\alpha\}_{\alpha\in\Delta}$ of subsets of a metric space is said to have Finite Intersection Property (F.I.P.) if the intersection of every finite subcollection of $\{K_\alpha\}_{\alpha\in\Delta}$ is non-empty.

**5.1.8 Theorem :** The intersection of any collection of compact subsets of a metric space with finite intersection property is non-empty.

**Proof :** Let $\{K_\alpha\}_{\alpha\in\Delta}$ be a collection of compact subsets of metric space with finite intersection property. Suppose

$$\bigcap_{\alpha\in\Delta} K_\alpha = \phi$$

Fix $\alpha_0$ in $\Delta$. By theorem 5.1.6, each $K_\alpha$ is closed and hence each $G_\alpha = K_\alpha^c$ is open (by Corollary 4.1.24.1). Now,

$$\phi = \bigcap_{\alpha\in\Delta} K_\alpha \Rightarrow X = \phi^c = \bigcup_{\alpha\in\Delta} K_\alpha^c = \bigcup_{\alpha\in\Delta} G_\alpha$$

$\Rightarrow \{G_\alpha\}_{\alpha\in\Delta}$ is an open cover for $K_{\alpha_0}$ (since $K_{\alpha_0} \subseteq X$)

$\Rightarrow$ there exist $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n$ in $\Delta - \{\alpha_0\}$ such that

$$K_{\alpha_0} \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots\ldots\ldots \cup G_{\alpha_n} \quad \text{(since } K_{\alpha_0} \text{ is Compact)}$$

$\Rightarrow K_{\alpha_0} \cap K_{\alpha_1} \cap \ldots\ldots\ldots\ldots \cap K_{\alpha_n} = \phi$, a contradiction to our assumption that $\{K_\alpha\}_{\alpha \in \Delta}$ has F.I.P. Hence the theorem.

**5.1.8.1 Corollary :** If $\{K_n\}_{n=1,2,\ldots}$ is a decreasing sequence of non-empty compact sets in a metric space $X$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \phi .$$

**Proof :** Let $\{K_n\}_{n=1,2,\ldots}$ be a decreasing sequence of compact sets in a metric space $X$. By theorem 5.1.8, it is enough if we prove that the family $\{K_n\}_{n=1,2,\ldots}$ has F.I.P. Let $\mathcal{F}$ be a finite sub collection of $\{K_n\}_{n=1,2,\ldots}$. With out loss of generality, we can write

$$\mathcal{F} = \{K_{n_1}, K_{n_2}, \ldots\ldots\ldots\ldots, K_{n_\ell}\}$$

with $n_1 < n_2 < \ldots\ldots\ldots < n_\ell$. So,

$$K_{n_1} \supseteq K_{n_2} \supseteq \ldots\ldots\ldots\ldots \supseteq K_{n_\ell}$$

Now,

$$\cap \{F / F \in \mathcal{F}\} = K_{n_\ell} \neq \phi$$

Hence $\{K_n\}_{n=1,2,\ldots}$ has F.I.P. Hence the Corollary.

**5.1.9 Theorem :** If $E$ is an infinite subset of a compact set $K$, in a metric space. Then $E$ has a limit point in $K$.

**Proof :** Let $E$ be an infinite subset of a compact set $K$ in a metric space $X$. Suppose no point of $K$ is a limit point of $K$.

$p \in K \Rightarrow p$ is not a limit point of $E$.

$\Rightarrow$ there exists neighborhood $N_p$ of $p$ such that $N_p \cap E - \{p\} = \phi$

i.e. $N_p \cap E \subseteq \{p\}$

Thus, $\{N_p\}_{p\in K}$ is an open cover for $K$. Since $K$ is compact, there exist $p_1, p_2, \ldots\ldots\ldots\ldots, p_n$ in $K$ such that

$$K \subseteq \bigcup_{i=1}^{n} N_{p_i}$$

Now, $$E = E \cap K$$

$$\subseteq E \cap \left( \bigcup_{i=1}^{n} N_{p_i} \right)$$

$$= \bigcup_{i=1}^{n} \left( E \cap N_{p_i} \right)$$

$$\subseteq \bigcup_{i=1}^{n} \{p_i\}$$

$$= \{p_1, p_2, \ldots\ldots\ldots, p_n\}$$

So, $E$ is finite, a contradiction. Hence $E$ has a limit point in $K$.

**5.1.9.1 Note :** The property mentioned in Theorem 5.1.9 is called Bolzano-Weierstress property.

**5.1.10 Theorem :** If $\{I_n\}_{n=1,2,\ldots.}$ is a sequence of intervals in $\mathbb{R}'$ such that

$$I_n \supseteq I_{n+1} \qquad (n = 1, 2, \ldots\ldots\ldots)$$

Then $$\bigcap_{n=1}^{\infty} I_n \neq \phi .$$

**Proof :** Let $\{I_n\}_{n=1,2,\ldots\ldots\ldots}$ be a sequence of intervals in $\mathbb{R}'$ such that

$$I_n \supseteq I_{n+1} \qquad (n = 1, 2, \ldots\ldots\ldots.)$$

Let $I_n = [a_n, b_n]$ $(n = 1, 2, \ldots\ldots\ldots)$. Put

$$E = \{a_n / n \geq 1\}$$

For $n=1,2,\ldots\ldots$

$$I_n \subseteq I_1 \Rightarrow a_1 \le a_n \le b_n \le b_1.$$

So, $E$ is bounded above by $b_1$. Let $x = \text{lub}\, E$ ( which exists)

For any integers $m, n$

$$a_n \le a_{n+m} \le b_{n+m} \le b_m.$$

So, each $b_m$ is an upper bound of $E$. Since $x = \text{lub}\, E$; $x \le b_n$ for all $n$. But $a_n \le x$ for all $n$. Hence $x \in I_n$ for all $n$. Hence the conclusion.

We know that a $k-$ cell in $\mathbb{R}^k$ is the Cartesian product of $k$ bounded closed intervals of $\mathbb{R}$ (See Definition 4.1.10 and Note 4.1.10.1(i)).

**5.1.11 Theorem :** Let $k$ be a positive integer. If $\{I_n\}_{n=1,2,\ldots\ldots}$ is a decreasing sequence of $k-$cells, then

$$\bigcap_{n=1}^{\infty} I_n \ne \phi.$$

**Proof :** Let $\{I_n\}_{n=1,2,\ldots\ldots}$ be a decreasing sequence of $k-$cells. Then we can write each $I_n$ as

$$I_n = \prod_{i=1}^{k} I_{n_i}$$

where $I_{ni} = \left[a_{n_i}, b_{n_i}\right]$ $(i=1,2,\ldots,k)$. Clearly, $I_n \supseteq I_{n+1} \Leftrightarrow I_{n_i} \supseteq I_{n+1_i}$ $(i=1,2,\ldots\ldots,k)$

Fix $i$ such that $1 \le i \le k$. Now $\{I_{ni}\}_{n=1,,\ldots}$ is a decreasing sequence of intervals in $\mathbb{R}'$. By theorem 5.1.10,

$$\bigcap_{n=1}^{\infty} I_{n_i} \ne \phi$$

Choose $x_i$ in this set. Thus, we have $x_1, x_2, \ldots\ldots\ldots, x_k$. Clearly $x = (x_1, x_2, \ldots\ldots, x_k)$ lies in each $I_n$. Hence the conclusion.

**5.1.12 Theorem :** Every $k-$ cell is compact.

**Proof :** Let $I$ be a $k-$ cell. So,

$$I = \prod_{i=1}^{k} I_i,$$

the product of the intervals $I_i = [a_i, b_i]$ $(i = 1, 2, ...., k)$. Put

$$\delta = \left\{ \sum_{i=1}^{k} (b_i - a_i)^2 \right\}^{\frac{1}{2}}$$

Clearly, $x, y \in I$ implies $\|x - y\| \le \delta$. Suppose $I$ is not compact. So, there exists an open cover $\{G_\alpha\}_{\alpha \in \Delta}$ for $I$ which has no finite sub cover of $I$. Put $C_i = {(a_i + b_i)}/{2}$. The intervals $[a_i, c_i]$ and $[c_i, b_i]$ then determine $2^k$ $k-$ cells $Q_i$ whose union is $I$. Since $\{G_\alpha\}_{\alpha \in \Delta}$ has no finite sub cover for $I$, at least one of these $Q_i$s cell it $I_1$ is not covered by any finite sub cover of $\{G_\alpha\}$.

We, now divide $I_1$ and continue the process. We obtain a sequence $\{I_n\}$ of $k-$cells with the following properties.

(a) $I \supseteq I_1 \supseteq I_2 \supseteq .........;$

(b) $I_n$ is not covered by any finite sub collection of $\{G_\alpha\}$;

(c) $x \in I_n,\ y \in I_n$ implies $\|x - y\| \le z^{-n} \delta$.

By (a) and Theorem 5.3.11, there exists a point $x^*$ in $\mathbb{R}^k$ such that $x^* \in I_n$ for all $n$. Since $\{G_\alpha\}$ is an open cover of $I$, $x^* \in G_\alpha$ for some $\alpha$. Since $G_\alpha$ is open, there exists $r > 0$ such that $\|y - x^*\| < r$ implies $y \in G_\alpha$ (i.e. $N_r(x^*) \subseteq G_\alpha$). Choose a positive integer $n$ such that $2^{-n} \delta < r$ (of course this is possible). Clearly, $I_n \subseteq N_r(x^*) \subseteq G_\alpha$, a contradiction to the property (b). Hence $I$ is compact.

**5.1.12.1 Note :** When $k=1$, theorem 5.1.12 is called Heine-Borel Theorem. Thus, Heine Borel Theorem is - "Every closed and bounded interval on the real line $\mathbb{R}$ is compact". Inview of theorem 5.1.5 we can state the Heine - Borel Theorem as "Every closed and bounded subset of the real line is compact".

**5.1.13 Theorem :** Every bounded subset of $\mathbb{R}^k$ is contained in a $k-$cell.

**Proof :** Let $E$ be a bounded subset of $\mathbb{R}^k$. So, there exists $q=(q_1, q_2, \ldots\ldots\ldots\ldots, q_k)$ in $\mathbb{R}^k$ and a positive real $M$ such that

$$x=(x_1, x_2, \ldots\ldots\ldots\ldots, x_k) \in E \Rightarrow \|q-x\| \leq M$$

$$\Rightarrow |q_i - x_i| \leq \|q-x\| \leq M \text{ for } (i=1,2,\ldots\ldots\ldots,k)$$

$$\Rightarrow q_i - M \leq x_i \leq q_i + M \quad (i=1,2,\ldots\ldots\ldots,k)$$

$$\Rightarrow x_i \in [q_i - M,\ q_i + M] \ (= I_i \text{ say})$$

$$(i=1,2,\ldots\ldots\ldots,k)$$

$$\Rightarrow x \in \prod_{i=1}^{k} I_i \ (= I \text{ say})$$

Clearly, $I$ is a $k-$cell and $E \subseteq I$. Hence every bounded set in $\mathbb{R}^k$ is contained in a $k-$cell.

**5.1.14 Theorem :** Let $E$ be a subset of $\mathbb{R}^k$. The following are equivalent.

(a) $E$ is closed and bounded

(b) $E$ is compact

(c) Every infinite subset of $E$ has a limit point in $E$.

**Proof :** $(a) \Rightarrow (b)$ : Assume (a) i.e. $E$ is closed and bounded. By Theorem 5.1.13, there exists a $k-$ cell $I$ such that $E \subseteq I$ (since $E$ is bounded). By theorem 5.1.12, $I$ is compact. Thus $E$ is a closed subset of the compact set $I$. By Theorem 5.1.5, $E$ is compact.

$(b) \Rightarrow (c)$: Follows from Theorem 5.1.9.

$(c) \Rightarrow (a)$: Assume (c) i.e. every infinite subset of $E$ has a limit point in $E$.

Suppose $E$ is not bounded. So, to each positive integer $n$, there exists a point $x_n$ in $E$

such that $\|x_n\|>n$. Let $S=\{x_n/n\geq 1\}$. Clearly, $S$ is an infinite subset of $E$ and has no limit point in $\mathbb{R}^k$ and hence in $E$, a contradiction to our assumption. So, $E$ is bounded.

Let $x_0 \in \mathbb{R}^k$ is a limit point of $E$. Suppose $x_0 \notin E$. For $n=1,2,....$, there exist points $x_n$ in $E$ such that $\|x_n - x_0\| < \dfrac{1}{n}$. Let $S$ be the set of these points $x_n$. Then, $S$ is infinite (otherwise $|x_n - x_0|$ is a constant positive value for infinitely many $n$) and clearly, $x_0$ is a limit point of $S$. Now, we observe that $S$ has no limit point other than $x_0$. For if $y \in \mathbb{R}^k$, $Y \neq x_0$ then

$$\|x_n - y\| \geq \|x_0 - y\| - \|x_n - x_0\|$$

$$\geq \|x_0 - y\| - \frac{1}{n} \geq \frac{1}{2}\|x_0 - y\|$$

for all but finitely many $n$. So, $y$ is not a limit point of $S$. Hence $x_0$ is the only limit point of $S$. By our assumption, $x_0 \in E$. Thus, $E$ contains all of its limit points i.e. $E$ is closed.

**5.1.15 Theorem (Weierstrass) :** Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

**Proof :** Let $E$ be a bounded infinite subset of $\mathbb{R}^k$. By Theorem 5.1.12, there exists a $k-$ cell $I$ such that $E \subseteq I$. By Theorem 5.1.12, $I$ is compact. By Theorem 5.1.9, $E$ has a limit point in $I$ and hence in $\mathbb{R}^k$.

## 5.2 SHORT ANSWER QUESTIONS

**5.2.1 :** Prove that every finite subset of a metric space is compact.

**5.2.2 :** Is every compact set finite ?

**5.2.3 :** Is $(0,1)$ a compact set in $\mathbb{R}$ ?

**5.2.4 :** We know that every compact subset of a metric space is closed and bounded. Is the converse of this statement true ?

**5.2.5 :** Is every closed subset of a metric space compact ?

**5.2.6 :** In every bounded subset of a metric space compact ?

**5.2.7 :** What is the speciality of Heine-Borel Theorem ?

**5.2.8 :** Is every closed subset of compact set compact ?

**5.2.9 :** State Bolzano - Weierstrass property

**5.2.10 :** Give example of a compact set of real numbers whose limit points form a countable set

**5.2.11 :** Define $k-$ cell.

**5.2.12 :** Prove that every bounded set in $\mathbb{R}^k$ is contained in a $k-$ cell.

## 5.3 MODEL EXAMINATION QUESTIONS :

**5.3.1 :** Prove that a metric space $X$ is compact if and only if $X$ has Bolzano - Weierstrass property i.e. every infinite subset of $X$ has a limit point in $X$.

**5.3.2 :** Characterize discrete compact metric spaces.

**5.3.3 :** Let $X$ be the set of all rational numbers with metric $d$ on $X$ defined by $d(x,y)=|x-y|$.

Let $E=\{x \in X / 2 < x^2 < 3\}$

Prove that $E$ is bounded, but not compact.

**5.3.4 :** Let $X$ be a metric space. Let $K \subseteq Y \subseteq X$. Prove that $K$ is compact relative to $X$ if and only if $K$ is compact relative to $Y$.

**5.3.5 :** Prove that every $k-$cell is compact.

**5.3.6 :** Prove that every infinite subset of $\mathbb{R}^k$.

**5.3.7 :** Prove that every closed subset of a compact set is compact.

**5.3.8 :** Prove that every compact set is closed and bounded. Is converse true ? Justify your answer.

**5.3.9 :** Let $E$ be a subset of $\mathbb{R}^k$. Prove that the following are equivalent.

(a) $E$ is closed and bounded

(b) $E$ is compact

(c) Every infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

## 5.4 EXERCISES :

**5.4.1 :** Define compact set. Prove that every finite open interval in $\mathbb{R}'$ is not compact (i.e. if $a,b$ are real numbers such that $a<b$, prove that $(a,b)$ is not compact).

(Hint : See the open cover in Example 5.3.2 for $(0,1)$. Try to imitate).

**5.4.2 :** Prove that every closed subset of a compact set is compact.

**5.4.3 :** Prove that any compact set in a metric space is closed and bounded. Is converse true? Justify your answer (For converse : consider any infinite set $X$ together with discrete metric. Clearly $X$ is closed and bounded, but not compact.)

**5.4.4 :** Prove that every $k-$ cell is compact.

**5.4.5 :** Prove that a subset $K$ of a metric space is compact if and only if every infinite subset of $K$ has a limit point in $K$.

**5.4.6 :** Defnite finite intersection property. Prove that a metric space $X$ is compact if and only if every collection of closed sets in $X$ with finite intersection property has non-empty intersection.

**5.4.7 :** Let $E$ be a subset of $\mathbb{R}^k$. Prove that the following statements are equivalient :

(a) $E$ is closed and bounded

(b) $E$ is compact

(c) Every infinite subset of $E$ has a limit point in $E$.

**5.4.8 :** Let $X$ be a metric space.

(i) A real number $a>0$ is called a Lebesque number for an open cover $\{G_i\}$ of $X$ if every subset of $X$ whose diameter is less than a is contained in at least one $G_i$.

Prove that in a sequentially compact metric space, every open cover has a Lebesque number.

(ii) Let $\in>0$. A finite subset $A$ of $X$ is called an $\in-net$ if $X=\bigcup\limits_{a\in A} S_{\in}(a)$.

$X$ is said to be totally bounded if $X$ has an $\in-net$ for every $\in>0$.

Prove that every totally bounded set is bounded. Is converse true ? Justify your answer.

(iii) Prove that every sequentially compact metric space is totally bounded.

(iv) Prove that every sequentially compact metric space is compact.

**5.4.9 :** Let $(X,d)$ be a metric space. Prove that the following statements are equivalent.

(a) $X$ is compact

(b) $X$ is sequentially compact.

(c) $X$ has Bolzano - Weierstrass property.

**5.4.10 :** Let $A$ be a subset of a metric space $X$. Prove that $A$ is totally bounded if and only if $\overline{A}$ is totally bounded.

**5.4.11 :** Prove that a subset of $\mathbb{R}^n$ is bounded if and only if it is totally bounded.

**5.4.12 :** Prove that a compact metric space is separable.

**5.4.13 :** Let $X$ be a closed and bounded subset of $\mathbb{R}^n$. Prove that every infinite subset of $X$ has a limit point in $X$.

## 5.5 ANSWERS TO S.A.Q.s

**5.2.1 :** Let $K$ be a finite subset of a metric space $X$. Let $K=\{x_1, x_2, ......, x_n\}$. Let $\{G_\alpha\}_{\alpha\in\Delta}$ be an open cover of $K$. So, each point in $K$ is in some $G_\alpha$. Thus, to each $i=1,2,.........n$, there exist $\alpha_1, \alpha_2, ...., \alpha_n$ in $\Delta$ such that

$$x_i \in G_{\alpha_i}$$

for $i=1,2,..........,n$. Clearly,

$$K \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

Hence $K$ is compact.

**5.2.2 :** By Heine-Borel Theorem, $[0,1]$ is a compact subset of $\mathbb{R}$. By theorem $[0,1]$ is uncountable. Thus, every compact subset of a metric space is not finite.

**5.2.3 :** Consider the open cover $\{G_n\}_{n=3,4,....}$ of $E=(0,1)$ in $\mathbb{R}$, where

$$G_n = \left(\frac{1}{n}, \cdot - \frac{1}{n}\right) (n=3,4,........)$$

Suppose $E$ is compact. So, there exist positive integers $n_1, n_2, .........., n_k$ such that $n_1 < n_2 < ........ < n_k$ and

$$E \subseteq \bigcup_{i=1}^{k} G_{n_i} = G_{n_k}$$

i.e. $(0,1) \subseteq \left(\frac{1}{n_k}, 1-\frac{1}{n_k}\right)$ a contradiction. So, $(0,1)$ is not compact.

**5.2.4 :** No. consider the open interval in $\mathbb{R}'$. Clearly, $(0,1)$ is bounded (since, for any $x$ in $(0,1)$ $|x-0|<2$). We know that $(0,1)$ is not compact.

**5.2.5 :** Consider the metric space $(X,d)$, where $X=\mathbb{R}$, $d(x,y)=|x-y|$. Clearly $\mathbb{R}$ is closed but not compact since the open cover

$\{G_n=(-n, n)\}_{n=1,2,......}$ has no finite subcover.

**5.2.6 :** Clearly $(0,1)$ is a bounded subset of $\mathbb{R}$ but not compact.

**5.2.7 :** We know that every compact subset of a metric space is closed and bounded. The converse of this statement "Every closed and bounded subset of a metric space is compact" is not true ingeneral; but it is true in the case of the Real line with usual metric.

**5.2.8 :** Yes. (See Theorem 5.1.5)

**5.2.9 :** See Note 5.3.9.1

**5.2.10 :** Consider the subset $E$ of $\mathbb{R}$ given by

$$E=\left\{\frac{1}{n}\Big/ n\geq 1\right\} \cup \{0\}.$$

Clearly, 0 is the only limit point of $E$ in $\mathbb{R}$. Now, we prove that $E$ is compact. Let $\{G_\alpha\}_{\alpha\in\Delta}$ be an open cover for $E$. So, $0\in G_{\alpha_0}$ for some $\alpha_0$ in $\Delta$. Since $G_{\alpha_0}$ is open, there exists $\in>0$ such that $(0-\in, 0+\in)\subseteq G_{\alpha_0}$ i.e. $(-\in, \in)\subseteq G_{\alpha_0}$. Choose a positive integer $N$ such that $\frac{1}{N}<\in$. For $n\geq N$,

$$\left|\frac{1}{n}\right|=\frac{1}{n}\leq\frac{1}{N}<\in \Rightarrow \frac{1}{n}\in(-\in, \in)\subseteq G_{\alpha_0}.$$

To each positive integer $i = 1, 2, ...., N-1$, there exist $\alpha_1, \alpha_2, ........, \alpha_{N-1}$ in $\Delta$ such that

$$\frac{1}{i} \in G_{\alpha_i} \quad (i = 1, 2, ....., N-1)$$

Clearly,

$$E \subseteq \bigcup_{i=0}^{N-1} G_{\alpha_i}$$

So, $E$ is compact.

**5.2.11 :** See definition 4.1.10.

**5.2.12 :** See Theorem 5.1.13

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Principles of Mathematical Analysis, Third Edition, McGraw - Hill International Editions : Walter Rudin

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8

Lesson - 6

# PERFECT AND CONNECTED SETS

## 6.0 INTRODUCTION

In this lesson, we study the concept of a perfect set and observe that every non-empty perfect set in $\mathbb{R}^k$ is uncountable (see Theorem 6.1.3). We study the construction of Cantor set and prove that Cantor set is a perfect uncountable set of measure 0.

Further, we study the concept of a connected set in a metric space and characterize the connected subsets of the real line $\mathbb{R}$ (See Theorem 6.2.4).

## 6.1 PERFECT SETS

**6.1.1 Definition :** Let $X$ be a metric space. A subset $E$ of $X$ is called a perfect set if $E$ is closed and every point of $E$ is a limit point of $E$.

**6.1.1.1 Note :** Clearly a subset $E$ of a metric space is perfect if and only if $E = E'$.

**6.1.2 Examples :**

(i) Consider the metric space $(\mathbb{R}', d)$. Every interval $[a, b]$ where $a, b \in \mathbb{R}$ with $a < b$ is perfect.

(ii) Consider the metric space $(\mathbb{R}^2, d)$. The set $\{ z \in \mathbb{C} = \mathbb{R}^2 / |z| \leq 1\}$ is perfect.

In fact, $N_r[z]$ is perfect for any $z \in \mathbb{C}$ and real $r > 0$.

**6.1.3 Theorem :** Every non-empty perfect set in $\mathbb{R}^k$ is uncountable.

**Proof :** Let $P$ be a non-empty perfect set in $\mathbb{R}^k$. So, $P$ is closed and every point of $P$ is a limit point. Since $P \neq \phi$, $P$ has atleast one limit point. By Theorem 4.1.19, $P$ is infinite.

Assume that $P$ is countable. So, $P$ can be written as

$$P = \{x_1, x_2, \ldots\ldots\ldots\}$$

Let $V_1 = N_r(x_1)$ be a neighbourhood of $x_1$. Let $\overline{V_1}$ be the closure of $V_1$ i.e. $\overline{V_1} = N_r[x_1]$. $x_1 \in P$ and $P$ is perfect implies $x_1$ is a limit point of $P$. So,

$$V_1 \cap P - \{x_1\} \neq \phi.$$

Without loss of generality, we can assume that

$$x_2 \in V_1 \cap P - \{x_1\}$$

Choose a neighborhood $V_2$ of $x_2$ such that

$$\overline{V_2} \subseteq V_1, \ x_1 \notin \overline{V_2}.$$

Since $x_2$ is a limit point of $P$,

$$V_2 \cap P - \{x_2\} \neq \phi$$

without loss of generality, we can assume that

$$x_3 \in V_2 \cap P - \{x_2\}$$

Choose a neighborhood $V_3$ of $x_3$ such that

$$\overline{V_3} \subseteq V_2, \ x_2 \notin \overline{V_3}$$

Continuing this process, suppose we have a neighborhood $V_n$ of $x_n$ such that

$$V_n \cap P - \{x_n\} \neq \phi.$$

Without loss of generality, we can assume that

$$x_{n+1} \in V_n \cap P - \{x_n\}$$

Now, we choose a neighborhood $V_{n+1}$ of $x_{n+1}$ such that

$$\overline{V_{n+1}} \subseteq V_n, \ x_n \notin \overline{V_{n+1}}$$

Thus, we have a sequence $\{V_n\}$ of neighbourhoods such that

$$V_n \cap P \neq \phi$$

for all $n$. Let $K_n = \overline{V_n} \cap P$. Since $x_n \notin \overline{V_{n+1}}$, $x_n \notin K_{n+1}$.

Since each $K_n \subseteq P$, no point of $P$ is in $\bigcap_{n=1}^{\infty} K_n$ i.e. $\bigcap_{n=1}^{\infty} K_n = \phi$.

Since $P$ is closed, each $K_n$ is closed. Since each $\overline{V_n}$ is bounded, each $K_n$ is bounded. By theorem 5.1.14, each $K_n$ is compact. Further,

$$K_n \supseteq K_{n+1} \qquad (n=1,2,........)$$

By Corollary 5.2.8.1,

$$\bigcap_{n=1}^{\infty} K_n \neq \phi, \text{ a Contradiction.}$$

Hence $P$ is uncountable.

**6.1.3.1 Corollary :** Every interval $[a, b]$ $(a,b \in \mathbb{R},\ a<b)$ in $\mathbb{R}$ is uncountable. In particular, the set of all real numbers is uncountable.

**Proof :** We know that $[a, b]$ is a non-empty perfect set in $\mathbb{R}$. By the above theorem 6.1.3, $[a, b]$ is uncountable. Since $[a, b] \subseteq \mathbb{R}$, $\mathbb{R}$ is uncountable.

**6.1.4 The Cantor set :** The set which we are now going to construct shows that there exist perfect sets in $\mathbb{R}'$ which contain no segment.

Let $E_0 = [0,1]$. Let $E_1$ be the subset of $E_0$ obtained by removing the middle one third segment $\left(\frac{1}{3}, \frac{2}{3}\right)$ i.e.,

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Let $E_2$ be the subset of $E_1$ by removing the middle one third segments namely $\left(\frac{1}{9}, \frac{2}{9}\right)$ of $\left[0, \frac{1}{3}\right]$ and $\left(\frac{7}{8}, \frac{8}{9}\right)$ of $\left[\frac{2}{3}, 1\right]$ i.e.,

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Continuing this process ( of removing the middle one third of intervals ) we have a sequence $\{E_n\}$ such that

(i) $E_n \supseteq E_{n+1}$ for all $n \in N$

(ii) each $E_n$ is the union of $2^n$ intervals each of length $3^{-n}$.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$ is called Cantor set.

**6.1.5 Properties of Cantor set :**

**(a) The Cantor set is non-empty :** Clearly, $0 \in P$. Hence $P$ is non-empty. Infact, for each $i = 0, 1, 2, \ldots\ldots..$, the end points of the closed intervals that are appearing in $E_i$ (as the union of $2^i$ closed intervals) are in the Cantor set $P$ i.e., the points

$$0,\ 1;$$

$$0, \frac{1}{3}; \frac{2}{3}, 1;$$

$$0, \frac{1}{9}; \frac{2}{9}, \frac{3}{9} = \frac{1}{3}; \frac{2}{3}, \frac{7}{9}; \frac{8}{9}, 1;$$

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are in $P$.

**(b) The Cantor set is compact :** We know that every $k-$ cell in $\mathbb{R}^k$ is compact. So, $E_0$ is compact. Each $E_i$ is closed, since $E_i$ is a finite union of closed intervals. We know that arbitrary intersection of closed sets is closed. So, $P$ is a closed subset of the compact set $E_0$. By theorem 5.1.5, $P$ is compact.

**(c) The Cantor set contains no segment :** Clearly, no segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

where $k$ and $m$ are positive integers, has a point in common with $P$. Let $(\alpha, \beta)$ be a segment. If we choose a positive integer $m$ such that

$$3^{-m} < \frac{\beta - \alpha}{6}$$

then $(\alpha, \beta)$ contains a segment of the form (above). So, $P$ contains no segment.

**(d) The Cantor set is perfect (and hence Uncountable by Theorem 6.2.3) :** Clearly, $P$ is closed. Now, we show that every point of $P$ is a limit point of $P$. Let $x \in P$. Let $S$ be a neighborhood $N_{\in}(x) = (x - \in, x + \in)$ of $x$. Choose a positive integer $n$ such that $3^{-n} < \in$. Now $x \in E_n$ (for this $n$). So, $x \in I_n$ where $I_n$ is the closed interval among the $2^n$ closed intervals whose union is equal to $E_n$. Clearly $\ell(I_n) = 3^{-n} < \in$. Now, for any $y \in I_n$, $|x - y| \leq$ length of $I_n = 3^{-n} < \in$ and hence $y \in (x - \in, x + \in)$. So, $I_n$ is contained in $(x - t, \ x + t)$. Choose that end of point of $z$ of $I_n$ such that $x \neq z$. Clearly, $z \in (x - t, \ x + t) \cap P - \{x\}$. Thus, every neighborhood of $x$ contains atleast one point of $P$ other than $x$. So, $x$ is a limit point of $P$. Thus, every point of $P$ is a limit point of $P$. Hence $P$ is perfect. By theorem 6.1.3, $P$ is uncountable.

**(e) Measure of $P$ is 0 (zero) :** Sum of the lengths of the open intervals removed is

$$\sum_{n=1}^{\infty} 2^{n-1} \cdot \frac{1}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{2} \times \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 1$$

So, measure of $P$ is 0.

**6.1.5.1 Note :** In measure theory, we know that the measure of any countable set is zero. The Cantor set stands as an example for an uncountable set of measure zero.

## 6.2 CONNECTED SETS

**6.2.1 Definition :** A set $E$ in a metric space $X$ is said to be connected if there do not exist two disjoint open subsets $A$ and $B$ of $X$ such that both $A$ and $B$ intersect $E$, and $E \subseteq A \cup B$.

**6.2.1.1 Note :** The above definition is infact connectedness of $E$ relative to $X$.

**6.2.2 Theorem :** A set $E$ in a metric space $X$ is connected if and only if $E$ is connected relative to $E$.

**Proof :** Let $E$ be a subset of a metric space $X$. Assume that $E$ is connected (relative to $X$). Suppose $E$ is not connected relative to $E$. So, there exist non-empty disjoint sets $G$, $H$, open relative to $E$ such that (both $G$ and $H$ intersect $E$) and $E \subseteq G \cup H$ i.e. $E = G \cup H$.

Since $G$ is open relative to $E$, to each $p \in G$, there exists $\delta_p > 0$ such that $q \in E,\ d(p,q) < \delta_p$ implies $q \in G$. Similarly, to each $q \in H$, there exists $\delta_q > 0$ such that $p \in E,\ d(p,q) < \delta_q$ implies $p \in H$.

For any $p \in G,\ q \in H$, both these inequalities fail and hence

$$d(p,q) \geq \frac{1}{2}\left(\delta_p + \delta_q\right).$$

For $p \in G,\ q \in H$, let $V_p$ be the set of all $x$ in $X$ such that $2d(p,x) < \delta_p$ and let $W_q$ be the set of all $x$ in $X$ such that $2d(q,x) < \delta_q$. Clearly, $V_p \cap W_q = \phi$ (otherwise we can choose $x \in V_p \cap W_q$ and hence

$$d(p,q) \leq d(p,x) + d(x,q) \leq \frac{1}{2}\left(\delta_p + \delta_q\right),$$ a contradiction). Put

$$A = \bigcup_{p \in G} V_p, \qquad B = \bigcup_{q \in H} W_q$$

Clearly, $A$ and $B$ are non-empty dis-joint open sets such that both $A$ and $B$ intersect $E$ and $E = G \cup H \subseteq A \cup B$. So, $E$ is not connected relative to $X$, a Contradiction. Hence $E$ is connected relative to $E$.

Conversely assume that $E$ is connected relative to $E$. Suppose $E$ is not connected (relative to $X$). So, there exist non-empty disjoint open subsets $A$ and $B$ of $X$ such that both $A$ and $B$ intersect $E$ and $E \subseteq A \cup B$. Put $G = A \cap E,\ H = B \cap E$. Clearly $G$ and $H$ are non-empty disjoint open subsets of $E$ (relative to $E$) such that

$$E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B) = G \cup H.$$

So, $E$ is not connected relative to $E$, a contradiction. So, $E$ is connected relative to $X$.

**6.2.3 Definition :** A subset $E$ of the real line $\mathbb{R}'$ is called an interval if $x \in E,\ y \in E$ and $x < z < y$ then $z \in E$.

**6.2.4 Theorem :** A subset $E$ of the real line $\mathbb{R}'$ is connected if and only if $E$ is an interval.

**Proof :** Let $E \subseteq \mathbb{R}'$. Assume that $E$ is connected. Suppose $E$ is not an interval. So, there exist real numbers. $x, y, z$ such that $x<z<y$, $x \in E$, $y \in E$ and $z \notin E$. Put $A=(-\infty, z)$, $B=(z, \infty)$. Clealry, $A, B$ are non empty (since $x \in A$, $y \in B$) disjoint opensets in $\mathbb{R}$, such that both $A$ and $B$ intersect $E$ (as $x \in A \cap E$ and $y \in B \cap E$) and

$$E \subseteq \mathbb{R}-\{z\}$$

$$=(-\infty, z) \cup (z, \infty)=A \cup B$$

So, $E$ is not connected, a contradiction to our assumption. Hence $E$ is an interval.

Conversely assume that $E$ is an interval. Suppose $E$ is not connected. So, there exist disjoint open sets $A$, $B$ such that both $A$ and $B$ intersect $E$ and $E \subseteq A \cup B$. Let $x \in A \cap E$, $y \in B \cap E$. With out loss of generality, we can assume that $x<y$. Let

$$S=A \cap [x, y]$$

Clearly $S$ is a non-empty set of reals (since $x \in S$) bounded above by $y$. So, lub $S$ exists say $z$. Since $S$ is bounded, $z \in \bar{A}$. Clearly, $x \leq z$ (since $x \in S$ and $z$ is an upper bound) $\leq y$ (since $y$ is an upper bound and $z=$lub $S$). By our assumption, $z \in E$.

$A \cap B=\phi \Rightarrow A \subseteq B^c \Rightarrow \bar{A} \subseteq \overline{B^c}=B^c$ (since $B$ is open, $B^c$ is closed and hence $\overline{B^c}=B^c$) $\Rightarrow \bar{A} \cap B=\phi$. Similarly, we have $A \cap \bar{B}=\phi$).

Since $z \in \bar{A}$, $z \notin B$ and hence $z \neq y$ i.e. $z<y$. Since $E \subseteq A \cup B$, $z \in A$. So, $z \notin \bar{B}$ (Since $A \cap \bar{B}=\phi$). So, $z$ is not a limit point of $B$ and hence there exists $\in>0$ such that

$$(z-\in, z+\in) \cap B-\{z\}=(z-\in, z+\in) \cap B \text{ (since } z \notin B) =\phi \quad \text{------- (1)}$$

Without loss of generality, we can assume that $z+\in<y$ (Choose $\in$ such that $0<\in<y-z$). Take $z_1$ such that

$$x \leq z<z_1<z+\in<y.$$

Then $z_1 \notin B$ (by (1)) and $z_1 \in [x, y]$. By our assumption, $z_1 \in E$. Since $E \subseteq A \cup B$, $z_1 \in A$ and hence $z_1 \leq z$, a contradiction to $z<z_1$. Hence $E$ is connected.

**6.2.4.1 Corollary :** A set $E$ in $\mathbb{R}'$ is connected if and only if $E$ is one of the following sets (where $a$ and $b$ are reals, $a \leq b$):

$(-\infty, b)\,(-\infty, b],\ (a, \infty), [a, \infty), (-\infty, \infty), (a,b), [a,b), (a,b], [a,b]$.

## 6.3 SHORT ANSWER QUESTIONS

**6.3.1 :** Define Perfect set.

**6.3.2 :** Is the set $(0,1)$ perfect in $\mathbb{R}'$ ?

**6.3.3 :** Is the set $[0,1]$ perfect in $\mathbb{R}'$ ?

**6.3.4 :** Is the set

$P=\{z \in \sigma / |z| \leq 1\}$ perfect in ($\mathbb{C}$, $d$) where $d(z_1, z_2) = |z_1 - z_2|$ ?

**6.3.5 :** Is every non-empty perfect set in $\mathbb{R}^k$ uncountable ?

**6.3.6 :** Is every finite non-empty set in a metric space $X$ perfect ?

**6.3.7 :** Construct cantor set.

**6.3.8 :** Define connected set.

**6.3.9 :** State precisely the connected subsets of the real line $\mathbb{R}$.

**6.3.10 :** What are the connected subsets of the discrete metric space ?

## 6.4 MODEL EXAMINATION QUESTIONS

**6.4.1 :** Define perfect set and prove that every non-empty perfect set in $\mathbb{R}^k$ is uncountable.

**6.4.2 :** Describe the construction of the Cantor set. Prove the following :

(a) Cantor set is compact.

(b) Cantor set is perfect.

(c) Cantor set is uncountable.

**6.4.3 :** Define connected set. Prove that a subset $E$ of $\mathbb{R}$ is connected if and only if "$x \in E$, $y \in E$, $z \in \mathbb{R}$ implies $z \in E$." (i.e. $E$ is an interval).

**6.4.4 :** Prove that a set $E$ in a metric space $X$ is connected if and only if $E$ is connected relative to $E$.

## 6.5 EXERCISES

**6.5.1 :** Prove that every interval $[a, b]$ $(a<b)$ in $\mathbb{R}$ is uncountable.

**6.5.2 :** What is Cantor set. Prove that Cantor set is a perfect set.

**6.5.3 :** Define connected set. Characterize the connected subsets of the real line $\mathbb{R}$.

**6.5.4 :** Call two subsets $A$ and $B$ of a metric space $X$ separated if $\bar{A} \cap B = \phi$ and $A \cap \bar{B} = \phi$.

(a) Prove that separated sets are disjoint.

(b) What can you say about the truth of the statement :

Disjoint sets are separated. (Hint : In $\mathbb{R}'$, take $A=(0,1)$ and $B=[1,2]$)

(c) Prove that disjoint open sets are separated.

**6.5.5 :** Let $X$ be a metric space. Let $E \subseteq X$. Prove that $E$ is connected if and only if $E$ cannot be written as a union of two non-empty separated sets.

## 6.6 ANSWERS TO SHORT ANSWER QUESTIONS :

**6.3.1 :** See definition 6.1.1

**6.3.2 :** No (since $(0,1)$ is not closed in $\mathbb{R}'$ as $0 \notin (0,1)$; but 0 is a limit point of $(0,1)$).

**6.3.3 :** Yes

**6.3.4 :** Yes.

**6.3.5 :** Yes (See Theorem 6.1.3).

**6.3.6 :** No. (Let $A$ be a nonempty finite subset of a metric space $X$. So, $A$ has no limit points. Hence $A$ is closed. Clearly, no point of $A$ is a limit point of $A$. So, $A$ is not perfect).

**6.3.7 :** See 6.1.4

**6.3.8 :** See definition 6.2.1.

**6.3.9 :** The connected subsets of $\mathbb{R}$ are precisely the intervals.

**6.3.10 :** Connected subsets of the discrete metric space are precisely single ton sets.

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition Mc Graw-Hill International Editions : Walter Rudin.

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Lesson - 7

# SEQUENCES IN METRIC SPACES

## 7.0 INTRODUCTION

In this lesson, we study the notion of convergence of a sequence (in a metric space) and its properties. In particular, we study relation between the convergence of a sequence in $\mathbb{R}^k$ and the convergence of its component sequences (in $\mathbb{R}$) (see theorem 7.1.10(a)); consequently the properties of convergent sequences in $\mathbb{R}^k$ (see theorem 7.1.10(b)). Further, we observe that every compact metric space is sequentially compact (see Theorem 7.1.7).

## 7.1 SEQUENCES IN METRIC SPACES

We start this section by recalling the definition of the sequence.

**7.1.1 Definition :** By a sequence in a set $A$, we mean any mapping $x : J \to A$ we denote $x(i)$ by $x_i$ and write $x = \{x_i\}$.

**7.1.2 Definition :** A sequence $\{p_n\}$ in a metric space $X$ is said to converge if there is a point $p \in X$ with the following property.

For every $\in > 0$ there exists a positive integer $N$ such that

$$n \geq N \Rightarrow d(p_n, p) < \in .$$

(Where $d$ is the distance in $X$). In this case we say that $\{p_n\}$ converges to $p$ or $p$ is the limit of $\{p_n\}$ and we write $p_n \to p$ or

$$\lim_{n} p_n = p .$$

If $\{p_n\}$ does not converge then we say that $\{p_n\}$ diverges.

**7.1.3 Definition :** If $\{p_n\}$ is a sequence then the set $E = \{p_n / n \geq 1\}$ is called the range of $\{p_n\}$.

**7.1.4 Definition :** A sequence $\{p_n\}$ is a metric space $X$ is said to be bounded if its range is bounded.

**7.1.5 Example :** Consider the metric space of complex numbers :

(a) Let $s_n = \frac{1}{n}$. Now, we prove that

$$\lim_n s_n = 0$$

Let $\in > 0$. Choose a positive integer $N$ such that $\frac{1}{N} < \in$.

Now,

$$n \geq N \Rightarrow |s_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \in$$

Hence, $s_n \to 0$. Clearly the range of $\{s_n\}$ is infinite and bounded (as $\left|\frac{1}{n}\right| \leq 1$ for all $n \geq 1$).

(b) Let $s_n = n^2$. Clearly the sequence is

(i) unbounded

(ii) divergent

(iii) with infinite range

(c) Let $s_n = 1 + \left[{(-1)^n}/{n}\right]$.

(i) $s_n \to 1$: Let $\in > 0$. Choose $N$ such that $\frac{1}{N} < \in$. For $n \geq N$,

$$|s_n - 1| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \in$$

Hence $s_n \to 1$.

(ii) Clearly the range of $\{s_n\}$ is infinite.

(iii) The range of $\{s_n\}$ is bounded since

$$|s_n| \le |1| + \left|\frac{(-1)^n}{n}\right| \le 1+1 = 2 \text{ for all } n.$$

(d) Let $s_n = i^n$. Then $s_1 = i,\ s_2 = -1,\ s_3 = -i,\ s_4 = 1$.

(i) Clearly, the range of $\{s_n\}$ is

$\{s_n / n \ge 1\} = \{i, -1, -i, 1\}$ is finite and hence bounded.

(ii) Clearly, the sequence is divergent.

(e) Let $s_n = 1$ for all $n \ge 1$. Clearly, $s_n \to 1$, the range of the sequence $\{s_n\}$ is $\{1\}$ which is finite and hence bounded.

(f) Let $s_n = (-1)^n$. The sequence $\{s_n\}$ is $-1, 1, -1, 1, \ldots\ldots\ldots\ldots$ in $(\mathbb{R}, d)$. So, the range of $\{s_n\}$ is the set $E = \{-1, 1\}$ which is clearly bounded. But $\{s_n\}$ is not convergent.

**7.1.6 Example :** Let $\{x_n\}$ be a constant sequence in a metric space. So, there exists an element $x$ in $X$ such that $x_n = x$ for all $n \ge 1$. Clearly, $d(x_n, x) = 0$ for all $n \ge 1$ and hence $x_n \to x$.

**7.1.6.1 Note :** Suppose that a sequence $\{x_n\}$ converges to a point $x$ in a metric space $(X, d)$ such that $x_n \ne x$ for all $n \ge 1$. Put $Y = X - \{x\}$. Clearly, $\{x_n\}$ is a sequence in $Y$. If $\{x_n\}$ converges to a point $y$ in $Y$ then $\{x_n\}$ converges to $y$ in $X$ and hence $x = y$ (by Theorem ), a contradiction. So, $\{x_n\}$ does not converge in the metric space $(Y, d)$. Thus, the convergence of a sequence depends on the metric space to which it belongs.

Now, we study some important properties of convergent sequences in metric spaces.

**7.1.7 Theorem :** Let $\{p_n\}$ be a sequence in a metric space $X$.

(a) $\{p_n\}$ converges to $p \in X$ if and only if every neighbourhood of $p$ contains infinitely many points (terms) of the sequence $\{p_n\}$.

(b) If $p \in X,\ p' \in X$ and if $\{p_n\}$ converges to both $p$ and $p'$ then $p = p'$.

(c) If $\{p_n\}$ converges then $\{p_n\}$ is bounded.

(d) If $E \subseteq X$ and if $p \in X$ is a limit point of $E$ then there is a sequence $\{p_n\}$ in $E$ such that $p_n \to p$.

(e) If the range $E$ of the sequence $\{p_n\}$ is infinite then $p_n \to p$ if and only if $p$ is the limit point of $E$.

**Proof :** (a) Assume that $p_n \to p$. Let $\in > 0$. Since, $p_n \to p$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow d(p_n, p) < \in$$

$$\text{i.e. } p_n \in N_{\in}(p).$$

Hence every neighbourhood of $p$ contains all but finitely many terms of $\{p_n\}$.

Conversely assume that every neighbourhood of $p$ coı ains all but finitely many terms of $\{p_n\}$. Let $\in > 0$. By our assumption, $N_{\in}(p)$ contains all but finitely many terms of $\{p_n\}$. So, there exists a positive integer $N$ ıch that

$$n \geq N \Rightarrow p_n \in N_{\in}(p)$$

$$\Rightarrow d(p_n, p) < \in.$$

Hence $p_n \to p$.

(b) Suppose $p, p' \in X$ such that $p_n \to p$ and $p_n \to p'$. Suppose $p \neq p'$. Put $2\in = d(p, p')$. Clearly $\in > 0$. Since $p_n \to p$, there exists a positive integer $N_1$ such that

$$n \geq N_1 \Rightarrow d(p_n, p) < \in.$$

Since $p_n \to p'$, there exists a positive integer $N_2$ such that $n > N_2 \Rightarrow d(p_n, p') < \in$.

Let $n$ be a positive integer such that $n > N_1$ and $n > N_2$. Now,

$2\in = d(p, p') \leq d(p, p_n) + d(p_n, p') < \in + \in = 2\in$, a contradiction. So, $p = p'$.

(c) Suppose $\{p_n\}$ converges. So, there exists $p \in X$ such that $p_n \to p$. So, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow d(p_n, p) < 1.$$

Put $M = \text{Max}\{1, d(p_1, p), d(p_2, p), \ldots\ldots\ldots, d(p_N, p)\}$

Clearly, $d(p_n, p) \leq M$ for all $n \geq 1$.

Hence, the range $E = \{p_n / n \geq 1\}$ is bounded i.e. the sequence $\{p_n\}$ is bounded.

(d) Let $E \subseteq X$ and let $p \in X$ be a limit point of $E$. To each positive integer $n$,

$$N_{\frac{1}{n}}(p) \cap (E - \{p\}) \neq \phi$$

Choose a point $p_n$ in this set. So, to each positive integer $n$, $p_n \in E$, $p_n \neq p$ and $d(p_n, p) < \frac{1}{n}$. Now we show that $p_n \to p$.

Let $\in > 0$. Choose a positive integer $N$ such that $\frac{1}{N} < \in$. For $n \geq N$,

$$d(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \in.$$

Hence $p_n \to p$.

(e) Suppose the range $E = \{p_n / n \geq 1\}$ of $\{p_n\}$ is infinite. By (a),

$p_n \to p \Leftrightarrow$ every neighbourhood of $p$ contains all but finitely many terms of $\{p_n\}$

o $\Leftrightarrow$ every neighbourhood of $p$ contains infinitely many points of $E$.

$\Leftrightarrow p$ is a limit point of $E$.

To study the sequences in $\mathbb{R}^k$, we study the relation between convergence on one hand and the algebraic operations on the other. First, we study the sequences of complex numbers.

**7.1.8 Theorem :** Let $\{s_n\}$ and $\{t_n\}$ be sequences of complex numbers such that $s_n \to s$ and $t_n \to t$. Then

(a) $s_n + t_n \to s + t$

(b) $s_n t_n \to st$

(c) $\dfrac{1}{s_n} \to \dfrac{1}{s}$, provided $s_n \neq 0 (n = 1, 2, .....)$ and $s \neq 0$.

**Proof :** (a) Let $\in > 0$. Since $s_n \to s$, there exists a positive integer $N_1$ such that

$$n \geq N_1 \Rightarrow |s_n - s| < \frac{\in}{2}.$$

Since, $t_n \to t$, there exists a positive integer $N_2$ such that

$$n \geq N_2 \Rightarrow |t_n - t| < \frac{t}{2}.$$

Put $N = \text{Max}\ \{N_1, N_2\}$. Now,

$$n \geq N \Rightarrow |(s_n + t_n) - (s + t)|$$

$$\Rightarrow |(s_n - s) + (t_n - t)|$$

$$\leq |s_n - s| + |t_n - t|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence, $s_n + t_n \to s + t$

(b) Let $\in > 0$. Choose $\in_1$ such that

$$0 < \in_1 < \text{Min} \left\{ \sqrt{\frac{\in}{2}}, \frac{\in}{2(|s| + |t| + 1)} \right\}$$

Since $s_n \to s$, there exists a positive integer $N_1$ such that

$$n \geq N_1 \Rightarrow |s_n - s| < \in_1$$

Since $t_n \to t$, there exists a positive integer $N_2$ such that

$$n \geq N_2 \Rightarrow |t_n - t| < \epsilon_1$$

Put $N = \text{Max}\{N_1, N_2\}$. Now,

$$n \geq N \Rightarrow \{s_n t_n - st\}$$

$$\Rightarrow |(s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)|$$

$$\leq |s_n - s|\,|t_n - t| + |s||t_n - t| + |t||s_n - s|$$

$$\leq \epsilon_1^2 + |s|\epsilon_1 + |t|\epsilon_1$$

$$< \epsilon_1^2 + (|s| + |t| + 1)\epsilon_1$$

$< \epsilon$ (by the choice of $\epsilon_1$).

Hence $s_n t_n \to st$

(c) Let $\epsilon > 0$. Choose $\epsilon_1$ such that $0 < \epsilon_1 < \dfrac{|s|^2 \epsilon}{2}$.

Since $s_n \to s$, there exists a positive integer $N_1$ and $N_2$ such that

$$n \geq N_1 \Rightarrow |s_n - s| < \epsilon_1$$

and $\quad n \geq N_2 \Rightarrow |s_n - s| < \dfrac{1}{2}|s|$

$$\Rightarrow |s| - \frac{1}{2}|s| < |s_n| < |s| + \frac{1}{2}|s|$$

$$\Rightarrow |s_n| \geq \frac{1}{2}|s|$$

Put $N = \text{Max}\{N_1, N_2\}$

$$n \geq N \Rightarrow \left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n \cdot s}\right|$$

$$\leq \frac{2\in_1}{|s|^2} < \in$$

Hence $\frac{1}{s_n} \to \frac{1}{s}$

The above theorem can also be stated as follows.

**7.1.9 Theorem :** Suppose $\{s_n\}, \{t_n\}$ are complex sequences and $\lim\limits_{n\to\infty} s_n = s, \lim\limits_{n\to\infty} t_n = t$. Then

(a) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t;$

(b) $\lim\limits_{n\to\infty} c\,s_n = c\,s, \lim\limits_{n\to\infty} (c + s_n) = c + s;$

(c) $\lim\limits_{n\to\infty} s_n t_n = st;$

(d) $\lim\limits_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0$ $(n = 1, 2, .....)$ and $s \neq 0$

**7.1.10 Theorem :** (a) Suppose $x_n \in \mathbb{R}^k$ $(n = 1, 2.....)$ and

$$x_n = (\alpha_{1,n}, \alpha_{2,n}, ..........., \alpha_{k,n}).$$

Then the sequence $\{x_n\}$ converges to $x = (\alpha_1, \alpha_2, ........., \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \,(j = 1, 2, ...., k)$$

(b) Suppose $\{x_n\}, \{y_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers and $x_n \to x$, $y_n \to y$, $\beta_n \to \beta$. Then

$$\lim_{n\to\infty} (x_n + y_n) = x + y, \quad \lim_{n\to\infty} x_n \cdot y_n = x\,y, \quad \lim_{n\to\infty} \beta_n x_n = \beta x$$

**Proof :** (a) : Assume that $x_n \to x$ in $\mathbb{R}^k$. Fix $j$ such that $1 \leq j \leq k$. Now, we show that

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j.$$

Let $\in > 0$. Since $x_n \to x$ in $\mathbb{R}^k$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \|x_n - x\| < \in$$

$$\text{i.e. } \left\{ \sum_{i=1}^{k} \left| \alpha_{i,n} - \alpha_i \right|^2 \right\}^{\frac{1}{2}} < \in$$

$$\Rightarrow \left| \alpha_{j,n} - \alpha_j \right| \leq \|x_n - x\| < \in$$

Hence

$$\lim_{n \to \infty} \alpha_{j,n} = \alpha_j$$

Conversely, assume that

$$\lim_{n \to \infty} \alpha_{j,n} = \alpha_j \text{ (for } j = 1, 2, \ldots\ldots, k)$$

Now, we show that $x_n \to x$. Let $\in > 0$. Put $\in_1 = \in / \sqrt{k}$. Clearly $\in_1 > 0$. By our assumption, to each $j$ with $1 \leq j \leq k$, there exists a positive integer $N_j$ such that

$$n \geq N_j \Rightarrow \left| \alpha_{j,n} - \alpha_j \right| < \in_1$$

Put $N = \text{Max}\{N_1, N_2, \ldots.., N_k\}$. Now,

$$n \geq N \Rightarrow \|x_n - x\|$$

$$= \left\{ \sum_{j=1}^{k} \left| \alpha_{j,n} - \alpha_j \right|^2 \right\}^{\frac{1}{2}} < \left( \sum_{j=1}^{k} \in_1^2 \right)^{\frac{1}{2}}$$

$$= \left\{ \sum_{j=1}^{k} \left( \in^2 / k \right) \right\}^{\frac{1}{2}} = \in$$

Hence $x_n \to x$ in $\mathbb{R}^k$.

(b) Let $x_n = (\alpha_{1,n}, \alpha_{2,n}, \ldots\ldots\ldots, \alpha_{k,n})$,

$y_n = (\gamma_{1,n}, \gamma_{2,n}, \ldots\ldots\ldots \gamma_{k,n})$, $x = (\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_k)$ and

$y = (\gamma_1, \gamma_2, \ldots\ldots\ldots, \gamma_k)$.

$x_n \to x,\ y_n \to y \Rightarrow \lim\limits_{n\to\infty} \alpha_{j,n} = \alpha_j,\ \lim\limits_{n\to\infty} \gamma_{j,n} = \gamma_j\ (j = 1, 2, \ldots\ldots, k)$

(by Theorem 7.1.10 (i))

(i) $\Rightarrow \lim\limits_{n\to\infty} (\alpha_{j,n} + \gamma_{j,n}) = \alpha_j + \gamma_j$

(ii) $\lim\limits_{x\to\infty} \beta_n\, \alpha_{j,n} = \beta\, \alpha_j$ (for $j = 1, 2, \ldots, k$)

(iii) $\lim\limits_{n\to\infty} \alpha_{j,n}\, \gamma_j = \alpha_j \gamma_j$ (for $j = 1, 2, \ldots, k$) and hence

$$\lim_{n\to\infty} \sum_{j=1}^{k} \alpha_{j,n} \cdot \gamma_{j,n} = \sum_{j=1}^{k} \alpha_j \gamma_j \text{ (By Theorem 7.1.9(i))}$$

$$\Rightarrow \lim_{n\to\infty} (x_n + y_n) = x + y,\ \lim_{n\to\infty} \beta_n\, x_n = \beta x$$

and $\lim\limits_{n\to\infty} x_n \cdot y_n = x \cdot y$

Now, we give the definition of a subsequence

**7.1.11 Definition :** Let $\{p_n\}$ be a sequence. Let $\{n_k\}$ be a sequence of positive integers such that $n_1 < n_2 < \ldots\ldots\ldots\ldots\ldots$ Now, the sequence $\{p_{n_k}\}$ is called a subsequence of $\{p_n\}$. If $\{p_{n_k}\}$ converges then its limit is called a subsequential limit of $\{p_n\}$.

**7.1.11.1 Note :** Every sequence is a subsequence of itself (clear). Consider the following example in the metric space $\mathbb{C}$ (or $\mathbb{R}^2$).

**7.1.12 Example :** Let $p_n = (-1)^n$. The sequence $\{p_n\}$ can be written as $-1, +1, -1, +1, \ldots\ldots\ldots$. Cleearly, the constant sequences $\{-1\}$ and $\{+1\}$ are the cnvergent subsequences of $\{p_n\}$. The set $E$ of all subsequential limits of $\{p_n\}$ is $E = \{-1, +1\}$.

**7.1.13 Example :** Let $p_n = i^n$. Clearly, $\{p_n\}$ has the convergent constant sequences $\{i\},\{-1\},\{-i\}$ and $\{1\}$. So, the set $E$ of all subsequential limits of $\{p_n\}$ is $E = \{1,-1,i,-i\}$.

**7.1.14 Theorem :** Let $(X,d)$ be a metric space. A sequence $\{p_n\}$ in $X$ converges to a point $p$ in $X$ if and only if every subsequence $\{p_n\}$ converges to $p$.

**Proof :** Let $\{p_n\}$ be a sequence in $X$. Assume that $p_n \to p$ in $X$. Let $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$. Now, we show that $p_{n_k} \to p$. Let $\in > 0$. Since $p_n \to p$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow d(p_n, p) < \in$$

Since $\{p_{n_k}\}$ is a subsequence of $\{p_n\}$, we have that

$$n_1 < n_2 < \text{.........}$$

Clearly, $n_1 \geq 1$. Now,

$$n_2 > n_1 \geq 1 \Rightarrow n_2 > 1 \Rightarrow n_2 \geq 2;$$

$$n_3 > n_2 \geq 2 \Rightarrow n_3 > 2 \Rightarrow n_3 \geq 3;$$

Continuing this process, by induction we have that

$n_k \geq k$ for each positive integer $k$. Now,

$$k \geq N \Rightarrow n_k \geq k \geq N$$

$$\Rightarrow d\left(p_{n_k}, p\right) < \in.$$

Hence $p_{n_k} \to p$. Converse is clear since $\{p_n\}$ is a subsequence of itself.

**7.1.15 Theorem :** Every sequence in a compact metric space contains a convergent subsequence.

**Proof :** Let $\{p_n\}$ be a sequence in a compact metric space $(X,d)$. Let $E$ be the range of the sequence $\{p_n\}$.

**Case (i) :** $E$ is finite : Then there exists a point $p$ in $E$ which is infinitely many times repeated in the sequence $\{p_n\}$. Thus, $\{p_n\}$ has the constant subsequence which is convergent.

**Case (ii) :** $E$ is infinite; Thus, $E$ is an infinite subset of the compact set $X$. By theorem 5.1.9, $E$ has a limit point $p$ say. So,

$$N_1(p) \cap E - \{p\} \neq \phi.$$

Let $p_{n_1}$ be a point in this set. So, $p_{n_1} \in E$, $p \notin p_{n_1}$, $d(p, p_{n_1}) < 1$.

Since $p$ is a limit point of $E$, $p$ is a limit point of $E - \{p_1, p_2, \ldots\ldots\ldots, p_{n_1}\}$.

So,

$$N_{\frac{1}{2}}(p) \cap \left(E - \{p_1, p_2, \ldots\ldots, p_{n_1}\}\right) - \{p\} \neq \phi$$

Let $p_{n_2}$ be a point in this set. So,

$$p_{n_2} \in E,\ p \neq p_{n_2},\ d(p, p_{n_2}) < \frac{1}{2}.$$

Clearly, $n_1 < n_2$. Continuing this process, after choosing $p_{n_1}, p_{n_2}, \ldots\ldots\ldots, p_{n_{i-1}}$ in $E$ such that

$$p \neq p_{n_j}\ (j = 1, 2, \ldots\ldots, i-1);$$

$$d(p, p_{n_j}) < \frac{1}{j}\ (j = 1, 2, \ldots\ldots, i-1)$$

$$n_1 < n_2 < \ldots\ldots\ldots < n_{i-1}$$

We choose $p_{n_i}$ as follows. Since $p$ is a limit point of $E$, $p$ is a limit point of $E - \{p_1, p_2, \ldots\ldots\ldots, p_{n_{i-1}}\}$. So,

$$N_{\frac{1}{i}}(p) \cap \left(E - \{p_1, p_2, \ldots\ldots\ldots, p_{n_{i-1}}\}\right) - \{p\} \neq \phi$$

Choose a $p_{n_i}$ in this set. So,

$$p \neq p_{n_i},\ d(p, p_{n_i}) < \frac{1}{i},\ n_{i-1} < n_i.$$

Thus, we have a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that

$$d(p, p_{n_i}) < \frac{1}{i} \text{ for all } i \geq 1.$$

Now, we show that $p_{n_i} \to p$. Let $\in > 0$. Choose a positive integer $k$ such that $\frac{1}{k} < \in$. Now,

$$i \geq k \Rightarrow d(p, p_{n_i}) < \frac{1}{i} \leq \frac{1}{k} < \in$$

Hence $p_{n_i} \to p$. Hence the theorem

**7.1.16 Definition :** A metric space $X$ is called sequentially compact if every sequence in $X$ contains a convergent subsequence.

In view of the definition 7.1.16, the Theorem 7.1.15 can be stated as follows.

**7.1.17 Theorem :** Every compact metric space is sequentially compact.

**7.1.18 Theorem :** Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

**Proof :** Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}^k$. So, the range $E = \{x_n / n \geq 1\}$ of $\{x_n\}$ is bounded. By Theorem 5.1.13, there exists a $k-$cell I such that $E \subseteq \text{I}$. By Theorem 5.1.12, I is compact. Thus, $\{x_n\}$ is a sequence in the compact metric space I. By Theorem 7.1.17, $\{x_n\}$ contains a convergent subsequence. Hence the theorem.

**7.1.19 Theorem :** The set of all subsequential limits of a sequence $\{p_n\}$ in a metric space is closed.

**Proof :** Let $E$ be the range of a sequence $\{p_n\}$ in a metric sapce $(X, d)$. Let $E^*$ be the set of all subsequential limits of $\{p_n\}$. To prove that $E^*$ is closed, we show that $E^*$ contains all of its limit points. Let $q \in X$ be a limit point of $E^*$. To prove that $q$ is in $E^*$ it is enough to prove that $q$ is a limit point of $E$ (by Theorem 7.1.7 (d)).

Let $\in > 0$. Since $q$ is a limit point of $E^*$, there is a point $p$ in $E^*$ such that

$$0 < d(p, q) < \frac{\in}{2}.$$

Since $p \in E^*$,

$$d(p, p_n) < d(p, q)$$

for some $p_n$. Clearly, $p_n \neq q$ and

$$0 < d(p_n, q) \leq d(p_n, p) + d(p, q) < 2d(p, q) < \in.$$

So, the set

$$N_{\in}(q) \cap E - \{q\} \neq \phi$$

as it contains $p_n$. Thus, every neighbourhood of $p$ contains a point of $E$ other than $q$. Hence $q$ is a limit point of $E$. Hence the theorem.

**7.1.20 Problem :** Let $\{p_n\}$, $\{q_n\}$ be two sequences in a metric space $(X, d)$ such that $p_n \to p$ and $q_n \to q$. Prove that $d(p_n, q_n) \to d(p, q)$ in the metric space $(\mathbb{R}, d)$.

**Solution :** Let $\in > 0$. Since $p_n \to p$ and $q_n \to q$, there exist positive integers $N_1$, $N_2$ such that

$$n \geq N_1 \Rightarrow d(p_n, p) < \frac{\in}{2} \text{ and}$$

$$n \geq N_2 \Rightarrow d(q_n, q) < \frac{\in}{2}.$$

Put $N = \text{Max}\{N_1, N_2\}$. Clearly,

$$d(p_n, q_n) - d(p, q) \leq d(p_n, p) + d(p, q) + d(q, q_n) - d(p, q)$$

$$= d(p_n, p) + d(q, q_n);$$

$$d(p, q) - d(p_n, q_n) \leq d(p, p_n) + d(p_n, q_n) + d(q_n, q) - d(p_n, q_n)$$

$$= d(p, p_n) + d(q, q_n)$$

and hence

$$|d(p_n, q_n) - d(p, q)| \leq d(p, p_n) + d(q, q_n).$$

Now it is clear that

$$n \geq N \Rightarrow |d(p_n, q_n) - d(p, q)| < \in .$$

Hence $d(p_n, q_n) \to d(p, q)$ in $(\mathbb{R}, d)$.

**7.1.21 Problem :** Let $\{p_n\}$ be a sequence in a closed subset $E$ of a metric space $X$. If $p_n \to p$ in $X$, then $p \in E$.

**Solution :** Suppose $p_n \to p$. Let $S$ be the range of $\{p_n\}$. If $S$ is finite, then $p$ repeats infinitely many times in the sequence $\{p\}$. So, $p \in E$.

Suppose $S$ is infinite. By Theorem 7.1.7(a) $p$ is a limit point of $S$ and hence $p$ is a limit point of $E$ (since $S \subseteq E$). Since $E$ is closed, $p \in E$.

## 7.2 SHORT ANSWER QUESTIONS

**7.2.1 :** Define convergent sequence.

**7.2.2 :** Define bounded sequence.

**7.2.3 :** Is every bounded sequence convergent ?

**7.2.4 :** Is every convergent sequence bounded ?

**7.2.5 :** Give an example of an unbounded sequence.

**7.2.6 :** Is the range of the sequence $\left\{\frac{1}{n}\right\}$ in $(\mathbb{R}, d)$ finite ?

**7.2.7 :** Is the range of any constant sequence finite ?

**7.2.8 :** Prove or disprove the statement - The range of any convergent sequence is finite.

**7.2.9 :** Prove or disprove the statement - the range of any convergent sequence is infinite.

**7.2.10 :** Let $E$ be the range of the sequence $\{p_n\}$ in a metric space $X$. Under what circumstances, does the limit of the sequence $\{p_n\}$ coincide with the limit point of $E$.

**7.2.11 :** Prove or disprove the statement - The convergence of a sequence depends on the metric space to which it belongs.

**7.2.12 :** Prove that the sequence $\{p_n\}$ where $p_n = (-1)^n$ in $(\mathbb{R}, d)$ is not convergent.

**7.2.13 :** Prove that the sequence $\{p_n\}$ where $p_n = (-1)^n$ in $(\mathbb{C}, d)$ is not convergent.

**7.2.14 :** Prove that the sequence $\{p_n\}$ where $p^n = i^n$ in $\mathbb{C}$ is not convergent.

## 7.3 MODEL EXAMINATION QUESTIONS

**7.3.1 :** Define convergence of a sequence. Prove that the limit of a convergent sequence is unique.

**7.3.2 :** Prove that every convergent sequence is bounded. Is the converse true ? Justify your answer.

**7.3.3 :** Define the range of a sequence. Let $\{p_n\}$ be a sequence in a metric space $X$. Let $E$ be the range of sequence $\{p_n\}$. Let $p \in X$. If $E$ is infinite, prove that $p_n \to p$ as $n \to \infty$ if and only if $p$ is the limit point of $E$.

**7.3.4 :** Prove that in a compact metric space, every sequence contains a convergent subsequence.

**7.3.5 :** Prove that the set of all subsequential limits of a sequence $\{p_n\}$ in a metric space is closed.

**7.3.6 :** Prove that every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.

## 7.4 EXERCISE

**7.4.1 :** Prove that for the sequence $\{s_n\}$ of real (complex) numbers,

$$\lim_{n\to\infty} s_n = s \Rightarrow \lim_{n\to\infty} |s_n| = |s|$$

**7.4.2 :** Compute $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$

**7.4.3 :** Show that the sequence $\{x^n\}$ in $\mathbb{R}$ is convergent if and only if $-1 < x \le 1$.

**7.4.4 :** For any sequence $\{s_n\}$ of reals, consider the arithmetic sequence $\{t_n\}$, where

$$t_n = \frac{s_1 + s_2 + \cdots\cdots + s_n}{n}$$

Prove that $s_n \to s$ as $n \to \infty$ implies $t_n \to s$ as $n \to \infty$. Further prove that there are divergent sequences $\{s_n\}$ which in this manner give rise to convergent sequence $\{t_n\}$.

**7.4.5 :** If $\{a_n\}$ is a sequence of real numbers such that

$$\lim_{n\to\infty}\left(a_{n+1}-a_n\right)=\ell,$$

prove that

$$\lim_{n\to\infty}\frac{a_n}{n}=\ell$$

**7.4.6 :** If $a_n>0$ and $\lim\limits_{n\to\infty} a_n=\ell$, show that $\lim\limits_{n\to\infty}\left(a_1\, a_2\,..........a_n\right)^{\frac{1}{n}} = \log\ell$.

**7.4.7 :** If $x_{n+1}=\sqrt{k+x_n}$, where $k, x_1$ are positive, prove that the sequence $\{x_n\}$ is increasing or decreasing according as $x_1$ is less than or greater than the positive root of the equation $x^2-x-k=0$ and has in either case this root as its limit.

**7.4.8 :** If $s_1=\sqrt{2}$, and $s_{n+1}=\sqrt{2+\sqrt{s_n}}$ $(n=1,2,........)$, prove that $\{s_n\}$ converges and that $s_n<2$ for $n=1,2,.....$

**7.4.9 :** Let $\{x_n\}$ be a sequence of reals such that $x_1>0,\ x_2>0$ and $x_{n+1}=\frac{1}{2}\left(x_n+x_{n-1}\right)$.

Prove that the sequences

$$x_1,\ x_3,\ x_5,.....\ \text{and}\ x_2,\ x_4,...........$$

are one a decreasing and the other an increasing sequence, and they converge to the same limit $\frac{1}{3}\left(x_1+2x_2\right)$.

**7.4.10 :** Let $\{p_n\}$ be a sequence in a metric space $(X,d)$. Let $p\in X$. Prove :

$p_n\to p$ as $n\to\infty$ if and only if $d(p_n,p)\to 0$ as $n\to\infty$ in $\mathbb{R}$.

## 7.5 ANSWERS TO SHORT ANSWER QUESTIONS

**7.2.1 :** See Definition 7.1.2

**7.2.2 :** Yes (See Theorem 7.1.7 (c))

**7.2.3 :** No (See Example 7.1.5 (f), 7.1.5 (d))

**7.2.4 :** See Definition 7.1.4

**7.2.5 :** See Example 7.1.5 (b)

**7.2.6 :** No

**7.2.7 :** Yes. The range of any constant sequence is a singleton set.

**7.2.8 :** "The range of any convergent sequence is finite" is false. See Example 7.1.5 (a).

**7.2.9 :** "The range of any convergent sequence is infinite" is false.

Example : any constant sequence.

**7.2.10 :** When the range $E$ of $\{p_n\}$ is infinite.

**7.2.11 :** Yes. The sequence $\left\{\frac{1}{n}\right\}$ converges in the metric space $(\mathbb{R}, d)$ to 0; but it does not converge in the metric space $(\mathbb{R}-\{0\}, d)$.

**7.2.12 :** Suppose $\{p_n\}$ converges in $(\mathbb{R}, d)$ to $p$ say. Let $0<\in<1$. Then there exists a positive integer $N$ such that

$$n \geq N \Rightarrow d(p_n, p)=|p_n - p|<\in$$

So, $|1-p|<\in$ and $|1+p|<\in$. Now,

$2=1-p+1+p \leq |1-p|+|1+p|<2\in \Rightarrow 1<\in$, a contradiction to $\in<1$. Hence $\{p_n\}$ is not convergent.

**7.2.13 :** Answer is same as that of 7.2.12

**7.2.14 :** Assume that $\{p_n\}$ converges to $p$ say. Let $0<\in<1$. So, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow |p_n - p| < \in.$$

So, $|1-p|<\in$, $|1+p|<\in$, $|i+p|<\in$ and $|p-i|<\in$.

Now, $2=|2|=|1-p+1+p| \leq |1-p|+|1+p|<2\in$

$\Rightarrow 1<\in$, a contradiction to $\in<1$.

Hence $\{p_n\}$ does not converge.

## 7.6 REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition : McGraw - Hill International Editions : Walter Rudin

Lesson writer :

*Prof. P. Ranga Rao*

Lesson - 8

# CAUCHY SEQUENCES AND COMPLETE METRIC SPACES

## 8.0 INTRODUCTION

In this lesson, we study the notion of Cauchy sequence; and we observe that every convergent seqoence is a Cauchy sequence (see Theorem 8.1.2). Furhter, we study the concept of a complete metric space and we observe that the metric space $\mathbb{R}^k$ is complete (see Theorem 8.1.5). We also study Cantor's Intersection Theorem.

## 8.1 CAUCHY SEQUENCES :

**8.1.1 Definition :** A sequence $\{p_n\}$ in a metric space $(X, d)$ is called a Cauchy sequence if the following condition (called Cauchy ndition) is satisfied.

For every $\in > 0$, there exists a positive integer $N$ such that

$$n \geq N,\ m \geq N \Rightarrow d(p_n,\ p_m) < \in.$$

The following theorem gives several examples of Cauchy sequences.

**8.1.2 Theorem :** In a metric space, every convergent sequence is a Cauchy sequence.

**Proof :** Let $\{p_n\}$ be a convergent sequence in a metric space $X$. So, there exists $p \in X$ such that $p_n \to p$. Let $\in > 0$. So, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow d(p_n, p) < \frac{\in}{2}$$

Now,

$$n \geq N,\ m \geq N \Rightarrow d(p_n,\ p_m) \leq d(p_n, p) + d(p, p_m)$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence $\{p_n\}$ is a Cauchy sequence.

**8.1.2.1 Note :** The converse of the Theorem 8.1.2 is not true i.e. every Cauchy sequence need not be convergent. We know that the sequence $\{p_n\}$ where $p_n = \frac{1}{n}$ in $(\mathbb{R}, d)$ converges to 0. By Theorem 8.1.2, $\{p_n\}$ is a Cauchy sequence. So, $\{p_n\}$ is a Cauchy sequence in $(0, 1]$ but not convergent in $(0, 1]$.

**8.1.3 Theorem :** A Cauchy sequence in a metric space is convergent if and only if it has a convergent subsequence.

**Proof :** Let $\{p_n\}$ be a Cauchy sequence in a metric space $(X, d)$. Assume that $\{p_n\}$ has a convergent subsequence $\{p_{n_k}\}$ say. Let $p_{n_k} \to p$. Now, we show that $p_n \to p$. Let $\in > 0$. Since $\{p_n\}$ is Cauchy, there exists a positive integer $N_1$ such that

$$n \geq N_1,\ m \geq N_1 \Rightarrow d(p_n, p_m) < \frac{\in}{2}.$$

Since $p_{n_k} \to p$, there exists a positive integer $N_2$ such that

$$k \geq N_2 \Rightarrow d(p_{n_k}, p) < \frac{\in}{2}$$

Put $N = \text{Max}\{N_1, N_2\}$. Let $n \geq N$. Let $k$ be such that $k \geq N$.

Now, $$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p)$$

$$< \frac{\in}{2} + \frac{\in}{2} \text{ (since } n_k \geq k \geq N)$$

Hence $p_n \to p$. Converse is obvious (as $\{p_n\}$ is a subsequence of itself).

**8.1.4 Theorem :** Every Cauchy sequence is bounded.

**Proof :** Let $\{p_n\}$ be a Cauchy sequence in a metric space $(X, d)$. So, there exists a positive integer $N$ such that

$$n \geq N,\ m \geq N \Rightarrow d(p_n, p_m) < 1$$

Put $$M = \text{Max}\{1, d(p_N, p_1), d(p_N; p_2), \ldots\ldots\ldots d(p_N, p_{N-1})\}$$

Clearly,

$$d(p_N, p_n) \le M \text{ for all } n \ge 1.$$

Hence the sequence $\{p_n\}$ is bounded.

In note 8.1.2.1, we have seen that the converse of theorem 8.1.2. i.e. "Every Cauchy sequence is convergent" is not true; but it is true in the Euclidean space $\mathbb{R}^k$.

**8.1.5 Theorem :** In the Euclidean space $\mathbb{R}^k$, every Cauchy sequence is convergent.

**Proof :** Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Let $S$ be the range of the sequence $\{p_n\}$. If $S$ is finite, then, there is a term $p$ say in the sequence $\{p_n\}$ such that $p_n = p$ for infinitely many $n$. Thus, $\{p_n\}$ has the constant subsequence $\{p\}$ which is convergent. By theorem 8.1.3, $\{p_n\}$ is convergent.

Suppose $S$ is infinite. By Theorem 8.1.4, $S$ is bounded. We know that every bounded infinite subset of $\mathbb{R}^k$ has a limit point (by Theorem 5.1.15). So, $S$ has a limit point $p$ say in $\mathbb{R}^k$ Now, we show that $p_n \to p$. Let $\in > 0$. Since $\{p_n\}$ is a Cauchy sequence, there exists a positive integer $N$ such that

$$n \ge N,\ m \ge N \Rightarrow d(p_n, p_m) < \frac{\in}{2}.$$

Fix $n$ such that $n \ge N$. Since $p$ is a limit point of $S$, $N_{\frac{\in}{2}}(p)$ contains infinitely many points of $S$ and hence we can choose a point $p_m$ in $N_{\frac{\in}{2}}(p)$ with $m \ge N$. So,

$$d(p_n, p) \le d(p_n, p_m) + d(p_m, p)$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence $p_n \to p$. Thus, every Cauchy sequence in $\mathbb{R}^k$ converges.

Now, we recall the definition of diameter of a set. (Definition 4.1.37)

**8.1.6 Definition :** Let $E$ be a subset of a metric space $(X, d)$. Let $S = \{d(x, y) / x, y \in E\}$. The diameter of $E$, denoted by diam $E$, is defined as the Supremum or least upper bound of $S$.

$$\text{i.e. diam } E = \sup S \left(\text{or lub } S\right)$$

**8.1.7 Example :** Consider the metric space $\left(\mathbb{R}^2, d\right)$.

(i) Let $E = \left\{x = (x_1, x_2) / \|x\| \le 1\right\}$

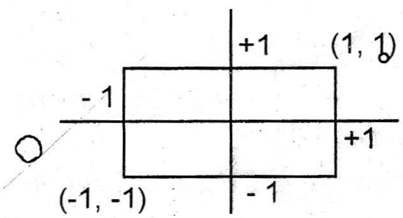
For any $x, y$ in $E$,

$$d(x,y) = \|x-y\| \le \|x\| + \|y\| \le 1+1 = 2$$

Further, $x=(-1, 0)$, $y=(1,0)$ are in $E$ such that $d(x, y)=2$.

So, diam $E$ = 2.

(ii) Take $E = \left\{x = (x_1, x_2) / |x_1| \le 1, |x_2| \le 1\right\}$




For any $x$, $y$ in $E$,

$$d(x,y) = \|x-y\|$$

$$= \|(x_1 - y_1, x_2 - y_2)\|$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$\le \sqrt{4+4} \text{ (since } |x_1 - y_1| \le 2 \text{ and } |x_2 - y_2| \le 2$$

$$= 2\sqrt{2}$$

Clearly, $x=(-1,-1)$ and $y=(1,1)$ are in $E$ and $d(x,y)$ = distance between $x$ and $y$ is $2\sqrt{2}$.

**8.1.8 Theorem :** For any subset $E$ of a metric space $(X,d)$, diam $E$ = diam $\overline{E}$

where $\overline{E}$ is the closure of $E$.

**Proof :** Let $E$ be a subset of a metric space $(X,d)$. Since $E \subseteq \overline{E}$, diam $E \le$ diam $\overline{E}$. (since the set $S = \left\{d(x,y)/x, y \in E\right\}$ is contained in the set $T = \left\{d(x,y)/x, y \in \overline{E}\right\}$).

Let $\in > 0$. Let $p, q \in \overline{E}$ . By Theorem 4.1.35, there exist points $x, y$ in $E$ such that

$$d(x,p) < \frac{\in}{2} \text{ and } d(y,q) < \frac{\in}{2}.$$

So, $d(p,q) \le d(p,x) + d(x,y) + d(y,q)$

$$< \in + \operatorname{diam} E$$

Hence $\operatorname{diam} \overline{E} \le \in + \operatorname{diam} E$

Since $\in$ is arbitrary,

$$\operatorname{diam} \overline{E} \le \operatorname{dian} E.$$

Hence the Theorem.

**8.1.9 Theorem :** Let $\{x_n\}$ be a sequence in a metric space $(X, d)$. For each positive integer $n$, write

$$E_n = \{x_m / m \ge n\}.$$

Then, $\{x_n\}$ is a Cauchy sequence if and only if

$$\lim_n \operatorname{diam} E_n = 0$$

**Proof :** Assume that $\{x_n\}$ is Cauchy sequence. Let $\in > 0$. Choose $\in_1$ such that $0 < \in_1 < \in$. Since $\{x_n\}$ is Cauchy sequence, there exists a positive integer $N$ such that

$$n \ge N,\ m \ge N \Rightarrow d(x_n, x_m) < \in_1.$$

Fix $n$ such that $n \ge N$. Now,

$$\ell \ge n,\ m \ge n \Rightarrow d(x_\ell, x_m) < \in_1$$

and hence

$$\operatorname{diam} E_n \le \in_1 < \in.$$

Thus

$$n \ge N \Rightarrow \operatorname{diam} E_n < \in$$

Hence $\lim_{n\to\infty} \operatorname{diam} E_n = 0$

Conversely assume that $\lim_{n} \operatorname{diam} E_n = 0$. Let $\in > 0$. So, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \operatorname{diam} E_n < \in.$$

For $n \geq N$, $m \geq N$,

$$d(x_n, x_m) \leq \operatorname{diam} E_n < \in.$$

Hence $\{x_n\}$ is Cauchy sequence.

**8.1.10 Theorem :** Let $\{K_n\}$ be a decreasing sequence of non-empty compact sets in a metric space $X$ such that

$$\lim_{n\to\infty} \operatorname{diam} K_n = 0.$$

Then $\bigcap_{n=1}^{\infty} K_n$ contains exactly one point.

**Proof :** Clearly, the sequence $\{K_n\}$ has $F.I.P.$ By Corollary 5.1.8.1, $K = \bigcap_{n=1}^{\infty} K_n \neq \phi$ Suppose $K$ contains more than one point. Then $\operatorname{diam} K > 0$ (Let $x, y \in K$ be such that $x \neq y$. So, $0 < d(x, y) \leq \operatorname{diam} K$. Since $\lim_{n} \operatorname{diam} K_n = 0$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \operatorname{diam} K_n < \operatorname{diam} K,$$

a contradiction (since $K \subseteq K_n$ for all $n$ implies $\operatorname{diam} K \leq \operatorname{diam} K_n$ for all $n$). Hence $\bigcap_{n=1}^{\infty} K_n$ contains exactly one point.

## 8.2 COMPLETE METRIC SPACES

**8.2.1 :** A metric space $(X, d)$ is called complete if every Cauchy sequence in $X$ converges.

**8.2.2 :** Let $(X,d)$ be a metric space. Let $Y(\neq \phi)\subseteq X$. $Y$ is called complete if $(Y,d)$ is complete.

**8.2.3 Example :** By theorem 8.1.5, $\mathbb{R}^k$ is complete. In particular $\mathbb{R}$ and $\mathbb{C}\left(=\mathbb{R}^2\right)$ are complete.

**8.2.4 Theorem :** Every compact metric space is complete.

**Proof :** Let $X$ be a compact metric space. Let $\{p_n\}$ be a Cauchy sequence in $X$. Let $E_n=\{p_m / m\geq n\}$. By Theorem 8.1.9,

$$\lim_{n} \operatorname{diam} E_n = 0$$

By Theorem 8.1.8, $\lim\limits_{n} \operatorname{diam} \overline{E_n} = 0$.

Clearly, each $\overline{E_n}$ is a closed subset of $X$. By Theorem 5.1.5, each $\overline{E_n}$ is compact. Furhter,

$$E_n \supseteq E_{n+1} \Rightarrow \overline{E_n} \supseteq \overline{E}_{n+1}$$

for all $n\geq 1$. By theorem 8.1.10,

$$\bigcap_{n=1}^{\infty} \overline{E_n}$$

contains exactly one point $p$ say. Now, we show that $p_n \to p$.

Let $\in > 0$. Since

$$\lim_{n} \operatorname{diam} \overline{E_n} = 0,$$

there exists a positive integer $N$ such that

$$n\geq N \Rightarrow \operatorname{diam} \overline{E_n} < \in.$$

Now, $n\geq N \Rightarrow p_n \in E_n \subseteq \overline{E_n}$,

$\Rightarrow d(p_n, p) \leq \operatorname{diam} \overline{E_n} < \in$ (since $p \in \overline{E_n}$ for all $n\geq 1$)

$< \in$

Thus, every compact metric space is complete.

**8.2.5 Theorem :** Any closed subset of a complete metric space is complete.

**Proof :** Let $Y$ be a closed subset of a complete metric space $X$. Let $\{p_n\}$ be a Cauchy sequence in $Y$. So, $\{p_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, $p_n \to p$ for some $p$ in $X$. Since $Y$ is closed, $p \in Y$, (by problem 7.1.21). Thus $p_n \to p$ in $Y$. Hence $Y$ is complete.

In view of theorems 8.2.4 and 8.2.5, we have many examples of complete metric spaces. The following theorem analogous to theorem 8.1.10.

**8.2.6 Theorem (Cantor's Intersection Theorem) :** Let $\{F_n\}$ be a decreasing sequence of non-empty closed sets in a complete metric space $(X,d)$ such that $\operatorname{diam} F_n \to 0$. Then

$\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

**Proof :** To each positive integer $n$, choose a point $x_n$ in $F_n$.

$\{x_n\}$ is a Cauchy sequence. Let $\in > 0$. Since $\operatorname{diam} F_n \to 0$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \operatorname{diam} F_n < \in,$$

Now,

$$n \geq N,\ m \geq N \Rightarrow x_n \in F_N, x_m \in F_N$$

$$\Rightarrow d(x_n, x_m) \leq \operatorname{diam} F_N < \in$$

Hence $\{x_n\}$ is a Cauchy sequence.

Since $X$ is complete, $x_n \to x$ for some $x$ in $X$. Fix $n \geq 1$.

Clearly, the sequence $\{x_m\}_{m=n,n+1,\ldots..}$ is in the closed set $F_n$ and converges to $x$ in $X$. By problem 7.1.21, $x \in F_n$. Thus $x$ is in every $F_n$ i.e.

$$x \in \bigcap_{n=1}^{\infty} F_n$$

Now, we show that

$$\bigcap_{n=1}^{\infty} F_n$$

contains exactly one point. Let $x, y$ be points in

$$\bigcap_{n=1}^{\infty} F_n$$

such that $x \neq y$. So, $d(x,y)>0$ and

$$d(x,y) \leq \operatorname{diam} F_n \text{ for all } n \geq 1$$

Since $\operatorname{diam} F_n \to 0$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \operatorname{diam} F_n < d(x,y)\text{, a contradiction to } d(x,y) \leq \operatorname{diam} F_n \text{ for all } n.$$

Hence, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

**8.2.6.1 Note :** Cantor's Intersection Theorem fails if we drop the hypothesis $\operatorname{diam} F_n \to 0$. See the following example.

**8.2.7 Example :** Consider the metric space $(\mathbb{R}, d)$. We know that this is complete. To each positive integer $n$, let $F_n = [n, \infty]$. Clealry, $\{F_n\}$ is a decreasing sequence of closed subsets of $\mathbb{R}$ and $\operatorname{diam} F_n = \infty$ (for any $n \geq 1$). Clearly, $\operatorname{diam} F_n \to 0$ fails and

$$\bigcap_{n=1}^{\infty} F_n = \phi.$$

So, in the hypothesis of Cantor's intersection theorem, the condition $\operatorname{diam} F_n \to 0$ can not be dropped.

## 8.3 SHORT ANSWER QUESTIONS

**8.3.1 :** Is every convergent sequence a Cauchy sequence ?

**8.3.2 :** Is every Cauchy sequence convergent ?

**8.3.3 :** Is every Cauchy sequence bounded ?

**8.3.4 :** Is every convergent sequence bounded ?

**8.3.5 :** The diameter of the set

$$E=\{x=(x_1, x_2)\in \mathbb{R}^2 / |x_1|\le 2, |x_2|\le 3\}$$ in the metric space $(\mathbb{R}^2, d)$ is________

**8.3.6 :** Let $s_n=(-1)^n\ (n\ge 1)$. Is $\{s_n\}$ a Cauchy sequence in $\mathbb{R}$.

**8.3.7 :** Is every metric space complete ?

**8.3.8 :** Is every complete metric space compact ?

## 8.4 MODEL EXAMINATION QUESTIONS :

**8.4.1 :** Prove that every Cauchy sequence in $\mathbb{R}^k$ is convergent (or) prove that $\mathbb{R}^k$ is complete.

**8.4.2 :** Let $\{x_n\}$ be a sequence in the metric space $(X,d)$. To each positive integer $n$, let

$$E_n=\{x_m / m\ge n\}.$$

Prove that $\{x_n\}$ is a Cauchy sequence if and only if $\operatorname{diam} E_n \to 0$ as $n\to\infty$.

**8.4.3 :** State and prove Cantor's Intersection Theorem.

## 8.5 EXERCISES

**8.5.1 :** Prove : If $\{E_n\}$ is a sequence of closed and bounded sets in a complete metric space and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0, \text{ then } \bigcap_{n=1}^{\infty} E_n \text{ contains exactly one point.}$$

**8.5.2 :** Suppose that $X$ is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of $X$. Prove Baire's Theorem; namely, $\bigcap_{1}^{\infty} G_n$ is non-empty. (In fact, it is dense in $X$). (Hint : Find a shrinking sequence of closed neighbourhoods $E_n$ such that $E_n \subseteq G_n$ and apply Exercise 8.5.1).

**8.5.3 :** Suppose that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space $X$. Prove :

(a) The sequence $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

(b) The sequence $\{d(p_n, q_n)\}$ converges (since $\mathbb{R}$ is complete).

**8.5.4 :** Let $(X, d)$ be a metric space. Let $\mathscr{C}$ be the set of all Cauchy sequences in $X$.

(a) Define a relation ~ on $\mathscr{C}$ by : $\{p_n\}, \{q_n\} \in \mathscr{C}$,

$\{p_n\} \sim \{q_n\}$ if and only if $\lim\limits_{n\to\infty} d(p_n, q_n) = 0$. Prove that ~ is an equivalence relation on $\mathscr{C}$.

(b) Let $X^*$ be the set of all equivalence classes. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$$

which exists (by Exercise 8.5.3). Show that $\Delta(P, Q)$ is independent of the choice of $\{p_n\} \in P$ and $\{q_n\} \in Q$ (i.e. if $\{p_n\}, \{p_n'\} \in P$ and $\{q_n\}, \{q_n'\} \in Q$ then

$$\lim_n d(p_n, q_n) = \lim_n d\ (p_n', q_n')\Big).$$

Prove that $\Delta$ is a metric on $X^*$.

(c) Prove that $(X^*, \Delta)$ is a complete metric space.

(d) To each $p \in X$, let $\{p\}$ be the constant sequence (and hence Cauchy sequence in $X$) so that $\{p\} \in \mathscr{C}$. Let $P_p$ be the element in $X^*$ containing the constant sequence $\{p\}$. Prove that $\Delta(P_p, P_q) = d(p, q)$ for all $p, q$ in $X$. In other words the mapping $\phi : X \to X^*$ defined by $\phi(p) = P_p$ is an isometry (i.e. a distance preserving mapping of $X$ into $X^*$).

(e) Prove that $\phi(X)$ is dense in $X^*$ and that $\phi(X) = X^*$ if $X$ is complete.

By (d), we may identify $X$ by $\phi(X)$ and thus, $X$ can be embeded in $X^*$. We call $X^*$ as the completion of $X$.

(f) If $(X_1, d_1)$ is a metric space such that $(X,d)$ can be isometrically embedded as a dense subspace in $(X_1, d_1)$, prove that $(X^*, \Delta)$ is isometric to $(X_1, d_1)$.

**8.5.5 :** Let $X$ be the set of all rational numbers; define $d: X \times X \to \mathbb{R}$ by $d(x,y) = |x-y|$. Then $d$ is a metric on $X$. What is the completion of $X$ ?

## 8.6 ANSWERS TO SHORT ANSWER QUESTIONS

**8.3.1 :** Yes. See Theorem 8.1.2

**8.3.2 :** No. See the Note 8.1.2.1

**8.3.3 :** Yes. See Theorem 8.1.4

**8.3.4 :** Yes. See Theorem 7.1.

**8.3.5 :** $\sqrt{13}$

**8.3.6 :** No. To each $n \geq 1$, let $E_n = \{s_m / m \geq 1\} = \{-1, 1\}$. Clearly $\operatorname{diam} E_n = 2$ for all $n$. So, $\lim\limits_n \operatorname{diam} E_n = 2 \neq 0$. By Theorem 8.1.9, $\{s_n\}$ is not a Cauchy sequence.

**Direct Proof :** Suppose $\{s_n\}$ is Cauchy. Let $0 < \in < 1$. So there exists a positive integer $N$ such that

$$n, m \geq N \Rightarrow |s_n - s_m| < \in .$$

We can choose $n, m \geq N$ such that $s_n = -1,\ s_m = 1$. So, $|s_n - s_m| < \in$ means $|-1\ -1| < \in$. i.e. $2 < \in$ a contradiction. So, $\{s_n\}$ is not Cauchy.

**8.3.7 :** No. Consider the examples $(0,1]$, $[0,1)$ (with respect to the usual metric $d$).

**8.3.8 :** No. Consider the example $\mathbb{R}$. This is complete but not compact as there is no finite sub cover for the open cover $\{(-n, n)\}_{n=1,2,.....}$

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition : McGraw - Hill International Editions : Walter Rudin.

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Lesson - 9

# NUMERICAL SEQUENCES

## 9.0 INTRODUCTION

In this lesson, we study the notion of a monotone sequence and the necessary and sufficient condition for a monotone sequence to be convergent (see Theorem 9.1.3). We also study the upper and lower limits of a sequence (see Definition 9.1.5); their properties (see Theorems 9.1.6 and 9.1.7) and their relation when the sequence is convergent (see Theorem 9.1.10). Further, we study the convergence of some special sequences (see Theorem 9.1.12).

## 9.1 NUMERICAL SEQUENCES :

**9.1.1 Definition :** Let $\{s_n\}$ be a sequence of real numbers. We say that $\{s_n\}$ is called.

(i) monotonically increasing if $s_n \leq s_{n+1}$ for all $n$;

(ii) strictly monotonically increasing if $s_n < s_{n+1}$ for all $n$;

(iii) monotonically decreasing if $s_n \geq s_{n+1}$ for all $n$;

(iv) strictly monotonically decreasing if $s_n > s_{n+1}$ for all $n$;

(v) monotonic if either it is monotonically increasing or monotonically decreasing.

**9.1.1.1 Note :** Clearly, any monotonically increasing sequence $\{s_n\}$ is bounded below by $s_1$; any monotonically decreasing sequence $\{s_n\}$ is bounded above by $s_1$.

**9.1.2 Example :** (i) If $s_n = n$ then $\{s_n\}$ is a strictly monotonically increasing sequence since

$n < n+1$ i.e. $s_n < s_{n+1}$ for all $n$;

(ii) If $s_n = \dfrac{1}{n^2}$ then $\{s_n\}$ is a strictly monotonically decreasing sequence since

$s_n > s_{n+1}$ for all $n$.

**9.1.3 Theorem :** Let $\{s_n\}$ be a monotonic sequence. Then $\{s_n\}$ is convergent if and only if it is bounded.

**Proof :** We know that every convergent sequence is bounded (See Theorem 7.1.7(c)). Now, we prove the converse in two cases.

**Case (i) :** Suppose $\{s_n\}$ is monotonically increasing i.e. $s_n \le s_{n+1}$ for all $n$. Assume that $\{s_n\}$ is bounded. So, it is bounded by $k$. i.e. the range $E = \{s_n / n \ge 1\}$ is bounded above by $k$. Clearly, $E \ne \phi$. By the least upper bound property of $\mathbb{R}$, $\text{lub}\, E$ exists. Let $s$ be the $\text{lub}\, E$. Now, we show that $s_n \to s$.

Let $\in > 0$ Clearly $s - \in < s$. Since $s$ is $\text{lub}\, E$, $s - \in$ is not an upper bound of $E$. So, there exists $s_N$ such that $s_N \not\le s - \in$ i.e. $s - \in < s_N$. Now

$$n \ge N \Rightarrow s - \in < s_n \le s_n \le s < s + \in$$

$$\Rightarrow |s_n - s| < \in$$

Hence $s_n \to s$.

**Case (ii) :** Suppose $\{s_n\}$ is monotonically decreasing i.e. $s_n \ge s_{n+1}$ for all $n$. Assume that $\{s_n\}$ is bounded i.e. the range $E = \{s_n / n \ge 1\}$ is bounded. So, $E$ is bounded below. Thus, $E$ is a non-empty subset of $\mathbb{R}$ bounded below. By the greatest lower bound property of $\mathbb{R}$, $\text{glb}\, E$ exists in $\mathbb{R}$. Let $t = \text{glb}\, E$. Now, we leave $s_n \to t$ as an exercise.

**9.1.4 Definition :** Let $\{s_n\}$ be a sequence of real numbers. We say that

(i) $\lim_n s_n = +\infty$ (or $s_n \to +\infty$) if for every real $M$, there exists an integer $N$ such that

$$n \ge N \Rightarrow s_n > M$$

(ii) $\lim_n s_n = -\infty$ (or $s_n \to -\infty$) if for every real $M$, there exists an integer $N$ such that

$$n \ge N \Rightarrow s_n < M.$$

**9.1.4.1 Note :** Let $\{s_n\}$ be a sequence of real numbers. In view of theorem 7.1.7(a), $s_n \to s$ in $\mathbb{R}$ is equivalent to every neighbourhood of $s$ contains all but finitely many of the terms of the sequence $\{s_n\}$. Actually, we have no right to define neighbourhood of $+\infty$ in $\mathbb{R}$ or neighbourhood of $-\infty$ in

$\mathbb{R}$ (since $+\infty \notin \mathbb{R}$ and $-\infty \notin \mathbb{R}$). Roughly speaking, if we define the neighbourhood of $+\infty$ in $\mathbb{R}$ as the interval

$$(a, +\infty) = \{x \in \mathbb{R} / a < x\}$$

where $a \in \mathbb{R}$, then, the above definition of $s_n \to +\infty$ is equivalent to "every neighbourhood of $+\infty$ in $\mathbb{R}$ contains all but finitely many of the terms of $\{s_n\}$".

**9.1.5 Definition :** Let $\{s_n\}$ be a sequence of real numbers. Let

$$E = \left\{x \in \mathbb{R} \cup \{-\infty, +\infty\} / s_{n_k} \to x \quad \text{for some subsequence } \left\{s_{n_k}\right\} \text{of } \{s_n\}\right\}$$

Put $s^* = \text{lub}\, E$,

$s_* = \text{glb}\, E$.

The numbers $s^*$, $s_*$ are called the upper and lower limits of the sequence $\{s_n\}$. We use the notation

$$\limsup_{n \to \infty} s_n = s^*, \quad \liminf_{n \to \infty} s_n = s_*$$

**9.1.6 Theorem :** Let $\{s_n\}$ be a sequence of real numbers. Let $E$ and $s^*$ have the same meaning as in Definition 9.1.5. Then $s^*$ has the following properties :

(i) $s^* \in E$

(ii) If $x > s^*$, there is an integer $N$ such that $n \geq N \Rightarrow s_n < x$

More over, $s^*$ is the only number with the properties (a) and (b).

**Proof : (a) Case (i) :** $s^* = +\infty$ : Then $E$ is not bounded above; hence the sequence $\{s_n\}$ is not bounded above. So, there is a subsequence $\left\{s_{n_k}\right\}$ of $\{s_n\}$ such that $s_{n_k} \to +\infty$. So, $s^* = +\infty \in E$.

**Case (ii) :** $s^*$ is real : Then $E$ is bounded above, and atleast one subsequential limit exists $B$.

**Case (iii) :** $s^* = -\infty$ : Then $E$ contains only one element namely $-\infty$, and there is no subsequential limit. Hence, for any real $M$, $s_n > M$ holds for atmost finite number of values of $n$.

So, $s_n \to -\infty$. So, $s^* \in E$. Thus we have proved that $s^* \in E$ in all cases.

**(b) :** Suppose there is a number $x$ such that $x > s^*$. Suppose the conclusion does not hold. So, $s_n \geq x$ for infinitely many $n$. To each positive integer $k$, there is a natural number $n_k$ such that $s_{n_k} \geq x$. Without loss of generality, we can assume that $n_1 < n_2 < \dots\dots\dots$. Thus, $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$. Let

$$y = \lim_k s_{n_k}$$

Then $y \in E$. So, $y \leq s^*$ Since each $s_{n_k} \geq x$, we have that $y \geq x\left(> s^*\right)$, a contradiction Hence there exists a positive integer $N$ such that

$$n \geq N \Rightarrow x > s_n.$$

**Uniqueness of $s^*$ with the properties (a) and (b) :** Clearly $s^*$ has the properties (a) and (b). Suppose $z$ has properties (a) and (b). Since $z$ has property (a) i.e. $z \in E$, we have that $z \leq s^*$. Suppose $z \neq s^*$ i.e. $z < s^*$. Choose $x$ such that $z < x < s^*$. Since $z$ has property (b), there exists a positive integer $N$ such that

$$n \geq N \Rightarrow s_n \leq x.$$

So, every sub sequential limit of $\{s_n\}$ is less than or equal to $x$ i.e. $x$ is an upper bound of $E$. Since $s^* = \text{lub}\, E$, $s^* \leq x$, a contradiction to $x < s^*$. Hence $z = s^*$.

Now, we state a theorem similar to the above theorem corresponding to $s_*$ and we leave the proof as an exercise.

**9.1.7 Theorem :** Let $\{s_n\}$ be a sequence of real numbers. Let $E$ and $s_*$ have the same meaning as in Definition. Then $s_*$ has the following properties.

(a) $s_* \in E$.

(b) If $x < s_*$, then there is a positive integer $N$ such that

$$n \geq N \Rightarrow x < s_n.$$

Moreover, $s_*$ is the unique number with properties (a) and (b).

**9.1.8 :** Consider the example 7.1.5(f). The set $E$ of all subsequential limits of $\{s_n\}$ is $E=\{-1, 1\}$. So $s^*=1,\ s_*=-1$. Here $\{s_n\}$ is not convergent.

**Proof :** Exercise.

**9.1.9 Example :** Let $\{s_n\}$ be the sequence of all rational numbers in the metric space $(\mathbb{R}', d)$. Then, the set of all subsequential limits of $\{s_n\}$ is $E=\mathbb{R}$. So, $s^*=+\infty$, $s_*=-\infty$. Here, $\{s_n\}$ is not convergent.

**9.1.10 Theorem :** A sequence $\{s_n\}$ of reals is convergent if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{x\to\infty} s_n \text{ i.e. } s^* = s_*.$$

**Proof :** Let $\{s_n\}$ be a sequence of real numbers. Assume that $\{s_n\}$ is convergent i.e. $s_n \to s$ for some $s \in \mathbb{R}$. So, every subsequence of $\{s_n\}$ converges to $s$ only. So, the set $E$ of all subsequential limits is $E=\{s\}$. So, $\sup E=\inf E=s$. That is

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} = s$$

Conversely assume that

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s \text{ (say).}$$

i.e. $s^* = s_* = s$.

Now, we show that $s_n \to s$ as $n \to \infty$.

Let $\in > 0$. Now, $s-\in < s = s_*$. By Theorem 9.1.7, there exists a positive integer $N_1$ such that

$$n \geq N_1 \Rightarrow s-\in < s_n$$

Now, $s+\in > s = s^*$. By Theorem 9.1.6, there exists a positive integer $N_2$ such that

$$n \geq N_2 \Rightarrow s+\in > s_n$$

Put $N=\text{Max}\{N_1, N_2\}$. Clearly,

$$n \geq N \Rightarrow s - \in < s_n < s + \in$$

i.e. $|s_n - s| < \in$.

Hence $s_n \to s$ as $n \to \infty$

**9.1.11 Theorem :** If $s_n \leq t_n$ for $n \geq N$, where $N$ is fixed, then

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n,$$

$$\limsup_{n \to \infty} s_n \leq \limsup_{n \to \infty} t_n$$

**Proof :** Let $s_n \leq t_n$ for $n \geq N$. Let

$$\alpha = \liminf_{n \to \infty} s_n, \qquad \beta = \liminf_{n \to \infty} t_n.$$

We have to prove that $\alpha \leq \beta$. Suppose $\alpha \not\leq \beta$ i.e. $\alpha > \beta$. Put $\in = \frac{\alpha - \beta}{2}$. Now,

$\alpha - \in < \alpha \Rightarrow \exists$ a positive integer $N_1 \ni$

$n \geq N_1 \Rightarrow \alpha - \in < s_n$

i.e. $\frac{\alpha + \beta}{2} < s_n$. (By Theorem 9.1.7).

Since $\beta = \liminf_{n} t_n$, there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $t_{n_k} \to \beta$ as $k \to \infty$. So, there is a natural number $N_2$ such that

$$k \geq N_2 \Rightarrow |t_{n_k} - \beta| < \in$$

i.e. $\beta - \in < t_{n_k} < \beta + \in = \frac{\alpha + \beta}{2}$.

Put $k > \max\{N, N_1, N_2\}$. Now,

$s_{n_k} \leq t_{n_k} < \frac{\alpha + \beta}{2}$ and $s_{n_k} > \frac{\alpha + \beta}{2}$ a contradiction. So $\alpha \leq \beta$.

Similarly, we can prove the other.

## SOME SPECIAL SEQUENCES

### 9.1.12 Theorem :

(a) If $p>0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p}=0$

(b) If $p>0$, then $\lim\limits_{n\to\infty} \sqrt[n]{p}=1$

(c) $\lim\limits_{n\to\infty} \sqrt[n]{n}=1$

(d) If $p>0$ and $\alpha$ is real, then

$$\lim_{n\to\infty} \frac{n^{\alpha}}{(1+p)^n}=0$$

(e) If $|x|<1$, then $\lim\limits_{n\to\infty} x^n=0$

**Proof : (a) :** Let $p>0$. Let $\in>0$. Choose a positive integer $N$ such that

$$N>\left(\frac{1}{\in}\right)^p$$

which is possible by the archimedian property of real numbers. For $n\geq N$,

$$\left|\frac{1}{n^p}-0\right|=\frac{1}{n^p}\leq\frac{1}{N^p}<\in.$$

Hence, the conclusion.

**(b) :** Let $p>0$. If $p=1$, then the conclusion is obvious since each term of the sequence is 1.

**Case (i) : Suppose** $p>1$ : Put $x_n=\sqrt[n]{p}-1$. Then $x_n>0$ and by binomial theorem,

$$p=(1+x_n)^n\geq 1+n\,x_n$$

$$\Rightarrow 0<x_n\leq\frac{p-1}{n}.$$

Since $\lim\limits_{n\to\infty}\frac{1}{n}=0$, we have that $\lim\limits_{n\to\infty}\frac{p-1}{n}=0$,

$$0\le \lim_{n\to\infty} x_n \le \lim_{n\to\infty}\frac{p-1}{n}=0,$$

Hence, $\lim\limits_{n} x_n = 0$ i.e. $\lim\limits_{n\to\infty}\sqrt[n]{p} = 1$

**Case (ii) : Suppose** $p<1$ : So, $\frac{1}{p}>1$. By case (i),

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{p}\right)}=1 \text{ i.e. } \lim_{n\to\infty}\frac{1}{\sqrt[n]{p}}=1$$

and hence

$$\lim_{n\to\infty}\sqrt[n]{p}=1$$

**(c)** : Put $x_n=\sqrt[n]{n}-1$. Clearly, $x_n\ge 0$ for all $n$. Now,

$$n=(1+x_n)^n$$

$$= 1+n\,x_n+\frac{n(n-1)}{2}x_n^2+\cdots\cdots \text{ (by binomial theorem)}$$

$$\ge \frac{n(n-1)}{2}x_n^2$$

So, $0\le x_n\le\sqrt{\frac{2}{n-1}}$ $\quad (n\ge 2)$.

Clearly, $\sqrt{\frac{2}{n-1}}\to 0$ as $n\to\infty$. Hence $x_n\to 0$ as $n\to\infty$

i.e. $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$

{Now, we directly prove that $x_n\to 0$ as $n\to\infty$. Let $\in>0$. Choose a positive integer $N(>2)$ such that

$$\sqrt{\frac{2}{N-1}} < \in \text{ i.e. } N > \frac{2}{\in^2} + 1$$

Now,

$$n \geq N \Rightarrow |x_n| = x_n \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{N-1}} < \in.$$

Thus, $x_n \to 0$ as $n \to \infty$}.

**(d) :** Let $p > 0$ and $\alpha$ be real. Let $k$ be a positive integer such that $k > \alpha$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)(n-2)\cdots\cdots(n-k+1)}{k!} p^k.$$

$$> \frac{n^k p^k}{2^k k!}$$

Hence $0 < \dfrac{n^\alpha}{(1+p)^n} < \dfrac{2^k k!}{p^k} n^{\alpha-k} \qquad (n > 2k).$

Since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ as $n \to \infty$ (by (a)).

Hence, the conclusion.

**(e) :** Let $|x| < 1$. Then $\dfrac{1}{|x|} > 1$. So, $\dfrac{1}{|x|} = 1 + p$ for some $p > 0$ and hence

$$\frac{1}{(1+p)^n} = |x|^n.$$

Now, $\lim\limits_{n\to\infty} |x|^n = \lim\limits_{n\to\infty} \dfrac{1}{(1+p)^n} = 0$ (taking $\alpha = 0$ in (d))

Hence the conclusion.

## 9.2 SHORT ANSWER QUESTIONS :

**9.2.1 :** Define a monotonic sequence.

**9.2.2 :** Give an example of a monotonic sequence which is not bounded.

**9.2.3 :** Suppose a monotonically increasing sequence converges. What is the relation between th limit of the sequence and its range ?

**9.2.4 :** Consider the sequence $\{s_n\}$ given by

$$1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots\ldots\ldots\ldots$$

write $s^*$ and $s_*$.

**9.2.5 :** Consider the sequence $\{s_n\}$

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots\ldots\ldots$$

Find $s^*$ and $s_*$.

**9.2.6 :** Let $\{s_n\}$ be a sequence such that $\lim\limits_{n\to\infty} s_n = s$. Write the values of $s^*$ and $s_*$.

**9.2.7 :** Let $\{s_n\}$ be the sequence of all rational numbers in $(0, 1)$. Write $s^*$ and $s_*$.

**9.2.8 :** $\lim\limits_{n\to\infty} \sqrt[n]{n} =$

**9.2.9 :** If $p>0$ and $\alpha$ is real then $\lim\limits_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} =$

## 9.3 MODEL EXAMINATION QUESTIONS

**9.3.1 :** Define a monotonic sequence. Prove that a monotone sequence is convergent if and only if it is bounded.

**9.3.2 :** Let $\{s_n\}$ be a sequence of real numbers. Define the upper limit $s^*$ of the sequence $\{s_n\}$. Prove the following :

(a) There exists a subsequence of $\{s_n\}$ converging to $s^*$.

(b) $x>s^*$ implies there exists a positive integer $N$ such that

$n \geq N \Rightarrow x > s_n$.

(c) $s^*$ is the unique number with properties (a) and (b).

**9.3.3 :** Prove that a sequence $\{s_n\}$ of reals converges if and only if both the upper limit $s^*$ and the lower limit $s_*$ of the sequence $\{s_n\}$ exist and are equal.

**9.3.4 :** Prove the following :

(a) $\lim\limits_{n\to\infty} \sqrt[n]{p} = 1 \quad (p>0)$

(b) If $p>0$ and $\alpha$ is real then

$$\lim_{n\to\infty} \frac{n^{\alpha}}{(1+p)^n} = 0$$

## 9.4 EXERCISES

**9.4.1 :** Let $\{s_n\}$ be the sequence defined by $s_1 = 0,\ s_{2m} = \dfrac{s_{2m-1}}{2},\ s_{2m+1} = \dfrac{1}{2} + s_{2m}$.

**9.4.2 :** Find the upper and lower limits of the sequence $\{s_n\}$.

**9.4.3 :** Let $\{a_n\}$ and $\{b_n\}$ be two real sequences. Prove that

$$\lim_{n\to\infty} \sup\,(a_n + b_n) \le \lim_{n\to\infty} \sup\, a_n + \lim_{n\to\infty} \sup\, b_n;$$

$$\lim_{n\to\infty} \inf\,(a_n + b_n) \ge \lim_{n\to\infty} \inf\, a_n + \lim_{n\to\infty} \inf\, b_n$$

**9.4.4 :** Prove theorem 9.1.7.

## 9.5 ANSWERS TO SELF ASSESSMENT QUESTIONS

**9.2.1 :** See definition 9.1.1

**9.2.2 :** Put $s_n = n$. Then $\{s_n\}$ is monotonically increasing; but not bounded.

**9.2.3 :** The limit of the sequence is lub of its range.

**9.2.4 :** $s^* = s_* = 0$

**9.2.5 :** $s^* = s_* = 0$

**9.2.6 :** $s^* = s = s_*$

**9.2.7 :** $s^* = 1, \ s_* = 0$

**9.2.8 :** 1

**9.2.9 :** 0

## 9.6 REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, McGraw - Hill International Editions Walter Rudin

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Lesson - 10

# SERIES

## 10.0 INTRODUCTION

In this lesson, we study the convergence of a series, Cauchy criterion for and the tests of convergence of a series-namely, comparison test, Root test and Ratio test.

In this lesson, we study the convergence of a series, Cauchy criterion for convergence of a series and the tests of convergence of a series-namely, comparison test, Root test and Ratio test. Further, we define the number $e$ and observe $e$ as a limit of a sequence (see Theorem 10.1.14) and we prove that $e$ is irrational. Furhter, we study the Leibnitz theorem. We also study the absolute convergence of a series and Riemann's theorem.

## 10.1 SERIES

**10.1.1 Definition :** Let $\{a_n\}$ be a sequence of reals. If $p$ and $q$ are integers with $p \le q$, we write

$$\sum_{n=p}^{q} a_n \text{ for } a_p + a_{p+1} + \cdots\cdots\cdots + a_q .$$

**10.1.2 Definition :** The expression

$$a_1 + a_2 + a_3 + \cdots\cdots\cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n \text{ is called an infinite series or just a series.}$$

**10.1.3 Definition :** Consider the series

$$\sum_{n=1}^{\infty} a_n .$$

Put $s_n = a_1 + a_2 + \cdots\cdots + a_n = \sum_{k=1}^{n} a_k$

(i) The sequence $\{s_n\}$ is called the partial sum sequence of the series

$$\sum_{n=1}^{\infty} a_n .$$

(ii) If $\{s_n\}$ converges to some $s$, then we say that the series

$\sum_{n=1}^{\infty} a_n$ converges to $s$ and we write

$$\sum_{n=1}^{\infty} a_n = s .$$

If $\{s_n\}$ is not convergent, then we say that the series

$\sum_{n=1}^{\infty} a_n$ diverges.

**10.1.4 Theorem :** If $\sum_{n=1}^{\infty} a_n$ converges thus $\lim_n a_n = 0$

**Proof :** Suppose $\sum_{n=1}^{\infty} a_n$ converges. So, the partial sum sequence $\{s_n\}$ (where $s_n = a_1 + a_2 + \cdots\cdots + a_n$) converges (to $s$ say). So,

$$\lim_n a_n = \lim_n (s_n - s_{n-1}) = n$$

$$= \lim_n s_n - \lim_n s_{n-1} = s - s = 0$$

**10.1.5 Theorem (Cauchy's Criterion) :** The series $\sum_{n=1}^{\infty} a_n$ converges if and oly if for every $t > 0$, ere is a positive integer $N$ such that

$$m \geq n \geq N \Rightarrow \left| \sum_{k=n}^{m} a_k \right| < t .$$

**Proof :** Assume that the series $\sum_{n=1}^{\infty} a_n$ converges. So, the partial sum sequence $\{s_n\}$(where $s_n = a_1 + a_2 + \cdots\cdots + a_n$) converges. So, $\{s_n\}$ is a Cauchy sequence. So, for every $t > 0$ there is a positive integer $N_1$ such that

$$m \geq N_1,\ n \geq N_1 \Rightarrow |s_m - s_n| < t.$$

Put $N = N_1 + 1$. Then

$$m \geq n \geq N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| = |s_m - s_{n-1}|$$

$$< t \text{ (since } n \geq N = N_1 + 1 \Rightarrow n - 1 \geq N_1)$$

Conversely assume the condition. Let $t > 0$. So, there exists a positive integer $N \ni$.

$$m \geq n \geq N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| < \frac{t}{2}$$

Now,

$$m \geq n \geq N \Rightarrow |s_m - s_n| = \left|\sum_{k=n+1}^{m} a_k\right|$$

$$= \left|\sum_{k=n}^{m} a_k - a_n\right| \leq \left|\sum_{k=n}^{m} a_k\right| + |a_n| < \frac{t}{2} + \frac{t}{2} = t.$$

Thus, $\{s_n\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, $\{s_n\}$ converges i.e. $\sum_{n=1}^{\infty} a_n$ converges.

**10.1.5.1 Note :** Theorem 10.1.4 can also be proved using Theorem 10.1.5. Suppose $\sum_{n=1}^{\infty} a_n$ converges. By Theorem 10.1.5, for every $t > 0$ there is a positive integer $N$ such that

$$m \geq n \geq N \Rightarrow \left|\sum_{k=n}^{m} a_k\right| < t$$

and hence $n \geq N \Rightarrow |a_n| < t$ (taking $n = m \geq N$).

Hence, $a_n \to 0$ as $n \to \infty$.

**10.1.6 Example :** Consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$. Clearly, $\frac{1}{n}\to 0$ as $n\to\infty$. But $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (i.e. not convergent) (See Theorem 10.1.11).

In view of Theorem 9.1.3 concerning monotonic sequences, we have the following.

**10.1.7 Theorem :** A series of non negative terms converges if and only if the partial sum sequence is bounded.

**Proof :** Let $\sum_{n=1}^{\infty} a_n$ be a series of non-negative terms (i.e. $a_n \geq 0$ for all $n$). Let $\{s_n\}$ be the partial sum sequence of $\sum_{n=1}^{\infty} a_n$. So, $s_n = \sum_{k=1}^{n} a_k$. Since $a_n \geq 0$ for all $n$, the sequence $\{s_n\}$ is monotonically increasing. Now,

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \{s_n\} \text{ converges}$$

$$\Leftrightarrow \{s_n\} \text{ is bounded (by Theorem 9.1.3)}$$

Now, we prove the comparison test.

**10.1.8 Theorem :** (a) If $|a_n| \leq c_n$ for $n \geq N_0$, where $N_0$ is some fixed positive integer, and if $\sum c_n$ converges then $\sum a_n$ converges (infact $\sum |a_n|$ converges).

(b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges then $\sum a_n$ diverges.

**Proof : (a) :** Assume that $|a_n| \leq c_n$ for $n \geq N_0$ and suppose $\sum_{n=1}^{\infty} c_n$ converges. Now, we prove that $\sum a_n$ converges.

Let $t > 0$. Since $\sum_{n=1}^{\infty} c_n$ converges, there exists a positive integer $N$ such that

$$m \geq n \geq N \Rightarrow \left|\sum_{k=n}^{m} c_k\right| < t \quad \text{i.e.} \quad \sum_{k=n}^{m} c_k < t$$

$$\Rightarrow \left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k| \le \sum_{k=n}^{m} c_k < t.$$

Hence, $\sum_{n=1}^{\infty} a_n$ converges (of course $\sum_{n=1}^{\infty} |a_n|$ converges) (by Theorem 10.1.5).

**(b) :** Let $a_n \ge d_n \ge 0$ for $n \ge N_0$. Suppose $\sum d_n$ diverges. If $\sum a_n$ converges, then $\sum d_n$ converges (by (a)). a Contradiction. So, $\sum a_n$ diverges.

Now, we consider the geometric series.

**10.1.9 Theorem :** If $0 \le x < 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \ge 1$, the series diverges.

**Proof :** Suppose $0 \le x < 1$ : Let $s_n = 1 + x + x^2 + \cdots\cdots\cdots + x^n$. Then

$$s_n = \frac{1-x^{n+1}}{1-x}$$

Since $x^n \to 0$ as $n \to \infty$, we have that

$$\lim_{n\to\infty} s_n = \frac{1}{1-x}$$

So, $\sum_{n=0}^{\infty} x^n$ converges and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Suppose $x = 1$ : Then, $s_n = 1 + x + x^2 + \cdots\cdots\cdots + x^n = n+1$. Since $s_n \to \infty$ as $n \to \infty$, the series $\sum_{n=0}^{\infty} x^n$ diverges.

Suppose $x>1$ : Then $s_n = 1+x+x^2+\cdots\cdots+x^n \geq n+1$ for all $n$. Since $n+1 \to \infty$ as $n \to \infty$,

$$\lim_n s_n = \infty$$

So, $\{s_n\}$ diverges and hence $\sum_{n=0}^{\infty} x^n$ diverges.

**10.1.10 Theorem :** Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots\cdots \geq 0$. Then the series

$\sum_{n=1}^{\infty} a_n$ converges if and only if the series.

$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots\cdots$ converges.

**Proof :** Let $s_n = a_1 + a_2 + \cdots\cdots + a_n$,

$t_n = a_1 + 2a_2 + \cdots\cdots + 2^n a_{2^n}$.

For $n < 2^k$, $s_n \leq a_1 + (a_2 + a_3) \cdots\cdots + \left(a_{2^k} + \cdots\cdots + a_{2^{k+1}-1}\right)$

$$\leq a_1 + 2a_2 + 2^2 a_4 + \cdots\cdots + 2^k a_{2^k}$$

$$= t_k \quad \text{---------------- (1)}$$

For $n > 2^k$, $s_n \geq a_1 + a_2 + (a_3 + a_4) + \cdots\cdots \left(a_{2^{k-1}+1} + \cdots\cdots + a_{2^k}\right)$

$$\geq \frac{1}{2} a_1 + a_2 + 2a_4 + \cdots\cdots + 2^{k-1} a_{2^k}$$

$$= \frac{1}{2} t_k .$$

i.e. $2s_n \geq t_k$ --------- (2)

Now,

$\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \{s_n\}$ is bounded (by Thoerem 9.1.3)

$$\Leftrightarrow \{t_k\} \text{ is bounded (by (1) and (2))}$$

$$\Leftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges (by Theorem 9.1.3)}$$

**10.1.11 Theorem :** $\sum \frac{1}{n^p}$ converges if $p>1$ and diverges if $p\le 1$.

**Proof :** If $p\le 0$ then $\lim\limits_{n} \frac{1}{n^p} = \infty$ and hence $\sum \frac{1}{n^p}$ diverges (by Theorem 10.1.4). Suppose $p>0$. Clearly,

$$\frac{1}{1^p} > \frac{1}{2^p} > \frac{1}{3^p} > \cdots\cdots \ge 0.$$

Consider the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{\left(2^k\right)^p} = \sum_{k=0}^{\infty} 2^{k(1-p)}$$

Clearly $2^{1-p} < 1$ if and only if $1-p<0$ i.e. $p>1$. By Theorem 10.1.10, $\sum \frac{1}{n^p}$ converges if and only if $p>1$.

**10.1.12 Theorem :** If $p>1$, then the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$ converges; if $p\le 1$, the series diverges.

**Proof :** Clealry,

$$\frac{1}{n(\log n)^p} > \frac{1}{(n+1)\left(\log(n+1)\right)^p} \ (\text{for all } n) > 0.$$

Consider the series

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k\left(\log 2^k\right)^p} = \sum_{k=1}^{\infty} \frac{1}{k^p(\log 2)^p}$$

$$=\frac{1}{(\log 2)^p}\sum_{k=1}^{\infty}\frac{1}{k^p}$$

The given series converges if and only if the series.

$\frac{1}{(\log 2)^p}\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges (by Thorem 1.1.10).

if and only if $p>1$ (by Theorem 1.1.11)

## THE NUMBER $e$ :

**10.1.13 Definition** : $e=\sum_{n=0}^{\infty}\frac{1}{n!}$

where $n!=1\cdot 2\cdot 3\cdots\cdots\cdots n(\text{if } n\geq 1)$ and $0!=1$.

**Note :** Let

$$s_n=\sum_{k=0}^{n}\frac{1}{k!}=1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots\cdots\cdots\cdots+\frac{1}{1\cdot 2\cdot 3\cdots\cdots n}<1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{n-1}}$$

$<1+1+1=3$.

Clearly, $\sum_{n=0}^{\infty}\frac{1}{n!}$ is a series of non negative terms. So, its partial sum sequence $\{s_n\}$ is monotonically increasing. Since $s_n<3$ for all $n$, we have that $\{s_n\}$ is bounded. By Theorem 9.1.3, $\{s_n\}$ converges and hence the series $\sum_{n=0}^{\infty}\frac{1}{n!}$ converges, we denote this by $e$. Clearly, $2<e\leq 3$.

In the following theorem, we prove that $e$ can be defined by means of another limit process.

**10.1.14 Theorem :** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$.

**Proof :** Let $s_n=\sum_{k=0}^{n}\frac{1}{k!}$ and $t_n=\left(1+\frac{1}{n}\right)^n$.

By Binomial Theorem,

$$t_n = 1^n + n_{c_1} \cdot 1^{n-1} \cdot \frac{1}{n} + n_{c_2} \cdot 1^{n-2} \cdot \left(\frac{1}{n}\right)^2 + \cdots\cdots + n_{c_n}\left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{n-1}{n}\right)$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots + \frac{1}{n!} = s_n .$$

So, $\limsup_{n\to\infty} t_n \leq \lim_{n\to\infty} s_n = e$ (by Theorem 9.1.11).

If $n \geq m$,

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots\cdots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{m-1}{n}\right)$$

Letting $n \to \infty$, keeping $m$ fixed, we have

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + \cdots\cdots + \frac{1}{m!} = s_m .$$

Now, letting $m \to \infty$, we have that

$$e \leq \liminf_{n\to\infty} t_n .$$

Thus, $e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e$.

So, that $e = \liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n$.

Hence, $\{t_n\}$ converges and

$\lim_n t_n = e$ (by Theorem 9.1.10).

**10.1.14.1 Note :** Let $\{s_n\}$ be the partial sum sequence of $\sum_{n=0}^{\infty} \frac{1}{n!}$. Since $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, we have that

$\lim_n s_n = e$. Now,

$$0 < e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\cdots\cdots$$

$$< \frac{1}{(n+1)!} + \left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\cdots\right]$$

$$= \frac{1}{n!n}$$

**10.1.15 Theorem :** $e$ is irrational

**Proof :** Suppose $e$ is rational. So, $e = \frac{p}{q}$ where $p, q$ are positive integers, $q \neq 0$. By the above note 10.1.14.1.

$$0 < q!\,(e - s_q) < \frac{1}{q}.$$

By our assumption, $q\,!e$ is an integer. Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots\cdots + \frac{1}{q!}\right)$$

is an integer, we have that $q!\,(e - s_q)$ is an integer which lies between 0 and $\frac{1}{q}$ i.e. between 0 and 1, a contradiction. Hence $e$ is irrational.

**10.1.16 Theorem (Root Test) :** Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then

(i) if $\alpha < 1$, $\sum a_n$ converges

(ii) if $\alpha > 1$, $\sum a_n$, diverges

(iii) if $\alpha = 1$, the test gives no information.

**Proof :** (i) Let $\alpha < 1$. Choose $\beta$ such that $\alpha < \beta < 1$. Since $\beta > \alpha$, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \beta > |a_n|^{\frac{1}{n}}$$

i.e. $\beta^n > |a_n|$.

Since $0<\beta<1$, the series $\sum \beta^n$ converges (geometric series). By comparison test, $\sum a_n$ converges.

(ii) Let $\alpha>1$. Since $\alpha = \lim\limits_{n\to\infty} \sup |a_n|$, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $\{a_{n_k}\}^{\frac{1}{n_k}} \to \alpha$ as $k\to\infty$. Since $\alpha>1$, $|a_n|>1$ holds for infinitely many $n$. So, $\lim\limits_{n\to\infty} a_n \neq 0$. By Theorem 10.1.4 $\sum a_n$ diverges (i.e. not Convergent).

(iii) Consider the series $\sum \frac{1}{n^2}$ and $\sum \frac{1}{n}$. Clearly $\alpha$ for both the series is 1. But $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent (see Theorem $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$). So, this test gives no information when $\alpha=1$.

**10.1.[illegible] Theorem (Ratio Test) :** The series $\sum a_n$

(a) converges if $\lim\limits_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$ where $n_0$ is some fixed positive integer.

(c) If $\lim\limits_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \lim\limits_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$, the test gives no information.

**Proof :** (a) : Assume that $\lim\limits_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$. Choose a number $\beta$ such that

$$\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < \beta < 1$$

[illegible]eorem, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < \beta$$

*In particular,*

$$|a_{N+1}| < \beta |a_N|,$$

$$|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|$$

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$$|a_{N+p}| \leq \beta^p |a_N|.$$

That is, $n \geq N \Rightarrow |a_n| < |a_N| \beta^{-N} \cdot \beta^n$

Since $0 < \beta < 1$, $\sum_{n=1}^{\infty} \beta^n$ converges. By Comparison Test (by Theorem 10.1.8), $\sum a_n$ converges.

(b) : Assume that there exists a positive integer no such that

$$n \geq n_0 \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| \geq 1 \text{ i.e. } |a_{n+1}| \geq |a_n|.$$

Now, we show that $a_n \not\to 0$ as $n \to \infty$. Suppose $a_n \to 0$ as $n \to \infty$ (Exercise 7.4.1). So, $|a_n| \to |0| = 0$ as $n \to \infty$ (Exercise 7.4.1). So, the monotonically increasing sequence $|a_{n_0}|, |a_{n_0+1}|, \cdots\cdots$ convergence to 0. We know that every monotonically increasing sequence of reals if converges, converged to the lub of its range. So, $|a_n| = 0$ i.e. $a_n = 0$ for $n \geq n_0$. This is a Contradiction. So, $a_n \not\to 0$ as $n \to \infty$. Hence the series diverges.

(c) : We know that the series

(i) $\sum \frac{1}{n}$ is divergent and

(ii) $\sum \frac{1}{n^2}$ is convergent.

For both the series, it is clear that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$$

So, this test gives no information in this case.

**10.1.18 Example :** Consider the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots\cdots\cdots\cdots$$

Here $$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0,$$

$$\liminf_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}};$$

$$\limsup_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}},$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{3}{2}\right)^n = +\infty.$$

The root test indicates the convergence; the ratio test does not apply.

**10.1.19 Example :** Consider the series

$$\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\cdots\cdots\cdots$$

where $$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{8},$$

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n} = 2, \text{ but}$$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \frac{1}{2}$$

Root test indicates the convergence, where as ratio test does not apply.

**Note :** The ratio test is easier to apply than the root test as it is easier to compute ratios than $n^{\text{th}}$ roots. Inview of the following theorem 10.1.20, whenever the ratio test shows convergence, the root test also shows convergence; whenever root test is inconclusive, the ratio test is also inconclusive.

**10.1.20 Theorem :** For any sequence $\{C_n\}$ of positive numbers

$$\liminf_{n\to\infty} \frac{C_{n+1}}{C_n} \leq \liminf_{n\to\infty} \sqrt[n]{C_n},$$

$$\limsup_{n\to\infty} \sqrt[n]{C_n} \leq \limsup_{n\to\infty} \frac{C_{n+1}}{C_n}.$$

**Proof :** Now, we prove the second inequality. Let

$$\alpha = \limsup_{n\to\infty} \frac{C_{n+1}}{C_n}.$$

If $\alpha = +\alpha$ then it is clear. Suppose $\alpha$ is finite. Choose $\beta$ such that $\beta > \alpha$. So, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow \frac{C_{n+1}}{C_n} \leq \beta.$$

In particular,

$$C_{N+1} \leq \beta C_N,$$

$$C_{N+2} \leq \beta C_{N+1} \leq \beta^2 C_N,$$

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$$C_{N+p} \leq \beta^p C_N \text{ (for any } p > 0).$$

Now, $n \geq N \Rightarrow C_n \leq C_N \beta^{-N} \beta^N$

$$\Rightarrow \sqrt[n]{C_n} \leq \sqrt[n]{C_N \beta^{-N}} \cdot \beta,$$

so that $\limsup_{n\to\infty} \sqrt[n]{C_n} \leq \beta$ -------------------- (1)

by Theorem 9.1.11.

Since (1) holds for every $\beta > \alpha$, we have that

$$\limsup \sqrt[n]{C_n} \leq \alpha.$$

Similarly, we can prove the other.

## SUMMATION BY PARTS

**10.1.21 Theorem :** Given two sequences $\{a_n\}, \{b_n\}$, put

$$A = \sum_{k=0}^{n} a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n\, b_n = \sum_{n=p}^{q-1} A_n\left(b_n - b_{n-1}\right) + A_q\, b_q - A_{p-1}\, b_p.$$

**Proof :** $\displaystyle\sum_{n=p}^{q} a_n\, b_n = \sum_{n=p}^{q-1} \left(A_n - A_{n-1}\right) b_n + A_q\, b_q$

$$= \sum_{n=p}^{q-1} A_n\, b_n - \sum_{n=p}^{q-1} A_{n-1}\, b_n + A_q\, b_q$$

$$= \sum_{n=p}^{q-1} A_n\, b_n - \sum_{n=p-1}^{q-1} A_n\, b_{n+1} + A_q\, b_q$$

$$= \sum_{n=p}^{q-1} A_n\, b_n - \sum_{n=p}^{q-1} A_n\, b_{n+1} - A_{p-1}\, b_p + A_q\, b_q$$

$$= \sum_{n=p}^{q} A_n\left(b_n - b_{n+1}\right) - A_{p-1}\, b_p + A_q\, b_q.$$

**10.1.21.1 Note :** The conclusion of the above theorem 10.1.21 is called "partial summation formula". This is useful in the investigation of the series of the form $\sum a_n\, b_n$, particularly when $\{b_n\}$ is monotonic.

**10.1.22 Theorem :** Suppose

(a) The partial sums $A_n$ of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$;

(c) $\lim_{n} b_n = 0$

Then $\sum a_n\, b_n$ converges.

**Proof :** Since $\{A_n\}$ is bounded, there exists a positive number $M$ such that $|A_n| \le M$ for all $n$.

Let $t > 0$. Since $\lim_{n\to\infty} b_n = 0$, there exists a positive integer $N$ such that

$$n \ge N \Rightarrow |b_n| < t/(2M).$$

Now, $q \ge p \ge N \Rightarrow \left| \sum_{n=p}^{q} a_n\, b_n \right|$

$$= \left| \sum_{n=p}^{q-1} A_n \left(b_n - b_{n+1}\right) + A_q b_q - A_{p-1} b_p \right|$$

$$\le M \left| \sum_{n=p}^{q-1} \left(b_n - b_{n+1}\right) + b_q + b_p \right|$$

$$= 2M\, b_p < t.$$

By Cauchy criterion(10.1.5), $\sum a_n\, b_n$ converges.

**10.1.23 Theorem (Leibnitz) :** Suppose

(a) $|C_1| \ge |C_2| \ge \cdots\cdots\cdots\cdots\cdots\cdots;$

(b) $C_{2m-1} \ge 0,\ C_{2m} \le 0 \quad (m = 1, 2, \cdots\cdots\cdots\cdots)$

(c) $\lim_{n\to\infty} C_n = 0$. Then $\sum C_n$ converges.

**Proof :** Take $a_n = (-1)^n$, $b_n = |C_n|$. Now, the partial sum sequence $\{A_n\}$ of $\sum a_n$ is bounded (since $A_n = -1$ or $1$). Clearly,

$$b_0 \ge b_1 \ge \cdots\cdots\cdots\cdots$$

Since $\lim_{n\to\infty} C_n = 0$,

we have that

$$\lim_{n\to\infty} b_n = 0.$$

By Theorem 10.1.23

$\sum a_n b_n$ converges

i.e. $\sum c_n$ converges.

## ABSOLUTE CONVERGENCE

**10.1.24 Definition :** The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges

**Theorem :** If $\sum a_n$ converges absolutely then $\sum a_n$ converges.

**Proof :** It is clear that

$$\left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k|$$

Now the theorem follows from Cauchy's criterion.

or

$\sum a_n$ converges absolutely

$\Rightarrow \sum |a_n|$ converges (by definition)

$\Rightarrow$ for every $t > 0$, there exists a positive integer $N$ such that

$$"m \ge n \ge N \Rightarrow \left|\sum_{k=n}^{m} |a_k|\right| < t \quad \text{(by Cauchy criterion)}$$

$$\Rightarrow \left|\sum_{k=n}^{m} a_k\right| \le \sum_{k=n}^{m} |a_k| < t"$$

$\Rightarrow \sum a_n$ converges (by Cauchy criterion 10.1.5)

**Note :** The converse of this theorem is not true. We know that the series $\sum \frac{1}{n}$ is divergent. By Leibnitz theorem, $\sum \frac{(-1)^n}{n}$ converges.

## ADDITION AND MULTIPLICATION OF SERIES

**10.1.25 Theorem :** If $\sum a_n = A$, and $\sum b_n = B$, then

$$\sum (a_n + b_n) = A + B$$

$$\sum C\, a_n = C\, A \text{ for any fixed } c.$$

**Proof :** Let $\sum a_n = A$, $\sum b_n = B$, $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$.

So, $\lim_{n\to\infty} A_n = A$, $\lim_{n\to\infty} B_n = B$

Then, $\lim_{n\to\infty} (A_n + B_n) = \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n = A + B$ and

$\lim_{n\to\infty} C\, A_n = C\, A$.

So, $\sum (a_n + b_n) = A + B$ and $\sum C\, a_n = CA$.

as $\{A_n + B_n\}$ and $\{C\, A_n\}$ are partial sum sequences of $\sum (a_n + b_n)$ and $\sum C\, a_n$ respectively.

**10.1.26 Definition :** Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, put

$$C_n = \sum_{k=0}^{n} a_k\, b_{n-k} \qquad (n = 0, 1, 2, \cdots\cdots\cdots).$$

We call the series $\sum_{n=0}^{\infty} C_n$ as the product of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

**10.1.26.1 Note :** Consider the product of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ i.e.

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n$$

$$=\left(a_0+a_1 z+a_2 z^2+\cdots\cdots\right)\left(b_0+b_1 z+b_2 z^2+\cdots\cdots\right)$$

$$=a_0 b_0+\left(a_0 b_1+a_1 b_0\right) z+\left(a_0 b_2+a_1 b_1+a_2 b_0\right) z^2+\cdots\cdots$$

$$=c_0+c_1 z+c_2 z^2+\cdots\cdots\cdots$$

The product defined in the definition can be obtained from this by taking $z=1$ here.

**10.1.26.2 Note :** Consider the series $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$.

Let $A_n=\sum_{k=0}^{n} a_k$, $B_n=\sum_{k=0}^{n} b_k$ $(n=0,1,2,\ldots\ldots)$. Let $\sum_{n=0}^{\infty} c_n$ be the product of $\sum_{k=0}^{\infty} a_n$ and $\sum_{k=0}^{\infty} b_n$. Suppose $\sum_{n=0}^{\infty} a_n=A$ and $\sum_{n=0}^{\infty} b_n=B$ i.e. $\lim_n A_n=A$ and $\lim_n B_n=B$. We do not have $C_n=\sum_{k=0}^{n} C_k \neq A_n B_n$. Now, the question is - Is $\sum_{n=0}^{\infty} C_n=AB$? Now, we show that the series $\sum_{n=0}^{\infty} C_n$ may diverge. Consider the following.

**10.1.27 Example :** Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\cdots\cdots$$

By Theorem 10.1.23 (i.e. Leibnitz's Theorem), this series is convergent. Consider the product of this series with itself and we obtain

$$\sum_{n=0}^{\infty} C_n=\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}\right)$$

$$=1-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}\sqrt{2}}+\frac{1}{\sqrt{3}}\right)$$

$$-\left(\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3}\sqrt{2}}+\frac{1}{\sqrt{2}\sqrt{3}}+\frac{1}{\sqrt{4}}\right)+\cdots\cdots\cdots$$

Here, $C_n=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{(n-k+1)(k+1)}}$

Since $(n-k+1)(k+1)=\left(\frac{n}{2}+1\right)^2-\left(\frac{n}{2}-k\right)^2\le\left(\frac{n}{2}+1\right)^2$

We have $|C_n|\ge\sum_{k=0}^{n}\frac{2}{n+2}=\frac{2(n+1)}{n+2}=2\left(1-\frac{1}{n+2}\right)$

So, $C_n\nrightarrow 0$ as $n\to\infty$. Hence, the series diverges.

**10.1.28. Theorem :** Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely,

(b) $\sum_{n=0}^{\infty} a_n=A$,

(c) $\sum_{n=0}^{\infty} b_n=B$

(d) $C_n=\sum_{k=0}^{n} a_k b_{n-k}\quad (n=0,1,2,\cdots\cdots\cdots)$

Then $\sum_{n=0}^{\infty} C_n=AB$

(briefly, the product of two convergent series is convergent whenever the convergence of at least one of these two series is absolute).

**Proof :** Put

$$A_n = \sum_{k=0}^{n} a_k,\ B_n = \sum_{k=0}^{n} b_k,\ C_n = \sum_{k=0}^{n} c_k,\ \beta_n = B_n - B$$

Now,

$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots\cdots + (a_0 b_n + a_1 b_{n-1} + \cdots\cdots + a_n b_0)$$

$$= a_0 B_n + a_1 B_{n-1} + \cdots\cdots + a_n B_0$$

$$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots\cdots + a_n (B + \beta_0)$$

$$= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots\cdots + a_n \beta_0$$

$$= A_n B + \gamma_n$$

where $\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots\cdots + a_n \beta_0$.

We have to show that $C_n \to AB$. Since $A_n \to A$, we have that $A_n B \to AB$. To prove the conclusion, it is enough if we prove that

$$\lim_{n\to\infty} \gamma_n = 0$$

Put $\alpha = \sum_{n=0}^{\infty} |a_n|$.

Let $t > 0$. Since $\sum_{n=0}^{\infty} b_n = B,$ we have

$$\lim_{n\to\infty} B_n = B \text{ i.e. } \lim_{n\to\infty} \beta_n = 0$$

So, there exists a positive integer $N$ such that

$$n \geq N \Rightarrow |\beta_n| < t.$$

Now, $n \geq N \Rightarrow |\gamma_n| \leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0|$

$$\leq |\beta_0 a_n + \cdots\cdots + \beta_N a_{n-N}| + |\beta_{N+1}|\,|a_{n-N-1}| + \cdots + |\beta_n|\,|a_0|$$

$$\leq |\beta_0 a_n + \cdots\cdots + \beta_n a_{n-N}| + t|a_{n-N-1}| + \cdots + t|a_0|$$

$$= |\beta_0 a_n + \cdots\cdots + \beta_N a_{n-N}| + t\left(|a_{n-N-1}| + \cdots + |a_0|\right)$$

$$\leq \left|\beta_0\, a_n + \cdots\cdots\cdots + \beta_N\, a_{n-N}\right| + t\alpha$$

Keeping $N$ fixed, and letting $n \to \alpha$, we have

$$\lim_{n\to\infty} \sup|\gamma_n| \leq t\alpha ,$$

since $a_k \to 0$ as $k \to \infty$. Since $t$ is arbitrary, we have

$$\lim_{n\to\infty} \sup|r_n| = 0$$

and hence $\lim_{n} \gamma_n = 0$

We now state the theorem (following) due to Abel without proof.

**10.1.29 Theorem (Abel) :** If the series $\sum a_n$, $\sum b_n$, $\sum C_n$ converge to $A, B, C$ respectively and $C_n = a_0\, b_n + \cdots\cdots + a_n\, b_0 \quad (n=0,1,2,\cdots\cdots\cdots)$ then $C = AB$.

## REARRANGEMENTS

**10.1.30 Definition :** Let $\{k_n\}$, $n=1,2,\cdots\cdots\cdots\cdots$ be a sequence of positive integers in which every positive integer appears exactly once. Put $a_n^1 = a_{k_n} \quad (n=1,2,\ldots\ldots\ldots)$. We say that $\sum_{n=1}^{\infty} a_n^1$ is a rearrangement of $\sum a_n$.

**10.1.30.1 Note :** Let $\sum a_n^1$ be rearrangement of $\sum a_n$. Let $\{s_n\}$, $\{s_n^1\}$ be partial sum sequences of $\sum a_n$ and $\sum a_n^1$ respectively. These two partial sum sequences are entirely different. Thus, we are led to the problem of determining - under what conditions all the rearrangements of a convergent series will converge and whether the sums are same.

**10.1.31 Example :** Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots\cdots\cdots\cdots\cdots$$

Clearly, this series is convergent by Leibnitz theorem and we know that this series is not absolutely convergent. Consider the rearrangement of this series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots\cdots\cdots$$

in which two positive terms are always followed by one negative term. Let $s=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$.

Then, $s<1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, since $\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}>0$ for $k\geq 1$. So,

$$s_3^1<s_6^1<s_9^1<\cdots\cdots\cdots$$

where $s_n^1$ is the partial sum of the rearrangement. So,

$$\lim_{n\to\infty}\sup s_n^1>s_3^1=\frac{5}{6}$$

Thus, this rearrangement - even if it converges to $t$ then $t\neq s$ ( infact $t\geq\frac{5}{6}>s$ )

**10.1.32 Riemann's Theorem :** Let $\sum a_n$ be a sequence of real numbers which converges but not absolutely. Suppose

$$-\infty\leq\alpha\leq\beta\leq+\infty$$

Then there exists a rearrangement $\sum a_n^1$ with partial sums $s_n^1$ such that

$$\lim_{n\to\infty}\inf s_n^1=\alpha, \lim_{n\to\infty}\sup s_n^1=\beta$$

**Proof :** Let $p_n=\frac{|a_n|+a_n}{2}$, $q_n=\frac{|a_n|-a_n}{2}$ $(n=1,2,\cdots\cdots\cdots)$.

Then $p_n-q_n=a_n$, $p_n+q_n=|a_n|$, $p_n\geq 0$, $q_n\geq 0$. The series $\sum p_n$, $\sum q_n$ - both must diverge.

Otherwise atleast one of these two must converge.

Suppose both $\sum p_n$ and $\sum q_n$ are convergent. So, $\sum|a_n|=\sum(p_n+q_n)=\sum p_n+\sum q_n$ is convergent, a Contradiction.

Suppose $\sum p_n$ diverges and $\sum q_n$ converges. Since

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n,$$

and the divergence of $\sum p_n$, we have that $\sum a_n$ diverges, a Contradiction. Similarly, even if $\sum p_n$ converges and $\sum q_n$ diverges we have a Contradiction.

Hence, both $\sum p_n$ and $\sum q_n$ must diverge.

Let $P_1, P_2, \ldots\ldots$ denote the non negative terms of $\sum a_n$ in the order in which they occur, and let $Q_1, Q_2, \ldots\ldots$ be the absolute value of the negative terms of $\sum a_n$, also in their original order. The series $\sum P_n$ and $\sum Q_n$ differ from $\sum p_n, \sum q_n$ only by zero terms and hence divergent.

Now, we construct sequences $\{m_n\}, \{k_n\}$ such that

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots Q_{k_1} + P_{m_1+1} + \cdots + p_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots$$

which is clearly a rearrangement of $\sum a_n$ satisfying the requirement.

Choose real valued sequences $\{\alpha_n\}, \{\beta_n\}$ such that

$$\alpha_n \to \alpha,\ \beta_n < \beta,\ \alpha_n < \beta_n,\ \beta_1 > 0.$$

Let $m_1, k_1$ be the smallest positive integers such that

$$P_1 + P_2 + \cdots + P_{m_1} > \beta_1,$$

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - Q_2 - \cdots - Q_{k_1} < \alpha_1;$$

Let $m_2, k_2$ be the smallest positive integers such that

$$P_1 + P_2 + \cdots P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$

$$P_1 + P_2 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2,$$

and continue this way. This is possible since $\sum P_n$ and $\sum Q_n$ diverge.

Let $x_n, y_n$ denote the partial sums of the series

$$P_1+P_2+\cdots\cdots+P_{m_1}-Q_1-Q_2\cdots\cdots-Q_{k_1}+P_{m_1+1}+\cdots\cdots+P_{m_2}-Q_{k_1+1}-\cdots\cdots-Q_{k_2}+\cdots\cdots\cdots$$

which is a rearrangement of $\sum a_n$ ending with the terms $P_{m_n}, -Q_{k_n}$ respectively. Then

$$|x_n-\beta_n|\le P_{m_n},\ |y_n-\alpha_n|\le Q_{k_n} \text{ ------------(1)}$$

(since $m_n$ and $k_n$ are least positive integers with their respective choices).

Since $\sum a_n$ converges, $a_n \to 0$ as $n \to \infty$. Hence $p_n \to 0$ and $Q_n \to 0$ as $n \to \infty$. Since $\beta_n \to \beta$ and $\alpha_n \to \alpha$ as $n \to \infty$, we have that $x_n \to \beta$ and $y_n \to \alpha$ (by (1)

It is clear that no number less than $\alpha$ or greater than $\beta$ can be a subsequential limit of the partial sum sequence $\{s_n^1\}$ of the series.

Clearly, $\{x_n\}$ and $\{y_n\}$ re subsequences of $\{s_n^1\}$ and hence

$$\liminf_{n\to\infty} s_n^1 = \alpha,\quad \limsup_{n\to\infty} s_n^1=\beta$$

**10.1.33 Theorem :** If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges and they all converge to the same sum.

**Proof :** Let $\sum a_n$ be a sequence of complex numbers converging absolutely. Let $\sum a_n^1$ be a rearrangement of $\sum a_n$. Let $s_n$ and $s_n'$ be the $n^{\text{th}}$ partial sums of the series $\sum a_n$ and $\sum a_n'$ respectively. Let $t>0$. Since $\sum a_n$ converges absolutely there exists a positive integer $N$ such that

$$m \ge n \ge N \Rightarrow \sum_{i=n}^{m} |a_i| < t$$

Since $\sum a_n^1$ is a rearrangement, we have that $a_n^1 = a_{k_n}$ $(n=1,2,\cdots\cdots)$. Choose $p$ such that the integers $1, 2, \ldots\ldots\ldots, N$ are all contained in $k_1, k_2, \ldots\ldots\ldots, k_p$. Then, if $n>p$, the numbers $a_1,\cdots\cdots\cdots,a_N$ will cancel in $s_n-s_n^1$ and hence $|s_n-s_n^1|\le t$. Since $\{s_n\}$ converges, we have that

$s_n^1$ also converges and converges to the same limit of $\{s_n\}$.

## 10.2 SHORT ANSWER QUESTIONS

**10.2.1 :** For what values of $p$, the series $\sum \frac{1}{n^p}$ converges ?

**10.2.2 :** Does $\lim_n a_n = 0$ imply the convergance of $\sum a_n$ ?

**10.2.3 :** State Root test

**10.2.4 :** State Ratio test

**10.2.5 :** State Comparison test.

**10.2.6 :** When $0 \le x < 1$, $\sum_{n=1}^{\infty} x^n = ?$

**10.2.7 :** Define the number $e$.

**10.2.8 :** Prove or disprove the following statement : Every series converges absolutely.

## 10.3 MODEL EXAMINATION QUESTIONS

1. State and Prove Cauchy criterion for series.

2. If $\sum a_n$ converges, prove that $\lim_n a_n = 0$. What can you say about the Converse ? Justify.

3. Define the Convergence of series. If $0 \le x < 1$, prove that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

4. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

5. Define the number $e$ and prove that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

6. State and Prove Root test.

7. State and Prove Ratio test.

8. State Root test and Ratio test. Prove that the convergence of a series by Ratio test implies the convergence of the same by Ratio test. Give an example of a series which is convergent by Root test and the Ratio test is in conclusive.

9. State and Prove Leibnitz's Theorem.

10. Define - rearrangmenet of a series. If $\sum a_n$ converges absolutely, Prove that all rearrangements of $\sum a_n$ converge and converge to the sum $\sum a_n$.

11. Let $\sum a_n$ be a series of real numbers which converges but not absolutely. Let $\alpha, \beta$ be numbers such that

$$-\infty \leq \alpha < \beta \leq \infty$$

Prove that there exists a rearrangement $\sum a_n^1$ with partial sums $s_n^1$ such that

$$\lim_{n\to\infty} \inf s_n^1 = \alpha, \quad \lim_{n\to\infty} \sup s_n^1 = \beta$$

## 10.4 EXERCISES

**10.4.1 :** Test the convergence of the series $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

(b $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

(c) $a_n = \left(\sqrt[n]{n} - 1\right)^n$

**10.4.2 :** If $\sum a_n$ is a series of nonnegative terms such that $\sum a_n$ Converges, prove that the series $\sum \frac{\sqrt{a_n}}{n}$ Converges.

**10.4.3 :** If $\sum a_n$ converges and if the sequence $\{b_n\}$ is monotonic and bounded prove that $\sum a_n b_n$ converges.

**10.4.4 :** Suppose $\sum a_n$ diverges, $a_n > 0$ and put $s_n = \sum_{i=1}^{n} a_i$. Prove the following :

(a) (i) $\sum \frac{a_n}{1+a_n}$ diverges.

(ii) $\sum \frac{a_n}{s_n}$ diverges

(iii) $\sum \frac{a_n}{s_n^2}$ converges

(b) What can you say about

$$\sum \frac{a_n}{1+n\,a_n} \text{ and } \sum \frac{a_n}{1+n^2\,a_n} ?$$

**10.4.5 :** Suppose $\sum a_n$ converges, $a_n > 0$ and put

$$r_n = \sum_{m=n}^{\infty} a_m$$

Prove the following :

(a) $\sum \frac{a_n}{r_n}$ diverges

(b) $\sum \frac{a_n}{\sqrt{r_n}}$ Converges.

Prove that the Cauchy product of two absolutely convergent series converges absolutely.

## 10.5 ANSWERS TO SHORT ANSWER QUESTIONS

**10.2.1 :** $p > 1$

**10.2.2 :** No. Clearly, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and the series $\sum \frac{1}{n}$ diverges.

**10.2.3 :** See statement of Theorem.

**10.2.4 :** See statement of Theorem

**10.2.5 :** See statement of Theorem

**10.2.6 :** $x/(1-x)$

**10.2.7 :** See definition

**10.2.8 :** No, Consider the $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. We know that this series converges (by Leibnitz theorem) and this series does not converge absolutely (since $\sum\frac{1}{n}$ is divergent).

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third edition, McGraw - Hill International Editions : Walter Rudin

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Lesson : 11

# LIMITS OF FUNCTIONS AND CONTINUOUS FUNCTIONS ON METRIC SPACES

## 11.0 INTRODUCTION

In this lesson the notion of limit of a function from one metric space into another is introduced. If $X$ and $Y$ are metric spaces and $E \subseteq X$ and $f$ maps $E$ into $Y$ and $p$ is a limit point of $E$, then $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E$ such that $p_n \neq p$ for all $n$ and $\lim_{n \to \infty} p_n = p$ is proved. Next the continuity of a function from a metric space into a metric space is defined. It has also been proved that if $X$ and $Y$ are metric spaces, $E \subseteq X$ and $f$ maps $E$ into $Y$ and if $p \in E$ is a limit point of $E$, then $f$ is continuous at $p$ if and only if $\lim_{x \to p} f(x) = f(p)$. Further it is proved that a mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ if and only if $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.

## 11.1 LIMITS OF FUNCTIONS :

**11.1.1 Definition :** Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces; suppose $E \subseteq X$; $f$ maps $E$ into $Y$ and $p$ is a limit point of $E$. If there is a point $q \in Y$ with the property that for every $\in > 0$, there exists a $\delta > 0$ such that $d_2(f(x), q) < \in$ for all points $x \in E$ for which $0 < d_1(x, p) < \delta$, then we write $f(x) \to q$ as $x \to p$, or $\lim_{x \to p} f(x) = q$.

**11.1.2 Note :** Suppose $X = Y = \mathbb{R}$ and $d_1(x, y) = d_2(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ and also suppose $E \subseteq \mathbb{R}$, $p$ is a limit point of $E$. Then $f : E \to \mathbb{R}$ is said to have a limit as $x \to p$, if there is a $q \in \mathbb{R}$ satisfying the condition : for every $\in > 0$, there is a $\delta > 0$ such that $|f(x) - q| < \in$ for all $x \in E$ with $0 < |x - p| < \delta$.

**11.1.3 Example :** Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x)=\begin{cases} x+2 & \text{if } x\neq 2 \\ 0 & \text{if } x=2. \end{cases} \quad \text{Then } \lim_{x\to 2} f(x)=4.$$

Let $\in>0$. Take $\delta=\in$. Then for any $x$ with $0<|x-2|<\delta$

$$|f(x)-4|=|x+2-4|=|x-2|<\delta=\in.$$

$$\therefore \lim_{x\to 2} f(x)=4$$

**11.1.4 Theorem :** Let $(X,d_1)$ and $(Y,d_2)$ be metric spaces and $E\subseteq X$ and $f$ maps $E$ into $Y$ and $p$ be a limit point of $E$. Then $\lim_{x\to p} f(x)=q$ if and only if $\lim_{x\to\infty} f(p_n)=q$ for every sequence $\{p_n\}$ in $E$ such that $p_n\neq p$ for all $n$ and $\lim_{x\to\infty} p_n=p$.

**Proof :** Given that $X$, $Y$ are metric spaces and $E\subseteq X$ and $f$ maps $E$ into $Y$ and $p$ is a limit point of $E$.

Suppose $\lim_{x\to p} f(x)=q$

Let $\{p_n\}$ be any sequence in $E$ such that $p_n\neq p$ and $\lim_{n\to\infty} p_n=p$

Let $\in>0$. Since $\lim_{x\to p} f(x)=q$, there exists a $\delta>0$ such that $d_2(f(x),q)<\in$ if $x\in E$ and $0<d_1(x,p)<\delta$ --------------- (1)

Since $p_n\neq p$ and $\lim_{n\to\infty} p_n=p$, there exists a positive integer $N$ such that $0<d_1(p_n, p)<\delta$ for all $n\geq N$.

Then, by (1), $d_2(f(p_n), q)<\in$ for all $n\geq N$.

$$\therefore \lim_{n\to\infty} f(n)=q.$$

Conversely suppose that $\lim_{x\to p} f(x)\neq q$.

Now we will show that there exists a sequence $\{p_n\}$ of points in $E$ such that

$p_n \neq p,\ \lim\limits_{n\to\infty} p_n = p$ does not imply $\lim\limits_{n\to\infty} f(p_n) = q$.

Since $\lim\limits_{x\to p} f(x) \neq q$, there exists $\in > 0$ such that for every $\delta > 0$, there exists a point $x \in E$ (depending on $\delta$) with $d_2(f(x), q) \geq \in$ but $0 < d_1(x, p) < \delta$. This implies for each $\delta_n = \frac{1}{n}$ $(n = 1, 2, \ldots\ldots\ldots)$, there exists a point $p_n \in E$ such that $d_2(f(p_n), q) \geq \in$ but $0 < d_1(p_n, p) < \delta_n$. Consequently $\lim\limits_{n\to\infty} f(p_n) \neq q$.

Now we will show that $p_n \neq p$ for all $n$ and $\lim\limits_{n\to\infty} p_n = p$.

Since $0 < d_1(p_n, p) < \frac{1}{n}$, we have $p_n \neq p$ for $n = 1, 2, \ldots\ldots\ldots$

Let $\in > 0$. Choose a positive integer $N$ such that $\frac{1}{N} < \in$. Now for all $n \geq N$, consider $d_1(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \in$.

This implies $d_1(p_n, p) < \in$ for all $n \geq N$ and hence $\lim\limits_{n\to\infty} p_n = p$.

Thus there exists a sequence $\{p_n\}$ of points in $E$ such that $p_n \neq p,\ \lim\limits_{n\to\infty} p_n = p$ but $\lim\limits_{n\to\infty} f(p_n) \neq q$.

**11.1.5 Corollary :** Suppose $f$ is mapping of a metric space $(X, d_1)$ into a metric space $(Y, d_2)$. If $\lim\limits_{x\to p} f(x)$ exists in $Y$, then it is unique.

**Proof :** Suppose $\lim\limits_{x\to p} f(x)$ exists in $Y$.

Suppose $\lim\limits_{x\to p} f(x) = q_1$ and $\lim\limits_{x\to p} f(x) = q_2$, where $q_1, q_2 \in Y$.

**Claim :** $q_1 = q_2$.

Let $\{p_n\}$ be any sequence of points in $X$ such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$. Then by the above theorem, $\lim_{n\to\infty} f(p_n) = q_1$ and $\lim_{n\to\infty} f(p_n) = q_2$. So $\{f(p_n)\}$ is 0, sequence of points in $Y$ such that $\lim_{n\to\infty} f(p_n) = q_1$ and $\lim_{n\to\infty} f(p_n) = q_2$. Since limit of a sequence is unique, we have $q_1 = q_2$.

**11.1.6 Definition :** Let $X$ be a metric space and let $f$ and $g$ be complex valued functions defined on $X$. Now we define $f \pm g,\ f\,g,\ f/g$ as follows :

Let $x \in X$. Define $(f \pm g)(x) = f(x) \pm g(x)$

$$fg(x) = f(x)\,g(x)$$

and $$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \text{ if } g(x) \neq 0.$$

**11.1.7 Definition :** Let $f$ and $g$ be functions defined from metric space $X$ into $\mathbb{R}^k$. Then we define

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

and $(\lambda f)(x) = \lambda f(x)$ for any real $\lambda$ and for all $x \in X$. If $f$ and $g$ are real valued functions and if $f(x) \geq g(x)$ for all $x \in X$, we write $f \geq g$.

**11.1.8 Theorem :** Suppose $(X,d)$ is a metric space and $f, g$ are complex valued functions defined on $E \subseteq X$. Suppose $p$ is a limit point of $E$. If $\lim_{x\to p} f(x) = A$ and $\lim_{x\to p} g(x) = B$, then

$\lim_{x\to p} g(x) = B$, then

(i) $\lim_{x\to p} (f+g)(x) = A + B$

(ii) $\lim_{x\to p} (f\,g)(x) = A\,B$

(iii) $\lim\limits_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ provided $B \neq 0$

**Proof :** Since $\lim\limits_{x \to p} f(x) = A$, by Theorem 11.1.4, we have $\lim\limits_{n \to \infty} f(p_n) = A$ for any sequence $\{p_n\}$ of points in $E$ with $\lim\limits_{n \to \infty} p_n = p$ and $p_n \neq p$ for all $n$.

Since $\lim\limits_{x \to p} g(x) = B$, by Theorem 11.1.4, we have

$\lim\limits_{n \to \infty} g(p_n) = B$ for any sequence $\{p_n\}$ of points in $E$ with $\lim\limits_{n \to \infty} p_n = p$ and $p_n \neq p$ for all $n$.

(i) Suppose $\{p_n\}$ is a sequence of points in $E$ such that $\lim\limits_{n \to \infty} p_n = p$ and $p_n \neq p$ for all $n$.

Consider $\lim\limits_{n \to \infty} (f+g)(p_n) = \lim\limits_{n \to \infty} (f(p_n) + g(p_n))$

$$= \lim_{n \to \infty} f(p_n) + \lim_{n \to \infty} g(p_n) = A + B.$$

Therefore $\lim\limits_{x \to p} (f+g)(x) = A + B$

(ii) Suppose $\{p_n\}$ is any sequence of points in $E$ such that $\lim\limits_{n \to \infty} p_n = p$ and $p_n \neq p$ for all $n$.

Consider $\lim\limits_{n \to \infty} (f\,g)(p_n) = \lim\limits_{n \to \infty} (f(p_n)\, g(p_n))$

$$= \lim_{n \to \infty} f(p_n) \cdot \lim_{n \to \infty} g(p_n) = AB.$$

Therefore $\lim\limits_{x \to p} (f\,g)(x) = AB$

(iii) Suppose $\{p_n\}$ is any sequence of points in $E$ such that $\lim\limits_{n \to \infty} p_n = p$ and $p_n \neq p$ for all $n$.

Consider $\lim\limits_{n\to\infty}\left(\frac{f}{g}\right)(p_n) = \lim\limits_{n\to\infty}\frac{f(p_n)}{g(p_n)} = \frac{\lim\limits_{n\to\infty} f(p_n)}{\lim\limits_{n\to\infty} g(p_n)}$

$= \frac{A}{B}$ where $B \neq 0$.

Therefore $\lim\limits_{x\to\infty}\left(\frac{f}{g}\right)(x) = \frac{A}{B}$.

## 11.2 CONTINUOUS FUNCTIONS

**11.2.1 Definition :** Suppose $(X, d_1)$ and $(Y, d_2)$ are metric spaces, $E \subseteq X$, $p \in E$ and $f$ maps $E$ into $Y$. Then $f$ is said to be continuous at $p$ if for every $\in > 0$. There exists a $\delta > 0$ such that $d_2(f(x), f(p)) < \in$ for all points $x \in E$ for which $d_1(x, p) < \delta$. If $f$ is continous at every point of $E$, then $f$ is said to be continuous on $E$.

**11.2.2 Definition :** Let $(X, d)$ be a metric space and $E \subseteq X$. A point $p \in E$ is said to be an isolated point of $E$ if there is a neighbourhood $N_\delta(p)$ of $p$ such that $N_\delta(p)$ has just one point $p$ of the set $E$.

That is $N_\delta(p) = \{x \in E / d(x, p) < \delta\} = \{p\}$ and

$$\{x \in E / 0 < d(x, p) < \delta\} = \phi$$

Therefore if $p$ is an isolated point of $E$, then the condition, in definition 11.2 1, $d_2(f(x), f(p)) < \in$ for all $x \in E$ with $d_1(x, p) < \delta$ holds obviously. Hence if $p \in E$ is an isolated point of $E$, then $f$ is continuous at $p$.

**11.2.3 Example :** Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} x+2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$$

Then $\lim\limits_{x\to 2} f(x) = 4$. But $f(2) = 0$. So $f$ is not continuous at $x = 2$.

**11.2.3 Example :** Define $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=x+2$ for all $x\in\mathbb{R}$. Then $\lim_{x\to 2} f(x)=4$ and is equal to $f(2)$. So $f$ is continuous at $x=2$.

**11.2.4 Theorem :** Let $(X,d_1)$ and $(Y,d_2)$ be metric spaces, $E\subseteq X$, and $f$ maps $E$ into $Y$. If $p\in E$ is a limit point of $E$, then $f$ is continuous at $p$ if and only if $\lim_{x\to p} f(x)=f(p)$.

**Proof :** Consider $f$ is continuous at $p$ if and only if for each $\in>0$, there exists a $\delta>0$ such that $d_2(f(x), f(p))<\in$ for all points $x\in E$ for which $d_1(x,p)<\delta$ if and only if $\lim_{x\to p} f(x)=f(p)$ ($\because$ $p$ is a limit point of $E$).

**11.2.5 Theorem :** Suppose $(X,d_1)$, $(Y,d_2)$ and $(Z,d_3)$ are metric spaces, $E\subseteq X$, $f$ maps $E$ into $Y$, $g$ maps the range of $f$, $f(E)$, into $Z$ and $h$ is the mapping of $E$ into $Z$ defined by $h(x)=g(f(x))$ for all $x\in E$. If $f$ is continuous at a point $p\in E$ and if $g$ is continuous at the point $f(p)$, then $h$ is continuous at $p$.

**Proof :** Suppose $f$ is continuous at $p\in E$ and $g$ is continuous at the point $f(p)$.

Let $\in>0$. Since $g$ is continuous at $f(p)$, there exists an $\eta>0$ such that $d_3(g(y),g(f(p)))<\in$ whenever $d_2(y,f(p))<\eta$ and $y\in f(E)$ --------- (1)

Since $f$ is continuous at $p$, there exists a $\delta>0$ such that $d_2(f(x), f(p))<\eta$ whenever $d_1(x,p)<\delta$ and $x\in E$ --------- (2).

Suppose $x\in E$ such that $d_1(x,p)<\delta$. Then consider

$d_3(h(x), h(p)) = d_3(g(f(x)), g(f(p)))<\in$ (from (1) and (2)).

Thus for $\in>0$, there exists a $\delta>0$ such that

$d_3(h(x), h(p))<\in$ whenever $d_1(x,p)<\delta$.

Therefore $h$ is continuous at $p$.

In the above theorem, $h$ is called the composition of $f$ and $g$ and we write $h = g \circ f$.

**11.2.6 Theorem :** Suppose $(X,d)$ is a metric space and $f$, $g$ are complex valued functions defined on $X$. If $f$ and $g$ are both continuous at $p \in X$, then $f+g$, $fg$ and $f/g$ (if $g(p) \neq 0$) are continuous at $p \in X$.

**Proof :** Suppose $(X,d)$ is a metric space and $f,g$ are complex valued functions defined on $X$ is continuous at $p$. So $f+g$, $fg$ and $\frac{f}{g}$ are continuous at $p$.

**Case (ii) :** Suppose $p$ is a limit point of $X$.

By theorem 11.2.4, $f$ is continuous at $p$ if and only if $\lim_{x \to p} f(x) = f(p)$ and $g$ is continuous at $p$ if and only if

$\lim_{x \to p} g(x) = g(p)$. Then by theorem 11.1.8,

$$\lim_{x \to p} (f+g)(x) = f(p) + g(p) = (f+g)(p)$$

$\therefore f+g$ is continuous at $p$. (By theorem 11.2.4)

Consider $\lim_{x \to p} (fg)(x) = f(p)g(p) = (fg)(p)$ (By theorem 11.1.8)

By theorem 11.2.4, $fg$ is continuous at $p$.

Suppose $g(p) \neq 0$.

Consider $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{f(p)}{g(p)} = \left(\frac{f}{g}\right)(p)$ (By Theorem 11.1.8)

By theorem 11.2.4, $\frac{f}{g}$ is continuous at $p$.

**11.2.7 Theorem (a) :** Let $f_1, f_2, \cdots, f_k$ be real functions on a metric space $X$, and let $f$ be the mapping of $X$ into $\mathbb{R}^k$ defined by $f(x) = (f_1(x), f_2(x), \cdots, f_k(x))$ $(x \in X)$;

then $f$ is continuous if and only if each of the functions $f_1, f_2, \cdots, f_k$ is continuous.

**(b) :** If $f$ and $g$ are continuous mappings of $X$ into $\mathbb{R}^k$, then $f+g$ and $f \cdot g$ are continuous on $X$.

**Proof :** Given that $f$ is mapping of a metric space $(X,d)$ into $\mathbb{R}^k$ defined by $f(x)=(f_1(x), f_2(x), \cdots, f_k(x))$ where $f_1, f_2, \cdots\cdots, f_k$ are real valued functions defined on $X$.

**(a) :** Assume $f$ is continuous on $X$.

Let $x \in X$ and let $\in > 0$. Since $f$ is continuous at $x$, there exists a $\delta > 0$ such that $|f(x)-f(y)| < \in$ whenever $d(x,y) < \delta$.

$$\Rightarrow \left( \sum_{i=1}^{k} |f_i(x) - f_i(y)|^2 \right)^{1/2} < \in \text{ for } d(x,y) < \delta.$$

$$\Rightarrow |f_i(x) - f_i(y)| < \in \text{ for } d(x,y) < \delta \text{ and for } 1 \le i \le K.$$

$$\Rightarrow f_i \text{ is continuous at } x \text{ for } 1 \le i \le n.$$

Since $x \in X$ is orbitrary, $f_i$ is continuous for $1 \le i \le k$.

Now, we will show that $f$ is continuous on $X$.

Let $x \in X$ and let $\in > 0$. Since each $f_i$ is continuous at $x$, there exists a $\delta_i > 0$ such that $|f_i(x) - f_i(y)| < \frac{\in}{\sqrt{k}}$ whenever $d(x,y) < \delta_i$ for $1 \le i \le k$.

Take $\delta = \min\{\delta_1, \delta_2, \cdots, \delta_k\}$.

Suppose $d(x,y) < \delta$. Then $d(x,y) < \delta_i$ for $1 \le i \le k$.

$$\Rightarrow |f_i(x) - f_i(y)| < \frac{\in}{\sqrt{k}} \text{ for } 1 \le i \le k.$$

Consider $|f(x)-f(y)|^2 = \left( \sum_{i=1}^{k} |f_i(x) - f_i(y)|^2 \right) < k \cdot \frac{\in^2}{k} = \in^2$

$\Rightarrow |f(x)-f(y)|<\in$.

Therefore $f$ is continuous at $x$.

Since $x\in X$ is arbitrary, $f$ is continuous on $X$.

**(b) :** Suppose $f$ and $g$ are continuous mappings of $X$ into $\mathbb{R}^k$, where $f$ and $g$ are defined by $f(x)=(f_1(x), f_2(x), \cdots\cdots, f_k(x))$ and

$g(x)=(g_1(x), g_2(x),\cdots\cdots, g_k(x))$ with $f_1, f_2,\cdots, f_k;\ g_1, g_2,\cdots, g_k$ are real valued functions defined on $X$.

Since $f$ and $g$ are continuous on $X$, by (a), each $f_i$ is continuous on $X$ and each $g_i$ is continuous on $X$. Then by Theorem 11.2.6, $f_i+g_i$ and $f_i\, g_i$ are continuous on $X$ for $1\le i\le k$. Since $(f+g)(x)=((f_1+g_1)(x), (f_2+g_2)(x),\cdots,(f_k+g_k)(x))$ for all $x\in X$, by (a), $f+g$ is continuous on $X$. Since $f_i\, g_i$ is continuous on $X$ for $1\le i\le k$, we have $\sum_{i=1}^{k} f_i\, g_i$ is continuous on $X$ and hence $f\cdot g$ is continuous on $X$.

**11.2.8 Example :** Every polynomial with complex coefficients is continuous at every point of $\mathbb{C}$.

For, let $p(x)=\alpha_0+\alpha_1 x+\cdots\cdots+\alpha_n x^n$ where $\alpha_0, \alpha_1,\cdots\cdots\cdots,\alpha_n$ are complex numbers.

Consider $p:\mathbb{C}\to\mathbb{C}$ as a function.

Define $I:\mathbb{C}\to\mathbb{C}$ as $I(x)=x$ for all $x\in\mathbb{C}$. Then $I$ is continuous at every point of $\mathbb{C}$ for if $\in>0$ is given, taking $\delta=\in$, for all $x\in\mathbb{C}$ with $0<|x-a|<\delta$ we have

$|I(x)-I(a)| = |x-a|<\delta = \in$. $\Rightarrow I$ is continuous.

$\Rightarrow I^2(x)=I(x)\,I(x)=x^2$ is continuous.

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$I^n(x)=x^n$ is continuous.

It is easy to verify that every constant function is continuous.

Therefore $f_0(x)=\alpha_0$

$$f_1(x)=\alpha_1 x = \alpha_1 I(x)$$

$$f_2(x)=\alpha_2 x^2 = \alpha_2 I^2(x)$$

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$f_n(x)=\alpha_n x^n = \alpha_n I^n(x)$ are all continuous on $\mathbb{C}$.

Hence $f_0(x)+f_1(x)+\cdots\cdots\cdots+f_n(x)=p(x)$ is continuous on $\mathbb{C}$.

Thus every polynomial is a continous function.

**11.2.9 Definition :** Suppose $f:A\to B$ is a mapping where $A$ and $B$ are any two sets. For any $T\subset A$, $f(T)=\{f(x)/x\in T\}$ is called the image of $T$ under $f$. For any $V\subset B$, the set $\{x\in A/f(x)\in V\}$ is called the inverse image of $V$ under $f$ and is denoted by $f^{-1}(V)$. That is $f^{-1}(V)=\{x\in A/f(x)\in V\}$.

**11.2.10 Theorem :** Suppose $f:A\to B$ is a mapping. Then for every set $V\subset B$,

(i) $f^{-1}(V^c)=\left[f^{-1}(V)\right]^c$

(ii) $f\left(f^{-1}(V)\right)\subseteq V$

**Proof :** (i) Consider $x\in f^{-1}(V^c)\Leftrightarrow x\in A$ and $f(x)\in V^c$

$\Leftrightarrow x\in A$ and $f(x)\notin V\Leftrightarrow x\notin f^{-1}(V)$

$$\Leftrightarrow x\in\left(f^{-1}(V)\right)^c$$

$$\therefore f^{-1}(V^c)=\left(f^{-1}(V)\right)^c$$

(ii) Suppose $t\in f\left(f^{-1}(V)\right)=\left\{f(x)/x\in f^{-1}(V)\right\}$.

$\Rightarrow t=f(x_0)$ for some $x_0 \in f^{-1}(V)$

$\Rightarrow t=f(x_0)$ for some $x_0 \in A$ with $f(x_0)\in V$

$\Rightarrow t \in V$.

$\therefore f\left(f^{-1}(V)\right) \subseteq V$.

**11.2.11 Theorem :** A mapping $f$ of a metric space $(X,d_1)$ into a metric space $(Y,d_2)$ is continous on $X$ if and only if $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.

**Proof :** Let $f:X\to Y$ be a mapping.

Suppose $f$ is continuous on $X$.

Let $V$ be an open set in $Y$.

Now we will show that every point in $f^{-1}(V)$ is an interior point of it.

Let $x \in f^{-1}(V)$. Then $f(x)\in V$. Since $V$ is an open set in $Y$, there exists $\in >0$ such that $N_{\in}(f(x))\subseteq V$. Since $f$ is continuous on $X$, $f$ is continuous at $x$. Then there exists a $\delta>0$ such that $d_2(f(z), f(x))<\in$ whenever $d_1(z,x)<\delta$. This implies $f(z)\in N_{\in}(f(x))$ whenever $z\in N_\delta(x)$. That is, $f(z)\in V$ whenever $z\in N_\delta(x)$ That is $z\in f^{-1}(V)$ whenever $z\in N_\delta(x)$ and hence $N_\delta(x)\subseteq f^{-1}(V)$

Thus there exists $\delta>0$ such that $x\in N_\delta(x)\subseteq f^{-1}(V)$.

$\therefore x$ is an interior point of $f^{-1}(V)$. Hence $f^{-1}(V)$ is open in $X$.

Thus $f^{-1}(V)$ is open in $X$ whenever $V$ is open in $Y$.

Conversely suppose that $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.

Now we will show that $f$ is continuous at every point of $X$.

Let $p\in X$ and let $\in>0$. Now $N_{\in}(f(p))$ is an open set in $Y$. By our supposition,

$f^{-1}\left(N_{\in}\left(f(p)\right)\right)$ is an open set in $X$ and $p \in f^{-1}\left(N_{\in}\left(f(p)\right)\right)$. Then there exists $\delta > 0$ such that $N_{\delta}(p) \subseteq f^{-1}\left(N_{\in}\left(f(p)\right)\right)$. This implies $f\left(N_{\delta}(p)\right) \subseteq N_{\in}\left(f(p)\right)$. That is, if $d_1(x,p) < \delta$, then $d_2\left(f(x), f(p)\right) < \in$. This shows that $f$ is continuous at $p$. Since $p \in X$ is arbitrary, $f$ is continuous on $X$.

Thus $f$ is continuous on $X$ if and only if $f^{-1}(V)$ is open in $X$ whenever $V$ is open in $X$.

**11.2.12 Corollary :** A mapping of a metric space $X$ into a metric space $Y$ is continuous if and only if $f^{-1}(V)$ is closed in $X$ for every closed set $V$ in $Y$.

**Proof :** Let $f : X \to Y$ be a function. Let $V$ be any closed set in $Y$. Consider $f$ is continuous on $X$ if and only if $f^{-1}\left(V^c\right)$ is open in $X$ (by Theorem 11.2.11) if and only if $\left(f^{-1}(V)\right)^C$ is open in $X$ $\left(\because f^{-1}\left(V^c\right) = \left(f^{-1}(V)\right)^C\right)$ if and only if $f^{-1}(V)$ is closed in $X$

Thus $f$ is continuous on $X$ if and only if $f^{-1}(V)$ is closed in $X$ for every closed set $V$ in $Y$.

**11.2.13 Problem :** If $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$, prove that $f\left(\overline{E}\right) \subseteq \overline{f(E)}$ for any subset $E$ of $X$.

**Solution :** Suppose $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$ and $E \subseteq X$. Now $\overline{f(E)}$ is a closed subset of $Y$ containing $f(E)$. Since $f$ is continuous on $X$, by corollary 11.2.12, $f^{-1}\left(\overline{f(E)}\right)$ is a closed set in $X$ and $E \subseteq f^{-1}\left(\overline{f(E)}\right)$. Since $\overline{E}$ is the smallest closed set containing $E$, we have $\overline{E} \subseteq f^{-1}\left(\overline{f(E)}\right)$. This implies $f\left(\overline{E}\right) \subseteq \overline{f(E)}$. Thus for any sub set $E$ of $X$, $f\left(\overline{E}\right) \subseteq \overline{f(E)}$.

**11.2.14 Problem :** Let $f$ be a continuous real function on a metric space $X$. Let $Z(f)$ (the zero set of $f$) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

**Solution :** Given that $f$ is a continuous real function on a metric space $X$ and $Z(f) = \{p \in X / f(p) = 0\}$.

**Claim :** $Z(f)$ is a closed set.

Let $y$ be a limit point of $Z(f)$ in $X$. Then by a known theorem, there exists a sequence $\{x_n\}$ of points in $Z(f)$ such that $x_n \to y$. Since $f$ is continuous, by Theorem 11.1.4, and Theorem 11.2.4, we have $f(x_n)$ converges to $f(y)$. This implies $f(y)=\lim_n f(x_n)=0$ ($\because x_n \in Z(f)$ for all $n$) and hence $y \in Z(f)$. This shows that $Z(f)$ is a closed set in $X$.

**11.2.15 Problem :** Let $f$ and $g$ be continuous mappings of a metric space $X$ into a metric space $Y$ and let $E$ be a dense subset of $X$. Prove that $f(E)$ is dense in $f(X)$. If $g(p)=f(p)$ for all $p \in E$, prove that $g(p)=f(p)$ for all $p \in X$ (In other words, a continuous mapping is determined by its values on a dense subset of its domain).

**Solution :** Given that $f$ and $g$ are continuous mappings of a metric space $X$ into a metric space $Y$ and $E$ is a dense subset of $X$.

**Claim :** $f(E)$ is dense in $f(X)$. That is, $\overline{f(E)}=f(X)$. Clearly $\overline{f(E)}\subseteq f(X)$.

Let $y \in f(X)$. If $y \in f(E)$, then $y \in \overline{f(E)}$.

Suppose $y \notin f(E)$. In this case we will show that $y$ is a limit point of $f(E)$.

Since $y \in f(X)$, $y=f(x)$ for some $x \in X$. Then $x \notin E$.

Since $E$ is dense in $X$, $x$ is a limit point of $E$. Then by a known result, there exists a sequence $\{x_n\}$ of points in $E$ such that $\{x_n\}$ converges to $x$. Since $f$ is continuous and $\{x_n\}$ converges $x$, by a known result, $\{f(x_n)\}$ converges to $f(x)$. Now $\{f(x_n)\}$ is a sequence of points in $f(E)$ such that $\{f(x_n)\}$ converges to $y$. This implies $y \in \overline{f(E)}$. This shows that $f(X)\subseteq \overline{f(E)}$ and hence $\overline{f(E)} = f(X)$.

Suppose $f(p)=g(p)$ for all $p \in E$.

Now we will show that $f(x)=g(x)$ for all $x \in X$.

Let $x \in X$. Since $E$ is dense in $X$, there exists a sequence $\{x_n\}$ of points in $E$ such that

$\{x_n\}$ converges to $x$. Since $f$ and $g$ are continuous on $X$, we have $\{f(x_n)\}$ converges to $f(x)$ and $\{g(x_n)\}$ converges to $g(x)$. Consider $f(x) = \lim_n f(x_n) = \lim_n g(x_n) = g(x)$

$(\because x_n \in E$ for all $n$ and $f(x_n) = g(x_n))$

$\therefore f(x) = g(x)$ for all $x \in X$.

## 11.3 SELF ASSESSMENT QUESTIONS

**11.3.1 :** When do you say that a function $f$ from a metric space into a metric space is continuous?

**11.3.2 :** Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 2$ for all $x \in \mathbb{R}$ is continuous at $x = 2$.

**11.3.3 :** Let $f$ be a continuous real function on a metric space $X$. Let $Z(f)$ be the set of al $p \in X$ at which $f(p) = 0$. Show that $Z(f)$ is closed.

## 11.4 MODEL EXAMINATION QUESTIONS

**11.4.1 :** If $(X, d_1)$ and $(Y, d_2)$ are metric spaces and $E \subseteq X$ and if $f$ maps $E$ into $Y$ and $p$ is a limit point of $E$, then show that $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E$ such that $p_n \neq p$ for all $n$ and $\lim_{n \to \infty} p_n = p$.

**11.4.2 :** Suppose $X, Y$ and $Z$ are metric spaces and $f$ maps $X$ into $Y$ and $g$ maps $Y$ into $Z$ and $h$ is the mapping of $X$ into $Z$ defined by $h(x) = g(f(x))$ for all $x \in X$. If $f$ is continuous on $X$ and $g$ is continuous on $Y$, then show that $h$ is continuous from $X$ into $Z$.

**11.4.3 :** Show that a mapping $f$ of a metric space $X$ into a metric space $Y$ is continuous on $X$ if and only if $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$

**11.4.4 :** Let $f$ and $g$ be continuous mappings of a metric space $X$ into a metric space $Y$ and let $E$ be a dense subset of $X$. Prove that $f(E)$ is dense in $f(X)$. If $g(p)=f(p)$ for all $p\in E,$ then prove that $g(p)=f(p)$ for all $p\in X$.

## 11.5 EXERCISES

**11.5.1 :** Suppose $f$ is a real function defined on $\mathbb{R}$ which satisfies $\lim\limits_{h\to 0}\left[f(x+h)-f(x-h)\right]=0$ for every $x\in\mathbb{R}$. Does this imply that $f$ is continuous ?

**11.5.2 :** If $f$ is a real continuous function defined on a closed set $E\subset\mathbb{R}$. Prove that there exist continuous real functions $g$ on $\mathbb{R}$ such that $g(x)=f(x)$ for all $x\in E$.

## 11.6 ANSWERS TO SHORT ANSWER QUESTIONS

For 11.3.1 see definition 11.2.1

For 11.3.2, see example 11.2.3

For 11.3.3, see problem 11.2.14

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin

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Lesson - 12

# CONTINUITY, COMPACTNESS AND CONNECTEDNESS

## 12.0 INTRODUCTION

In this lesson the behaviour of continuous functions when they are defined on compact sets or connected sets is discussed. It is proved that if $f$ is a continuous mapping of a compact metric space $X$ into a metric space $Y$, then $f(X)$ is compact. It has also been proved that a continuous 1 - 1 mapping of a compact metric space onto a metric space is a homemorphism. Further the uniform continuity of a function from a metric space into another metric sapce is defined. It is also proved that a continuous mapping of a compact metric space into a metric space is uniformly continuous. Further it is proved that continuous image of a connected set is connected.

## 12.1 CONTINUITY AND COMPACTNESS :

**12.1.1 Definition :** A mapping $f$ of a metric space $X$ into $\mathbb{R}^k$ is said to be bounded if there exists a real number $M$ such that $|f(x)| \leq M$ for all $x \in X$.

That is $f : X \to \mathbb{R}^k$ is bounded if the image $f(X)$ is a bounded set in $\mathbb{R}^k$

**12.1.2 Theorem :** Suppose $f$ is a continuous mapping of a compact metric space $X$ into a metric space $Y$. Then $f(X)$ is compact.

**Proof :** Suppose $X$ is a compact metric space and $f : X \to Y$ is a continuous mapping. Let $\{V_\alpha\}_{\alpha \in \Delta}$ be an open cover of $f(X)$ in $Y$. Then $f(X) \subseteq \bigcup\limits_{\alpha \in \Delta} V_\alpha$. Since $f$ is continuous on $X$ and $V_\alpha$ is open in $Y$ for each $\alpha \in \Delta$, the inverse image $f^{-1}(V_\alpha)$ is open in $X$ for each $\alpha \in \Delta$. Also it is clear that $X \subseteq \bigcup\limits_{\alpha \in \Delta} f^{-1}(V_\alpha)$. This implies that $\left\{f^{-1}(V_\alpha)\right\}_{\alpha \in \Delta}$ is an open cover for $X$. Since $X$ is compact, there exists $\alpha_1, \alpha_2, \cdots\cdots, \alpha_n \in \Delta$ such that $X \subseteq \bigcup\limits_{i=1}^{n} f^{-1}\left(V_{\alpha_i}\right)$. This implies

$f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha_i}$. Therefore $f(X)$ is compact.

This theorem can also be stated as "The image of a compact metric space under a continuous mapping is a compact metric space or the continuous image of a compact metric space is compact".

**12.1.3 Theorem :** If $f$ is a continuous mapping of a compact metric space $X$ into $\mathbb{R}^k$, then $f(X)$ is closed and bounded. Thus, $f$ is bounded.

**Proof :** Suppose $f$ is a continuous mapping of a compact metric space $X$ into $\mathbb{R}^k$. Then by theorem 12.1.2, $f(X)$ is a compact sub set of $\mathbb{R}^k$. Since every compact subset of $\mathbb{R}^k$ is closed and bounded, $f(X)$ is closed and bounded.

This implies there exists a real number $M$ such that $|f(x)| \leq M$ for all $x \in X$. Therefore $f$ is bounded.

**12.1.4 Corollary :** If $X$ is a compact metric space and $f$ is a continuous real valued function on $X$, then $f(X)$ is bounded.

**Proof :** Taking $k=1$, the corollary follows.

**12.1.5 Theorem :** Suppose $f$ is a continuous real function on a compact metric space $X$ and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

**Proof :** Let $X$ be a compact metric space and $f$ be a continuous real function on $X$. Then by theorem 12.1.3, $f(X)$ is closed and bounded. Since $f(X)$ is bounded, we have $\sup f(x)$ and $\inf f(x)$ exist in $\mathbb{R}$. Since $f(X)$ is closed in $\mathbb{R}$, by a known theorem, $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$. This implies $\sup_{x \in X} f(x) = f(p)$ and $\inf_{x \in X} f(x) = f(q)$ for some $p, q \in X$. Thus there exist $p, q \in X$ such that $M = f(p)$ and $m = f(q)$.

**12.1.6 Note :** The notation in the above theorem means that $M$ is the least upper bound of the set of all numbers $f(p)$, where $p$ ranges over $X$ and that $m$ is the greatest lower bound of this set of numbers.

**12.1.7 Note :** The conclusion in the above theorem may also be stated as follows. There exist points $p$ and $q$ in $X$ such that $f(q) \le f(x) \le f(p)$ for all $x \in X$, that is, $f$ attains its maximum $(\text{at } p)$ and minimum $(\text{at } q)$.

**12.1.8 Theorem :** Suppose $f$ is a continuous $1-1$ mapping of a compact metric space $X$ onto a metric space $Y$. Then the inverse mapping $f^{-1}$ defined on $Y$ by $f^{-1}(f(x)) = x \ (x \in X)$ is a continuous mapping of $Y$ onto $X$.

**Proof :** Suppose $f$ is a continuous $1-1$ mapping of a compact metric space $X$ onto a metric space $Y$. To show $f^{-1}$ is continuous, by theorem 11.2.11, it is enough if we show that $f(V)$ is open in $Y$ for every open set $V$ in $X$. Let $V$ be any open set in $X$. Then $V^c$ is a closed subset of $X$. Since every closed subset of a compact metric space is compact, we have $V^c$ is a compact subset of $X$. Since $f$ is continuous on $X$, by theorem 12.1.2, $f\left(V^c\right)$ is a compact subset of $Y$.

Since every compact subset of a metric space is closed, we have $f\left(V^c\right)$ is closed in $Y$. Since $f$ is $1-1$ and onto, $f(V) = \left(f\left(V^c\right)\right)^C$. This implies $f(V)$ is open in $Y$. Thus $f^{-1}$ is continuous.

**12.1.9 Definition :** A one - one, onto function $f$ of a metric space $X$ onto a metric space $Y$ is said to be a homemorphism if both $f$ and $f^{-1}$ are continuous.

**12.1.10 Note :** By theorem 12.1.8, a one-one, onto continuous function $f$ on a compact metric space is always a homemorphism.

**12.1.11 Definition :** Let $f$ be a mapping of a metric space $(X, d_1)$ into a metric space $(Y, d_2)$. We say that $f$ is uniformly continuous on $X$ if for every $\in > 0$ there exists a $\delta > 0$ such that $d_2(f(p), f(q)) < \in$ for all $p$ and $q$ in $X$ for which $d_1(p, q) < \delta$.

**12.1.12 Example :** Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = 2x$ for all $x \in \mathbb{R}$. Then $f$ is uniformly continuous.

For, let $\in > 0$. Take $\delta = \frac{\in}{2}$. Suppose $x, y \in \mathbb{R}$ such that $|x - y| < \delta$.

Consider $|f(x), \ f(y)| = |2x - 2y| = 2|x - y| < 2 \cdot \delta \ = 2 \cdot \frac{\in}{2} = \in$

$\Rightarrow |f(x)-f(y)| < \in$ whenever $|x-y| < \delta$ $\therefore f$ is continuous.

**12.1.13 Note :** Every uniformly continuous function is continuous but the converse need not be true.

For, suppose $f$ is a uniformly continuous function from a metric space $(X, d_1)$ into a metric space $(Y, d_2)$.

Let $\in > 0$. Since $f$ is uniformly continuous on $X$, there exists a $\delta > 0$ such that $d_2(f(x), f(y)) < \in$ whenever $d_1(x, y) < \delta$. ------------ (1)

Let $x \in X$. Let $y \in X$ such that $d_1(x, y) < \delta$. Then by (1), $d_2(f(x), f(y)) < \in$. Therefore $f$ is continuous at $x$. Since $x \in X$ is arbitrary, we have $f$ is continuous on $X$. Thus every uniformly continuous function is continuous.

In general the converse is not true. For, consider the following example.

Define $f:(0,1) \to \mathbb{R}$ as $f(x) = \dfrac{1}{x}$ for all $x \in (0,1)$. First we show that $f$ is continuous.

Let $\in > 0$ and $x \in (0,1)$. Choose a $\delta > 0$ such that $\delta < \dfrac{\in x^2}{1+\in x}$

Consider $\delta < \dfrac{\in x^2}{1+\in x} \Leftrightarrow \delta(1+\in x) < \in x^2 \Leftrightarrow \delta < \in x^2 - \delta \in x$

$$\Leftrightarrow \delta < \in (x-\delta)x \Leftrightarrow \frac{\delta}{x(x-\delta)} < \in \text{ ----------- (1)}$$

Suppose $y \in (0,1)$ such that $|x-y| < \delta$. Then $x-\delta < y < x+\delta$.

$$\Leftrightarrow \frac{1}{x+\delta} < \frac{1}{y} < \frac{1}{x-\delta} \text{ ------------- (2)}$$

Consider $|f(x)-f(y)| = \left|\dfrac{1}{x} - \dfrac{1}{y}\right| = \left|\dfrac{y-x}{xy}\right| = \dfrac{|y-x|}{xy} < \dfrac{\delta}{xy}$

$$< \frac{\delta}{x(x-\delta)} < \in \text{ (by (1) and (2))}.$$

This shows that $f$ is continuous at $x$ and hence $f$ is continuous on $(0,1)$.

Now we will show that $f$ is not uniformly continuous on $(0, 1)$.

If possible suppose that $f$ is uniformly continuous on $(0,1)$.

Then for $\in = 1$, there exists $\delta > 0$ such that

$|f(x)-f(y)|<1$ whenever $|x-y|<\delta$ ----------- (1)

Since $\delta > 0$, there exists a positive integer $N$ such that $\frac{1}{N} < \delta$. Consider

$$\left|\frac{1}{N} - \frac{1}{N+1}\right| = \frac{1}{N(N+1)} < \frac{1}{N} < \delta$$

Now $\frac{1}{N}, \frac{1}{N+1} \in (0,1)$ such that $\left|\frac{1}{N} - \frac{1}{N+1}\right| < \delta$.

Then by (1), $\left|f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right)\right| < 1$

$\Rightarrow |N-(N+1)| < 1 \Rightarrow |-1| < 1 \Rightarrow 1 < 1$, a contradiction.

So, $f$ is not uniformly continuous.

Thus $f$ is a continuous function but not uniformly continuous.

**12.1.14 Theorem :** Let $f$ be a continuous mapping of a compact metric space $(X, d_1)$ into a metric space $(Y, d_2)$. Then $f$ is uniformly continuous on $X$.

**Proof :** Given that $f$ is a continuous mapping of a compact metric space $X$ into a metric space $Y$.

Let $\in > 0$. Since $f$ is continuous on $X$, for each $p \in X$ there exists a positive number $\delta_p$ such that $q \in X$ with $d_1(p, q) < \delta_p$ implies that $d_2(f(p), f(q)) < \in/2$.

Write $V_p = \left\{ q \in X / d_1(p,q) < \frac{\delta p}{2} \right\}$. Then $V_p$ is a neighbourhood of $p$ and hence an open subset of $X$.

Now $\mathscr{S} = \{ V_p / p \in X \}$ is a class of open sets in $X$. It is clear that $\mathscr{S}$ is an open cover for $X$. Since $X$ is compact, there exists $p_1, p_2, \cdots, p_n \in X$ such that $X \subseteq \bigcup_{i=1}^{n} V_{p_i}$ ----- (1)

Take $\delta = \frac{1}{2} \min \{ \delta_{p_1}, \delta_{p_2}, \cdots\cdots, \delta_{p_n} \}$. Then $\delta > 0$.

Now let $p, q \in X$ be such that $d_1(p,q) < \delta$. By (1), there exists an integer $m$ with $1 \le m \le n$ such that $p \in V_{p_m}$. This implies $d_1(p, p_m) < \frac{\delta p_m}{2}$. Also $d_1(q, p_m) \le d_1(p,q) + d_1(p, p_m)$

$< \delta + \frac{\delta p_m}{2} \le \delta p_m$. Then $d_2(f(p), f(p_m)) < \epsilon/2$ and $d_2(f(q), f(p_m)) < \epsilon/2$.

Consider $d_2(f(p), f(q)) \le d_2(f(p), f(p_m)) + d_2(f(p_m), f(q))$

$$< \epsilon/2 + \epsilon/2 = \epsilon \Rightarrow d_2(f(p), f(q)) < \epsilon.$$

This shows that $f$ is uniformly continuous on $X$.

**12.1.15 Theorem :** Let $E$ be a non-compact set in $\mathbb{R}$. Then (a) there exists a continuous function on $E$ which is not bounded.

(b) there exists a continuous and bounded function on $E$ which has no maximum.

If, in addition, $E$ is bounded, then

(c) there exists a continuous function on $E$ which is not uniformly continuous.

**Proof :** Given that $E$ is a non-compact subset of $\mathbb{R}$. Since $E$ is a non-compact subset of $\mathbb{R}$, then either $E$ is bounded and $E$ is not closed or $E$ is closed and $E$ is not bounded or $E$ is not closed and not bounded.

**Case (i) :** Suppose $E$ is bounded and $E$ is not closed. Since $E$ is not closed, there exists a point $x_0 \in \mathbb{R}$ such that $x_0$ is a limit point of $E$ and $x_0 \notin E$.

Define $f : E \to \mathbb{R}$ as $f(x) = \frac{1}{x - x_0}$ for all $x \in E$.

Then $f$ is continuous on $E$.

Now we will show that $f$ is not bounded. That is, $f(E)$ is not bounded. Since $x_0$ is a limit point of $E$, there exists a sequence $\{x_n\}$ of points in $E$ such that $x_n \to x_0$ as $n \to \infty$. This implies $x_n - x_0 \to 0$ as $n \to \infty$ and consequently $\dfrac{1}{x_n - x_0} \to \infty$ as $n \to \infty$.

Let $M > 0$. Since $\dfrac{1}{x_n - x_0} \to \infty$ as $x \to \infty$, there exists a positive integer $N$ such that $\dfrac{1}{x_n - x_0} > M$ for all $n \geq N$. This implies $f(x_n) > M$ for all $n \geq N$. Therefore $f(E)$ is not bounded; i.e. $f$ is not bounded.

Next we will show that $f$ is not uniformly continuous on $X$. First we show that $f(N_\delta(x_0) \cap E)$ is not bounded for all $\delta > 0$. Let $\delta > 0$ be any real number. It is clear that $N_\delta(x_0) \cap E$ is bounded. Now we will show that $x_0$ is a limit point of $N_\delta(x_0) \cap E$. Let $r > 0$. Put $r_1 = \min\{r, \delta\}$.

Consider $N_{r_1}(x_0) \cap (E \cap N_\delta(x_0)) \setminus \{x_0\} = N_{r_1}(x_0) \cap \setminus \{x_0\} \neq \phi$

$(\because x_0 \text{ is a limit point of } E)$

But $\phi \neq N_{r_1}(x_0) \cap (E \cap N_\delta(x_0)) \setminus \{x_0\} \subseteq N_r(x_0) \cap (E \cap N_\delta(x_0)) \setminus \{x_0\}$

This implies that $N_r(x_0) \cap (E \cap N_\delta(x_0)) \setminus \{x_0\} = \phi$ and hence $x_0$ is a limit point of $N_\delta(x_0) \cap E$.

Since $x_0 \notin E$, we have $x_0 \notin E \cap N_\delta(x_0)$. So $E \cap N_\delta(x_0)$ is a bounded set and $x_0$ is a limit point of $N_\delta(x_0) \cap E$ such that $x_0 \notin N_\delta(x_0) \cap E$. Therefore by the above arguement, $f(N_\delta(x_0) \cap E)$ is not bounded. Since $\delta > 0$ is arbitrary, $f(N_\delta(x_0) \cap E)$ is not bounded for all $\delta > 0$.

Let $\in > 0$ and $\delta > 0$. Let $x \in N_\delta(x_0) \cap E$. Then $x \in E$ and $|x - x_0| < \delta$ and $|x - x_0| > 0$

$(\because x_0 \notin E)$.

Take $r = \delta - |x - x_0|$. Since $f\left(N_r(x_0) \cap E\right)$ is not bounded, there exists $t \in N_r(x_0) \cap E$ such that $|f(t)| \geq \in + \dfrac{1}{|x - x_0|}$.

Now $|t - x_0| < r$ and $t \in E$. This implies that $|t - x_0| + |x - x_0| < \delta$ and hence $|x - t| < \delta$.

Also $|f(t)| - |f(x)| \geq t$. Thus there exist $x, t \in E$ such that $|x - t| < \delta$ and $|f(x) - f(t)| \geq \in$.

Therefore $f$ is not uniformly continuous on $E$.

So (a) and (c) are proved.

(b) Define $g : E \to \mathbb{R}$ as $g(x) = \dfrac{1}{1 + (x - x_0)^2}$ for all $x \in E$.

Then $g$ is continuous on $E$. Also $0 < g(x) < 1$ for all $x \in E$.

This implies $g$ is bounded.

Now we will show that $\sup\limits_{x \in E} g(x) = 1$.

Clearly 1 is an upper bound of $\{g(x) / x \in E\}$.

Let $p$ be any upper bound of $\{g(x) / x \in E\}$.

Now we will show that $p \geq 1$.

If possible suppose that $p < 1$. Then $0 < p < 1$. Now we will show that there exists $x \in E$ such that $g(x) > p$. Take $\in = \sqrt{\dfrac{1}{p} - 1}$. Since $x_0$ is a limit point of $E$, $N_\in(x_0) \cap E \setminus \{x_0\} \neq \phi$. Choose $x \in N_\in(x_0) \cap E \setminus \{x_0\}$. Then $x \in E$ and $|x - x_0| < \in = \sqrt{\dfrac{1}{p} - 1} \Rightarrow |x - x_0|^2 < \dfrac{1}{p} - 1$

$$\Rightarrow \frac{1}{p} > 1 + |x - x_0|^2$$

$$\Rightarrow \frac{1}{1+(x-x_0)^2} > p \Rightarrow g(x) > p$$

Thus there exists $x \in E$ such that $g(x) > p$, which is a contradiction to the fact that $p$ is an upper bound of the set $\{g(x)/x \in E\}$. Therefore $p \geq 1$. Hence $\sup\limits_{x \in E} g(x) = 1$.

This shows that $g$ has no maximum.

Thus if $E$ is bounded, then $(a),(b)$ and $(c)$ are proved.

**Case (ii) :** Suppose $E$ is not bounded.

(a) Define $f : E \to \mathbb{R}$ as $f(x) = x$ for all $x \in E$. Then $f$ is continuous on $E$ and $f$ is not bounded on $E$.

So (a) is proved.

(b) Define $h : E \to \mathbb{R}$ as $h(x) = \frac{x^2}{1+x^2}$ for all $x \in E$.

Then $h$ is continuous on $E$. Since $h(x) < 1$ for all $x \in E$, $h$ is bounded. Now we will show that $h$ has no maximum. For this we will show that $\sup\limits_{x \in E} h(x) = 1$.

Since $h(x) < 1$ for all $x \in E$, we have 1 is an upper bound of $\{h(x)/x \in E\}$. Let $p$ be any upper bound of $\{h(x)/x \in E\}$. If possible suppose that $p < 1$. Then $0 < p < 1$.

Now we will show that there exists $x \in E$ such that $h(x) > p$.

Since $E$ is not bounded, there exists $x \in E$ such that

$$|x| > \sqrt{\frac{p}{1-p}} \Rightarrow x^2 > \frac{p}{1-p} \Rightarrow (1-p)x^2 > p$$

$$\Rightarrow x^2 - px^2 > p \Rightarrow x^2 > p + px^2 = p\left(1 + x^2\right)$$

$\Rightarrow \dfrac{x^2}{1+x^2} > p \Rightarrow h(x) > p$, which is a contradiction to the fact that $p$ is an upper bound of $\{h(x)/x \in E\}$.

Therefore $1 \le p$ and hence $\sup\limits_{x \in E} h(x) = 1$

Thus $h$ is no maximum.

**12.1.16 Note :** (c) Would be false if boundedness were omitted from the hypothesis.

**Example :** Let $E$ be the set of all integers. Then $E$ is a non-compact subset of $\mathbb{R}$ which is not bounded. Then every function defined on $E$ is uniformly continuous. For, let $f$ be any function from $E$ into $\mathbb{R}$. Let $\in > 0$. Choose $\delta$ such that $0 < \delta < 1$. Suppose $x, y \in E$ such that $|x - y| < \delta$. Then $x = y$. This implies $|f(x) - f(y)| = 0 < \in$. Hence $f$ is uniformly continuous on $E$.

## 12.2 CONTINUITY AND CONNECTEDNESS :

**12.2.1 Theorem :** If $f$ is a contnuous mapping of a metric space $X$ into a metric space $Y$ and if $E$ is a connected subset of $X$, then $f(E)$ is connected.

**Proof :** Suppose $f$ is a continuous mapping of a metric space $X$ into a metric space $Y$ and $E$ is a connected subset of $X$.

**Claim :** $f(E)$ is a connected subset of $Y$.

If possible suppose that $f(E)$ is not connected. Then there exist non-empty subsets $A$ and $B$ of $Y$ such that $f(E) = A \cup B$ and $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$.

Write $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$.

Since $A$ and $B$ are non-empty, we have $G \neq \phi$ and $H \neq \phi$. Now consider

$$G \cup H = \left[E \cap f^{-1}(A)\right] \cup \left[E \cap f^{-1}(B)\right]$$

$$= E \cap \left[f^{-1}(A) \cup f^{-1}(B)\right] = E \cap f^{-1}(A \cup B) = E$$

$\therefore E = G \cup H$

Now we will show that $G \subseteq f^{-1}(\overline{A})$

Let $x \in G \Rightarrow x \in E \cap f^{-1}(A) \Rightarrow x \in E$ and $f(x) \in A$

$\Rightarrow f(x) \in \overline{A} \ (\because A \subseteq \overline{A}) \Rightarrow x \in f^{-1}(\overline{A})$

Therefore $G \subseteq f^{-1}(\overline{A})$.

Since $\overline{A}$ is a closed set in $Y$ and since $f$ is continuous, by corollary 11.2.12, $f^{-1}(\overline{A})$ is a closed set in $X$.

Since $f^{-1}(\overline{A})$ is a closed set containing $G$ and $\overline{G}$ is the smallest closed set containing $G$ we have $\overline{G} \subseteq f^{-1}(\overline{A})$.

This implies $f(\overline{G}) \subseteq \overline{A}$.

Next we will show that $f(H) = B$.

Let $y \in f(H)$. Then $y = f(x)$ for some $x \in H$.

$x \in H \Rightarrow x \in E$ and $x \in f^{-1}(B) \Rightarrow f(x) \in B \Rightarrow y \in B$.

So $f(H) \subseteq B$.

Let $y \in B \Rightarrow y \in f(E) \Rightarrow y = f(x)$ for some $x \in E$.

$\Rightarrow x \in f^{-1}(B)$ and $x \in E \Rightarrow x \in E \cap f^{-1}(B)$

$\Rightarrow x \in H \Rightarrow f(x) \in f(H) \Rightarrow y \in f(H)$

So $B \subseteq f(H)$ and hence $f(H) = B$.

Next we will show that $\overline{G} \cap H = \phi$.

If possible suppose that $\overline{G} \cap H \neq \phi$. Then choose $x \in \overline{G} \cap H$

$\Rightarrow x \in \overline{G}$ and $x \in H \Rightarrow x \in \overline{G}$ and $f(x) \in f(H) = B$.

$\Rightarrow f(x) \in f(\bar{G})$ and $f(x) \in B \Rightarrow f(x) \in \bar{A}$ and $f(x) \in B$ $\left(\because f(\bar{G}) \subseteq \bar{A}\right)$.

$\Rightarrow f(x) \in \bar{A} \cap B \Rightarrow \bar{A} \cap B \neq \phi$, a contradiction.

So $\bar{G} \cap H = \phi$.

Similarly we can show that $G \cap \bar{H} = \phi$.

Therefore $E = G \cup H$ such that $\bar{G} \cap H = \phi$ and $G \cap \bar{H} = \phi$.

Thus $E$ is the union of two separated sets; which is a contradiction to the fact that $E$ is connected. This contradiction arises due to our supposition $f(E)$ is not connected. Hence $f(E)$ is connected.

**12.2.2 Theorem :** Let $f$ be a real continuous function on the closed interval $[a,b]$. If $f(a) < f(b)$ and if $c$ is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a,b)$ such that $f(x) = c$.

**Proof :** Given that $f$ is a continuous real function on the closed interval $[a,b]$. Suppose $f(a) < f(b)$ and $c$ is a number such that $f(a) < c < f(b)$.

By a known theorem, $[a,b]$ is connected. Since $f$ is continuous, by theorem 12.2.1, $f[a, b]$ is connected subset of $R$. Then by a known theorem, $f[a, b]$ is an interval. Since $f(a) < c < f(b)$ and $f(a), f(b) \in f[a, b]$, we have $c \in f[a,b] \Rightarrow c = f(x)$ for some $x \in [a, b]$.

**12.2.3 Note :** Theorem 12.2.2 holds if $f(a) > f(b)$.

**12.2.4 Definition :** If $f$ is defined on $E$, then the set $\{(x, f(x)) / x \in E\}$ is called the graph of $f$.

**12.2.5 Problem :** If $f$ is a real valued function defined on a set $E$ of real numbers and if $E$ is compact, then show that $f$ is continuous on $E$ if and only if the graph of $f$ is compact.

**Solution :** Suppose $f$ is a real valued function defined on a set $E$ of real numbers and also suppose that $E$ is compact.

**Claim :** $f$ is continuous on $E$ if and only if the graph of $f$ is compact.

Suppose $f$ is continuous on $E$. Then by theorem 12.1.2, $f(E)$ is compact. Since the product of a non-empty family of compact sets is compact, we have $E\,X\,f(E)$ is compact. Since every closed subset of a compact set is compact, to show the graph of $f$ is compact, it is enough if we show that the graph of $f$ is a closed subset of $E\,X\,f(E)$.

Write $G=\{(x,f(x))/x\in E\}$. Then $G$ is the graph of $f$. Let $(x,y)\in E\,X\,f(E)$ be a limit point of $G$. Then there exist a sequence $\{(x_n,f(x_n))\}$ of points in $G$ such that $\lim\limits_n (x_n,f(x_n))=(x,y)$. This implies $\lim\limits_n x_n=x$ and $\lim\limits_n f(x_n)=y$. Since $f$ is continuous and $\lim\limits_n x_n=x$, we have $\lim\limits_n f(x_n)=f(x)$. Since the limit of a sequence is unique, we have $f(x)=y$.

Therefore $(x,y)=(x,f(x))\in G$. This shows that $G$ contains all of its limit points and hence $G$ is a closed subset of $E\,X\,f(E)$. Consequently $G$ is compact. That is, the graph of $f$ is compact.

Conversely suppose that the graph $G$ of $f$ is compact.

We will show that $f$ is continuous.

Since $G$ is compact, by a known result, $G$ is closed and bounded. Let $x\in E$. Let $\{x_n\}$ be a sequence of points in $E$ such that $\{x_n\}$ converges to $x$. Now $\{(x_n,\ f(x_n))\}$ is a sequence of points in $G$. Since $G$ is bounded, $\{(x_n,f(x_n))\}$ is bounded. This implies that $\{f(x_n)\}$ is bounded. Then $\lim\sup f(x_n)$ and $\lim\inf f(x_n)$ exist. So let $\infty=\lim\sup f(x_n)$. Then there exists a sub sequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ such that $f(x_{n_k})$ converges to $\infty$.

Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{x_n\}$ converges to $x$, we have $\{x_{n_k}\}$ converges to $x$. Then $\{x,\alpha\}=\lim\limits_k (x_{n_k},\ f(x_{n_k}))$. Now $\{(x_{n_k},\ f(x_{n_k}))\}$ is a sequence of points in $G$ such that $(x,\alpha)=\lim\limits_k (x_{n_k},f(x_{n_k}))$. This implies that $(x,\alpha)$ is a limit point of $G$. Since $G$ is closed, $(x,\alpha)\in G$ and hence $(x,\alpha)=(x,f(x))$.

Therefore $(x, f(x)) = \lim\limits_{n} \sup(x_n, f(x_n))$.

Similarly we can show that $(x, f(x)) = \lim\limits_{n} \inf(x_n, f(x_n))$

Therefore $(x, f(x)) = \lim\limits_{n}(x_n, f(x_n))$. Consequently

$\lim\limits_{n} f(x_n) = f(x)$. So $f$ is continuous at $x$. Since $x \in E$ is arbitrary, $f$ is continuous on $E$.

Thus $f$ is continuous on $E$ if and only if the graph of $f$ is compact.

**12.2.6 Problem :** Let $I = [0, 1]$ be the closed unit interval. Suppose $f$ is a continuous mapping of $I$ into $I$. Prove that $f(x) = x$ for atleast one $x \in I$.

**Solution:** Given that $I = [0, 1]$ be the closed unit interval and $f$ is a continuous mapping of $I$ into $I$.

Define $g: I \to \mathbb{R}$ as $g(x) = x - f(x)$ for all $x \in [0,1]$

Since $f$ is a continuous function, we have $g$ is also a continuous function.

Consider $g(0) = 0 - f(0) \le 0$ and $g(1) = 1 - f(1) \ge 0$ $(\because 0 \le f(0)$ and $f(1) \le 1)$.

$\therefore g(0) \le 0 \le g(1)$

If $g(0) = 0$, then $0 - f(0) = 0 \Rightarrow f(0) = 0$.

If $g(1) = 0$, then $1 - f(1) = 0 \Rightarrow f(1) = 1$.

Suppose $g(0) < 0 < g(1)$. Then, by theorem 12.18, there exists $x \in (0,1)$ such that $g(x) = 0$ This implies $x - f(x) = 0$ and hence $f(x) = x$.

Thus, in any case, $f(x) = x$ for some $x \in I$.

**12.2.7 Problem :** Show that a uniformly continuous function of a uniformly continuous function is uniformly continuous.

**Solution :** Let $(X, d_1)$, $(Y, d_2)$ and $(Z, d_3)$ be metric spaces. Suppose $f: X \to Y$ and $g: Y \to Z$

are uniformly continuous functions.

**Claim :** $g\,o\,f : X \to Z$ is uniformly continuous.

Let $\in > 0$. Since $g : Y \to Z$ is uniformly continuous, there exists $\eta > 0$ such that $d_3\left(g(y_1), g(y_2)\right) < \in$ whenever $d_2(y_1, y_2) < \eta$, ----------- (1)

Since $f : X \to Y$ is uniformly continuous, there exists a $\delta > 0$ such that $d_2\left(f(x_1), f(x_2)\right) < \eta$ whenever $d_1(x_1, x_2) < \delta$.

Suppose $x_1, x_2 \in X$ such that $d_1(x_1, x_2) < \delta$ ------------- (2)

Then from (1) and (2), $d_3\left(g\,o\,f(x_1), g\,o\,f(x_2)\right) < \in$.

Therefore $g\,o\,f : X \to Z$ is uniformly continuous.

**12.2.8 Problem :** If $E$ is a non-empty subset of a metric space $(X, d)$, define the distance from $x \in X$ to $E$ by

$$p_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that $p_E(x) = 0$ if and only if $x \in \bar{E}$

(b) Prove that $p_E$ is a uniformly continuous function on $X$, by showing that $|P_E(x) - P_E(y)| \le d(x, y)$ for all $x \in X,\ y \in X$.

**Solution :** Suppose $E$ is a non-empty subset of a metric space $(X, d)$.

Define $P_E(x) = \inf_{z \in E} d(x, z)$ for all $x \in X$

(a) : To show $P_E(x) = 0$ if and only if $x \in \bar{E}$.

Suppose $x \in \bar{E}$.

Now $d(x, x) = 0$

If $x \in E$, then $0 \le P_E(x) \le d(x, x) = 0 \Rightarrow P_E(x) = 0$.

Suppose $x \notin E$. Since $x \in \bar{E}$, $x$ is a limit point of $E$. Then there exists a sequence $\{x_n\}$

of points in $E$ such that $\lim\limits_{n\to\infty} x_n = x$.

Let $\in > 0$. Since $\lim\limits_{n\to\infty} x_n = x$, there exists a positive integer $N$ such that $d(x_n, x) < \in$ for all $n \geq N$.

Now $0 \leq P_E(x) \leq d(x_n, x) < \in$ for all $n \geq N$.

$\Rightarrow 0 \leq P_E(x) < \in$.

Since $\in > 0$ is arbitary. We have $P_E(x) = 0$.

Thus if $x \in \bar{E}$, then $P_E(x) = 0$

Conversely suppose that $P_E(x) = 0$.

Let $\in > 0$. Since $P_E(x) = 0$ there exists $y \in E$ such that $d(x, y) < \in$. This implies $y \in N_\in(x)$.

Therefore $N_\in(x) \cap E \neq \phi$.

This shows that $N_\in(x) \cap E \neq \phi$ for any $\in > 0$ and hence $x \in \bar{E}$.

Thus $x \in \bar{E}$ if and only if $P_E(x) = 0$.

(b) To show $P_E$ is uniformly continuous on $X$.

Let $\in > 0$. Take $\delta = \in$. Suppose $x, y \in X$ such that $d(x, y) < \delta$.

Consider $P_E(x) \leq d(x, z)$ for all $z \in E$

$\leq d(x, y) + d(y, z)$ for all $z \in E$.

$\Rightarrow P_E(x) - d(x, y) \leq d(y, z)$ for all $z \in E$.

$\Rightarrow P_E(x) - d(x, y)$ is a lower bound of $\{d(y, z) / z \in E\}$.

$\Rightarrow P_E(x) - d(x, y) \leq P_E(y) \Rightarrow P_E(x) - P_E(y) \leq d(x, y) < \delta = \in$.

Similarly $P_E(y)-P_E(x)<\in$

Therefore $|P_E(x)-P_E(y)|<\in$ whenever $d(x,y)<\delta$.

Hence, $P_E$ is uniformly continuous on $X$.

## 12.3 SHORT ANSWER QUESTIONS

**12.3.1 :** When do you say that a mapping $f$ of a metric space $X$ into $\mathbb{R}^k$ is bounded ?

**12.3.2 :** Define a homemorphism of a metric space into another metric space.

**12.3.3 :** When do you say that a function $f$ of a metric space $X$ into a metric space $Y$ is uniformly continuous ?

**12.3.4 :** Is every uniformly continuous function a continuous function ? Justify your answer.

**12.3.5 :** Is every continuous function a uniformly continuous function ? Justify your answer.

## 12.4 MODEL EXAMINATION QUESTIONS

**12.4.1 :** If $f$ is a continuous mapping of a compact metric space $X$ into a metric space $Y$, then show that $f(X)$ is compact. (Equivalently show that continuous image of a compact metric space is compact).

**12.4.2 :** Show that a continuous $1-1$ mapping of a compact metric space $X$ onto a metric space $Y$ is a homemorphism.

**12.4.3 :** Show that a continuous mapping of a compact metric space $X$ into a metric space $Y$ is uniformly continuous.

**12.4.4 :** Let $E$ be a non-compact set in $\mathbb{R}$. Then show that

(a) there exists a continuous function on $E$ which is not bounded.

(b) there exists a continuous and bounded function on $E$ which has no maximum.

(c) If, in addition, $E$ is bounded, then show that there exists a continuous function on $E$ which is not uniformly continuous.

**12.4.5 :** Show that continuous image of a connected set is connected.

**12.4.6 :** Let $f$ be a real continuous function on the closed interval $[a,b]$. If $f(a)<f(b)$ and if $c$ is

a number such that $f(a)<c<f(b)$, then show that there exists a point $x\in(a,b)$ such that $f(X)=C$.

**12.4.7 :** If $f$ is a real valued function defined on a set $E$ of real numbers and if $E$ is compact, then show that $f$ is continuous on $E$ if and only if the graph of $f$ is compact.

## 12.5 EXERCISES

**12.5.1 :** Let $f$ be a real uniformly continuous function on the bounded set $E$ in $\mathbb{R}$. Prove that $f$ is bounded on $E$. Show that the conclusion is false if boundedness of $E$ is omitted from the hypothesis.

**12.5.2 :** Suppose $f$ is a uniformly continuous mapping of a metric $X$ into a metric space $Y$. Then prove that $\{f(x_n)\}$ is a Cauchy sequence in $Y$ for every Cauchy sequence $\{x_n\}$ in $X$.

**12.5.3 :** Let $E$ be a dense subset of a metric space $X$ and let $f$ be a uniformly continuous real function defined on $E$. Prove that $f$ has a continuous extension from $E$ to $X$.

**12.5.4 :** Call a mapping $f$ of a metric space $X$ into a metric space $Y$ open if $f(V)$ is an open set in $Y$ whenever $V$ is an open set in $X$. Prove that every continuous open mapping of $\mathbb{R}$ is monotonic.

## 12.6 ANSWERS TO SHORT ANSWER QUESTIONS :

For 12.3.1, see definition 12.1.1

For 12.3.2, see definition 12.1.9

For 12.3.3, see definition 12.1.11

For 12.3.4, see note 12.1.13

For 12.3.5, see note 12.1.13

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions Walter Rudin

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Lesson - 13

# DISCONTINUITIES OF REAL FUNCTIONS

## 13.0 INTRODUCTION

Through out this lesson $f(x)$ denotes a real valued function of real variable. In this leson the discontinuity of first kind and the discoutinuity of second kind are defined. It is proved that if $f$ is a monotonically increasing function defined on $(a,b)$, then $f(x+)$ and $f(x-)$ exist at every point $x$ of $(a,b)$. It is also proved that if $f$ is monotonic on $(a,b)$, then the set of points at which $f$ is discontinuous is at most countable.

## 13.1 DISCONTINUITIES

**13.1.1 Definition :** Let $f$ be a function from a metric space $X$ into a metric space $Y$. If $f$ is not continuous at a point $x \in X$, then we say that $f$ is discontinuous at $x$.

**13.1.2 Definition :** Let $f$ be a real valued function defined on $(a,b)$. Let $x$ be a point such that $a \leq x < b$. A number $q$ is called the right hand limit of $f$ at $x$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in $(x,b)$ such that $t_n \to x$ and we write $f(x+) = q$.

**13.1.3 Definition :** Let $f$ be a real valued function defined on $(a,b)$. Let $x$ be a point such that $a < x \leq b$. A number $p$ is called the left hand limit of $f$ at $x$ if $f(t_n) \to p$ as $n \to \infty$, for all sequences $\{t_n\}$ in $(a,x)$ such that $t_n \to x$ and we write $f(x-) = p$.

**13.1.4 Note :** If $x \in (a,b)$, then $\lim\limits_{t \to x} f(t)$ exists if and only if $f(x+) = f(x-) = \lim\limits_{t \to x} f(t)$.

**13.1.5 Definition :** Let $f$ be a real valued function defined on $(a,b)$. If $f$ is discontinuous at a point $x \in (a,b)$ and if $f(x+)$ and $f(x-)$ exist, then $f$ is said to have a discountinuity of the first kind or a simple discontinuity at $x$. In this case either $f(x+) \neq f(x-)$ (in which case the value of $f(x)$ is immaterial) or $f(x+) = f(x-) \neq f(x)$.

**13.1.6 Definition :** Let $f$ be a real valued function defined on $(a,b)$. If $f$ is discountinuous at $x \in (a,b)$ and if either $f(x+)$ or $f(x-)$ does not exist, then $f$ is said to have discountinuity of second kind.

**13.1.7 Definition :** Let $f$ be a real valued function defined on $(a,b)$. Then $f$ is said to be monotonically increasing on $(a,b)$ if $a<x<y<b$ implies that $f(x) \le f(y)$ and $f$ is said to be monotonically decreasing on $(a,b)$ if $a<x<y<b$ implies that $f(y) \le f(x)$. $f$ is said to be a monotonic function if it is either monotonically increasing or monotonically decreasing.

**13.1.8 Theorem :** Let $f$ be a monotonically increasing function defined on $(a,b)$. Then $f(x+)$ and $f(x-)$ exist at every point $x$ of $(a,b)$. More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t)$$

Furthermore, if $a<x<y<b,$ then $f(x+) \le f(y-)$

**Proof :** Let $f$ be a monotonically increasing function defined on $(a,b)$.

Let $x \in (a,b)$. Since $f$ is monotonically increasing, we have $f(t) \le f(x)$ for all $t$ such that $a<t<x.$ This implies $\{f(t)/a<t<x\}$ is bounded above by $f(x)$. Since $\mathbb{R}$ has least upper bound property, $\{f(t)/a<t<x\}$ has a least upper bound, say $A$. Then $A \le f(x)$.

Now we will show that $A = f(x-)$.

Let $\in > 0$. Then $A - \in$ is not an upper bound of $\{f(t)/a<t<x\}$. This implies there exists $t_0$ such that $a<t_0<x$ and

$$A - \in < f(t_0) \le A \text{ --------------(1)}$$

Take $\delta = x - t_0$. Then $\delta > 0$. Suppose $t_0 < t < x$. Since $f$ is monotonically increasing, we have

$$f(t_0) \le f(t) \le A \text{ ------------ (2)}$$

From (1) and (2), we have $A - \in < f(t) < A + \in$ when ever $x - \delta < t < x.$ This implies $|f(t) - A| < \in$ for all $t$ such that $x - \delta < t < x$ and hence $\lim_{t \to x-} f(t) = A.$ Thus $f(x-) = A$.

That is, $f(x-)=\sup\limits_{a<t<x} f(t)$.

Next we will show that $f(x+)=\inf\limits_{x<t<b} f(t)$

Since $f$ is monotonically increasing, $f(x)\le f(t)$ for all $t$ such that $x<t<b$. This implies that $\{f(t)/x<t<b\}$ is bounded below by $f(x)$. Since $\mathbb{R}$ has greatest lower bound property, $\{f(t)/x<t<b\}$ has a greatest lower bound, say $A$.

Then $f(x)\le A$.

Now, we will show that $A=f(x+)$

Let $\in>0$. Then $A+\in$ is not a lower bound of $\{f(t)/x<t<b\}$. This implies that there exists $t_0$ such that $x<t_0<b$ and

$$A\le f(t_0)<A+\in \text{ --------------------- (3)}$$

Take $\delta=t_0-x$. Then $\delta>0$. Suppose $x<t<t_0$.

Since $f$ is monotonically increasing, we have

$$A\le f(t)\le f(t_0) \text{ ------------------- (4)}$$

From (3) and (4), we have $A-\in<f(t)<A+\in$ whenever $x<t<x+\delta$. This implies $|f(t)-A|<\in$ for all $t$ such that $x<t<x+\delta$ and hence $\lim\limits_{t\to x+} f(t)=A$. i.e. $f(x+)=A$. Thus $f(x+)=\inf\limits_{x<t<b} f(t)$.

Hence $\sup\limits_{a<t<x} f(t)=f(x-)\le f(x)\le f(x+)=\inf\limits_{x<t<b} f(t)$

Next we will show that $f(x+)\le f(y-)$ if $a<x<y<b$.

Suppose $a<x<y<b$. Then by the above

$$f(x+)=\inf_{x<t<b} f(+)=\inf_{x<t<y} f(t) \text{ -------------- (5)}$$

and $$f(y-)=\sup_{a<t<y} f(t)=\sup_{x<t<y} f(t) \text{ ----------- (6)}$$

From (5) and (6), we have $f(x+) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = f(y-)$

Thus if $a<x<y<b$, then $f(x+) \leq f(y-)$.

**13.1.9 Note :** The above theorem also holds for monotonically decreasing functions.

**13.1.10 Corollary :** Monotonic functions have no discountinuities of the second kind.

**Proof :** Let $f$ be a monotonic function defined on $(a,b)$. Then by theorem 13.1.8 (if $f$ is monotonically increasing) and by note 13.1.9 (if $f$ is monotonically decreasing), $f(x+)$ and $f(x-)$ exist at every point $x \in (a,b)$. So $f$ has no discountinuities of second kind.

**13.1.11 Theorem :** Let $f$ be monotonic on $(a,b)$. Then the set of points of $(a,b)$ at which $f$ is discontinuous is atmost countable.

**Proof :** Given that $f$ is monotonic on $(a,b)$. Suppose $f$ is monotonically increasing. Let $E$ be the set of points at which $f$ is discontinuous. If $E$ is empty or finite, then $E$ is atmost countable.

Suppose $E$ is not finite. In this case we will show that $E$ is countable.

Let $x \in E$. Then $f$ is discountinuous at $x$. Since $f$ is monotonic, by corollary 13.10, $f$ has discontinuities of first kind. This implies $f(x+)$, $f(x-)$ exist and $f(x-) < f(x+)$. Then choose a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$. Thus if $x \in E$, then there exists a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$.

Write $T = \{r(x) / x \in E\}$. Then $T \subseteq Q$, the set of rational numbers. Since $Q$ is countable, $T$ is also countable.

Now define $f : E \to T$ as $f(x) = r(x)$ for all $x \in E$. Then clearly $f$ is a function.

Suppose $x_1, x_2 \in E$ such that $x_1 \neq x_2$. Assume $x_1 < x_2$.

Then by theorem 13.1.8, $f(x_1+) \leq f(x_2-)$. This implies that

$$f(x_1-) < r(x_1) < f(x_1+) \leq f(x_2-) < r(x_2) < f(x_2+)$$

$\therefore r(x_1) \neq r(x_2)$ and hence $f(x_1) \neq f(x_2)$.

Thus $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$

Consequently $f$ is one - one.

Clearly $f$ is onto.

Therefore $f:E\to T$ is a bijection and hence $E$ is countable ($\because T$ is countable).

So $E$ is atmost countable.

Now if $f$ is a monotonicaly decreasing function, then $-f$ is a monotonically increasing function, then the set of discontinuities of $-f$ are the same, we have the set of discontinuities of $f$ is atmost countable. Thus the set of discontinuities of a monotonic function is atmost countable.

**13.1.12 Definition :** For any real $c$, the set of real numbers $x$ such that $x>c$ is called a neighbourhood of $+\infty$ and is written $(c,+\infty)$. For any real $c$, the set of real numbers $x$ such that $x<c$ is called a neighbourhood of $-\infty$ and is written $(-\infty,c)$.

**13.1.13 Definition :** Let $f$ be a real function defined on $E$. We say that $f(t)\to A$ as $t\to x$, where $A$ and $x$ are in the extended real number system, if for every neighbourhood $U$ of $A$ there is a neighbourhood $V$ of $x$ such that $E\cap V$ is non-empty and such that $f(t)\in U$ for all $t\in E\cap V, t\neq x$.

**13.1.14 Theorem :** $\lim\limits_{t\to x} f(t)=A,$ where $A$ and $x$ are extended real numbers if and only if $\lim\limits_{n\to\infty} f(t_n)=A$ for all sequences $\{t_n\}$ in $E$ such that $t_n\neq x$ and $t_n\to x$.

**Proof :** Suppose $\lim\limits_{t\to x} f(t)=A$.

Let $\{t_n\}$ be any sequence in $E$ such that $t_n\neq x$ and $t_n\to x$.

Let $U$ be any neighbourhood of $A$. Since $\lim\limits_{t\to x} f(t)=A,$ there exists a neighbourhood $V$ of $x$ such that $V\cap E\neq\phi$ and $f(t)\in U$ for all $t\in V\cap E$ and $t\neq x$. Since $t_n\to x$, there exists a positive integer $N$ such that $t_n\in V$ for all $n\geq N$. This implies $f(t_n)\in U$ for all $n\geq N$ and hence $\lim\limits_{n\to\infty} f(t_n)=A$.

Conversely suppose that $\lim\limits_{n\to\infty} f(t_n)=A$ for all sequences $\{t_n\}$ in $E$ such that $t_n\neq x$ and $t_n\to x$.

If possible suppose that $\lim\limits_{t\to x} f(t)\neq A$. Then there exists a neighbourhood $u$ of $A$ such that for every neighbourhood $V$ of $x$, there exists a point $t\in E$ for which $f(t)\notin U$ and $t\in V$.

**Case (i) :** Suppose $x=+\infty$. Let $n$ be a positive integer. Now $(n,\infty)$ is a neighbourhood of $\infty$. Then there exists $t_n \in E$ such that $f(t_n) \notin U$ and $t_n \in (n,\infty)$. Therefore $\{t_n\}$ is a sequence of points in $E$ such that $t_n \to \infty, t_n \neq \infty$ and $\lim\limits_{n\to\infty} f(t_n) \neq A$.

**Case (ii) :** Suppose $x=-\infty$. Let $n$ be any positive integer. Now, $(-\infty,-n)$ is a neighbourhood of $-\infty$. Then there exists $t_n \in E$ such that $f(t_n) \notin U$ and $t_n \in (-\infty,-n)$. Therefore $\{t_n\}$ is a sequence of points in $E$ such that $t_n \to -\infty$ as $n \to \infty$ and $t_n \neq -\infty$ and $\lim\limits_{n\to\infty} f(t_n) \neq A$.

**Case (iii) :** Suppose $x$ is a real number. Let $n$ be any positive integer. Now $\left(x-\frac{1}{n}, x+\frac{1}{n}\right)$ is a neighbourhood of $x$. Then there exists $t_n \in E$ such that $f(t_n) \notin U$ and $t_n \in \left(x-\frac{1}{n}, x+\frac{1}{n}\right)$. Therefore $\{t_n\}$ is a sequence of points in $E$ such that $t_n \to x$ as $n \to \infty$ and $t_n \neq x$ and $\lim\limits_{n\to\infty} f(t_n) \neq A.$

Thus in any case there exists a sequence $\{t_n\}$ of points in $E$ such that $t_n \neq x$ and $t_n \to x$ and $\lim\limits_{n\to\infty} f(t_n) \neq A$; which is a contradiction to our supposition. This contradiction arises due to our assumption $\lim\limits_{t\to x} f(t) \neq A$.

Hence $\lim\limits_{t\to x} f(t) = A.$

**13.1.15 Problem :** Define $f:(0,2)\to \mathbb{R}$ as $f(x)=1$ if $0<x\leq 1$ and $f(x)=2$ if $1<x<2$. Then show that $f$ is continuous at every point $x \neq 1$ and $f$ has a discontinuity of first kind at $x=1$.

**Solution :** First we show that $f$ is continuous at every $x \in (0,2)$ such that $x \neq 1$.

Let $x \in (0,2)$ such that $x \neq 1$ and let $\in > 0$.

Then $0<x<1$ or $1<x<2$

Suppose $0<x<1$. Choose $\delta$ such that $0<\delta<\min\{x, 1-x\}$. Then $0<x-\delta<x<x+\delta<1$.

Suppose $y \in (0,2)$ such that $|x-y|<\delta$. Then $x-\delta<y<x+\delta$. This implies $0<y<1$.

Consider $|f(x)-f(y)| = |1-1| = 0 < \in$.

So, in this case, $f$ is continuous at $x$.

Suppose $1<x<2$. Choose $\delta$ such that $0<\delta<\min\{x-1,\ 2-x\}$

Then $1<x-\delta<x<x+\delta<2$.

Suppose $y\in(0,2)$ such that $|x-y|<\delta$. Then $x-\delta<y<x+\delta$

This implies $1<y<2$.

Consider $|f(x)-f(y)|=|2-2|=0<\in$

So, in this case also $f$ is continuous at $x$.

Thus $f$ is continuous at every point $x\in(0,2)$ such that $x\neq 1$.

Next we will show that $f$ is discontinuous at $x=1$.

Let $\{t_n\}$ be any sequence in $(1,2)$ such that $t_n\to 1$.

Then $f(1+)=\lim\limits_{n\to\infty} f(t_n)=\lim\limits_{n\to\infty} 2=2$. So $f(1+)=2$.

Let $\{t_n\}$ be any sequence of points in $(0,1)$ such that $t_n\to 1$.

Then $f(1-)=\lim\limits_{n\to\infty} f(t_n)=\lim\limits_{n\to\infty} 1=1$. So $f(1-)=1$

Therefore $f(1+)$ and $f(1-)$ exist and $f(1+)\neq f(1-)$. So $f$ has a discontinuity of first kind at $x=1$.

**13.1.16 Problem :** Define $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=1$ if $x$ is a rational number and $f(x)=0$ if $x$ is an irrational numbers. Then show that $f$ has a discontinuity of second kind at every point $x\in\mathbb{R}$.

**Solution :** First we show that $f$ is discontinuous at every point $x\in\mathbb{R}$.

Let $x\in\mathbb{R}$ and let $0<\in<1$.

Let $\delta$ be any real number such that $\delta>0$.

**Case (i) :** Suppose $x$ is a rational number.

Choose an irrational number $y$ such that $x-\delta<y<x+\delta$. Then $|x-y|<\delta$.

Consider $|f(x)-f(y)|=|1-0|=1>\in$

**Case (ii) :** Suppose $x$ is an irrational number.

Choose a rational number $y$ such that $x-\delta<y<x+\delta$. Then $|x-y|<\delta$.

Consider $|f(x)-f(y)|=|0-1|=1>\epsilon$

Thus in any case, for $0<\epsilon<1$, for any $\delta>0$, there exists $y\in(x-\delta, x+\delta)$ such that $|f(x)-f(y)|>\epsilon$.

This shows that $f$ is discountinuous at $x$.

Hence $f$ is discontinuous at every point $x\in\mathbb{R}$.

Next we will show that $f$ has a discontinuity of second kind at every point $x\in\mathbb{R}$.

Let $x\in\mathbb{R}$. For each positive integer $n$, consider $\left(x, x+\frac{1}{n}\right)$, Choose a rational number $r_n$ in $\left(x, x+\frac{1}{n}\right)$. Then $\{r_n\}$ is a sequence of rational numbers such that $r_n \to x$ (since $\lim_{n\to\infty}\left(x+\frac{1}{n}\right)=x$ and $x<r_n<x+\frac{1}{n}$).

Consider $\lim_{n\to\infty} f(r_n)=\lim_{n\to\infty} 1=1$. So $\{r_n\}$ is a sequence of rational numbers in $(x,\infty)$ such that $r_n \to x$ as $n\to\infty$ and $\lim_{n\to\infty} f(r_n)=1$. Let $\{s_n\}$ be a sequence of irrational numbers such that $x<s_n<x+\frac{1}{n}$. Then $\{s_n\}$ is a sequence of irrational numbers in $(x,\infty)$ such that $s_n \to x$ as $n\to\infty$ and $\lim_{n\to\infty} f(s_n)=\lim_{n\to\infty} o=0$.

Thus $\{r_n\}$ and $\{s_n\}$ are two different sequences in $(x,\infty)$ such that $r_n \to x$ and $s_n \to x$ as $n\to\infty$ but $\lim_{n\to\infty} f(s_n)=0\neq 1=\lim_{n\to\infty} f(r_n)$. This shows that $f(x+)$ does not exist and hence $f$ has a discontinuity of second kind at $x$. Hence $f$ has a discountinuity of second kind at every point in $\mathbb{R}$.

**13.1.17 Problem :** Define $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=x$ if $x$ is rational and $f(x)=0$ if $x$ is irrational. Then show that $f$ is continuous at $x=0$ and has a discontinuity of the second kind at every other point in $\mathbb{R}$.

**Solution :** First we show that $f$ is continuous at $x=0$.

Let $\epsilon>0$. Take $\delta=\epsilon$.

Suppose $y \in \mathbb{R}$ such that $|y-0|<\delta \Rightarrow |y|<\in$

Consider $|f(y)-f(0)| = |f(y)-0| = |f(y)| = |y|$ or 0 according as $y$ is rational or $y$ is irrational. This implies that $|f(y)-f(0)|<\in$.

Therefore $f$ is continuous at $x=0$.

Suppose $x \in \mathbb{R}$ such that $x \neq 0$. For each positive integer $n$, consider $\left(x, x+\frac{1}{n}\right)$. Choose a rational number $r_n$ in $\left(x, x+\frac{1}{n}\right)$. Then $\{r_n\}$ is a sequence of rational numbers in $(x,\infty)$ such that $r_n \to x$ as $n \to \infty$.

Consider $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = x$

For each positive integer $n$, choose an irrational number $s_n$ in $\left(x, x+\frac{1}{n}\right)$. Then $\{s_n\}$ is a sequence of irrational numbers in $(x,\infty)$ such that $s_n \to x$ as $n \to \infty$ and $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 0 = 0.$

So $\{r_n\}$ and $\{s_n\}$ are two different sequences in $(x,\infty)$ such that $r_n \to x$ and $s_n \to x$ as $n \to \infty$ but $\lim_{n\to\infty} f(s_n) = 0 \neq x = \lim_{n\to\infty} f(r_n)$. This shows that $f(x+)$ does not exist and hence $f$ has a discontinuity of second kind at $x$. Thus $f$ is continuous at $x=0$ and $f$ has a discontinuity of second kind at every point $x \neq 0$.

## 13.2 SHORT ANSWER QUESTIONS

**13.2.1 :** When do you say that a real valued function $f$ defined on $(a,b)$ has a discontinuity of first kind ?

**13.2.2 :** When do you say that a real valued function $f$ defined on $(a,b)$ has a discontinuity of second kind ?

**13.2.3 :** Define $f:\mathbb{R} \to \mathbb{R}$ as $f(x)=x$ if $x$ is rational and $f(x)=0$. If $x$ is irrational. Then show that $f$ is continuous at $x=0$.

## 13.3 MODEL EXAMINATION QUESTIONS

**13.3.1 :** Let $f$ be a monotonically increasing function defined on $(a,b)$. Then show that $f(x+)$ and $f(x-)$ exist at every point $x$ of $(a,b)$. More precisely,

$$\sup_{a<t<x} f(t)=f(x-)\le f(x)\le f(x+)=\inf_{x<t<b} f(t)$$

**13.3.2 :** Let $f$ be monotonic on $(a,b)$. Then show that the set of points of $(a,b)$ at which $f$ is discontinuous is atmost countable.

**13.3.3 :** Define $f:(0,2)\to\mathbb{R}$ as $f(x)=|\text{if } 0<x\le 1|$ and $f(x)=2$ if $1<x<2$. Then show that $f$ is continuous at every point $x\ne 1$ and $f$ has a discontinuity of first kind at $x=1$.

**13.3.4 :** Define $f:\mathbb{R}\to\mathbb{R}$ as $f(x)=1$ if $x$ is a rational number and $f(x)=0$ if $x$ is an irrational number. Then show that $f$ has a discontinuity of second kind at every point $x\in\mathbb{R}$.

## 13.4 EXERCISES

**13.4.1 :** Suppose $X$, $Y$ and $Z$ are metric spaces and $Y$ is compact. Let $f$ map $X$ into $Y$; let $g$ be a continuous one - to - one mapping of $Y$ into $Z$, and put $h(x)=g(f(x))$ for all $x\in X$. Prove that $f$ is uniformly continuous if $h$ is uniformly continuous.

## 13.5 ANSWERS TO SHORT ANSWER QUESTIONS :

1. For 13.2.1, see definition 13.1.5
2. For 13.2.2, see definition 13.1.6
3. For 13.2.3, see definition 13.1.7

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin

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Lesson - 14

# THE RIEMANN - STIELTJES INTEGRAL - THE DEFINITION AND EXISTANCE OF THE INTEGRAL

## 14.0 INTRODUCTION

In this lesson, the Riemann integral of a bounded real valued function is defined. A necessary and sufficient condition that a function to be Riemann integrable is proved. It is also proved that every continuous function defined on a closed interval $[a,b]$ is integrable over $[a,b]$. Further it is proved that if $f$ is monotonic on $[a,b]$ and if $\alpha$ is monotonically increasing and continuos on $[a,b]$, then $f \in R(\alpha)$.

## 14.1 THE DEFINITION AND EXISTANCE OF THE INTEGRAL

**14.1.1 Definition :** Let $[a,b]$ be an interval. By a partition $P$ of $[a,b]$, we mean a finite set $P$ of points $x_0, x_1, x_2, \cdots\cdots, x_n$ such that $a = x_0 < x_1 < \cdots\cdots < x_{n-1} < x_n = b$.

Put $\Delta x_i = x_i - x_{i-1},\ 1 \le i \le n$.

Clearly, $\Delta x_i$ is the length of the sub interval $[x_{i-1}, x_i]$

**14.1.2 Definition :** Let $f$ be a bounded real valued function defined on $[a,b]$. Corresponding to each partition $P = \{x_0, x_1, \ldots\ldots\ldots, x_{n-1}, x_n\}$ of $[a,b]$, we put $M_i = Sup\{f(x) / x_{i-1} \le x \le x_i\}$, and $m_i = Inf\{f(x) / x_{i-1} \le x \le x_i\},\ 1 \le i \le n$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i ;\quad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\text{Put } \overline{\int_a^b} f\, dx = Inf\ U(p,f) \text{ ---------------- (1)}$$

and $\underline{\int_a^b} f\, dx = Sup\ L(p,f)$ --------------- (2)

where the $Inf$ and the $Sup$ are taken over all partitions $P$ of $[a,b]$. $\overline{\int_a^b} f\, dx$ is called the upper Riemann integral of $f$ and $\underline{\int_a^b} f\, dx$ is called the lower Riemann integral of $f$ over $[a,b]$. If $\overline{\int_a^b} f\, dx = \underline{\int_a^b} f\, dx$, then we say that $f$ is Riemann integrable over $[a,b]$ and we denote the set of all Riemann integrable functions by $\mathscr{R}$ and we denote the common value of (1) and (2) by $\int_a^b f\, dx$ or by $\int_a^b f(x)\, dx$.

**14.1.3 Theorem :** The upper and lower Riemann integrals always exist for every bounded function.

**Proof :** Let $f$ be a bounded real valued function defined on $[a,b]$. Then there exist two numbers $m$ and $M$ such that

$$m \le f(x) \le M \text{ for all } x \in [a,b].$$

Let $P = \{x_0, x_1, \cdots\cdots\cdots, x_n\}$ be any partition of $[a,b]$ and

put $m_i =$ Infimum $\{f(x)/x_{i-1} \le x \le x_i\}$ and

$M_i =$ Supremum $\{f(x)/x_{i-1} \le x \le x_i\}$ for $1 \le i \le n$.

Then $m \le m_i \le M_i \le M$ for $1 \le i \le n$. This implies

$\sum_{i=1}^{n} m \Delta x_i \le \sum_{i=1}^{n} m_i\, \Delta x_i \le \sum_{i=1}^{n} M_i\, \Delta x_i \le \sum_{i=1}^{n} M\, \Delta x_i$ and hence

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

This shows that $\{L(P,f)/P$ is a partition of $[a,b]\}$ and $\{U(P,f)/P$ is a partition of $[a,b]\}$ are bounded sets. Therefore

$Sup\{L(P,f)/P$ is a partition of $[a,b]\}$ and

$Inf\{U(P,f)/P$ is a partition of $[a,b]\}$ exist. That is

$\underline{\int_{a}^{b}} f\,dx$ and $\overline{\int_{a}^{b}} f\,dx$ exist.

Thus the lower and upper Riemann integrals of a bounded function always exist.

**14.1.4 Definition :** Let $f$ be a bounded real valued function defined on $[a,b]$ and let $\alpha$ be a monotonically increasing function on $[a,b]$ (Then $\alpha$ is bounded on $[a,b]$). For each partition $P=\{x_0,x_1,x_2,\cdots\cdots,x_n\}$ of $[a,b]$, we write $\Delta_{\alpha_i}=\alpha(x_i)-\alpha(x_{i-1})$. Since $\alpha$ is monotonicaly increasing on $[a,b]$, $\Delta\alpha_i \geq 0$ for $1\leq i\leq n$.

Define $M_i = Sup\{f(x)/x\in[x_{i-1},x_i]\}$ and

$m_i = Inf\{f(x)/x\in[x_{i-1},x_i]\}$ for $1\leq i\leq n$.

$$U(P,f,\alpha)=\sum_{i=1}^{n} M_i\,\Delta\,\alpha_i \text{ and } L(P,f,\alpha)=\sum_{i=1}^{n} m_i\,\Delta\,\alpha_i$$

The sums $U(P,f,\alpha)$ and $L(P,f,\alpha)$ are respectively called the upper and lower Riemann - Stieltjes sums of $f$ with respect to $\alpha$ corresponding to the partition $P$.

Now, define $\overline{\int_{a}^{b}} f\,d\alpha = Inf\{U(P,f,\alpha)/P$ is a partition of $[a,b]\}$

and $\underline{\int_{a}^{b}} f\,d\alpha = Sup\{L(P,f,\alpha)/P$ is a partition of $[a,b]\}$

Then $\overline{\int_{a}^{b}} f\,d\alpha$ is called the upper Riemann - Stieltjes integral of $f$ with respect to $\alpha$ over

16.

$[a,b]$ and $\underline{\int_a^b} f\, d\alpha$ is called the lower Riemann - Stieltjes integral of $f$ with respect to $\alpha$ over $[a,b]$.

If $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$, we denote the common value by $\int_a^b f\, d\alpha$ or $\int_a^b f(x)\, d\alpha(x)$. $\int_a^b f\, d\alpha$ is called Riemann - Stieltjes integral of $f$ with respect to $\alpha$ over $[a,b]$. If $\int_a^b f\, d\alpha$ exists, that is, $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$, we say that $f$ is integrable with respect to $\alpha$, in the Riemann sense.

We denote the set of all Riemenn - Stieltjes integrable functions with respect to $\alpha$ by $\mathscr{R}(\alpha)$.

Note that, by taking $\alpha(x) = x$ for all $x \in [a, b]$, the Riemann integral is seen to be a special case of the Riemann - Stieltjes integral.

**14.1.5 Definition :** Let $P$ be a partition of $[a,b]$. A partition $P^*$ of $[a,b]$ is called a refinement of $P$ if $P^*$ contains $P$ (i.e., if every point of $P$ is a point of $P^*$).

Given two partitions $P_1$ and $P_2$ of $[a,b]$, we say that $P^*$ is their common refinement if $P^* = P_1 \cup P_2$.

**14.1.6 Theorem :** If $P^*$ is a refinement of $P$, then $L(P,f,\alpha) \le L(P^*,f,\alpha)$ and $U(P^*,f,\alpha) \le U(P,f,\alpha)$.

**Proof :** Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$ and $P^*$ be a refinement of $P$.

First suppose that $P^*$ contains just one point more than $P$. Let this extra point be $x^*$ and suppose $x_{i-1} < x^* < x_i$ for some $i$ such that $1 \le i \le n$. Then $P^* = \{x_0, x_1, \cdots, x_{i-1}, x^*, x_i, \cdots, x_n\}$.

Write $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$

$$W_1 = Inf\left\{f(x)/x \in \left[x_{i-1}, x^*\right]\right\}$$

and $W_2 = Inf\left\{f(x)/x \in \left[x^*, x_i\right]\right\}$

Then clearly $W_1 \geq m_i$ and $W_2 \geq m_i$.

Consider $L\left(P^*, f, \alpha\right) - L(P, f, \alpha)$

$$= m_1 \Delta\alpha_1 + m_2 \Delta\alpha_2 + \cdots + m_{i-1} \Delta_{\alpha_{i-1}} + W_1\left[\alpha\left(x^*\right) - \alpha\left(x_{i-1}\right)\right] +$$

$$W_2\left[\alpha\left(x_i\right) - \alpha\left(x^*\right)\right] + \cdots + m_n \Delta\alpha_n - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \left(W_1 - m_i\right)\left[\alpha\left(x^*\right) - \alpha\left(x_{i-1}\right)\right] + \left(W_2 - m_i\right)\left[\alpha\left(x_i\right) - \alpha\left(x^*\right)\right] \geq 0$$

That is $L\left(P^*, f, \alpha\right) - L(P, f, \alpha) \geq 0$ and hence $L(P, f, \alpha) \leq L\left(P^*, f, \alpha\right)$.

If $P^*$ contains $k$ points more than $P$, we repeat this reasoning $k$ times and hence we have $L(P, f, \alpha) \leq L\left(P^*, f, \alpha\right)$.

Similarly we can show that $U\left(P^*, f, \alpha\right) \leq U(P, f, \alpha)$.

**14.1.7 Theorem :** $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha$.

**Proof :** For any partition $P$ of $[a, b]$, $L(P, f, \alpha) \leq U(P, f, \alpha)$

Let $P^*$ be the common refinement of two partitions $P_1$ and $P_2$ of $[a,b]$. By theorem 14.1.6,

$\because L(P_1, f, \alpha) \leq L\left(P^*, f, \alpha\right) \leq U\left(P^*, f, \alpha\right) \leq U(P_2, f, \alpha)$

Then $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$ -------- (1)

If $P_2$ is fixed and the Supremum is taken over all $P_1$ in (1), we have

$\underline{\int_a^b} f\, d\alpha \leq U(P_2, f, \alpha)$ ------------ (2)

If the infimum is taken over all $P_2$ in (2), we have

$$\underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha .$$

**14.1.8 Theorem :** $f \in \mathscr{R}(\alpha)$ on $[a,b]$ if and only if for every $\in > 0$ there exits a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$.

**Proof :** Assume that for each $\in > 0$, there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$.

Let $\in > 0$. Then there exists a partition $P$ of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \in \text{ ------------ (1)}$$

By theorem 14.1.7, $L(P,f,\alpha) \leq \underline{\int_a^b} f\, d\alpha \leq \overline{\int_a^b} f\, d\alpha \leq U(P,f,\alpha)$

Then $0 \leq \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha) < \in$ (By (1))

This implies $0 \leq \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha < \in$

Since $\in > 0$ is arbitrary, we have $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$

Therefore $f \in \mathscr{R}(\alpha)$.

Conversely assume that $f \in \mathscr{R}(\alpha)$

Then $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha$.

Let $\in>0$. Then $\int_a^b f\,d\alpha+\in/2$ is not a lower bound of the set $\{U(P,f,\alpha)/P$ is a partition of $[a,b]\}$.

Then there exists a partition $P_1$ of $[a,b]$ such that $U(P_1,f,\alpha)<\int_a^b f\,d\alpha+\in/2$. --------- (2)

Now $\int_a^b f\,d\alpha-\in/2$ is not an upper bound of the set $\{L(P,f,\alpha)/P$ is a partition of $[a,b]\}$.

Then there exists a partition $P_2$ of $[a,b]$ $\int_a^b f\,d\alpha-\in/2<L(P_2,f,\alpha)$. This implies that

$\int_a^b f\,d\alpha<L(P_2,f,\alpha)+\in/2$ ------ (3).

Let $P$ be the common refinement of $P_1$ and $P_2$. Then by theorem 14.1.6, and by (2) and (3), we have

$$U(P,f,\alpha)\le U(P_1,f,\alpha)<\int_a^b f\,d\alpha+\in/2<L(P_2,f,\alpha)+\in\le L(P,f,\alpha)+\in.$$

This implies that $U(P,f,\alpha)-L(P,f,\alpha)<\in$.

Thus for given $\in>0$, there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha)-L(P,f,\alpha)<\in$.

**14.1.9 Theorem :** If $U(P,f,\alpha)-L(P,f,\alpha)<\in$ for some partition $P$ of $[a,b]$ and for some $\in>0$, then $U(P^*,f,\alpha)-L(P^*,f,\alpha)<\in$ for any refinement $P^*$ of $P$.

**Proof :** Suppose $P$ is a partition of $[a,b]$ such that

$U(P,f,\alpha)-L(P,f,\alpha)<\in$ for some $\in>0$.

Let $P^*$ be any refinement of $P$.

Then by theorem 14.1.6,

$$L(P,f,\alpha)\le L(P^*,f,\alpha)\le U(P^*,f,\alpha)\le U(P,f,\alpha)$$

This implies that $U(P^*,f,\alpha)-L(P^*,f,\alpha)<\in$

**14.1.10 Theorem :** If $U(P, f, \alpha)-L(P,f,\alpha)<\in$ for a partition $P=\{x_0, x_1, x_2, \ldots\ldots, x_n\}$ of $[a,b]$ and for some $\in>0$ and if $s_i, t_i$ are arbitrary points in $[x_{i-1}, x_i]$, then $\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i<\in$.

**Proof :** Suppose $U(P,f,\alpha)-L(P,f,\alpha)<\in$ for a partition $P=\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and for some $\in>0$. Let $s_i, t_i$ be arbitrary points in $[x_{i-1}, x_i]$.

Let $m_i = Inf\{f(x)/x\in[x_{i-1}, x_i]\}$ and

$M_i = Sup\{f(x)/x\in[x_{i-1}, x_i]\}$ for $1\le i\le n$.

Then $m_i \le f(s_i) \le M_i$ and $m_i \le f(t_i) \le M_i$. This implies that $f(s_i), f(t_i)\in[m_i, M_i]$ for $1\le i\le n$. This implies that $|f(s_i)-f(t_i)| \le M_i - m_i$ for $1\le i\le n$.

Consider $\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i \le \sum_{i=1}^{n}(M_i-m_i)\Delta\alpha_i$

$$=\sum_{i=1}^{n}M_i\,\Delta\alpha_i - \sum_{i=1}^{n}m_i\,\Delta\alpha_i = U(p,f,\alpha)-L(p,f,\alpha)<\in$$

Therefore $\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i<\in$.

**14.1.11 Theorem :** If $f\in\mathscr{R}(\alpha)$ and $U(P,f,\alpha)-L(P,f,\alpha)<\in$ for a partition $P=\{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and for some $\in>0$ and if $t_i$ is an arbitrary point in $[x_{i-1}, x_i]$ for $1\le i\le n$, then

$$\left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right|<\in.$$

**Proof :** Suppose $f\in\mathscr{R}(\alpha)$.

Assume $U(P,f,\alpha)-L(P,f,\alpha)<\in$ for a partition $P=\{x_0, x_1, \ldots\ldots\ldots, x_n\}$ of $[a,b]$ and for some $\in>0$ and $t_i\in[x_{i-1}, x_i]$ for $1\le i\le n$.

Write $m_i = Inf\,\{f(x)/x\in[x_{i-1}, x_i]\}$ and $M_i = Sup\{f(x)/x\in[x_{i-1}, x_i]\}$ for $1\le i\le n$.

Now $m_i \le f(t_i) \le M_i$ for $1 \le i \le n$.

Then $\sum_{i=1}^{n} m_i \, \Delta\alpha_i \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le \sum_{i=1}^{n} M_i \, \Delta\alpha_i$. This implies that

$$L(P,f,\alpha) \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le U(P,f,\alpha) \text{ ---------- (1)}$$

Since $f \in \mathcal{R}(\alpha)$, we have $\underline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \, d\alpha = \int_a^b f \, d\alpha$. This implies that

$$L(P,f,\alpha) \le \int_a^b f \, d\alpha \le U(P,f,\alpha) \text{ ----------- (2)}$$

From (1) and (2), we have

$$\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f \, d\alpha \le U(P,f,\alpha) - L(P,f,\alpha) < \in \text{ (By assumption)}$$

and $$\int_a^b f \, d\alpha - \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \le U(P,f,\alpha) - L(P,f,\alpha) < \in \text{ (By assumption)}$$

Therefore $-\in < \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f \, d\alpha < \in$ and hence

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f \, d\alpha \right| < \in$$

**14.1.12 Theorem :** If $f$ is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

**Proof :** Suppose $f$ is continuous on $[a,b]$. Let $\in > 0$.

Since $a \le b$ and $\alpha$ is monotonically increasing on $[a,b]$, we have $\alpha(a) \le \alpha(b)$. This implies that $\alpha(b) - \alpha(a) \ge 0$.

Put $\eta_0 = \dfrac{\in}{\alpha(b)-\alpha(a)+1}$. Then $\eta_0 > 0$.

Since $f$ is continuous on $[a,b]$ and since $[a,b]$ is compact, by Theorem 12.1.14, $f$ is uniformely continuous on $[a,b]$. Then there exists $\delta > 0$ such that $|f(x)-f(t)| < \eta_0$ --------- (1),

whenever $x, t \in [a,b]$ and $|x-t| < \delta$.

Since $\delta > 0$, by Archimedean principle, there exists a positive integer $n$ such that $n\delta > b-a$.

Write $x_i = a + \dfrac{i(b-a)}{n}$ for $0 \le i \le n$.

Then $P = \{x_0, x_1, ..........., x_n\}$ is a partition of $[a,b]$ such that $\Delta x_i = x_i - x_{i-1} < \delta$ for $1 \le i \le n$.

For any $x, t \in [x_{i-1}, x_i]$, we have $|x-t| \le \Delta x_i < \delta$.

Then by (1), $|f(x)-f(t)| < \eta_0$ ----------- (2)

Write $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$

and $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Since $f$ is continuous on $[a,b]$, $f$ is also continuous on $[x_{i-1}, x_i]$. Then by theorem 12.1.5, there exists $p_i, q_i \in [x_{i-1}, x_i]$ such that $f(p_i) = m_i$ and $f(q_i) = M_i$ for $1 \le i \le n$

Since $p_i, q_i \in [x_{i-1}, x_i]$, by (2), we have

$$|f(p_i) - f(q_i)| < \eta_0 \text{ -------- (3)}$$

Consider $|M_i - m_i| = |f(p_i) - f(q_i)| < \eta_0$ for $1 \le i \le n$ (By (3))

$\Rightarrow M_i - m_i \le \eta_0$ for $1 \le i \le n$ ---------- (4)

Consider $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \le \sum_{i=1}^{n} \eta_0 \Delta\alpha_i = \eta_0 \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \eta_b(\alpha(b)-\alpha(a)) = \frac{\in(\alpha(b)-\alpha(a))}{\alpha(b)-\alpha(a)+1} < \in.$$

So for given $\in > 0$ there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$. Then by theorem 14.1.8, $f \in \mathscr{R}(\alpha)$

Thus every continuous function on $[a,b]$ is Riemann Stieltjes integrable over $[a,b]$.

**14.1.13 Theorem :** If $f$ is monotonic on $[a,b]$ and if $\alpha$ is monotnonically increasing and continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$

**Proof :** Suppose $f$ is monotonic on $[a,b]$ and $\alpha$ is monotonically increasing and continuous on $[a,b]$.

First we show that to each positive integer $n$, there exists a partition $P = \{x_0, x_1, \ldots\ldots, x_n\}$ of $[a,b]$ such that $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b)-\alpha(a)}{n}$ for $1 \le i \le n$.

Let $n$ be a positive integer.

Put $\delta = \frac{\alpha(b)-\alpha(a)}{n}$,

Write $C_i = \alpha(a) + i\delta$ for $1 \le i \le n$.

Then $C_1 = \alpha(a)+\delta; C_2 = \alpha(a) + 2\delta, \cdots\cdots\cdots\cdots$

$C_n = \alpha(a)+n\delta = \alpha(a)+\alpha(b)-\alpha(a) = \alpha(b)$

Now $\alpha(a) < C_1 < C_2 < \cdots\cdots\cdots < C_n = \alpha(b)$.

Since $\alpha$ is continuous on $[a,b]$ and $\alpha(a) < C_1 < \alpha(b)$, by Theorem 12.1.18, there exists $x_1 \in (a,b)$ such that $\alpha(x_1) = C_1$.

Now $C_1 = \alpha(x_1) < C_2 < \alpha(b)$. Again by Theorem 12.1.8, there exists $x_2 \in (x_1, b)$ such that $\alpha(x_2) = C_2$.

Continuing in this way for $i = 3, 4, \cdots\cdots, n-1$, we have $x_3, x_4, \cdots\cdots, x_{n-1}$ such that $a < x_1 < x_2 < \cdots\cdots < x_{n-1} < b$ and $\alpha(x_i) = C_i$ for $1 \le i \le n-1$.

Put $x_0 = a$ and $x_n = b$. Then $P = \{x_0, x_1, \cdots\cdots, x_n\}$ is a partition of $[a,b]$ and

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$$= C_i - C_{i-1} = \delta = \frac{\alpha(b) - \alpha(a)}{n}.$$

Therefore $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ for $1 \le i \le n$.

So for each positive integer $n$, we have a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \text{ for } 1 \le i \le n \text{ ------------- (1)}$$

Let $\epsilon > 0$.

Since $f$ is monotonic on $[a,b]$, we have either $f$ is monotonically increasing or monotonically decreasing.

**Case (i) :** Suppose $f$ is monotonically increasing. Then $f(a) \le f(b)$.

Since $\epsilon > 0$, by Archimedean principle, there exists a positive integer $n$ such that $n\epsilon > (\alpha(b) - \alpha(a))(f(b) - f(a))$.

This implies $\dfrac{(\alpha(b) - \alpha(a))}{n}(f(b) - f(a)) < \epsilon$ -------------- (2)

For this positive integer $n$, by (1), there exists a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ such that $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ for $1 \le i \le n$.

Put $m_i = Inf\{f(x) / x \in [x_{i-1}, x_i]\}$ and $M_i = Sup\{f(x) / x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Since $f$ is monotonically increasing, we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$ for $1 \le i \le n$.

Consider $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$

$$= \sum_{i=1}^{n} \Delta\alpha_i (M_i - m_i) = \sum_{i=1}^{n} \left(\frac{\alpha(b) - \alpha(a)}{n}\right)\left(f(x_i) - f(x_{i-1})\right)$$

$$=\left(\frac{\alpha(b)-\alpha(a)}{n}\right)\cdot\sum_{i=1}^{n}\left(f(x_i)-f(x_{i-1})\right)$$

$$=\left(\frac{\alpha(b)-\alpha(a)}{n}\right)\left(f(b)-f(a)\right)<\in \text{ (by (2))}$$

Therefore $U(P,f,\alpha)-L(P,f,\alpha)<\in$.

**Case (ii) :** Suppose $f$ is monotonically decreasing.

Then $f(b)\leq f(a)$.

Since $\in>0$, by Archimedean principle, there exists a positive integer $n$ such that $\frac{(\alpha(b)-\alpha(a))}{n}(f(a)-f(b))<\in$ --------- (3)

For this positive integer $n$, by(1) there exists a partition $P=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ such that $\Delta\alpha_i=\frac{\alpha(b)-\alpha(a)}{n}$ for $1\leq i\leq n$.

Since $f$ is monotonically decreasing, $m_i=f(x_i)$ and $M_i=f(x_{i-1})$ for $1\leq i\leq n$.

Consider $U(P,f,\alpha)-L(P,f,\alpha)=\sum_{i=1}^{n}M_i\,\Delta\alpha_i-\sum_{i=1}^{n}m_i\,\Delta\alpha_i$

$$=\sum_{i=1}^{n}\Delta\alpha_i\,(M_i-m_i)=\sum_{i=1}^{n}\left(\frac{\alpha(b)-\alpha(a)}{n}\right)\left(f(x_{i-1})-f(x_i)\right)$$

$$=\left(\frac{\alpha(b)-\alpha(a)}{n}\right)\cdot\sum_{i=1}^{n}\left(f(x_{i-1})-f(x_i)\right)$$

$$=\left(\frac{\alpha(b)-\alpha(a)}{n}\right)\left(f(a)-f(b)\right)<\in \text{ (by (3))}$$

Thus in any case, for $\in>0$, there exists a partition $P$ of $[a,b]$ such that

$U(P,f,\alpha)-L(P,f,\alpha)<\in$

Then by theorem 14.1.8, $f\in\mathscr{R}(\alpha)$

**14.1.14 Theorem :** Suppose $f$ is bounded on $[a,b]$, $f$ has only finitely many points of discontinuity on $[a,b]$ and $\alpha$ is continuous at every point at which $f$ is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

**Proof :** Suppose $f$ is bounded on $[a,b]$ and $f$ has only finitely many points of discontinuity on $[a,b]$ and $\alpha$ is continuous at every point at which $f$ is discountinuous.

Let $\in > 0$. Put $M = Sup\{|f(x)| / x \in [a,b]\}$.

Let $E$ be the set of points at which $f$ is discountinuous. Then $E$ is finite. So let $E = \{c_1, c_2, \cdots\cdots, c_k\}$ and assume that $c_1 < c_2 < \cdots\cdots < c_k$.

Write $\in_1 = \dfrac{\in}{\alpha(b) - \alpha(a) + 4kM + 1}$. Then $\in_1 > 0$.

Since $\alpha$ is continuous at $c_j$, there exists $\delta_j > 0$ such that $|\alpha(c_j) - \alpha(x)| < \in_1$ whenever $|c_j - x| < \delta_j$ for $j = 1, 2, \cdots, k$ --------- (1)

Take $\delta_0 < \min\{\delta_j, c_{j+1} - c_j / 1 \le j \le k\}$

Now choose $u_j$ and $v_j$ such that $c_j - \dfrac{\delta_0}{2} < u_j < c_j < v_j < c_j + \dfrac{\delta_0}{2}$ for $1 \le j \le n$.

Now we will show that $[u_j, v_j]$'s are disjoint intervals.

For this, it is enough if we show that $v_j < u_{j+1}$ for $1 \le j \le k$.

Now consider $c_{j+1} - c_j > \delta_0$. Then $c_{j+1} - \dfrac{\delta_0}{2} > c_j + \dfrac{\delta_0}{2}$.

This implies $v_j < c_j + \dfrac{\delta_0}{2} < c_{j+1} - \dfrac{\delta_0}{2} < u_{j+1}$ and hence $v_j < u_{j+1}$.

This shows that $[u_j, v_j]$'s are disjoint.

Since $|c_j - v_j| < \delta_j$ and $|c_j - u_j| < \delta_j$, by (1), we have

$$|\alpha(c_j) - \alpha(u_j)| < \in_1 \text{ and } |\alpha(c_j) - \alpha(v_j)| < \in_1 \text{ for } 1 \le j \le k.$$

This implies that $|\alpha(v_j) - \alpha(u_j)| < 2\in_1$ for $1 \le j \le k$.

Consider $\sum_{i=1}^{k}\left(\alpha(v_j)-\alpha(u_j)\right)<\sum_{i=1}^{k}2\in_1=2k\in_1$

So, $\{[u_j, v_j]/1\le j\le k\}$ is a finite class of disjoint intervals such that $[u_j, v_j]\subseteq[a,b]$ and this class covers $E$ and the sum of the corresponding differences $\alpha(v_j)-\alpha(u_j)$ is less than $2k\in_1$. Also it is clear that every point of $E\cap[a,b]$ lies in the interior of some $[u_j, v_j]$.

Write $K=[a,b]\setminus\left(\bigcup_{i=1}^{k}(u_j, v_j)\right)$

Then $K=[a,u_1]\cup[v_1,u_2]\cup[v_2,u_3]\cup\cdots\cdots\cup[v_k,b]$

It is clear that $K$ is compact and $f$ is continuous on $K$.

By theorem 12.1.14, $f$ is uniformly continuous on $K$. Then there exists a $\delta>0$ such that $|f(s)-f(t)|<\in_1$ whenever $s,t\in K$ with $|s-t|<\delta$.

Now form a partition $P=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ as follows : Each $u_j$ occurs in $P$. Each $v_j$ occurs in $P$. No point of any segment $(u_j, v_j)$ occurs in $P$. If $x_{i-1}$ is not one of the $u_j$, then $\Delta x_i=x_i-x_{i-1}<\delta$.

Write $M_i=Sup\{f(x)/x\in[x_{i-1},x_i]\}$ and $m_i=Inf\{f(x)/x\in[x_{i-1},x_i]\}$ for $1\le i\le n$.

Assume $x_{ij}=v_j$ for $1\le j\le k$.

If $x_{i_1}=v_1$ and $x_{i_2}=v_2$, by the definition of $P$, $u_1=x_{i_1}-1$ and $u_2=x_{i_2}-1,\cdots, etc.$

Therefore for any $r\in\{1,2,\cdots\cdots,n\}$, $x_r\ne x_{ij}$ implies that $x_r\ne v_j$ and $x_{r-1}\ne u_j$.

Also for any $r\in\{1,2,\cdots,n\}$ $-M\le m_r\le M_r\le M$. This implies that $M_r-m_r\le 2M$.

Let $r\in\{1,2,\cdots\cdots,n\}\setminus\{i_1,i_2,\cdots\cdots\cdots,i_k\}$. Then $x_{r-1},x_r\in K$ and $|x_r-x_{r-1}|<\delta$ (by the definition of $P$).

Since $f$ is continuous on $[x_{r-1},x_r]$, by theorem 12.1.5, there exist $s_r,t_r\in[x_{r-1},x_r]$ such that $f(s_r)=M_r$ and $f(t_r)=m_r$. Consider $|s_r-t_r|\le|x_r-x_{r-1}|<\delta$. This implies that $|f(s_r)-f(t_r)|<\in_1$. Consequently $M_r-m_r=|M_r-m_r|<\in_1$.

Consider $\Delta\alpha_{i_1} = \alpha\left(x_{i_1}\right) - \alpha\left(x_{i_1} - 1\right) = \alpha(v_1) - \alpha(u_1) < 2\in_1$

$$\Delta\alpha_{i_2} = \alpha\left(x_{i_2}\right) - \alpha\left(x_{i_2} - 1\right) = \alpha(v_2) - \alpha(u_2) < 2\in_1$$

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$$\Delta\alpha_{i_k} = \alpha(v_k) - \alpha(u_k) < 2\in_1$$

So for any $r \in \{i_1, i_2, \cdots, i_k\}$, $\Delta\alpha_r < 2\in_1$

Consider $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i - \sum_{i=1}^{n} m_i\,\Delta\alpha_i$

$$= \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \sum_{r\in\{1,2,\cdots,n\}\setminus\{i_1,i_2,\cdots,i_k\}}(M_r - m_r)\Delta\alpha_r +$$

$$+\sum_{r\in\{i_1,\, i_2,\ldots,i_k\}}(M_r - m_r)\Delta\alpha_r < \in_1 \cdot \sum_{i=1} \Delta\alpha_i + K.2M.2\in_1$$

$$= \in_1(\alpha(b) - \alpha(a)) + 4kM\in_1 < \in_1\left((\alpha(b) - \alpha(a)) + 4kM + 1\right) = \in$$

Thus for $\in > 0$, there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$. Then by theorem 14.1.8, $f \in \mathscr{R}(\alpha)$.

**14.1.15 Note :** If $f$ and $\alpha$ have a common point of discontinuity, then $f$ need not be in $\mathscr{R}(\alpha)$.

**Example :** Define $\alpha:[-1, 1] \to \mathbb{R}$ by $\alpha(x) = 0$ if $x < 0$ and $\alpha(x) = 1$ if $x > 0$ and $\alpha(0) = \frac{1}{2}$. Let $f$ be a bounded function on $[-1,1]$ such that $f$ is not continuous at 0.

Now we will show that $f \notin \mathscr{R}(\alpha)$ on $[-1,1]$.

It is easy to verify that $\alpha$ is discontinuous at 0. So $\alpha$ and $f$ are discontinuous at 0.

If possible suppose that $f \in R(\alpha)$ on $[-1, 1]$.

Let $\in > 0$.

Since $f \in \mathcal{R}(\alpha)$, there exists a partition $P=\{x_0, x_1, ...., x_n\}$ of $[-1, 1]$ such that

$$U(P,f,\alpha)-L(P,f,\alpha) < \in/2 \text{ ------------ (1)}$$

Now, either $0 \in P$ or $0 \notin P$.

Suppose $0 \notin P$. Then $x_{i-1} < 0 < x_i$ for some $i$ such that $1 \le i \le n$.

Then $\Delta\alpha_j = 0$ for $1 \le j \le i-1$; $\Delta\alpha_j = 0$ for $i+1 \le j \le n$.

and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = 1 - 0 = 1$

write $M_j = Sup\{f(x)/x \in [x_{j-1}, x_j]\}$ and $m_j = Inf\{f(x)/x \in [x_{j-1}, x_j]\}$ for $1 \le j \le n$.

Consider $U(P,f,\alpha) = \sum_{j=1}^{n} M_j \Delta\alpha_j = M_i$

and $L(P,f,\alpha) = \sum_{j=1}^{n} m_j \Delta\alpha_j = m_i$.

By (1), we have $M_i - m_i = U(P,f,\alpha) - L(P,f,\alpha) < \in/2$ ---------- (2)

Choose $\delta$ such that $0 < \delta < \min\{x_i, -x_{i-1}\}$. Then $x_{i-1} < -\delta < \delta < x_i$.

Suppose $x \in [-1, 1]$ such that $|x-0| < \delta$. Then $-\delta < x < \delta$. Consequently $x_{i-1} < x < x_i$. This implies that $m_i \le f(x) \le M_i$ ------------- (3)

Since $x_{i-1} < 0 < x_i$, we have $m_i \le f(0) \le M_i$ ------------ (4)

From (2), (3) and (4), $|f(x) - f(0)| \le M_i - m_i < \in$.

Therefore $f$ is continuous at 0, which a contradiction to that fact that $f$ is not continuous at 0.

Suppose $0 \in P$. Then $x_i = 0$ for some $i$ such that $1 \le i \le n$.

For $1 \le j \le i-1$, $\Delta\alpha_j = 0$ and for $i+2 \le j \le n$, $\Delta\alpha_j = 0$

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha(0) - \alpha(x_{i-1}) = \frac{1}{2} - 0 = \frac{1}{2}$$

$\Delta\alpha_{i+1}=\alpha(x_{i+1})-\alpha(x_i)=1-\frac{1}{2}=\frac{1}{2}$.

Consider $U(P,f,\alpha)=\sum_{j=1}^{n} M_j \Delta\alpha_j=\frac{1}{2}M_i+\frac{1}{2}M_{i+1}=\frac{1}{2}(M_i+M_{i+1})$

and $L(P,f,\alpha)=\frac{1}{2}(m_i+m_{i+1})$

Consider $\frac{1}{2}(M_i+M_{i+1})-\frac{1}{2}(m_i+m_{i+1})=U(P,f,\alpha)-L(P,f,\alpha)<\in/2$ (By (1))

This implies that $(M_i-m_i)+(M_{i+1}-m_{i+1})<\in$ ---------- (5)

Choose $\delta$ such that $0<\delta<\min\{-x_{i-1},x_{i+1}\}$. Then $x_{i-1}<-\delta<\delta<x_{i+1}$.

Suppose $x\in[-1, 1]$ such that $|x-0|<\delta$. Then $-\delta<x<\delta$. This implies that $x_{i-1}<x<x_{i+1}$.

If $x_{i-1}<x<0=x_i$, then $m_i\le f(x)\le M_i$ and $m_i\le f(0)\le M_i$. This implies that $|f(x)-f(0)|$

$\le M_i-m_i\le(M_i-m_i)+(M_{i+1}-m_{i+1})<\in$ (By (5))

If $x_i=0<x<x_{i+1}$, then $m_{i+1}\le f(x)\le M_{i+1}$ and $m_{i+1}\le f(0)\le M_{i+1}$. This implies that

$|f(x)-f(0)|\le M_{i+1}-m_{i+1}<\in$ (By (5)).

Therefore $f$ is continuous at 0, which is a contradiction. So in any case we have a contradiction.

Hence $f\notin\mathcal{R}(\alpha)$ on $[-1, 1]$.

**14.1.16 Theorem :** Suppose $f\in\mathcal{R}(\alpha)$ on $[a,b]$, $m\le f(x)\le M$ for all $x\in[a,b]$, $\phi$ is continuous on $[m,M]$ and $h(x)=\phi(f(x))$ on $[a,b]$. Then $h\in\mathcal{R}(\alpha)$ on $[a,b]$.

**Proof :** Suppose $f\in\mathcal{R}(\alpha)$ on $[a,b]$, $m\le f(x)\le M$ for all $x\in[a,b]$, $\phi$ is continuous on $[m, M]$ and $h(x)=\phi(f(x))$ on $[a,b]$.

Let $\in>0$.

Since $\phi$ is continuous on $[m,M]$, we have $\phi$ is bounded on $[m,M]$. So put

$k = Sup\{\phi(t)/t \in [m,M]\}$. Write $\epsilon_1 = \dfrac{\epsilon}{\alpha(b)-\alpha(a)+2k+1}$.

Since $\phi$ is continuous on $[m, M]$ and since $[m, M]$ is compact, we have $\phi$ is uniformly continuous on $[m, M]$. Then there exists $\delta > 0$ such that $|\phi(s)-\phi(t)| < \epsilon_1$ whenever $s,t \in [m,M]$ with $|s-t| < \delta_0$.

Choose $\delta$ such that $0 < \delta < \min\{\delta_0, \epsilon_1\}$

Then for any $s,t \in [m,M]$ with $|s-t| < \delta$, we have

$|\phi(s)-\phi(t)| < \epsilon_1$ ----------- (1)

Since $f \in \mathscr{R}(\alpha)$ on $[a,b]$, there exists a partition $P = \{x_0, x_1, ..........., x_n\}$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$ ---------- (2)

Write $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$

and $M_i^* = Sup\{h(x)/x \in [x_{i-1}, x_i]\}$ and $m_i^* = Inf\{h(x)/x \in [x_{i-1}, x_i]\}$

for $1 \le i \le n$ ( since $h$ is bounded).

Put $A = \{i \in \{1,2,......,n\} / M_i - m_i < \delta\}$ and

$B = \{i \in \{1,2,........,n\} / M_i - m_i \ge \delta\}$. Then $A \cup B = \{1,2,......,n\}$.

First we show that $|f(x)-f(y)| \le M_i - m_i$ for all $x, y \in [x_{i-1}, x_i]$ for $1 \le i \le n$.

Let $x, y \in [x_{i-1}, x_i]$. Then $m \le m_i \le f(x) \le M_i \le M$ and $m \le m_i \le f(y) \le M_i \le M$. This implies that $f(x), f(y) \in [m,M]$ and $|f(x)-f(y)| \le M_i - m_i$.

Next we will show that $i \in A$ implies that $M_i^* - m_i^* \le \epsilon_1$. Suppose $i \in A$ and $x, y \in [x_{i-1}, x_i]$

Then $|f(x)-f(y)| \le M_i - m_i < \delta$ and $f(x), f(y) \in [m,M]$

This implies that $|\phi(f(x)) - \phi(f(y))| < \epsilon_1$ (By (1))

Consequently $|h(x)-h(y)| < \epsilon_1$ ------ (3)

Consider $M_i^* - m_i^* = Sup\{h(x)/x \in [x_{i-1}, x_i]\} - Inf\{h(y)/y \in [x_{i-1}, x_i]\}$

$$= Sup\{h(x)/x \in [x_{i-1}, x_i]\} + Sup\{-h(y)/y \in [x_{i-1}, x_i]\}$$

$$= Sup\{h(x) - h(y)/x, y \in [x_{i-1}, x_i]\} \leq \epsilon_1 \text{ (By (3))}$$

So $i \in A$ implies that $M_i^* - m_i^* \leq \epsilon_1$ ---------- (4)

Next we will show that $i \in B$ implies that $M_i^* - m_i^* \leq 2k$. Suppose $i \in B$.

For any $x, y \in [x_{i-1}, x_i]$, consider $|h(x) - h(y)|$

$$= |\phi(f(x)) - \phi(f(y))| \leq |\phi(f(x))| + |\phi(f(y))| \leq k + k = 2k$$

Therefore $M_i^* - m_i^* = Sup\{h(x) - h(y)/x, y \in [x_{i-1}, x_i]\} \leq 2k$ ------ (5)

Consider $\delta \sum_{i \in B} \Delta\alpha_i = \sum_{i \in B} \delta \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i$

$$\leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i = U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \text{ ( By (2)).}$$

This implies that $\delta \sum_{i \in B} \Delta\alpha_i < \delta^2$ and hence $\sum_{i \in B} \Delta\alpha_i < \delta$ --------- (6)

Now consider $U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i=1}^{n} \left(M_i^* - m_i^*\right)\Delta\alpha_i$

$$= \sum_{i \in A} \left(M_i^* - m_i^*\right)\Delta\alpha_i + \sum_{i \in B} \left(M_i^* - m_i^*\right)\Delta\alpha_i$$

$$\leq \epsilon_1 \sum_{i \in A} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i \quad \text{(by (4) and (5))}$$

$$\leq \epsilon_1 \cdot \sum_{i=1}^{n} \Delta\alpha_i + 2k \cdot \sum_{i \in B} \Delta\alpha_i < \epsilon_1 (\alpha(b) - \alpha(a)) + 2k\delta \quad \text{(By (6))}$$

$$< \epsilon_1 (\alpha(b) - \quad 2k\,\epsilon_1 = \epsilon_1 (\alpha(b) - \alpha(a) + 2k)$$

$<\in_1 (\alpha(b)-\alpha(a)+2k+1)=\in$.

So for given $\in>0$, there exists a partition $P$ of $[a,b]$ such that

$U(P,h,\alpha)-L(P,h,\alpha)<\in$ and hence $h\in\mathcal{R}(\alpha)$ on $[a,b]$.

**14.1.17 Problem :** If $f(x)=0$ for all irrational $x$ and $f(x)=1$ for all rational $x$, prove that $f\notin\mathcal{R}$ on $[a,b]$ for any $a<b$.

**Solution :** Let $a,b$ be real numbers such that $a<b$.

Let $f:[a,b]\to\mathbb{R}$ be the function defined by

$f(x)=0$ if $x$ is irrational and

$f(x)=1$ if $x$ is rational.

Let $P=\{x_0,x_1,.........,x_n\}$ be any partition of $[a,b]$.

Write $M_i=Sup\{f(x)/x\in[x_{i-1},x_i]\}$ and $m_i=\inf\{f(x)/x\in[x_{i-1},x_i]\}$ for $1\le i\le n$.

Then $M_i=1$ and $m_i=0$ for $1\le i\le n$.

Consider $U(P,f)=\sum_{i=1}^{n} M_i\,\Delta x_i=b-a$ and $L(P,f)=\sum_{i=1}^{n} m_i\,\Delta x_i=0$

Then $\underline{\int_a^b} f\,dx=Sup\{L(P,f)/P \text{ is a partition of } [a,b]\}=0$

and $\overline{\int_a^b} f\,dx=Inf\{U(P,f)/P \text{ is a partition of } [a,b]\}=b-a$

Therefore $\underline{\int_a^b} f\,dx\neq\overline{\int_a^b} f\,dx$ and hence $f\notin\mathcal{R}$ on $[a,b]$

**14.1.18 Problem :** Suppose $f\ge 0$, $f$ is continuous on $[a,b]$ and $\int_a^b f(x)\,dx=0$. Prove that $f(x)=0$ for all $x\in[a,b]$.

**Solution :** Suppose $f \geq 0$, $f$ is continuous on $[a,b]$ and $\int_a^b f(x)dx = 0$.

If possible suppose that $f(c) \neq 0$ for some $c \in [a,b]$. Then $f(c) > 0$. Since $f$ is continuous on $[a,b]$, $f$ is continuous at $c$. Then there exists $\delta > 0$ such that $|f(x) - f(c)| < f(c)$ --------- (1), whenever $x \in [a,b]$ with $|x - c| < \delta$.

Now we will show that $f(x) \neq 0$ for all $x \in (c-\delta, c+\delta)$. If possible suppose that $f(x) = 0$ for some $x \in (c-\delta, c+\delta)$. Then $|x-c| < \delta$ and by (1), $|f(x) - f(c)| < f(c)$.

Since $f(x) = 0$, we have $f(c) < f(c)$, a contradiction.

So $f(x) \neq 0$ for all $x \in (c-\delta, c+\delta)$.

Since $f \geq 0$ on $[a,b]$, we have $f(x) > 0$ for all $x \in (c-\delta, c+\delta)$

This implies $\int_{c-\delta}^{c+\delta} f(x)dx > 0$ and hence $\int_a^b f(x)dx \neq 0$; a contradiction. So $f(x) = 0$ for all $x \in [a,b]$.

**14.1.19 Problem :** Suppose $\alpha$ increases on $[a,b]$ and $a < s < b$ and $\alpha$ is continuous at $s$, $f(s) = 1$ and $f(x) = 0$ if $x \neq s$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int_a^b f\,d\alpha = 0$.

**Solution :** Let $P = \{x_0, x_1, \ldots\ldots, x_n\}$ be any partition of $[a,b]$. Then $x_{i-1} < s \leq x_i$ for some $i$ such that $1 \leq i \leq n$. Write $M_j = Sup\{f(x)/x \in [x_{j-1}, x_j]\}$ and

$$m_j = Inf\{f(x)/x \in [x_{j-1}, x_j]\} \text{ for } 1 \leq j \leq n.$$

Then $M_j = 0$ for $1 \leq j \leq n$ and $j \neq i$ and $M_i = 1$ and $m_j = 0$ for $1 \leq j \leq n$.

Now $L(P, f, \alpha) = 0$ and $U(P, f, \alpha) = M_i \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$

Therefore $\int_a^b f\,d\alpha = Sup\{L(P, f, \alpha)/P \text{ is a partition of } [a, b]\} = 0$

and $\overline{\int_a^b} f\, d\alpha = Inf\{U(P,f,\alpha)/\ P \text{ is a partition of } [a,b]\} \geq 0$

Now we will show that $\overline{\int_a^b} f\, d\alpha = 0$.

If possible suppose that $\overline{\int_a^b} f\, d\alpha > 0$. Choose $\in$ such that $0 < \in < \overline{\int_a^b} f\, d\alpha$.

Since $\alpha$ is continuous at $s$, there exists $\delta > 0$ such that $a < s - \delta < s < s + \delta < b$

and $|\alpha(s) - \alpha(x)| < \frac{\in}{2}$ ------- (1)

whenever $|s - x| < \delta$.

Take $x_0 = a,\ x_1 = s - \delta/2,\ x_2 = s + \frac{\delta}{2},\ x_3 = b$. Then $p = \{x_0, x_1, x_2, x_3\}$ is a partition of $[a,b]$ and $x_1 < s < x_2$.

Clearly $M_1 = 0, M_2 = 1, M_3 = 0$.

Consider $|s - x_1| = \left|s - \left(s - \frac{\delta}{2}\right)\right| = \frac{\delta}{2} < \delta$.

Then by (1), $|\alpha(s) - \alpha(x_1)| < \in/2$.

Consider $|s - x_2| = \left|s - \left(s + \frac{\delta}{2}\right)\right| = \frac{\delta}{2} < \delta$

Then by (1), $|\alpha(s) - \alpha(x_2)| < \in/2$

Consider $\alpha(x_2) - \alpha(x_1) = |\alpha(x_2) - \alpha(x_1)|$

$$\leq |\alpha(x_2) - \alpha(s)| + |\alpha(s) - \alpha(x_1)| < \in/2 + \in/2 = \in$$

Therefore $\alpha(x_2) - \alpha(x_1) < \in$ --------- (2)

Consider $U(P,f,\alpha)=\sum_{i=1}^{3} M_i \Delta\alpha_i = M_2\,\Delta\alpha_2 = \alpha(x_2)-\alpha(x_1) < \in$ (By (2))

Thus there exists a partition $P$ of $[a,b]$ such that

$$U(p,f,\alpha) < \overline{\int_a^b} f\,d\alpha, \text{ which is a contradiction.}$$

So, $\overline{\int_a^b} f\,d\alpha = 0$ and hence $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = 0$

Consequently $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = 0$

## 14.2 SHORT ANSWER QUESTIONS

**14.2.1 :** Define the upper Riemann integral and lower Riemann integral of a bounded function $f$ defined on $[a,b]$.

**14.2.2 :** Show that $\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha$

**14.2.3 :** If $f(x)=0$ for all irrationals $x$ and $f(x)=1$ for all rationals $x$, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a<b$.

## 14.3 MODEL EXAMINATION QUESTIONS

**14.3.1 :** Show that $f \in \mathscr{R}(\alpha)$ on $[a,b]$ if and only if for every $\in > 0$ there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$.

**14.3.2 :** If $f$ is continuous on $[a,b]$ then show that $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

**14.3.3 :** If $f$ is monotonic on $[a,b]$, and if $\alpha$ is monotonically increasing and continuous on $[a,b]$, then show that $f \in \mathscr{R}(\alpha)$.

**14.3.4 :** Suppose $f$ is bounded on $[a,b]$, $f$ has finitely many points of discontinuity on $[a,b]$ and $\alpha$ is continuous at every point at which $f$ is discontinuous, then show that $f \in \mathscr{R}(\alpha)$.

**14.3.5 :** Suppose $\alpha$ increases on $[a,b]$ and $a<s<b$ and $\alpha$ is continuous at $s$, $f(s)=1$ and $f(x)=0$ if $x \neq s$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int_a^b f\, d\alpha = 0$.

## 14.4 EXERCISES

**14.4.1 :** Define $\beta : [-1, 1] \to \mathbb{R}$ as $\beta(x)=0$ if $x<0$ and $\beta(x)=1$ if $x>0$. Let $f$ be a bounded function defined on $[-1,1]$. Show that $f \in \mathscr{R}(\beta)$ if and only if $f(0+)=f(0)$ and that then $\int_{-1}^{1} f\, d\beta = f(0)$.

## 14.5 ANSWERS TO SHORT ANSWER QUESTIONS

For 14.2.1, see definition 14.1.2

For 14.2.2, see theorem 14.1.7

For 14.2.3, see problem 14.1.17

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin

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Lesson - 15

# PROPERTIES OF RIEMANN - STIELTJES INTEGRAL

## 15.0 INTRODUCTION

In this lesson the properties of Riemann - Stieltjes integral are studied. If $\mathscr{R}_\alpha(a, b)$ denotes the set of all real-valued functions $f$ defined on $[a,b]$ such that $f \in R(\alpha)$ on $[a,b]$, then it is proved that $f+g$ and $cf$ are in $\mathscr{R}_\alpha(a, b)$ for any $f, g \in \mathscr{R}_\alpha(a, b)$ and for any real number $c$. This shows that $\mathscr{R}_\alpha(a, b)$ is a vector space over the field of real numbers. Furhter it is proved that if $a<s<b$, $f$ is bounded on $[a, b]$, $f$ is continuous at $s$ and $\alpha(x)=I(x-s)$, then $\int_a^b f\, dx = f(s)$.

## 15.1 PROPERTIES OF INTEGRAL

**15.1.1 Theorem :** If $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in \mathscr{R}(\alpha)$ and $\int_a^b (f_1+f_2)d\alpha$

$$= \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

**Proof :** Suppose $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$.

Put $f = f_1 + f_2$.

Let $\in > 0$.

Since $f_1 \in \mathscr{R}(\alpha)$ and $f_2 \in \mathscr{R}(\alpha)$, by Theorem 14.1.8, there exist partitions $P_1$ and $P_2$ of $[a,b]$ such that

$$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \in/2 \text{ and } U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \in/2$$

Let $P$ be the common refinement of $P_1$ and $P_2$. Then by Theorem 14.1.9,

$$U(P,f_1,\alpha) - L(P,f_1,\alpha) < \epsilon/2 \text{ ------- (1)}$$

and $$U(P,f_2,\alpha) - L(P,f_2,\alpha) < \epsilon/2 \text{ -------- (2)}$$

Suppose $P=\{x_0, x_1, \cdots\cdots\cdots, x_n\}$. Then

$$Inf\{f(x)/x \in [x_{i-1}, x_i]\} \geq Inf\{f_1(x)/x \in [x_{i-1}, x_i]\} + Inf\{f_2(x)/x \in [x_{i-1}, x_i]\} \text{ --------- (3)}$$

and $Sup\{f(x)/x \in [x_{i-1}, x_i]\} \leq Sup\{f_1(x)/x \in [x_{i-1}, x_i] + Sup\{f_2(x)/x \in [x_{i-1}, x_i]\}\}$ ------- (4)

From (3) and (4), we have

$$L(P,f_1,\alpha) + L(P,f_2,\alpha) \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq U(P,f_1,\alpha) + U(P,f_2,\alpha)$$

This implies $U(P,f,\alpha) - L(P,f,\alpha) \leq U(P,f_1,\alpha) - L(P,f_1,\alpha)$

$$+ U(P,f_2,\alpha) - L(P,f_2,\alpha) < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{(By (1) and (2))}.$$

Thus for $\epsilon > 0$, there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. There fore $f \in \mathscr{R}(\alpha)$ That is $f_1 + f_2 \in \mathscr{R}(\alpha)$.

Next we will show that $\int_a^b (f_1+f_2)d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$

Let $\epsilon$ be an arbitrary positive real number.

Since $f_1, f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$, $\underline{\int_a^b} f_1\, d\alpha = \overline{\int_a^b} f_1\, d\alpha + \int_a^b f_1 d\alpha$ and

$$\underline{\int_a^b} f_2\, d\alpha = \overline{\int_a^b} f_2\, d\alpha = \int_a^b f_2\, d\alpha$$

For $j=1,2$, $\int_a^b f_j\, d\alpha + \epsilon/2$ is not a lower bound of the set $\{U(P,f_j,\alpha)/P \text{ is a partition of } [a,b]\}$.

Then $U(P_j, f_j, \alpha) < \int_a^b f_j\, d\alpha + \epsilon/2$ for some partitions $P_j$ of $[a,b]$ for $j=1, 2$. -------- (5)

Let $P$ be the common refinement of $P_1$ and $P_2$.

Then $U(P,f_j,\alpha) \le U(P_j,f_j,\alpha) < \int_a^b f_j\, d\alpha + \epsilon/2$ for $j=1,2$.

Now $\int_a^b f\, d\alpha \le U(P,f,\alpha) \le U(P,f_1,\alpha) + U(P,f_2,\alpha)$

$$\le U(P_1,f_1,\alpha) + U(P_2,f_2,\alpha) < \int_a^b f_1\, d\alpha + \epsilon/2 + \int_a^b f_2\, d\alpha + \epsilon/2 \quad \text{-------- (5)}$$

This implies $\int_a^b f\, d\alpha \le \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha + \epsilon$.

Since $\epsilon > 0$ is arbitrary, we have $\int_a^b f\, d\alpha \le \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$ --------- (6)

For $j=1, 2,$ $\int_a^b f_j\, d\alpha - \epsilon/2$ is not an upper bound of the set

$\{L(P,f_j,\alpha)/P \text{ is a partition of } [a,b]\}$. Then $\int_a^b f_j\, d\alpha - \epsilon/2 < L(P_j,f_j,\alpha)$ for some partitions $P_j$ of $[a,b]$ for $j=1,2$ ------------ (7).

Let $P$ be the common refinement of $P_1$ and $P_2$. Then

$$\int_a^b f_j\, d\alpha - \epsilon/2 < L(P_j,f_j,\alpha) \le L(P,f_j,\alpha) \text{ for } j=1, 2, \ldots$$

This implies that $\int_a^b f_1\, d\alpha - \epsilon/2 + \int_a^b f_2\, d\alpha - \epsilon/2 < L(P,f_1,\alpha) + L(P,f_2,\alpha) \le L(P,f,\alpha) \le \int_a^b f\, d\alpha$

Therefore $\int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha - \epsilon < \int_a^b f\, d\alpha$

Since $\in>0$ is arbitrary, we have

$$\int_a^b f_1 d\alpha + \int_a^b f_2\, d\alpha \le \int_a^b f\, d\alpha \text{ ----------- (8)}$$

From (6) and (8), $\int_a^b f\, d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2\, d\alpha$

Thus $\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$

**15.1.2 Theorem :** If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $c$ is any constant, then $c\,f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $\int_a^b c\,f\, d\alpha = c\int_a^b f\, d\alpha$.

**Proof :** Suppose $f \in \mathscr{R}(\alpha)$ and $c$ is any constant.

If $c=0$, then clearly $c\,f \in \mathscr{R}(\alpha)$.

Let $\in>0$.

Suppose $c>0$.

Since $f \in \mathscr{R}(\alpha)$ on $[a,b]$, there exists a partition $P=\{x_0, x_1, \cdots\cdots, x_n\}$ of $[a,b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \in/c$.

Write $Sup\{f(x)/x \in [x_{i-1}, x_i]\} = M_i$

and $Inf\{f(x)/x \in [x_{i-1}, x_i]\} = m_i$ for $1 \le i \le n$.

Consider $Sup\{(c\,f)(x)/x \in [x_{i-1}, x_i]\} = Sup\{c\; f(x)/x \in [x_{i-1}, x_i]\}$

$$c\; Sup\{(c\,f)(x)/x \in [x_{i-1}, x_i]\} = c M_i$$

Similarly, $Inf\{(c\,f)(x)/x \in [x_{i-1}, x_i]\} = c\; m_i$ for $1 \le i \le n$

Consider $U(P, c\,f, \alpha) - L(P, c\,f, \alpha) = \sum_{i=1}^{n} c\, M_i\, \Delta\alpha_i - \sum_{i=1}^{n} c\, M_i\, \Delta\alpha_i$

$$=c\left\{\sum_{i=1}^{n} M_i\,\Delta\alpha_i - \sum_{i=1}^{n} m_i\,\Delta\,\alpha_i\right\} = c\,\{U(P,f,\alpha)-L(P,f,\alpha)\} < c\cdot\frac{\in}{2} = \in$$

Therefore $U(P,cf,\alpha)-L(P,cf,\alpha)<\in$ for some partition $P$ of $[a,b]$ and hence $cf\in\mathscr{R}(\alpha)$.

So in this case $cf\in\mathscr{R}(\alpha)$

Suppose $c<0$ then $-c>0$.

Since $f\in\mathscr{R}(\alpha)$, there exists a partition $P=\{x_0, x_1, \ldots\ldots, x_n\}$ of $[a,b]$ such that $U(P,f,\alpha)-L(P,f,\alpha)<\frac{\in}{-c}$.

Consider $Sup\{(cf)(x)/x\in[x_{i-1}, x_i]\}$

$$=Sup\{cf(x)/x\in[x_{i-1},x_i]\} = -c\;Sup\{-f(x)/x\in[x_{i-1},x_i]\}$$

$$=-c\cdot -Inf\{f(x)/x\in[x_{i-1}, x_i]\} = c.m_i$$

This implies $U(P,cf,\alpha)=c\,L(P,f,\alpha)$

Similarly we can show that $L(P,cf,\alpha)=c\,U(P,f,\alpha)$

Consider $U(P,cf,\alpha)-L(P,cf,\alpha)=c\,L(P,f,\alpha)-c\,U(P,f,\alpha)$

$$=-c[U(P,f,\alpha)-L(P,f,\alpha)] < -c\cdot\frac{\in}{-c} = \in$$

Therefore $U(P,cf,\alpha)-L(P,cf,\alpha)<\in$ for some partition $P$ of $[a,b]$ and hence $cf\in\mathscr{R}(\alpha)$

Thus in any case $cf\in\mathscr{R}(\alpha)$

Next we will show that $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$

Since $f\in\mathscr{R}(\alpha)$, we have $\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

If $C=0$, then clearly $\int_a^b c\, f\, d\alpha = c\int_a^b f\, d\alpha$

Since $c\, f \in \mathcal{R}(\alpha)$, we have $\int_a^b c\, f\, d\alpha = \underline{\int_a^b} c\, f\, d\alpha = \overline{\int_a^b} c\, f\, d\alpha$

Suppose $c>0$,

Then $U(P, c\, f, \alpha) = c \cdot U(P, f, \alpha)$ and $L(P, c\, f, \alpha) = c \cdot L(p, f, \alpha)$ for any partition $P$ of $[a,b]$.

Consider $\int_a^b c\, f\, d\alpha = \overline{\int_a^b} c\, f\, d\alpha = Inf\ \{U(P, c\, f, \alpha)/P \text{ is a partition of } [a, b]\}$

$= Inf\ \{c\, U(P, f, \alpha)/P \text{ is a partition of } [a, b]\}$

$= c\ Inf\{U(P, f, \alpha)/P \text{ is a partition of } [a, b]\} = c\int_a^b f\, d\alpha$

So, in this case $\int_a^b c\, f\, d\alpha = c\int_a^b f\, d\alpha$

Suppose $c<0$

Then $U(P, c\, f, \alpha) = c\, L(P, f, \alpha)$ and $L(P, c\, f, \alpha) = c\, U(P, f, \alpha)$ for any partition $P$ of $[a,b]$.

Consider $\int_a^b c\, f\, d\alpha = \underline{\int_a^b} c\, f\, d\alpha = Sup\{L(P, c\, f, \alpha)/P \text{ is a partition of } [a, b]\}$

$= Sup\{c \cdot U(P, f, \alpha)/P \text{ is a partition of } [a, b]\}$

$= -c \cdot\ Sup\{-c\, U(P, f, \alpha)/P \text{ is a partition of } [a, b]\}$

$= -c \cdot -Inf\{\, U(P, f, \alpha)/P \text{ is a partition of } [a, b]\}$

$= c \cdot Inf\{\, U(P, f, \alpha)/P \text{ is a partition of } [a, b]\} = c \cdot \int_a^b f\, d\alpha$

Therefore $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$

Thus in any case $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$

**15.1.3 Theorem :** If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f_1(x) \le f_2(x)$ on $[a,b]$, then $\int_a^b f_1\,d\alpha \le \int_a^b f_2\,d\alpha$.

**Proof :** Suppose $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ and $f_1(x) \le f_2(x)$ on $[a,b]$.

Let $P = \{x_0, x_1, \cdots, x_n\}$ be any position of $[a,b]$.

Write $M_i = Sup\{f_1(x) / x \in [x_{i-1}, x_i]\}$ and

$N_i = Sup\{f_2(x) / x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Since $f_1(x) \le f_2(x)$ for $x \in [a,b]$, we have $f_1(x) \le f_2(x)$ for all $x \in [x_{i-1}, x_i]$ for $1 \le i \le n$.

Then $M_i \le N_i$ for $1 \le i < n$. This implies that

$U(P, f_1, \alpha) \le U(P, f_2, \alpha)$

Consider $\int_a^b f_1\,d\alpha = \overline{\int_a^b} f_1\,d\alpha \le U(P, f_1, \alpha) \le U(P, f_2, \alpha)$

This shows that $\int_a^b f_1\,d\alpha$ is a lower bound of $\{U(P, f_2, \alpha) / P$ is a partition of $[a,b]\}$

Therefore $\int_a^b f_1\,d\alpha \le \underline{\int_a^b} f_2\,d\alpha = \int_a^b f_2\,d\alpha$ (since $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$)

Thus $\int_a^b f_1\,d\alpha \le \int_a^b f_2\,d\alpha$

**15.1.4 Theorem :** If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$$

**Proof :** Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $a<c<b$.

Let $\in>0$

First we show that there exists a partition $P$ of $[a, b]$ such that

$c \in P$ and $U(P,f,\alpha)-L(P,f,\alpha)<\in$.

Since $f \in \mathscr{R}(\alpha)$ on $[a, b]$, there exists a partition $Q=\{x_0, x_1, \ldots\ldots\ldots, x_n\}$ of $[a, b]$ such that $U(Q, f, \alpha) - L(Q, f, \alpha) < \in$.

Since $c \in [a, b]$, we have either $c = x_i$ or $x_{i-1} < c < x_i$ for some $i$ such that $1 \le i \le n$.

If $c=x_i$, then $c \in Q$. So $Q$ is a partition of $[a, b]$ such that $c \in Q$ and $U(Q,f,\alpha) - L(Q, f, \alpha) <\in$.

Suppose $x_{i-1} < c < x_i$. Then $P=\{x_0, x_1, \ldots\ldots, x_{i-1}, c, x_i, \ldots\ldots\ldots, x_n\}$ is a partition of $[a, b]$ which is a refinement of $Q$.

Then, by theorem 14.1.6. $L(Q, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(Q, f, \alpha)$

This implies that $U(P, f, \alpha) - L(P, f, \alpha) \le U(Q, f, \alpha) - L(Q, f, \alpha) < \in$.

So there exists a partition $P$ of $[a, b]$ such that $c \in P$ and $U(P, f, \alpha) - L(P, f, \alpha) < \in$.

Assume that the above partition $P=\{x_0, x_1, \ldots\ldots, x_n\}$ and $x_{i_0} = c$ for some $i_0$ such that $1 \le i_0 \le n$.

Write $Q_1 = \{x_0, x_1, \ldots, x_{i0}\}$ and $Q_2 \{x_{i_0}, x_{i_0+1}, \ldots\ldots, x_n\}$.

Then $Q_1$ is a partition of $[a, c]$ and $Q_2$ is a partition of $[c, b]$.

Write $f/[a, C] = f_1$ and $f/[c,b] = f_2$.

Write $.M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Consider $U(P, f\, \alpha) = \sum_{i=1}^{n} M_i\, \Delta\alpha_i = \sum_{i=1}^{i_0} M_i\, \Delta\alpha_i + \sum_{i=i_0+1}^{n} M_i\, \Delta\alpha_i$

$= U(Q_1, f_1, \alpha) + U(Q_2, f_2, \alpha)$

Similarly $L(P,f,\alpha) = L(Q_1, f_1,\alpha) + L(Q_2, f_2,\alpha)$

Consider $U(P,f,\alpha) - L(P,f,\alpha) < \in$. This implies that

$U(Q_1, f_1, \alpha) - L(Q_1, f_1, \alpha) < \in$ and

$U(Q_2, f_2, \alpha) - L(Q_2, f_2, \alpha) < \in$.

That is $U(Q_1, f, \alpha) - L(Q_1, f, \alpha) < \in$ on $[a, c]$ and

$U(Q_2, f, \alpha) - L(Q_2, f, \alpha) < \in$ on $[c, b]$.

By theorem 14.1.8, $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and $f \in \mathscr{R}(\alpha)$ on $[c, b]$.

Next we will show that $\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$

Let $\in > 0$

Since $f \in \mathscr{R}(\alpha)$, there exists a partition $P = \{x_1, x_2, \ldots\ldots, x_n\}$ of $[a, b]$ such that

$U(P, f, \alpha) - L(P, f, \alpha) < \in$ ---------- (1)

without loss of generality, we may assume that $c \in P$ and suppose $x_{i_0} = c$ for some $i_0$ such that $1 \le i_0 \le n$.

Write $Q_1 = \{x_0, x_1, \ldots\ldots, x_{i_0}\}$ and $Q_2 = \{x_{i_0}, x_{i_0+1}, \ldots\ldots, x_n\}$.

Then $Q_1$ is a partition of $[a, c]$ and $Q_2$ is a partition of $[c, b]$ such that $P = Q_1 \cup Q_2$.

Now $\int_a^b f\, d\alpha \le U(P, f, \alpha) < L(P, f, \alpha) + \in$ (By (1))

$= L(Q_1, f, \alpha) + L(Q_2, f, \alpha) + \in \le \int_a^c f\, d\alpha + \int_c^b f\, d\alpha + \in$

This implies that $\int_a^b f\, d\alpha < \int_a^c f\, d\alpha + \int_c^b f\, d\alpha + \in$

Since $\in>0$ is arbitrary, we have $\int_a^b f\,d\alpha \leq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ --------- (2)

Consider $\int_a^b f\,d\alpha \geq L(P, f,\alpha) > U(P, f,\alpha) - \in$

$= U(Q_1, f,\alpha) + U(Q_2, f,\alpha) - \in$

$\geq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha - \in$

This implies that $\int_a^b f\,d\alpha \geq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha - \in$

Since $\in>0$ is arbitrary, we have $\int_a^b f\,d\alpha \geq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ ---- (3)

From (2) and (3), $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$.

**15.1.5 Theorem :** If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \leq M$ on $[a,b]$, then $\left|\int_a^b f\,d\alpha\right| \leq M[\alpha(b)-\alpha(a)]$.

**Proof :** Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $|f(x)| \leq M$ on $[a,b]$. Since $f \in \mathscr{R}(\alpha)$ on $[a,b]$, we have $\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$.

Let $P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}$ be a partition of $[a,b]$.

Write $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and

$m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$.

Since $|f(x)| \le M$ for all $x \in [a, b]$, we have $-M \le f(x) \le M$ for all $x \in [a, b]$. This implies that $-M \le f(x) \le M$ for all $x \in [x_{i-1}, x_i]$ for $1 \le i \le n$ and hence $-M \le m_i \le M_i \le M$ for $1 \le i \le n$.

Consider $L(P, f, \alpha) \le \underline{\int_a^b} f\, d\alpha = \int_a^b f\, d\alpha = \overline{\int_a^b} f\, d\alpha \le U(P, f, \alpha)$ ---------------- (1)

Consider $L(P, f, \alpha) = \sum_{i=1}^{n} m_i\, \Delta\alpha_i \ge \sum_{i=1}^{n} (-M)\Delta\alpha_i = -M \cdot \sum_{i=1}^{n} \Delta\alpha_i$

$= -M(\alpha(b) - \alpha(a))$

This implies $L(P, f, \alpha) \ge -M.(\alpha(b) - \alpha(a))$ ----------- (2)

Consider $U(P, f, \alpha) = \sum_{i=1}^{n} M_i\, \Delta\alpha_i \le \sum_{i=1}^{n} M\, \Delta\alpha_i = M \cdot \sum_{i=1}^{n} \Delta\alpha_i = M(\alpha(b) - \alpha(a))$

This implies $U(P, f, \alpha) \le M(\alpha(b) - \alpha(a))$ ----------------- (3)

From (1), (2) and (3), we have

$$-M(\alpha(b) - \alpha(a)) \le \int_a^b f\, d\alpha \le M(\alpha(b) - \alpha(a)) \text{ and hence}$$

$$\left|\int_a^b f\, d\alpha\right| \le M(\alpha(b) - \alpha(a))$$

**15.1.6 Theorem :** If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ on $[a, b]$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ on $[a, b]$ and

$$\int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 .$$

**Proof :** Suppose $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ on $[a, b]$. Let $\epsilon > 0$.

Since $f \in \mathcal{R}(\alpha_j)$ on $[a, b]$ for $j = 1, 2$ there exist partitions $P_j$ of $[a, b]$ such that

$U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \epsilon/2$ ----------------- (1)

Let $P$ be the common refinement of $P_1$ and $P_2$.

Then by theorem 14.1.6, $L(P_j, f, \alpha_j) \le L(P, f, \alpha_j) \le U(P, f, \alpha_j) \le U(P_j, f, \alpha_j)$ for $j=1, 2$. . This implies that

$$U(P, f, \alpha_j) - L(P, f, \alpha_j) \le U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \in/2 \text{ for } j=1, 2 \text{ -------------- (2)}$$

Assume $P = \{x_0, x_1, \ldots\ldots\ldots\ldots, x_n\}$

write $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Consider $U(P, f, \alpha_1 + \alpha_2) = \sum_{i=1}^{n} M_i \; \Delta(\alpha_1 + \alpha_2)$

$$= \sum_{i=1}^{n} M_i \left((\alpha_1 + \alpha_2)(x_i) - (\alpha_1 + \alpha_2)(x_{i-1})\right)$$

$$= \sum_{i=1}^{n} M_i \left[\alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})\right]$$

$$= \sum_{i=1}^{n} M_i \, \Delta\alpha_{1i} + \sum_{i=1}^{n} M_i \, \Delta\alpha_{2i} = U(P, f, \alpha_1) + U(P, f, \alpha_2)$$

Therefore $U(P, f, \alpha_1 + \alpha_2) = U(P, f, \alpha_1) + U(P, f, \alpha_2)$.

Similarly $L(P, f, \alpha_1 + \alpha_2) = L(P, f, \alpha_1) + L(P, f, \alpha_2)$

Now consider $U(P, f, \alpha_1 + \alpha_2) - L(P, f, \alpha_1 + \alpha_2) =$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2) - L(P, f, \alpha_1) - L(P, f, \alpha_2) < \in/2 + \in/2 = \in \text{ (By (2))}$$

So, for $\in > 0$ there exists a partition $P$ of $[a, b]$ such that

$$U(P, f, \alpha_1 + \alpha_2) - L(P, f, \alpha_1 + \alpha_2) < \in \text{ and hence } f \in \mathscr{R}(\alpha_1 + \alpha_2).$$

Next we will show that $\int_a^b f \, d(\alpha_1 + \alpha_2) = \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2$.

Since $f \in \mathscr{R}(\alpha_1+\alpha_2)$, we have $\int_a^b f\,d(\alpha_1+\alpha_2) = \int_{\underline{a}}^{b} f\,d(\alpha_1+\alpha_2) = \int_a^{\overline{b}} f\,d(\alpha_1+\alpha_2)$

Consider $\int_a^b f\,d(\alpha_1+\alpha_2) = Inf\{U(P, f, \alpha_1+\alpha_2)/P \text{ is a partition of } [a, b]\}$

$= Inf\{U(P, f, \alpha_1) + U(P, f, \alpha_2)/P \text{ is a partition of } [a, b]\}$

$\geq Inf\{U(P, f, \alpha_1)/P \text{ is a partition of } [a, b]\}$

$+ Inf\{U(P, f, \alpha_2)/P \text{ is a partition of } [a, b]\}$

$= \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2.$

Therefore $\int_a^b f\,d(\alpha_1+\alpha_2) \geq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$ ------------------ (3)

Consider $\int_a^b f\,d(\alpha_1+\alpha_2) = Sup\{L(P, f, \alpha_1+\alpha_2)/P \text{ is a partition of } [a, b]\}$

$= Sup\{L(P, f, \alpha_1) + L(P, f, \alpha_2)/P \text{ is a partition of } [a, b]\}$

$\leq Sup\{L(P, f, \alpha_1)/P \text{ is a partition of } [a, b]\}$

$+ Sup\{L(P, f, \alpha_2)/P \text{ is a partition of } [a, b]\}$

$= \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$

Therefore $\int_a^b f\,d(\alpha_1+\alpha_2) \leq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$ ------------- (4)

From (3) and (4),

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2.$$

**15.1.7 Theorem :** If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $c$ is a positive constant, then $f \in \mathcal{R}(c\,\alpha)$ and

$$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha.$$

**Proof :** Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $c$ is a positive constant. Since $c > 0$ and $\alpha$ is monotonically increasing, $C\alpha$ is also monotonically increasing.

Let $\in > 0$

Since $f \in \mathcal{R}(\alpha)$, there exists a partition $P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}$ of $[a,b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\in}{c} \quad \text{----------} \ (1)$$

Write $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

Consider $U(P, f, c\alpha) = \sum_{i=1}^{n} M_i\ \Delta\, c\alpha_i = \sum_{i=1}^{n} M_i\left(C\alpha(x_i) - c\alpha(x_i - 1)\right)$

$$= c\sum_{i=1}^{n} M_i\left(\alpha(x_i) - \alpha(x_{i-1})\right) = c\ \sum_{i=1}^{n} M_i\,\Delta\alpha_i = c\,U(P, f, \alpha)$$

Therefore $U(P, f, c\alpha) = cU(P, f, \alpha)$

Similarly $L(P, f, c\alpha) = c\ L(P, f, \alpha)$

Consider $U(P, f, c\alpha) - L(P, f, c\alpha) = c\left[U(P, f, \alpha) - L(P, f, \alpha)\right]$

$$< c\frac{\in}{c} = \in \ \text{(By (1))}$$

So for $\in > 0$, there exists a partition $P$ of $[a,b]$ such that

$U(P, f, c\alpha) - L(P, f, c\alpha) < \in$ and hence $f \in \mathcal{R}(c\alpha)$

Next we will show that $\int_a^b f\ d(c\,\alpha) = c\int_a^b f\ d\alpha$.

Since $f \in \mathcal{R}(c\,\alpha)$, we have $\int_a^b f\, d(c\alpha) = \underline{\int_a^b} f\, d(c\alpha) = \overline{\int_a^b} f\, d(c\alpha)$

Consider $\int_a^b f\, d(c\alpha) = \underline{\int_a^b} f\, d(c\alpha) = Sup\left[L(P, f, c\alpha)/P \text{ is a partition of } [a, b]\right]$.

$$= Sup\{c \cdot L(P, f, \alpha)/P \text{ is a partition of } [a, b]\}$$

$$= c \cdot Sup\{L(P, f, \alpha)/P \text{ is a partition of } [a, b]\} = c\int_a^b f\, d\alpha$$

So, $\int_a^b f\, d(c\alpha) = c.\int_a^b f\, d\alpha$

**15.1.8 Theorem :** If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, then

(a) $fg \in \mathcal{R}(\alpha)$

(b) $|f| \in \mathcal{R}(\alpha)$ and $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$.

**Proof :** Suppose $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$

**(a)** First we show that $f^2 \in \mathcal{R}(\alpha)$, Since $f$ is bounded, we have $m \le f(x) \le M$ for all $x \in [a, b]$ for some real numbers $m$ and $M$.

Define $\phi : [m, M] \to \mathbb{R}$ as $\phi(t) = t^2$ for all $t \in [m, M]$.

Then $\phi$ is continuous on $[m, M]$.

Write $h = \phi \circ f$. Then $h(x) = \phi(f(x)) = (f(x))^2 = f^2(x)$ for all $x \in [a, b]$.

This implies $h = f^2$

By Theorem 14.1.16, $h \in \mathcal{R}(\alpha)$ and hence $f^2 \in \mathcal{R}(\alpha)$.

Since $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$, by theorem 15.1.1, $f + g \in \mathcal{R}(\alpha)$.

By theorem 15.1.1 and by theorem 15.1.2, $f-g \in \mathcal{R}(\alpha)$.

Therefore $(f+g)^2 + (f-g)^2 \in \mathcal{R}(\alpha)$ and hence $fg \in \mathcal{R}(\alpha)$

**(b)** Define $\phi:[m, M] \to \mathbb{R}$ as $\phi(t) = |t|$ for all $t \in [m, M]$

Then $\phi$ is continuous on $[m, M]$.

Write $h = \phi \circ f$. Then $h(x) = \phi(f(x)) = |f(x)| = |f|(x)$ for all $x \in [a, b]$. This implies $h = |f|$

But by theorem 14.1.16, $h \in \mathcal{R}(\alpha)$ and hence $|f| \in \mathcal{R}(\alpha)$. Choose $c = \pm 1$

so that $c\int_a^b f\,d\alpha \geq 0$

$$\left[\text{For, if } \left|\int_a^b f\,d\alpha\right| = \int_a^b f\,d\alpha, \text{ take } c=1 \text{ and if } \left|\int_a^b f\,d\alpha\right| = -\int_a^b f\,d\alpha, \text{ take } c=-1\right]$$

Therefore $\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha = \int_a^b c f\,d\alpha \leq \int_a^b |f|\,d\alpha$ ( Since $cf \leq |f|$)

So, $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

**15.1.9 Definition :** The unit step function $I$ is defined by $I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$.

**15.1.10 Note :** $I$ is continuous at every point $x \neq 0$. $I$ is not continuous at $x = 0$.

**15.1.11 Theorem :** If $a < s < b$, $f$ is bounded on $[a, b]$, $f$ is continuous at $s$ and $\alpha(x) = I(x-s)$,

then $\int_a^b f\,d\alpha = f(s)$.

**Proof :** Suppose $a < s < b$, $f$ is bounded on $[a, b]$ and $f$ is continuous at $s$ and $\alpha(x) = I(x-s)$, for all $x \in [a, b]$.

If $x \leq s$, then $\alpha(x) = I(x-s) = 0$

If $x>s$, then $\alpha(x)=I(x-s)=1$.

Clearly $\alpha$ is not continuous at $x=s$.

First we show that $f\in\mathscr{R}(\alpha)$ on $[a,b]$.

Let $\in>0$. Put $\in_1=\dfrac{\in}{4}$.

Since $f$ is continuous at $s$, there exists a $\delta>0$ such that $|f(t)-f(s)|<\in_1$ whenever $t\in(a,b)$ with $|s-t|<\delta$. That is $|f(t)-f(s)|<\in_1$ whenever $a<s-\delta<t<s+\delta<b$ ------------ (1)

Write $x_0=a,\ x_1=s,\ x_2=s+\dfrac{\delta}{2},\ x_3=b$. Then

$P=\{x_0,x_1,x_2,x_3\}$ is a partition of $[a,b]$.

Consider $\alpha(x_0)=\alpha(a)=I(a-s)=0$

$$\alpha(x_1)=\alpha(s)=I(s-s)=0$$

$$\alpha(x_2)=\alpha\left(s+\frac{\delta}{2}\right)=I\left(s+\frac{\delta}{2}-s\right)=I\left(\delta/2\right)=1$$

$$\alpha(x_3)=\alpha(b)=I(b-s)=1$$

Consider $\Delta\alpha_1=\alpha(x_1)-\alpha(x_0)=0$

$$\Delta\alpha_2=\alpha(x_2)-\alpha(x_1)=1$$

$$\Delta\alpha_3=\alpha(x_3)-\alpha(x_2)=1-1=0$$

Write $M_i=Sup\{f(x)/x\in[x_{i-1},x_i]\}$ and $m_i=Inf\{f(x)/x\in[x_{i-1},x_i]\}$ for $i=1,2,3$.

Then $U(P,f,\alpha)=M_2$ and $L(P,f,\alpha)=m_2$

Consider $-L(P,f,\alpha)=-m_2=-Inf\{f(y)/y\in[x_1,x_2]\}$

$$=Sup\{-f(y)/y\in[x_1,x_2]\}$$

Now consider $U(P,f,\alpha)-L(P,f,\alpha)=$

$$=Sup\{f(x)/x\in[x_1,x_2]\}+Sup\{-f(y)/y\in[x_1,x_2]\}$$

$=Sup\{f(x)-f(y)/x,\ y\in[x_1,x_2]\}\leq\in_1+\in_1$ (by (1))

$=2\in_1=\in/2<\in$ --------------- (2)

Thus for $\in>0$, there exists a partition $P$ of $[a,b]$ such that

$U(P,f,\alpha)-L(P,f,\alpha)<\in$ and hence $f\in\mathscr{R}(\alpha)$.

Next we will show that $\int_a^b f\,d\alpha=f(s)$.

Let $P$ be the partition as above. Then $U(P,f,\alpha)=M_2$ and $L(P,f,\alpha)=m_2$.

Consider $L(P,f,\alpha)=m_2=Inf\{f(x)/x\in[x_1,x_2]\}\leq f(s)$

$\leq Sup\{f(x)/x\in[x_1,x_2]\}=U(P,f,\alpha)$

This implies $L(P,f,\alpha)\leq f(s)\leq U(P,f,\alpha)$ ------------ (3)

Also we have $L(P,f,\alpha)\leq\int_a^b f\,d\alpha\leq U(P,f,\alpha)$ ---------------- (4)

From (3) and (4), we have $\left|f(s)-\int_a^b f\,d\alpha\right|\leq U(P,f,\alpha)-L(P,f,\alpha)<\in$ (By (2)).

This implies that $\left|f(s)-\int_a^b f\,d\alpha\right|<\in$.

Since $\in>0$ is arbitrary, we have $\int_a^b f\,d\alpha=f(s)$.

**15.1.12 Theorem :** Suppose $C_n\geq0$ for $n=1,2,.........$, $\sum_{n=1}^{\infty}C_n$ converges, $\{s_n\}$ is a sequence of distinct points in $(a,b)$ and $\alpha(x)=\sum_{n=1}^{\infty}C_n\,I(x-s_n)$. Let $f$ be continuous on $[a,b]$.

Then $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} C_n f(s_n)$.

**Proof :** Frist we will show that $\sum_{n=1}^{\infty} C_n I(x-s_n)$ converges.

For any $x \in [a,b]$, $|C_n\, I(x-s_n)| = |C_n|\,|I(x-s_n)| \le |C_n| = C_n$ for $n=1,2,3,\ldots\ldots\ldots$.

Since $\sum_{n=1}^{\infty} C_n$ converges, by the comparison test, $\sum_{n=1}^{\infty} C_n I(x-s_n)$ converges.

Next we will show that $\alpha$ is monotonically increasing. Suppose $x, y \in [a, b]$ such that $x \le y$. Then $x-s_n \le y-s_n$ for all $n$. This implies $I(x-s_n) \le I(y-s_n)$ and hence

$$\sum_{n=1}^{\infty} C_n I(x-s_n) \le \sum_{n=1}^{\infty} C_n\, I(x-s_n)$$

That is, $\alpha(x) \le \alpha(y)$. So $\alpha$ is monotonically increasing. Since $f$ is continuous on $[a, b]$ by theorem 14.1.12, $f \in \mathscr{R}(\alpha)$.

Since $a < s_n$ for all $n$, $\alpha(a) = \sum_{n=1}^{\infty} C_n I(a-s_n) = 0$

Since $s_n < b$ for all $n, \alpha(b) = \sum_{n=1}^{\infty} C_n I(b-s_n) = \sum_{n=1}^{\infty} C_n$.

Since $f$ is continuous on $[a, b]$, $f$ is bounded on $[a, b]$.

So put $M = Sup\{|f(x)| / x \in [a, b]\}$.

Now we will show that $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} C_n f(s_n)$. That is, we have to show that the sequence of partial sums of the series $\sum_{n=1}^{\infty} C_n f(s_n)$ converges to $\int_a^b f\,d\alpha$.

Let $\in > 0$. Write $\in_1 = \dfrac{\in}{M+1}$

Since $\sum\limits_{n=1}^{\infty} C_n$ converges, there exists a positive integer $N$ such that

$$\sum_{n=N+1}^{\infty} C_n < \in_1 \quad \text{----------------} \quad (1)$$

Put $\alpha_1(x) = \sum\limits_{i=1}^{N} C_i I(x-s_i)$ and $\alpha_2(x) = \sum\limits_{i=N+1}^{\infty} C_i I(x-s_i)$ for all $x \in [a,b]$. Then $\alpha = \alpha_1 + \alpha_2$ and $\alpha_1, \alpha_2$ are monotonically increasing on $[a,b]$.

Since $f$ is continuous on $[a,b]$, by theorem 14.1.12, $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$.

For $i = 1, 2, \ldots\ldots\ldots\ldots$ Put $I_i(x) = I(x-s_i)$ for all $x \in [a,b]$.

Since $a < s_i < b$ and $f$ is continuous at $s_i$ and $f$ is bounded on $[a,b]$, by theorem 15.1.11,

$$\int_a^b f\, dI_i = f(s_i) \text{ for } i = 1, 2, \ldots\ldots\ldots \quad \text{----------------} \quad (2).$$

By theorem 15.1.6 and theorem 15.1.7

$$\int_a^b f d\alpha_1 = \int_a^b fd\left(\sum_{i=1}^{N} C_i I_i\right) = \sum_{i=1}^{N} \int_a^b f d(C_i I_i) = \sum_{i=1}^{N} C_i \int_a^b f\, d\, I_i = \sum_{i=1}^{N} C_i f(s_i) \text{ (By (2)).}$$

Consider $\alpha_2(b) - \alpha_2(a) = \sum\limits_{i=N+1}^{\infty} C_i I(b-s_i)$

$$= \sum_{i=N+1}^{\infty} C_i < \in_1 \text{ (By (1))}$$

By theorem 15.1.5, $\left|\int_a^b f\, d\alpha_2\right| \le M(\alpha_2(b) - \alpha_2(a)) < M \in_1$

Therefore $\left|\int_a^b f\,d\alpha - \sum_{i=1}^{N} C_i f(s_i)\right| = \left|\int_a^b f\,d\alpha - \int_a^b f\,d\alpha_1\right|$.

$$= \left|\int_a^b f\,d\alpha_2\right| < M \in_1 < (M+1)\in_1 = \in.$$

This implies that $\left|\int_a^b f\,d\alpha - \sum_{i=1}^{n} C_i f(s_i)\right| < \in$ for all $n \geq N$.

Thus for given $\in > 0$, there exists a positive integer $N$ such that $\left|\int_a^b f\,d\alpha - \sum_{i=1}^{n} C_i f(s_i)\right| < \in$ for all $n \geq N$.

This shows that the sequence of partial sums of the series $\sum_{n=1}^{\infty} C_n f(s_n)$ converges to $\int_a^b f\,d\alpha$.

Hence $\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} C_n f(s_n)$.

**15.1.13 Note :** Let $f:[a,b] \to \mathbb{R}$ be defined by $f(x) = k$ for some constant $k$ and for all $x \in [a,b]$.

Then $f \in \mathscr{R}$ on $[a,b]$ and $\int_a^b f(x)\,dx = k(b-a)$.

For, let $P = \{x_0, x_1, \ldots\ldots\ldots\ldots\ldots, x_n\}$ be a partition of $[a,b]$. Write $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \leq i \leq n$. Then $M_i = k$ and $m_i = k$ for $1 \leq i \leq n$. Consider $U(p,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} k \Delta x_i = k \cdot \sum_{i=1}^{n} \Delta x_i = k(b-a)$.

Similarly $L(P,f) = k(b-a)$

Therefore $U(P,f) - L(P;f) = 0 < \in$ for any $\in > 0$ and hence $f \in \mathscr{R}$ on $[a,b]$.

Consider $k(b-a) = L(P,f) \leq \int_a^b f(x)\,dx \leq U(P,f) = k(b-a)$.

Therefore $\int_a^b f(x)dx = k(b-a)$

**15.1.14 Theorem :** Assume $\alpha$ increases monotonically on $[a, b]$ and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let $f$ be a bounded real function defined on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case

$$\int_a^b f\,d\alpha = \int_a^b f(x)\,\alpha'(x)\,dx.$$

**Proof :** Suppose $\alpha$ is monotonically increasing on $[a, b]$ and $\alpha' \in \mathscr{R}$ on $[a, b]$ and also assume that $f$ is a bounded real function defined on $[a, b]$.

Let $\in > 0$.

Since $\alpha' \in \mathscr{R}$, there exists a partition $P=\{x_0, x_1, \ldots\ldots\ldots\ldots, x_n\}$ of $[a, b]$ such that $U(P, \alpha') - L(P, \alpha') < \in$ -------------- (1).

Since $\alpha'$ exists, $\alpha$ is differentiable on $[a, b]$. Then $\alpha$ is continuous on $[a, b]$ and $\alpha$ is differentiable on $(a,b)$. This implies $\alpha$ is continuous on $[x_{i-1}, x_i]$ and $\alpha$ is differentiable on $(x_{i-1}, x_i)$ for $1 \le i \le n$. So by mean value theorem, there exists a point $t_i \in (x_{i-1}, x_i)$ such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1}) \text{ for } 1 \le i \le n.$$

That is, $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$ for $1 \le i \le n$.

Since $f$ is bounded on $[a, b]$. Put $M = \sup\{|f(x)| / x \in [a, b]\}$.

Now we will show that $U(P, f, \alpha) \le U(P, f\alpha') + M\in$

$$U(P, f\alpha') \le U(P, f, \alpha) + M\in$$

$$L(P, f, \alpha) \le L(P, f\alpha') + M\in$$

$$L(P, f\alpha') \le L(P, f, \alpha) + M\in$$

Let $s_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$. Then by theorem 14.1.10 and by (1),

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\,\Delta x_i < \in \text{ --------------- (2)}$$

Consider $\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right|$

$$= \left| \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$$

$$= \left| \sum_{i=1}^{n} f(s_i)\left[\alpha'(t_i)-\alpha'(s_i)\right]\Delta x_i \right|$$

$$\leq \sum_{i=1}^{n} |f(s_i)|\,|\alpha'(t_i)-\alpha'(s_i)|\Delta x_i \leq \sum_{i=1}^{n} M|\alpha'(t_i)-\alpha'(s_i)|\Delta x_i$$

$$= M \cdot \sum_{i=1}^{n} |\alpha'(t_i)-\alpha'(s_i)|\,\Delta x_i < M\in \quad \text{(by (2))}$$

This implies that $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n} (f\alpha')(s_i)\Delta x_i + M\in$ ------------ (3)

and $\sum_{i=1}^{n} (f\alpha')(s_i)\Delta x_i \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + M\in$ ------------- (4)

Write $M_i^* = Sup\{(f\alpha')(x)/x\in[x_{i-1},x_i]\}$ for $1\leq i\leq n$.

Then from (3), $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n} (f\alpha')(s_i)\Delta x_i + M\in$

$$\leq \sum_{i=1}^{n} M_i^* \Delta x_i + M\in = U(P, f\alpha') + M\in$$

This implies that $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq U(P, f\alpha') + M\in$ --------------- (5)

Write $M_i = Sup\{f(x)/x\in[x_{i-1},x_i]\}$ and $m_i = Inf\{f(x)/x\in[x_{i-1}, x_i]\}$ for $1\leq i\leq n$.

Then from (3), $L(P, f, \alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i$

$$\leq \sum_{i=1}^{n} (f\alpha')(s_i)\,\Delta x_i + M \in$$

This implies that $L(P, f,\alpha) - M \in \leq \sum_{i=1}^{n} (f\alpha')(s_i)\,\Delta x_i$ -------------- (6)

Therefore inequalities (5) and (6) are true for any $s_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$.

Consider $U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i = M_1\,\Delta\alpha_1 + M_2\,\Delta\alpha_2 + \cdots\cdots\cdots + M_n\,\Delta\alpha_n$.

$$= Sup\{f(x)/x \in [x_0, x_1]\}\,\Delta\alpha_1 + \ldots\ldots\ldots\ldots + Sup\{f(x)/x \in [x_{n-1}, x_n]\}\,\Delta\alpha_n$$

$$= \sum_{i=1}^{n} Sup\{f(x)\Delta\alpha_i / x \in [x_{i-1}, x_i]\}$$

$$= Sup\left\{\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \Big/ s_i \in [x_{i-1}, x_i]\right\}$$

Therefore $U(P, f\alpha) = Sup\left\{\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \Big/ s_i \in [x_{i-1}, x_i]\right\}$ --------------- (7)

Similarlay $L(P, f\alpha') = Inf\left\{\sum_{i=1}^{n} (f\alpha')(s_i)\Delta\alpha_i \Big/ s_i \in [x_{i-1}, x_i]\right\}$ --------- (8)

From (5), $U(P, f\alpha') + M \in$ is an upper bound of

$\left\{\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \Big/ s_i \in [x_{i-1}, x_i]\right\}$ and

from (6), $L(P,f,\alpha) - M \in$ is a lower bound of the

$\left\{\sum_{i=1}^{n} (f\alpha')(s_i)\Delta x_i \Big/ s_i \in [x_{i-1}, x_i]\right\}$,

From (7) and (8), $U(P,f,\alpha) \leq U(P,f\alpha') + M \in$ and $L(P, f,\alpha) - M \in \leq L(P, f\alpha')$

Therefore $U(P,f,\alpha) \leq U(P,f\alpha') + M \in$ -------------- (9)

$$L(P, f, \alpha) \leq L(P, f\alpha') + M \in \text{------------- (10)}$$

Similarly from (4), we can show that

$$U(P, f\alpha') \leq U(P, f, \alpha) + M \in \text{-------------- (11)}$$

and $L(P, f\alpha') \leq L(P, f, \alpha) + M \in$ -------------- (12)

Now we will show that $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if $f\alpha' \in \mathscr{R}$ on $[a, b]$.

Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$

Let $\in > 0$. Put $\in_1 = \dfrac{\in}{2M+1}$

Since $\alpha' \in \mathscr{R}$ on $[a, b]$, there exists a partition $P_1$ of $[a, b]$ such that

$$U(P_1, \alpha') - L(P_1, \alpha') < \in_1$$

Since $f \in \mathscr{R}(\alpha)$ on $[a, b]$, there exists a partition $P_2$ of $[a, b]$ such that

$$U(P_2, f, \alpha) - L(P_2, f, \alpha) < \in_1 .$$

Write $P = P_1 \cup P_2$. Then $P$ is a partition of $[a, b]$ and $P$ is the common refinement of $P_1$ and $P_2$. Then by theorem 14.1.6,

$U(P, f, \alpha) - L(P, f, \alpha) < \in_1$ and $U(P, \alpha') - L(P, \alpha') < \in_1$

This implies that $P$ satisfies (9), (10), (11) and (12) for $\in_1$.

Consider $U(P, f\alpha') \leq U(P, f, \alpha) + M \in_1$

$$L(P, f, \alpha) \leq L(P, f\alpha') + M \in_1$$

From the above two inequalities, $U(P, f\alpha') - L(P, f\alpha')$

$$\leq U(P, f, \alpha) - L(P, f\alpha) + M \in_1 + M \in_1 < 2M \in_1 + \in_1 = \in$$

Therefore, for $\in > 0$, there exists a partition $P$ of $[a, b]$ such that $U(P, f\alpha') - L(P, f\alpha') < \in$ and hence $f\alpha' \in \mathscr{R}$ on $[a, b]$.

Conversely suppose that $f\alpha' \in \mathscr{R}$ on $[a, b]$.

Let $\in > 0$. Put $\in_1 = \dfrac{\in}{2M+1}$

Since $\alpha' \in \mathscr{R}$ on $[a, b]$, there exists a partition $P_1$ of $[a, b]$ such that

$$U(P_1, \alpha') - L(P_1, \alpha') < \in_1$$

Since $f\alpha' \in \mathscr{R}$ on $[a,b]$, there exists a partition $P_2$ of $[a,b]$ such that

$$U(P_2, f\alpha') - L(P_2, f\alpha') < \in_1$$

Write $P = P_1 \cup P_2$. Then $P$ is the common refinement of $P_1$ and $P_2$. By theorem 14.1.6, $U(P,\alpha') - L(P,\alpha') < \in_1$ and $U(P, f\alpha') - L(P, f\alpha') < \in_1$.

This implies that $P$ satisfies (9), (10), (11) and (12) for $\in_1$.

Now consider $U(P,f,\alpha) - L(P,f,\alpha) \le U(P, f\alpha') - L(P, f\alpha')$

$$+M \in_1 + M \in_1 < 2M \in_1 + \in_1 = \in$$

Thus, for $\in > 0$, there exists a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$ and hence $f \in \mathscr{R}(\alpha)$ on $[a,b]$.

Now we will show that $\int_a^b f\, d\alpha = \int_a^b (f\alpha')(x)\,dx = \int_a^b f(x)\, \alpha'(x)\,dx$

Let $\in > 0$. Put $\in_1 = \dfrac{\in_1}{M+1}$

Since $\alpha' \in \mathscr{R}$ on $[a,b]$, there exists a partition $Q$ of $[a,b]$ such that $U(Q,\alpha') - L(Q,\alpha') < \in_1$.

Let $S$ be any partition of $[a,b]$. Put $P = SUQ$. Then $P$ is the common refinement of $S$ and $Q$ and $U(P,\alpha') - L(P,\alpha') \le U(Q,\alpha') - L(Q,\alpha') < \in_1$.

Now $P$ satisfies (9), (10), (11) and (12) for $\in_1$.

Consider $\int_a^b f\, d\alpha \le U(P,f,\alpha) \le U(P, f\alpha') + M \in_1$

$$\le U(S, f\alpha') + M \in_1 < U(S, f\alpha') + M \in_1 + \in_1$$

$$= U(s, f\alpha') + \in.$$

This implies that $\int_a^b f\, d\alpha < U(S, f\alpha') + \in$ for any partition $S$ of $[a,b]$.

Consider $\int_a^b f\, d\alpha \ge L(P,f,\alpha) \ge L(P, f\alpha') - M \in_1$

$$\geq L(S, f\alpha') - M\in_1 > L(S, f\alpha') - M\in_1 - \in_1$$

$$= L(S, f\alpha') - \in$$

Therefore $\int_a^b f\,d\alpha > L(S, f\alpha') - \in$ for any partition $S$ of $[a, b]$

Now $\int_a^b f\,d\alpha - \in \leq Inf\{U(S, f\alpha')/S \text{ is a partition of } [a, b]\} = \int_a^b (f\alpha')(x)\,dx$

and $\int_a^b f\,d\alpha + \in \geq Sup\{L(S, f\alpha')/S \text{ is a partition of } [a, b]\} = \int_a^b (f\alpha')(x)\,dx$

Therefore $\int_a^b f\,d\alpha - \in \leq \int_a^b (f\alpha')(x)\,dx \leq \int_a^b f\,d\alpha + \in$

This implies that $\left|\int_a^b f\,d\alpha - \int_a^b (f\alpha')(x)\,dx\right| \leq \in$

Since $\in > 0$ is arbitrary, we have $\int_a^b f\,d\alpha = \int_a^b (f\alpha')(x)\,dx$

**15.1.15 Theorem (change of variable) :** Suppose $\phi$ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose $\alpha$ is monotonically increasing on $[a, b]$ and $f \in \mathscr{R}(\alpha)$ on $[a, b]$. Define $\beta$ and $g$ on $[A, B]$ by

$$\beta(y) = \alpha(\phi(y)),\ g(y) = f(\phi(y)).$$

Then $g \in \mathscr{R}(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$

**Proof :** Since $f \in \mathscr{R}(\alpha)$, $f$ is a bounded function and so $f[a, b]$ is bounded.

Since $\phi$ is onto, $g[A, B] = f(\phi[A, B]) = f[a, b]$. This implies that $g[A, B]$ is bounded and hence $g$ is bounded.

Let $y_1, y_2 \in [A, B]$ be such that $y_1 \leq y_2$. Since $\phi$ is increasing on $[A, B]$, $\phi(y_1) \leq \phi(y_2)$. Since $\alpha$ is increasing on $[a, b]$, we have $\alpha(\phi(y_1)) \leq \alpha(\phi(y_2))$. This implies that $\beta(y_1) \leq \beta(y_2)$

and hence $\beta$ is monotonically increasing on $[A, B]$.

Next we will prove that $\phi(A)=a$ and $\phi(B)=b$.

Clearly $\phi(A)\in[a,b]$. This implies that $a\leq\phi(A)$.

Since $\phi$ is onto and $a\in[a,b]$, there exists $y\in[A,B]$ such that $\phi(y)=a$

If $A<y$, then $\phi(A)<\phi(y)$ (since $\phi$ is strictly increasing).

This implies that $\phi(A)<a$, a contradiction.

So, $A=y$ and hence $\phi(A)=a$.

Similarly we can show that $\phi(B)=b$.

Let $Q=\{y_0,y_1,...,y_n\}$ be a partition of $[A, B]$. Then $y_0=A, y_n=B$ and $y_0\leq y_1\leq......\leq y_n$.

This implies that $\phi(A)\leq\phi(y_1)\leq.........\leq\phi(y_n)=\phi(B)$

Take $x_i=\phi(y_i)$ for $0\leq i\leq n$. Then

$a=x_0\leq x_1\leq..........\leq x_n=b$. So $P=\{x_0,x_1,.......,x_n\}$ is a partition of $[a, b]$ such that $\phi(y_i)=x_i$ for $0\leq i\leq n$.

Conversely let $P=\{x_0,x_1,.......,x_n\}$ be a partition of $[a, b]$.

Then $a=x_0\leq x_1\leq.......\leq x_n=b$.

Since $\phi$ is onto, for each $x_i$, there exists $y_i\in[A,B]$ such that $\phi(y_i)=x_i$. This implies that $\phi(y_0)=a$ and $\phi(y_n)=b$. Since $\phi$ is strictly increasing, we have $\phi$ is one - one.

Since $\phi(A)=a$ and $\phi(B)=b$, we have $A=y_0, B=y_n$.

Also $A=y_0\leq y_1\leq......\leq y_n=B$

So $Q=\{y_0,y_1,...,y_n\}$ is a partition of $[A,B]$ such that $\phi(y_i)=x_i$ for $0\leq i\leq n$.

Next we will prove that $f[a,b]=g[A,B]$.

Let $x\in f[a, b]$. Then $x=f(y)$ for some $y\in[a, b]$. Since $\phi$ is onto, there exists $t\in[A,B]$ such that $\phi(t)=y$.

Consider $g(t)=f(\phi(t))=f(y)=x$. This implies that $x\in g[A, B]$.

So $f[a, b]\subseteq g[A,B]$

Let $y\in g[A, B]$. Then $y=g(t)$ for some $t\in[A,B]$.

Now $\phi(t)\in[a,b]$. This implies that $f(\phi(t))\in f[a,b]$

Since $g(t)=f(\phi(t))$, we have $g(t)\in f[a,b]$ and so $y\in f[a,b]$.

Hence $f[a,b]=g[A,B]$

Let $P=\{x_0,x_1,\ldots\ldots,x_n\}$ be any partition of $[a,b]$. Then there exists a partition $Q=\{y_0,y_1,\ldots\ldots,y_n\}$ of $[A,B]$ such that $\phi(y_i)=x_i$ for $0\le i\le n$. This implies that $f[x_{i-1},x_i]=g[y_{i-1},y_i]$ for $1\le i\le n$.

Write $M_i=Sup\{f(x)/x\in[x_{i-1},x_i]\}$ and $m_i=Inf\{f(x)/x\in[x_{i-1},x_i]\}$ and $N_i=Sup\{g(y)/y\in[y_{i-1},y_i]\}$ and $n_i=Inf\{g(y)/y\in[y_{i-1},y_i]\}$ for $1\le i\le n$.

For $1\le i\le n$, consider $M_i=Sup\{f(x)/x\in[x_{i-1},x_i]\}=Sup\,f[x_{i-1},x_i]=Sup\,g[y_{i-1},y_i]=N_i$.

This implies that $M_i=N_i$ for $1\le i\le n$.

Similarly $m_i=n_i$ for $1\le i\le n$.

Consider $U(P,f,\alpha)=\sum_{i=1}^{n}M_i\,\Delta\alpha_i=\sum_{i=1}^{n}M_i\left(\alpha(x_i)-\alpha(x_{i-1})\right)$

$$=\sum_{i=1}^{n}N_i\left(\alpha(\phi(y_i))-\alpha(\phi(y_{i-1}))\right)=\sum_{i=1}^{n}N_i\left(\beta(y_i)-\beta(y_{i-1})\right)=U(Q,g,\beta)$$

Therefore $U(P,f,\alpha)=U(Q,g,\beta)$.

Similarly we can show that $L(P,f,\alpha)=L(Q,g,\beta)$

Let $\in>0$

Since $f\in\mathscr{R}(\alpha)$, there exists a partition $P$ of $[a,b]$ such that

$U(P,f,\alpha)-L(P,f,\alpha)<\in$ ------ (1)

Since $P$ is a partition of $[a,b]$, by the above facts, we have a partition $Q$ of $[A,B]$ such that $U(P,f,\alpha)=U(Q,g,\beta)$ and $L(P,f,\alpha)=L(Q,g,\beta)$.

Then by (1), $U(Q,g,\beta)-L(Q,g,\beta)<\in$.

Therefore $g\in\mathscr{R}(\beta)$ on $[A,B]$.

Consider $\int_a^b f\, d\alpha = Sup\{L(P,f,\alpha)/P$ is a partition of $[a,b]\}$

$= Sup\{L(Q,g,\beta)/Q$ is a partition of $[A,B]\}$

$= \int_A^B g\, d\beta$

Hence $\int_a^b f\, d\alpha = \int_A^B g\, d\beta$

## 15.2 SHORT ANSWER QUESTIONS

**15.2.1 :** If $f_1, f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$ and $f_1(x) \le f_2(x)$ on $[a,b]$, then show that $\int_a^b f_1\, d\alpha \le \int_a^b f_2\, d\alpha$.

**15.2.2 :** Define the unit step function $I$ and show that $I$ is continuous at every point $x \neq 0$.

**15.2.3 :** Let $f:[a,b] \to \mathbb{R}$ be defined by $f(x) = k$ for some constant $k$ and for all $x \in [a,b]$. Then show that $f \in R$ on $[a,b]$ and $\int_a^b f(x)dx = k(b-a)$.

## 15.3 MODEL EXAMINATION QUESTIONS

**15.3.1 :** If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then show that $f \in \mathscr{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and $\int_a^c f\, d\alpha + \int_c^b f\, d\alpha = \int_a^b f\, d\alpha$.

**15.3.2 :** If $a<s<b$, $f$ is bounded on $[a,b]$, $f$ is continuous at $s$ and $\alpha(x) = I(x-s)$, then show that $\int_a^b f\, d\alpha = f(s)$.

**15.3.3 :** Suppose $\phi$ is a strictly increasing continuous function that maps an interval $[A,B]$ onto

$[a,b]$ and $\alpha$ is monotonically increasing on $[a,b]$ and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Define $\beta$ and $g$ on $[A,B]$ by $\beta(y)=\alpha(\phi(y)),\ g(y)=f(\phi(y))$. Then show that $g \in \mathscr{R}(\beta)$ and

$$\int_A^B g\,d\beta = \int_a^b f\,d\alpha.$$

## 15.4 EXERCISES

**15.4.1 :** Suppose $f$ is a bounded real function on $[a,b]$ and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

## 15.5 ANSWERS TO SHORT ANSWER QUESTIONS

For 15.2.1, see theorem 15.1.3

For 15.2.2, see definition 15.1.9

For 15.2.3, see note 15.1.13

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin

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Lesson - 16

# FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS AND RECTIFIABLE CURVES

## 16.0 INTRODUCTION

In this lesson, it has been shown that integration and differentiation are, in certain sense, inverse operations. The fundamental theorem of calculus and integration by parts are proved. Also the integration of vector valued function is studied. Further rectifiable curve is defined and it is proved that every continuously differentiable curve on $[a, b]$ is rectifiable.

## 16.1 INTEGRATION AND DIFFERENTIATION

**16.1.1 Theorem :** Let $f$ be a real valued function defined on $[a, b]$ such that $f \in \mathscr{R}$ on $[a, b]$. For $a \leq x \leq b$, put $F(x) = \int_a^x f(t)\,dt$. Then $F$ is continuous on $[a, b]$; further more, if $f$ is continuous at a point $x_0$ of $[a, b]$, then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.

**Proof :** Given that $f$ is a real valued function defined on $[a, b]$ such that $f \in \mathscr{R}$ on $[a, b]$. Also given that for $a \leq x \leq b$, $F(x) = \int_a^x f(t)\,dt$.

Since $f \in \mathscr{R}$ on $[a, b]$, $f$ is bounded on $[a, b]$. Then there exists an $M$ such that $|f(t)| \leq M$ for all $t \in [a, b]$.

Let $\in > 0$. Write $\delta = \dfrac{\in}{M+1}$. Then $\delta > 0$.

Let $x, y \in [a, b]$ such that $x < y$ and $|x - y| < \delta$.

Consider $|F(x) - F(y)| = \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right|$

$$=\left|-\int_x^y f(t)\,dt\right| = \left|\int_x^y f(t)\,dt\right| \le M(y-x) \text{ (By theorem 15.5)}$$

$$=M|x-y| < M\delta < (M+1)\delta = \in$$

So for $\in > 0$, there exists $\delta > 0$ such that $|F(x)-F(y)| < \in$, whenever $|x-y| < \delta$.

This implies that $F$ is uniformly continuous and hence $F$ is continuous on $[a,b]$. Suppose $f$ is continuous at a point $x_0 \in [a,b]$. Now we will show that $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$. Define $h(t) = \dfrac{F(t)-F(x_0)}{t-x_0}$ for all $t$ such that $a<t<b$ and $t \ne x_0$.

Now we show that $\lim\limits_{t \to x_0} h(t) = f(x_0)$.

Let $\in > 0$. Since $f$ is continuous at $x_0$, there exists a $\delta > 0$ such that $|f(x_0)-f(t)| < \in$ whenever $t \in [a,b]$ with $|x_0 - t| < \delta$. -------------------- (1)

Suppose $0 < |t-x_0| < \delta$. Then $x_0 - \delta < t < x_0 + \delta$. This implies that either $x_0 - \delta < x_0 < t < x_0 + \delta$ or $x_0 - \delta < t < x_0 < x_0 + \delta$.

Suppose $x_0 - \delta < t < x_0 < x_0 + \delta$.

Consider $|h(t) - f(x_0)| = \left|\dfrac{F(x_0)-F(t)}{x_0 - t} - f(x_0)\right|$

$$=\frac{1}{x_0 - t}\left|\int_a^{x_0} f(u)\,du - \int_a^t f(u)\,du - f(x_0)(x_0 - t)\right|$$

$$=\frac{1}{x_0 - t}\left|\int_t^{x_0} f(u)\,du - f(x_0)(x_0 - t)\right|$$

$$=\frac{1}{x_0 - t}\left|\int_t^{x_0} f(u)\,du - \int_t^{x_0} f(x_0)\,du\right| = \frac{1}{x_0 - t}\left|\int_t^{x_0} \left(f(u) - f(x_0)\right)du\right|$$

$< \frac{1}{x_0 - t} \in (x_0 - t) = \in$ (By (1))

Therefore $|h(t) - f(x_0)| < \in$

Similarly we can show that if $x_0 - \delta < x_0 < t < x_0 + \delta$, then $|h(t) - f(x_0)| < \in$.

So $\lim_{t \to x_0} h(t) = f(x_0)$. That is $\lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0)$.

This shows that $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.

**16.1.2 Theorem (The fundamental theorem of Calculus) :** If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function $F$ on $[a, b]$ such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

**Proof :** Suppose $f \in \mathscr{R}$ on $[a, b]$ and suppose $F$ is a differentiable function on $[a, b]$ such that $F' = f$.

Let $\in$ be any positive real number.

Since $f \in \mathscr{R}$ on $[a, b]$, there exists a partition $P = \{x_0, x_1, ......., x_n\}$ of $[a, b]$ such that $U(p, f) - L(p, f) < \in$ ------------ (1).

Since $F$ is differentiable on $[a, b]$, $F$ is differentiable on $[x_{i-1}, x_i]$ for $1 \le i \le n$. This implies that $F$ is differentiable on $(x_{i-1}, x_i)$ and $F$ is continuous on $[x_{i-1}, x_i]$ for $1 \le i \le n$. By Mean value theorem, there exists $t_i \in (x_{i-1}, x_i)$ such that

$F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1})$ for $1 \le i \le n$.

Since $F' = f$ on $[a, b]$, we have $F(x_i) - F(x_{i-1}) = f(t_i) \Delta x_i$ for $1 \le i \le n$.

Now $\sum_{i=1}^{n} f(t_i) \Delta x_i = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a)$.

Therefore $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(t_i) \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i = U(P, f)$, where $M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}$ and $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}$ for $1 \le i \le n$.

So $L(P, f) \le F(b) - F(a) \le U(P, f)$ ---------------- (2)

Also $L(P,f) \le \int_a^b f(x)dx \le U(P,f)$ ------------------ (3)

From (1), (2) and (3), $\left| F(b) - F(a) - \int_a^b f(x)dx \right| < \in$

Since $\in > 0$ is arbitrary, $\int_a^b f(x)dx = F(b) - F(a)$.

**16.1.3 Theorem (Integration by parts)** : Suppose $F$ and $G$ are differentiable functions on $[a,b]$, $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$. Then $\int_a^b F(x)\, g(x)\, dx = F(b)G(b) - F(a)G(a)$ $- \int_a^b f(x)\, G(x)\, dx$.

**Proof :** Suppose $F$ and $G$ are differentiable functions on $[a, b]$ and $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$.

Define $H$ on $[a, b]$ as $H(x) = F(x)G(x)$ for all $x \in [a, b]$.

Since $F$ and $G$ are differentiable on $[a, b]$, $H$ is also differentiable on $[a, b]$ and $H' = F'G + G'F = fG + gF$.

Since $G$ is differentiable on $[a, b]$. $G$ is continuous on $[a, b]$. Then by theorem 14.1.12, $G \in R$. Therefore $fG \in \mathcal{R}$. Similarly $gF \in \mathcal{R}$. By theorem 15.1.1. $fG + gF \in R$. That is, $H' \in \mathcal{R}$.

Put $h = H'$. By theorem 16.1.2, $\int_a^b h(x)dx = H(b) - H(a)$.

But $\int_a^b h(x)dx = \int_a^b (f(x)\, G(x) + g(x)\, F(x))\, dx$

$$= \int_a^b f(x)\,G(x)\,dx + \int_a^b g(x)\,F(x)\,dx.$$

Therefore $\int_a^b f(x)\,G(x)\,dx + \int_a^b g(x)\,F(x)\,dx = F(b)G(b) - F(a)G(a)$

and hence $\int_a^b g(x)\,F(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)\,G(x)\,dx$.

**16.1.4 Definition :** Let $f_1, f_2, \ldots\ldots, f_k$ be real valued functions on $[a, b]$ and let $f = (f_1, f_2, \ldots\ldots\ldots\ldots, f_k)$ be the corresponding vector valued function of $[a, b]$ into $\mathbb{R}^k$. Let $\alpha$ be monotonically increasing function on $[a, b]$. We say that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if $f_j \in \mathcal{R}(\alpha)$ on $[a, b]$ for $1 \le j \le k$. If this is the case, we define

$$\int_a^b f\,d\alpha = \left( \int_a^b f_1\,d\alpha, \ldots\ldots\ldots, \int_a^b f_k\,d\alpha \right)$$

**16.1.5 Theorem :** If $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$, then

(i) $f + g \in \mathcal{R}(\alpha)$

(ii) $cf \in \mathcal{R}(\alpha)$ on $[a, b]$ for every constant $c$ and

$$\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha \text{ and } \int_a^b c\,f\,d\alpha = c\int_a^b f\,d\alpha.$$

**Proof :** Suppose $f = (f_1, f_2, \ldots\ldots, f_k)$ and $g = (g_1, g_2, \ldots\ldots, g_k)$ are vector valued functions of $[a, b]$ into $\mathbb{R}^k$ and $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$. Then $f_i \in \mathcal{R}(\alpha)$ on $[a, b]$ for $1 \le i \le k$ and $g_i \in \mathcal{R}(\alpha)$ on $[a, b]$ for $1 \le i \le k$.

By theorem 15.1.1, $f_i + g_i \in \mathcal{R}(\alpha)$ on $[a, b]$ for $1 \le i \le k$ and

$$\int_a^b (f_i + g_i)\,d\alpha = \int_a^b f_i\,d\alpha + \int_a^b g_i\,d\alpha \text{ for } 1 \le i \le k.$$

Since $f+g=(f_1+g_1, f_2+g_2, \ldots\ldots, f_k+g_k)$ and $f_i+g_i \in R(\alpha)$ on $[a,b]$ for $1 \le i \le k$, we have $f+g \in R(\alpha)$ on $[a,b]$ and

$$\int_a^b (f+g)\,d\alpha = \left( \int_a^b (f_1+g_1)\,d\alpha, \int_a^b (f_2+g_2)\,d\alpha, \ldots\ldots, \int_a^b (f_k+g_k)\,d\alpha \right)$$

$$= \left( \int_a^b f_1\,d\alpha + \int_a^b g_1\,d\alpha,\ \int_a^b f_2\,d\alpha + \int_a^b g_2\,d\alpha \ldots\ldots\ldots\ldots, \int_a^b f_k\,d\alpha + \int_a^b g_k\,d\alpha \right) = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$$

Thus we have proved (i)

Let $c$ be any constant

By theorem 15.1.2, $c f_i \in \mathscr{R}(\alpha)$ on $[a,b]$ and $\int_a^b c f_i\,d\alpha = c\int_a^b f_i\,d\alpha$ for $1 \le i \le k$.

Since $cf = (cf_1, cf_2, \ldots\ldots, cf_k)$, we have $cf \in R(\alpha)$ on $[a,b]$

and $\int_a^b c f\,d\alpha = \left( \int_a^b c f_1\,d\alpha, \int_a^b c f_2\,d\alpha, \ldots\ldots\ldots\ldots, \int_a^b c f_k\,d\alpha \right)$

$$= \left( c\int_a^b f_1\,d\alpha, c\int_a^b f_2\,d\alpha, \ldots\ldots\ldots\ldots, c\int_a^b f_k\,d\alpha \right) = c\int_a^b f\,d\alpha$$

Thus we have proved (ii)

Similarly we prove the following Theorem by using theorem 15.1.4, Theorem 15.1.6 and Theorem 15.1.7.

**16.1.6 Theorem :** Let $f$ be a vector - valued function of $[a,b]$ into $\mathbb{R}^k$.

(i) If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $a<c<b$, then $f \in \mathscr{R}(\alpha)$ on $[a,c]$ and $f \in \mathscr{R}(\alpha)$ on $[c,b]$ and $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

(ii) If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$ on $[a,b]$, then $f \in \mathscr{R}(\alpha_1+\alpha_2)$ and

$$\int_a^b f\,d(\alpha_1+\alpha_2)=\int_a^b f\,d\alpha_1+\int_a^b f\,d\alpha_2$$

(iii) If $f\in\mathscr{R}(\alpha)$ on $[a,b]$ and $c$ is a positive constant, then $f\in\mathscr{R}(c\alpha)$ and

$$\int_a^b f\,d(c\,\alpha)=c\int_a^b f\,d\alpha$$

Theroem 16.1.1 is also true for vector-valued functions.

**16.1.7 Theorem :** If $f$ and $F$ map $[a,b]$ into $\mathbb{R}^k$, if $f\in\mathscr{R}$ on $[a,b]$ and if $F'=f$, then $\int_a^b f(t)\,dt=F(b)-F(a)$.

**Proof :** Suppose $f=(f_1,f_2,\ldots\ldots,f_k)$ and $F=(F_1,F_2,\ldots\ldots,F_k)$ map $[a,b]$ into $\mathbb{R}^k$ and $f\in\mathscr{R}$ on $[a,b]$ and $F'=f$.

Then $f_i\in\mathscr{R}$ on $[a,b]$ and $F_i'=f_i$ for $1\le i\le k$.

By Theorem 16.1.2, $\int_a^b f_i(x)dx=F_i(b)-F_i(a)$ for $1\le i\le k$.

Therefore $$\int_a^b f(x)\,dx=\left(\int_a^b f_1(x)dx,\int_a^b f_2(x)dx,\ldots\ldots\ldots,\int_a^b f_k(x)dx\right)$$

$$=\left(F_1(b)-F_1(a),F_2(b)-F_2(a),\ldots\ldots\ldots,F_k(b)-F_k(a)\right)$$

$$=F(b)-F(a)$$

Thus $$\int_a^b f(x)dx=F(b)-F(a)$$

**16.1.8 Theorem :** If $f$ maps $[a,b]$ into $\mathbb{R}^k$ and if $f\in\mathscr{R}(\alpha)$ for some montonically increasing function $\alpha$ on $[a,b]$, then $|f|\in\mathscr{R}(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right|\le\int_a^b |f|d\alpha$.

**Proof :** Suppose $f=(f_1, f_2, \ldots\ldots\ldots, f_k)$ maps $[a, b]$ into $\mathbb{R}^k$ and suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$ for some monotonically increasing function $\alpha$ on $[a, b]$. Then $f_i \in \mathscr{R}(\alpha)$ on $[a, b]$ for $1 \le i \le k$ and

$|f| = \left(f_1^2 + f_2^2 + \cdots\cdots\cdots + f_k^2\right)^{1/2}$.

Since $f_i \in \mathscr{R}(\alpha)$ on $[a, b]$, by theorem 15.1.8, $f_i^2 \in \mathscr{R}(\alpha)$ for $1 \le i \le k$. Then by theorem 15.1.1, $\sum\limits_{i=1}^{k} f_i^2 \in \mathscr{R}(\alpha)$.

Since $x^2$ is a continuous function of $x$, by a known theorem, the square root function is continuous on $[O, M]$ for every positive real number $M$.

Since $|f| = \left(f_1^2 + f_2^2 + \cdots\cdots\cdots + f_k^2\right)^{1/2}$, by theorem 14.1.16 we have $|f| \in \mathscr{R}(\alpha)$ on $[a, b]$.

Now, we will show that $\left|\int\limits_a^b f\, d\alpha\right| \le \int\limits_a^b |f|\, d\alpha$

Put $y_j = \int\limits_a^b f_j\, d\alpha$ for $1 \le j \le k$ and write $y=(y_1, y_2, \ldots\ldots\ldots, y_k)$

Then we have $y = \int\limits_a^b f\, d\alpha$ and $|y|^2 = \sum\limits_{j=1}^{k} y_j^2 = \sum\limits_{j=1}^{k} y_j \int\limits_a^b f_j\, d\alpha = \int\limits_a^b \left(\sum\limits_{j=1}^{k} y_j f_j\right) d\alpha$

By the Schwarz inequality, $\sum\limits_{j=1}^{k} y_j f_j(t) \le |y|\,|f(t)|$ for all $t \in [a, b]$.

By theorem 15.1.3 $|y|^2 \le |y| \int\limits_a^b |f|\, d\alpha$ -------- (1).

If $y=0$, then trivially $\left|\int\limits_a^b f\, d\alpha\right| \le \int\limits_a^b |f|\, d\alpha$.

If $y \ne 0$, then divide (1) by $|y|$ on both sides. Then we have

$|y| \leq \int_a^b |f| \, d\alpha$. That is, $\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$.

## 16.2 RECTIFIABLE CURVES

**16.2.1 Definition :** A continuous mapping $r$ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a curve in $\mathbb{R}^k$. In this case we some times say that $r$ is a curve on $[a, b]$.

If $r$ is one - to - one, $r$ is called an arc. If $r(a) = r(b)$, $r$ is said to be a closed curve.

We associate to each partition $P = \{x_0, x_1, ........, x_n\}$ of $[a, b]$ and to each curve $r$ on $[a, b]$ the number

$$\wedge(P, r) = \sum_{i=1}^{n} |r(x_i) - r(x_{i-1})|.$$

The $i$ th term in this sum is the distance $\left(\text{in } \mathbb{R}^k\right)$ between the points $r(x_{i-1})$ and $r(x_i)$. Hence $\wedge(P,r)$ is the length of a polygonal path with vertices at $r(x_0), r(x_1), ......, r(x_n)$ in this order. This polygon approaches the range of $r$ if $|P| \to 0$. Hence the following definition is reasonable.

**16.2.2 Definition :** Let $r$ be a curve on $[a, b]$. We define the length of $r$, denoted by $\wedge(r)$, as

$$\wedge(r) = \sup\{\wedge(P, r)/P \text{ is a partition of } [a, b]\}$$

We say that $r$ is rectifiable, if $\wedge(r)$ is finite.

In the case of continuously differentiable curves, i.e. for curves $r$ whose derivative $r'$ is continuous, $\wedge(r)$ is given by a Riemann integral.

**16.2.3 Theorem :** If $r$ is continuously differentiable on $[a, b]$, then $r$ is rectifiable and

$$\wedge(r) = \int_a^b |r'(t)| \, dt.$$

**Proof :** Suppose $r$ is continuously differetentiable on $[a, b]$, Let $P = \{x_0, x_1, ............, x_n\}$ be any partition of $[a, b]$. Consider $\left|r(x_i)-r(x_{i-1})\right| = \left|\int\limits_{x_{i-1}}^{x_i} r'(t)dt\right| \le \int\limits_{x_{i-1}}^{x_i} \left|r'(t)\right|dt$ for $1 \le i \le n$. (By theorem 15.1.8) This implies that $\wedge(P,r) = \sum\limits_{i=1}^{n}\left|r(x_i)-r(x_{i-1})\right| \le \sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i} \left|r'(t)\right| dt = \int\limits_{a}^{b}\left|r'(t)\right| dt$

So for any partition $P$ of $[a, b]$, $\wedge(P,r) \le \int\limits_{a}^{b}\left|r'(t)\right| dt$.

Consequently $\wedge(r) \le \int\limits_{a}^{b}\left|r'(t)\right|dt$ ------------ (1).

Let $\in > 0$. Write $\in_1 = \dfrac{\in}{2((b-a)+1)}$

Since $r$ is continuously differentiable on $[a, b]$, $r'$ is continuous on $[a, b]$, Since $[a, b]$ is compact and $r'$ is continuous on $[a, b]$, $r'$ is uniformly continuous on $[a, b]$. Then there exits $\delta > 0$ such that $\left|r'(s) - r'(t)\right| < \in_1$ whenever $s, t \in [a, b]$ with $|s-t| < \delta$ ------------ (2)

Let $P = \{x_0, x_1, \ldots\ldots\ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta$ for $1 \le i \le n$ (See theorem 14.1.12).

If $t \in [x_{i-1}, x_i]$, then $\left|r'(t)\right| = \left|r'(t) - r'(x_i) + r'(x_i)\right|$

$$\le \left|r'(t)-r'(x_i)\right| + \left|r'(x_i)\right| < \left|r'(x_i)\right| + \in_1 \text{ ----- (By (2))}$$

This implies that $\int\limits_{x_{i-1}}^{x_i} \left|r'(t)\right|dt \le \left|r'(x_i)\right|\Delta x_i + \in_1 \Delta x_i$

$$= \left|\int\limits_{x_{i-1}}^{x_i} r'(x_i)dt\right| + \in_1 \Delta x_i = \left|\int\limits_{x_{i-1}}^{x_i} \left[r'(t)+r'(x_i)-r'(t)\right] dt\right| + \in_1 \Delta x_i$$

$$\le \left|\int\limits_{x_{i-1}}^{x_i} r'(t)dt\right| + \left|\int\limits_{x_{i-1}}^{x_i} \left[r'(x_i) - r'(t)\right] dt\right| + \in_1 \Delta x_i$$

$$\le \left|r(x_i)-r(x_{i-1})\right| + 2\in_1 \Delta x_i \quad \text{------------- (By (2))}$$

Consider $\int_a^b \left|r'(t)\right| dt = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left|r'(t)\right| dt$

$$\le \sum_{i=1}^{n} \left|r(x_i) - r(x_{i-1})\right| + 2\in_1 \sum_{i=1}^{n} \Delta x_i = \wedge(P, r) + 2\in_1 (b-a)$$

$$\le \wedge(r) + 2\in_1 (b-a) = \wedge(r) + \frac{2\in(b-a)}{2((b-a)+1)} < \wedge(r) + \in$$

Therefore $\int_a^b \left|r'(t)\right| dt \le \wedge(r) + \in$

Since $\in > 0$ is arbitrary, $\int_a^b \left|r'(t)\right| dt \le \wedge(r)$ --------- (3)

From (1) and (3), $\int_a^b \left|r'(t)\right| dt = \wedge(r)$.

Hence $r$ is rectifiable and $\wedge(r) = \int_a^b \left|r'(t)\right| dt$

## 16.3 SHORT ANSWER QUESTIONS

**16.3.1 :** State the fundamental theorem of calculus.

**16.3.2 :** Define a curve - when do you say that a curve is rectifiable ?

## 16.4 MODEL EXAMINATION QUESTIONS

**16.4.1 :** State and prove the fundamental theorem of calculus.

**16.4.2 :** Suppose $F$ and $G$ are differentiable functions on $[a, b]$, $F' = f \in \mathcal{R}$ and $G' = g \in \mathcal{R}$.

Show that $\int_a^b F(x) g(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x) G(x) dx$.

**16.4.3 :** If $r$ is continuously differentiable on $[a, b]$, then show that $r$ is rectifiable and

$$\wedge(r) = \int_a^b |r'(t)| \, dt$$

## 16.5 EXERCISES

**16.5.1 :** Let $r_1, r_2, r_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by

$$r_1(t) = e^{it}, \; r_2(t) = e^{2it}, \; r_3(t) = e^{2\pi it \sin(1/t)}$$

Show that these three curves have the same range, that $r_1$ and $r_2$ are rectifiable, that the length of $r_1$ is $2\pi$, that the length of $r_2$ is $4\pi$ and that $r_3$ is not rectifiable.

## 16.6 ANSWERS TO SHORT ANSWER QUESTIONS

For 16.3.1, see theorem 16.1.2

For 16.3.2, see definition 16.2.1 and definition 16.2.2.

## REFERENCE BOOK :

Principles of Mathematical Analysis, Third Edition, Mc Graw - Hill International Editions : Walter Rudin

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Lesson - 17

# UNIFORM CONVERGENCE - I

## 17.0 INTRODUCTION

Uniform convergence is the fundamental notion required for study of spaces of continuous functions. If $K$ is a compact metric space, then $C(K)$, the linear space of all (bounded) complex valued continuous functions on $K$, is a complete metric (hence Banach) space with respect to the uniform metric defined by

$$d(f,g) = \sup\{|f(x)-g(x)|/x \in K\}.$$

Convergence in this metric is nothing but uniform convergence. This notion is also connectd with compactness in $C(K)$.

This lesson provides an introduction to such a fundamental concept in Analysis. We define uniform convergence of sequences and series of functions on a set $E$, obtain Cauchy general principle for uniform convergence of sequences and series of functions, derive a sufficient condition for uniform convergence of a series of functions, namely the famous Weierestrass M-test and provide a number of examples that will be useful for further Analysis of the topic.

Let us recall that a sequence $\{a_n\}$ in $\mathbb{C}$ is convergent if there exists a number "$a$", called the limit of $\{a_n\}$ and denoted by $\lim a_n$, if for every positive real number $\in$ there corresponds a positive integer $N(\in)$ depending possibly on $\in$ such that

$$|a_n - a| < \in \text{ for } n \geq N(\in).$$

We further make the observation that the inequalities $<$ and $\leq$ is the above definiton may be replaced by <u>any</u> of $<$ and $\leq$.

When the $a_n$'s are the values $f_n(x)$ where $\{f_n\}$ is a sequence of functions defined on a common domain $E$ and $x \in E$, the limit depends naturally on $x$ and thus defines a function of $x$. The $N(\in)$ also depends on $x$ and is now to be written as $N(\in, x)$ as it depends upon $\in > 0$ as well as $x \in N$. There is another possibility as well. The $N(\in)$ may not change with $x$. In this context we thus have to distinguish between pointwise convergence in which case the $N(\in)$ is dependent on $x$ and uniform convergence where $N(\in)$ is independent of $x$. We now state the definitions of these two types of convergence. In what follows, we mean by a function we mean a (real or) complex valued function.

## 17.2 DEFINITIONS

Let $E$ be a set, $\{f_n\}$ a sequence of functions defined on $E$ and $f$ be a function defined on $E$ and $f$ be a function defined on $E$. We say that

(a) $\{f_n\}$ converges to $f$ pointwise or converges pointwise to $f$ on $E$ if for every $x \in E$, $\lim f_n(x) = f(x)$ i.e.

if for every positive number $\in$ and $x \in E$, there corresponds a positive integer $N(\in, x)$ depending on $\in$ and $x$ as well, such that

$$|f_n(x) - f(x)| < \in \text{ whenever } n \geq N(\in, x).$$

In this case we say that $f$ is the pointwise limit of $\{f_n\}$ on $E$ and write $\lim f_n = f$ (pointwise).

When there is a $f$ defined on $E$ such that $\{f_n\}$ converges pointwise to $f$ on $E$ we simply say that $\{f_n\}$ converges pointwise - without explicitly mentioning the limit function $f$.

(b) Let $s_n(x) = f_1(x) + \cdots\cdots\cdots + f_n(x)$ for $x \in E$. If the sequence of functions (called the partial sums of $\sum_{n=1}^{\infty} f_n(x)$) converges pointwise to the function $f(x)$, we say that the series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f$ and write

$$\sum_{n=1}^{\infty} f_n = f \qquad \text{(p.w.) or (pointwise)}$$

(c) we say that $\{f_n\}$ converges uniformly to $f$ on $E$ if for every positive number $\in$ there corresponds a positive integer $N(\in)$ such that

$|f_n(x) - f(x)| < \in$ for $n \geq N(\in)$ and all $x \in E$ and all $x \in E$.

In this case we say that $f$ is the uniform limit of $\{f_n\}$ and write $\lim f_n(x) = f(x)$ on $E$.

or $\lim f_n = f$ uniformly on $E$.

(d) We say that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$ on $E$ if the sequence of partial sums $\{s_n(x)\}$ defined in (b) above converges uniformly on $E$ to $f(x)$.

i.e. for every positive number $\in$ there corresponds a positive integer $N(\in)$ such that

$$|s_n(x) - f(x)| < \in \text{ whenever } n \geq N(\in) \text{ and } x \in E.$$

**17.3 REMARKS**

It is clear that if $\{f_n\}$ converges uniformly to $f$ on $E$, then $\{f_n\}$ converges to $f$ pointwise on $E$. However pointwise convergence does not imply uniform convergence. Let us keep in mind that for substanticiating failure of uniform convergence of a sequence $\{f_n\}$ to $f$ on a set $E$ we have to search for and find an $\in > 0$ such that for every positive integer $K$, there is a positive integer $n_K > K$ such that

$$|f_{n_K}(x) - f(x)| \geq \in$$

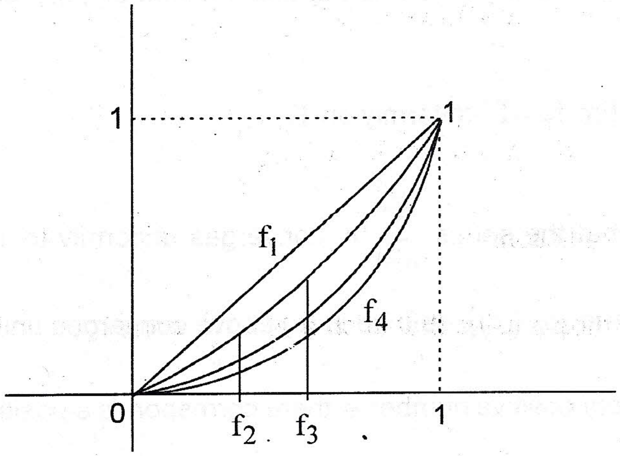
for at least one $x$ in $E$.

**17.4 EXAMPLE :**

Define $f_n(x) = x^n$ on $[0, 1]$. Then $f_n(0) = 0$ and $f_n(1) = 1$ for every $n \geq 1$. So $\lim f_n(0) = 0$ and $\lim f_n(1) = 1$.

If $0 < x < 1$ and $y = \frac{1}{x}$ then $y > 1$ so $y - 1 = h > 0$ and

$$y^n = (1+h)^n = 1+nh+\binom{n}{2}h^2+\cdots\cdots+h^n > 1+nh\cdot$$




Thus $x^n < \frac{1}{nh} < \in$ for all $n \geq N(\in, x)$.

If $\in > 0 \frac{1}{nh} < \in$ when $n > \frac{1}{h\in}$. Thus if $N(\in, x)$ is any fixed integer $> \frac{1}{h\in}$ and $n \geq N(\in, x)$ then

$$x^n < \frac{1}{nh} < \in \text{ for all } n \geq N(\in, x).$$

This implies that $\lim f_n(x) = \lim x^n = 0$ if $0<x<1$.

Write $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x=1 \end{cases}$

it is now clear that $\{f_n\}$ converges pointwise to $f$ on $[0, 1]$. However the convergence is not uniform because if it were there would exist a positive integer $N_{\frac{1}{2}}$ corresponding to $\in = \frac{1}{2}$ so that $|f_n(x) - f(x)| < \frac{1}{2}$ for all $n \geq \frac{1}{N_{\frac{1}{2}}}$ and all $x \in [0,1]$. We write $N$ for $N_{\frac{1}{2}}$. Then

$$|f_N(x) - f(x)| < \frac{1}{2} \text{ for all } x \text{ in } [0, 1]$$

$$\Rightarrow \left|x^N\right| = x^N < \frac{1}{2} \text{ for all } x \text{ in } [0, 1)$$

$$\Rightarrow \forall x \ni 0 \le x < 1, \quad x^N < \frac{1}{2}$$

$$\Rightarrow \forall x \ni 0 \le x < 1, \quad x < \left(\frac{1}{2}\right)^N.$$

This is a contradiction.

This contradiction implies that such a $N = \frac{N_1}{2}$ corresponding to $\in = \frac{1}{2}$ does not exist.

**17.5 EXAMPLE :**

Let $f_n(x) = \frac{x^2}{\left(1+x^2\right)^{n-1}}$ for $n \ge 1$ for $x \in \mathbb{R}$ and $s_n(x) = \sum_{K=1}^{n} f_K(x)$.

For $x \neq 0$ $\qquad s_n(x) = \sum_{K=1}^{n} \frac{x^2}{\left(1+x^2\right)^{K-1}}$

$$= x^2 \sum_{K=1}^{n} \frac{1}{\left(1+x^2\right)^{K-1}}$$

$$= \left(1+x^2\right)\left\{1 - \frac{1}{\left(1+x^2\right)^n}\right\}$$

Since $0 < \frac{1}{1+x^2} < 1$, $\lim_n \left(\frac{1}{1+x^2}\right)^n = 0$

So $\lim s_n(x) = 1 + x^2$ if $x \neq 0$

Clearly $\lim s_n(0)=0$.

If we define f by $f(x)=\begin{cases}0 \text{ if } x=0\\ 1+x^2 \text{ if } x\neq 0\end{cases}$

then $\sum_{n=1}^{\infty} f_n(x)=\lim_n s_n(x)=f(x)$.

Thus the series converges pointwise on $\mathbb{R}$ to $f$. We prove that the convergence is not uniform on $\mathbb{R}$. Suppose on the contrary that the convergence is uniform on $\mathbb{R}$. Then $\{s_n(x)\}$ would converge uniformly to $f$ on $\mathbb{R}$. Then corresponding to $\in=\frac{1}{2}$ there would exist a positive integer $N$ such that for $n\geq N$.

$$|s_N(x)-f(x)|<\frac{1}{2} \text{ for all } x\in\mathbb{R}.$$

In particular when $n=N$ and $x\neq 0$, $s_N(x)=\frac{1}{\left(1+x^2\right)^N}$ and $f(x)=0$.

so that $\frac{1}{(1+x)^N}<\frac{1}{2}$ for all $x\neq 0$.

$$\Rightarrow \left(1+x^2\right)^N>2 \text{ for } x\neq 0$$

$$\Rightarrow x^2>2^{\frac{1}{N}}-1 \text{ for } x\neq 0$$

$$\Rightarrow |x|>\left(2^{\frac{1}{N}}-1\right)^{1/2} \text{ for all } x\neq 0.$$

$$\Rightarrow \mathbb{R}-\{0\}\subseteq\left\{x/|x|>\left(2^{\frac{1}{N}}-1\right)^{\frac{1}{2}}\right\}$$

This is impossible.

*Hence the convergence is not uniform on $\mathbb{R}$.*

We now prove an analogue of Cauchy's general principle for uniform convergence.

**17.6 Theorem :** Let $\{f_n\}$ be a sequence of functions defined on $E$. Then $\{f_n\}$ converges uniformly to some function defined on $E$ if and only if corresponding to every positive number $\in$ there exists a positive integer $N(\in)$ such that

for all positive integers $n, m$ each $\geq N(\in)$ and all $x \in E$

$$|f_n(x) - f_n(x)| < \in.$$

**Proof** $(\Rightarrow)$: If $\{f_n\}$ converges uniformly to $f$ on $E$ and $\in > 0$ there corresponds a positive integer $N(\in)$ depending on $\frac{\in}{2}$ so that $|f_n(x) - f(x)| < \frac{\in}{2}$ for all $n \geq N(\in)$ and all $x \in E$

$\Rightarrow$ for all $n \geq N(\in)$ and $m \geq N(\in)$ and all $x \in E$

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in.$$

Conversely assume that the condition holds.

That is for every positive number $\in$ there corresponds a positive integer $N(\in)$ such that for all $n$, $m$ both $\geq N(\in)$ and all $x \in E$

$$|f_n(x) - f_m(x)| < \in$$

Then $\forall\ x \in E \cdot \{f_n(x)\}$ is a sequence of numbers that satisfies Cauchys general principle for convergence. So there exists a number depending on $x$ to which $\{f_n(x)\}$ converges. We denote this number by $f(x)$ as this depends on $x$. Clearly $f(x)$ is uniquely fixed with $x$. So $x \to f(x)$ defines a function on $E$ such that $f(x) = \lim_n f_n(x) \forall\ x \in E$.

We show that the convergence is uniform.

Since $f(x) = \lim f_n(x)$, for every positive integer $m$

$$|f(x)-f_m(x)| = \lim_n |f_n(x)-f_m(x)|.$$

Now let $\in > 0$. By the converse hypothesis there corresponds a positive integer $N(\in)$ such that

$$|f_n(x)-f_m(x)| < \in \text{ for all } x \in E \text{ and } n \geq N(\in) \text{ and } m \geq N(\in)$$

Hence $\lim_n |f_n(x)-f_m(x)| < \in$ for all $x \in E$ and $m \geq N(\in)$

$\Rightarrow |f(x)-f_m(x)| \leq \in$ for all $x \in E$ and $m \geq N(\in)$.

This implies that $\{f_m\}$ converges uniformly to $f$ on $E$

**17.6 COROLLARY :**

The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E If and only if for every positive number $\in$ there corresponds a positive integer $N(\in)$ such that $|f_{n+1}(x)+\cdots\cdots+f_m(x)| < \in$ for all $m > n \geq N(\in)$ and all $x \in E$.

**17.7 PROOF :**

If $m > n$ and $x \in E$

$$s_m(x)-s_n(x) = f_{n+1}(x)+f_{n+2}(x)+\cdots\cdots+f_m(x).$$

The rest is direct application of 17.6 to $\{s_n(x)\}$.

**17.8 REMARK :**

**(i)** If $\{f_n\}$ converges uniformly on $E$ to $f$ and $A \subseteq E$ then $\{f_n\}$ converges uniformly to $f$ on $A$.

**(ii)** If $\{f_n\}$ converges uniformly to $f$ on $E$ and also on $F$ then $\{f_n\}$ converges uniformly to $f$ on $E \cup f$.

**17.9 THEOREM :**

A sequence of functions $\{f_n\}$ defined on a set $E$ converges uniformly to a function $f$ defined on $E$ if and only if the sequence of numbers $\{M_n\}$ defined by

$$M_n = \sup\{|f_n(x) - f(x)| / x \in E\} \text{ converges to 0.}$$

**Proof :** If $\lim M_n = 0$ and $\in > 0$ there exists a positive integer $N(\in)$ such that $0 \le M_n < \in$ for $n \ge N(\in)$.

If $x \in E$ and $n \ge N(\in)$ $|f_n(x) - f(x)| \le M_n < \in$

Also this holds good for all $x \in E$ and $n \ge N(\in)$ $\{f_n\}$ converges uniformly to $f$ on $E$.

Conversely if $\{f_n\}$ converges uniformly to $f$ on $E$ and $\in > 0$ there exists a positive integer $N(\in)$ such that $|f_n(x) - f(x)| < \frac{\in}{2}$ for all $n \ge N(\in)$ and all $x \in E$.

Hence for $n \ge N_{\in}$, $0 \le M_n = \sup\{|f_n(x) - f(x)|\} \le \frac{\in}{2} < \in$. This is true for every $\in > 0$ so that $\lim M_n = 0$.

**17.10 COROLLARY :**

A sequence $\{f_n\}$ of functions defined on a set $E$ converges to a function $f$ on $E$ uniformly if there is a sequence of numbers $\{A_n\}$ such that

$$|f_n(x) - f(x)| < A_n \text{ for all } x \in E \text{ and all } n \text{ and } \lim A_n = 0.$$

**17.11 WEIERSTRASS M-TEST FOR UNIFORM CONVERGENCE :**

Let $\{f_n\}$ be a sequence of functions defined on a set $E$ and $\{M_n\}$ be a sequence of numbers such that $|f_n(x)| \le M_n$ for all $x \in E$ and all positive integers $n$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $E$ if $\sum_{n=1}^{\infty} M_n$ converges.

**Proof :** Let $s_n(x) = f_1(x)+f_2(x)+\cdots\cdots+f_n(x)$ for $n \ge 1$ and $x \in E$ and $S_n = M_1+\cdots\cdots+M_n$ for $n \ge 1$. Since $\sum_{n=1}^{\infty} M_n$ converges, given $\in > 0$ there exists a positive integer $N(\in)$ such that for $m > n \ge N(\in)$ $|S_m - S_n| < \in$

Thus for $m > n \ge N(\in)$ and $x \in E$

$$|s_m(x)-s_n(x)| = |f_{n+1}(x)+\cdots\cdots+f_m(x)|$$

$$\le |f_{n+1}(x)|+\cdots\cdots+|f_m(x)|$$

$$\le M_{n+1}+\cdots\cdots+M_m$$

$$= |S_m - S_n| < \in$$

Hence by 17.7 $\sum f_n(x)$ converges uniformly on $E$ to some function defined on $E$.

**17.12 EXAMPLE :**

Let $f_n(x) = (-1)^n \dfrac{x^2+n}{n^2}$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on every bounded interval but does not converge absolutely for any $x$.

If $I$ is any bounded interval, there is a positive real number $K$ such that $I \subseteq [-K, K]$ so that $|x| \le K \ \forall \ x \in I$.

$f_n(x) = \dfrac{(-1)^n x^2}{n^2} + \dfrac{(-1)^n}{n}$. Since $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ is a convergent series of constant terms, uniform convergence of $\sum_{n=1}^{\infty} \dfrac{(-1)^n x^2}{n^2}$. On $I$ $|x| \le K$ so that $\dfrac{\left|(-1)^n x^2\right|}{n^2} \le \dfrac{K^2}{n^2}$. The series $\sum_{n=1}^{\infty} \dfrac{K^2}{n^2}$ is convergnet, hence by Weierstrass M - test 17.11 the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n x^2}{n^2}$ converges uniformly on $I$. This implies that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $I$.

For any $x \in \mathbb{R}$ $|f_n(x)| = \frac{x^2}{n^2} + \frac{1}{n}$.

Since $\sum_{n=1}^{\infty} \frac{x^2}{n^2}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} |f_n(x)|$ diverges.

**17.13 :** Let $f_n(x) = \frac{x}{1+n\,x^2}$. Then $\{f_n\}$ converges uniformly.

For any $x \in \mathbb{R}$, and positive integer $n$,

$$\left(1-\sqrt{n}\,|x|^2\right) \geq 0.$$

$$\Rightarrow |f_n(x)| = \frac{|x|}{1+n x^2} \leq \frac{1}{2\sqrt{n}}$$

Since $\lim \frac{1}{2\sqrt{n}} = 0$, by 17.9 $\{f_n(x)\}$ converges uniformly to 0.

**17.14 EXAMPLE :**

$$\text{Let } f_n(x) = \begin{cases} 0 \text{ if } x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} \text{ if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 \text{ if } x > \frac{1}{n} \end{cases}$$

Then $\{f_n\}$ does not converge uniformly but converges pointwise to a function and $\sum |f_n|$ converges absolutely but not uniformly.

If $x \leq 0$ $f_n(x)=0 \forall n$ so $f(x) = \lim f_n(x)=0$. I $x>0$ there is a pointwise integer $N$ such that $\frac{1}{N}<x$ so that $\frac{1}{n} \leq \frac{1}{N} < x$ for $n \geq N$. For such $n$ $f_n(x)=0$ hence $f(x)=\lim f_n(x)=0$ for $x>0$. Thus $\{f_n\}$ converges pointwise to the zero function on $\mathbb{R}$.

If the convergence were uniform, for $\in = \frac{1}{2}$ there would exist a positive integer $N$ such that $|f_n(x)|< \frac{1}{2}$ for $n \geq N$ and all $x \in \mathbb{R}$

In particular $|f_N(x)| < \frac{1}{2}$ for all $x \in \mathbb{R}$. However for $x=\frac{1}{N+\frac{1}{2}}$, $\frac{1}{N+1} < x < \frac{1}{N}$ so

$$f_N(x) = \sin^2 \frac{\pi}{2}(2N+1)=1.$$

If $x>1$ $f_n(x) = 0 \ \forall \ n \Rightarrow \sum|f_n(x)| = 0$

If $x=1$ $f_1(x)= 0$ and also $f_n(x)=0$ for $n>1$.

If $0<x<1$ there is a unique positive integer $m$ such that $\frac{1}{m+1} \leq x < \frac{1}{m}$ for this $m$ $f_m(x) = \sin^2 \frac{\pi}{x}$ while $f_n(x) = 0$ for other $n$.

In this case $\sum\limits_{n=1}^{\infty} |f_n(x)| = \sin^2 \frac{\pi}{x}$.

Thus the series $\sum\limits_{n=1}^{\infty} |f_n(x)|$ converges absolutely for every $x$, but $\sum\limits_{n=1}^{\infty} f_n(x)$ does not converge uniformly as this would imply uniform convergence of the sequence $\{f_n\}$ to 0.

**17.15 PROPOSITION :**

If $\{f_n\}$ and $\{g_n\}$ are uniformly convergent sequences of functions defined on a set $E$, then $\{f_n + g_n\}$ and $\{\alpha f_n\}$ for any number $\alpha$ are uniformly convergent on $E$.

**Proof :** By Cauchy's general principle for uniform convergence, for $\in > 0$ the correspond positive integers $N_1, N_2$ such that $|f_n(x) - f_m(x)| < \frac{\in}{2}$ for $n > m \geq N_1$ and all $x \in E$ and $|g_n(x) - g_m(x)| < \frac{\in}{2}$ for $n > m \geq N_2$ and all $x \in E$. If $N(\in) = \max\{N_1, N_2\}$ then for $n > m \geq N(\in)$ and all $x \in E$

$$|(f_n + g_n)(x) - (f_m + g_m)(x)|$$

$$\leq |f_n(x) - f_m(x)| + |g_n(x) - g_m(x)|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in.$$

Hence, again by Cauchy's general principle for uniform convergence, $\{f_n + g_n\}$ converges uniformly on $E$.

Also $|\alpha f_n(x) - \alpha f_m(x)|$

$$= |\alpha| |f_n(x) - f_m(x)|$$

$$< (1 + |\alpha|) \in \text{ for } n > m \geq N(\in) \text{ and } x \in E.$$

Since $\in > 0$ is arbitrary, it follows that $\{\alpha f_n\}$ converges uniformly on $E$.

**17.16 EXAMPLE :**

The sequence $\left\{\frac{x}{n}\right\}$ converges pointwise to the zero function on $\mathbb{R}$ but the convergence is not uniform on $\mathbb{R}$.

If $\in > 0$ and $\left|\frac{x}{n}\right| < \in$ whenever $n > \frac{|x|}{\in}$, this gives pointwise convergence to the zero function.

On the other hand, given $\in > 0$ and any positive integer $N$, $\left|\frac{x}{N}\right| > \in$ if $|x| > N \cdot \in$. Hence the convergence is not uniform on $\mathbb{R}$.

**17.17 PROPOSITION :**

Suppose $\{f_n\}$ and $\{g_n\}$ converge uniformly on $E$ and there exists a $M > 0$ such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n$ and $x \in E$. Then $\{f_n\, g_n\}$ converges uniformly on $E$.

Given $\in > 0$ $\exists$ positive integers $N_1, N_2$ such that

$$|f_n(x) - f_m(x)| < \frac{\in}{2M} \text{ for } n > m \geq N, \text{ and } x \in E \text{ and}$$

$$|g_n(x) - g_m(x)| < \frac{\in}{2M} \text{ for } n > m \geq N_2 \text{ and } x \in E.$$

If $N(\in) = \max\{N_1, N_2\}$ and $n \geq m \geq N(\in)$ and $x \in E$

$$|f_n(x) g_n(x) - f_m(x), g_m(x)|$$

$$\leq |f_n(x)(g_n(x) - g_m(x))| + |g_m(x)(f_n(x) - f_m(x))|$$

$$= |f_n(x)| \cdot |g_n(x) - g_m(x)| + |g_m(x)|\, |f_n(x) - f_m(x)|$$

$$< \frac{\in}{2M} M + \frac{\in}{2M} M = \in.$$

Hence $\{f_n\, g_n\}$ converges uniformly on $E$.

**17.18 SHORT ANSWER QUESTIONS**

**17.18.1 :** For a sequence of numbers $\{a_n\}$ prove that the following are equivalent.

For every positive number $\in$ there corresponds a positive integer $N(\in)$.

(a) Such that $|a_n| < \in$ whenever $n \geq N(\in)$

(b) Such that $|a_n| \leq \in$ whenever $n \geq N(\in)$

(c) such that $|a_n| \le \in$ whenever $n > N(\in)$

(d) such that $|a_n| \le \in$ whenever $n > N(\in)$.

**17.18.2 :** Show that uniform convergence of $\{f_n\}$ and $\{g_n\}$ on a set $E$ does not necessarily imply that of $\{f_n\ g_n\}$

**17.18.3 :** If $\{f_n\}$ converges uniformly on $E$ and $F \subseteq E$, $\{f_n\}$ converges uniformly on $F$.

**17.18.4 :** If $\{f_n\}$ converges uniformly on $A$ and also on $B$ then $\{f_n\}$ converges uniformly on $A \cup B$.

**17.18.5 :** Let $f_n(x) = \dfrac{(nx)}{n^2}$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$ where, for any $a \in \mathbb{R}$.

$[a]$ is the largest integer not exceeding $a$ and $(a) = a - [a]$.

Show that $\sum\limits_{n=1}^{\infty} f_n(x)$ is uniformly convergent.

## 17.19 MODEL EXAMINATION QUESTIONS :

**17.19.1 :** Define pointwise and uniform a convergence of a sequence of functions. Show that $\left\{\dfrac{1}{nx+1}\right\}$ converges pointwise, but not uniformly on $(0, 1)$.

**17.19.2 :** Show that a sequence of functions $\{f_n\}$ defined on a set $E$ converges uniformly to a function $f$ defined on $E$ if and only if

$$\sup\left\{|f_n(x) - f(x)| / x \in E\right\} \to 0$$

**17.19.3 :** State and prove Weierstarss M - test on uniform convergence of a series of functions.

**17.19.4 :** Discuss (i) uniform convergence and (ii) absolute convergence of

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$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2} \quad \text{for } x \in (-\infty, \infty)$$

**17.19.5 :** Discuss the uniform convergence of $\{f_n\}$ on $[0, 1]$ where $f_n(x) = x^n$ for $n \geq 1$ and $x \in [0,1]$.

## 17.20 ANSWERS TO SHORT ANSWER QUESTIONS :

**17.20.1 :** $a \Rightarrow b \Rightarrow c$: clear.

For $c \Rightarrow d$, given $\in > 0$ choose $N\left(\frac{\in}{2}\right) \ni |a_n| \leq \frac{\in}{2}$ for $n \geq N\left(\frac{\in}{2}\right)$. Clearly $|a_n| \leq \frac{\in}{2} < \in$ for $n \geq N\left(\frac{\in}{2}\right)$, this $N\left(\frac{\in}{2}\right)$ works. For $d \Rightarrow a$ given $\in > 0$, for $N_1(\in)$ satisfying $d$ and write $N(\in) = 1 + N_1(\in)$. This new $N(\in)$ works.

**17.20.2 :** Let $f_n(x) = x \ \forall\ n \in \mathbb{N}$ and $x \in \mathbb{R}$ and $g_n(x) = \frac{1}{n} \forall\ n \in \mathbb{N}$ and $x \in \mathbb{R}$.

See 17.16.

**17.20.3 :** If $\in > 0$ and $N(\in)$ is a positive integer such that

$$|f_n(x) - f_m(x)| < \in \text{ for } n \geq N(\in) \text{ and } x \in E$$

then since $F \subset E$ the above inequality holds good for $x \in F$ and $n \geq N(\in)$

**17.20.4 :** If $\in > 0$ and $N_1$, $N_2$ are positive integers such that

$$|f_n(x) - f_m(x)| < \in \text{ for } x \in A \text{ and } n > m \geq N_1$$

and $$|f_n(x) - f_m(x)| < \in \text{ for } x \in B \text{ and } n > m > N_2$$

then we write $N(\in) = \max\{N_1, N_2\}$.

For $n > m \geq N(\in)$ and $x \in A \cup B$ then either $x \in A$ or $x \in B$. In the first case $|f_n(x) - f_m(x)| < \in$ since $N(\in) \geq N_1$ and in the second case the inequality holds such $N(\in) \geq N_2$.

**17.20.5 :** Clearly $0 \le f_n(x) \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Apply Weierstrass M - test.

## 17.21 EXERCISES :

**17.21.1 :** Let $f_n(x) = \frac{x^2}{\left(1+x^2\right)^n}$ $x \in \mathbb{R},\ n \in \mathbb{N}$

and $f(x) = \begin{cases} 0 \text{ if } x=0 \\ 1+x^2 \text{ if } x \ne 0 \end{cases}$

Show that $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f$ on $\mathbb{R}$, but not uniformly.

**17.21.2 :** Let $\left.\begin{aligned} f_n(x) &= \frac{\sin nx}{\sqrt{n}} \\ \text{and}\quad g_n(x) &= \sqrt{n}\cos nx \end{aligned}\right\} x \in \mathbb{R} \text{ and } n \in \mathbb{N}$

Does $\{f_n\}$ converge uniformly on $\mathbb{R}$ ?

Does $\{g_n\}$ converge uniformly on $\mathbb{R}$ ?

**17.21.3 :** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded. More precisely. Let $\{f_n\}$ converge uniformly on $E$. If $\forall n \exists a\ M_n > 0$ such that $|f_n(x)| \le M_n\ \forall\ n$ show that $\exists a\, M > 0$ such that $|f_n(x)| \le M\ \forall\ n \in \mathbb{N}$ and $x \in E$.

**17.21.4 :** Construct sequences $\{f_n\}\ \{g_n\}$ which converge uniformly on $E$ but such that $\{f_n\ g_n\}$ does not converge uniformly.

**17.21.5 :** Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$.

Show that the series converges pointwise if $0 < x < \infty$.

Show that the series converges absolutely and pointwise if $-1 < x < \infty$ and $x \ne 0$.

Show that the series converges pointwise if $-\frac{1}{n^2} < x < \frac{-1}{(n+1)^2} \ \forall \ n \in N$.

Find $E \subset \mathbb{R}$ consisting of all $x \ni \lim_{n} \frac{1}{1+n^2 x} = 0$.

Does this sequence converge uniformly on $E$ ?

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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Lesson - 18

# UNIFORM CONVERGENCE - II

## 18.1 INTRODUCTION

One natural question that arises in connection with sequences of functions is that if a property is possessed by each member of a sequence $\{f_n\}$ and $f(x) = \lim_n f_n(x) \forall x$ does $f$ have the property ? This property may be, for example boundedness, continuity, integrability or differentiability.

In this lesson we study this aspect. Pointwise convergence is not strong enough for inheritance of these properties by the limit function where as boundedness, continuity and integrability can be obtained by $f$ under uniform convergence. Differentiation needs more assumptions besides uniform convergence. We provide a number of examples that focus more light on various possibilities.

## UNIFORM CONVERGENCE AND CONTINUITY :

### 18.2 THEOREM :

Let $(X, d)$ be a metric space, $E \subseteq X$ and $\{f_n\}$ be a sequence of continuous functions defined on $E$. If $\{f_n\}$ converges uniformly to $f$ on $E$ then $f$ is continuous on $E$.

**Proof :** We show that $f$ is continuous at every point of $E$. Let $x \in E$ and $\in > 0$. Since $\{f_n\}$ converges uniformly to $f$, there is a positive integer $N(\in)$ such that

$$|f_n(y) - f(y)| < \frac{\in}{3} \text{ for } n \geq N(\in) \text{ and all } y \in E \text{ --------- (1)}$$

Since $f_{N(\in)}$ is continuous at $x$, there a $\delta > 0$ such that

$$\left|f_{N(\in)}(x) - f_{N(\in)}(y)\right| < \frac{\in}{3} \text{ for } y \in E \ni d(x, y) < \delta \text{ --------- (2)}$$

If $y \in E$ and $d(x, y) < \delta$

$$|f(x)-f(y)| \le |f(x)-f_{N(\in)}(x)| + |f_{N(\in)}(x)-f_{N(\in)}(y)| + |f_{N(\in)}(y)-f(y)|$$

$$< \frac{\in}{3}+\frac{\in}{3}+\frac{\in}{3} = \in \qquad \text{by (1) and (2)}$$

Hence $f$ is continuous at $x$.

This is true for every $x \in E$ hence $f$ is continuous on $E$.

**18.3 COROLLARY :**

Let $(X, d)$ be a metric space, $E \subseteq X$ and $\{f_n\}$ be a sequence of continuous functions defined on E. If the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$ on E then $f$ is continuous on E.

**Proof :** Let $s_n(x) = f_1(x) + \cdots\cdots + f_n(x)$ for $n \ge 1$ and $x \in E$. Then $\{s_n\}$ is a sequence of continuous functions defined on $E$, which converges uniformly to $f$ on $E$.

By 18.2 $f$ is continuous on E.

**18.4 EXAMPLE :**

Consider the series $\sum f_n$ of exercise 5 in lesson 17. Where $f_n(x) = \dfrac{1}{1+n^2 x}$ $(x \in \mathbb{R})$

Since $n^2 x + 1 = 0 \Leftrightarrow x = -\dfrac{1}{n^2}$ the common domain for the sequence of functions $\{f_n\}$ is

$D = \mathbb{R} \setminus \left\{ -\dfrac{1}{n^2} \,\middle|\, \begin{matrix} |n| \ge 1 \\ n \in z \end{matrix} \right\}$. If $k \in \mathbb{R}$ and $k > 1$ write $\Delta_k = \mathbb{R} - (-k, k) = \{x \in \mathbb{R} / |x| \ge k\}$.

Since $|1+n^2 x| \ge n^2 |x| - 1 \ge n^2 k - 1 \ge n^2 (k-1)$ for $x \in \Delta_k$ and $n \ge 1$.

$$|f_n(x)| \le \frac{1}{n^2 (k-1)} \text{ for } x \in \Delta_k.$$

Since the series $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} f_n$ converges uniformly and absolutely on $\Delta_k$

Since $f_n$ is continuous on $\Delta_k$ $\forall\, n \geq 1$, the sum of the series $\sum_{n=1}^{\infty} f_n$ is continuous on $\Delta_k$ $\forall\, k>1$.

Clearly the series diverges when $x=0$.

Let $E$ be any compact subset of $D\setminus\{0\}\cap[-1, 1]$. Then there exist positive numbers a and b such that $a<b$ and $a \leq |x| \leq |b|$ if $x \in E$.

If $x \in E$, $\left|1+n^2 x\right| \geq n^2 |x| - 1 \geq n^2 a - 1 \geq n^2 \frac{a}{2}$ for $n \geq \sqrt{\frac{2}{a}}$.

Since $\sum_{n=1}^{a} \frac{1}{n^2}$ is convergent,

$\sum f_n(x)$ conveges uniformly and absolutely in $E$. Since each $f_n$ is continuous in $E$, so is the sum function.

That uniform convergence is only a sufficient condition for inheriting continuity by the limit functions from a sequence of functions is evident from the following.

**18.5 EXAMPLE :**

Let $f_n(x) = \frac{1}{nx+1}$ for $0<x<1$.

If $0<x<1$ and $\in>0$.

$$\left|f_n(x)\right| = \frac{1}{nx+1} \leq \frac{1}{n(x+1)} < \in \text{ if } n > \frac{1}{(x+1)\in}$$

so that $\lim f_n(x) = 0$ (p.w.)

The sequence $\{f_n\}$ of functions as well as the limit function are continuous. However the convergence is not uniform for, if it were so for $\in = \frac{1}{2}$ there would crrespond a positive integer $N$ such that $0 \leq f_n(x) < \frac{1}{2}$ for all $x$ such that $0<x<1$ and all $n \geq N$.

In particular we would have $\frac{1}{Nx+1} < \frac{1}{2} \forall x \in (0, 1)$ which would imply that $(0, 1) \subseteq \left(\frac{1}{N}, 1\right)$.

which is impossible.

**18.6 DINI'S THEOREM :**

Let $E$ be a compact subset of a metric space $(X, d)$ $f$ and $f_n (n \geq 1)$ be continuous functions defined on $E$ and suppose that $f_n(x) \geq f_{n+1}(x)$ for $x$ belonging to $E$ and $n \geq 1$ and $\lim_n f_n(x) = f(x)$ for every $x \in E$. Then $\{f_n\}$ converges uniformly to $f$ on $E$.

**Proof :** Let $g_n(x) = f_n(x) - f(x)$ for $x \in E$ and $n \geq 1$.

$\{f_n\}$ converges to $f$ uniformly on $E$ if and only if $\{g_n\}$ converges to 0 uniformly. Moreever each $g_n$ is continuous on $E$ and $\lim_n g_n(x) = 0 \; \forall \; x \in E$. Thus it is enough if we show that $\{g_n\}$ converges to 0 uniformly on $E$.

Let $\in > 0$. Since $\lim g_n(x) = 0$ when $x \in E, \exists$ a positive integer $N(\in, x)$ such that $0 \leq g_n(x) < \frac{\in}{2}$ for $n \geq N(\in, x)$. We write $N(x)$ for $N(\in, x)$. Then $g_{N(x)} < \frac{\in}{2}$. Since $g_{N(x)}$ is continuous at $x$, there exists a $\delta > 0$ (depending on $x$) such that

$$\left|g_{N(x)}(y) - g_{N(x)}(x)\right| < \frac{\in}{2} \text{ for } y \in E \text{ satisfying } d(x, y) < \delta.$$

The set $J(x) = \{y / y \in E \text{ and } d(x, y) < \delta\}$ is clearly open in $E$. The family $\{J(x) / x \in E\}$ is an open cover of the compact space $E$ so that there exist finitely many $x$ in $E$, say $x_1, x_2, \cdots\cdots, x_r$ such that $E = \bigcup_{i=1}^{r} J(x_i)$.

Let $N(\in) = \text{maximum } \{N(x_1), \cdots\cdots, N(x_r)\}$. Clearly $N(\in)$ depends on $\in$ only. If $y \in E$ $\exists a \, i \, (1 \leq i \leq r)$ such that $y \in J(x_i)$.

Hence $\left|g_{N(x_i)}(y) - g_{N(x_i)}(x_i)\right| < \frac{\in}{2}$

$$\Rightarrow g_{N(x_i)}(x_i) - \frac{\in}{2} < g_{N(x_i)}(y) < g_{N(x_i)}(x_i) + \frac{\in}{2}$$

In particular $g_{N(x_i)}(y) < g_{N(x_i)} + \frac{\in}{2} < \in$.

Since $g_n(y) \le g_{N(x_i)}(y)$ for $n \ge N(x_i)$

it follows that

$$g_n(y) < \in \text{ for } n \ge N(x_i).$$

In particular if $n \ge N(\in), n \ge N(x_i) \forall\, i,\ 1 \le i \le r$ so that for all $y \in E$ $0 \le g_n(y) < \in$ whenever $n \ge N(\in)$.

Thus $\{g_n\}$ conveges uniformly to 0 on $E$.

This completes the proof.

## CHANGE OF ORDER OF TAKING LIMITS

**18.7 THEOREM :**

Let $(X, d)$ be a metric space, $E \subseteq X$, $x$ a limit point of $E$, $\{f_n\}$ a sequence of functions defined on $E$, converging uniformly to a function $f$ defined on $E$ and suppose that for each $n$ $\lim_{t \to x} f_n(t) = A_n$. Then $\{A_n\}$ converges and

$$\lim_{n \to \infty} A_n = \lim_{t \to x} f(t).$$

In other words $\lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t)$.

We divide the proof into four steps.

**Step 1 :** $\{A_n\}$ is a Cauchy's sequence and hence converges.

**Proof of Step 1 :** Given $\in > 0$, by Cauchy's general principle for uniform convergence, there exists a positive integer $N_1$, such that

$$|f_n(t) - f_m(t)| < \frac{\in}{2}$$

for $n \geq N_1$, $m \geq N_1$ and $t \in E$.

Since $\lim\limits_{t \to x} f_n(t) = A_n \forall n$ it follows that for $n \geq N_1$ and $m \geq N_1$

$$|A_n - A_m| = \lim_{t \to x} |f_n(t) - f_m(t)| \leq \frac{\in}{2} < \in.$$

Hence $\{A_n\}$ is a Cauchy sequence and hence converges.

**Step 2 :** If $A = \lim A_n$ and $\in > 0$ there exists a positive integer $N$ and a $\delta(x) > 0$ such that

(i) $|f_N(t) - f(t)| < \frac{\in}{3}$ for all $t \in E$

(ii) $|A_n - A| < \frac{\in}{3}$ and

(iii) $|f_N(t) - A_n| < \frac{\in}{3}$ if $t \in E$ and $d(x, y) < \delta(x)$

**Proof :** Choose positive integers $N_2, N_3$ such that

$$|f_n(t) - f(t)| < \frac{\in}{3} \text{ for } n \geq N_2 \text{ and all } t \in E$$

and $|A_n - A| < \frac{\in}{3}$ for $n \geq N_3$

Let $N = \max = \{N_1, N_2\}$. For this $N$ (i) and (ii) hold.

Since $\lim\limits_{t \to x} f_N(t) = A_N$, there exists a $\delta(x) > 0$, such that (iii) holds.

**Step 3 :** $\underset{t \to x}{\text{Limit}} \; f(t) = A$.

**Proof :** Given $\in > 0$ choose $\delta(x) > 0$ and a positive integer $N$ satisfying (i), (ii) and (iii) of step 2.

If $0 < d(t, x) < \delta(x)$ and $t \in E$.

$$|f(t)-A| \le |f(t)-f_N(t)| + |f_N(t)-A_N| + |A_N - A|$$

$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in \text{ by step 2.}$$

Hence $\underset{t\to x}{\text{limit}}\, f(t) = A$.

**18.8 COROLLARY :**

If $(X, d)$ is a metric space, $E \subset X$ and $\{f_n\}$ converges uniformly on $E$ to $f$ and $f_n$ is continuous at $x \in E$ for every positive integer $n$ then $f$ is continuous at $x$.

**Proof :** We may assume that $x$ is a limit point of $E$.

$$f(x) = \lim_n f_n(x) = \lim_n \lim_{t\to x} f_n(t)$$

$$= \lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{t\to x} f_n(t)$$

**18.9 EXAMPLE :** Let $f_n(x) = \dfrac{1}{(1+x^2)^n}$.

For $x \neq 0$ $\lim_n f_n(x) = 0$ since $0 < \dfrac{1}{1+x^2} < 1$.

when $x = 0$, $\lim_n f_n(0) = 1$

Hence $f(x) = \lim f_n(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

For every $n$, $\lim_{t\to 0} f_n(t) = 1$ (since $f_n$ is continuous on $\mathbb{R}$)

so that $\lim_{n\to\infty} \lim_{t\to 0} f_n(t) = 1$

Also $\lim_{t\to 0} \lim_{n\to\infty} f_n(t) = \lim_{t\to 0} f(t) = 0$.

## UNIFORM CONVERGENCE & INTEGRATION

### 18.10 THEOREM :

If $f_n (n \geq 1)$ are R.S. integrable on $[a, b]$ with respect a monotonically increasing function $\alpha$ defined on $[a, b]$ and if $\{f_n\}$ converges uniformly to $f$ on $[a, b]$ then $f$ is R.S integrable with respect to $\alpha$ on $[a, b]$ and

$$\lim_n \int_a^b f_n \, d\alpha = \int_a^b f \, d\alpha$$

**Proof : Step I :** $f$ is bounded.

Corresponding to $\in = 1,$ there is a positive integer $N$ such that

$|f_N(x) - f(x)| < 1$ for all $x \in [a, b]$.

$$\Rightarrow |f(x)| - |f_N(x)| \leq |f(x) - f_N(x)| < 1 \ \forall \ x \in [a, b]$$

$$\Rightarrow |f(x)| < 1 + |f_N(x)| \leq 1 + M \ \forall \ x \in [a, b]$$

where $M$ is an upper bound of $|f_N|$ on $[a, b]$.

**Step II :** If $g$ and $h$ are bounded functions and

$g(x) \leq h(x) \forall \ x \in [a, b]$ then

$$\underline{\int_a^b} g \, d\alpha \leq \underline{\int_a^b} h \, d\alpha \text{ and } \overline{\int_a^b} g \, d\alpha \leq \overline{\int_a^b} h \, d\alpha$$

For any partition $P = \{a = x_0 < \cdots\cdots < x_n = b\}$ of $[a, b]$ with

$m_i = \text{g.l.b}\{f(x)/x_{i-1} \leq x \leq x_i\}$

$M_i = \text{l.u.b}\{h(x)/x_{i-1} \leq x \leq x_i\}$

$m_i' = \text{g.l.b}\{h(x)/x_{i-1} \leq x \leq x_i\}$ and

$M_i' = \text{l.u.b.}\{h(x)/x_{i-1} \le x \le x_i\}$ for $1 \le i \le n$, we have $m_i \le m_i'$ and $M_i \le M_i'$.

Hence $L(P, g, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} m_i' \Delta\alpha_i = L(P, h, \alpha)$ and

$$U(P, g, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i' \Delta\alpha_1 = U(P, h, \alpha)$$

This is true for every partition $p$ of a $[a, b]$ so

$$\underline{\int_a^b} g\, d\alpha = \sup\{L(P, g, \alpha)/P\} \le \sup\{L(P, h, \alpha)/P\} = \underline{\int_a^b} h\, d\alpha$$

and similarly $\overline{\int_a^b} g\, d\alpha \le \overline{\int_a^b} f\, d\alpha$.

**Step III :** $f \in R(\alpha)$ and $\lim \int_a^b f_n\, d\alpha = \int_a^b f\, d\alpha$.

Since $\{f_n\}$ converges uniformly to $f$ on $[a, b]$, given $\in > 0$ there corresponds a positive integer $N(\in)$ such that

$$|f_n(x) - f(x)| < \frac{\in}{2(1+\alpha(b)-\alpha(a))} \text{ for } n \ge N(\in) \text{ and } x \in [a, b]$$

$$\Rightarrow f(x) - \frac{\in}{2(1+\alpha(b)-\alpha(a))} < f_N(x) < f(x) + \frac{\in}{2(1+\alpha(b)-\alpha(a))} \text{ for } n \ge N(\in) \text{ and } x \in [a, b]$$

$$\Rightarrow \text{ by step II, } \underline{\int_a^b}\left(f - \frac{\in}{2(1+\alpha(b)-\alpha(a))}\right) d\alpha \le \underline{\int_a^b} f_n\, d\alpha \le \underline{\int_a^b}\left\{f + \frac{\in}{2(1+\alpha(b)-\alpha(a))}\right\} d\alpha$$

for $n \ge N(\in)$

$$\Rightarrow \underline{\int_a^b} f\, d\alpha - \epsilon < \underline{\int_a^b} f_n\, d\alpha < \underline{\int_a^b} f\, d\alpha + \frac{\epsilon}{2} \quad \text{for } n \geq N(\epsilon).$$

$$\Rightarrow \left| \underline{\int_a^b} f_n\, d\alpha - \underline{\int_a^b} f\, d\alpha \right| < \epsilon \quad \text{for } n \geq N(\epsilon).$$

$$\Rightarrow \lim \underline{\int_a^b} f_n\, d\alpha = \underline{\int_a^b} f\, d\alpha.$$

By symmetry $\lim \overline{\int_a^b} f_n\, d\alpha = \overline{\int_a^b} f\, d\alpha$.

Since $\underline{\int_a^b} f_n\, d\alpha = \overline{\int_a^b} f_n\, d\alpha$, the limits on the l.h.s. are equal. So,

$$\underline{\int_a^b} f\, d\alpha = \lim \underline{\int_a^b} f_n\, d\alpha = \lim \overline{\int_a^b} f_n\, d\alpha = \overline{\int_a^b} f\, d\alpha,$$

$\Rightarrow f \in R(\alpha)$ and

$$\int_a^b f\, d\alpha = \lim \int_a^b f_n\, d\alpha$$

**18.11 EXAMPLE :**

Let $\mathbb{Q} \cap [0, 1] = \{r_1, r_2, \cdots\cdots, r_n, \cdots\}$ be an enumeration of the set of rational numbers in $[0, 1]$.

Define $f_n$ on $[0, 1]$ by

$$f_n(x) = \begin{cases} 0 \text{ if } x \in \{r_1, \cdots\cdots, r_n\} \\ 1 \text{ otherwise} \end{cases}$$

and $f(x)=\begin{cases}0 \text{ if } x\in\{r_1, r_2, \cdots, r_n, \cdots\} \text{ and} \\ 1 \text{ otherwise}\end{cases}$

Since $f_n$ has a finite number of discontinuities with value 0 namely at $r_1, \cdots, r_n$ and is 1 at other $x$, $f_n$ is Riemann integrable on $[0, 1]$ and $\int_0^1 f_n\,dx = 1$.

If $x\in[0, 1]$ and is not rational, $f_n(x)=1\ \forall\ n$, so $f(x)=\lim_n f_n(x)=1$

If $x\in[0, 1]$ and is rational, say $x=r_K$, $f_n(r_K)=0$ for $n\geq K$ so

$$f(r_K)=\lim f_n(r_K)=0.$$

Thus $f(x)=\begin{cases}1 \text{ if } x \text{ is irrational} \\ \text{and} \\ 0 \text{ if } x \text{ is rational}\end{cases}$

It is known that $f$ is not Riemann integrable.

**Conclusion :** If $\{f_n\}$ converges to $f$ p.w. but not uniformly it is possible that $f\notin R(\alpha)$ even though $f_n\in R(\alpha)\forall n$ where $\alpha$ is any monotonically increasing function on $[a,b]$.

**18.12 EXAMPLE :**

$f_n(x)=nx\left(1-x^2\right)^n$ for $n\geq 1$ and $0\leq x\leq 1$. Clearly $f_n$ is continuous and

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{n}{2}\, y^n\,dx = \frac{n}{2n+2}. \text{ So}$$

$$\lim \int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

$$0<x<1,\ 0<1-x^4<1 \Rightarrow 0<\left(1-x^2\right)^n<\frac{1}{\left(1+x^2\right)^n}$$

Since $\left(1+x^2\right)^n = 1+\binom{n}{1}x^2+\binom{n}{2}x^4+\cdots\cdots+x^{2n} > \binom{n}{2}x^4 = \frac{n(n-1)}{2}x^4$

$$\frac{1}{\left(1+x^2\right)^n} < \frac{2}{n(n-1)}x^{-4}$$

$$\Rightarrow 0 \le f_n(x) \le \frac{2}{n(n-1)x^3}. \text{ for } 0<x<1 \text{ and } n \ge 2.$$

$$\Rightarrow \lim f_n(x) = 0\ \forall\ x \in [0,1].$$

**Conclusion :** If $\lim f_n(x) = f(x)$ p.w. but not uniformly it is a possible that $f \in R(\alpha)$ when $f_n \in \mathbb{R}(\alpha)\ \forall\ n$ but yet

$$\int_a^b f\,d\alpha \ne \lim_n \int_a^b f_n\,d\alpha$$

**18.13 EXAMPLE :**

$f_n(x) = n^2x\left(1-x^2\right)^n$ for $n \ge 1$ and $0 \le x \le 1$. As in example 18 above for each n, $f_n$ continuous, hence Riemann integrable and

$$\int_a^b f_n(x)dx = \frac{n^2}{2n+2} \text{ so that } \lim \int_a^b f_n\,dx = \infty$$

Also $\lim_n f_n(x) = 0$ for every $x$ so $\lim f_n$ is Riemann integrable.

**Conclusion :** If $\{f_x\}$ converges to $f$ p.w. but not uniformly, it is possible that $f \in \mathbb{R}(\alpha)$ when $f_n \in \mathbb{R}(\alpha)\ \forall\ n$ even though $\lim_n \int_a^b f_n\,d\alpha$ does not exist in $\mathbb{R}$.

**18.14 COROLLARY :**

If $f_n \in R(\alpha)$ on $[a,b]$ for $n \geq 1$ and the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$ on $[a, b]$ then $f \in R(\alpha)$ on $[a, b]$ and

$$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha.$$

**Proof :** Write $s_n(x) = f_1(x) + \cdots + f_n(x)$ for $x \in [a, b]$ and $n \geq 1$. Then $s_n \in \mathbb{R}(\alpha)$ for $n \geq 1$ and $\{s_n\}$ converges uniformly to $f$ on $[a, b]$. Then by 18.16 $f \in \mathbb{R}(\alpha)$ on $[a, b]$ and

$$\int_a^b f\, d\alpha = \lim_n \int_a^b s_n\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha.$$

**18.15 SHORT ANSWER QUESTIONS :**

**18.15.1 :** Find $\lim_{n\to\infty} \lim_{x\to 0} \frac{1}{nx+1}$ and $\lim_{x\to 0} \lim_{n\to\infty} \frac{1}{nx+1}$

**18.15.2 :** Let $(x) = x - [x]$, the fractional part of $x$ where $[x]$ is the largest integer not exceeding $x$.

Discuss the uniform convergence of $f(x) = \sum_{n=0}^{\infty} \frac{(nx)}{n^2}$ $(x \in \mathbb{R})$. Find all discontinuities of $f$.

**18.15.3 :** Let $f_n(t) = nt$ $(t \in \mathbb{R})$.

Does $\{f_n\}$ converge uniformly on

(a) $\mathbb{R}$ (b) $[\alpha, \beta]$ where $\alpha < \beta$ ?

**18.15.4 :** **Power Series :** Show that if $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely in $(-R, R)$ then the series converges uniformly and absolutely in $[-R+\in, R-\in]$ for $0<\in<2R$ and that the sum function is continuous in $(-R, R)$.

## 18.16 MODEL EXAMINATION QUESTIONS :

**18.16.1 :** Define uniform convergence of a sequence $\{f_n\}$ of functions to a function defined on a set $E$.

Give examples of a sequence of functions that converges

(a) uniformly (b) pointwise but not uniformly.

**18.16.2 :** Show that if $\{f_n\}$ is a sequence of functions defined on $E$ and $f$ is a function defined on $E$ then $\{f_n\}$ converges uniformly to $f$ on $E$.

If and only if $\{\sup\{|f_n(x)-f(x)|/x\in E\}\}$ converges to zero.

**18.16.3 :** State and prove Weierstrass' M-test.

**18.16.4 :** Discuss the uniform convergence and absolute convergence of $\sum(-1)^n \frac{x^2+n}{n^2}$

**18.16.5 :** Show that if each $f_n\ (n\geq 1)$ is continuous on a metric space $X$ and $\{f_n\}$ converges uniformly to $f$ on $X$ then $f$ is continuous on $X$.

**18.16.6 :** Show that if each $f_n\ (n>1)\in R(\alpha)$ where $\alpha$ is a monotonically increasing function on $[a, b]$ and $\{f_n\}$ converges uniformly to $f$, then $f\in R(\alpha)$ on $[a, b]$ and

$$\lim \int_a^b f_n\, d\alpha = \int_a^b f\, d\alpha .$$

**18.16.7 :** The sequence of functions $\left\{\frac{1}{nx+1}\right\}$ converges to 0 on $(0, 1)$.

Is this convergence uniform ? Justify your answer.

**18.16.8 :** Let $(X,d)$ be a metric space; $E\subset X$, $x$ a limit point of $E$, and $\{f_n\}$ a sequence of functions defined on $E$. State a set of sufficient conditions under which

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

holds and prove the result.

## 18.17 ANSWERS TO SAQ'S :

**18.17.1 :** $\lim_{x\to 0}\frac{1}{nx+1} = 1 \Rightarrow \lim_{n\to\infty}\lim_{x\to 0}\frac{1}{nx+1} = 1$

$$\lim_{n\to\infty}\frac{1}{nx+1} = 0 \Rightarrow \lim_{x\to 0}\lim_{n\to\infty}\frac{1}{nx+1} = 0.$$

**18.17.2 :** Since the set of discontinuities of $[x]$ is the set $Z$ of integers, the set of discontinuities of $[n, x]$ for each $n \geq 1$ is the set $\left\{\frac{m}{n} / m \in Z\right\}$. Moreover $0 \leq (nx) \leq 1 \ \forall\ n \in \mathbb{N}$ and $x \in \mathbb{R}$. Hence $\sum_{n=0}^{\infty}\frac{(nx)}{n^2}$, converges uniformly. The sum function f is continuous at the points of continuity of each $f_n$. Hence the set of discontinuities of f is precisely the set $\mathbb{Q}$ of rational numbers.

**18.17.3 :** For $t \in \mathbb{R}$ $\{|(n-m)t|/n \geq 1, m \geq 1\} = \left\{K|t| \big/ \begin{matrix} K \geq 0 \\ K \in \mathbb{Z} \end{matrix}\right\}$.

It is impossible to find a $N(\in)$ corresponding to any $\in > 0$ such that $K|t| < \in$ for $K \geq N(\in)$.

**18.16.4 :** If $0 < \in < 2R$, and $|x| \leq R - \in$, by comparison test $\sum|a_n|(R-\in)^n$ converges. So, $\sum a_n x^n$ converges uniformly and absolutely in $[-R+\in, R-\in]$.

By 18.2 continuity follows in $[-R+\in, R-\in]$.

If $|x| < R$, $\exists \in > 0 \ni |x| \leq R - \in$ so continuity follows in $(-R, R)$.

## 18.17 EXERCISES :

**18.17.1 :** Discuss the uniform convergence of the sequence $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{nx+1} \quad (0 < x < \infty)$$

**18.17.2 :** Let $s_{m,n} = \dfrac{m}{m+n}$. Find $\lim\limits_{m\to\infty} \lim\limits_{n\to\infty} s_{m,n}$ and $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} s_{m,n}$.

**18.17.3 :** Show that every uniformly convergent of bounded functions is uniformly bounded i.e. $|f_n(x)| \le M$ for a fixed $M > 0$ and all $x$ and $n \ge 1$.

**18.17.4 :** Suppose $\{f_n\}$ is uniformly convergent on $E$ and each $f_n$ is a bounded function. Show that $\{f_n^2\}$ converges uniformly on $E$.

**18.17.5 :** Suppose $\{f_n\}$ is a sequence of bounded functions and $\{f_n^2\}$ converges uniformly. Does it imply that

(i) $\{f_n\}$ converges uniformly to $f$

(ii) $\{|f_n|\}$ converges uniformly to $f$. Justify

**18.17.6 :** Suppose $f_n \to f$ uniformly on $\mathbb{R}$ and $g_n \to g$ uniformly on $\mathbb{R}$. Does it follow that $g_n o f_n \to g o f$ uniformly on $\mathbb{R}$? Why ?

**18.17.7 :** Let $I(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$ and $\{x_n\}$ be a sequence of distinct real numbers in $(a, b)$ and $\sum\limits_{n=1}^{\infty} |c_n|$ be a convergent series.

Prove that $\sum\limits_{n=1}^{\infty} c_n \, I(x - x_n)$ converges uniformly and that the sum function is continuous outside $(a, b)$.

**18.17.8 :** Show that the function $f$ defined in SAQ 2 is Riemann integrable in $[a, b]\ \forall\ a < b$. (Hint : First prove for $[n, n+1]$ where $n$ is an integer).

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

**Lesson writer :**

***Prof. I. Ramabhadra Sarma***

Lesson - 19

# UNIFORM CONVERGENCE - III

## 19.1 INTRODUCTION

In this leson we concentrate our attention on differentiability of a sequence / series of differentiable functions. It is interesting to note that uniform convergence is not enough for the limit function to inherit differentiability from the sequence of functions. Every term of a uniformly convergent sequence of functions may be differentiable. Neverthless the limit function may not possess derivative at all. Even if the limit function is differetiable it is possible that $\lim f_n' \neq (\lim f_n)'$.

In this lesson we establish the result $\lim f_n' \neq (\lim f_n)'$ under extra hypothesis. Using some results of previous lessons on uniform convergence we also establish the existence of a nowhere differentiable continuous function on $\mathbb{R}$.

## UNIFORM CONVERGENCE AND DIFFERENTIATION :

We now consider the following question.

Do differentiation and limiting process commute under uniform convergence ? More specifically suppose $\{f_n\}$ is a sequence of functions defined in an open interval $I$ and $f_n \to f$ uniformly on $I$. If each $f_n$ is differentiable at a does it follow that $f$ is differentiable at $a$ and $f'(a) = \lim f_n'(a)$? The following examples throw some light on this aspect.

**19.2 EXAMPLE :**

$$f_n(x) = \frac{\sin nx}{\sqrt{n}} \quad x \in \mathbb{R} \text{ and } n \geq 1.$$

Since $\lim\limits_{n} \frac{1}{\sqrt{n}} = 0$ and $|\sin nx| \leq 1 \forall x \in \mathbb{R} \,\&\, n \geq 1$ $\{f_n\}$ converges uniformly to 0. (by 17.9).

Also $f_n'(x) = \sqrt{n} \cos nx$

when $x = 2K\pi$ where $K$ is a fixed integer, $f_n'(x) = \sqrt{n}$

so $\lim f_n'(x) = \infty$.

Thus $f'(0) \neq \lim\limits_{n} f_n'(0)$

**19.3 EXAMPLE :**

$$f_n(x) = \frac{x}{1+nx^2}, \; x \in \mathbb{R}$$

In 17.13 we proved that $\{f_n\}$ converges uniformly to 0 on $\mathbb{R}$. Clearly $f_n'(x) = \dfrac{1-nx^2}{(1+nx)^2}$,

so that $\lim\limits_{n} f_n'(0) = 1 \neq f'(0)$.

**19.4 EXAMPLE :**

Define a sequence of polynomials recursively as follows on $[-1, 1]$

$$P_0(x) = |x|.$$

$$P_{n+1}(x) = \frac{x^2}{2} + P_n(x) - \frac{P_n{}^2(x)}{2} \quad \text{for } n \geq 0$$

1. $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x| \leq 1$

2. $|x| - P_n(x) \leq |x| \cdot \left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$ if $|x| \leq 1$.

3. $\{P_n(x)\}$ converges uniformly to $|x|$ on $[-1, 1]$

**Proof (1) :** $|x| - P_{n+1}(x)$

$$= |x| - \frac{|x|^2}{2} - P_n(x) + \frac{P_n{}^2(x)}{2}$$

$$= |x| - P_n(x) - \frac{1}{2}(|x| - P_n(x))(|x| + P_n(x))$$

$$=\left[|x|-P_n(x)\right]\left[1-\frac{|x|+P_n(x)}{2}\right] \text{ ---------------------- } (*)$$

(1) holds when $n=0$. Assume for $n$.

Since $x^2 \geq P_n^2(x)$, $P_n(x) \leq P_{n+1}(x)$.

Since $P_n(x) \leq |x| \leq 1$,

$$P_n(x) \leq \frac{|x|+P_n(x)}{2} \leq |x| \leq 1$$

so that $1-\dfrac{|x|+P_n(x)}{2} \geq 0$, so that

$$|x|-P_{n+1}(x) \geq 0.$$

Thus (1) holds for all $n$ and $x$ in $[-1, 1]$.

**(2)** : From (1) $1 \geq |x| \geq P_{n+1}(x) \geq P_n(x) \geq 0$

$$\Rightarrow 1 \geq \frac{|x|+P_{n+1}(x)}{2} \geq \frac{|x|}{2}$$

$$\Rightarrow 1-\frac{|x|+P_{n+1}(x)}{2} \leq 1-\frac{|x|}{2}$$

$$\Rightarrow \left[|x|-P_n(x)\right]\left[1-\frac{|x|+P_{n+1}(x)}{2}\right] \leq \left[|x|-P_n(x)\right]\left(1-\frac{|x|}{2}\right)$$

Thus if $|x|-P_n(x) \leq |x|\left(1-\dfrac{|x|}{2}\right)^n$ then by $(*)$.

$$|x|-P_{n+1}(x) < |x|\left(1-\frac{|x|}{2}\right)^n$$

As the inequality holds good trivially when $n=0$, it holds for all $n \geq 0$ by induction.

For the second part of the inequality put $\frac{|x|}{2}=a$.

We have $1 \geq \left(1-a^2\right)^n = (1-a)^n \ (1+a)^n$

$$\geq \left(1-a^n\right)\left(1+\binom{n}{1}a + \cdots\cdots + a^n\right) \text{ (binomial theorem)}$$

$$\geq (1-a)^n (1+na)$$

$$>(1-a)^n \ a(1+n)$$

so that $(1-a)^n < \frac{1}{(n+1)a}$ and hence

$$2a(1-a)^n < \frac{2}{n+1}.$$

i.e. $|x|\left(1-\frac{|x|}{2}\right)^n < \frac{2}{n+1}$

**(3)** : The sequence of polymomials $\{P_n(x)\}$ converges uniformly to $|x|$ on $[-1, 1]$.

**Proof :** By (2) $0 \leq |x| - P_n(x) < \frac{2}{n+1}$ for all $n \geq 0$ and $x \in [-1, 1]$.

Since $\lim \frac{2}{n+1} = 0$ by 17.10, $\{P_n(x)\}$ converges uniformly to 0 on $[-1, 1]$

**19.5 REMARK :**

We now make the following observations. The sequences of functions under consideration in examples 19.2, 19.3, and 19.4. Converge uniformly on their respective domains. All these are differentiable.

In example 19.2 the limit function is differentiable at 0, but $f'(0) = \lim_n f_n'(0)$ does not hold because $\lim f_n'(0)$ exist. In example 19.3, $\lim f_n'(0)$ and $f'(0)$ exist but $\lim f_n'(0) \neq f'(0)$. In example 19.4 the limit function is not differentiable at 0.

In view of these observations one thing is clear. Inheritence of differentiability the limit function and deduction of $f'(x) = \lim f_n'(x)$ require stronger hypothesis than mere uniform convergence.

We now prove the following theorem.

**19.6 THEOREM :**

Suppose $\{f_n\}$ is a sequence of functions each of which is differentiable on $[a, b]$, the sequence of derivatives $\{f_n'\}$ converges uniformly to a function $g$ on $[a, b]$ and for some $x$ in $[a, b]$, the sequence $\{f_n(x)\}$ converges.

Then $\{f_n\}$ converges uniformly on $[a, b]$.

If $f = \lim f_n$ then $f$ is differentiable on $[a, b]$ and $g = f'$, i.e. $\lim f_n'(x) = (\lim f_n(x))'$.

We divide the proof into three steps.

Our proof is based on 18.7.

**Step 1 :** $\{f_n\}$ converges uniformly on $[a, b]$.

**Proof :** Since $\{f_n(x)\}$ converges, given $\in > 0$ there exists a positive integer $N$, such that

$$|f_n(x) - f_m(x)| < \frac{\in}{2} \text{ for } n \geq N_1 \text{ and } m \geq N_1 \quad \text{---------- (1)}$$

Since $\{f_n'\}$ converges uniformly on $[a, b]$ there exists a positive integer $N_2$ such that for $t \in [a, b], n \geq N_2$ and $m \geq N_2$.

$$|f_n'(t) - f_m'(t)| < \frac{\in}{2(b-a)} \quad \text{---------------- (2)}$$

If $a \leq y < z \leq b$, by the mean value theorem applied to $f_n - f_m$ on $[y, z]$, $\exists\, t \in (y, z) \ni$

$$(f_n - f_m)(z) - (f_n - f_m)(y) = (z - y)\,(f_n - f_m)'(t)$$

so that

$$\left|\{f_n(z)-f_n(y)\}-\{f_m(z)-f_m(y)\}\right| = |z-y|\ \left|f_n'(t)-f_m'(t)\right|$$

$$\leq |z-y|\cdot\frac{\in}{2(b-a)} \leq \frac{\in}{2} \text{ for } n \geq N_2, m \geq N_2 \text{ ----------- (3)}$$

Let $N=\max\{N_1, N_2\}$. From (1) and (3) we get for $t\in[a,b]$ and $n \geq N,\ m \geq N$.

$$|f_n(t)-f_m(t)| \leq \left|\{f_n(t)-f_m(t)\}+\{f_n(x)-f_m(x)\}\right|+\left|f_n(x)-f_m(x)\right|$$

$$<\frac{\in}{2}+\frac{\in}{2}=\in$$

Hence by Cauchy's general principle for uniform convergence, $\{f_n\}$ converges uniformly on $[a,b]$.

Let $f(t)=\lim_n f_n(t)$ for $t\in[a, b]$.

**Step 2 :** Fix $y\in[a, b]$. Let $E=[a, b]-\{y\}$.

Define $\phi_n$ and $\phi$ on $E$ by

$$\phi_n(t)=\frac{f_n(t)-f_n(y)}{t-y} \text{ and } \phi(y)=\frac{f(y)-f(y)}{t-y}$$

Then $\{\phi_n\}$ converges uniformly to $\phi$ on $E$.

**Proof :** Given $\in>0$ choose a positive integer $N$ as in (3) of step (1). For $t\in E$, $n \geq N$ and $m \geq N$

$$|\phi_n(t)-\phi_m(t)|=\frac{\left|\{f_n(t)-f_m(t)\}-\{f_n(y)-f_m(y)\}\right|}{|t-y|}$$

$$\leq\frac{\in}{2(b-a)}.$$

Thus $\{\phi_n\}$ satisfies Cauchy's criterion for uniform convergence and hence converges uniformly on $E$.

Since $\lim f_n(t) = f(t)$ for $t \in [a, b]$

$\lim \phi_n(t) = \phi(t)$ for $t \in E$ and the convergence is uniform on $E$.

**Step 3 :** $f$ is differentiable on $[a, b]$ and $g = f'$.

**Proof :** Our proof makes use of 18.7.

Again fix $y \in [a, b]$ and let $E = [a, b] \setminus \{y\}$. Clearly $y$ is a limit point of $E$. The sequence $\{\phi_n\}$ and the function $\phi$ defined in step 2 satisfy condition of uniform convergence :

$\{\phi_n\}$ converges uniformly to $\phi$ on $E$. Also for every $n \geq 1$.

$$f_n'(y) = \lim_{t \to y} \frac{f_n(t) - f_n(y)}{t - y} = \lim_{t \to y} \phi_n(t)$$

By 18.7 $\lim_{n \to \infty} f_n'(y)$ exists and

$$g(y) = \lim_{n} f_n'(y) = \lim_{t \to y} \phi(t) = \lim_{t \to y} \frac{f(t) - f(y)}{t - y}$$

Hence $f$ is differentiable at $y$ and $f'(y) = g(y)$.

This is true for every $y \in [a, b]$ and $f'(y) = g(y)$.

### 19.7 COROLLARY :

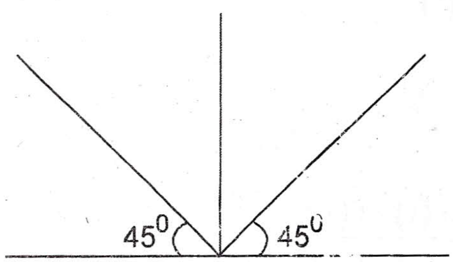
Suppose $f_n\ (n \geq 1)$ is differentiable on $[a, b]$ and the series $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$ with sum $g$. If for some $x \in [a, b]$ the series $\sum_{n=1}^{\infty} f_n(x)$ converges, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$, the sum function $f(x)$ of this series is differentiable and $f'(x) = g(x)$ for all $x$ in $[a, b]$.

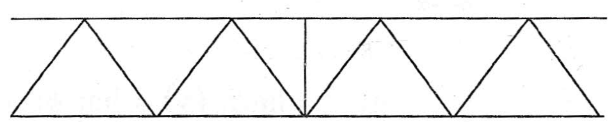
**Proof :** Let $s_n(t) = f_1(t) + \cdots\cdots + f_n(t)$ for $t \in [a, b]$ and $n \geq 1$. Then each $s_n$ is differentiable on $[a, b]$, $\{s'_n\}$ converges uniformly to $g$ on $[a, b]$ and $\{s_n(x)\}$ converges.

Hence by 18.14 $\{s_n\}$ converges uniformly on $[a, b]$ and the sum function $f$ is differentiable on $[a, b]$ and satisfies $f'(x) = g(x)$ for $x \in [a, b]$. From this the corollary follows.

**19.8 EXAMPLE OF A NOWHERE DIFFERENTIABLE CONTINUOUS FUNCTION :**

Is every continuous function differentiable ? The absolute value function defined by $\phi(x) = |x|$ is a handy example for a negative answer. After all zero is the only point at which the function is not differentiable. Could this set of points of nondifferentiability be infinite. The answer is yes ! Extend the above function $\phi$ periodically with period to $\mathbb{R}$.



$\phi(x+2) = \phi(x)$

Then what can you say about the set of points of nondifferentiability of a continuous function? A lot could be said. We will be content with providing an example of a nowhere differentiable continuous function $\mathbb{R}$. This, we do by making use of uniform convergence. We first prove the following.

**19.9 LEMMA :**

Define $\phi(x) = |x|$ if $-1 \leq x \leq 1$ and

$$\phi(x+2) = \phi(x) \text{ if } x \in \mathbb{R}.$$

Then $|\phi(s) - \phi(t)| \leq |s - t|$ for $s, t$ in $\mathbb{R}$.

**Proof :** Let $k$ be an integer.

If $s = 2k$ $\phi(s-2k) = \phi(0) = 0$ while

If $s = 2k+1, \phi(s) = \phi(s-2k) = \phi(1) = 1$

If $s \in (2k-1,\ 2k+1)$, $\phi(s) = \phi(s-2k) = s-2k$ since $0 < s-2k < 1$

If $s \in (2k-1,\ 2k)$ $\phi(s) = \phi(s-2k) = 2k-s$ since $-1 < s-2k < 0$

Since $0 \le \phi(s) \le 1 \forall s$, $|\phi(s)-\phi(t)| \le 1 \ \forall$ s, t in $\mathbb{R}$

in particular, if $|s-t| > 1$, $|\phi(s)-\phi(t)| \le 1 < |s-t|$

Thus it is enough to consider the case $|s-t| \le 1$.

Let $s \in \mathbb{R}, t \in \mathbb{R}$ and $s < t$, $0 \le t-s = |t-s| \le 1$.

Then both $s$ and $t$ may lie in between two consecutive integers or may have one and only one integer in between them, we consider various possibilities that arise in this context. The $K$ that occurs hereunder is an integer.

| | Position of $s$ and $t$ | $\lvert\phi(s) - \phi(t)\rvert$ |
|---|---|---|
| (i) | $2k-1 \le s < t \le 2k$ | $\lvert(2k-s)-(2k-t)\rvert = \lvert s-t\rvert$ |
| (ii) | $2k-1 \le s < 2k < t \le 2k+1$ | $\lvert(2k-s)-(t-2k)\rvert = \lvert(t+s-4k)\rvert$ |
| (iii) | $2k < s < t \le 2k+1$ | $\lvert(s-2k)-(t-2k)\rvert = \lvert s-t\rvert$ |
| (iv) | $2k \le s < 2k+1 < t \le 2k+2$ | $\lvert(s-2k)-(2k+2-t)\rvert = \lvert s+t-4k-2\rvert$ |

In case (ii) $s < 2k < t \Rightarrow t-s = |s-t|$.

$$\Rightarrow |\phi(s)-\phi(t)| = |(t-2k)-(2k-s)| < t-s = |s-t|.$$

In case (iv)

$$|\phi(s)-\phi(t)| = |-(2k+1-s)+(t-2k+1)| < |t-s|.$$

Thus in all the possible cases we have $|\phi(s)-\phi(t)| \le |s-t|$.

Among the above four cases, which exhaust all possibilities, equality occurs when $s, t$ lie between consecutive integers while the inequality is strict when one and only one integer lies in between $s$ and $t$.

**Notation :** Fix $x \in \mathbb{R}$ and an integer $m$.

Clearly atmost one of the intervals $\left(4^m x - \frac{1}{2}, 4^m x\right)$ and $\left(4^m x, 4^m x + \frac{1}{2}\right)$ contains an integer. Let $\delta_m$ be $\pm\frac{1}{2} 4^{-m}$ so that $4^m x$ and $4^m (x+\delta_m)$ do not have an integer in between them.

For each $n \geq 0$ write

$$\gamma_n = \frac{\phi\left(4^n (x+\delta_m) - \phi\left(4^n x\right)\right)}{\delta_m}$$

We now prove the following

**19.10 LEMMA :**

$$\gamma_n = \begin{cases} 0 \text{ if } n > m \\ 4^m \text{ it } n = m \end{cases}$$

and $\quad |\gamma_n| \leq 4^n$ if $0 \leq n < m$

**Proof :** If $n > m$ $4^n \delta_m = \pm\frac{4^{n-m}}{2}$ which is an even integer so that

$$\phi\left(4^n (x + \delta_m)\right) = \phi\left(4^n x \pm \frac{4^{n-m}}{2}\right) = \phi\left(4^n x\right), \text{ hence } \gamma_n = 0.$$

If $n = m$ $\phi\left(4^m (x + \delta_m)\right) - \phi\left(4^m x\right) = \phi\left(4^m x \pm \frac{4^m}{2}\right)$.

Since there is no integer between $4^m x$ and $4^m x \pm \frac{4^m}{2}$,

$$\left|\phi\left(4^m (x + \delta_m)\right) - \phi\left(4^m x\right)\right| = \frac{1}{2} \text{ and } |\gamma_m| = 4^m.$$

If $0 \le n < m$, $\left|4^n(x+\delta_m)-4^n x\right| = 4^n|\delta_m|$ so that

$$\left|\phi\left(4^n(x+\delta_m)\right)-\phi\left(4^n x\right)\right| = 4^n|\delta_m|$$

and $\quad |\gamma_n| = \frac{1}{2}4^{n-m} < 4^n$.

This completes the proof of lemma 2.

## THE FUNCTION f AND ITS CONTINUITY ON $\mathbb{R}$ :

The geometric series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$ is convergent. Since $|\phi(x)| \le 1 \,\forall\, x \in \mathbb{R}$, by Weierstrass M-test the series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi\left(4^n x\right)$ converges uniformly on $\mathbb{R}$. Since $\phi$ is continuous, $\phi\left(4^n x\right)$ is continuous for every $n \ge 0$ so that by 18.2 the sum function $f$ defined by

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi\left(4^n x\right) \text{ is continuous on } \mathbb{R}.$$

## NON DIFFERENTIABILITY OF f :

**19.11 Lemma :** Let $x \in \mathbb{R}$. Then $\lim_{m\to\infty}\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \infty$

$$\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$= \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$\ge \left(\frac{3}{4}\right)^m |\gamma_m| - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n |\gamma_n|.$$

$$=3^m - \sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n \cdot 4^n \text{ by}$$

$$=\frac{3^m+1}{2}.$$

Since $\lim\limits_{m} \frac{3^m+1}{2} = \infty$, $\lim\limits_{m}\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \infty$.

**Consequence :** Since $\lim\limits_{m} \delta_m = 0$,

$\lim\limits_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right|$ does not exist as a real number.

Hence $f$ is not differentiable at $x$. Since this holds $\forall x \in \mathbb{R}$, $f$ is not differentiable at any point of $\mathbb{R}$, i.e. $f$ is no where differentiable.

## 19.12 SHORT ANSWER QUESTIONS :

**19.12.1 :** Show that if $\sum\limits_{n=1}^{\infty} f_n$ is a series of functions, each $f_n$ is differentiable on $[a, b]$, $\sum\limits_{n=1}^{\infty} f_n(x)$ converges for some $x \in [a, b]$ and $\sum\limits_{n=1}^{\infty} f_n'$ converges uniformly on $[a,b]$ then $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ and

$$\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'.$$

**19.12.2 :** Let $f_n(x) = \frac{x^2}{\left(1+x^2\right)^n}$ $x \in \mathbb{R}$ $n \geq 0$ and $f(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1+x^2 \text{ if } x \neq 0 \end{cases}$.

Show that $\sum\limits_{n=0}^{\infty} f_n(x) = f(x)$ $\forall$ $x \in \mathbb{R}$; Each $f_n$ is differentiable but $f$ is not differentiable at 0.

**19.12.3 :** *Derive $g = f'$ in 19.6 using 18.10 under the assumption that* $f_n'$ is continuous for every $n$.

**19.13 MODEL EXAMINATION QUESTIONS :**

**19.13.1 :** Show that if $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $[a, b]$ $\lim f_n = f$ and $\lim f_n' = g$ then $f$ is differentiatiable on $[a, b]$ and $f' = g$.

**19.13.2 :** Show that if $\sum\limits_{n=0}^{\infty} a_n x^n$ converges in $(-R, R)$ and has sum $f(x)$ then $f$ is differentiable and $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ for $x \in (-R, R)$.

**19.13.3 :** For $n \geq 1$ and $x \in \mathbb{R}$ put $f_n(x) = \dfrac{x}{1+nx^2}$.

Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$.

Is it true that $(\lim f_n)' = \lim f_n'$ on $\mathbb{R}$ ? Justify your answer.

## 19.14 ANSWERS TO SAQ'S :

**19.14.1 :** Apply 19.6 to the sequence $\{s_n\}$ of partial sums : $s_n = \sum\limits_{i=1}^{n} f_i$.

**19.14.2 :** Take the limit with respect to $n$ to the sequence $\{s_n\}$ of partial sums, $s_n = \dfrac{\left(1+x^2\right)^n - 1}{\left(1+x^2\right)^{n-1}}$

**19.14.3 :** Since each $f_n'$ is continuous, by 18.2, $g$ is continuous By 18.10 $\forall y \in [a, b]$.

$$\lim \int_x^y f_n'(t)\,dt = \int_x^y g(t)\,dt \quad (*)$$

If $f(y) = \lim f_n(y)\ \forall\ y \in [a,b]$ then from $(*)$

$$f(y)=f(x)+\int_{x}^{y} g(t)\,dt \qquad (**)$$

Since the R.H.S. in $(**)$ is differentiable so is L.H.S. and we get $f'(y)=g(y)$.

**19.15 EXERCISES :**

**19.15.1 :** Show that if $f(x)=\sum_{n=0}^{\infty} a_n x^n \ (-R<x<R)$ then $f$ possesses derivatives of all orders and that

$$f^{(k)}(x)=\sum_{n=k}^{\infty} b_n x^{n-k} \text{ where } b_n=n(n-1)\cdots\cdots(n-k+1)a_n.$$

In particular show that $a_K=\dfrac{f^{(k)}(0)}{\lfloor k}\ \forall k\geq 0$.

**19.15.2 :** Show that the series $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\cdots\cdots\cdots\dfrac{(-1)^n}{(2n+1)!}x^{2n+1}+\cdots\cdots$

converges uniformly in $\mathbb{R}$. If the sum is denoted by $s(x)$ sow that

$$s'(x)=C(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\cdots+\frac{(-1)^n}{(2n)!}x^{2n}+\cdots$$

**19.15.3 :** Show that $C^2(x)+S^2(x)=1$ for all $x$.

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

**Lesson writer :**

***Prof. I. Ramabhadra Sarma***

## Lesson - 20

# SPACES OF CONTINUOUS FUNCTIONS

### 20.1 INTRODUCTION

The space $\mathscr{C}(X, \mathbb{R})$ of all real valued continuous functions is one basic example and model in Functional Analysis and measure theory. This lesson is devoted to learn some basic properties of $\mathscr{C}(X, K)$ where $X$ is a compact metric space and $K=\mathbb{C}$ or $\mathbb{R}$. The notonion of uniform convergence is closely connected with convergence in $\mathscr{C}(X, \mathbb{K})$. We also study equicontinuity of a family of functions which is helpful in characterizing compact subsets of $\mathscr{C}(X, \mathbb{R})$.

We finally learn Weirstrass approximation theorem for continuous functions by polynomials which are comparatively easy to handle because of their smoothness.

Let $X$ be a set. We consider the collection $\mathcal{F}$ of all complex valued functions on $X$ and define pointwise operations as follows. Let $f \in \mathcal{F}$, $g \in \mathcal{F}$ and $\alpha \in \mathbb{C}$.

Pointwise addition $f+g$: define for $x \in X$

$$(f+g)(x)=f(x)+g(x)$$

Pointwise multiplication $fg$ : Define for $x \in X$

$$(fg)(x)=f(x)\,g(x)$$

Pointwise scalar multiplication : Define for $x \in X$

$$(\alpha f)(x)=\alpha f(x).$$

It is easy to verify that, $\mathcal{F}$ is a vector space over $\mathbb{C}$ with pointwise addition and scalar multiplication. Moreover $\mathcal{F}$ is a commutative ring with unity with respect to pointwise addition and multiplication.

When $f$ and $g$ are continuous so are $f+g$, $fg$ and $\alpha f$ for every $\alpha \in \mathbb{C}$ so that the space consisting of all complex valued continuous functions on a metric space $X$ is a subspace of $\mathcal{F}$.

When $f, g \in \mathcal{F}$ and $\alpha \in \mathbb{C}$ and $f, g$ are bounded so are $f+g$, $fg$ and $\alpha f$ so we have the following.

**20.2 LEMMA :**

If $X$ is a metric space, the space $\mathscr{C}(X)$ of all complex valued bounded functions on $X$ is a vector space.

When $X$ is a compact metric space, every continuous function is bounded. In this case, $\mathscr{C}(X)$ is precisely the vector space of all complex valued functions on $X$ and the word 'bounded' becomes redundant. In what follows $X$ is a compact metric space.

For $f \in \mathscr{C}(X)$ define $\|f\| = \sup\{f|x|/x \in X\}$ and call $\|f\|$ by norm of $f$ or simply norm $f$.

**20.3 LEMMA :**

For $f, g \in \mathscr{C}(X)$, $\alpha \in \mathbb{C}$

(i) $\|f\| \geq 0$ with equality if and only if $f(x) = 0 \forall x \in X$

(ii) $\|f + g\| \leq \|f\| + \|g\|$

(iii) $\|\alpha f\| = |\alpha| \, \|f\|$

**Proof :** Since $|f(x)| \geq 0 \forall x \in X, \|f\| \geq 0$.

Since $0 \leq |f(x)| \leq \|f\| \forall x \in X$, $\|f\| = 0 \Leftrightarrow f(x) = 0 \forall x \in X$

Also for $x \in X$

$$|(f \in g)(x)| = |f(x) + g(x)|$$

$$\leq |f(x)| + |g(x)|$$

$$\leq \|f\| + \|g\|$$

Since this is true for every $x \in X$, $\|f + g\| \leq \|f\| + \|g\|$

Finally when $\alpha = 0$ $\alpha f = 0$ and so $\|\alpha f\| = \|\alpha\| \, \|f\| = 0$.

In the general case $\|\alpha f\| = \sup\{|\alpha f(x)|/x \in X\}$

$$= \sup\{|\alpha|\,|f(x)| / x \in X\}$$

$$= |\alpha| \sup\{|f(x)| / x \in X\}$$

$$= |\alpha| \|f\|.$$

The completes the proof of lemma.

**LEMMA** : The space $\mathscr{C}(X)$ is a metric space with respect to the function defined by

$$d(f, g) = \|f - g\|.$$

**Proof :** From (i) of lemma 19.3 $\|f - g\| \geq 0$ with equality if and only if $f - g = 0$ i.e. $f = g$ so that

$d(f, g) \geq 0$ and $d(f, g) = 0$ if and only if $f = g$.

From (ii) of lemma 19.4 for any $f, g, h$ in $\mathscr{C}(X)$, as $f - g$ and $g - h$ belong to $\mathscr{C}(X)$.

$$d(f, g) + d(g, h) = \|f - g\| + \|g - h\| \geq \|f - g + g - h\| = \|f - h\| = d(f, h)$$

This implies triangle inequality for $d$.

Finally for $f, g$ in $\mathscr{C}(X)$ by (iii) of lemma 19.4

$$d(f, g) = \|f - g\| = \|-1(g - f)\| = \|g - f\| = d(g, f)$$

Hence $d$ is a metric on $\mathscr{C}(X)$.

**20.4 THEOREM :**

Let $X$ be a compact metric space and $d$ be the metric on $\mathscr{C}(X)$ defined by

$$d(f, g) = \|f - g\| = \sup\{|f(x) - g(x)| / x \in X\}.$$

Let $\{f_n\}$ be a sequence in $\mathscr{C}(X)$ and $f \in \mathscr{C}(X)$. Then

(i) $\lim_n d(f_n, f) = 0$ if and only if $\{f_n\}$ converges to f uniformly on $X$.

(ii) $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$ if and only if $\{f_n\}$ satisfies Cauchy's criterion for uniform convergence.

**Proof :** Assume that $d(f_n, f) \to 0$ as $n \to \infty$

Given $\epsilon > 0$ there is a positive integer $N(\epsilon)$ such that

$$d(f_n, f) < \epsilon \text{ for } n \geq N(\epsilon).$$

$$\Rightarrow \sup\{|f_n(x) - f(x)| / x \in X\} < \epsilon \text{ for } n \geq N(\epsilon)$$

$$\Rightarrow |f_n(x) - f(x)| < \epsilon \text{ for } n \geq N(\epsilon) \text{ and } \cancel{\text{all}}\ x \in X$$

$\Rightarrow \{f_n\}$ converges uniformly to $f$ on $X$.

Conversely suppose $\{f_n\}$ converges uniformly to $f$ on $X$.

Given $\epsilon > 0$ there corresponds a positive integer $N(\epsilon)$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ for } n \geq N(\epsilon) \text{ and all } x \in X.$$

$$\Rightarrow \sup\{|f_n(x) - f(x)| / x \in X\} \leq \frac{\epsilon}{2} < \epsilon \text{ for } n \geq N(\epsilon)$$

$$\Rightarrow d(f_n, f) < \epsilon \text{ for } n \geq N(\epsilon).$$

$$\Rightarrow \lim_{n} d(f_n, f) = 0.$$

Assume that $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$. Given $\epsilon > 0$ there is a positive integer $N(\epsilon)$ such that

$$d(f_n, f_m) < \epsilon \text{ for } n > m \geq N(\epsilon)$$

$$\Rightarrow \sup\{|f_n(x) - f_m(x)| / x \in X\} < \epsilon \text{ for } n > m \geq N(\epsilon).$$

$$\Rightarrow |f_n(x) - f_m(x)| < \epsilon \text{ for } n > m \geq N(\epsilon) \text{ and all } x \in X.$$

$\Rightarrow \{f_n\}$ satisfies Cauchy's criterion for uniform convergence.

Conversely assume that $\{f_n\}$ satisfies Cauchy's criterion for uniform convergence. Then given $\in > 0$ there is a positive integer $N(\in)$ such that

$$|f_n(x) - f_m(x)| < \frac{\in}{2} \text{ for } n > m \geq N(\in) \text{ and all } x \in X.$$

$$\Rightarrow d(f_n, f_m) = \sup\{|f_n(x) - f_m(x)| / x \in X\} \leq \frac{\in}{2} < \in \text{ for } n > m \geq N(\in).$$

**20.5 DEFINITION :**

The metric $d$ on $\mathscr{C}(X)$ defined in 20.4 is called the uniform metric.

**20.6 THEOREM :**

The metric space $\mathscr{C}(X)$ of all bounded continuous functions on a metric space $X$ is complete with respects to the uniform metric defined by

$$d(f, g) = \sup\{|f(x) - g(x)| / x \in X\}$$

**Proof :** Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. We prove

(i) $\forall x \in X \{f_n(x)\}$ converges.

(ii) the limit function $f(x)$ is continuous on $X$ and bounded.

(iii) the convergence of $\{f_n\}$ to $f$ is uniform on $X$.

**Proof of (i) :** Since $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$ given $\in > 0$ there is a positive integer $N$ depending on $\in$ such that $d(f_n, f_m) < \in$ for $n > m \geq N$

$$\Rightarrow \forall x \in X \; |f_n(x) - f_m(x)| \leq d(f_n, f_m) < \in \text{ for } n > m \geq N \text{ ---------- (1)}$$

$\Rightarrow \{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$, hence converges.

**Proof of (ii) :** From (1) $|f_n(x) - f_N(x)| < \in$ for $n \geq N$ and all $x \in X$. Letting $n \to \infty$ we get $|f(x) - f_N(x)| \leq \in$. Since $f_N$ is continuous, $f_N$ is uniformly continuous hence there exists a $\delta > 0$ such that $|f_N(x) - f_N(y)| < \in$ if $x$, $y$ belong to $X$ and $d(x, y) < \delta$. For such $x$, $y$.

$$|f(x)-f(y)| \le |f(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f(y)|$$

$$<\epsilon+\epsilon+\epsilon = 3\epsilon.$$

Hence $f$ is in fact uniformly continuous on $X$.

**Proof of (iii) :** $\{f_n\}$ converges uniformly to $f$ on $X$.

From (1) given $\epsilon > 0$ there is a positive integer $N$ such that $|f_n(x)-f_m(x)| < \epsilon$ for $m > n \ge N$ keeping $x$ and $n$ fixed and letting $m \to \infty$ we get

$$|f(x)-f_n(x)| \le \epsilon.$$

Since this is true for every $x \in X$ and $n \ge N$ it follows that $\{f_n\}$ converges uniformly to $f$ on $X$. Since uniform convergence is equaivalent to convergence in $\mathscr{C}(X)$ with respect to $d$ it follows that $\{f_n\}$ converges to $f$ in $(\mathscr{C}(X), d)$

**20.7 DEFINITION :**

Let $\{f_n\}$ be a sequence of functions defined on a set $E$. We say that $\{f_n\}$ is pointwise bounded on $E$ if for every $x \in E$, there is a positive and real number $\phi(x)$ depending on $E$ such that $|f_n(x)| \le \phi(x)\ \forall\ n \ge 1$; i.e. $\forall x \in E$ the sequence of numbers $\{f_n(x)\}$ is bounded.

We say that $\{f_n\}$ is uniformly bounded on $E$ if there is a positive number $M$ such that

$|f_n(x)| \le M$ for all $x \in E$ and $n \ge 1$.

**20.8 THEOREM :**

Let $E$ be a countable set and $\{f_n\}$, a sequence of functions defined on $E$ such that $\{f_n(x)\}$ is bounded for every $x$ in $E$. Then there is a subsequence $\{f_{n_K}\}$ such that $\{f_{n_K}(x)\}$ converges for every $x$ in $E$.

**Proof :** We use the following :

(a) Every bounded sequence in $\mathbb{C}$ has a convergent subsequence

(b) Every subsequence of a convergent sequence is convergent.

Arrange the elements of E in a sequence $\{x_n\}$. *Obtain a sequence of sequences* $\{f_{n,0}\}, \{f_{n,1}\}, \cdots\cdots, \{f_{n,K}\}\cdots\cdots$ *with the following properties :*

(c) $\{f_{n,0}\} = f_n \ \forall n$.

(d) $\{f_{n,K+1}\}$ is a subsequence of $\{f_{n,K}\} \ \forall K \geq 0$ and

(e) $\{f_{n,K}(x_K)\}$ converges for every positive integer $K$.

This can be achieved by induction as follows. When $K=0$, $\{f_{n,0}(x_1)\}$ being a bounded sequence, contains a convergent subsequence $\{f_{n,1}(x_1)\}$. if $\{f_{n,K-1}\}$ is a subsequence of its predecessor such that $\{f_{n,K-1}(x_{K-1})\}$ converges replace $x_{K-1}$ by $x_K$. Then $\{f_{n,K-1}(x_K)\}$ being a subsequence of the bounded sequence $\{f_{n,K-1}(x_K)\}$ is itself bounded, hence contains a convergent subsequence $\{f_{n,K}(x_K)\}$.

By induction this holds good for all $K$. We thus have a sequence of sequences $\{f_{n,K}/n \geq 1\}$ satisfying (c) (d) (e).

Write $S_K$ for the sequence $\{f_{1,K}, f_{2,K}, \cdots\cdots, f_{n,K}\cdots\cdots\}$.

Let $S$ be the sequence $[f_{1,1}, f_{2,2}, f_{3,3}, \cdots\cdots, f_{K,K}, \cdots\cdots]$.

obtained by picking up the $K$ th function from $S_K$.

We show that the sequence

$$S(x_K) = \{f_{1,1}(x_K), f_{2,2}(x_K)\cdots\cdots f_{K,K}(x_K)\cdots\cdots\}$$

converges for all $x_K$ in $E$.

Clearly $S_2$ is a subsequence of $S_1$, $S_3$ of $S_2$ and so on and in general $S_K$ is a subsequence of $S_{K-1}$. Hence

$$\{f_{K,K}, f_{K+1,K+1}, \cdots\cdots\}$$

is a subsequence of $\{f_{K,K}, f_{K+1,K}, f_{K+2,K} \cdots\cdots\}$

Now since $S_1(x_1):\{f_{1,1}(x_1),\cdots\cdots f_{n,1}(x_1)\cdots\cdots\}$

is convergent, the subsequence

$S(x_1):\{f_{1,1}(x_1), f_{2,2}(x_1),\cdots\cdots\}$ is convergent.

Again since $S_2(x_2):\{f_{1,2}(x_2), f_{2,2}(x_2),\cdots\cdots f_{n,2}(x_2)\cdots\cdots\}$ is convergent

$\{f_{2,2}(x_2)\; f_{3,3}(x_2),\cdots\cdots f_{K,K}(x_2)\cdots\cdots\}$ being its subsequence, is convergent hence

$S(x_2):\{f_{1,1}(x_2), f_{2,2}(x_2)\cdots\cdots\}$ is convergnet.

The arrangement hold goods for every $K$.

$$\{f_{K,K}(x_K), f_{K+1,K+1}(x_K),\cdots\cdots\cdots\}\cdots\cdots(*)$$

being a subsequence of $S_K(x_K)=\{f_{1,K}(x_K), f_{2,K}(x_K),\cdots\cdots\}$

which is convergent, is itself convergent and hence

$$S(x_K)=\{f_{1,1}(x_K), f_{2,2}(x_K),\cdots\cdots, f_{K-1,K-1}(x_K), f_{K,K}(x_K),\cdots\cdots\}$$

obtained by adjoining $f_{1,1}(x_K),\cdots\cdots, f_{K-1,K-1}(x_K)$ at the beginning to $(*)$ is convergent.

As this holds for all $K$, the proof is complete.

**20.9 DEFINITION :**

A family $\mathscr{F}$ of complex valued functions defined on a subset $E$ of a metric space $(X,d)$ is said to be equicontinuous if for every $\in > 0$ there corresponds a $\delta > 0$ (depending on $\in$) such that $|f(x)-f(y)| < \in$ for all $f \in \mathscr{F}$ and $x, y$ in $E$.

**REMARK :** If $\mathscr{F}$ is an equicontinuous family defined on $E$ every $f \in \mathscr{F}$ is uniformly continuous on $E$.

**20.10 EXAMPLE :**

Let $f:\mathbb{R} \to \mathbb{R}$ be any continuous function.

Define $f_n(x) = \dfrac{x}{n}$ for $n \geq 1$ and $x \in \mathbb{R}$.

and $g_n(x) = nx$ for $n \geq 1$ and $x \in \mathbb{R}$.

The sequence $\{f_n\}$ is equicontinuous on $\mathbb{R}$

since $|f_n(x) - f_n(y)| \leq |x - y|$ for all $n \geq 1$ while $\{g_n\}$ is not equicontinuous because for any $\delta > 0$ and $n > \dfrac{2}{\delta}$.

$$|g_n(x) - g(y)| = n|x - y| = \frac{n\delta}{2} > 1 \text{ when } y = x + \frac{\delta}{2}.$$

## UNIFORM CONVERGENCE IMPLIES EQUICONTINUITY FOR SEQUENCES :

### 20.11 THEOREM :

Let $K$ be a complete metric space and $\{f_n\}$ be a sequence of continuous functions on $K$. If $\{f_n\}$ converges uniformly on $K$ then $\{f_n\}$ is an equicontinuous family in $\mathscr{C}(X)$.

**Proof :** By uniform convergence $\forall \in > 0$ there corresponds a positive integer $N(\in)$ such that

$|f_n(x) - f_m(x)| < \dfrac{\in}{3}$ for $n \geq m \geq N(\in)$ and $x \in K$. Fix $N = N(\in)$. Then $|f_n(x) - f_N(x)| < \dfrac{\in}{3}$ for $n \geq N$ and $x \in K$. Since $f_N$ is continuous on $K$, $f_N$ is uniformly continuous as K is compact so that there is a $\delta_0 > 0$ such that

$|f_N(x) - f_N(y)| < \dfrac{\in}{3}$ whenever $x, y$ belong to $K$ and $|x - y| < \delta_0$. For such $x, y$ and $n \geq N$.

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3}.$$

We are left with $f_1, f_2, \cdots f_{N-1}$.

Choose positive numbers $\delta_1, \delta_2, \cdots \delta_{N-1}$ such that

$|f_i(x) - f_i(y)| < \epsilon$ for $d(x, y) < \delta_i$ and $1 \le i \le N-1$. If $\delta =$ minimum of $\{\delta_1, \delta_2, \cdots, \delta_{N-1} \text{ and } \delta_0\}$ then

$$|f_i(x) - f_i(y)| < \epsilon \text{ for all } i \text{ and } x, y \text{ in } X \ni d(x, y) < \delta.$$

This implies equicontinuity of $\{f_n\}$ on $K$. The proof is complete.

**Question :** Does equicontinuity imply uniform convergence for sequences in $\mathscr{C}(K)$ ?

that the answer is no is evident from SAQ 4.

**20.12 THEOREM :**

If $(K, d)$ is a compact metric space and $\{f_n\}$ is a pointwise bounded sequence in $\mathscr{C}(K)$ which is equicontinuous on $K$, then $\{f_n\}$ is uniformly bounded and contains a subsequence which converges uniformly on $K$.

**Proof Uniform boundness :** Since $\{f_n\}$ is continuous on $K$, corresponding to $\epsilon = 1$ there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < 1 \text{ for all } n \text{ and } x, y \text{ in } K \ni d(x, y) < \delta.$$

If $V_x = \{y / y \in K \text{ and } d(x, y) < \delta\}$, the family $\{V_x / x \in K\}$ is an open cover for the compact space, hence contains a finite subcover say $V_{x_1}, V_{x_2}, \cdots\cdots, V_{x_r}$.

Since for each $i, 1 \le i \le r$, $\{f_n(x_i)\}$ is bounded $\exists M_i > 0 \ni$

$$|f_n(x_i)| \le M_i \text{ for all } n.$$

Since $K \subset \bigcup_{i=1}^{r} V_{x_i}$, $x \in K \Rightarrow \exists \text{ a } i \ni x \in V_{x_i}$

Since $d(x, x_i) < \delta$, $|f_n(x) - f_n(x_i)| < 1$ for all $n$.

$\Rightarrow |f_n(x)| - |f_n(x_i)| < 1$ for all $n$

$\Rightarrow |f_n(x)| < 1 + |f_n(x_i)| \leq 1 + M_i$. Let $M = Max(1 + M_i)$. Hence $\forall x \in K$, and $n \geq 1$.

$$|f_n(x)| \leq M$$

This implies that $\{f_n\}$ is uniformly bounded.

**UNIFORM CONVERGENCE OF A SUBSEQUENCE :**

Since $K$ is a compact metric space, $K$ has a countable dense subset $E$.

If $\in > 0$ by equicontinuiy of $\{f_n\}$ $\exists$ a $\delta > 0 \ni$

$$|f_n(x) - f_n(y)| < \frac{\in}{3} \text{ if } x, y \in K \text{ and } d(x, y) < \delta \text{ ---------- (1)}$$

Since $E$ is dense in $X$ for every $x$ in $K$ $\exists$ a $p \in E \ni d(x, p) < \delta$. Thus the collection of neighborhoods each of radius $\delta$ and centered at a point $p$ of $E$ is an open cover of $K$ and hence has a finite subcover $\{S_{p_1}, \cdots S_{p_r}\}$ where

$$S_{p_i} = \{y / y \in K \ni d(y, p_i) < \delta\}, (1 \leq i \leq r).$$

Since $E$ is countable and $\{f_n\}$ is pointwise bounded there is a subsequence $\{g_i\}$ of $\{f_n\}$ such that $\{g_i(p)\}$ converges for every $p \in E$. In particular $\{g_n(p_i)\}$ converges. Hence given $\in > 0$ there is a positive integer $N(\in)$ such that for $n \geq m \geq N(\in)$ and $1 \leq i \leq r$.

$$|g_n(x_i) - g_m(p_i)| < \frac{\in}{3}$$

If $x \in K \; \exists$ a $i \ni 1 \leq i \leq r$ and $x \in S_{p_i}$

$$\Rightarrow |g_n(x) - g_n(p_i)| < \frac{\in}{3} \text{ for } n \geq 1.$$

For $n \geq N(\epsilon)$ and $m \geq N(\epsilon)$

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(p_i)| + |g_n(p_i) - g_m(p_i)| + |g_m(p_i) - g_m(x)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Since this is true for all $x \in K$ $\{g_n\}$ converges uniformly on $K$.

## WEIERSTRASS' APPROXIMATION THEOREM :

### 20.13 THEOREM :

If $f$ is a complex valued continuous function on $[a, b]$, there exists a sequence $\{P_n\}$ of polynomials such that

$$\lim_n P_n(x) = f(x)$$

uniformly on $[a, b]$. If $f$ is real, the $P_n$ may be taken real.

**Proof :** We divide the proof into 4 steps.

**Step 1 :** We may assume that $a = 0, b = 1$ and $f(0) = f(1) = 0$.

**Proof of Step 1 :** Suppose for every complex valued continuous function $g$ on $[0, 1]$ satisfying $g(0) = g(1) = 0$, there is a sequence $\{Q_n\}$ of polynomials such that $\lim_n Q_n(x) = g(x)$ uniformly on $[0, 1]$.

Let $f : [a, b] \to \mathbb{C}$ be any continuous function such that $f(a) = f(b) = 0$

$$\begin{tikzcd} {[0, 1]} \arrow[r, "\phi"] \arrow[rr, bend right, "g"'] & {[a, b]} \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

Define $\phi(t) = (b - a)t + a$

$\phi : [0, 1] \to [a, b]$ is continuous, one - one and onto.

and its $\psi:[a, b]\to[0, 1]$ defined by $\psi(t)=(b-a)t+a$ is also one - one, onto and continuous.

$g=fo\phi:[0, 1]\to\mathbb{C}$ is a continuous function satisfying

$g(0)=f(\phi(0))=f(a)=0$ and $g(1)=f(\phi(1))=f(b)=0$

So there is a sequence $\{Q_n\}$ of polynomials such that

$\lim\limits_{n} Q_n(x)=g(x)$ uniformly on $[0, 1]$

$\Rightarrow \forall \in>0$ there corresponds a positive integer $N(\in)$ such that

$|Q_n(x)-g(x)|<\in$ for $n\geq N(\in)$ and $x\in[0, 1]$

If $t\in[a, b]$, $\psi(t)=x\in[0, 1]$ so

$|Q_n(\psi(t))-g(\psi(t))|<\in$ for $n\geq N(\in)$ and $t\in[a, b]$

Since $\psi=\phi^{-1}$ and $f o\phi=g$, $f=g o\psi$. We now write $Q_n o\psi=P_n$.

Clearly each $P_n$ is a polynomial on $[a, b]$ and

$\lim\limits_{n} P_n(t)=f(t)$ uniformly on $[a, b]$.

Now let $f:[a, b]\to\mathbb{C}$ be any continuous function.

Then the function $g:[a, b]\to\mathbb{C}$ defined by

$$g(x)=(f(x)-f(a))-(x-a)\cdot\frac{f(b)-f(a)}{b-a}$$

is defined on $[a, b]$, continuous and $g(a)=g(b)=0$.

Also $f(x)=f(a)+(x-a)\dfrac{f(b)-f(a)}{b-a}+g(x)$.

If $\{Q_n(x)\}$ is a sequence of polynomials such that

$$\lim_n Q_n(x) = g(x) \text{ uniformly on } [a, b]$$

and $$P_n(x) = f(a) + (x-a)\frac{f(b)-f(a)}{b-a} + Q_n(x)$$

then $|P_n(x) - f(x)| = |Q_n(x) - g(x)|$

so that $\lim_n P_n(x) = f(x)$ is uniformly on $[a, b]$.

This completes the proof of step 1.

**Step 2 :** If $Q_n(x) = C_n\left(1-x^2\right)^n$ $(n \geq 1)$ and $\int_{-1}^{1} Q_n(x)\,dx = 1$, then $C_n < \sqrt{n}$

**Proof of Step 2 :** For any positive integer $n$ and $x \in [-1, 1]$

$$\left(1-x^2\right)^{n+1} = \left(1-x^2\right)^n\left(1-x^2\right) \text{ so that if } \left(1-x^2\right)^n \geq 1 - n x^2,$$

$$\left(1-x^2\right)^{n+1} \geq \left(1-nx^2\right)\left(1-x^2\right) = 1-(n+1)x^2 + n x^4 \geq 1-(n+1)x^2$$

so by the principle of mathematical induction, it follows that

$$\left(1-x^2\right) \geq 1 - n x^2 \text{ for } n \geq 1 \text{ and } x \in [-1, 1]$$

Hence $$1 = \int_{-1}^{1} Q_n(x)\,dx = C_n \cdot \int_{-1}^{1} \left(1-x^2\right)^n dx$$

$$= 2C_n \int_{0}^{1} \left(1-x^2\right)^n dx$$

$$\geq 2C_n \int_{0}^{1} 1 - n x^2\, dx$$

$$> 2C_n \int_0^{1/\sqrt{n}} \left(1 - nx^2\right) dx$$

$$= C_n \frac{4}{3\sqrt{n}} > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow \sqrt{n} > C_n \ \forall\, n.$$

If $0 < \delta < 1$ and $E(\delta) = \{x / \delta \le |x| \le 1\}$

$0 \le 1 - x^2 \le 1 - \delta^2$ so that

$$0 \le Q_n(x) \le C_n \left(1-\delta^2\right)^n < \sqrt{n}\left(1-\delta^2\right)^n$$

since $\left(1-\delta^4\right)^n < 1,$

$$\left(1-\delta^2\right)^n < \frac{1}{\left(1+\delta^2\right)^n}$$

since $\left(1+\delta^n\right) = 1 + n\delta^2 + \frac{n(n-1)}{1 \cdot 2}\delta^4 + \cdots\cdots + \delta^{2n} > n\delta^2$

$$\frac{1}{\sqrt{n}\left(1+\delta^2\right)^n} < \frac{1}{n^{3/2}\delta^2}$$

$$\Rightarrow \lim_n \frac{1}{\sqrt{n}\left(1+\delta^2\right)^n} = 0$$

$\Rightarrow \lim_n Q_n(x) = 0$ uniformly on $E(\delta)$.

This completes the proof of step 2.

**Step 3 :** Construction of the sequence of polynomials $\{P_n\}$.

Define $f(x)=0$ if $x \le 0$ or $x \ge 1$. Clearly $f$ is continuous on $\mathbb{R}$.

Write $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$, for $x \in [0, 1]$.

Then $P_n(x) = \int_{-1}^{-x} f(x+t)Q_n(t)dt + \int_{-x}^{1-x} f(x+t)Q_n(t)dt + \int_{1-x}^{1} f(x+t)Q_n(t)dt$.

If $-1 \le t \le -x$, $x+t \le 0$ so $f(x+t)=0$

If $1-x \le t \le 1$, $x+t \ge 1$ so $f(x+t)=0$. Hence

$$P_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)dt = \int_{0}^{1} f(s)Q_n(s-x)ds.$$

If $Q_n(s-x) = a_0(s)+a_1(s)x^2 + \cdots\cdots + a_{2n}(s)x^{2n}$

Where $a_0(s), a_1(s), \cdots\cdots a_{2n}(s)$ are polynomials in $s$ involving binomial coefficients and powers of $s$ only,

$$P_n(x) = \sum_{i=0}^{2n} \alpha_i x^i \text{ where } \alpha_i = \int_{0}^{1} a_i(s)\, f(s)ds.$$

Thus $P_n(x)$ is a polynomial in $x$ and the coefficients are real when $f$ is real valued.

**Step 4 :** Uniform convergence of $\{P_n\}$ to $f$ on $[0, 1]$.

**Proof of Step 4 :** Let $M>0$ be $\ni |f(x)| < M \, \forall x \in [0, 1]$. If $\in > 0$ choose $\delta \ni 0<\delta<1$ and $|f(x)-f(y)| < \frac{\in}{2}$ if $x, y \in [0, 1]$ and $|x-y| < \delta$.

Since $\lim_{n} Q_n(x)=0$ uniformly on $E(\delta)$,

$\exists$ a positive integer $N(\in) \ni |Q_n(t)| < \frac{\in}{8M}$ for $n \ge N(\in)$ and $t \in E(s)$.

Also $0 \le \int_{-\delta}^{\delta} Q_n(t)\,dt \le \int_{-1}^{1} Q_n(t)\,dt = 1$.

Now for $0 \le x \le 1$.

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) Q_n(t)\,dt - \int_{-1}^{1} f(x) Q_n(t)\,dt \right|$$

$$= \left| \int_{-1}^{1} \{f(x+t) - f(x)\} Q_n(t)\,dt \right|$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)| Q_n(t)\,dt$$

$$= \int_{-1}^{-\delta} |f(x+t) - f(x)| Q_n(t)\,dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)| Q_n(t)\,dt + \int_{\delta}^{1} |f(x+t) - f(x)| Q_n(t)\,dt$$

$$\le 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \int_{-\delta}^{\delta} \frac{\in}{2} Q_n(t)\,dt + 2M \int_{\delta}^{1} Q_n(t)\,dt.$$

$$< 2M \cdot \frac{\in}{8M}(1-\delta) + \frac{\in}{2} \int_{-1}^{1} Q_n(t)\,dt + 2M \cdot \frac{\in}{8M}(1-\delta)$$

$$< \frac{\in}{4} + \frac{\in}{2} + \frac{\in}{4} = \in \qquad (\because 0 < \delta < 1)$$

This is true for all $x \in [0, 1]$ and $n \ge N(\in)$.

Hence $\lim_{n} P_n(x) = f(x)$ uniofrmly on $[a, b]$.

## 20.14 SHORT ANSWER QUESTIONS :

**20.14.1 :** If $\{f_n\}$ is a sequence of continuous functions converging uniformly to a function $f$ on a metric space $(X, d)$ and $\lim x_n = x$ then $\lim f_n(x_n) = f(x)$.

**20.14.2 :** Give an example of a sequence of continuous functions which are bounded but not equicontinuous.

**20.14.3 :** Give an example of a sequence of bounded functions that converge pointwise but not uniformly.

**20.14.4 :** Does equicontinuity imply uniform convergence for sequences of functions ?

**20.14.5 :** Show that if $f$ is continuous on $[0, 1]$ and $\int_0^1 x^n f_n(x)\,dx = 0 \ \forall \ n \geq 0$ then $f(x)=0$ for all $x$ in $[0, 1]$.

**20.14.6 :** Using Weierstrass approximation theorem prove that $\forall$ real number $a > 0$ there is a sequence of polynomials $\{P_n(x)\}$ such that $\{P_n\}$ converges uniformly to $|x|$ on $[-a, a]$ and $P_n(0)=0$.

**20.14.7 :** Find $C_n$ such that $\int_{-1}^{1} C_n \left(1-x^2\right)^n dx = 1$.

## 20.15 MODEL EXAMINATION QUESTIONS :

**20.15.1 :** If $K$ is a compact metric space, $f_n \in \mathscr{C}(K)$ $(n \geq 1)$ and $\{f_n\}$ converges uniformly then $\{f_n\}$ is equicontinuous on $K$.

**20.15.2 :** If $K$ is a compact metric space, $f_n \in \mathscr{C}(K)$ and if $\{f_n\}$ is pointwise bounded and equicontinuous then $\{f_n\}$ is uniformly bounded on $K$.

**20.15.3 :** If $K$ is a compact metric space, $f_n \in \mathscr{C}(K)$ and if $\{f_n\}$ is equicontinuous and uniformly bounded then prove that $\{f_n\}$ contains a uniformly convergent subsequence $\{f_{n,K}\}$.

**20.15.4 :** If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set $E$ then $\{f_n\}$ has a subsequence $\{f_{n,K}\}$ that converges for all $x \in E$.

**20.15.5 :** State and prove Weiestrass approximation theorem

**20.15.6 :** If $f$ is continuous on $[0, 1]$ and $\int_0^1 x^n \, dx = 0$ for $n \geq 0$, prove that $f(x) = 0$.

## 20.16 ANSWERS TO SHORT ANSWER QUESTIONS IN 20.14

**20.14.1 :** Given $\in > 0$ choose a positive integer $N(\in)$ and $\delta > 0 \ni |f_n(y) - f(y)| < \frac{\in}{2}$ for $n \geq N(\in)$ and $y \in X$ and $|f(x) - f(y)| < \frac{\in}{2}$ for $y \in X$ and $d(x, y) < \delta$.

Choose $N \ni d(x_n, x) < \delta$ for $n \geq N$.

For $n \geq N$, $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| < \in$.

**20.14.2 :** Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$ $\qquad (0 \leq x \leq 1, \; n \geq 1)$

$|f_n(x)| \leq 1 \, \forall \, n \geq 1$ and $0 \leq x \leq 1$.

$$\lim_n f_n(x) = \lim_n \frac{1}{1 + \left(\frac{1}{x} - n\right)^2} = 0 \text{ for } 0 \leq x \leq 1 \text{ and}$$

$$\lim_n f_n\left(\frac{1}{n}\right) = 1$$

**20.14.3 :** Let $f_n(x) = n^2 x \left(1 - x^2\right)^n$.

For $0 \leq x \leq 1$ $\lim_n f_n(x) = 0$

Also $\{f_n(x) / n \geq 1\}$ is bounded $\forall \, x \in [0, 1]$ (see 18.18).

**20.14.4 :** See example 17.17.

**20.14.5 :** If $f$ is continuous on $[0, 1]$ and if $\int_0^1 f(x) x^n \, dx = 0$ for $n \geq 0$, then $f(x) = 0$ on $[0, 1]$.

**Proof :** Since $f$ is continuous, so is $|f|$.

By Weierstrass approximation theorem $\exists$ a sequence $\{P_n\}$ of polynomials such that $\lim\limits_n P_n = \bar{f}$ uniformly on $[0, 1]$. Since each $P_n$ is bounded on $[0, 1]$ and $\bar{f}$ is bounded. $P_n\, f \to |f|^2$ uniformly on $[0, 1]$.

Hence $\int_0^1 P_n f \to \int_0^1 |f|^2$.

Now if $P_n(t) = a_0 + a_1 t + \cdots\cdots + a_K t^K$,

$$\int_0^1 P_n(t) f(t)\,dt = a_0 \int_0^1 f(t)\,dt + a_1 \int_0^1 t f(t)\,dt + \cdots\cdots + a_K \int_0^1 t^K f(t)\,dt$$

$= 0$ by hypothesis

Hence $0 = \lim\limits_n \int_0^1 P_n(t) f(t)\,dt = \int_0^1 |f(t)|^2\,dt$.

Since $|f(t)|^2 \geq 0 \;\forall\; t \in [0, 1]$ and is continuous

it now follows that $f(t) = 0$ on $[0, 1]$.

**20.14.6 :** By Weierstrass approximation there is a sequence of polynomials $P_n^*$ such that $\{P_n^*(x)\}$ converges uniformly on $[-a, a]$ to $|x|$. Then $\lim\limits_n P_n^*(0) = P^*(0)$.

If $P_n(x) = P_n^*(x) - P_n^*(0)$, $\{P_n\}$ converges uniformly on $[-a, a]$ to $|x|$ and $P_n(0) = 0$.

**20.14.7 :** Let $Q_n(t) = C_n\left(1 - x^2\right)^n$. Find $C_n$ if $\int_{-1}^1 Q_n(x) = 1$.

**Solution :** $Q_n(x) = C_n\left(1-\binom{n}{1}x^2+\binom{n}{2}x^4+\cdots\cdots+(-1)^n x^{2n}\right)$

$$\text{So } 1 = \int_{-1}^{1} Q_n = C_n\left\{\int_{-1}^{1} 1\,dx - \binom{n}{1}\int_{-1}^{1} x^2\,dx + \cdots\cdots+(-1)^n \int_{-1}^{1} x^{2n}\,dx\right\}.$$

$$\int_{-1}^{1} x^{2K}\,dx = \left[\frac{x^{2K+1}}{2K+1}\right]_{-1}^{1} = \frac{2}{2K+1}$$

$$\Rightarrow \frac{1}{C_n} = 2-\binom{n}{1}\frac{2}{3}+\binom{n}{2}\frac{2}{5}+\cdots\cdots+(-1)^K\binom{n}{K}\frac{2}{2K+1}+\cdots\cdots+(-1)^n\frac{2}{2n+1}$$

$$\Rightarrow \frac{C_n}{2} = \left\{1-\frac{1}{3}\binom{n}{1}+\frac{1}{5}\binom{n}{2}+\cdots\cdots+\frac{(-1)^n}{2n+1}\right\}$$

## 20.17 EXERCISES :

**20.17.1 :** If $f$ is a real continuous function on $\mathbb{R}$ and the sequence $\{f_n\}$ defined by $f_n(t)=f(nt)$ is equicontinuous on $[0, 1]$ what conclusion can you draw about $f$ ?

**20.17.2 :** Let $\{f_n\}$ be uniformly bounded on $[a, b]$ which are Riemann integrable. Put

$$F_n(t) = \int_a^t f_n(x)\,dx.$$

Prove that there is . subsequence $\{F_{n_K}\}$ which converges uniformly on $[a, b]$.

**20.17.3 :** Let $\{f_n\}$ be a equicontinuous sequence of functions on a compact set $K$ which is pointwise bounded. Prove that $\{f_n\}$ converges uniformly on $K$.

**20.17.4 :** Define the notion of uniform convergence for mappings on a metric space $X$ into a metric space $Y$. Prove that $\{f_n:(X,d)\to(Y,\rho)\}$ converge uniformly iff $\lim M_n=0$ where $M_n = \sup_{x\in X} \rho(f_n(x), f(x))$.

**20.17.5 :** Show that if $f_n : (X, d) \to (Y, p)$ is continuous for $n \geq 1$ and $\{f_n\}$ converges uniformly to $f : X \to Y$ show that $f$ is continuous.

**20.17.6 :** Extend the notion of equicontinuity when the codomain is a metric space. Prove that if $\{f_n\}$ is any sequence of continuous functions defined on a compact a metric space $K$ into the Euclideun space $\mathbb{R}^K$ that converges uniformly on $K$ then $\{f_n\}$ is equicontinuous.

**20.17.7 :** Using SAQ 7 evaluate

$$R_n(x) = \int_0^1 t\, Q_n(t - x)\,dx \text{ for } n = 0, 1, 2, 3, 4.$$

Also evaluate $P_n$ for $n = 0, 1, 2, 3, 4$ where

$P_n$ is defined as in example 19.3.

**20.17.8 :** Find $S_n(x) = \int_0^2 t\, Q_n(t - x)\,dx$ for $n = 0, 1, 2, 3, 4$.

The sequence $S_n \,(n \geq 1)$ converges to $|x|$ on $[-2, 2]$.

The sequence $R_n \,(n \geq 1)$ of exercise 7 above converges to $|x|$ on $[-1, 1]$.

We know these facts from Weierstrass approximation. Is $S_n$ restricted to $[-1, 1]$ equal to $R_n$?

**20.17.9 :** Show that if $f$ is continuous on $[0, 1]$ and $\int_0^1 x^n f(x)\,dx = 0$ for $n \geq 1$ then $f(x) = 0$.

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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**Lesson - 21**

# ADDITIVE SET FUNCTIONS

## 21.1 INTRODUCTION

The Lebesgue integral is developed to overcome some difficulties that arise in Riemann's theory of the integral, some of which are pointed out in the lessons on uniform convergence.

In this lesson we develop some tools required for the development of Lebesgue Theory on $\mathbb{R}^p (p \geq 1)$.

We begin with the notion of a ring of sets and an additive set function. We consider the family $\mathscr{E}_p$ of elementary sets in $\mathbb{R}^p$, generated by the intervals in $\mathbb{R}^p$, show this is a ring and then define the Lebesgue measure on $\mathscr{E}_p$. We also show that the Lebesgue measure is regular.

In this lesson and the subsequent lessons we write O for the empty set and 0 for the zero element in $\mathbb{R}'$.

For any sets $A, B$, $A - B = \{x \,/\, x \in A \text{ and } x \notin B\}$

If $X$ is any set and $A \subseteq X$, $A^C$ stands for $X - A$.

## ELEMENTARY SETS IN $\mathbb{R}^P$:

Let us recall that by an interval in $\mathbb{R}'(=\mathbb{R})$, we mean a subset $I$ of $\mathbb{R}'$ with the following property :

$$x \in I,\ y \in I,\ x < z < y \Rightarrow z \in I.$$

It is clear that the empty set O and the singleton set $\{a\}$, for any $a \in \mathbb{R}'$ are intervals. If an interval $I$ is bounded and infimum $I = a$ while supremum $I = b$ then $I$ is one of the sets $(a, b), (a, b], [a, b)$ and $[a, b]$, where each of these sets consists of all $x$ between a and b and the inclusion of a and b is indicated by the appropriate closed bracket [ or ] while the exclusion is indicated by ( or ).

In the sequel we consider these four types of intervals only. As such by an interval in $\mathbb{R}'$ we mean one of above four sets.

**21.1 SAQ :**

If $I$ and $J$ are intervals in $\mathbb{R}'$ show that $I \cap J$ is an interval in $\mathbb{R}'$ and $I - J$ is the union of atmost two intervals in $\mathbb{R}'$.

**21.2 DEFINITION :**

A subset of $\mathbb{R}^p$ is called an interval if it is of the form $I_1 \times I_2 \times \cdots\cdots \times I_p$ where each $I_i \ (1 \leq i \leq p)$ is an interval in $\mathbb{R}'$.

**21.3 SAQ :**

If $I$ and $J$ are intervals in $\mathbb{R}^p$ then so is $I \cap J$, and $I - J$ is the union of a finite number of intervals in $\mathbb{R}^p$.

**21.4 DEFINITION :**

A subset $E$ of $\mathbb{R}^p$ is called an elementary set in $\mathbb{R}^p$ if $E$ is the union of a finite number of intervals in $\mathbb{R}^p$. The collection of elementary sets in $\mathbb{R}^p$ is denoted by $\mathscr{E}_p$ and when there is no ambiguity we drop the suffix $p$ and simply write $\mathscr{E}$ for $\mathscr{E}_p$.

**Examples :** $(1, 2] \cup (3, 4) \cup (-6, -5)$ is an elementary set in $\mathbb{R}'$.

$(0, 1) \times \left[\frac{1}{5}, 2\right]$ is an interval in $\mathbb{R}^2$.

$(3, 7) \times (5, 6) \cup (-5, -4) \times [10, 11]$ is an elementary set in $\mathbb{R}^2$ which is not an interval.

**21.5 REMARK :**

Clearly the collection $\mathscr{I}$ of all intervals in $\mathbb{R}^p$ is contained in $\mathscr{E}$. From SAQ 21.3, it follow that $\mathscr{E}$ contains finite intersections of members of $\mathscr{I}$, finite unions of members of $\mathscr{I}$ and th difference of any two members of $\mathscr{I}$.

**21.6 PROPOSITION :**

(i) $\emptyset \in \mathscr{E}$

(ii) $\mathscr{E}$ is closed under finite unions and finite intersections

(iii) $A \in \mathscr{E}$ and $B \in \mathscr{E} \Rightarrow A - B \in \mathscr{E}$.

**Proof of (i) :** The empty set O is an interval in $\mathbb{R}'$ hence $O = O \times O \times \cdots \times O$ ( p time

is an interval in $\mathbb{R}^p$, so belongs to $\mathscr{E}$.

**(ii) :** If $A_1, \cdots\cdots, A_n$ are in $\mathscr{E}$ and $A_j = \bigcup_{r=1}^{m_j} I_{j_r} \quad (1 \le j \le n)$, where

where $n, m_1, \cdots\cdots, m_n$ are positive integers

then $\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n}\left(\bigcup_{r=1}^{m_j} I_{j_r}\right)$ is in $\mathscr{E}$,

and $\bigcap_{j=1}^{n} A_j = \bigcap_{j=1}^{n}\left(\bigcup_{r=1}^{m_j} I_{j_r}\right)$

$= \bigcup_{r=1}^{K} C_r$ where each $C_r$ is the intersection of a finite number of

intervals chosen from the representation of each $A_j$ and

$K = m_1 \cdots\cdots m_n$. Hence $\bigcap_{j=1}^{n} A_j \in \mathscr{E}$.

**(iii) :** Finally if $A \in \mathscr{E}$ and $B \in \mathscr{E}$ there exist intervals $I_1, \cdots\cdots, I_n$ and $J_1, \cdots\cdots, J_m$ in $\mathbb{R}^p$ such that

$A = \bigcup_{r=1}^{n} I_r$ and $B = \bigcup_{s=1}^{m} J_s$ so that

$$A - B = A - \bigcup_{s=1}^{m} J_s = \bigcap_{s=1}^{m}(A - J_s) = \bigcap_{s=1}^{m}\ \bigcup_{r=1}^{n}(I_r - J_s)$$

By SAQ 3, $I_r - I_s \in \mathscr{E}$ so by (ii) $A - B \in \mathscr{E}$.

**21.7 :** If $A \in \mathscr{E}_p$ then $A$ is the union of a finite number of pointwise disjoint intervals in $\mathbb{R}^p$.

Let $A = \bigcup_{j=1}^{n} I_j$,

$I_j = I_{j_1} \times \cdots\cdots \times I_{j_p}$ for $1 \le j \le n$ where each $I_{jk}$ is an interval in $\mathbb{R}'$ with end points, say $a_{jk} \le b_{jk}$. Arrange these end points in increasing order. For each $k, 1 \le k \le p$ these $2n$ consecutive numbers generate $(2n-1)$ intervals, which become nonoverlaping by allowing the common end points of adjacent intervals into one of them only, say for definiteness into the left interval. These $(2n-1)$ intervals in the $r^{th}$ coordinate axis for $1 \le r \le p$ generate $(2n-1)^p$ intervals in $\mathbb{R}^p$ which are disjoint because the common edges are included in only one of the adjacent intervals in $\mathbb{R}^p$. Clearly $A$ is the union of a subcollection of these $(2n-1)^p$ pairwise disjoint intervals in $\mathbb{R}^p$.

## SYMMETRIC DIFFERENCE :

For any sets $A, B$ the symmetric difference

$$S(A, B) \text{ is the set } (A-B) \cup (B-A).$$

**21.8 SAQ :** For any sets $A, B, C$

(i) $S(A, A)=0$

(ii) $S(A, B)=S(B, A)=S(A^C, B^C)$

(iii) $S(A, C) \subseteq S(A, B) \cup S(B, C)$.

**21.9 SAQ :** For any sets $A_1, A_2, B_1$ and $B_2$ each of the following sets is a subset of $S(A_1, B_1) \cup S(A_2, B_2)$

(i) $S(A_1 \cup A_2, B_1 \cup B_2)$

(ii) $S(A_1 \cap A_2, B_1 \cap B_2)$ and

(iii) $S(A_1 - A_2, B_1 - B_2)$

## RING OF SETS :

**21.10 Definition :** A family $\mathscr{R}$ of sets is said to be a ring if $A \in \mathscr{R}, B \in \mathscr{R} \Rightarrow A \bigcup B \in \mathscr{R}$ and $A - B \in \mathscr{R}$

$\mathscr{R}$ is said called a $\sigma$ – ring if $\mathscr{R}$ is a ring of sets and is closed under countable unions.

i.e. $A_n \in \mathscr{R}$ for $n \geq 1 \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$

**21.11 SAQ :**

If $\mathscr{R}$ is a ring and $A \in \mathscr{R}$ and $B \in \mathscr{R} \Rightarrow A \bigcap B \in \mathscr{R}$. Further $\mathscr{R}$ is closed under finite unions, i.e. for any positive integer n and any sets $A_i, 1 \leq i \leq n$ in $\mathscr{R}$, $\bigcup_{n=1}^{\infty} A_i \in \mathscr{R}$.

**21.12 SAQ :** If $\mathscr{R}$ is a $\sigma$ ring then $\mathscr{R}$ is closed under countable intersections; i.e. if $A_n \in \mathscr{R}$ for $n \geq 1$, $\bigcap_{n=1}^{\infty} A_n \in \mathscr{R}$.

**REMARK :** The definitions do not guarantee that a ring or a $\sigma$ ring of sets is necessarily non empty. However if $\mathscr{R}$ is nonempty, $\mathscr{R}$ must contain the empty set because for any set $A \in \mathscr{R}$, $A - A = \phi$.

Since the empty ring or empty $\sigma$ ring is of no interest for us, we consider nonempty rings and nonempty $\sigma$ rings only. Thus we may assume that all our rings ( $\sigma$ rings) contain the empty set.

**21.13 PROPOSITION :**

The collection $\mathscr{E}$ of elementary sets is a ring.

**Proof :** Follows from proposition 21.6.

**21.14 SAQ :**

$\mathscr{E}_2$ is not a $\sigma$ ring.

**21.15 PROPOSITION :**

Let $\mathscr{R}$ be a ring of sets and $\{A_n\}$ be a sequence of sets in $\mathscr{R}$. Then there exists a sequence $\{B_n\}$ of sets in $\mathscr{R}$ such that

(i) $B_n \subseteq A_n \ \forall \ n \geq 1$

(ii) $B_n \cap B_m = O$ if $n \neq m$ and

(iii) $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$

**Proof :** Write $B_1 = A_1$ and for $n > 1$ $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$. Clearly $B_n \in \mathscr{R}$ and $B_n \subseteq A_n$ for every $n \geq 1$. If $n < m$, $x \in B_n \Rightarrow x \in A_n \Rightarrow x \in \bigcup_{i=1}^{m-1} A_i \Rightarrow x \notin A_m$. Thus $B_n \cap B_m = O$ if $n < m$ and hence when $n \neq m$.

Finally to prove (iii) it is enough to show that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$. If $x \in A_1$, $x \in B_1$.

If $x \in A_n$ for some $n > 1$ then we choose the smallest such $n$, so that $x \notin A_i$ for $1 \leq i < n$.

$\Rightarrow x \in B_n$. This shows that $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} B_n$.

## ADDITIVE SET FUNCTIONS :

### 21.16 DEFINITION :

If $\mathscr{R}$ is a ring and $\phi : \mathscr{R} \to \mathbb{R} \cup \{\pm \infty\}$ is a function, $\phi$ is called a set function on $\mathscr{R}$.

A set function $\phi$ defined on a ring of sets $\mathscr{R}$ is said to be additive if $\phi(A \cup B) = \phi(A) + \phi(B)$

Whenever $A$ and $B$ are disjoint sets in $\mathscr{R}$

$\phi$ is said to be countably additive if for every sequence $\{A_n\}$ of sets in $\mathscr{R}$ which are pairwise disjoint and whose union $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$,

$$\phi\left(\bigcup_{n=1}^{\infty}, A_n\right) = \sum_{n=1}^{\infty} \phi(A_n).$$

**21.17 REMARKS :**

(1) If $\phi$ is an additive set function defined on a ring $\mathscr{R}$ and $A, B$ are disjoint then $\phi(A)+\phi(B)$ is defined so that $\phi(A)$, $\phi(B)$ are, if infinite, both $+\infty$ or both $-\infty$.

(2) If $\phi$ is countably additive and $\{A_n\}$ is any pairwise disjoint sequence of sets in $\mathscr{R}$ whose union $\bigcup_{n=1}^{\infty} A_n$ is in $\mathscr{R}$, then since $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}\left(A_{\sigma_n}\right)$ for every rearrangement of $\{A_{\sigma_n}\}$ of $\{A_n\}$, $\sum_{n=1}^{\infty}\phi(A_n) = \sum_{n=1}^{\infty}\phi\left(A_{\sigma_n}\right) = \phi\left(\bigcup_{n=1}^{\infty} A_n\right)$

so that the series converges for all rearrangements which implies that the series converges absolutely or else diverges to $+\infty$ or $-\infty$.

(3) We shall assume that the range of an additive set function $\phi$ contains atmost one of $+\infty$ and $-\infty$ and is not $\{\infty\}$ or $\{-\infty\}$.

**21.18 THEOREM :**

Suppose $\phi$ is an additive set function defined on a ring $\mathscr{R}$. Then

(a) $\phi(O)=O$

(b) $\phi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n}\phi(A_i)$ for every finite collection of sets $\{A_1,\cdots\cdots,A_n\}$ in $\mathscr{R}$ which are pairwise disjoint.

(c) $A_1 \subseteq A_2$, $A_1, A_2$ belong to $\mathscr{R} \Rightarrow \phi(A_1)\le\phi(A_2)$ if $\phi(A)\ge 0\ \forall A \in \mathscr{R}$.

(d) $\phi(A_1 \cup A_2)+\phi(A_1 \cap A_2) = \phi(A_1)+\phi(A_2)\,\forall A_1$ and $A_2$ in $\mathscr{R}$.

**Proof :**

**(a)** Since $\phi$ is additive $\phi(O) = \phi(O \cup O)\phi(O)+\phi(O)$. Since the range of $\phi$ is not $\{\infty\}$ or $\{-\infty\}$, it follows that $\phi(O) = O$.

**(b)** The proof is by induction on $n$. The statement holds good when $n=1$. Assume that (b) is valid for $n-1$. For any $n$ pairwise disjoint sets $A_1,\cdots\cdots,A_n$ in $\mathscr{R}$,

$$\phi\left(\bigcup_{i=2}^{n} A_i\right) = \sum_{i=2}^{n} \phi(A_i).$$

By additivity of $\phi$

$$\phi\left(\bigcup_{i=1}^{n} (A_i)\right) = \phi\left(A_1 \cup \left(\bigcup_{i=2}^{n} A_i\right)\right)$$

$$= \phi(A_1) + \phi\left(\bigcup_{i=2}^{n} A_i\right)$$

$$= \phi(A_1) + \sum_{i=2}^{n} \phi(A_i)$$

$$= \sum_{i=1}^{n} \phi(A_i)$$

By induction, (b) holds for all $n$.

**(c):** If $A_1 \in \mathscr{R}$, $A_2 \in \mathscr{R}$ then $A_2 - A_1 \in \mathscr{R}$ and $A_1 \cap (A_2 - A_1) = 0$, so

$\phi(A_2) = \phi(A_1 \cup (A_2 - A_1)) = \phi(A_1) + \phi(A_2 - A_1)$

If $\phi(A) \geq 0$ for all $A \in \mathscr{R}$, $\phi(A_1) = \phi(A_2)$

If $\phi(A_2) = \infty$ then $\phi(A_1) \leq \phi(A_2)$

If $0 \leq \phi(A_2) < \infty$, then $\phi(A_1), \phi(A_2), \phi(A_2 - A_1)$ are finite hence $\phi(A_2) - \phi(A_1) = \phi(A_2 - A_1)$.

**(d):** If $A_1 \in \mathscr{R}$ and $A_2 \in \mathscr{R}$ then $A_2 - A_1$, $A_2 \cap A_1$ are in $\mathscr{R}$ and by the additivity of $\phi$ we have

$\phi(A_1 \cup A_2) = \phi(A_1) + \phi(A_2 - A_1)$ and

$\phi(A_2 - A_1) + \phi(A_2 \cap A_1) = \phi(A_2)$ so that

$$\phi(A_1 \cup A_2)+\phi(A_1 \cap A_2)+\phi(A_2 - A_1) = \phi(A_1)+\phi(A_2)+\phi(A_2 - A_1)$$

*If one of* $\phi(A_1)$, $\phi(A_2)$ *is* $+\infty$ *(d) holds since* $\phi(A_1 \cup A_2)=\infty$.

If $-\infty<\phi(A_1)<\infty$ and $-\infty<\phi(A_2)<\infty$, then

$\phi(A_2 - A_1)>\phi(A_2 \cap A_1)$ are both finite so $\phi(A_1 \cup A_2)<\infty$, hence cancelling $\phi(A_2 - A_1)$ we get (d).

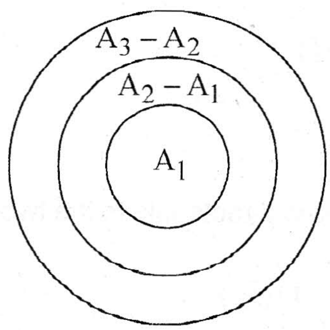
**21.19 THEOREM :**

Suppose $\phi$ is a countably additive set function defined on a ring $\mathscr{R}$ of sets and $\{A_n\}$ is a sequence in $\mathscr{R}$ such that

(i) $A_n \subseteq A_{n+1}\ \forall n$ and (ii) $A = \bigcup\limits_{n=1}^{\infty} A_n \in \mathscr{R}$. Then

$$\lim_{n} \phi(A_n) = \phi(A)$$

**Proof :** Write $B_1 = A_1$ and $B_n = A_n - A_{n-1}$ for $n>1$. Clearly $B_n \in \mathscr{R}$, $\forall\ B_n \subset A_n$, $B_n \cap B_m = O$ if $n \neq m$, $\bigcup\limits_{i=1}^{n} B_i = \bigcup\limits_{i=1}^{n} A_i = A_n \forall n$ and $m \neq n$. Also $\bigcup\limits_{n=1}^{\infty} B_n = \bigcup\limits_{n=1}^{\infty} A_n = A$.




Hence $\phi(A) = \phi\left(\bigcup\limits_{n=1}^{\infty} B_n\right)$

$$= \sum_{n=1}^{\infty} \phi(B_n)$$

$$= \lim_{n} \sum_{i=1}^{n} \phi(B_i)$$

$$= \lim_{n} \phi\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n} \phi\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= \lim_{n} \phi(A_n)$$

This completes the proof.

## LEBESGUE MEASURE :

### 21.20 DEFINITION :

If $I$ is an interval in $\mathbb{R}'$ with end points $a, b$ where $a < b$. We define measure of $I$ as its length :

$$m(I) = b - a$$

If $I = I_1 \times I_2 \times \cdots\cdots \times I_p$ is an interval in $\mathbb{R}^p$ we define

$$m(I) = \pi_{j=1}^{p} m(I_j).$$

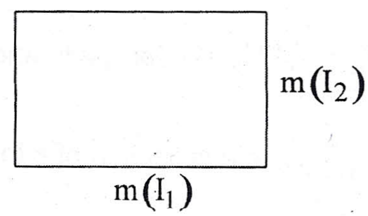
### 21.21 REMARKS :

(i) When $p=2$, $I = I_1 \times I_2$ is a rectangle in the two dimensional Euclidian plane anc

$$m(I) = m(I_1)\, m(I_2)$$

= Area of the rectagle $I$.

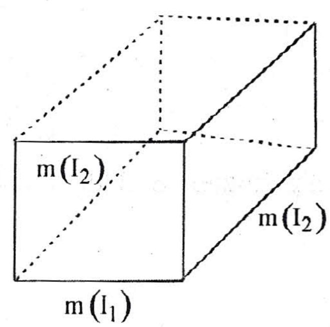
**(ii)** When $p=3$, $I = I_1 \times I_2 \times I_3$ is a rectangular parallelopiped and its measure $m(I) = m(I_1)\, m(I_2)\, m(I_3)$ is the volume of $I$.




**(iii)** If an interval $I$ in $\mathbb{R}^p$ is divided into a finite number of pairwise disjoint intervals in $\mathbb{R}^p$, say

$I = I_1 \cup I_2 \cup \cdots\cdots \cup I_n$ where $I_r \cap I_s = O$ if $r \neq s$ then

$$m(I) = \sum_{r=1}^{m} m(I_r)$$




**21.22 DEFINITION :**

If $A \in \mathscr{E}$ is the union of pairwise disjoint intervals $I_1, \cdots\cdots, I_n$ in $\mathbb{R}^p$ we define the Lebesgue measure of $A$ by

$$m(A) = \sum_{j=1}^{n} m(I_j)$$

**21.23 SAQ :**

$m(A)$ is independent of the choice of the decomposition $A = \bigcup_{j=1}^{n} I_j$.

**21.24 PROPOSITION :**

The Lebesgue measure is additive on $\mathscr{E}$.

**Proof :** Let $A \in \mathscr{E}$, $B \in \mathscr{E}$ and $A \cap B = O$. There exist collections $\{I_1, \cdots, I_n\}$ and $\{J_1, \cdots, J_m\}$

of intervals in $\mathbb{R}^p$, such that $I_j \cap I_k = J_r \cap J_s = O$ for $j \neq k$ and $r \neq s$, $A = \bigcup_{j=1}^{n} I_j$ and $B = \bigcup_{k=1}^{m} J_k$

Since $A \cap B = O$, $\{I_1, \cdots, I_n, J_1, \cdots, J_m\}$ is a collection of intervals in $\mathbb{R}^p$ every pair in which is disjoint and whose union is $A \cup B$. So

$$m(A \cup B) = m(I_1) + \cdots + m(I_n) + m(J_1) + \cdots + m(J_m)$$

$$= m(A) + m(B).$$

**21.25 SAQ :**

**(i)** If $\{A_1, \cdots, A_n\}$ is a pairwise disjoint collection in $\mathscr{E}$

$$m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i)$$

**(ii)** If $\{A_1, \cdots, A_n\}$ is any collection of sets in $\mathscr{E}$ then $m\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m(A_i)$

**21.26 DEFINITION :**

A nonnegative additive set function $\phi$ defined on $\mathscr{E}$ is said to be regular if for every $A \in \mathscr{E}$ and $\epsilon > 0$ there correspond sets $F$ and $G$ such that $F \subseteq A \subseteq G$, $F$ is closed and $G$ is open in $\mathbb{R}^p$ and

$$m(G) - \epsilon \leq m(A) \leq m(F) + \epsilon.$$

**21.27 PROPOSITION :**

The set function $m$ defined in 21.26 is regular.

**Proof :** That $m$ is additive is proved in 24.

**(i)** Let $I$ be any interval in $\mathbb{R}^p$. Then there exist intervals $I_1, \cdots, I_p$ in $\mathbb{R}'$ such that $I = I_1 \times \cdots \times I_p$. Let $a_j$, $b_j$ be the end points of $I_j$ and $a_j \leq b_j$. Write $c_j = b_j - a_j$.

Define $f$ and $g$ on $[0, \infty)$ by

$$f(x) = (c_1 + x)(c_2 + x) \cdots (c_p + x) - m(I)$$

and $\quad g(x) = m(I) - (c_1 - x) \cdots\cdots (c_p - x)$

Clearly f and g are continuous on $[0, \infty)$ and

$$f(0) = g(0) = 0 \text{ since } m(I) = c_1 c_2 \cdots\cdots c_p$$

Since f and g are continuous at 0 (from the right) it now follows that given $\in > 0$ there exists a $\delta > 0$ and a $\eta > 0$ such that $|f(x)| < \in$ if $0 \le x \le \delta$ and $|g(y)| < \in$ if $0 \le y \le \eta$.

Let $J_r = \left(a_r - \frac{\delta}{2}, b_r + \frac{\delta}{2}\right)$ and $K_r = \left[a_r + \frac{\delta}{2}, b_r - \frac{\delta}{2}\right]$,

$G = J_1 \times \cdots\cdots \times J_p$ and $F = K_1 \times \cdots\cdots K_p$.

Clearly $G$ is open and $F$ is closed in $\mathbb{R}^p$, $G, F$ are intervals and since $F_r \subseteq I_r \subseteq G_r$, $F \subseteq I \subseteq G$.

Further $m(G) = \prod_{j=1}^{p} (c_j + \delta) = f(\delta) + m(I)$

so that $m(G) - m(I) = f(\delta) < \in$

and similarly $m(F) = \prod_{j=1}^{p} (c_j - \eta) = g(\eta) + m(I)$

so that $m(I) - m(F) = + g(\eta) < \in$.

Thus, $m(G) - \in < m(I) < m(F) + \in$.

Hence m satisfies the regularity condition for intervals.

**(ii)** Now let $A$ be any element of $\mathscr{E}$, so that $A$ can be written as the disjoint union of intervals $I_1, \cdots\cdots, I_n$ in $\mathbb{R}^p$. If $\in > 0$, for $1 \le j \le n$ there exist, sets $G_j$, $F_j$ such that $G_j$ is open, $F_j$ is closed, $F_j \subset I_j \subset G_j$ and $m(G_j) - \frac{\in}{2^n} < m(I_j) < m(F_j) + \frac{\in}{2^n}$.

$G = \bigcup_{j=1}^{n} G_j$ is an open set, $F = \bigcup_{j=1}^{n} F_j$ is a closed set.

Since $F_j \subseteq I_j \subseteq G_j \; \forall j,\; F \subseteq A \subseteq G$

$$m(G) = m\left(\bigcup_{j=1}^{n} G_j\right) \le \sum_{j=1}^{n} m(G_j) < \sum_{j=1}^{n} m(I_j) + \frac{\in}{2}$$

$$\Rightarrow m(G) - \in < m(G) - \frac{\in}{2} < \sum_{j=1}^{n} m(I_j) = m(A).$$

Similarly

$$m(F) + \in > m(F) + \frac{\in}{2} = \left\{\sum_{j=1}^{n} m(F_j)\right\} + \frac{\in}{2} \qquad \text{(since } F_i \cap F_j = 0 \text{ if } i \ne j\text{)}.$$

$$= \sum_{j=1}^{n}\left(m(F_j) + \frac{\in}{2^n}\right)$$

$$> \sum_{j=1}^{n} m(I_j) = m(A).$$

This completes the proof.

**21.28 EXAMPLE :**

Let $\alpha$ be a monotonically increasing continuous function defined on $\mathbb{R}$. For any interval $I$ with end points $a, b, a \le b$ define

$$\mu(I) = \alpha(b) - \alpha(a)$$

If $A \in \mathscr{E}_1$ and $A$ is the disjoint union of intervals $I_1, \cdots, I_n$, define $\mu(A) = \sum_{r=1}^{n} \mu(I_j)$.

**(1)** $\mu$ is an additive set function on $\mathscr{E}_1$.

Let $A, B$ be disjoint elementary sets in $\mathscr{E}_1$, $\exists$ disjoint intervals $I_1, \cdots, I_n$ such that

$A=\bigcup_{j=1}^{n} I_j$ and disjoint intervals $J_1,\cdots,J_m$ such that $B=\bigcup_{k=1}^{m} J_k$. Since $A\cap B=O$, $I_j\cap J_k=O$ for $1\le j\le n, 1\le k\le m$. So $\mu(A\cup B)=\sum_{i=1}^{n}\mu(I_j)+\sum_{k=1}^{m}\mu(J_k)=\mu(A)+\mu(B)$

**(2)** $\mu$ is regular : Let $I$ be any nonempty interval with end points $a$, $b$ where $a\le b$. Then $\bar{I}=[a, b]$. Since $\alpha$ is uniformly continuous on $[a, b]$ given $\in>0$ there exists $\delta>0$ such that

$$x\in\bar{I},\ y\in\bar{I},\ |x-y|<\delta>0\Rightarrow|\alpha(x)-\alpha(y)|<\frac{\in}{2}.$$

Let $F=\left[a+\frac{\delta}{2},\ b-\frac{\delta}{2}\right]$, and $G=\left(a-\frac{\delta}{2},\ b+\frac{\delta}{2}\right)$

Then $F$ is a compact set, $G$ is an open set, $F\subseteq\bar{I}\subseteq G$ and

$$\mu(F)=\alpha\left(b-\frac{\delta}{2}\right)-\alpha\left(a+\frac{\delta}{2}\right),$$

$$\mu(G)=\alpha\left(b+\frac{\delta}{2}\right)-\alpha\left(a-\frac{\delta}{2}\right)$$

and $\mu(I)=\alpha(b)-\alpha(a)$.

Hence $\mu(G-I)=\alpha\left(b+\frac{\delta}{2}\right)-\alpha(b)-\left(\alpha\left(a-\frac{\delta}{2}\right)-\alpha(a)\right)$

$$<\left\{\alpha\left(b+\frac{\delta}{2}\right)-\alpha(b)\right\}+\left\{\alpha\left(a-\frac{\delta}{2}\right)-\alpha(a)\right\}<\in$$

Similarly $\mu(I-F)<\in$

If $A\in\mathscr{E}_1$ and $A$ is the disjoint union of intervals $I_1,\cdots,I_n$ and $\in>0$, for $1\le j\le n$. Choose open interval $G_j$ and closed interval $F_j$ such that $F_j\subseteq I_j\subseteq G_j$ and

$$\mu(G_j) \leq \mu(I_j) + \frac{\in}{2^n},\ \mu(I_j) \leq \mu(F_j) + \frac{\in}{2^n}$$

$F = \bigcup_{j=1}^{n} F_j$ is a compact set, $G = \bigcup_{j=1}^{n} G_j$ is an open set

$F \subseteq A \subseteq G$ and $\mu(G - A) \leq \mu\left(\bigcup_{j=1}^{n} G_j - I_j\right)$

$$\leq \sum_{j=1}^{n} \mu(G_j) - \mu(I_j) \leq \frac{\in}{2} < \in$$

Also $\mu(A - F) \leq \sum_{j=1}^{n} \mu(I_j - F_j) \leq \frac{\in}{2} < \in$ This completes the proof.

**21.29 SOLUTIONS TO SAQ'S :**

**SAQ 1 :** If $I$ and $J$ are intervals in $\mathbb{R}'$ with end points $a, b$ and $c, d$ respectively $a \leq b$, $c \leq d$ then the end points of $I \cap J$ are max $\{a, c\}$ and $\min\{b, d\}$.

When $I \subseteq J$, $I - J = O$

When $I \cap J = O$, $I - J = I$

When $J \subseteq I$ $I - J$, $a \leq c \leq d \leq b$ so $I - J = A \cup B$.

Where $A, B$ are intervals with end points $a, c$ and $d, b$ respectively. The remaining cases can be handled in a similar way.

**SAQ 3 :** Let $I = I_1 \times \cdots\cdots \times I_p$, $J = J_1 \times \cdots\cdots \times J_p$, where $I_r$, $J_r$ are intervals in $\mathbb{R}'$.

$$I \cap J = (I_1 \cap J_1) \times \cdots\cdots\cdots \times (I_r \cap J_r)$$

$I - J = \bigcup_{r=1}^{p} A_r$ where

$$A_r = A_{r_1} \times \cdots\cdots \times A_{r_p}$$

$A_{r_r} = I_r - J_r$ and $A_{r_s} = I_s$ for $s \neq r$ (see exercise 2)

**SAQ 8 :**

$$S(A, A) = (A-A) \cup (A-A) = O \cup O = O$$

$$S(A, B) = (A-B) \cup (B-A) = S(B, A).$$

Since $A-C \subseteq (A-B) \cup (B-C)$ it follows that

$$S(A,C) \subseteq S(A,B) \cup S(B, C).$$

**SAQ 9 :** Verify $(A_1 \cup A_2)-(B_1 \cup B_2) \subseteq (A_1-B_1) \cup (A_2-B_2)$

$$(A_1 \cap A_2)-(B_1 \cap B_2) \subseteq (A_1-B_1) \cup (A_2-B_2)$$

$$(A_1-A_2)-(B_1-B_2) \subseteq (A_1-B_1) \cup (A_2-B_2)$$

The required inclusions follow from the above inclusions.

**SAQ 11 :** $A \in \mathcal{R},\ B \in \mathcal{R} \Rightarrow A \cap B = A-(A-B) \in \mathcal{R}$.

If $A \in \mathcal{R}$, $A_1 = A_1 \cup A_1 \in \mathcal{R}$. If $A_1 \in \mathcal{R}$ and $A_2 \in \mathcal{R}$ by definition $A_1 \cup A_2 \in \mathcal{R}$. Assume that for any positive integer $n>1$ and any $(n-1)$ sets $A_1, \cdots, A_{n-1}$ in $\mathcal{R}$, $\bigcup_{i=1}^{n-1} A_i \in \mathcal{R}$. Now if $A_1, \cdots, A_n$ are any $n$ sets in $\mathcal{R}$ $\bigcup_{i=1}^{n} A_i = A_1 \cup \left( \bigcup_{i=2}^{n} A_i \right) \in \mathcal{R}$ because $\bigcup_{i=2}^{n} A_i \in \mathcal{R}$ by assumption. By mathematical induction the union of any $n$ sets in $\mathcal{R}$ is in $\mathcal{R}$.

**SAQ 12 :** $\bigcap_{n=1}^{\infty} A_n = A_1 - \bigcup_{n=1}^{\infty} A_1 - A_1$

**SAQ 13 :** $\mathscr{E}_2$ is not a $\sigma$ ring of sets : Let $A = \mathbb{N} \times \{1\}$, where $\mathbb{N}$ is the set of natural numbers. If $A_n = \{(n, 1)\}$, $A_n \in \mathscr{E}_2\ \forall n$ and $A = \bigcup_{n=1}^{\infty} A_n$. We show that $A \notin \mathscr{E}_2$, there would exist a finite number of intervals $I_1, \cdots, I_n$ in $\mathscr{E}_2$ such that $A = \bigcup_{j=1}^{n} I_j$ we may also assume that each $I_j$ is nonempty and $I_j = B_j \times C_j$ where $B_j$ and $C_j$ are intervals in $\mathbb{R}$. Since $I_j$ is nonempty $B_j$ and $C_j$

are both nonempty intervals in $\mathbb{R}$ so that either both $B_j$ and $C_j$ are singletons or at least one of $B_j$ and $C_j$ is uncountable. If both $B_j$ and $C_j$ are singletons for every $j$, $I_j$ would be a singleton set for every $j$ so that $A$ would be finite, which is false, If for some $j$ one of $B_j$ and $C_j$ is uncountable, the corresponding $I_j$ would be uncountable and hence $A$ would be uncountable and this is also false. Thus $A$ cannot be written as a finite union of intervals in $\mathbb{R}^2$ so that $A \notin \mathscr{E}_2$.

The proof for $\mathscr{E}_p \ (p \geq 3)$ is similar.

**SAQ 23 :** If $A \in \mathscr{E}$, $A = \bigcup_{j=1}^{n} I_j = \bigcup_{r=1}^{m} J_r$

where $I_1, \cdots I_n$ are pairwise disjoint intervals and $J_1, \cdots, J_m$ are disjoint intervals in $\mathbb{R}^p$, for each $j, 1 \leq j \leq n$

$$I_j = I_j \cap A = \bigcup_{r=1}^{m} \left(I_j \cap J_r\right)$$

For each $r$, $I_j \cap J_r$ is an interval in $\mathbb{R}^p$ and since the $J_r$'s are pairwise disjoint, so are $I_j \cap J_r$'s

By (iii) of remark

$$m\left(I_j\right) = \sum_{r=1}^{m} m\left(I_j \cap J_r\right)$$

$$\Rightarrow \sum_{j=1}^{n} m\left(I_j\right) = \sum_{j=1}^{n} \sum_{r=1}^{m} m\left(I_j \cap J_r\right).$$

Similarly $\sum_{r=1}^{m} m\left(J_r\right) = \sum_{r=1}^{m} \sum_{j=1}^{n} m\left(I_j \cap J_r\right).$

Since the sums on the r.h.s. are equal it follows that

$$\sum_{j=1}^{n} m\left(I_j\right) = \sum_{r=1}^{m} m\left(J_r\right).$$

**SAQ 25 :** Follows from theorem 18.

## 21.30 MODEL EXAMINATION QUESTIONS :

**21.30.1 :** *Define an elementary set in* $\mathbb{R}^p$ $(p \geq 1)$. *Show that the class* $\mathscr{E}$ *of all elementary sets in* $\mathbb{R}^p$ is a ring.

**21.30.2 :** Define the Lebesgue measure on $\mathscr{E}_p$ . Show that this measure is additive.

**21.30.3 :** Show that the Lebesgue measure defined on the elementary sets in $\mathbb{R}^p$ is regular.

**21.30.4 :** Define a regular measure. Show that $\mu(I) = \alpha(b) - \alpha(a)$ where $I$ is any interval in $\mathbb{R}'$ with end points $a$, $b(a \leq b)$ induces a measure $\mu$ on $\mathscr{E}_p$ and that $\mu$ is regular.

**21.30.5 :** Show that if $\phi$ is a non-negative additive set function defined on a ring of sets $\mathscr{R}$ and $A_1$, $A_2$ belong to $\mathscr{R}$ .

$$\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2).$$

## 21.31 EXERCISES :

**21.31.1 :** Show that $\mathscr{E}_p$ $(p \geq 2)$ is not $\sigma$ ring.

**21.31.2 :** Show that $(A_1 \times A_2 \times \cdots \times A_n) - (B_1 \times \cdots \times B_n) = \bigcup_{i=1}^{n} C_i$

where $C_i = C_{i_1} \times \cdots \times C_{i_n}$, $C_{i_i} = A_i - B_i$ and $C_{i_j} - A_j$ for $i \neq j$.

**21.31.3 :** Show that if $a \in \mathbb{R}^p$ , $m(\{a\}) = 0$. Deduce that

$m(F) = 0$ if $F$ is any finite subset of $\mathbb{R}^p$.

**21.31.4 :** Let $\delta(A)=0$ if $0 \notin A$ and 1 if $0 \in A$ where $A \subseteq \mathbb{R}'$. Show that $\delta$ defines a measure on the ring $\mathscr{E}_p$ of elementary sets in $\mathbb{R}^p$.

**21.31.5 :** Show that the empty set $\phi$ is an interval in $\mathbb{R}^p$.

**21.31.6 :** Let $\alpha$ be monotonically increasing on $\mathbb{R}'$. Define

$$\mu([a, b]) = \alpha(b+)-\alpha(a-)$$

$$\mu([a, b]) = \alpha(b-)-\alpha(a-)$$

$$\mu([a, b]) = \alpha(b-)-\alpha(a+)$$

and $$\mu((a, b)) = \alpha(b-)-\alpha(a+)$$

Show that $\mu$ indcues a measure in a natural way on $\mathscr{E}_1$ which is regular (see example 28).

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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# Lesson - 22

# OUTER MEASURE AND MEASURABLE SETS

## 22.0 INTRODUCTION

This lesson is devoted to study some properties of set functions. Starting wtih an nonnegative additive finite regular set function $\mu$ on the ring $\mathscr{E}$ of elementary sets in $\mathbb{R}^p$ we define the outer measure $\mu^*$ for every subset $A$ of $\mathbb{R}^p$. We study some properties of this outer measure $\mu^*$ that are natural consequences of some properties of $\mu$. We then identify some subsets of $\mathbb{R}^p$ with special properties and call them measurable sets. We show that the class of measurable sets is a $\sigma$ algebra, containing all open sets, hence closed sets and consequently "Borel sets" which constitute the smallest $\sigma$ algebra containing all open sets. We also show that the restriction of $\mu^*$ to the $\sigma$ algebra $m(\mu)$ of all measurable sets is a regular measure.

Let $\mu$ be a finite nonnegative finitely additive and regular set function defined on the ring $\mathscr{E}$. By a countable open cover of a set $A \subseteq \mathbb{R}^p$ we mean a countable collection of sets $\{E_n\}$ in $\mathscr{E}$ such that $E_n$ is open for every $n \geq 1$ and $A \subseteq \bigcup_{n=1}^{\infty} E_n$.

## 22.1 DEFINITION :

The outer measure induced by $\mu$ is defined by

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu(E_n) \Big/ \{E_n\}, \text{ a countable open cover of A from } \mathscr{E} \right\}$$

## 22.2 PROPOSITION :

(i) $\mu^*(A) \geq 0$ if $A \subseteq \mathbb{R}^p$ (non negativity)

(ii) If $A \subseteq B$, $\mu^*(A) \leq \mu^*(B)$ (monotonicity)

(iii) If $A \in \mathscr{E}$, $\mu^*(A) = \mu(A)$ (Extension)

(iii) If $\{A_n\}$ is any countable collection of sets in $\mathbb{R}^p$

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n) \qquad \text{(countable subadditivity)}$$

**Proof : (i)** $\forall E \in \mathscr{E}$, $\mu(E) \ge 0$ so $\mu^*(A) \ge 0$.

**(ii)** If $\{E_n\}$ is any countable open cover for B from $\mathscr{E}$ and $A \subset B$ then so is it for $A$ so that.

$\mu^*(A) \le \sum_{n=1}^{\infty} \mu(E_n)$. This is true for every such $\{E_n\}$ hence $\mu^*(A) \le \mu^*(B)$.

**(iii)** Suppose $A \in \mathscr{E}$. If $\in > 0$, by the regularity of $\mu$, there is an open set $G$ and a closed set $F \ni F \subseteq A \subseteq G$ and $F \in \mathscr{E}$, $G \in \mathscr{E}$ $\mu(G) \le \mu(A) + \in$. Since $A \subseteq G \in \mathscr{E}$, $\mu^*(A) \le \mu(G) \le \mu(A) + \in$. Since this is true for every $\in > 0$ it follows that $\mu^*(A) \le \mu(A)$, we now show that $\mu(A) \le \mu^*(A)$. ----------- (a)

If $\in > 0$ there is a countable open cover $\{E_n\}$ of $A$ from $\mathscr{E}$, such that $\mu^*(A) + \frac{\in}{2} > \sum_{n=1}^{\infty} \mu(E_n)$.

By the regularity of $A$, there is a closed set $F$ in $\mathscr{E} \ni F \subset A$ and $\mu(F) + \frac{\in}{2} > \mu(A)$. Since $F \subset A$, $F$ is bounded, hence compact. Since $F \subset A \subset \bigcup_{n=1}^{\infty} E_n$, there is a finite subcover of $F$ say $\{E_1, \cdots\cdots, E_k\}$.

Since $F \subseteq \bigcup_{n=1}^{k} E_k$, by countable subadditivity.

$$\mu(F) \le \sum_{n=1}^{k} \mu(E_n) \le \sum_{n=1}^{\infty} \mu(E_k) \le \mu^*(A) + \frac{\in}{2}.$$

Hence $\mu(A)<\mu(F)+\frac{\in}{2}\leq\mu^*(A)+\in$

This is true for every $\in>0$ so $\mu(A)\leq\mu^*(A)$ ----------- (b)

Hence $\mu(A)=\mu^*(A)$

**(iv)** Let $\{A_n\}$ be any countable collection of sets in $\mathbb{R}^p$ and $A=\bigcup_{n=1}^{\infty} A_n$. If $\mu^*(A_n)=\infty$ for some $n$ then $\sum_{n=1}^{\infty}\mu^*(A_n)=\infty\geq\mu^*(A)$.

Assume that $\mu^*(A_n)<\infty$ for every $n$. If $\in>0$ for every $n\geq 1$, there is a countable open cover of $E_n$ from $\mathscr{E}$, $\{E_{n,k}/k\geq 1\}$, such that $\sum_{k=1}^{\infty}\mu^*(E_{n,k})<\mu^*(A_n)+\frac{\in}{2^n}$

Since $A=\bigcup_{n=1}^{\infty} A_n\subseteq\bigcup_{n=1}^{\infty}\left\{\bigcup_{k=1}^{\infty} E_{n,k}\right\}$

the collection $\{E_{n,k}/n\geq 1,\ k\geq 1\}$ is a countable open cover of $A$ from $\mathscr{E}$ so

$$\mu^*(A)\leq\sum_{n,k}\mu^*(E_{n,k})$$

$$=\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty}\mu^*(E_{n,k})\right\}\leq\sum_{n=1}^{\infty}\left\{\mu^*(A_n)+\frac{\in}{2^n}\right\}$$

$$=\sum_{n=1}^{\infty}\mu^*(A_n)+\sum_{n=1}^{\infty}\frac{\in}{2^n}$$

$$=\sum_{n=1}^{\infty}\mu^*(A_n)+\in$$

Since this is true for every $\in>0$ it follows that

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Recall that in lesson 21 we defined the symmetric difference of S by $S(A, B) = (A - B) \cup (B - A)$ and proved that for any subsets $A, B, C$ of $\mathbb{R}^p$

$$S(A, A) = 0$$

$$S(A, B) = S(B, A) \text{ and}$$

$$S(A, C) \subseteq S(A, B) \cup S(B, C).$$

We have also established that for any sets $A_1, A_2, B_1, B_2$ in $\mathbb{R}^p$, each of the sets $S(A_1 \cup A_2, B_1 \cup B_2)$, $S(A_1 \cap A_2, B_1 \cap B_2)$ and $S(A_1 - A_2, B_1 - B_2)$ is a subset of $S(A_1, B_1) \cup S(A_2, B_2)$.

We now define the distance between $A, B$ with respect to $\mu^*$ by $d(A, B) = \mu^*(S(A, B))$

**22.3 SAQ :**

If $A, B, C$ are subsets of $\mathbb{R}^p$ then

(i) $d(A, A) = 0$

(ii) $d(A, B) = d(B. A)$ and

(iii) $d(A, C) \leq d(A, B) + d(B, C)$

**22.4 SAQ :**

If $A_1, A_2, B_1$ and $B_2$ are subsets of $\mathbb{R}^p$ then each of $d(A_1 \cup A_2, B_1 \cup B_2)$, $d(A_1 \cap B_1, A_2 \cap B_2)$ and $d(A_1 - A_2, B_1 - B_2)$ is less than or equal to $d(A_1 \cup B_1, A_2 \cup B_2)$.

**22.5 PROPOSITION :**

If $A \subseteq \mathbb{R}^p$ and $B \subseteq \mathbb{R}^p$ and at least one of $\mu^*(A)$ and $\mu^*(B)$ is finite then

$$\left|\mu^*(A) - \mu^*(B)\right| \le d(A, B)$$

**Proof :** Assume $0 \le \mu^*(B) \le \mu^*(A)$

Then $\mu^*(A) = \mu^*(S(A, O))$

$$\le \mu^*(S(A,B)) + \mu^*(S(B,O)) \qquad \text{(SAQ 22.3)}$$

$$= d(A, B) + \mu^*(B)$$

$$\Rightarrow \mu^*(A) - \mu^*(B) \le d(A,B)$$

$$\Rightarrow \left|\mu^*(A) - \mu^*(B)\right| \le d(A,B).$$

**22.6 PROPOSITION :**

The class $\mathfrak{M}_F(\mu)$ is a ring.

**Proof :** Let $A \in \mathfrak{M}_F(\mu)$ and $B \in \mathfrak{M}_F(\mu)$

Then $\exists \{A_n\}$ and $\{B_n\}$ in $\mathscr{E}$ such that

$$\lim_n d(A_n, A) = \lim_n d(B_n, B) = 0.$$

Since $\mathscr{E}$ is a ring $A_n \cup B_n \in \mathscr{E}\ \forall n$.

By 22.4, $0 \le d(A_n \cup B_n, A \cup B)$

$\le d(A_n, A) + d(B_n, B)$. Hence $\lim_n d((A_n \cup B_n), (A \cup B)) = 0$.

Hence $A \cup B \in \mathfrak{M}_F(\mu)$ ------------------- (1)

$$d(A_n - B_n, A - B) \le d(A_n, A) + d(B_n, B)$$

$$\Rightarrow \lim_n d(A_n - B_n, A - B) = 0$$

$\Rightarrow A-B \in \mathfrak{M}_F(\mu)$ -------------------- (2)

Hence from (1) and (2) $\Rightarrow \mathfrak{M}_F(\mu)$ is a ring.

**22.7 PROPOSITION :**

$\mu^*$ is additive on $\mathfrak{M}_F(\mu)$.

**Proof :** Let $A \in \mathfrak{M}_F(\mu)$, $B \in \mathfrak{M}_F(\mu)$ and $A \cap B = 0$

$\exists$ sequences $\{A_n\}$, $\{B_n\}$ in $\mathscr{E}$ such that

$$\lim_n d(A_n, A) = \lim_n d(B_n, B) = 0$$

Since $\left|\mu^*(A) - \mu^*(B)\right| \le d(A, B)$ for any $A, B$ in $\mathbb{R}^p$ it follows that

$$\lim_n \mu^*(A_n) = \mu^*(A) \text{ and } \lim_n \mu^*(B_n) = \mu^*(B).$$

Also since $\lim_n d(A_n \cup B_n, A \cup B) = \lim_n d(A_n \cap B_n, A \cap B) = 0$

We get as above

$$\lim_n \mu^*(A_n \cup B_n) = \mu^*(A \cup B) \text{ and } \lim_n \mu^*(A_n \cap B_n) = \mu^*(A \cap B)$$

Since $\mathfrak{M}_F(\mu)$ is a ring $\{A_n \cup B_n\}$ and $\{A_n \cap B_n\}$ are sequences in $\mathscr{E}$.

Since $\mu^*(A_n \cup B_n) + \mu^*(A_n \cap B_n) = \mu^*(A_n) + \mu^*(B_n)$

and $\mu = \mu^*$ on $\mathscr{E}$ we get by taking limits as $n$ tends to $\infty$,

$$\mu^*(A \cup B) + \mu^*(A \cap B) = \mu^*(A) + \mu^*(B).$$

Since $A \cap B = O$ wet $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

**22.8 PROPOSITION :**

$\mathfrak{M}(\mu)$ is a $\sigma$ ring and $\mu^*$ is countably additive on $m$

**Proof :** Let $A \in \mathfrak{M}(\mu)$. Then $A$ can be written as a countable union of disjoint sets in $\mathfrak{M}_F(\mu)$ say $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathfrak{M}_F(\mu)$ for every $n$ and $A_n \cap A_m = O$ when $n \neq m$. Since $\mu^*$ is countably subadditive,

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \text{ ------------ (a)}$$

Since $\bigcup_{i=1}^{n} A_i \subseteq A$ for every $n \geq 1$, and $\{A_i\}$ are disjoint,

$$\mu^*(A) \geq \mu^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu^*(A_i)$$

This is true $\forall n$, so $\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(A_n)$ --------- (b)

From (a) and (b), $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$ -------------- (1)

Now suppose that $\mu^*(A) < \infty$ and $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \cap A_m = O$ if $n \neq m$ and each $A_n \in \mathfrak{M}_F(\mu)$.

Put $B_n = \bigcup_{i=1}^{n} A_i$. Then $S(A, B_n) = A - B_n = \bigcup_{i=n+1}^{\infty} A_i$

Therefore $d(A, B_n) = \mu^*(S(A, B_n))$

$$= \mu^*\left(\bigcup_{i=n+1}^{\infty} A_i\right)$$

$$= \sum_{i=n+1}^{\infty} \mu^*(A_i)$$

Since $\mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A_i)$ converges it follows that

$\lim_n d(A, B_n) = 0$. Hence $A \in \mathfrak{M}_F(\mu)$ ------------- (2)

countable additivity of $\mu^*$ on $\mathfrak{M}(\mu)$ :

Let $A_n \in \mathfrak{M}(\mu)\ \forall n$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathfrak{M}(\mu)$ and $A_n \cap A_m = O$ if $n \neq m$.

Then by (1) $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$.

If $\mu^*(A_n) < \infty$ for some $n$ then l.h.s = r.h.s. = $\infty$.

If $\mu^*(A_n) = \infty$ for every $n$, then by (2) $A_n \in \mathfrak{M}_F(\mu)\ \forall n$ hence by

(i) $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$. Hence $\mu^*$ is countably additive on $\mathfrak{M}(\mu)$ --------------- (3)

Finally to prove that $\mathfrak{M}(\mu)$ is a $\sigma$ ring, let $\{A_n\}$ be any countable collection of sets in $\mathfrak{M}(\mu)$. Then for every $n$, $A_n = \bigcup_{k=1}^{\infty} A_{n,k}$. Where $A_{n,k} \in \mathfrak{M}_F(\mu)\ \forall\ k \geq 1$. Hence $A = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{n,k}$.

The collection $\{A_{n,k} / n \geq 1, k \geq 1\}$ is a countable family of sets in $\mathfrak{M}_F(\mu)$, hence

$A \in m(\mu)$ ------------ (4)

We now show that $A \in \mathfrak{M}(\mu)$ and $B \in \mathfrak{M}(\mu)$ and $\Rightarrow A - B \in \mathfrak{M}(\mu)$.

Let $A = \bigcup_{n=1}^{\infty} A_n$ & $B = \bigcup_{n=1}^{\infty} B_n$. Where $A_n \in \mathfrak{M}_F(\mu) \cup B_n \in \mathfrak{M}_F(\mu)\ \forall n \geq 1$. Then $A_n \cap B_m \in \mathfrak{M}_F(\mu)\ \forall m\ \&\ n$ so that

$$A_n \cap B = \bigcup_{m=1}^{\infty} (A_n \cap B_m) \in \mathfrak{M}(\mu) \forall n$$

Since $\mu^*(A_n \cap B) \leq \mu^*(A_n) < \infty$, $A_n \cap B \in \mathfrak{M}_F(\mu)$.

Since $A_n$ and $A_n \cap B$ belong to $\mathfrak{M}_F(\mu)$, $A_n - B = A_n - A_n \cap B \in \mathfrak{M}_F(\mu)$.

Thus $A - B = \bigcup_{n=1}^{\infty} (A_n - B) \in \mathfrak{M}(\mu)$ ------------ (5)

From (4) and (5) it follows that $m(\mu)$ is a $\sigma$ ring.

**22.9 PROPOSITION :**

$\mu$ is regular.

**Proof :** Let $A \in \mathfrak{M}(\mu)$ and $\in > 0$. By the definition of $\mu^*(A)$, there is a countable collection $\{A_n\}$ of elementary open sets such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\mu^*(A) + \in > \sum_{n=1}^{\infty} \mu^*(A_n)$. If $G = \bigcup_{n=1}^{\infty} A_n$, $G$ is open, $A \subseteq G$ and $\mu^*(G) = \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n) < \mu^*(A) + \in$.

Since $A \in \mathfrak{M}(\mu)$, $A^c \in \mathfrak{M}(\mu)$. So $\exists$ an open set $G \supseteq A^c$ such that $\mu^*(G) < \mu^*(A^c) + \in$

Since $G \supset A^c$, $F = G^c \subseteq A$ and $F$ is a closed set.

Since $\mu^*(F^c) < \mu^*(A^c) + \in$, it follows that

$$\mu^*(A - F) = \mu^*(A \cap F^c) = \mu^*(F^c - A^c) < \in.$$

**22.10 SAQ :** $\mathfrak{M}(\mu)$ contains all open sets

**22.11 SAQ :** The intersection of any family of $\sigma$ rings of subsets of a set $X$ is a $\sigma$ ring.

## THE BOREL FIELD :

The smallest $\sigma$ ring $\mathbb{B}$ containing the class $G$ of all open sets in $\mathbb{R}^p$ is called the Borel field in $\mathbb{R}^p$. Elements of $\mathbb{B}$ are called Borel sets. Every interval $I = I_1 \times \cdots \times I_p$ where $I_j = (a_j, b_j)$ for every $j$ is an open set. Hence $\mathbb{B}$ contains all intervals of this type.

Further $\mathbb{B}$ contains all singleton sets. Hence $\mathbb{B}$ contains all intervals hence every elementary set is a Borel set. Further every Borel set is $\mu$ measurable for every $\mu$ because $m(\mu)$ is a $\sigma$ algebra containing every open set in $\mathbb{R}^p$. Since $\mathbb{R}^p$ is an open set, $\mathbb{R}^p \in \mathbb{B}$. Since $\mathbb{B}$ is a $\sigma$ algebra it now follows that every closed set in $\mathbb{R}^p \in \mathbb{B}$. Since $\mathbb{B}$ is a $\sigma$ algebra it now follows that every closed set is in $\mathbb{B}$.

**22.12 PROPOSITION :**

If $A \in \mathfrak{M}(\mu)$ then there exist Borel sets $F$ and $G$ in $\mathbb{R}^p$ such that $F \subseteq A \subseteq G$ and $\mu(G - A) = \mu(A - F) = 0$.

**Proof :** Since $\mu$ is regular, $\forall$ positive integer $n$, there exists $F_n, G_n$ such that $F_n$ is a closed set, $G_n$ is an open set $F_n \subseteq A \subseteq G_n$ and $\mu(G_n - A) < \frac{1}{n}$ and $\mu(A - F_n) < \frac{1}{n}$. Clearly $F = \bigcup_{n=1}^{\infty} F_n \in \mathbb{B}$ and $G = \bigcap_{n=1}^{\infty} G_n \in \mathbb{B}$ and $A - F = A - \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} A - F_n$ so that $0 \le \mu(A - F) \le \mu(A - F_n) < \frac{1}{n} \forall n \ge 1$. Hence $\mu(A - F) = 0$. Similarly $G - A = \bigcap_{n=1}^{\infty} (G_n - A) \le G_n - A \forall n$ so that $\mu(G - A) \le \mu(G_n - A) < \frac{1}{n}$. As above we get $\mu(G - A) = 0$.

**22.13 SAQ :** If $A \in m(\mu)$, $A$ is the union of a Borel set and a set of $\mu$ measure 0.

**22.14 SAQ :** The Cantor set $\rho$ has Lebesgue measure zero. (This is an example of an uncountable set whose measure is zero).

### 22.15 MODEL EXAMINATION QUESTIONS :

**22.15.1 :** Show that the outer measure $\mu^*$ induced by a non-negative additive finite set function $\mu$ on $E$ is countably subadditive.

**22.15.2 :** *Describe the class $\mathfrak{M}_F(\mu)$ when $\mu$ is a finite nonnegative additive set function on $\mathscr{E}$ and show that $\mathfrak{M}_F(\mu)$ is a ring.*

**22.15.3 :** Show that the outer measure $\mu^*$ corresponding to a finite nonnegative additive set function is additive on $\mathfrak{M}_F(\mu)$.

**22.15.4 :** Describe the class $\mathfrak{M}(\mu)$ and show that $\mathfrak{M}(\mu)$ is a $\sigma$ ring.

**22.16 SOLUTIONS TO SAQ'S :**

**22.3 :** (i) $S(A, A) = 0 \Rightarrow d(A, A) = \mu^*(S(A,A)) = \mu^*(O) = 0$

(ii) $S(A,B) = S(B,A) \Rightarrow d(A,B) = \mu^*(S(A, B)) = \mu^*(S(B,A)) = d(B,A)$ and

(iii) $d(A, C) = \mu^*(S(A,C))$

$$\leq \mu^*(S(A, B) \cup S(B,C))$$

$$\leq \mu^*(S(A, B)) + \mu^*(S(B,C))$$

$$= d(A, B) + d(B,C)$$

**22.4 :** $d(A_1 \cup A_2, B_1 \cup B_2)$

$$= \mu^*(S(A_1 \cup A_2, B_1 \cup B_2)) \leq \mu^*(S(A_1, B_1) \cup S(A_2, B_2))$$

$$\leq \mu^*(S(A_1, B_1)) + \mu^*(S(A_2, B_2))$$

$$= d(A_1, B_1) + d(A_2, B_2)$$

$$d(A_1 \cap A_2, B_1 \cap B_2) = \mu^*(S(A_1 \cap A_2, B_1 \cap B_2))$$

$$\leq \mu^*(S(A_1, B_1) \cup S(A_2 \cup B_2))$$

$$\leq \mu^*(S(A_1, B_1)) + \mu^*(S(A_2, B_2))$$

$$= d(A_1, B_1) + d(A_2, B_2)$$

$$d(A_1 - A_2, B_1 - B_2) = \mu^*(S(A_1 - A_2, B_1 - B_2))$$

$$\leq \mu^*\left(S(A_1,B_1)\cup S(A_2,B_2)\right)$$

$$\leq \mu^*\left(S(A_1,B_1)\right)+\mu^*\left(S(A_2,B_2)\right)$$

$$= d(A_1,B_1)+d(A_2+B_2).$$

**22.10 :** It is clear that $\mathfrak{M}(\mu)$ contains all intervals $I=I_1\times \cdots\times I_p$ and in particular all intervals $I$ where each $I_j$ is of the form $\left(a_j,\ b_j\right)$, $a_j\leq b_j$. It is a fact in topology that every open set in $\mathbb{R}^p$ is a countable union of intervals of the above type, i.e. of the type $I_1\times\cdots\cdots\times I_p$ where each $I_j=\left(a_j,\ b_j\right)$. Hence $m(\mu)$ contains all open sets.

**22.11 :** Let $\{A_\alpha/\alpha\in\Delta\}$ is any family of $\sigma$ rings of subsets of a set $X$ and

$$A=\bigcap_{\alpha\in\Delta} A_\alpha,$$

$A\in\mathscr{A}$, $B\in\mathscr{A}\Rightarrow A\in\mathscr{A}_\alpha,\ B\in\mathscr{A}_\alpha$ for every $\alpha\in\Delta\Rightarrow$

$A\cup B\in\mathscr{A}_\alpha$ and $A-B\in\mathscr{A}_\alpha\ \forall\ \alpha\in\Delta$

$\Rightarrow A\cup B\in\mathscr{A}$ and $A-B\in\mathscr{A}$.

For any countable collection $\{A_n\}$ in $\mathscr{A}$, $A_n\in\mathscr{A}_\alpha\ \forall\ n\geq 1$ and $\alpha\in\Delta$, so that $\bigcap_{n\geq 1} A_n\in\mathscr{A}_\alpha\ \forall\ \alpha\in\Delta$ and hence $\bigcap_{n\geq 1} A_n\in\mathscr{A}$.

Hence $\mathscr{A}$ is a $\sigma$ ring.

**22.13 :** Let $A\in\mathfrak{M}(\mu)$ and $F$ a Borel set such that $F\subseteq A$ and $\mu(A-F)=0$, we have $A=F\cup(A-F)$.

**22.14 :** We know that the Cantor set $P=\bigcap_{n=1}^{\infty} E_n$ where each $E_n$ is a disjoint union of $2^n$ closed intervals each of length $\frac{1}{3^n}$. Thus $m(E_n)=\left(\frac{2}{3}\right)^n$.

Hence $0\leq m(P)\leq m(E_n)<\left(\frac{2}{3}\right)^n$. Since $\lim_n\left(\frac{2}{3}\right)^n=0$ it follows that $m(P)=0$.

## 22.17 EXERCISES :

**22.17.1 :** Show that every open set in $\mathbb{R}'$ is a countable union of pairwise disjoint open intervals.

**22.17.2 :** Extend 1 to $\mathbb{R}^p$ where $p \geq 1$.

**22.17.3 :** Given a set $X$ and a collection $\mathscr{S}$ of subsets of $X$, Show that there is a "smallest" $\sigma$ algebra $\sigma(\mathscr{S})$ containing $\mathscr{S}$ in the sense that

(i) $\sigma(\mathscr{S})$ is a $\sigma$ algebra containing $\mathscr{S}$ and

(ii) $\sigma(\mathscr{S}) \supseteq \mathscr{A}$ for every $\sigma$ algebra $\mathscr{A}$ containing $\mathscr{S}$.

**Hint :** $\sigma(\mathscr{S}) = \cap\{\mathscr{A}_\alpha / \alpha \in \Delta\}$ where $\{\mathscr{A}_\alpha / \alpha \in \Delta\}$ is the collection of all $\sigma$ algebras containing $\mathscr{S}$.

**22.17.4 :** Show that $\mathbb{B}$ contains all singleton sets. Deduce that every interval in $\mathbb{R}^p$ is a Borel set. consequently show that every elementary set is a Borel set.

**22.17.5 :** Show that $\mathbb{B}$ is the smallest $\sigma$ algebra containing all closed sets in $\mathbb{R}^p$.

**22.17.6 :** Show that for every $\mu$, the collection $z=\{A/\mu(A)=0\}$ is a $\sigma$ ring.

**22.17.7 :** Show that every countable set in $\mathbb{R}^p$ has measure zero.

**22.17.8 :** Let $\delta$ be the measure defined on $\mathscr{E} \subseteq \mathbb{R}^p$ by

$$\delta(A) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}.$$

Find $\delta^*$ $m_{F(\delta)}$ and $m(\delta)$.

## REFERENCE BOOK :

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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## Lesson - 23

# INTEGRAL OF A MEASURABLE FUNCTION

### 23.0 INTRODUCTION :

In this lesson we introduce the notion of a measurable function on a measurable space and develp the theory of integral of a measurable function. We also study some elementary properties of the integral and integrable functions.

### 23.2 DEFINITION :

Let $X$ bea set, $\mathscr{M}$ a $\sigma$ ring of subsets of $X$ and $\mu$ a countably additive, non negative set function defined on $\mathscr{M}$ . Then $(X, \mathscr{M}, \mu)$ is called a measure space. If $(X, \mathscr{M}, \mu)$ is a measure space and $X \in \mathscr{M}$, $(X, \mathscr{M}, \mu)$ is called a measurable space. A set $E \subset X$ is said to be measurable if $E \in \mathscr{M}$ and $\mu(E)$ is called the measure of $E$ with respect to $\mu$.

**Examples :**

(1) If m is the Lebesgue measure on $\mathbb{R}'$ and $\mathfrak{M}$ is the class of all Lebesgue measurable functions then $(R, \mathfrak{M}, m)$ is a measurable space.

(2) If $X$ is an uncountable set, $\mathscr{C}$ the collection of all atmost countable sets (i.e., finite or coutnable sets in $X$ and $\mathscr{C}$ is a $\sigma$ ring of subsets of $X$. Define for $E \subset X$,

$\mu(E) =$ number of elements in $E$ if $E$ is finite,

$= \infty$ if $E$ is infinite.

Then $(X, \mathscr{C}, \mu)$ is a measure space but not a measurable space.

### 23.3 PROPOSITION :

Let $(X, \mathscr{M}, \mu)$ be a measurable space and $f$ be an extended real valued function defined on $X$. The following are equivalent.

(1) for every $a \in \mathbb{R}'$, $\{x / f(x) > a\}$ is measurable.

(2) for every $a \in \mathbb{R}'$ $\{x / f(x) \geq a\}$ is measurable.

(3) for every $a \in \mathbb{R}'$ $\{x/f(x) < a\}$ is measurable.

(4) for every $a \in \mathbb{R}'$ $\{x/f(x) \leq a\}$ is measurable.

**Proof :** $(1) \Rightarrow (2)$ : Let a be any real number. For any $x \in X,\ f(x) \geq a$ if and only if $f(x) > a - \frac{1}{n}$ for every positive integer $n$ so that

$$\{x/f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x/f(x) > a - \frac{1}{n}\right\}$$

Since $\left\{x/f(x) > a - \frac{1}{n}\right\}$ is measurable $\forall$ positive integer $n$ and $\mathscr{M}$ is a $\sigma$- ring $\bigcap_{n=1}^{\infty} \left\{x/f(x) > a - \frac{1}{n}\right\}$ is measurable. Hence $\{x/f(x) \geq a\}$ is measurable. Thus $(1) \Rightarrow (2)$.

$(2) \Rightarrow (3)$ : Let a be any real number. Clearly

$$\{x/f(x) < a\} = X - \{x/f(x) \geq a\}$$

Since $\{x/f(x) \geq a\}$ is measurable and $X$ is measurable. Thus r.h.s. is measurable. That is l.h.s is measurable. Thus $(2) \Rightarrow (3)$.

$(3) \Rightarrow (4)$ : for any $a \in \mathbb{R}'$,

$$\{x/f(x) \leq a\} = \bigcap_{n=1}^{\infty} \left\{x/f(x) < a + \frac{1}{n}\right\}$$

for each $n \geq 1$, $\{x/f(x) < a + 1/n\}$ is measurable. Thus $(3) \Rightarrow (4)$.

$(4) \Rightarrow (1)$ : for any real number a,

$$\{x/f(x) > a\} = X - \{x/f(x) \leq a\}$$

Since $\{x/f(x) \leq a\}$ and $X$ are measurable and $(X, \mathscr{M}, \mu)$ is a measurable space, $X \in \mathscr{M}$ and so $X - \{x/f(x) \leq a\}$ is measurable.

Thus $(4) \Rightarrow (1)$. This completes the proof.

**23.4 DEFINITION :**

An extended real valued function $f$ defined on a set is said to be measurable with respect to the measurable space $(X, \mathcal{M}, \mu)$ (or simply measurable with respect to $\mu$) if for every $a \in \mathbb{R}'$ $\{x/f(x) > a\}$ is measurable. A measurable function $f$ on $X$ with respect to $(X, \mathcal{M}, \mu)$ is also called simply measurable.

**Note :** When a measurable space $(X, \mathcal{M}, \mu)$ is fixed we use the words measurable set and measurable function without specifying the measurable space.

In the sequel $(X, \mathcal{M}, \mu)$ is a measurable space.

**23.5 SAQ :**

If $f$ is measurable, so are $|f|$ and $-f$.

**Proposition :** If $\{f_n\}$ is a sequence of measurable functions $X$ the functions $f$ and F defined by

$$f(x) = \inf_{n \geq 1} f_n(x) \text{ and } F(x) = \sup_{n \geq 1} f_n(x) \text{ are measurable.}$$

**Proof :** For any $a \in \mathbb{R}'$, $f(x) < a \Leftrightarrow f_n(x) < a$ for some $n \geq 1$.

Hence $\{x/f(x) < a\} = \bigcup_{n=1}^{\infty} \{x/f_n(x) > a\}$

Since $f_n$ is measurable $\forall\ n \geq 1$, $\{x/f_n(x) > a\}$ is measurable, hence the set on the r.h.s. is measurable.

So $\{x/f(x) < a\}$ is measurable. This being true for every real number a, it follows that $f$ is measurable.

For any real number a,

$F(x) > a$ if and only if $f_n(x) > a$ for some $n$. Hence $\{x/F(x) > a\} = \bigcup_{n=1}^{\infty} \{x/f_n(x) > a\}$ since $\{x/f_n(x) > a\}$ is measurable for every $n$, the set on the r.h.s., hence the set on the l.h.s. is measurable. Hence F is measurable.

**23.7.1 COROLLARY :**

*If $\{f_n\}$ is a sequence of measurable functions defined on a mesurable space $(X, \mathcal{M}, \mu)$,* $f = \lim\inf f_n$ and $F = \lim\sup f_n$ then $f$ and $F$ are measurable.

**Proof :** By definition $f(x) = \inf\limits_{n\geq 1} \sup\limits_{k\geq n} f_k(x)$. For each $n$ and $x \in X$. Let $g_n(x) = \sup\limits_{k\geq n} f_k(x)$. Then by 23.6 $g_n$ is measurable. Hence $f = \inf g_n$ is measurable by 23.6 again. By symmetry F is measurable.

**23.7.2 COROLLARY :** If $\{f_n\}$ is a sequence of measurable functions on a measurable space $(X, \mathcal{M}, \mu)$ and $f(x) = \lim\limits_{n} f_n(x)$ for $x \in X$ then $f$ is measurable.

**Proof :** Follows from corollary 23.7.1 and the fact that

$$f(x) = \inf_{n\geq 1} \sup_{k\geq n} f_n(x) = \sup_{n\geq 1} \inf_{k\geq n} f_k(x)$$

**COROLLARY :** If $f$ is measurable so are $f^+$ and $f^-$.

**Proof :** Recall that $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\}$. Now it is clear from 23.6 that $f^+$ and $f^-$ are measurable.

**23.8 SAQ :**

Prove directly that $f$ is measurable so are $f^+$ and $f^-$.

**23.9 THEOREM :**

If $f$ and $g$ are real valued measurable functions on a measurable space $(X, \mathcal{M}, \mu)$ and $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous then the function $h$ defined on $X$ by $h(x) = F(f(x), g(x))$ is measurable.

**Proof :** We use the fact that every open set in $\mathbb{R}^2$ is a countable union of "open" intervals. i.e. intervals of the type $I \times J$ where I, J are open intervals in $\mathbb{R}'$.

For every $a \in \mathbb{R}$, the set $G = \{(x_1, x_2)/F(x_1, x_2) > a\}$ is an open set in $\mathbb{R}' \times \mathbb{R}'$ since F is continuous. Since every open set is a countable union of "open" intervals in $\mathbb{R}^2$, $\exists$ sequence of open intervals in $R'\{I_n\}$ and $\{J_n\}$ such that $G = \bigcup\limits_{n=1}^{\infty} I_n \times J_n$.

Now $h(x) > a \Leftrightarrow F(f(x), g(x)) > a$

$$\Leftrightarrow (f(x), g(x)) \in G$$

$$\Leftrightarrow (f(x), g(x)) \in I_n \times J_n \text{ for some } n.$$

$$\Leftrightarrow x \in f^{-1}(I_n) \cap g^{-1}(J_n) \text{ for some } n.$$

$$\Leftrightarrow x \in \bigcup_{n=1}^{\infty} f^{-1}(I_n) \cap g^{-1}(I_n)$$

Thus $\{x/h(x) > a\} = \bigcup_{n=1}^{\infty} f^{-1}(I_n) \cap g^{-1}(J_n)$

Since $f$ and $g$ are measurable, by exercise $f^{-1}(I_n)$ and $g^{-1}(J_n)$ are measurable for every $n$, so that $\bigcup_{n=1}^{\infty} f^{-1}(I_n) \cap {}^{-1}(I_n)$ is measurable. Hence $\{x/h(x) > a\}$ is measurable $\forall a \in \mathbb{R}'$. Hence $h$ is measurable. This completes the proof of the theorem.

**23.10 COROLLARY :**

If $f$ and $g$ are real valued measurable functions defined on a measurable space $(X, \mathcal{M}, \mu)$ then so are $f+g$ and $fg$.

**Proof :** The functions $F, G$ defined on $\mathbb{R}' \times \mathbb{R}'$ by $F(x_1, x_2) = x_1 + x_2$ and $G(x_1, x_2)$ are continuous. Hence $F(f, g) = f + g$ and $G(f, g) = fg$ are measurable.

**23.11 SAQ :**

Prove directly that if $f$ and $g$ are real valued measurable functions on a measurable space $(X, \mathcal{M}, \mu)$ then so are $f+g$ and $f \cdot g$.

**Definition :** For any set $E \subset X$, the characteristic function $\chi_E$ (called Kai E) is defined by

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E. \end{cases}$$

## 23.12 PROPOSITION :

If $E \subset X$, $\chi_E$ is measurable if and only if E is measurable.

**Proof :** For any real number $a < 0$

$\chi_E(x) > a \Leftrightarrow x \in X$ so that $\{x/\chi_E(x) < a\} = X$. If $0 \le a < 1$, $\chi_E(x) > a \Leftrightarrow x \in E$ so that $\{x/\chi_E(x) > a\} = E$ and if $a > 1$, $\chi_E(x) \le a \forall x \in X$ so that $\{x/\chi_E(x) > a\} = 0$. Since the sets 0, E and X are measurable $\chi_E$ is measurable conversely if $\chi_E$ is measurable then for $0 \le a \le 1$, $\{x/\chi_E(x) > a\}$ is measurable so that E is measurable.

**DEFINITION :** A real valued function $s$ defined on a measurable space $X$ is said to be simple if its range is finite.

## 23.13 PROPOSITION :

A real valued function $s$ is a simple function if and only if three exists a positive integer $n$ and measurable sets $E_1, \cdots\cdots, E_n$ and $C_1, \cdots\cdots, C_n$ in $\mathbb{R}'$ such that

$$s = \sum_{i=1}^{n} C_i \chi_{E_i}$$

**Proof :** If $s$ is asimple function with range $\{C_1, \cdots\cdots, C_n\}$ and $E_i = \{x/s(x) = C_i\}$, then it is clear that $E_i \cap E_j = 0$ if $i \ne j$ and $X = \bigcup_{n=1}^{n} E_i$ and $s(x) = \sum_{i=1}^{n} C_i \chi_{E_i}(x)$ for every $x \in X$.

Conversely if each $E_i$ is measurable $s = \sum_{i=1}^{n} C_i \chi_{E_i}$ is simple.

## 23.14 DEFINITION :

If $s$ is simple function and $s = \sum_{i=1}^{n} C_i \chi_{E_i}$ where $\bigcup_{n=1}^{n} E_i = X$ then the above representation of $s$ is called the canonical representation of $s$.

## 23.15 PROPOSITION :

A simple function $s$ is measurable iff the sets $E_1, \cdots, E_n$ in the canonical representation $s = \sum_{1}^{n} C_i \chi_{E_i}$ is measurable.

**Proof :** If $s$ has range $\{C_1, C_2, \cdots\cdots, C_n\}$ and $E_i = \{x / s(x) = C_i\}$ then $s$ is measurable $\Rightarrow E_i$ is measurable $\forall i$. Conversely if each $E_i$ is measurable $\chi_{E_i}$ is measurable for every $i$ hence $s$ is measurable.

**23.16 PROPOSITION :**

Let $f$ be a real valued function defined on a measurable space $X$. Then there is a sequence $\{s_n\}$ of simple functions such that $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. If $f$ is measurable, $\{s_n\}$ may be choosen to be a sequence of measurable functions. If $f$ is non-negative, $\{s_n\}$ may be choosen to be monotonically increasing.

**Proof :** First suppose that $f(x) \geq 0$ for every $x \in X$. For each $n \geq 1$ and $1 \leq i \leq 2^n \cdot n$. Write

$$E_{n_i} = \left\{ x \middle/ \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\} \text{ and } F_n = \{x / f(x) \geq n\} \text{ and define}$$

a sequence of functions $\{s_n\}$ by $s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n_i}} + n\chi_{F_n}$.

Clearlry $s_n$ assumes the value $\frac{i-1}{2^n}$ in $E_{n_i}$, $1 \leq i \leq n2^n$ and $n$ in $F_n$. Hence $s_n$ is simple for every $n$ since $E_{n_i}\ (1 \leq i \leq n2^n)$ & $F_n$ are measurable.

If $x \in E_{n_i}$, $\frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}$

$$\Rightarrow \frac{2i-2}{2^{n+1}} \leq f(x) < \frac{2i}{2^{n+1}}$$

$$\Rightarrow (i)\ x \in E_{n+1_{2i-1}} \text{ or (ii) } x \in E_{n+1_{2i}}$$

In case (i) $f(x) = \frac{2i-2}{2^{n+1}} = \frac{i-1}{2^n}$

so $s_n(x) = \frac{i-1}{2^n} = \frac{2i-2}{2^{n+1}} = s_{n+1}(x)$

In case (ii) $f(x)=\dfrac{2i-1}{2^{n+1}}$

$$\text{so } s_n(x)=\frac{i-1}{2^n}=\frac{2i-2}{2^{n+1}}=s_{n+1}(x)$$

If $x \in F_n$, $f(x) \geq n$ so $s_n(x)=n$

If $f(x) \geq n+1$, $s_{n+1}(x)=n+1>s_n(x)$

If $n \leq f(x)<n+1$, $s_{n+1}(x)=n+1>n=s_n(x)$

Thus $s_n(x) \leq s_{n+1}(x) \forall n$, so that $\{s_n\}$ is monotonically increasing.

If $x \in X$ there is a positive integer $N$ such that

$$N-1 \leq f(x)<N.$$

If $n \geq N$ and $\dfrac{i-1}{2^n} \leq f(x)<\dfrac{i}{2^n}$

$$|f(x)-s_n(x)|=\left|f(x)-\frac{i-1}{2^n}\right|<\frac{i}{2^n}-\frac{i-1}{2^n}=\frac{1}{2^n}$$

Since $\lim\limits_{n} \dfrac{1}{2^n}=0$ it follows that $\lim\limits_{n} s_n(x)=f(x)$.

If $f$ is measurable each $E_{n,i}$ and $F_n$ are measurable so that $s_n$ is measurable for every $n$.

If $f$ is an arbitrary function then $f$ is the difference of two non-negative functions $f^+$ and $f^-$; $f=f^+-f^-$. If $\{s_n\}$ and $\{t_n\}$ are sequences of simple functions such that $\lim\limits_{n} s_n=f^+$ and $\lim\limits_{n} t_n=f^-$ and $u_n=s_n-t_n$, then $\{u_n\}$ is a sequence of simple functions and

$$\lim_{n} u_n=\lim_{n}(s_n-t_n)=\lim_{n} s_n-\lim_{n} t_n$$

$$=f^+-f^-=f$$

This completes the proof.

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## INTEGRATION :

We consider a measurable space $(X, \mathcal{M}, \mu)$

### 23.17 DEFINITION :

(a) If $s$ is a non-negative simple measurable function assuming the values $C_1, C_2, \cdots\cdots, C_n$, where $C_i > 0 \ \forall i$ if $E_i = \{x / f(x) = C_i\}$. We define

$$I_E(s) = \sum_{i=1}^{n} C_i \, \mu(E_i \cap E) \text{ for every } E \in \mathcal{M}.$$

(b) If $f$ is a non-negative measurable function defined on $X$, we define the Lebesgue integral of $f$ with respect to $\mu$ over $E \in \mathcal{M}$ by

$$\int_E f \, d\mu = \sup I_E(s)$$

where the supremum is taken over all nonnegative simple measurable functions $s \leq f$.

(c) If $f$ is any measurable function, $E \in \mathcal{M}$ and if one of $\int_E f^+$ and $\int_E f^-$ is finite we define

$$\int_E f \, d\mu = \int_E f^+ - \int_E f^-$$

If both the integrals $\int_E f^+$ and $\int_E f^-$ are finite, we say that $f$ is integrable or summable on in the Lesbesgue sense with respect to $\mu$ and write $f \in \mathcal{L}(\mu)$ on $E$.

We write $\mathcal{L}$ for $\mathcal{L}(m)$, $m$ being the Lebesgue measure.

**23.18 PROPOSITION :** If $s = \sum_{i=1}^{n} C_i \, \chi_{E_i}$ where $C_i > 0 \ \forall \ i$ and $E_1, \cdots\cdots, E_n$ are pairwise disjoint measurable sets then for every $E \in \mathcal{M}, I_E(s) = \sum_{i=1}^{n} C_i \, \mu(E_i \cap E)$.

**Proof :** Let $s = \sum_{i=1}^{n} C_i \, \chi_{E_i}$ where $C_i > 0 \forall i$ and $E_i \cap E_j = 0$ for $i \neq j$ and each $E_i$ is measurable.

Let $\{d_1, \cdots\cdots, d_k\}$ be the subset of $\{C_1, \cdots\cdots, C_n\}$ such that $d_i \neq d_j$ if $i \neq j$ and each $C_i$ in some $d_j$ and viceversa so that Range $s = \{d_1, \cdots\cdots, d_k\}$. If $F_i = \{x / s(x) = d_i\}$ then $F_i$ is the union of those $E_j$ for which $c_j = d_j$. If this union is $F_i = E_{j_1} \cup \cdots\cdots \cup E_{j_n}$

$$\mu(F_i) = \mu\left(\bigcup_{\ell=1}^{r} E_{j_\ell}\right) = \sum_{\ell=1}^{r} \mu\left(E_{j_\ell}\right)$$

and $$\sum_{i=1}^{K} d_i \, \mu(F_i \cap E) = \sum_{i=1}^{K} d_i \sum_{\ell=1}^{r} \mu\left(E_{j_\ell} \cap E\right).$$

$$= \sum_{i=1}^{K} \sum_{\ell=1}^{r} C_{j_\ell} \, \mu\left(E_{j_\ell} \cap E\right) \qquad \left(\because d_i = C_{j_\ell}\right)$$

$$= \sum_{j=1}^{n} C_j \mu\left(E_j \cap E\right)$$

Thus $I_E(s) = \sum_{j=1}^{n} C_j \mu\left(E_j \cap E\right)$.

**REMARK :** The difference between definition 23.17.a and proposition 23.18 is that the $C_i$ in (a) are all distinct where as the $C_i$ in 23.18 are not necessarily distinct.

**23.20 PROPOSITION :**

Assume that $s_1, s_2$ are non-negative simple and measurable functions defined on $X$ and $s_1(x) \leq s_2(x) \ \forall x \in X$. Then for every $E \in \mathcal{M}$, $I_E(s_1) \leq I_E(s_2)$.

**Proof :** Let $s_1(x) = \sum_{i=1}^{n} C_i \chi_{E_i}(x)$ and $s_2(x) = \sum_{j=1}^{m} d_j \chi_{F_j}(x)$ be the canonical representations so that $c_i > 0$ and $d_j > 0 \, \forall i, j$; $E_i = \{x / s_1(x) = c_i\}$ and $F_j = \{x / s_2(x) = d_j\}$ are measurable sets. If

$E_0 = \left(\bigcup_{i=1}^{\infty} E_i\right)^c$ and $F_0 = \left(\bigcup_{j=1}^{m} F_j\right)^c$ then $s_1(x) = 0$ on $E_0$ and $s_2(x) = 0$ on $F_0$. Further

$E_i = \bigcup_{j=0}^{m} \left(E_i \cap F_j\right)$ and $F_j = \bigcup_{i=0}^{n} E_i \cap F_j$

For $i \geq 1$ and $j \geq 1$, $x \in E_i \cap F_j \Rightarrow s_1(x) \leq s_2(x)$

$$\Rightarrow c_i \leq d_j$$

Now $I_E(s_1) = \sum_{i=1}^{n} c_i \mu(E_i \cap E)$

$$= \sum_{i=1}^{n} c_i \, \mu\left(\bigcup_{j=0}^{m} E_j \cap F_j \cap E\right)$$

$$= \sum_{i=1}^{n} \sum_{j=0}^{m} c_i \, \mu(E_i \cap F_j \cap E)$$

$$\leq \sum_{i=1}^{n} \sum_{j=0}^{m} d_j \, \mu(E_i \cap F_j \cap E)$$

$$= \sum_{j=1}^{m} d_j \sum_{i=1}^{n} \mu(E_i \cap F_j \cap E) = \sum_{j=1}^{m} d_j \mu\left(\bigcup_{i=1}^{n} E_i \cap F_j \cap E\right)$$

$$= \sum_{j=1}^{m} d_j \, \mu(F_j \cap E)$$

$$= I_E(s_2)$$

**23.21 COROLLARY :**

If $s$ is a non-negative simple measurable function on $X$ then $\int_E s = I_E(s)$.

**Proof :** If $0 \leq s_1 \leq s$ and $s_1$ is a non-negative simple measurable function then $\forall E \in \mathcal{M}$

$$I_E(s_1) \leq I_E(s).$$

Hence $\int_E s \, d\mu = \sup_{s_1} I_E(s_1) \leq I_E(s)$

Further $I_E(s) \leq \int_E s \, d\mu$ since $s \leq s$

This gives $I_E(s) = \int_E s\,d\mu$

**23.22 SAQ :**

If $f$ is measurable and bounded on E and if $\mu(E) < \infty$ then $f \in \mathscr{L}(\mu)$ on E.

**23.23 SAQ :**

If $f$ and $g$ are in $\mathscr{L}(\mu)$ on E and $f(x) \le g(x)$ $\forall x \in E$ then $\int_E f\,d\mu \le \int_E g\,d\mu$.

**23.24 SAQ :**

If $f \in \mathscr{L}(\mu)$ on E then $cf \in \mathscr{L}(\mu)$ on E $\forall c \in \mathbb{R}$ and $\int_E cf\,d\mu = c\int_E f\,d\mu$.

**23.25 SAQ :**

If $\mu(E) = 0$ and $f$ is measurable then $\int_E f\,d\mu = 0$.

**23.26 SAQ :**

If $f \in \mathscr{L}(\mu)$ on E, $\mathscr{A} \in \mathscr{M}$ and $A \subseteq E$ then $f \in \mathscr{L}(\mu)$ on $A$.

**23.27 PROPOSITION :**

If $s$ is a nonnegative measurable simple function defined on a measurable space $(X, \mathscr{M}, \mu)$ and $\{A_n\}$ is a sequence of pairwise disjoint mesurable sets and $A = \bigcup_{n=1}^{\infty} A_n$ then

$$\int_A s\,d\mu = \sum_{n=1}^{\infty} \int_{A_n} s\,d\mu.$$

**Proof :** First assume that $s$ is a characteristic function, say $s = \chi_E$ where $E \in \mathscr{M}$. Then $\int_A s\,d\mu = \int_A \chi_E\,d\mu = \mu(A \cap E)$. Similarly $\forall n$ $\int_{A_n} s\,d\mu = \mu(A_n \cap E)$.

Since $A \cap E = \bigcup_{n=1}^{\infty} (A_n \cap E)$ and $\{A_n \cap E\}$ is a sequence of pairwise disjoint measurable sets,

$$\mu(A\cap E)=\sum_{n=1}^{\infty}\mu(A_n\cap E).$$

Hence $\int\limits_A s\,d\mu = \sum\limits_{n=1}^{\infty}\int\limits_{A_n} s\,d\mu$ ------------- (1)

Now let $s$ be a nonnegative simple measurable function assuming the values $C_1,\cdots\cdots,C_n$ on the disjoint measurable sets $E_1, E_2,\cdots\cdots,E_n$ respectively and $c_i > 0\ \forall i$ so that $s=\sum\limits_{i=1}^{n} c_i\,\chi_{E_i}$.

Then $\int\limits_A s\,d\mu = \sum\limits_{i=1}^{n} c_i \int\limits_A \chi_{E_i}\,d\mu$

$$=\sum_{i=1}^{n} c_i \sum_{j=1}^{\infty}\int_{A_j}\chi_{E_i}\,d\mu$$

$$=\sum_{j=1}^{\infty}\sum_{i=1}^{n} c_i \int_{A_j}\chi_{E_i}\,d\mu$$

$$=\sum_{j=1}^{\infty}\int_{A_j}\sum_{i=1}^{n} c_i\,\chi_{E_i}\,d\mu=\sum_{j=1}^{\infty}\int_{A_j} s\,d\mu$$

**23.28 THEOREM :**

If $f$ is a nonnegative measurable function defined on a measurable space $(X, \mathcal{M}, \mu)$ then the set function $\phi$ defined by $\phi(A)=\int\limits_A f\,d\mu$ is finitely additive.

**Proof :** Let $A_1, A_2,\cdots\cdots,A_n$ be pairwise disjoint measurable sets. If $\phi(A_j)=\infty$ for some $j$, then

$$\infty=\phi(A_j)=\int_{A_j} f\,d\mu \le \int_{\bigcup\limits_{i=1}^{n} A_i} f = \phi\left(\bigcup_{i=1}^{n} A_i\right)\le\infty.$$

Thus in this case

$$\phi\left(\bigcup_{i=1}^{n} \mathscr{A}_i\right) = \sum_{i=1}^{n} \phi(A_i) = \infty$$

Now assume that $\phi(A_i)<\infty$ for every $i$, $1 \le i \le n$. Given $\in > 0$ there exist nonnegative simple measurable functions $s_1, \cdots\cdots, s_n$ such that $0 \le s_i \le f$ and

$$\int_{A_i} s_i \, d\mu > \int_{A_i} f \, d\mu - \frac{\in}{n} \text{ for } 1 \le i \le n.$$

Let $s = \max\{s_1, \cdots\cdots, s_n\}$. Then $s$ is a nonnegative simple measurable function, $0 \le s \le f$ and

$$\int_{A_i} s \, d\mu \ge \int_{A_i} f \, d\mu - \frac{\in}{n} \text{ for } 1 \le i \le n.$$

Hence $\phi\left(\bigcup_{i=1}^{n} A_i\right) = \int_{\bigcup_{i=1}^{n} A_i} f \, d\mu \ge \int_{\bigcup_{i=1}^{n} A_i} s \, d\mu = \sum_{i=1}^{n} \int_{A_i} s \, d\mu$

$$> \sum_{i=1}^{n} \int_{A_i} f \, d\mu - \in$$

$$= \sum_{i=1}^{n} \phi(A_i) - \in$$

This being true for every $\in > 0$, it follows that

$$\phi\left(\bigcup_{n=1}^{\infty} A_i\right) \ge \sum_{i=1}^{n} \phi(A_i) \text{ --------- (1)}$$

We now show that $\phi\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \phi(A_i)$.

For any nonnegative simple measurable function $s \ni$

$$0 \le s \le f, \quad \int_{\bigcup_{i=1}^{n} A_i} s\, d\mu = \sum_{i=1}^{n} \int_{A_i} s\, d\mu$$

$$\le \sum_{i=1}^{n} \int_{A_i} f\, d\mu = \sum_{i=1}^{n} \phi(A_i)$$

This being true for every such $s$, it follows that

$$\phi\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \phi(A_i) \text{ ---------- (2)}$$

From (1) and (2) it follows that

$$\phi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \phi(A_i)$$

**22.29 THEOREM :**

If $f$ is a nonnegative measurable function defined on a measurable space $(X, \mathcal{M}, \mu)$ then the set function $\phi$ defined on $\mathcal{M}$ by $\phi(A) = \int_A f\, d\mu$ is countably additive.

**Proof :** Let $\{A_n\}$ be any sequence of pairwise disjoint sets in $\mathcal{M}$ and $A = \bigcup_{n=1}^{\infty} A_n$.

If $0 \le s \le f$ and $s$ is a nonnegative simple measurable function, $\int_A s\, d\mu = \sum_{i=1}^{\infty} \int_{A_i} s\, d\mu \le \sum_{i=1}^{\infty} \int_{A_i} f\, d\mu$.

This is true for every $A \in \mathcal{M}_0$ it follows that $\phi(A) = \int_A f\, d\mu = \sup \int_A s\, d\mu \le \sum_{i=1}^{\infty} \phi(A_i)$

$$= \sum_{i=1}^{\infty} \phi(A_i).$$

On the otherhand for every $n$, $\bigcup_{i=1}^{n} A_i \subseteq A$,

$$\int_A f \geq \int_{\bigcup_{i=1}^{n} A_i} f\, d\mu = \phi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \phi(A_i)$$

This being true for every $n$, we get

$$\phi(A) = \int_A f = \sum_{i=1}^{\infty} \phi(A_i)$$

Hence $\phi$ is countably additive.

**23.30 COROLLARY :**

If $f \in \mathscr{L}(\mu)$ then the set function $\phi$ defined on $\mathscr{M}$ by $\phi(A) = \int_A f\, d\mu$ is countably additive.

**Proof :** Since $f \in \mathscr{L}(\mu)$, $f^+$ and $f^-$ are nonnegative measurable functions such that $\int_A f^+$ and $\int_A f^-$ are both finite for every $A \in \mathscr{M}$.

If $\phi_1(A) = \int_A f^+ d\mu$ and $\phi_2(A) = \int_A f^- d\mu$ for $A \in \mathscr{M}$.

$\phi_1$ and $\phi_2$ are nonnegative, finite countably additive set functions on $\mathscr{M}$. Hence $\phi = \phi_1 - \phi_2$ is a finite countably additive set function.

**23.31 COROLLARY :**

If $(X, \mathscr{M}, \mu)$ is a measure space, $A \in \mathscr{M}$, $B \in \mathscr{M}$, $B \subset A$ and $\mu(A-B) = 0$ then for every $f \in \mathscr{L}(\mu)$,

$$\int_A f\, d\mu = \int_B f\, d\mu$$

**Proof :** By additivity of $\int_A f\, d\mu = \phi(A)$, we have

$$\int_A f\, d\mu = \int_B f\, d\mu + \int_{A-B} f\, d\mu$$

Since $\mu(A-B)=0, \quad \int_{A-B} f\,d\mu=0$

So $\int_A f\,d\mu = \int_B f\,d\mu$

**23.32 THEOREM :**

Let $(X, \mathcal{M}, \mu)$ be a measurable space, $E \in \mathcal{M}$ and $f \in \mathcal{L}(\mu)$ on E. Then $|f| \in \mathcal{L}(\mu)$ on E and

$$\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu$$

**Proof :** Let $A=\{x/x \in E, f(x) \ge 0\}$ and

$B=\{x/x \in E, \text{ and } f(x)<0\}$

Then $A \in \mathcal{M}$, $B \in \mathcal{M}$, $A \cap B = 0$ and $E = A \cup B$. Since $f$ is measurable, so is $|f|$. By the additivity of the integral $\int_E |f|\,d\mu = \int_A |f|\,d\mu + \int_B |f|\,d\mu = \int_A f^+\,d\mu + \int_A f^-\,d\mu < \infty$.

Hence $|f| \in \mathcal{L}(\mu)$ on E

Since $f \le |f|$ and $-f \le |f|$,

$$\int_E f\,d\mu \le \int_E |f|\,d\mu \text{ and } -\int_E |f|\,d\mu \le \int_E f\,d\mu$$

Hence $\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu$.

**23.33 THEOREM :**

Suppose $f$ is measurable on E and $g$ is measurable on E, $|f| \le g$ and $g \in \mathcal{L}(\mu)$ on E. Then $f \in \mathcal{L}(\mu)$ on E.

**Proof :** Since $f$ is measurable so are $f^+$, $f^-$ and $|f|$. Also $f^+ \le |f| \le g$ and $f^- \le |f| \le g$ so that $f^+$ and $f^-$ belong to $\mathcal{L}(\mu)$ on E.

## 23.34 SOLUTIONS TO SAQ'S :

**SAQ 5 :** $\{x/f(x)=\infty\} = \bigcap\limits_{n=1}^{\infty}\{x/f(x)>n\}$

**SAQ 6 :** $\{x/|f|(x)>a\} = \{x/|f(x)|>a\}$

$$=\{x/f(x)>a\}\cup\{x/f(x)<-a\}$$

$$\{x/(-f)(x)>a\} = \{x/f(x)<-a\}$$

**SAQ 8 :** For any $a$, $\{x/f^{-1}(x)>a\}=\{x/\max\{f(x),0\}>a\}$

$$=\{x/f(x)>0\} \text{ if } a<0$$

$$=\{x/f(x)>a\} \text{ if } a>0$$

$$=\{x/f(x)>a^+\}$$

This set is measurable since $f$ is measurable.

The proof for measurability of $f^-$ is similar.

**SAQ 11 :** For any $a\in\mathbb{R}'$,

$$\{x/f(x)+g(x)>a\} = \{x/f(x)>a-g(x)\}$$

$$=\bigcup_{q\in Q}\left(\{x/f(x)>q\}\cap\{x/g(x)>a-q\}\right)$$

Since Q is countable and $f,g$ are measurable it follows that $f+g$ is measurable.

To prove measurability of $f,g$ it is enough to prove measurability of $f^2$ since

$$fg=\frac{(f+g)^2-(f-g)^2}{4}$$

For any $a>0,\ f^2(x)>a \Leftrightarrow |f(x)|>\sqrt{a}$

$$\Leftrightarrow f(x) > \sqrt{a} \text{ or } f(x) - \sqrt{a}$$

Hence $\{x/f^2(x) > a\} = \{x/f(x) > \sqrt{a}\} \cup \{x/f(x) < -\sqrt{a}\}$

Measurability of $f^2$ is now clear.

**SAQ 22 :** With out loss of generality we may assume that $0 \le f(x) \le M \, \forall x \in E$. If $s$ is a non-negative simple measurable function and $0 \le s(x) \le f(x)$ for $x \in E$ then $0 \le s(x) \le M$ for $x \in E$. So by exercise and the hypothesis that $\mu(E)$ is finite $I_E(s) \le M\mu(E)$.

This is true for every such $s$ so

$$\int_E f \, d\mu \le M\mu(E)$$

Hence $f \in \mathscr{L}(\mu)$ on E.

**SAQ 23 :** Since $f^+ + g^- \le f^- + g^+$ we may assume that $0 \le f \le g$ on E. In this case every nonnegative simple measurable function $s \ni s(x) \le f(x) \forall x \in E$ satisfies $s(x) \le g(x)$ on E.

So that $I_E(s) \le \int_E g$. Hence $\int_E f \le \int_E g$

**SAQ 24 :** If $c > 0$ and $f \ge 0$ then $0 \le s \le cf \Leftrightarrow 0 \le \frac{s}{c} \le f$.

Thus $\int_E cf \, d\mu = \sup\{I_E(s)/0 \le s \le cf\}$ where the supremum is taken over all nonnegative simple functions.

$$= \sup\left\{cI_E\left(\frac{s}{c}\right) \Big/ 0 \le \frac{s}{c} \le f\right\}$$

$$= c \sup\{I(s)/0 \le s \le f\}$$

$$= c \int_E f \, d\mu \cdot$$

**SAQ 25 :** The statement is trivially true when $s$ is simple.

Extend for nonnegative measurable function first and then the general case.

**SAQ 26 :** Verify for nonnegative $f$.

## 23.35 MODEL EXAMINATION QUESTIONS :

1. Show that if $(X, \mathcal{M}, \mu)$ is a measurable space and $f$ is measurable then show that $|f|$ is measurable.

2. Show that if $f$ and $g$ are real measurable functions so is $f+g$.

3. If $f$ is a nonnegative measurable function defined on $(X, \mathcal{M}, \mu)$ show that the set function $A \to \int_A f\,d\mu$ is countably additive.

4. If $f$ and $g$ are nonnegative measurable functions defined on a measurable space $(X, \mathcal{M}, \mu)$ show that

$$\forall E \in \mathcal{M}, \int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu.$$

5. If $f \in \mathcal{L}(\mu)$ show that $|f| \in \mathcal{L}(\mu)$ and $\forall E \in \mathcal{M}$

$$\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu$$

## 23.36 EXERCISES :

1. If $X$ is an uncountable set, the collection $\mathscr{E}$ of all subsets E of $X$ such that E is atmost countable is a $\sigma$ algebra and if

$$\mu(E) = \text{the number of elements in E if E is finite}$$

$$= \infty \text{ otherwise}$$

$(X, \mathscr{C}, \mu)$ is a measure space which is not a measurable space.

2. Prove directly that if $f$ and $g$ are measurable then so are max $\{f,g\}$ and min $\{f,g\}$.

3. If $f$ is measurable show that $\forall$ interval $(a, b) \subseteq \mathbb{R}'$, $f^{-1}\{(a,b)\}=\{x/a<f(x)<b\}$ is measurable.

   Deduce that for every open set $0 \leq \mathbb{R}$, $f^{-1}(0)$ is measurable.

4. If $f$ is measurable and $c \in \mathbb{R}'$ then show that $cf$ is measurable.

5. If $0 \leq s(x) \leq M \forall x$ and $s$ is a nonnegative measurable simple function then $\forall E \in \mathcal{M}$, $I_E(s) \leq M\mu(E)$.

6. If $s$ is a nonnegative simple measurable function, $E \in \mathcal{M}$ and $c$ is any real number then

$$I_E(cs)=cI_E(s)$$

7. If $f$ is measurable on $X$, $f(x) \geq 0$ $A \in \mathcal{M}$ and $B \in \mathcal{M}$ and $A \subseteq B$ then $\int_A f\,d\mu \leq \int_B f\,d\mu$.

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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## Lesson - 24

# CONVERGENCE THEOREMS

### 24.0 INTRODUCTION :

In this lesson we prove some convergence theorems for Lebesgue integral which do not have counterparts in Riemann integral. We also obtain a criterion for a bounded function $f$ defined on $[a, b]$ to be Riemann integrable on $[a, b]$ and that every Riemann integrable function defined on $[a, b]$ is measurable and Lebesgue integrable.

### 24.1 LESBESGUE'S MONOTONE CONVERGENCE THEOREM :

Let $(X, \mathcal{M}, \mu)$ be a measurable space, $\{f_n\}$ a sequence of measurable functions such that $0 \le f_n(x) \le f_{n+1}(x) \forall n \ge 1$ and $x \in E$ where $E$ is a measurable set. If $\lim_n f_n(x) = f(x) \forall\, x \in E$ then $\lim_n \int_E f_n \, d\mu = \int_E f \, d\mu$.

**Proof :** By hypothesis $f_n(x) \le f_{n+1}(x) \forall x \in E$ so

$$\int_E f_n \, d\mu \le \int_E f_{n+1} \, d\mu$$

Since $\lim_n f_n(x) = f(x)$ for $x \in E,\ f_n(x) \le f(x) \forall x \in E$ so

$$\int_E f_n \, d\mu \le \int_E f \, d\mu$$

Since $\left\{ \int_E f_n \right\}$ is monotonically increasing and bounded above in the extended number system,

$$0 \le \lim_n \int_E f_n \le \int_E f \le \infty .$$

Let $\alpha = \lim_n \int_E f_n$ . Then $\alpha \le \int_E f$

To prove that $\int_E f \le \alpha$. Let $0 \le s \le f$ on E and $s$ be any simple measurable function.

For any $c \in (0,1)$ write $E_n = \{x : x \in E \text{ and } f_n(x) \ge cs(x)\}$

Since $f_{n+1}(x) \ge f_n(x)$, $E_n \subseteq E_{n+1} \forall n$ and since $\lim_n f_n(x) = f(x) = \sup_n f_n(x)$ and $0 < c < 1$, so that $\forall\ x \in E\ \exists N \ni cf(x) < f_N(x)$ so that $x \in E_n$ hence $E = \bigcup_{n=1}^{\infty} E_n$.

Since $E_n \subseteq E \forall n$

$$\int_E f_n\, d\mu \ge \int_{E_n} f_n\, d\mu \ge c \int_{E_n} s\, d\mu$$

$$\Rightarrow \alpha \ge c \lim_n \int_{E_n} s\, d\mu \cdot$$

Write $F_1 = E_1$ and $F_n = E_n - E_{n-1}$ for $n > 1$. Then $\{F_n\}$ is a sequence of pairwise disjoint measurable sets such that $\forall n\ E_n = \bigcup_{K=1}^{n} F_n$. Hence by countable additivity,

$$\int_E s\, d\mu = \sum_{n=1}^{\infty} \int_{F_n} s\, d\mu = \lim_n \sum_{i=1}^{n} \int_{F_i} s\, d\mu$$

$$= \lim_n \int_{\bigcup_{i=1}^{n} F_i} s\, d\mu$$

$$= \lim_n \int_{E_n} s\, d\mu$$

Thus $\alpha \ge c \int_E s\, d\mu$

This is true $\forall s$. So $\alpha \ge c \int_E f$

This is true $\forall c \ni 0<c<1$ so $\alpha \geq \int f\, d\mu$

This completes the proof.

**24.2 SAQ :**

If $s_1, s_2$ are nonnegative simple measurable functions and $E \in \mathcal{M}$ then

$$\int_E (s_1+s_2)\, d\mu = \int_E s_1\, d\mu + \int_E s_2\, d\mu$$

**24.3 SAQ :**

If $\{E_n\}$ is a sequence of measurable sets in $(X, \mathcal{M}, \mu)$, $E_n \subseteq E_{n+1} \forall n$ and if $E = \bigcup_{n=1}^{\infty} E_n$ then

$$\int_E f\, d\mu = \lim_n \int_{E_n} f\, d\mu$$

**24.4 PROPOSITION :**

If $f_1, f_2$ are non-negative measurable functions on $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$ then

$$\int_E (f_1+f_2)\, d\mu = \int_E f_1\, d\mu + \int_E f_2\, d\mu$$

**Proof :** When $s_1, s_2$ are simple the equality holds from SAQ 2.

In the general case let $\{s_n\}$ and $\{t_n\}$ be sequences of non negative simple measurable functions such that

(i) $0 \leq s_n \leq s_{n+1}$ on E and $\lim_n s_n(x) = f_1(x)$ for $x \in E$ and

(ii) $0 \leq t_n \leq t_{n+1}$ on E and $\lim_n t_n(x) = f_2(x)$ for $x \in E$

Write $u_n = s_n + t_n$. $\{u_n\}$ is a sequence of non negative simple measurable functions such that

$0 \leq u_n \leq u_{n+1}$ and $\lim_n u_n(x) = (f_1+f_2)(x)$ on E, so that by the monotone convergence theorem

$$\int_E (f_1 + f_2)\, d\mu = \lim_n \int_n u_n\, d\mu$$

$$= \lim_n \int_n s_n\, d\mu + \lim_n \int_n t_n\, d\mu$$

$$= \int_E f_1\, d\mu + \int_E f_2\, d\mu$$

**24.5 PROPOSITION :**

If $f_1, f_2$ belong to $\mathscr{L}(\mu)$ on E and $f = f_1 + f_2$ then $f \in \mathscr{L}(\mu)$ on E and

$$\int_E f\, d\mu = \int_E f_1\, d\mu + \int_E f_2\, d\mu$$

**Proof :** When $f_1$ and $f_2$ are both non negative the equality holds from proposition 24.4.

Suppose $f_1 \geq 0$ and $f_2 \leq 0$

Put $A = \{x/f(x) \geq 0\} \cap E$, $B = \{x/f(x) < 0\} \cap E$ on $A$, $f$, $f_1, -f_2$ are non negative. Hence by 24.4.

$$\int_A f_1\, d\mu = \int_A (f - f_2)\, d\mu = \int_A f\, d\mu + \int_A -f_2\, d\mu$$

$$= \int_E f\, d\mu - \int_A f_2\, d\mu$$

so that $\int_A f\, d\mu = \int_A f_1\, d\mu + \int_A f_2\, d\mu$ ----------------- (1)

on B, $-f$, $f_1$, $-f_2$ are non negative. Hence by (1)

$\int_B (-f_2)\, d\mu = \int_B f_1\, d\mu + \int_B (-f)\, d\mu$ ----------- (2)

From (1) and (2) we get the required equality.

In the general case we write for $i = 1$ and 2.

$$A_i = \{x/x \in E,\ f_i(x) \geq 0\}$$

$$B_i = \{x / x \in E,\ f_i(x) < 0\}$$

Let $E_1 = A_1 \cap B_1$, $E_2 = A_1 \cap B_2, E_3 = A_2 \cap B_1$ and $E_4 = A_2 \cap B_2$

Then in each $E_i, f_1$ has constant sign as well as $f_2$.

Hence $\int_{E_i} f\, d\mu = \int_{E_i} f_1\, d\mu + \int_{E_i} f_2\, d\mu$ for $i = 1, 2, 3, 4$

Adding these four equalities we get

$$\int_E f\, d\mu = \int_E f_1\, d\mu + \int_E f_2\, d\mu$$

**24.6 COROLLARY :**

If $\{f_n\}$ is a sequence of non negative measurable functins each defined on a measurable space $(X, \mathcal{M}, \mu)$, $E \in \mathcal{M}$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for $x \in E$ then

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_E f_n\, d\mu$$

**Proof :** Let $s_n(x) = f_1(x) + \cdots\cdots\cdots + f_n(x)\ (x \in E)$.

Then $\{s_n\}$ is a monotonically increasing sequence of nonnegative measurable functions converging to $f$ for $x \in E$. Hence by Monotone convergence theorem,

$$\int_E f = \lim_n \int_E s_n$$

$$= \lim_n \sum_{i=1}^{n} \int_E f_i\, d\mu$$

$$= \sum_{n=1}^{\infty} \int_E f_n\, d\mu$$

**24.7 FATOU'S LEMMA :**

Let $(X, \mathcal{M}, \mu)$ be a measurable space, $\{f_n\}$ a sequence of nonnegative measurable functions, $E \in \mathcal{M}$ and

$$f(x) = \liminf_{n} f_n(x) \quad (x \in E)$$

then $\int_E f\, d\mu \le \lim_n \inf \int f_n d$

**Proof :** For each positive integer $n$ and each $x \in E$ write $g_n(x) = \inf\{f_i(x)/i \ge n\}$. Then each $g_n$ is measurable and for $x \in E$ $g_n(x) \le g_{n+1}(x)$. Moreover $g_n(x) \le f_n(x)$ and $\liminf_n \int_E g_n\, d\mu \le \lim_n \inf \int_E f_n\, d\mu$.

**24.8 EXAMPLE :**

Of a sequence $\{f_n\}$ for which strict inequality holds in Fatou's lemma.

Let $X = [0,1]$ $E = \left(\frac{1}{2}, 1\right]$

$$g(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases}$$

$f_{2K}(x) = g(x)$ if $x \in X$ and

$f_{2K+1}(x) = g(1-x)$ if $x \in X$.

Then for $0 \le x < \frac{1}{2}$, $f_{2K}(x) = 0$

and $f_{2K+1}(x) = 1$

and for $\frac{1}{2} < x \le 1$ $\quad f_{2K}(x) = 1$

$f_{2K+1}(x) = 0$

while $f_{2K}\left(\frac{1}{2}\right) = f_{2K+1}\left(\frac{1}{2}\right) = 0 \; \forall K$.

For any $x \in X$, and $n \ge 1$, $\inf_{K \ge n} f_K(x) = 0$ so

$\liminf\limits_{n} f_K(x) = 0$. Hence $\int_0^1 \liminf\limits_{n} f_n(x)\,dx = 0$.

But $\int_0^1 f_n(x)\,dx = \int_{\frac{1}{2}}^1 f_n(x)\,dx = \frac{1}{2}$ if $n$ is even

$$\int_0^{\frac{1}{2}} f_n(x)\,dx = \frac{1}{2} \text{ if } n \text{ is odd.}$$

So that $\lim\limits_{n} \int_0^1 f_n(x)\,dx = \frac{1}{2}$

## 24.9 LEBESGUE'S DOMINATED CONVERGENCE THEOREM :

Let $(X, \mathcal{M}, \mu)$ be a measurable space, $E \in \mathcal{M}$, $\{f_n\}$ a sequence of measurable functions such that $\lim\limits_{n} f_n(x) = f(x)$ on E. If there exists a $g \in \mathcal{L}(\mu)$ on E such that

$|f_n(x)| \le g(x)$ on E for $n \ge 1$ then

$$\lim_{n} \int_E f_n = \int_E f$$

**Proof :** Since $|f_n(x)| \le g(x) \forall x \in E,\ f_n \in \mathcal{L}(\mu)$, $f_n + g \ge 0$ on E and also $\lim\limits_{n}(f_n + g) = f + g$.

Hence by Fatou's lemma

$$\int_E (f+g)\,d\mu \le \liminf_{n} \int_E f_n + \int_E g\,d\mu$$

$$\Rightarrow \int_E f\,d\mu \le \liminf_{n} \int_E f_n\,d\mu$$

Since $(g - f_n)$ is non negative, measurable and as above that $\lim\limits_{n}(g - f_n) = g - f$, it follows as above that $\int_E (g-f)\,d\mu \le \liminf\limits_{n} \int_E (g - f_n)\,d\mu$.

Hence $\int_E f\,d\mu \geq \limsup_n \int_E f_n\,d\mu$

Hence $\lim_n \int_E f_n\,d\mu = \int_E f\,d\mu$.

**24.10 COROLLARY :**

If $\mu(E) < \infty$, $\{f_n\}$ is a sequence of measurable functions which are uniformly bounded on E and $f_n \to f$ as $n \to \infty$ on E then $\lim_n \int_E f_n\,d\mu = \int_E f\,d\mu$.

**Proof :** By hypothesis $\exists M > 0 \ni |f_n(x)| \leq M$ for all $n \geq 1$ and $x \in E$.

Since $\mu(E) < \infty$, the constant function $g(x) = M (x \in E)$ is integrable on E. Hence by the dominated convergence theorem the conclusion follows.

**COMPARISION WITH RIEMANN INTEGRAL :**

We now make a comparison between the Riemann integral and the Lebesgue integral.

**24.11 EXAMPLE :**

The ruler function $\chi_E$ where E is the set of rational numbers in $[0, 1]$ is Lebesgue integrable because the set E is measurable but not Riemann integrable since for any partition P of $[0, 1]$.

$U(P, \chi_E) = 1$ while $L(P, \chi_E) = 0$

As is evident from the above example it is clear that the Lebesgue integral includes a larger class of functions as against the Riemann integral. Besides this limit operations can be handled with more ease in Lebesgue theory when compared to the Riemann integral.

We fix the measure space $[a, b]$ and consider the $\sigma$ additive algebra $\mathcal{M}$ of Lebesgue measurable sets in $[a, b]$ and the Lebesgue measure $m$ on $\mathcal{M}$. A notion that is of most importance in Lebesgue integration is what is known as "almost every where" which is simply denoted by a.e. we say that a property P holds almost everywhere in a set $E \subset X$, measurable with respect to a measurable space $(X, \mathcal{M}, \mu)$ if and only if the set A of all $x \in E$ for which P doesn't hold is of measure zero. For example we say that the measurable functions $f$ and $g$ are equal a.e. on E if $\mu(\{x / x \in E, f(x) \neq g(x)\}) = 0$. We can easily prove the following.

**Theorem A :** Let m be the Lebesgue measure on $\mathbb{R}'$ and $\{f_n\}$ an increasing sequence of nonnegative measurable functions such that $f_n(x) \le f_{n+1}(x) \forall x \in E$ where $E \in \mathcal{M}$

and $\lim\limits_n f_n(x) = f(x)$ a.e. on E.

then $\lim\limits_n \int\limits_E f_n \, dm = \int\limits_E f \, dm$

**Theorem B :** If $E \in \mathcal{M}$ and $\{f_n\}$ a sequence of measurable functions on $\mathbb{R}'$ and $f(x) = \liminf\limits_n f_n(x)$ a.e. on E then $\int\limits_E f \, dm \le \liminf\limits_n \int\limits_E f_n$.

**Theorem C :** If $E \in \mathcal{M}$ and $\{f_n\}$ a sequence of measurable functions such that $\lim\limits_n f_n(x) = f(x)$ a.e. on E and if there exists a $g \in \mathcal{L}(m)$ on E a.e. on E $|f_n(x)| \le g(x)$ a.e. on E then $\lim\limits_n \int\limits_E f_n \, dm = \int\limits_E f \, dm$.

**24.12 EXAMPLE :**

If $f(x) \ge 0$ and $\int\limits_E f \, d\mu = 0$, show that $f(x) = 0$ a.e.

Let $E_K = \left\{ x / x \in E \text{ and } f(x) > \frac{1}{K} \right\} \forall$ integer $K \ge 1$

and $E_0 = \bigcup\limits_{K=1}^{\infty} E_K$ so that $\mu(E_0) = \lim\limits_K \mu(E_K)$

We claim that $\mu(E_K) = 0 \forall K$. If $\mu(E_K) > 0$ for some K, we would have

$0 = \int\limits_E f \, d\mu \ge \int\limits_{E_K} f \, d\mu \ge \frac{1}{K} \mu(E_K) > 0$, a contradiction.

Since $\mu(E_K) = 0 \;\forall\; K \ge 1$, $\mu(E_0) = 0$. Clearly

$x \in E - E_0 \Rightarrow f(x) = 0$.

**24.13 (i) :** If $f$ is Riemann integrable on $[a, b]$ then $f \in \mathscr{L}(m)$ on $[a, b]$ and

$$R\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm$$

**(ii) :** If $f$ is a bounded real valued function on $[a, b]$ then $f$ is Riemann integrable on $[a, b]$ if and only if $f$ is continuous a.e. on $[a, b]$.

**Notation :** To distinguish the Riemann integral from the Lebesgue integral we fix R before Riemann integral.

Suppose that $f$ is bounded. Since

$$R\underline{\int_a^b} f\,dx = \sup_p L(p,f) \text{ and}$$

$$R\overline{\int_a^b} f\,dx = \operatorname{Inf}_p U(p,f),$$

for each positive integer K we can find a partition of $[a,b]$.

$$P_K : \left\{a = x_0^{(K)} < \cdots\cdots < x_{n_K}^{(K)} = b\right\} \ni$$

(i) $x_i^{(K)} - x_{i-1}^{(K)} \le \dfrac{1}{K}\ \forall i$

(ii) $P_{K+1}$ is a refinement of $P_K\ \forall K$ and

(iii) $0 \le R\underline{\int_a^b} f\,dx - L(P_K, f) \le \dfrac{1}{K}$ and

$$0 \le U(P_K, f) - \overline{\int_a^b} f\,dx \le \frac{1}{K}$$

so that $\lim\limits_{K} L(P_K f) = \underline{\int_a^b} f\, dx$ and $\lim\limits_{K} U(P_K f) = \overline{\int_a^b} f\, dx$

Write $L_K(a) = U_K(a) = f(a)$

and $U_K(x) = M_i^{(K)}$ and $L_K(x) = m_i^{(K)}$ for $x_{i-1}^{(K)} \le x \le x_i^{(K)}$

where $m_i^{(K)} = g.\ell.b\left\{f(x) / x_{i-1}^{(K)} < x \le x_i^{(K)}\right\}$ and

$$M_i^{(K)} = \ell.u.b\left\{f(x) / x_{i-1}^{(K)} < x \le x_i^{(K)}\right\}$$

Clearly $U_K = f(a)\chi_{\{a\}} + \sum\limits_{i=1}^{n} M_i^{(K)} \chi_{A_i}$ and

$$L_K = f(a)\chi_{\{a\}} + \sum_{i=1}^{n} m_i^{(K)} \chi_{A_i}$$

where $A_i = \left(x_{i-1}^{(K)}, x_i^{(K)}\right]$.

Since $\{a\}$ and $A_i$ are measurable.

$$\int L_K\, dm = f(a)\, m\{a\} + \sum_{i=1}^{n} m_i^{(K)}\, m(A_i)$$

$$= \sum_{i=1}^{n} m_i^{(K)} \Delta_i \text{ where } \Delta_i = \left(x_i^{(K)} - x_{i-1}^{(K)}\right)$$

$$= L(P_K, f)$$

Similarly $U_K$ is a simple measurable function and

$$\int_{[a,b]} U_K\, dm = \sum_{i=1}^{n} M_i^K \Delta_i = U(P_K, f)$$

For every $x \in [a, b]$ *and* $K \geq 1$

$$L_K(x) \leq L_{K+1}(x) \leq L(x) \leq f(x) \leq U(x) \leq U_{K+1}(x) \leq U_K(x)$$

where $L(x) = \lim\limits_{K} L_K(x)$ and $U(x) = \lim\limits_{K} U_K(x)$.

Clearly L, U are bounded and measurable.

By the monotone convergence theorem

$$\int_{[a,b]} L\,dm = R\underline{\int_a^b} f(x)\,dx \quad \text{and} \quad \int_{[a,b]} U\,dm = R\overline{\int_a^b} f(x)\,dx$$

If $f$ is Riemann integrable $L(x) = U(x) = f(x)$ a.e. so that $f$ is measurable (exercise 24.1) and in this case

$$\int f = \int L = \int U \text{ so that } \int (U - L)\,dm = 0$$

Hence $f$ is continuous a.e. on $[a, b]$

On the other hand if $f$ is continuous a.e. on $[a, b]$

$U = L = f$ a.e. on $[a, b]$ so that

$$\int_{[a,b]} f = \int_{[a,b]} L = \int_{[a,b]} U = R\underline{\int_a^b} f\,dx = R\overline{\int_a^b} f\,dx$$

so that $f$ is Riemann integrable.

## 24.14 SOLUTIONS TO SHORT ANSWER QUESTIONS :

**SAQ 2** : If $s_1, s_2$ are nonnegative measurable simple functions and $E \in M$ then $\int_E s_1\,d\mu + \int_E s_2\,d\mu = \int_E (s_1 + s_2)\,d\mu$.

**Proof** : Let $s_1$ assume the values $c_1, \cdots\cdots, c_n$ where $c_i \neq c_j \ \forall\ i \neq j, c_i > 0 \forall i$ and $E = \{x \in E / s_1(x) = C_i\}$.

Then $\int_E s_1\, d\mu = \sum_{i=1}^{n} c_i\, \mu(E_i \cap E)$

Similarly $\int_E s_2\, d\mu = \sum_{j=1}^{m} d_j\, \mu(F_j \cap E)$

Where $d_1, \cdots\cdots, d_n$ are the distinct values assumed by $s_2$, $d_j > 0 \forall j$ and $F_j = \{x \in E / s_2(x) = d_j\}$.

The sets $E_i \cap F_j$ are pairwise disjoint and

$$(s_1 + s_2)(x) = \sum_{i,j} (c_i + d_j)\, \chi_{E_i \cap F_j}.$$

This implies that $\int_E (s_1 + s_2)\, d\mu = \sum_{i,j} (c_i + d_j)\, \mu(E_i \cap F_j)$

$$= \sum_i c_i\, \mu(E_i) + \sum_j d_j \mu(F_j)$$

$$= \int_E s_1\, d\mu + \int_E s_2\, d\mu$$

**SAQ 3 :** If $\{E_n\}$ is a sequence of measurable sets in $(X, \mathscr{M}, \mu)$, $E_n \subseteq E_{n+1} \forall n$ and if $E = \bigcup_{n \geq 1} E_i$ then $\int_E f\, d\mu = \lim_n \int_{E_n} f\, d\mu$ for every nonnegative measurable function $f$.

The set function $\phi(A) = \int_A f\, d\mu$ is countably additive. If $E_n \subseteq E_{n+1} \forall n$, $E_n \in M \forall n$ and $E = \bigcup_{n=1}^{\infty} E_n$ write $F_1 = E_1$ and $F_n = E_n - E_{n-1}$ for $n > 1$. Then $F_n$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} F_i = E_n$ and $E = \bigcup_{i=1}^{\infty} F_i$.

Hence $\phi(E)=\sum_{n=1}^{\infty}\phi(F_n)$

$$= \lim_{n} \sum_{i=1}^{n}\phi(F_i)$$

$$= \lim_{n} \phi\left(\bigcup_{i=1}^{n} F_i\right)$$

$$= \lim_{n} \phi(E_n)$$

**24.15 MODEL EXAMINATION QUESTIONS :**

**24.15.1 :** State and prove Monotone convergence theorem.

**24.15.2 :** State and prove Lebesgues dominated convergence theorem.

**24.15.3 :** State and prove Fatou's Lemma.

**24.15.4 :** Show that if $f(x)\geq 0$ on E and $\int_E f\,d\mu = 0$ then $f=0$ a.e. on E.

**24.15.5 :** Show that in $\mathbb{R}^p$, if $A\subseteq B$ and $m(B)=0$ then A is measurable and $m(A)=0$

**24.15.6 :** If $f=f_1+f_2,\ f_1\in\mathscr{L}(\mu)$ on E and $f_2\in\mathscr{L}(\mu)$ on E show that $f\in\mathscr{L}(\mu)$ on E and

$$\int_E f\,d\mu = \int_E f_1\,d\mu + \int_E f_2\,d\mu$$

**24.16 EXERCISES :**

**24.16.1 :** If $E\subseteq\mathbb{R}'$ is any measurable set, $f;g$ functions defined on E, $f$ is measurable and $f=g$ a.e. on E show that $g$ is measurable.

**24.16.2 :** Prove Theorem A

**24.16.3 :** Prove Theorem B

**24.16.4 :** Prove Theorem C

**24.16.5 :** In 24.13 Let $E=\left\{x_i^{(K)}/0\leq i\leq n_K \text{ and } K\geq 1\right\}$. If $x\notin E$, $a\leq x\leq b$ show that $f$ is continuous at $x$ if and only if $U(x)=L(x)$.

**24.16.6 :** If $\{E_n\}$ is a sequence of pairwise disjoint measurable sets $E = \bigcup_{n=1}^{\infty} E_n$ and $f$ is a nonnegative measurable function, show that $\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$.

**24.16.7 :** If $\int_E f\,d\mu = 0$ for every measurable subset A of a measurable set E show that $f(x) = 0$ a.e. on E.

**24.16.8 :** If $\{f_n\}$ is a sequence of measurable functions on $\mathbb{R}'$ show that $\left\{x / \lim_n f_n(x) \text{ exists in } \mathbb{R}'\right\}$ is measurable.

**24.16.9 :** If $f \in \mathscr{L}(\mu)$ on E and $g$ is bounded and measurable on E show that $fg \in \mathscr{L}(\mu)$ on E.

**24.16.10 :** Let $f_n(x) = \begin{cases} \dfrac{1}{n} & \text{if } |x| \le n \\ 0 & \text{if } |x| > n \end{cases}$

show that $\lim_n f_n(x) = 0$ uniformly on $\mathbb{R}'$ and that $\int_{\mathbb{R}'} f_n\,dm = 2 \ \forall\ n \ge 1$

(compare with 24.9)

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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Lesson - 25

# THE SPACE $\mathscr{L}^2(\mu)$

## 25.0 INTRODUCTION :

Let $(X, \mathscr{M}, \mu)$ be a measurable space. The space $\mathscr{L}^2(\mu)$ consisting of all measurable functions of $X$ such that $\int_X |f|^2 d\mu < \infty$, called the space of square integrable functions, is the most fundamental object in $\mathscr{L}^p$-theory . This lesson provides an introduction of this space. We discuss how we can treat this as metric space, show that the continuous functions on $[-\pi, \pi]$ are dense in $\mathscr{L}^2$ on $[-\pi, \pi]$ and establish completeness of this metric space.

We define (i) orthogonal system (ii) complete orthogonal system and show that a complete orthogonal system acts like a "basis". In the process we prove Riesz-Fisher theorem and also take up its converse.

## 25.2 INTEGRATION OF COMPLEX VALUED FUNCTIONS :

Let $(X, \mathscr{M}, \mu)$ be a measurable space, $f$ a complex valued function defined on $X$, real part $f = u$ and imaginary part $f = v$. We say that $f$ is measurable if and only if $u$ and $v$ are measurable.

If $E \in \mathscr{M}$, we say that $f$ is integrable over E, in symbols $f \in \mathscr{L}(\mu)$ on E, provided $f$ is measurable and $\int_E |f| d\mu < \infty$. In this case $u$ and $v$ are measurable and $|u| \le \sqrt{u^2 + v^2} = |f|$ so that $u \in \mathscr{L}(\mu)$ on E and similarly $v \in \mathscr{L}(\mu)$ on E. The integral of $f$ on E is now defined by

$$\int_E f\, d\mu = \int_E u\, d\mu + i \int_E v\, d\mu$$

**25.3 SAQ :** If $f \in \mathscr{L}(\mu)$ on E then $|f| \in \mathscr{L}(\mu)$ on E and $\left| \int_E f\, d\mu \right| \le \int_E |f|\, d\mu$.

## 25.4 DEFINITION :

Let $(X, \mathscr{M}, \mu)$ be a measurable space. We write $\mathscr{L}^2(\mu)$ for the collection of all measurable functions $f : X \to \mathbb{C}$ such that $\int |f|^2 \, d\mu < \infty$. If $\mu$ is the Lebesgue measure we write $\mathscr{L}^2$ for $\mathscr{L}^2(\mu)$.

We write $\|f\| = \left\{ \int_X |f|^2 \, d\mu \right\}^{1/2}$

We call this, the $\mathscr{L}^2(\mu)$-norm of $f$ or simply norm of $f$.

**25.5 SCHWARZ INEQUALITY :**

If $f \in \mathscr{L}^2(\mu)$ and $g \in \mathscr{L}^2(\mu)$ then $fg \in \mathscr{L}^2(\mu)$ and

$$\int_X |f\,g| \, d\mu \leq \|f\| \, \|g\|$$

**Proof :** For every real number $\lambda$, $\left(|f(x)| + \lambda |g(x)|^2\right) \geq 0 \quad \forall x \in X$.

$$\text{So } 0 \leq \int_X \left(|f| + \lambda |g|^2\right) d\mu = \int_X |f|^2 \, d\mu + 2\lambda \int_X |f| \, |g| \, d\mu + \lambda^2 \int_X |g|^2 \, d\mu$$

$$= \|f\|^2 + 2\lambda \int_X |f\,g| \, d\mu + \lambda^2 \|g\|^2$$

$$\Rightarrow \left( \int_X |f\,g| \, d\mu \right)^2 \leq \|f\|^2 \, \|g\|^2$$

$$\Rightarrow \int_X |f\,g| \, d\mu \leq \|f\| \, \|g\|$$

**25.6 TRIANGLE INEQUALITY :**

If $f \in \mathscr{L}^2(\mu)$ and $g \in \mathscr{L}^2(\mu)$ then $f + g \in \mathscr{L}^2(\mu)$ and $\|f + g\| \leq \|f\| + \|g\|$.

**Proof :** $\|f + g\|^2 = \int_X |f + g|^2 \, d\mu$

$$= \int_X (f+g)(\bar{f}+\bar{g})\,d\mu$$

$$= \int_X f\bar{f}\,d\mu + \int_X (f\bar{g}+\bar{f}g)\,d\mu + \int_X g\bar{g}\,d\mu$$

$$= \|f\|^2 + 2\,\text{Real}\int f\bar{g}\,d\mu + \|g\|^2$$

$$= \|f\|^2 + 2\|f\|\,\|g\| + \|g\|^2 = = (\|f\| + \|g\|)^2$$

$$\Rightarrow \|f+g\| \le \|f\| + \|g\|$$

**25.7 PROPOSITION :**

Assume that $\mu(A)=0$ and $B \subseteq A \Rightarrow B \in \mathscr{M}$. If $f \in \mathscr{L}^2(\mu)$

(i) $\|f\| \ge 0$ with equality if and only if $f=0$ a.e.

(ii) $\|cf\| = |c|\,\|f\|\ \forall\ c \in \mathbb{C}$

**Proof :** (ii) is clear. In (i) we need only to verify that

$\|f\| = 0 \Rightarrow f(x) = 0$ a.e. on $X$

$\|f\| = 0 \Rightarrow \int_X |f|^2\,d\mu = 0 \Rightarrow f^2(x) = 0$ a.e. on $X$

$\Rightarrow f(x) = 0$ a.e.

**25.8 SAQ :** Define a relation ~ on $\mathscr{L}^2(\mu)$ by $f \sim g$ if and only if $f(x) = g(x)$ a.e. on $X$ (i) This defines an equivalence relation which satisfies

(ii) $f_1 \sim g_1$ and $f_2 \sim g_2 \Rightarrow f_1 + f_2 \sim g_1 + g_2$ and

(iii) $f \sim g \Rightarrow cf \sim cg\ \forall c \in \mathbb{C}$

**25.9 SAQ :** (i) $\mathscr{L}^2(\mu)$ is a vector space

(ii) $N = \{f / f \in \mathscr{L}^2(\mu) \text{ and } f \sim 0\}$ is a subspace of $\mathscr{L}^2(\mu)$.

(iii) For $f \in \mathscr{L}^2(u)$, $\|f+N\| = \inf\{\|f+g\| / g \in N\}$ defines a norm on the quotient space $\mathscr{L}^2(\mu)/N$

(iv) For $f, g \in \mathscr{L}^2(\mu)$, $\int |f-g|^2 \, d\mu = 0 \Leftrightarrow f = g$ a.e.

**25.10 REMARK :**

We identify functions in $\mathscr{L}^2(\mu)$ if they are equal almost every where; i.e. we consider the Quotient space $\mathscr{L}^2(\mu)/N$ and choose a representative function in each coset. With this identification $\mathscr{L}^2(\mu)$ is a metric space where $d(f, g) = \|f-g\|$.

**25.11 SAQ :**

Show that $\mathscr{L}^2(\mu)$ is a metric space with respect to the distance defined by

$$d(f,g) = \|f-g\|$$

**25.12 THEOREM :**

The continuous functions form a dense subspace of $\mathscr{L}^2$ on $[a,b]$ with respect to the metric d, defined in 25.10.

**Proof :**

**Step 1 :** We first show that if A is a closed subset of $[a, b]$ there is a sequence $\{g_n\}$ of continuous functions that converge to $\chi_A$ in the $\mathscr{L}^2$ metric defined in 25.11.

Defined $g$ on $[a, b]$ by $g(x) = \inf\{|x-y| / y \in A\}$.

If $x, y \in [a, b]$ and $z \in A$

$$|x-z| \le |x-y| + |y-z|$$

$$\Rightarrow g(x) - |x-y| \le |y-z| \ \forall \ z \in A$$

$$\Rightarrow g(x) - |x-y| \le g(y)$$

$$\Rightarrow g(x) - g(y) \le |x-y| \text{ By symmetry}$$

$$g(y)-g(x) \le |x-y|$$

Hence $|g(x)-g(y)| \le |x-y|$

Hence $g$ is continuous on $[a, b]$

We now define a sequence $\{g_n\}$ of continuous functions that converge to $\chi_A$ pointwise.

Write $g_n(x) = \dfrac{1}{1+nt(x)}$ $(x \in [a, b] \text{ and } n \ge 1)$

Since $t$ is continuous and $t(x) \ge 0$ on $[a, b]$ $g_n$ is continuous $0 \le g_n(x) \le 1$ for $n \ge 1$ and $x \in [a, b]$.

If $x \in A$, $t(x) = 0$ so $g_n(x) = 1$ and $\lim_n g_n(x) = 1 = \chi_A(x)$

if $x \notin A$, $t(x) > 0$ so for $0 < \in < 1$

$$0 < g_n(x) < \in \text{ when } n > \frac{1}{t(x)}\left(\frac{1}{\in} - 1\right)$$

so that $\lim_n g_n(x) = 0 = \chi_A(x)$ if $x \notin A$

$$\|g_n - \chi_A\|^2 = \int_{[a,b]} (g_n - \chi_A)^2 (x)\, dm$$

$$= \int_{[a,b]-A} g_n^2 \, dm = 0 \; (\because g_n = \chi_A = 1 \text{ if } x \in A \text{ and } \chi_A(x) = 0 \text{ if } x \notin A)$$

by the Dominated convergence theorem.

Hence $\lim_n \|g_n - \chi_A\| = 0$.

Thus $\chi_A$ is in the closure of the set of continuous functions on $[a, b]$ in the metric space $\mathscr{L}^2$.

**Step 2 :** We show that for every measurable set A, $\chi_A$ is in the closure of the set of continuous functions in $\mathscr{L}^2$ norm

If A is measurable and $\in > 0$ there is a closed set $F \subseteq A$ such that $m(A-F) < \frac{\in}{2}$.

Since F is a closed set, by step 1 $\exists$ a $g_n$ sequence $\{g_n\}$ of continuous functions such that $\lim\limits_n \|g_n - \chi_F\| = 0$ so that $\exists$ a positive integer $N(\in) \ni \|g_n - \chi_F\| < \frac{\in}{2}$ for $n \geq N(\in)$.

$$\|g_n - \chi_A\| \leq \|g_n - \chi_F\| + \|\chi_F - \chi_A\|$$

$$< \frac{\in}{2} + \mu(A-F) < \in \text{ for } n \geq N(\in).$$

Hence $\chi_A = \lim\limits_n g_n$ in $\mathscr{L}^2$. This completes the proof of step 2.

**Step 3 :** If $f$ is a simple measurable function then $f$ is in the closure of the collection of continuous functions on $[a, b]$ in $\mathscr{L}^2$.

**Proof :** There exist measurable sets $E_1, \cdots\cdots, E_n$ and scalars $c_1, \cdots\cdots, c_n$ such that $c_i \neq 0 \forall i$ and

$$f = \sum_{i=1}^{n} c_i \chi_{E_i}$$

For each i $\exists$ a sequence $\left\{g_n^{(i)}\right\}$ of continuous functions on $[a, b]$ such that $\lim\limits_n \left\|\chi_{E_i} - g_n^{(i)}\right\| = 0$.

If $g_n = \sum_{i=1}^{n} c_i g_n^{(i)}$, $g_n$ is continuous on $[a, b]$ and

$$\|f - g_n\| \leq \sum_{i=1}^{n} |c_i| \left\|g_n^{(i)} - \chi_{E_i}\right\|$$

Since the sequence on r.h.s. converges to zero,

$$\lim_n \|f - g_n\| = 0$$

The proof of step 3 is complete.

**Step 4 :** If $f \geq 0$ and $f \in \mathscr{L}^2$ then there is a monotonically increasing sequence of simple measurable functions such that $\{s_n(x)\}$ converges to $f$ pointwise on $[a, b]$. Since $f(x) = \operatorname{lub}_n s_n(x), 0 \leq |f(x) - s_n(x)|^2 \leq f^2(x)$ for $x \in [a, b]$.

Hence from Lebesgue's Dominated convergence theom it follows that

$$\lim_n \|f - s_n\| = 0$$

Proof of step 4 is complete.

**Step 5 :** If $f$ is a real valued function in $\mathscr{L}^2$ then $f$ is the limit of a sequence of continuous functions on $[a,b]$ in $\mathscr{L}^2$.

**Proof :** Write $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Then $f^+$ and $f^- \in \mathscr{L}^2$ and are nonnegative. So there exist sequences $\{u_n\}$ and $\{v_n\}$ of continuous functions on $[a,b]$ such that

$$\lim_n \|f^+ - u_n\| = \lim_n \|f^- - v_n\| = 0$$

Since $f = f^+ - f^-$, $\lim_n \|f - f_n\| = 0$ where $f_n = u_n - v_n$. The sequence of functions $\{f_n\}$ is clearly a sequence of continuous functions.

**Step 6 :** If $f$ is any complex valued function in $\mathscr{L}^2$ then $f$ is the limit of a sequence of continuous functions in $\mathscr{L}^2$.

**Proof :** Let $f_1$ = real part of $f$ and $f_2$ = imaginary part of $f$. Then $f_1 \in \mathscr{L}^2$ and $f_2 \in \mathscr{L}^2$. By step 5 there exist sequences of continuous functions $\{u_n\}, \{v_n\}$ which are real valued and

$\lim u_n = f_1$ and $\lim v_n = f_2$ in the $\mathscr{L}^2$ – metric.

If $g_n = u_n + i v_n$, $g_n$ is continuous and $\lim g_n = f$ in $\mathscr{L}^2$.

This completes the proof.

**25.13 DEFINITION :** We say that a sequence $\{\phi_n\}$ of complex valued functions defined o. measurable space $(X, \mathcal{M}, \mu)$ is orthonormal if

$$\int_X \phi_n \, \overline{\phi}_m \, d\mu = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

**Observation :** If $\{\phi_n\}$ is orthonormal, $\phi_n \in \mathscr{L}^2(\mu)$ on $X$ for every $n$.

**25.14 DEFINITION :**

If $f \in \mathscr{L}^2(\mu)$ and $\{\phi_n\}$ is an orthonormal sequence in $\mathscr{L}^2(\mu)$ the sequence $\{c_n\}$ defined by

$$c_n = \int_X f(t)\, \overline{\phi}_n(t)\, d\mu \quad \text{----------} \ (1)$$

is called the sequence of Fourier coefficients of $f$.

We write $f \sim \sum_{n=1}^{\infty} c_n \phi_n$

and call the series on the right hand side, the Fourier series of $f$ with respect to $\{\phi_n\}$.

**25.15 SAQ :** Let $\{\phi_n\}$ be an orthonormal set in $\mathscr{L}^2(\mu)$, $f \in \mathscr{L}^2(\mu)$ and $\{s_n\}$ be the sequence of partial sums of the Fourier series of $f$ :

$$s_n(x) = \sum_{i=1}^{n} c_i \, \phi_i(x) \text{ where } c_i \text{ is defined by (1)}$$

If for every choice of $\{r_n\}$ in $\mathbb{C}$ we write $t_n(x) = \sum_{i=1}^{n} r_i \, \phi_i(x)$

then $\|f - s_n\| \leq \|f - t_n\| \, \forall n$ with equality if and only if $c_n = r_n \forall n$.

**25.16 BESSELS INEQUALITY :**

If $\{\phi_n\}$ is an orthogonal sequence in $\mathscr{L}^2(\mu)$ on $X$ and $f \in \mathscr{L}^2(\mu)$ has the Fourier serie

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n$$

then $\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|^2$. In particular $\lim c_n = 0$

**Proof :** If $s_n = \sum_{i=1}^{n} c_i \phi_i$

$$\|s_n\|^2 = \sum_{\substack{i=1\\ j=1}}^{n} c_i \bar{c}_j \int_X \phi_i \bar{\phi}_j \, d\mu$$

$$= \sum_{i=1}^{n} |c_i|^2 \le \|f\|^2 \quad \int_X \phi_i \bar{\phi}_j \, d\mu = 0 \text{ if } i \ne j \text{ and}$$

$$= 1 \text{ if } i = j$$

Letting $n \to \infty$ we get

$$\sum_{n=1}^{\infty} |c_n|^2 \le \|f\|^2$$

In particular $\lim\limits_{n} c_n = 0$

**25.17 DEFINITION :**

$f : \mathbb{R} \to \mathbb{C}$ is said to be periodic with period $a$ if $f(x+a) = f(x)$ for all $x \in \mathbb{R}$.

**25.18 DEFINITION :**

By a trigonometric polynomial we mean a function of the form

$$P(t) = \sum_{K=-n}^{n} c_K \, e^{iKt}; (t \in \mathbb{R})$$

where $n \ge 0$ is an integer and $c_K \in \mathbb{C}$ for $-n \le K \le n$.

**25.19 DEFINITIONS :**

By an algebra of complex valued functions on a set E we mean a vector space $\mathscr{A}$ of functions on E satisfying the condition that $\mathscr{A}$ is closed under multiplication of functions i.e. if $f \in \mathscr{A}$, $g \in \mathscr{A}$ and $h(x) = f(x)g(x) \forall x \in E$, then $h \in \mathscr{A}$ : $\mathscr{A}$ is self adjoint if $f \in \mathscr{A} \Rightarrow \bar{f} \in \mathscr{A}$.

$\mathscr{A}$ separates points in E if $x \in E$, $y \in E$, $x \ne y \Rightarrow f(x) \ne f(y)$ for some $f \in \mathscr{A}$.

**25.20 REMARK :**

It is easy to verify tha the collection $\mathscr{T}$ of all trigonometric polynomials is an algebra which is self adjoint and separates points.

**25.21 SAQ :**

Suppose $f$ is Riemann integrable on $[a, b]$. Then for $\in > 0$ there corresponds a $g \in \mathscr{L}^2$ on $[a, b]$ such that $g$ is continuous and $\|f - g\|_2 < \in$ on $[a, b]$.

**25.22 SAQ :**

Suppose $g$ is continuous on $[-\pi, \pi]$. Then for $\in > 0$ there is a trigonometric polynomial $p(t)$ such that

$$|f(t) - p(t)| < \in \ \forall \ t \in [-\pi, \pi]$$

**25.23 Theorem :** Suppose $f \in \mathscr{L}^2$ on $[-\pi, \pi]$

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ and } s_n(x) = \sum_{K=-n}^{n} c_K e^{iKx} \text{ for } n \geq 0$$

Then (i) $\lim\limits_{n} \|f - s_n\| = 0$ and

(ii) $\sum\limits_{n=-\infty}^{\infty} |c_n|^2 = \dfrac{1}{2\pi} \int\limits_{-\pi}^{\pi} |f|^2 \, dx$

**Proof :** If $\in > 0$ there is a continuous function $g$ such that $\|f - g\| < \frac{\in}{2}$ and we may choose $g$ so that $g(-\pi) = g(\pi)$. By SAQ 25.22 there is a trigonometric polynomial P of degree, say, N such that

$$\|g - P\| < \frac{\in}{2}$$

By SAQ 25.15 $\|g - s_N\| \leq \|g - P\|$

Hence $\|f - s_N\| \leq \|f - g\| + \|g - s_N\|$

$$\leq \|f - g\| + \|g - P\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This completes the proof of (i)

Since $\left|\|f\| - \|s_n\|\right| \le \|f - s_n\|$, from (i) it follows that

$\lim_n \|s_n\| = \|f\|$, hence

$$\lim_n \|s_n\|^2 = \|f\|^2$$

Hence $\sum_{-\infty}^{\infty} |c_n|^2 = \lim_n \sum_{-N}^{N} |c_n|^2 = \lim_n \|s_n\|^2$

$$= \|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2 \, dt$$

This completes the proof of (ii).

**25.24 THEOREM :**

The space $\mathscr{L}^2(\mu)$ is complete, that is if $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$ there exists $f$ in $\mathscr{L}^2(\mu)$ such that $\{f_n\}$ converges to $f$ in $\mathscr{L}^2(\mu)$.

**Proof :** Let $\{f_n\}$ be any Cauchy sequence in $\mathscr{L}^2(\mu)$.

Choose positive integers $\{n_K\}$ such that

$$1 \le n_1 < n_2 < \cdots\cdots < n_K < n_{K+1} < \cdots\cdots$$

and $\left\|f_n - f_{n_K}\right\| < \frac{1}{2^K}$ for $n \ge n_K$

Since $n_{K+1} > n_K$ we have $\left\|f_{n_{K+1}} - f_{n_K}\right\| < \frac{1}{2}K \;\forall\, K \ge 1$. We show that the serie $\sum_{K=1}^{\infty} \left|f_{n_{K+1}} - f_{n_K}\right|$ converges a.e. on $X$ and consequently $\{f_{n_K}\}$ cnverges a.e. on $X$.

If $g \in \mathscr{L}^2(\mu)$, by Schwarz inequality.

$$\int_X \left|g(x)\left(f_{n_{K+1}}(x)-f_{n_K}(x)\right)\right| d\mu \leq \|g\| \left\|f_{n_{K+1}}-f_{n_K}\right\| < \frac{\|g\|}{2^K}$$

Hence $\sum_{K=1}^{\infty} \int_X \left|g\left(f_{n_{K+1}}-f_{n_K}\right)\right| d\mu \leq \|g\| \sum_{K=1}^{\infty} \frac{1}{2^K} = \|g\|$

By 24.6 $\sum_{K=1}^{\infty} |g(x)| \left|f_{n_{K+1}}(x)-f_{n_K}(x)\right|$ converges a.e. on $X$.

Choosing $g(x)$ to the characteristic function of a set E of finite measure $\mu(E)>0$ we conclude that

$\sum_{n=1}^{\infty} \left|f_{n_{K+1}}(x)-f_{n_K}(x)\right|$ converges every where on any set of positive finite measure.

This implies that the above series converges a.e.

**25.25 THE RIESZ-FISCHER THEOREM :**

Let $\{\phi_n\}$ be an orthonormal sequence on a measurable space $(X, \mathscr{M}, \mu)$ and $\{c_n\}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |c_n|^2$ is convergent. Then there is a $f$ in $\mathscr{L}^2(\mu)$ on $X$ such that $f \sim \sum_{n=1}^{\infty} c_n \phi_n$ and

if we put $s_n = c_1\phi_1 + \cdots\cdots + c_n\phi_n$, $\lim_n \|f - s_n\| = 0$

**Proof :** Clearly $\|s_n - s_m\|^2 = \left\|\sum_{i=n+1}^{m} c_i \phi_i\right\|^2$ for $m > n$.

$$= \sum_{i=n+1}^{m} \int_X \left|\sum_{i=n+1}^{m} c_i \phi_i\right|^2 d\mu$$

$$= \sum_{i=n+1}^{m} \int_X \left( \sum_{i=n+1}^{m} c_i \phi_i \right) \left( \sum_{j=n+1}^{m} \bar{c}_j \bar{\phi}_j \right)$$

$$= \sum_{i=n+1}^{m} |c_i|^2$$

Sicne $\sum_{n=1}^{\infty} |c_i|^2$ is convergent, the sequence of partial sums is a Cauchy sequence so that given $\in > 0 \ \exists \ N_\in \in N \ni$

$$\|s_n - s_m\|^2 \le \sum_{i=n+1}^{m} |c_i|^2 < \in^2 \text{ for } n > m \ge N_\in$$

and hence $\|s_n - s_m\| < \in$ for $n > m \ge N(\in)$.

Thus $\{s_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$ on $X$.

Since $\mathscr{L}^2(\mu)$ on $X$ is complete, $\exists \ f \in \mathscr{L}^2(\mu)$ such that

$$\lim_n \|f - s_n\| = 0.$$

By Cauchy criterion, given $\in > 0$ there is a positive integer $N(\in)$ such that

$$\|f_n - f_m\| < \in \text{ for } n \ge N(\in) \text{ and } m \ge N(\in) \quad (*)$$

Since $\lim f_{n_K} = f$ a.e. on $X$, $\liminf_{n_\ell} \int_X |f_{n_K} - f_{n_\ell}|^2 = |f_{n_K} - f|^2$ a.e. on $X$.

By Fotou's theorem

$$\int_X |f_{n_K} - f|^2 d\mu \le \liminf_{n_\ell} \int_X |f_{n_K} - f_{n_\ell}|^2 d\mu \le \in^2 \text{ for } n_K \ge N(\in)$$

$$\Rightarrow \|f_{n_K} - f\| \le \in \text{ for } n_K \ge N(\in) \text{------------- } (**)$$

$$\Rightarrow f - f_{n_K} \in \mathscr{L}^2(\mu) \text{ on } X \text{ for } n_K \ge N(\in)$$

Since $f_{n_K} \in \mathscr{L}^2(\mu)$ on $X$, it follows that $f \in \mathscr{L}^2(\mu)$ on $X$.

From $(**)$ $\{f_{n_K}\}$ converges to $f$ in $\mathscr{L}^2(\mu)$.

From $(*)$ and $(**)$, $\|f_n - f\| \le \|f_n - f_{n_K}\| + \|f_{n_K} - f\|$

$$\le \epsilon + \epsilon = 2\epsilon \text{ for } n \ge N(\epsilon)$$

Where $n_K$ is choosen to be so that $n_K \ge N(\epsilon)$.

Hence $\lim f_n = f$ in $\mathscr{L}^2(\mu)$ on $X$.

This completes the proof.

For any $n, K \in \mathbb{N}$ with $n > K$

$$\int_X s_n \overline{\phi}_K \, d\mu = \sum_{i=1}^{n} c_i \overline{\phi}_K \, d\mu = c_K$$

$$\Rightarrow \left| \int_X f \overline{\phi}_K \, d\mu - c_K \right| = \left| \int_X f \overline{\phi}_K \, d\mu - \int_X s_n \overline{\phi}_K \, d\mu \right|$$

$$= \left| \int_X (f - s_n) \overline{\phi}_K \, d\mu \right|$$

$$\le \|f - s_n\| \|\overline{\phi}_K\| \qquad \text{(Schwarz inequality)}$$

$$= \|f - s_n\|$$

Since $\lim_n \|f - s_n\| = 0$, it follows that

$$\int_X f \overline{\phi}_K \, d\mu = c_K \text{ and this holds } \forall\, K \in \mathbb{N}$$

This completes the proof.

**25.26 DEFINITION :**

An orthonormal set $\{\phi_n\}$ on a measurable space $(X, \mathscr{M}, \mu)$ is said to be complete if $f \in \mathscr{L}^2(\mu)$ and for every K, $\int_X f\,\bar{\phi}_K\, d\mu = 0$ then $\|f\| = 0$ i.e. $f(x) = 0$ a.e. on $X$.

**25.27 EXAMPLE :**

$\left\{ \dfrac{e^{int}}{\sqrt{2\pi}} \Big/ n \in z \right\}$ is complete $\mathscr{L}^2$ on $[-\pi, \pi]$.

**25.27 THEOREM :**

Let $(X, \mathscr{M}, \mu)$ be a measurable space, $\{\phi_n\}$ a complete orthonormal set on $X$ and $f \in \mathscr{L}^2(\mu)$.

If $f \sim \sum_{n=1}^{\infty} c_n\,\phi_n$ then

$$\sum_{n=1}^{\infty} |c_n|^2 = \int_X |f|^2\, d\mu = \|f\|^2$$

**Proof :** By Bessel's inequality, $\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|^2$.

Hence the series $\sum_{n=1}^{\infty} |c_n|^2$ converges.

Write $s_n = c_1\,\phi_1 + \cdots\cdots + c_n\phi_n$. Then $\|s_n\|^2 = |c_1|^2 + \cdots\cdots + |c_n|^2$.

By Riesz-Fischer theorem there exists $g \in \mathscr{L}^2(\mu)$ such that $g \sim \sum_{n=1}^{\infty} c_n\,\phi_n$ and $\{\|g - s_n\|\}$ converges to 0.

Since $\big|\|g\| - \|s_n\|\big| \leq \|g - s_n\|$

$$\lim_n \|s_n\| = \|g\|$$

$$\Rightarrow \sum_{n=1}^{\infty} |c_n|^2 = \lim_n \|s_n\|^2 = \|g\|^2 = \int_X |g|^2 \, d\mu$$

Since $f \sim \sum_{n=1}^{\infty} c_K \phi_K$ and $g \sim \sum_{n=1}^{\infty} c_K \phi_K$,

$$c_K = \int_X f \, \bar{\phi}_K \, d\mu = \int_X g \bar{\phi}_K \, d\mu$$

$$\Rightarrow \int_X (f-g)\bar{\phi}_K \, d\mu = 0 \ \forall \ K \geq 1$$

$$\Rightarrow \int_X f \bar{\phi}_K \, d\mu = \int_X g \, \bar{\phi}_K \, d\mu$$

Since $\{\phi_K\}$ is complete, it follows that $\|f-g\| = 0$

Since $\big| \|f\| - \|g\| \big| \leq \|f-g\| = 0, \ \|f\| = \|g\|$

so that $\int_X |f(t)|^2 \, d\mu = \int_X |g(t)|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2$

This completes the proof.

**Conclusion :** Let $(X, \mathcal{M}, \mu)$ be a measurable space and $\{\phi_n\}$ be a complete orthonormal system on $X$. If $f \in \mathscr{L}^2(\mu)$, by theorem 25.28 the series $\sum_{n=1}^{\infty} |c_n|^2 < \infty$ where $\{c_n\}$ is the sequence of Fourier coefficients. On the otherhand, if $\sum_{n=1}^{\infty} |c_n|^2$ is any convergent series of positive terms then by Riesz-Fischer theorem, there is a $f \in \mathscr{L}^2(\mu)$ such that $f \sim \sum_{n=1}^{\infty} c_n \phi_n$.

Moreover this correspondence between $\mathscr{L}^2(\mu)$ and the space $\ell^2$ consisting of all sequences $\{c_n\}$ such that $\sum_{n=1}^{\infty} |c_n|^2$ is convergent, is one-one and onto. Thus we may identify $\mathscr{L}^2(\mu)$ with $\ell^2$ which is called the infinite dimensional Hilbert space.

## 25.29 SOLUTIONS TO SHORT ANSWER QUESTIONS :

**SAQ 25.3 :** If $\int_E f\,d\mu = 0$ the inequality holds trivially. Otherwise let c be such that $c\int_E f\,d\mu = \left|\int_E f\,d\mu\right|$.

Then $|c| = 1$. If $g = cf = u + iv$ where $u$ and $v$ are real valued functions then

$$\left|\int_E f\,d\mu\right| = c\int_E f\,d\mu = \int_E cf\,d\mu$$

$$= \int_E g\,d\mu = \int_E u\,d\mu + i\int_E v\,d\mu = \int_E u\,d\mu$$

since $\left|\int_E f\,d\mu\right|$ is real.

But $\int_E u\,d\mu \le \int_E |u|\,d\mu \le \int_E \sqrt{u^2+v^2}\,d\mu = \int_E |cf|\,d\mu$

$$= \int_E |f|\,d\mu$$

Hence $\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu$

**SAQ 25.8 :** We verify that $f \sim g$ and $g \sim h \Rightarrow f \sim h$, the other two being clear.

$$f \sim g \Rightarrow \mu(\{x/f(x) \ne g(x)\}) = 0$$

$$g \sim h \Rightarrow \mu(\{x/g(x) \ne h(x)\}) = 0$$

$f(x) \ne h(x) \Rightarrow f(x) = g(x)$ and $g(x) \ne h(x)$ or

$f(x) \ne g(x)$ and $f(x) \ne h(x)$

so that $\{x/f(x) \ne h(x)\} \subseteq \{x/f(x) = g(x)\} \cap \{x/g(x) \ne h(x)\}$

$\cup \{x/f(x) \ne g(x)\} \cap \{x/f(x) \ne h(x)\}$.

Since both the sets on the r.h.s. are subsets of sets whose measure is zero, the set on the left side has measure we verify (ii). (iii) is clear.

$$f_1 \sim g_1 \text{ and } f_2 \sim g_2 \Rightarrow \mu(\{x/f_1(x) \neq g_1(x)\}) = \mu(\{x/f_2(x) \neq g_2(x)\}) \neq 0$$

$$f_1(x) + f_2(x) \neq g_1(x) + g_2(x) \Rightarrow \text{ either } f_1(x) \neq g_1(x) \text{ or } f_2(x) \neq g_2(x)$$

$$\Rightarrow \{x/f_1(x) + f_2(x) \neq g_1(x) + g_2(x)\} \subseteq \{x/f_1(x) \neq g_1(x)\} \cup \{x/f_2(x) \neq g_2(x)\}$$

$$\Rightarrow f_1 + f_2 \sim g_1 + g_2$$

**SAQ 25.9 :**

**(i)** From the triangle inequality and 25.7 (ii) it follows that $\mathscr{L}^2(\mu)$ is a vector space.

**(ii)** From SAQ 25.8 it follows that N is a linear subspace of $\mathscr{L}^2(\mu)$.

**(iii)** Follows from 25.6 and 25.7.

**SAQ 25.11 :** Triangle inequality and symmetry of the distance follow from SAQ 25.7 (ii). That $d(f,g) \geq 0$ is clear while $d(f,g) = 0$ if and only if $f \sim g$ i.e. $f = g$ in the sense of remark 25.10.

**SAQ 25.15 :** Let $\int g$ stand for $\int_X g\, d\mu$ and $\Sigma$ for $\sum_{i=1}^{n}$

Then $\int f\, \bar{t}_n\, d\mu = \int f \sum_{i=1}^{n} \bar{r}_i \bar{\phi}_i = \sum \bar{r}_i \int f \bar{\phi}_i = \sum c_i \bar{r}_i$

$$\int |t_n|^2 = \int t_n \bar{t}_n$$

$$= \int \sum r_i \phi_i \sum \bar{r}_j \bar{\phi}_j$$

$$= \sum \sum r_i \bar{r}_j \int \phi_i \bar{\phi}_j$$

$$= \sum |r_i|^2 \text{ since } \{\phi_n\} \text{ is orthonormal.}$$

Hence $\int |f - t_n|^2 = \int (f - t_n)(\bar{f} - \bar{t}_n)$

$$= \int |f|^2 - 2\,\text{Real} \int f\, E_n + \int |t_n|^2$$

$$= \int |f^2| - 2\,\text{Real} \sum c_i \bar{r}_i + \sum |r_i|^2$$

$$= \int |f|^2 - \sum |c_i|^2 + \sum |r_i - c_i|^2$$

$\Rightarrow \int |f - t_n|^2 \geq \int |f|^2 - \sum |c_i|^2$ while equality occurs if and only if $\sum |r_i - c_i|^2 = 0$, equivalently $r_i = c_i \forall i$.

Putting $r_i = c_i$ we get $\|f - s_n\|^2 = \int |f|^2 - \sum |c_i|^2$ so that $\|f - s_n\| \leq \|f - t_n\|$.

**SAQ 25.21 :** Choose a parition $P = \{a = x_0 < x_1 < \cdots\cdots < x_n = b\}$ such that $U(P,f) - L(P,f) < \dfrac{\epsilon^2}{2M}$ where $m > |f(x)| \forall x \in [a, b]$.

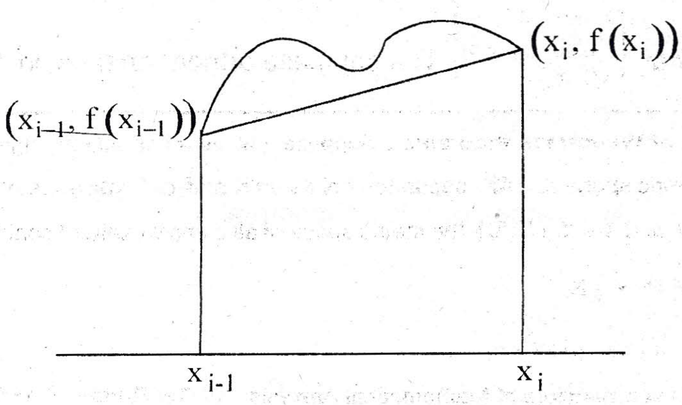
Let $M_i$ and $m_i$ be the bounds of $f$ in $[x_{i-1}, x_i]$ with $m_i \leq M_i$ $(1 \leq i \leq n)$ and $\Delta x_i = x_i - x_{i-1}$. Then the function $g$ defined by

$$g(t) = \frac{f(x_{i-1})}{\Delta x_i}(x_i - t) + \frac{f(x_i)}{\Delta x_i}(t - x_{i-1})$$

is continuous on $[x_{i-1}, x_i]$ since $(x_i - t)$ and $(t - x_{i-1})$

are continuous and $g(x_{i-1}+) = g(x_{i-1}-) = f(x_{i-1})$

Also $|f(t) - g(t)| = |f(x_i) - f(x_{i-1})| \leq M_i - m_i$ if $t \in [x_{i-1}, x_i]$



Hence $\int_{x_{i-1}}^{x_i} |f(t)-g(t)|^2 dt \le (M_i - m_i)^2 \Delta x_i$

$$\Rightarrow \int_a^b |f-g|^2 dt \le \sum_{i=1}^{n} (M_i - m_i)^2 \Delta x_i$$

$$\le 2M(U(P, f) - 2(P, f))$$

$$< \in^2$$

$$\Rightarrow \|f-g\| < \in .$$

**SAQ 25.22 :** $[-\pi, \pi]$ is a compact metric space. The trigonometric polynomials form a self adjoint algebra of continuous functions on $[-\pi, \pi]$ that separates points and vanishes at no point. Then by Stone's generalization of Weierstrass approximation theorem* given $\in > 0$ and $f$, there exists a trigonometric polynomial P such that for

$$t \in [-\pi, \pi], |f(t) - P(t)| < \in$$

## 25.30 MODEL EXAMINATION QUESTIONS :

**25.30.1 :** State and prove Schwarz inequality in $\mathscr{L}^2(\mu)$ on $X$.

**25.30.2 :** Show that the characteristic function $\chi_A$ of a measurable set A is in the closure of the set of continuous functions in $\mathscr{L}^2[a, b]$.

**25.30.3 :** Define (1) orthonormal set and (2) complete orthonormal set on a measurable space $X$. Show that $\left\{ \frac{e^{int}}{\sqrt{2\pi}} \Big/ n \in z \right\}$ is a complete orthonormal set in $\mathscr{L}^2[-\pi, \pi]$.

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* **Stone's generalization of Weirstrass theorems :** Suppose $\mathscr{A}$ is a self adjoint algebra of complex continuous functions on a compact metric space K, $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K. Then $\mathscr{A}$ is dense in K. i.e. given $\in > 0$ and $f \in \mathscr{C}(K, \mathbb{C})$ the metric space of all complex valued continuous functions on K, there is a $g \in \mathscr{A}$ such that for all $x \in K$.

$$|f(x) - g(x)| < \in$$

For further details see principle of Mathematical Analysis - Walter Ruddin P. 168

**25.30.4 :** If $\{\phi_n\}$ is an orthonormal set in $\mathscr{L}^2(\mu)$, $f \in \mathscr{L}^2(\mu)$, $f \sim \sum_n c_n \phi_n$ and $\{d_n\}$ any sequence of numbers. Show that for every $n$,

$$\left\| f - \sum_{i=1}^{n} c_i \phi_i \right\| \le \left\| f - \sum_{i=1}^{n} d_i \phi_i \right\|$$

**25.30.5 :** Derive Bessel's inequality.

**25.30.6 :** Suppose $f \in \mathscr{L}^2$ on $[-\pi, \pi]$ and $f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$

If $s_n(x) = \sum_{-n}^{n} c_K e^{iKx}$, show that

$\lim_n \|f - s_n\| = 0$ and $\sum_n |c_n|^2 = \|f\|^2$

**25.30.7 :** Show that the metric space $\mathscr{L}^2(\mu)$ on $X$ is complete.

**25.30.8 :** State and prove Riesz - Fischer theorem.

## 25.31 EXERCISES :

**25.31.1 :** Show that for $f, g$ in $\mathscr{L}^2$, $\big| \|f\| - \|g\| \big| \le \|f - g\|$.

**25.31.2 :** Prove that $\left\{ \frac{e^{int}}{\sqrt{2\pi}} \middle/ n \in z \right\}$ is a complete orthonormal set in $\mathscr{L}^2[-\pi, \pi]$.

**25.31.3 :** If $\mu(X) < \infty$ prove that, $\int_X |f|^2 \, d\mu < \infty \Rightarrow \int_X |f| \, d\mu < \infty$

**25.31.4 :** Let $E = \{\sin nx / n \ge 1\}$. Show that E is a closed and bounded subset of $\mathscr{L}^2$ on $[-\pi, \pi]$ but not compact.

**25.31.5 :** Prove that we may assume that $g(\pi) = g(-\pi)$ in 25.23.

**25.31.6 :** Prove $\left\{ \dfrac{e^{inx}}{\sqrt{2\pi}} \middle/ n \geq 0 \right\}$ is an orthonormal set in $\mathscr{L}^2[-\pi, \pi]$ which is not complete.

**25.31.7 :** Show that an orthogonal set in $\mathscr{L}^2$ on $X$ is linearly independent.

**REFERENCE BOOK :**

Principles of Mathematical Analysis - Walter Rudin (3rd Edition)

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