# COMPLEX ANALYSIS AND SPECIAL FUNCTIONS AND PARTIAL DIFFERENTIAL EQUATIONS

# (DM03)

# (MSC MATHEMATICS)

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# LESSON 1. LEGENDRE'S EQUATION

## 1.1 INTRODUCTION :

Many real world problems are represented approximately by differential equations which may not be solved exactly. Approximations might enter due to lack of knowledge of the exact natural laws governing the behaviour of the real world or due to the non-availability of exact methods of solving certain equations.

There are several exact methods available for solving linear ordinary differential equations with constant coefficients. But in the case of linear differential equations with variable coefficients the scope is limited. However, if the coefficients $a_0$, $a_1$, $a_2$ and b of the differential equation $a_0(x)\, y_{11} + a_1(x) + a_2(x)$, $y = b(x)$ have convergent power series expansion, then a solution in terms of power series can be ascertained. For example, let us consider the equation $(1-x^2)\, y^{11} - 2xy^1 + n(n+1)\, y = 0$, $n \in I^+$. This is known as the Legendre equation and it arose when the Laplace equation was solved in spherical coordinates by the method of seperation of variables. In this lesson we solve the legendre equation by the series solution method due to Frobenius.

## 1.2 Legendre's equation and its solution :

Consider the Legendre's differential equation $(1-x^2)\, y^{11} - 2xy^1 + n(n+1)\, y = 0$ - (1.1) where n is a non-negative integer. If we write this equation as $y^{11} + p(x)\, y^1 + Q(x)y = 0$, then $P(x) = \dfrac{-2x}{1-x^2}$, $Q(x) = \dfrac{n(n+1)}{1-x^2}$ and they are well defined at x = 0. Since P(x) and Q(x) are expressible in terms of power series about x = 0, we note that x = 0 is an ordinary point for the Legendre equation (1.1).

We can solve equation (1.1) in series either in ascending powers of x or in descending powers of x for -1<x<1. We shall now obtain the series solution in descending powers of x.

$$Let\ y = \sum_{r=0}^{\infty} a_r x^{k-r};\ a_0 \neq 0$$

$$\text{Then } y^1 = \frac{dy}{dx} = \sum_{r=0}^{\infty} (k-r)\, a_r x^{k-r-1}$$

$$y^{11} = \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (k-r)(k-r-1)\, a_r x^{k-r-2}$$

substituting these expressions in equation (1.1), we get

$$(1-x^2)\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)x^{k-r-2} - 2x \sum_{r=0}^{\infty} a_r (k-r)x^{k-r-1}$$

$$+ n(n+1) \sum_{r=0}^{\infty} a_r x^{k-r} = 0$$

$$\sum_{r=0}^{\infty} a_r [(k-r)(k-r-1)x^{k-r-2} + \{n(n+1)-(k-r)(k-r+1\}x^{k-r}] = 0$$

or

$$\sum_{r=0}^{\infty} a_r [(k-r)(k-r-1)x^{k-r-2} + \left\{n^2 (k-r)^2 + n - (k-r)^2\right\} x^{k-r}] = 0$$

or

$$\sum_{r=0}^{\infty} a_r [(k-r)(k-r-1)x^{k-r-2} + \{(n-k+r)(n+k-r+1\}x^{k-r}] = 0 \quad \text{...... (1.2)}$$

since equation (1.2) is an identity, we can equate the coefficients of different powers of x to zero.

Equating the coefficient of $x^k$, the highest power of x, to zero we get $a_0(n-k)(n+k+1) = 0$. This is called the indicial equation.

since $a_0 \neq 0$, $n - k = 0$ or $n + k + 1 = 0$.

If $n - k = 0$, then $n = k$.

If $n + k + 1 = 0$, then $k = -(n+1)$ ..... (1.3)

If we now equate the coefficient of $x^{k-1}$ to zero in equation (1.2), then $a_1(n-k+1)(n+k)=0$.

since $(n - k + 1) \neq 0$ and $(n + k) \neq 0$ by virtue of equation (1.3), we have $a_1 = 0$

Equating the coefficient of $x^{k-r}$ in equation (1.2) to zero, we get

$$a_{r-2}(k - r + 2)(k - r + 1) + (n - k + r)(n + k - r + 1)\,a_r = 0$$

$$\Rightarrow a_r = -\frac{(k - r + 2)(k - r + 1)}{(n - k + r)(n + k - r + 1)}\,a_{r-2} \quad \ldots\ldots\ldots (1.4)$$

putting r = 3 in equation (1.4) we get

$$a_3 = -\frac{(k - 1)(k - 2)}{(n - k + 3)(n + k - 2)}\,a_1 = 0, \text{ since } a_1 = 0$$

Similarly, putting r = 5, 7, ........ and noting $a_1 = a_3 = 0$ we find that $a_1 = a_3 = a_5 = \ldots\ldots = 0$ - (.15)

To obtain $a_2, a_4, a_6, \ldots\ldots$ we consider the following cases :

**Case 1 : k = n**

In this case, from equation (1.4), we have

$$a_r = -\frac{(n - r + 2)(n - r + 1)}{r(2n - r + 1)}\,a_{r-2}$$

we now put r = 2,4,6, ..... in succession to get

$$a_2 = -\frac{n(n-1)}{2(2n-1)}\,a_0$$

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}\,a_2 = \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}\,a_0$$

and so on,

Substituting these values of $a_n$'s together with the values of $a_n$'s given by equation (1.5) in the equation $y = \sum_{r=0}^{\infty} a_r x^{k-r}$

$$y = a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \ldots\ldots\right] \quad \ldots\ldots (1.6)$$

This forms one solution for the Legendre differential equation (1.1)

**Case 2 : k = - (n+1)**

In this case, from equation (1.4) we g:

$$a_r = -\frac{(n+r-1)(n+r)}{r(2n+r+1)} a_{r-2}$$

Putting r = 2, 4, 6, ..... in succession, we have

$$a_2 = -\frac{(n+1)(n+2)}{2(2n+3)} a_0$$

$$a_4 = -\frac{(n+3)(n+4)}{4(2n+5)} a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} a_0$$ and so on. In this case,

$$y = a_0 x^{-n-1} + a_2 x^{-n-3} + a_4 x^{-n-5} + ....$$

$$= a_0 \left[ x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + ... \right] \quad ... \quad \textbf{(1.7)}$$

This forms the other (second) solution for the Legendre's equation (1.1)

## 1.3 Legendre's function of the first kind and second kind :

The solutions of the Legendre equation are called the Legendre functions. We have two Legendre functions denoted by $P_n(x)$ and $Q_n(x)$ corresponding to the choice of $a_0$ and k.

If we choose $a_0 = \frac{1.3.5..........(2n-1)}{\angle n}$ and k = n, n is a positive integer,

then from equation (1.6) we have

$$P_n(x) = \frac{1.3.5..........(2n-1)}{\angle n} \left[ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + ....... \right]$$

$$\sum_{n=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$$

where, $N = \frac{n}{2}$ if n is even and $N = \frac{n-1}{2}$ if n is odd. In this case, $P_n(x)$ is a terminating series and gives Legendre polynomials for different values of n and $P_n(x)$ is called the Legendre function of the first kind.

If we choose $a_0 = \frac{n!}{1.3.5.......(2n+1)}$ and $k = -(n+1)$, n is a positive integer then the solution obtained from equation (1.6) gives the second solution denoted by $Q_n(x)$.

$$Q_n(x) = \frac{n!}{1.3.5............(2n+1)}\left[\begin{array}{l} x^{-n-1} + \dfrac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \\ \dfrac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} x^{-n-5} + ..... \end{array}\right] \quad ............ (1.9)$$

Since n is a positive integer, $Q_n(x)$ is a non-terminating series and is called the Legendre's function of the second kind. Since $Q_n(x)$ is a non-polynominal and $P_n(x)$ is a polynomial, we can conclude that $P_n(x)$ and $Q_n(x)$ are linearly independent and together form the two linearly independent solutions of Legendre equation (1.1). Hence the general solution of the equation (1.1) is

$y = C_1 P_n(x) + C_2 Q_n(x)$, where $C_1$ and $C_2$ are arbitrary constants

## 1.4 Generating function for Legendre's polynomials :

We now show that $P_n(x)$ is the coefficient of $h^n$ in the expansion of $(1 - 2xh + h^2)^{-1/2}$ in ascending powers of h when $|x| \leq 1$ and $|h| < 1$

We have the binomial expansion for $|t| < 1$,

$$(1-t)^{-\frac{1}{2}} = 1 + \frac{1}{2}t + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^2 + ..........$$

$$= = 1 + \frac{1}{2}t + \frac{1.3}{2.4}t^2 + \frac{1.3.5.}{2.4.6}t^3 + .......$$

since $|x| \leq 1$ and $|h| < 1$ we have

$$(1 - 2xh + h^2)^{-\frac{1}{2}} = [1 - (2xh - h^2)]^{-\frac{1}{2}}$$

$$= 1+\frac{1}{2}(2xh-h^2)+\frac{1.3}{2.4}\left(2xh-h^2\right)^2+.........+\frac{1.3......(2k-1)}{2.4.........2k}(2xh-h^2)^k+.........$$

$$= 1+\sum_{k=1}^{\infty}\frac{1.3.........(2k-1)}{2.4.........2k}(2xh-h^2)^k \quad ........(1.10)$$

Since $(2xh - h^2)^k = h^k(2x-h)^k$

$$= \quad h^k\left[(2x)^k - k(2x)^{k-1}h+\frac{k(k-1)}{2!}(2x)^{k-2}h^2+.......+(-1)^k h^k\right],$$

We have from equation (1.10)

$$(1-2xh-h^2)^{-1/2} = [\,1+\sum_{k=1}^{\infty}\frac{1.3......(2k-1)}{2.4......2k}+(2x)^k - k(2x)^{k-1}h$$

$$+\frac{k(k-1)}{2!}(2x)^{k-2}h^2+....+(-1)^k h^k\,] \;....... \textbf{(1.11)}$$

The coefficient of $h^n$ in the RHS of equation (1.11) is

$$\frac{1.3.....(2n-1)}{2.4....(2n)}(2x)^n-\frac{1.3.....(2n-3)}{2.4....(2n-2)}(n-1)(2x)^{n-2}$$

$$+\frac{1.3.....(2n-5)}{2.4....(2n-4)}(n-2)(n-3)(2x)^{n-4}+......$$

$$= \frac{1.3.....(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}+.......\right]=p_n(x)$$

We observe here that $P_1(x)$, $P_2(x)$, ..... are the coefficients of $h$, $h^2$, ...... in the expansion of $(1 - 2xh + h^2)^{-1/2}$ given by (1.10). Hence $(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n p_n(x)$

In fact we note that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = 1/2(3x^2-1)$, $P_3(x) = 1/2(5x^3-3x)$, $P_4(x) = 1/8\,(35x^4-30x^2+3)$. etc are first few Legendre polynomials. The function $(1 - 2xh + h^2)^{-1/2}$ is called the generating function of the Legendre polynomials. This is useful in obtaining the recurrance relations and in evaluating integrals involving $P_n(x)$ in the subsequent discussion.

## 1.5 Examples :

**1.5.1. :Example :** Show that $P_n(1)=1$ and $P_n(-x)=(-1)^n P_n(x)$. Hence deduce that $P_n(-1)=(-1)^n$

**Solution :**

From generating function, we have for $|x| \le 1, |h| < 1$

$$\sum_{n=0}^{\infty} h^n p_n(x) = (1 - 2xh + h^2)^{-1/2}.$$

putting x = 1, we obtain $\sum_{n=0}^{\infty} h^n p_n(1) = (1 - 2h + h^2)^{-1/2}$

$$= (1-h)^{-1} = 1 + h + h^2 + .... + h^n + ....$$

$$= \sum_{n=0}^{\infty} h^n$$

Equating the coefficient of $h^n$ on both sides of the above equation, we have $p_n(1)=1$.

Also $(1 + 2xh + h^2)^{-1/2} = [1 - 2x(-h) + (-h)^2]^{-1/2}$

$$= \sum_{n=0}^{\infty} (-h)^n P_n(x)$$

$$= \sum_{n=0}^{\infty} (-1)^n h^n P_n(x) \qquad ...... (1.12)$$

Again $(1 + 2xh + h^2)^{-1/2} = [1 - 2(-x)h + h^2]^{-1/2}$

$$= \sum_{n=0}^{\infty} h^n p_n(-x) \qquad ....... (1.13)$$

From equations (1.12) and (1.13) we have

$$\sum_{n=0}^{\infty} (-1)^n h^n P_n(x) = \sum_{n=0}^{\infty} h^n p_n(-x)$$

Equating the coefficient of $h^n$ on both sides of the equation we get,

$$P_n(-x) = (-1)^n P_n(x) \quad \text{....} \quad \textbf{(1.14)}$$

**Deduction :** Put x = 1 in equation (1.14) we get $P_n(-1) = (-1)^n$

**1.5.2. Example :**

Show that $P_n(0) = 0$, for n odd and $P_n(0) = \dfrac{(-1)^{n/2} n!}{2^n [(n/2)!]^2}$ for n even.

**Solution :**

We know that,

$$P_n(x) = \frac{1.3.5..........(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + .......\right]$$

When n is odd i.e., n = (2m + 1), we have

$$P_{2m+1}(x) = \frac{1.3.5..........[2(2m+1)-1)}{(2m+1)!}\left[x^{2m+1} - \frac{(2m+1)(2m+1-1)}{2\{2(2m+1)-1)\}}x^{2m+1-2} + .......\right]$$

Putting x = 0, we get $P_{2m+1}(0) = 0$

i.e., $P_n(0) = 0$ when n is odd

Also, we have $\sum_{n=0}^{\infty} h^n p_n(x) = (1 - 2xh + h^2)^{-1/2}$

$$\sum_{n=0}^{\infty} h^n p_n(0) = (1+h^2)^{-1/2} = [1 - (-h)^2]^{-1/2}$$

$$= 1 + \frac{1}{2}(-h)^2 + \frac{1.3}{2.4}(-h^2)^2 + \frac{1.3.5.}{2.4.6.}(-h^2)^3 + ..... + \frac{1.3.5....(2r-1)}{2..4....2r}(-h^2)^r + ....$$

It can be seen that all powers of h on the R.H.S. of the above equation are even. Hence equating the coefficient of $h^{2m}$ on both sides we get :

$$P_{2m}(0) = \frac{1.3.5....(2m-1)}{2.4.6.......2m}(-1)^m = \frac{(-1)^m(2m)!}{2^{2m}(m!)^2}$$

When n is even, i.e, n - 2m, then $P_n(0) = \dfrac{(-1)^{n/2}n!}{2^n[(n/2)!]^2}$

**1.5.3. Example :** $-1 \leq P_n(\cos\theta) \leq 1$

**Solution :** From the generating function of the Legendre's polynomial, we have

$$\sum_{n=0}^{\infty} Z^n p_n(x) = (1-2xZ+Z^2)^{-1/2}$$

Substituting $\cos\theta$ for x we get

$$\begin{aligned}\sum_{n=0}^{\infty} Z^n p_n(\cos\theta) &= (1-2(\cos\theta)Z+Z^2)^{-1/2}\\ &= [1-Z(e^{i\theta}+e^{-i\theta})+Z^2]^{-1/2}\\ &= [(1-Ze^{i\theta})(1-Ze^{-i\theta})]^{-1/2}\end{aligned}$$

which attains maximum when $\theta = 0$

Hence $\sum_{n=0}^{\infty} z^n p_n(1) = [(1-Z)(1-Z)]^{-1/2} = (1-Z)^{-1} = 1 + Z + Z^2 + ........ + Z^n + .......$

Equating the coefficient of $Z^n$ on both sides we get $P_n(1)=1$. Hence maximum value of $P_n(\cos\theta)=1$ Similary we can show that minimum value of $P_n(\cos\theta) = -1$. Hence the result.

**S.A.Qs :**

1. Prove that $P^{1}_{n}(1) = \dfrac{n(n+1)}{2}$

2. Prove that $P^{1}_{n}(-1) = (-1)^{n-1}\dfrac{n(n+1)}{2}$

3. Prove that $P_{2n}(0) = \dfrac{(-1)^{n}(2n)!}{2^{2n}(n!)^{2}}$.

## 1.6 SUMMARY :

In this lesson, the Legendre's differential equation was solved by the Frobenius series solution method. When n is an integer, one of the series solution has finitely many terms only. These solutions are known as Legendre polynomials. The other series solution is non-terminating. As such these two solutions are linearly independent and thus form a basis for the solution of the Legendre's equation. By choosing $a_0 = \dfrac{1.3.5....(2n-1)}{n!}$ and k = n, we obtained $P_n(x)$ called the Legendre function of the first kind. By choosing $a_0 = \dfrac{n!}{1.3.5.....(2n+1)}$ and k = -(n+1), we obtained $Q_n(x)$, called the Legendre function of second kind. The generating function for Legendre's polynomial was derived and this is useful for obtaining recursions and for evaluating integrals involving $P_n(x)$.

## 1.7 Model Examination questions :

1. Define Legendre's polynomial $P_n(x)$. Show that it satisfies the equation $(1-x^2)\, y^{11} - 2xy^{1} + n(n+1)\, y=0$.

2. Prove that $1+\dfrac{1}{2}P_1(\cos\theta)+\dfrac{1}{3}P_2(\cos\theta)+........ = \log\dfrac{(1+\sin\frac{\theta}{2})}{\sin(\frac{\theta}{2})}$ , $\theta \neq 2n\pi$

3. Prove that $\dfrac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n x$

4. Show that $\int_0^{\pi} P_n(\cos\theta)\cos n\theta = \dfrac{1.3.5.....(2n-1)}{2.4.......2n}$

5. P.T. $P_n(-\frac{1}{2}) = P_0(-\frac{1}{2})P_{2n}(\frac{1}{2})P_{2n}(\frac{1}{2}) + P_1(-\frac{1}{2})P_{2n-1}(\frac{1}{2}) + ........ + P_{2n}(-\frac{1}{2})P_0(\frac{1}{2})$

6. If n is a positive integer, show that $\int_0^{\pi} Pn(\cos\theta)\cos n\theta d\theta = \beta(n+\frac{1}{2}+\frac{1}{2})$

7. Show that the general solution of Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, where n is a positive integer is $y = A\,P_n(x) + B\,Q_n(x)$ where $P_n(x)$ and $Q_n(x)$ stand for their usual meanings

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# LESSON - 2 : LEGENDRE'S EQUATION

## 2.1 INTRODUCTION :

In the earlier lesson we have considered the Legendre's differential equation and obtained a series solution for it using the method of Frobenius. The solution of the Legendre equation denoted by $P_n(x)$ and $Q_n(x)$ were called the Legendre functions. In this lesson we shall derive the Laplaces first and second integral for $P_n(x)$ and establish the orthogonality property of Legendre polynomials. We shall also derive the Legendre series for f(x) when f is a polynomial and expand f(x) in a series of Legendre polynomials.

## 2.2 Laplaces first integral for $P_n(x)$ :

**Proposition :** If n is a positive integer, then $P_n(x) = \frac{1}{\pi}\int_0^\pi \left[x \pm \sqrt{x^2-1}.\cos\phi\right]^n d\phi$

**Proof :** We know from elementary calculus that ,

$$\int_0^\pi \left[\frac{1}{a \pm b\cos\phi}\right] d\phi = \frac{\pi}{\sqrt{a^2-b^2}}, \text{ where } a^2 > b^2 \;\;\; ......... (2.1)$$

Let $a = 1 - Zx$ and $b = Z\sqrt{x^2-1}$

Then $a^2 - b^2 = (1-Zx)^2 - Z^2(x^2-1) = 1 - 2xZ + Z^2$

Substituting these values in equation (2.1) and rearranging terms we get :

$$\pi(1 - 2xZ + Z^2)^{-1/2} = \int_0^\pi \left[(1-Zx) \pm Z\sqrt{x^2-1}\cos\phi\right]^{-1} d\phi$$

$$\text{or } \pi \sum_{n=0}^{\infty} Z^n P_n(x) = \int_0^\pi \left[(1-Zt)^{-1} d\phi\right] \text{ where } t = x \pm \sqrt{x^2-1}\cos\phi$$

$$\text{i.e., } \pi \sum_{n=0}^{\infty} Z^n P_n(x) = \int_0^\pi (1 + Zt + Z^2t^2 + ......)\, d\phi$$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} (Zt)^n d\phi = \sum_{n=0}^{\infty} Z^n \int_0^{\pi} t^n d\phi$$

$$\text{or } \pi \sum_{n=0}^{\infty} Z^n P_n(x) = \sum_{n=0}^{\infty} Z^n \int_0^{\pi} \left[ x \pm \sqrt{x^2-1}\cos\phi \right]^n d\phi$$

Equating the coefficient of $Z^n$ on both sides of this equation, we get

$$\pi P_n(x) = \int_0^{\pi} \left[ x \pm \sqrt{x^2-1}\cos\phi \right]^n d\phi$$

or

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[ x \pm \sqrt{x^2-1}\cos\phi \right]^n d\phi$$

Hence the result.

## 2.3 . Laplaces second integral for $P_n(x)$

**Proposition :** If n is a positive integer, Then

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{1}{\left[ x \pm \sqrt{x^2-1}\cos\phi \right]^{n+1}} d\phi \quad \ldots\ldots \quad (2.2)$$

**Proof :** From integral calculus, we know that

$$\int_0^{\pi} \frac{1}{a \pm b\cos\phi} d\phi = \frac{\pi}{\sqrt{a^2-b^2}}, \text{ if } a^2 > b^2 \quad \ldots\ldots \quad (2.2a)$$

Let a = Zx-1, and b = $Z\sqrt{x^2-1}$. Then $a^2-b^2 = (Zx-1)^2 - Z^2(x^2-1) = 1 - 2xZ + Z^2$

Substituting these values in equation (2.2a) we get

$$\pi\left(1-2xZ+Z^2\right)^{-1/2} = \int_0^{\pi} \left[ -1 + Zx \pm Z\sqrt{x^2-1}\cos\phi \right]^{-1} d\phi$$

or $\quad \dfrac{\pi}{Z}\left(1-2x\dfrac{1}{Z}+\dfrac{1}{Z^2}\right)^{-1/2} = \int_0^{\pi}\left[-1+Z\left\{x\pm\sqrt{x^2-1}\cos\phi\right\}\right]^{-1} d\phi$

$$= \int_0^{\pi}[-1+Z\omega]^{-1} d\phi \text{ where } \omega = x\pm\sqrt{x^2-1}\cos\phi$$

we know that $(1 - 2xZ + Z^2)^{-1/2} = \sum_{n=0}^{\infty} Z^n P_n(x)$

Hence $\dfrac{\pi}{Z}\sum_{n=0}^{\infty}\dfrac{1}{Z^n}P_n(x) = \int_0^{\pi}(-1+Z\omega)^{-1} d\phi$

$$= \int_0^{\pi}(Z\omega)^{-1}\left(1-\frac{1}{Z\omega}\right)^{-1} d\phi$$

$$= \int_0^{\pi}\frac{1}{Z\omega}\sum_{n=0}^{\infty}\left(\frac{1}{Z\omega}\right)^n d\phi$$

$$= \sum_{n=0}^{\infty}\frac{1}{Z^{n+1}}\int_0^{\pi}\frac{d\phi}{\omega^{n+1}}$$

Hence $\displaystyle\sum_{n=0}^{\infty}\frac{\pi}{Z^{n+1}}P_n(x) = \sum_{n=0}^{\infty}\frac{1}{Z^{n+1}}\int_0^{\pi}\frac{1}{\left[x\pm\sqrt{x^2-1}\cos\phi\right]^{n+1}} d\phi$

Equating the coefficient of $\dfrac{1}{Z^{n+1}}$ on both sides of this equation, we get,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{1}{\left[x\pm\sqrt{x^2-1}\cos\phi\right]^{n+1}} d\phi$$

**Note :** In equation (2.2) if we replace n by -(n+1) We get

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi} \frac{1}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{-n}} d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} (x \pm \sqrt{x^2-1}\cos\phi)^n d\phi$$

$= P_n(x)$ by Laplaces first integral.

Hence $P_n(x) = P_{-(n+1)}(x)$

## 2.4 Orthogonal properties of Legendre Polynomials

**Theorem :** Let m, n be integers. Then

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \text{ if } m \neq n$$

$$= \frac{2}{2n+1}, \text{ if } m = n$$

This result can also be stated as $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$ where $\delta_{mn}$ is the Kronecker delta defined by :

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

**Proof :**

**Case 1 :** $m \neq n$

Since $P_m(x)$ and $P_n(x)$ satisfy the Legendre's differential equation, we have

$$(1-x^2)P_m'' - 2x P_m' + m(m+1) P_m = 0 \qquad (2.3)$$

$$(1-x^2)P_n'' - 2x P_n' + n(n+1) P_n = 0 \quad ..... \quad (2.4)$$

Multiplying equation (2.3) by $P_n(x)$, equation (2.4) by $P_n(x)$ and subtracting the latter from the former,

$$(1-x^2)[P_n(x) P''_m(x) - P_m(x) P_n''(x)] - 2x [P_n(x) P'_m(x) - P_m(x)P'_n(x)] + [m(m+1) - n(n+1)] P_m(x)P_n(x) = 0$$

or $$(1-x^2) \frac{d}{dx}[P_n(x) P'_m(x) - P'_n(x) P_m(x)] - 2x [P_n(x) P'_m(x) - P_m(x) P'_n(x)] = [n(n+1) - m(m+1)] P_m(x) P_n(x)$$

i.e., $$\frac{d}{dx}(1-x^2)[P_n(x) P_m'(x) - P_n'(x) P_m(x)] = (n-m)(n+m+1) P_m(x) P_n(x)$$

Integrating both sides of this equation w.r.t. x between -1 and +1 we get,

$$(n-m)(n+m+1) \int_{-1}^{1} P_m(x)P_n(x)dx = [(1-x^2)\{P_n(x) P_m'(x) - P_m(x) P'_n(x)]'_{-1} = 0$$

Hence $$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \text{ if } m \neq n$$

<u>**Case 2 :**</u> Let m = n. In this case we show that

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

We have from the generating function :

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$$

$$(1 - 2xh + h^2)^{-1/2} = \sum_{m=0}^{\infty} h^m P_m(x)$$

From these two equations we get,

$$(1-2xh+h^2)^{-1} = \sum_{n=0}^{\infty} h^{2n}[P_n(x)]^2 + 2\sum_{\substack{n=0,\\ m=0\\ n\neq m}}^{\infty} h^{m+n}.P_m(x)P_n(x)$$

Integrating both sides of this equations w.r.t. x between -1 and +1 we get.

$$(1-2xh+h^2)^{-1}\,dx = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\int_{-1}^{1} P_m(x)P_n(x)h^{m+n}dx\right]$$

For m = n, this equation reduces to

$$\sum_{n=0}^{\infty}\int_{-1}^{1} h^{2n}[P_n(x)]^2 dx = \int_{-1}^{1}\frac{1}{(1-2xh+h^2)}dx$$

$$= \frac{-1}{2h}\log\left[\frac{1+h^2-2xh}{1}\right]_{-1}^{1}$$

$$= \frac{-1}{2h}\left[\log(1-h)^2-\log(1+h)^2\right]$$

$$= \frac{1}{h}\,[\log(1+h)-\log(1-h)]$$

$$= \frac{1}{h}\left[(h-\frac{h^2}{2}+\frac{h^3}{3}-\ldots\ldots.)-(-h-\frac{h^2}{2}-\frac{h^3}{3}-\ldots\ldots.)\right]$$

$$= \frac{2}{h}\left[(h+\frac{h^3}{3}+\frac{h^5}{5}+\ldots\ldots.)\right]$$

$$= \frac{2}{h}\sum_{n=0}^{\infty}\frac{h^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{2}{2n+1}h^{2n}$$

Equating the coefficient of $h^{2n}$ on both sides of this equation we get,

$$\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1}$$

## 2.5 Legendre series for f(x) when f is a polynomial :

**Theorem :** If f is a polynomial of degree n, then

$$f(x) = \Sigma C_k P_k(x) \text{ where } C_k = (k+\frac{1}{2}) \int_{-1}^{1} f(x) P_k(x)\, dx$$

**Proof :** Since f is a polynomial of degree n, we can write $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots. a_1 x + a_0;\ a_n \neq 0$

since $P_n(x)$, is also a polynomial of degree n,

Let $P_n(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots..+ b_1 x + b_0,\ b_n \neq 0$

We shall now consider $f(x) - \left(\frac{a_n}{b_n}\right) P_n(x)$ ......... (2.5)

If $f(x) - \left(\frac{a_n}{b_n}\right) P_n(x) = 0$, then $f(x) = \left(\frac{a_n}{b_n}\right) P_n(x)$ and the result follows from the orthogonality relation.

If $f(x) - \left(\frac{a_n}{b_n}\right) P_n(x) \neq 0$, then let it be equal to $g_{n-1}(x)$,

Where $g_{n-1}$ is a polynomial of degree n-1.

In this case let $C_n = \frac{a_n}{b_n}$. So that we can write $f(x) = C_n P_n(x) + g_{n-1}(x)$

If we take $g_{n-1}(x)$ in the place of f(x) in equation (2.6) and proceed as above, we obtain

$$g_{n-1}(x) = C_{n-1}P_{n-1}(x) + g_{n-2}(x) \qquad \text{........ (2.7)}$$

using equation (2.7) in equation (2.6) we get $f(x) = C_n P_n(x) + C_{n-1}P_{n-1}(x) + g_{n-2}(x)$
continuing this procedure recursively, finally we obtain $f(x) = C_n P_n(x) + C_{n-1}P_{n-1}(x) + ....... + C_0P_0(x)$

$$\text{or } f(x) = \sum_{k=0}^{n} C_k P_k(x) = \sum_{s=0}^{n} C_s P_s(x) \qquad \text{........(2.8)}$$

Multiplying equation (2.8) by $P_k(x)$ and then integrating w.r.t. x from -1 to +1, we get

$$\int_{-1}^{1} f(x)P_k(x)dx = \sum_{s=0}^{n} C_s \int_{-1}^{1} P_s(x)P_k(x)dx \qquad \text{.........(2.9)}$$

By the orthogonality relation of Legendre polynomials we know that

$$\int_{-1}^{1} P_s(x)P_k(x)dx = \frac{2}{2k+1}\delta_{sk}$$

Hence equation (2.9) reduces to $\int_{-1}^{1} f(x)P_k(x)dx = C_k\left(\frac{2}{2k+1}\right)$

$$\text{or } C_k = (k+\frac{1}{2})\int_{-1}^{1} f(x)P_k(x)dx \qquad \text{.......(2.10)}$$

**Special case :**

(i) Suppose f is an even function. We know that $P_k(x)$ is even when k is even and is odd when k is odd. As such f(x) $P_k(x)$ is even when k is even and is odd when k is odd.

Hence if k is odd, equation (2.10) gives $C_k=0$, since $\int_{-a}^{a} G(x)\,dx = 0$ if G is odd.

If k is even, only those $C_k$ with even suffixes remain or survive.

(ii) Suppose f is an odd function. In this case, by following a similar argument, it can be shown that only those $C_k$'s with odd suffixes survive.

## 2.6 Expansion of f(x) in a series of Legendre polynomials :

Let us assume that the given function f can be expanded in a series of Legendre polynomials

as $$f(x) = \sum_{k=0}^{\infty} C_k P_k(x) \qquad \ldots\ldots. \quad (2.11)$$

Where $C_0, C_1, C_2, \ldots.. C_s$ are constants. Multiplying both sides of equation (2.11) by $P_r(x)$ and integrating both side w.r.t. x from -1 to +1 we get

$$\int_{-1}^{1} f(x) P_r(x)\, dx = \sum_{r=0}^{\infty} C_k \int_{-1}^{1} P_k(x) P_r(x) dx \qquad \ldots\ldots \quad (2.12)$$

But by orthogonality property, we have

$$\int_{-1}^{1} P_k(x) P_r(x) dx = \frac{2}{2r+1} \delta_{kr}$$

Where $\delta_{kr} = 0$, if $k \neq r$

$= 1$, if $k = r$

Using this result in equation (2.12) we get ,

$$\int_{-1}^{1} f(x) P_r(x)\, dx = C_r \left( \frac{2}{2r+1} \right)$$

$$\Rightarrow c_r = (r + \frac{1}{2}) \int_{-1}^{1} f(x) P_r(x) dx \qquad \ldots\ldots \quad (2.13)$$

## 2.7 Examples :

**2.7.1 Example :** Expand $f(x) = x^2$ in terms of the Legendre polynomials $P_0(x), P_1(x)\ P_2(x)$, etc

**Solution :** Since $x^2$ is a polynomial of degree 2, from the Legendre series, we have

$$x^2 = \sum_{r=0}^{2} C_r P_r(x) = C_0P_0(x) + C_1P_1(x) + C_2P_2(x)$$

$$\text{where } C_r = \left(r+\frac{1}{2}\right)\int_{-1}^{1} x^2 P_r(x)dx \quad \ldots\ldots\ (2.14)$$

We know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = 1/2\ (3x^2-1)$ etc. Now, putting $r = 0, 1, 2$ successively in equation (2.14) we have :

$$C_0 = \frac{1}{2}\int_{-1}^{1} x^2 dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{3}$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} x^3 dx = \frac{3}{2}\left(\frac{x^4}{4}\right)_{-1}^{1} = 0$$

$$C_2 = \frac{5}{2}\int_{-1}^{1} \frac{x^2(3x^2-1)}{2}dx = \frac{5}{4}\left(\frac{3x^5}{5}-\frac{x^3}{3}\right)_{-1}^{1} = \frac{2}{3}$$

Substituting these values of $C_0$, $C_1$, $C_2$, in equation (2.14) we get

$$x^2 = \frac{1}{3}P_0(x) + 0 + \frac{2}{3}P_2(x) = \frac{1}{3}P_0(x) + \frac{2}{3}P_2(x)$$

**2.7.2. Example :** Expand $f(x) = |x|$ for $-1 \le x \le 1$ in a series of Legendre Polynomials

**Solution :** We have shown that f(x) can be expanded in a series of Legendre Polynomials as

$$f(x) = \sum_{r=0}^{\infty} C_r P_r(x) \quad \ldots\ \mathbf{(2.15)}$$

$$\text{where } C_r = \left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_r(x)$$

$$\text{e} \quad C_r = \left(r+\frac{1}{2}\right)\left[\int_{-1}^{0} f(x)P_r(x)dx + \int_{0}^{1} f(x)P_r(x)dx\right] \quad \ldots\ldots \quad (2.15)$$

Putting r = 0, 1, 2, 3, ....... in succession in equation (2.15) we get,

$$C_0 = \frac{1}{2}\left[\int_{-1}^{0}(-x)P_0(x)dx + \int_{0}^{1} xP_0(x)dx\right]$$

$$= \frac{1}{2}\left[-\int_{-1}^{0} x\,dx + \int_{0}^{1} x\,dx\right]$$

$$= \frac{1}{2}\left[-\left[\frac{x^2}{2}\right]_{1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}\right] = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{1}{2}$$

$$C_1 = \frac{3}{2}\left[-\int_{-1}^{0} x^2 dx + \int_{0}^{1} x^2 dx\right]$$

$$= \frac{3}{2}\left[-\left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{0}^{1}\right] = \frac{3}{2}\left(-\frac{1}{3}+\frac{1}{3}\right) = 0$$

$$C_2 = \frac{5}{2}\left[-\int_{-1}^{0} x\left(\frac{3x^2-1}{2}\right)dx + \int_{0}^{1} x\left(\frac{3x^2-1}{2}\right)dx\right]$$

$$= \frac{5}{2}\left[-\int_{-1}^{0} d\left(\frac{3x^3-x}{2}\right)dx + \int_{0}^{1} d\left(\frac{3x^3-x}{2}\right)dx\right]$$

$$= \frac{5}{4}\left[-\left(\frac{3x^4}{4}-\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{3x^4}{4}-\frac{x^2}{2}\right)_{0}^{1}\right] = \frac{5}{4}\left(\frac{3}{4}-\frac{1}{2}+\frac{3}{4}-\frac{1}{2}\right) = \frac{5}{8}$$

$$C_3 = \frac{7}{2}\left[\int_{-1}^{0}(-x)\left(\frac{5x^3-3x}{2}\right)dx + \int_{0}^{1} x\left(\frac{5x^3-3x}{2}\right)dx\right]$$

$$= \frac{7}{2}\left[-\int_{-1}^{0}\left(\frac{5x^4-3x^2}{2}\right)dx + \int_{0}^{1}\left(\frac{5x^4-3x^2}{2}\right)dx\right]$$

$$= \frac{7}{4}\left[-(x^5-x^3)_{-1}^{0} + (x^5-x^3)_{-1}^{0}\right] = 0$$

Similarly we can calculate $C_4$, $C_5$, .... and so on. Subsituting these values of C's in equation (2.15) we get $|x| = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x) + \ldots\ldots.$

**2.7.3 Example :** Show that $P_n(\cos\theta) = \frac{1}{\pi}\int_{0}^{\pi}(\cos\theta + i\sin\theta\cos\theta)^n d\theta$

Deduce that $P_1(x) = \frac{1}{\pi}\int_{0}^{\pi}\left\{x + \sqrt{x^2-1}\cos\theta\right\}^n d\theta$

**Solution :**

Let $x = \cos\theta$. Then $\sqrt{x^2-1} = \sqrt{\cos^2\theta - 1} = \sqrt{-(1-\cos^2\theta)} = i\sqrt{1-\cos^2\theta} = i\sin\theta$.

From the Laplaces first integral for $P_n(x)$ with positive sign we have

$$P_n(\cos\theta) = \frac{1}{\pi}\int_{0}^{\pi}(\cos\theta + i\sin\theta\cos\theta)^n d\theta$$

We now put n = 1 and take the positive sign in Laplaces first integral so that

$$P_1(x) = \frac{1}{\pi}\int_{0}^{\pi}\left\{x + \sqrt{x^2-1}\cos\theta\right\}d\theta$$

## SAQ s :

1. If f(x) is a Polynomial of degree less than k, prove that $\int_{-1}^{1} f(x)P_k(x)dx = 0$.

2. Expand $(x^4 - 3x^2 + x)$ in a series of the form $\Sigma C_r P_r(x)$

(Ans : $x^4 - 3x^2 + x = \frac{-4}{5}P_0(x) + P_1(x) - \frac{10}{7}P_2(x) + \frac{8}{35}P_4(x)$)

3. Expand $f(x) = 2x + 1, \ 0 < x \leq 1$
$= 0, \ -1 \leq x < 0$ in a series of Legendre polynomials

(Ans : $f(x) = P_0(x) + \frac{7}{14}P_1(x) + \frac{5}{8}P_2(x) - \frac{7}{16}P_n(x) + ....$)

## 2.8 Summary :

In this lesson, we have obtained the Laplaces first and the second integral for $P_n(x)$ when n is a positive integer and established the orthogonality properties of Legendre polynomials. We also derived an expansion of f(x) in a series of Legendre Polynomials and exemplified the same for $f(x) = x^2$ and $f(x) = |x|$, for $-1 \leq x \leq 1$.

## 2.9 Model Examination questions :

1. Expand f(x) in a series of Legendre polynomials of

(a) $f(x) = 1/2, \quad 0 < x < 1$
$= -1/2, \quad -1 < x < 1$

(Ans : $C_r = 0$, if r is even $C_r = \dfrac{(-1)^{r-\frac{1}{2}}(r+\frac{1}{2})(r-1)!}{2^r\left(\frac{r+1}{2}\right)!\left(\frac{r-1}{2}\right)!}$, if r is odd )

(b) $f(x) = x, \quad 0 < x < 1$
$= 0, \quad -1 < x < 0$

(Ans : $f(x) = \frac{1}{4} + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + \ldots\ldots\ldots + C_r P_r(x) + \ldots\ldots,$

where $C_r = \left(r + \frac{1}{2}\right)\int_0^1 x P_r(x)\,dx$).

2. Obtain the first three terms in the expansion of

$f(x) = 0$, if $-1 < x < 0$

$= x$, if $0 < x < 1$ in terms of Legendre's polynomials

(Ans : $f(x) = -\frac{1}{4}P_0(x) + \frac{1}{6}P_1(x) + \frac{1}{16}P_2(x)$)

3. If $f(x) = \sum_{n=0}^{\infty} C_r P_r(x)$, obtain the Parsval's identity : $\int_{-1}^{1} [f(x)]^2\,dx = \sum_{r=0}^{\infty} \frac{C_r^2}{(2r+1)}$

Illustrate this case, by taking $f(x) = x^4 - 3x^2 + x$.

# LESSON 3 LEGENDRE EQUATION -3
# RECURRENCE RELATIONS AND RODRIGUE'S FORMULA

## 3.1 INTRODUCTION :-

In the preceding discussion, we obtained two linearly independent solutions for the legendre differential equation which are termed as the Legendre function of the first kind ($P_n(x)$) and the legendre function of the second kind ($Q_n(x)$). Interms of these polynomials (solutions), we now establish a few recurrence relations and derive the Rodrigue's formula for $P_n(x)$ which serves as a representation for the Legendre polynomial.

## 3.2 Recurrence relations :-

We shall now establish some recurrence relations involving Legendre Polynomials :

**3.2.1 Prove that $(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$,** for $n \geq 1$.

Proof :- We have from the generating function

$$\left(1-2xh+h^2\right)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad \text{.........(3.1)}$$

Differentiating both sides w.r.t. h, we get

$$\frac{-1}{2}\left(1-2xh+h^2\right)^{-3/2}(-2x+2h) = \sum_{n=1}^{\infty} nh^{n-1} P_n(x)$$

$$i.e.,\ (x-h)\left(1-2xh+h^2\right)^{-1/2} = (1-2xh+h^2) = \sum_{n=1}^{\infty} nh_1^{n-1} P_n(x)$$

$$or\ (x-h)\sum_{n=1}^{\infty} h^n P_n(x) = \left(1-2xh+h^2\right)\sum_{n=1}^{\infty} nh_1^{n-1} P_n(x)$$

$$\text{i.e. } (x-h)\,[P_0(x)+hP_1(x)+.......+h^n P_n(x)..........] =$$

$$\left(1-2xh+h^2\right)[P_1(x)+2hP_2(x)+.......+n\,h^{n-1}P_n(x)+(n+1)h^n+P_{n+1}(x)+...]$$

Equating the co-efficient of $h^n$ on both sides of this equation we get.

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

or $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

Hence the result.

**3.2.2.** Prove that $nP_n(x) = x\,P^1_n(x) - P^1_{n-1}(x)$

**Proof :** Consider the equation (3.1) and differentiate both sides w.r.t. h. Then

$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} n\,h^{n-1}\,P_n(x) \quad ....(3.2)$$

Differentiating equation (3.1) w.r.t. X we get

$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P^1_n(x)$$

or $$h(x-h)(1-2xh+h^2)^{-3/2} = (x-h)\sum_{n=0}^{\infty} h^n P^1_n(x) \quad ... (3.3)$$

From equations **(3.2)** and **(3.3)** we have

$$h\sum_{n=0}^{\infty} nh^{n-1}P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P^1_n(x)$$

i.e. $h\left[P_1(x) + 2hP_2(x) + ........ + h^{n-1}nP_n(x)\right] =$

$$(x-h)\left[P^1_0(x) + hP^1_1(x) + ....... + h^n P^1_n(x) + .....\right]$$

Equating the co-efficient of $h^n$ on both sides of this equation, we obtain

$$nP_n(x) = x\,P_n{}^1(x) - P_{n-1}{}^1(x)$$

Hence the result.

**3.2.3 Prove that** $(2n+1)\ P_n(x) = P^1_{n+1}(x) - P^1_{n-1}(x)$

**Proof :-** From the recurrence equation --------- (3.2.1) we have

$$(2n+1)\, x\, P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating both sides of this equation w.r.t. x

$$(2n+1)x\, P_n^1(x) + (2n+1)P_n(x) = (n+1)\, P^1_{n+1}(x) + n.P^1_{n-1}(x) \quad \ldots\ldots \quad (3.4)$$

From the recurrence relation (3.2.2) we have

$$x\, P_n^{\,1}(x) = n.P_n(x) + P^1_{n-1}(x) \quad ---- \quad (3.5)$$

Eliminating $xP^1_n(x)$ from equations (3.4) and (3.5) we get

$$(2n+1)\,[nP_n(x) + P^1_{n-1}(x)] + (2n+1)\, P_n(x) = (n+1)\, P^1_{n+1}(x) + P^1_{n-1}(x)$$

or $$(2n+1)(n+1)P_n(x) = (n+1)P^1_{n+1}(x) + nP^1_{n-1}(x) - (2n+1)P^1_{n-1}(x)$$

i.e. $$(2n+1)\,(n+1)\, P_n(x) = (n+1)\, P^1_{n+1}(x) - (n+1)\, P^1_{n-1}(x)$$

or $$(2n+1)P_n(x) = P^1_{n+1}(x) - P^1_{n-1}(x)$$

Hence the result

**3.2.4. Prove that** $(n+1)P_n(x) = P^1_{n+1} - xP^1_n(x)$

**Proof :-** Let us consider the recurrence relations (3.2.2) and (3.2.3.)

$$n\, P_n(x) = x\, P_n^{\,1}(x) - P_{n-1}^{\,1}(x)$$
$$(2n+1)P_n(x) = P^1_{n+1}(x) - P^1_{n-1}(x)$$

Subtracting the first equation from the second one we get

$$(n+1)P_n(x) = P^1_{n+1} - xP^1_n(x)$$

Which is the required relation

**3.2.5 Prove that** $\left(1-x^2\right)P^1_n(x) = n(P_{n-1}(x) - xP_n(x))$

**Proof :-** Replacing n by (n-1) in the recurrence relation (**3.2.4)** we get

$$nP_{n-1}(x) = P_n^1(x) - xP_{n-1}^1(x) \quad \text{----------- (3.6)}$$

Multiplying both sides of the recurrence relation (**3.2.2.**) **with x** we get

$$nxP_n(x) = x^2 P_n^1(x) - xP_{n-1}^1(x) \quad \text{--------- (3.7)}$$

Subtracting equation (**3.6**) from equation Equation (**3.7**)

$$n\left(xP_n(x) - P_{n-1}(x)\right) = \left(x^2 - 1\right)P_n^1(x)$$

or $$\left(1 - x^2\right)P_n^1(x) = n\left(P_{n-1}(x) - xP_n(x)\right)$$

Hence the relation

**3.2.6 Prove that** $\left(1 - x^2\right)P_n^1(x) = (n+1)\left(xP_n(x) - P_{n+1}(x)\right)$

**Proof :-** From recurrence formula (3.2.1) we have

$$(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$$

$$or\ (n+1)\, x\, P_n(x) + nxP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

**i.e.** $$(n+1)\left(xP_n(x) - P_{n+1}(x)\right) = n\left(P_{n-1}(x) - xP_n(x)\right) \quad \text{----- (3.8)}$$

Also from recurrence relation (3.2.5) we have

$$\left(1 - x^2\right)P_n^1(x) = n\left(P_{n-1}(x) - xP_n(x)\right) \quad \text{---- (3.9)}$$

From equations (3.8) and (3.9) we obtain

$$\left(1 - x^2\right)P_n^1(x) = (n+1)\left(xP_n(x) - P_{n+1}(x)\right)$$

Which is the required relation

**<u>3.2.7.</u> <u>Beltrami's Result:-</u>**

Prove that $(2n+1)\left(x^2 - 1\right)P_n^1(x) = n(n+1)\left(P_{n+1}(x) - P_{n-1}(x)\right)$

**<u>Proof :-</u>** From recurrence relation (3.2.5) we have

$$\left(1-x^2\right)P_n^1(x)=n\left(P_{n-1}(x)-xP_n(x)\right)$$

$$\Rightarrow -xP_n(x)=\left[\left(\frac{1-x^2}{n}\right)P_n^1(x)\right]-P_{n-1}(x) \qquad \text{-------- (3.10)}$$

Reccurence relations (3.2.6) gives

$$\left(1-x^2\right)P_n^1(x)=(n+1)\left(xP_n(x)-P_{n+1}(x)\right) \qquad \text{----- (3.11)}$$

Using equation (3.10) in equation (3.11) we get

$$\left(1-x^2\right)P_n^1(x)=(n+1)\left[P_{n-1}(x)-\left(\frac{1-x^2}{n}\right)P_n^1(x)-P_{n+1}(x)\right]$$

or $$\left(1-x^2\right)\left[1+\frac{n+1}{n}\right]P_n^1(x)=(n+1)\left(P_{n-1}(x)-P_{n+1}(x)\right)$$

i.e. $$-\left(x^2-1\right)(2n+1)P_n^1(x)=n(n+1)\left(P_{n-1}(x)-P_{n+1}(x)\right)$$

or $$(2n+1)\left(x^2-1\right)P_n^1(x)=n(n+1)\left(P_{n+1}(x)-P_{n-1}(x)\right)$$

This Result is known as the Beltrami's Result.

## 3.3 Rodrigues formula for $P_n(x)$

We shall now establish a representation for the Legendre Polynomial $P_n(x)$ known as the Rodrigue's Formula

**Theorem :** Prove that $P(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$

**Proof :-** Let $y=\left(x^2-1\right)^n$ Then $\dfrac{dy}{dx}=n\left(x^2-1\right)^{n-1}(2x)$

or $\left(x^2-1\right)\dfrac{dy}{dx}=2nxy$

Differentiating this equation (n+1) times w.r.t. x using Leibnitz's rule, we get

$$y(x^2-1)+(n+1)y_{n+1}(2x)+\frac{n(n+1)}{1.2}y_n(2)-2n\{y_{n+1}x+(n+1)y_n\}=0$$

n+2

i.e. $$\left(1-x^2\right)y_{n+2}-2xy_{n+1}+n(n+1)y_n=0 \qquad \text{..... (3.11)}$$

where $y_n$ denotes $\dfrac{d^n y}{dx^n}$

Let $y_n = \upsilon(x)$, so that $y_{n+1}(x)=\dfrac{dv}{dx}$ and $y_{n+2}(x)\dfrac{d^2v}{dx^2}$

Substituting these results in equation (3.11) we get

$$\left(1-x^2\right)\frac{d^2v}{dx^2}-2x\frac{dv}{dx}+n(n+1)v=0$$

Observe here that this is the Legendre's Equation for which $P_n(x)$ is a solution.

Hence $V = C\,P_n(x)$, where C is a constant.

or $$\left(\frac{d^n y}{dx^n}\right)=C\,P_n(x)$$

Putting x =1. Since $P_n(1) = 1$, we have $\left(\dfrac{d^n y}{dx^n}\right)_{x=1} = C$ -------- (3.12)

Also $y=(x^2-1)^n=(x+1)^n(x-1)^n$

Differentiating this equation n times w.r.t. x, using the Leibnitz's rule we get,

$$\frac{d^n y}{dx^n}=(x-1)^n\frac{d^n}{dx^n}(x+1)^n+n\left\{\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\right\}$$

$$n(x-1)^{n-1}+..........+n\left(\frac{\mathrm{d}}{\mathrm{dx}}(x+1)^n\right)\frac{d^{n-1}}{dx^{n-1}}(x-1)^n+(x+1)^n\frac{d^n}{dx^n}(x-1)^n$$

$$(x-1)^n n! + n(n!)(x+1)n(x-1)^{n-1} + \dots\dots\dots + n.n(x+1)^{n-1} n!(x-1) + (x+1)^n n!$$

Putting x =1 in this equation,

$$\left(\frac{d^n y}{dx^n}\right)_{x=1} = 2^n n! = C, \quad \text{Since} \left(\frac{d^n y}{dx^n}\right)_{x=1} = C, \quad \text{by equation (3.12)}$$

Hence $$P_n(x) = \frac{1}{c}\frac{d^n y}{dx^n} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This formula for $P_n(x)$ is known as the Rodrigue's formula

## 3.4 Examples

**3.4.1 Example** :- If $P_n(x)$ is a Legendre Polynomial of degree n and $\alpha$ is such that $P_n(\alpha)=0$, then show that $P_{n-1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite signs.

**Solution :** From the recurrence relation 3.2.1. we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Since $P_n(\alpha)=0$ by Hypothesis, by putting $x = \alpha$ in the above equation, we obtain.

$$(2n+1)\alpha.0. = (n+1)P_{n+1}(\alpha) + nP_{n-1}(\alpha)$$

or $$\frac{P_{n+1}(\alpha)}{P_{n-1}(\alpha)} = -\frac{n}{n+1}. \qquad \text{---------- (3.13)}$$

Since n is a positive integer, the ratio on the RHS of equation (3.13) is always negative. Hence it follows that $P_{n+1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite signs.

**3.4.2 Example** : Show that

$$\sum_{r=0}^{m}(2r+1)P_r(x)P_r(y) = \frac{m+1}{x-y}\left[P_{m+1}(x)P_m(y) - P_m(x)P_{m+1}(y)\right]$$

**Solution** : From recurre-nce relation (3.2.1) we have

$$(2r+1)xP_r(x) = (r+1)P_{r+1}(x) + rP_{r-1}(x) \qquad \text{---- (3.14)}$$

and $$(2r+1)yP_r(y) = (r+1)P_{r+1}(y) + rP_{r-1}(y) \qquad \text{----- (3.15)}$$

Multplying equation (3.14) by $p_r(y)$ and équation (3.15) by $p_r(x)$ and subtracting one from the other we get

$$(2r+1)(x-y)p_r(x)p_r(y) =$$

$$(r+1)[P_{r+1}(x)P_r(y)-P_{r+1}(y)P_r(x)]+r[P_{r-1}(x)P_r(y)-P_{r-1}(y)P_r(x)].....(3.16)$$

Putting r =0 1,2............. m-1, m in succession in equation (3.16) and adding the equations we get

$$(x-y)\ \sum_{r=0}^{m}(2r+1)p_r(x)p_r(y) = (m+1)\left[P_{m+1}(x)P_m(y)-P_{m+1}(y)P_m(x)\right]$$

or $$\sum_{r=0}^{m}(2r+1)p_r(x)p_r(y) = \frac{m+1}{x-y}\left[P_{m+1}(x)P_m(y)-P_{m+1}(y)P_m(x)\right] \quad ..(3.17)$$

**Deduction** :- Putting y=1 in equation (3.17) and using the fact that

$P_m(1) = P_{m+1}(1) = P_r(1) = 1$ we obtain :

$$\sum_{r=0}^{m}(2r+1)p_r(x) = \frac{m+1}{x-y_1}\left\{P_{m+1}(x)-P_m(x)\right)$$

**3.4.3. Example :-** Show that $P_n^1(x)+P_{n+1}^1(x) = \sum_{r=0}^{m}(2r+1)p_r(x)$

Solution :- from the recurrence relation (3.2.3) we have

$$(2n+1)p_n(x) = P_{n+1}^1(x)-P_{n-1}^1(x)$$

Putting n =1,2,3,....... n is succession in this equation we obtain

$$3p_1(x) = P_2^1(x)-P_0^1(x)$$

$$5p_2(x) = P_3^1(x)-P_1^1(x)$$

$$7p_3(x) = P_4^1(x)-P_2^1(x)$$

...... .......

...... .......

$$(2n-1)P_{n-1}(x) = P_n^1(x)-P_{n-2}^1(x)$$

$$(2n+1)P_n(x) = P_{n+1}^1(x)-P_{n-1}^1(x)$$

Adding these relations we get

$$3P_1(x)+5P_2(x)+..........+(2n+1)P_n(x) = -P_0^1(x)-P_1^1(x)+P_n^1(x)+P_{n+1}^1(x)$$

Since $P_0(x)=1$, and $P_1(x)=x$, we have $P_0^1(x)=0,\ P_1^1(x)=1=P_0(x)$

Hence $3P_1(x)+5P_2(x)+..........+(2n+1)P_n(x) = -P_0(x)+P^1n(x)+P^1n+1(x)$

or $\quad P_0(x)+3P_1(x)+5P_2(x)+..........+(2n+1)P_n(x) = P_n^1(x)+P_{n+1}^1(x)$

i.e. $\quad \sum_{r=0}^{n}(2r+1)P_r(x)=P_n^1(x)+P_{n+1}^1(x)$

Hence the Result

**3.4.4. Example** : Obtain the first four Legendre Polynomials $P_0(x), P_1(x)\ P_2(x)$ and $P_3(x)$

**Solution :-** From Rodrigues formula we know that

$$P_n(x)=\frac{1}{2^n\angle n}\frac{d^n}{dx^x}(x^2-1)^n$$

Putting n =0, 1,2,3, in succession in the above formula we obtain $P_0(x)=1$,

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1)=x$$

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1)$$

$$= \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3\angle 3}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{2}(5x^3 - 3x)$$

Similarly for n =4, following the same procedure we get

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

**3.4.5. Example** :- Show that $\int_{-1}^{1} P_n(x)dx = 0,\ n \neq 0$ 3

**Solution :-** From that Rodrigues formula, we have

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^x}(x^2 - 1)^n$$

$$\int_{-1}^{1} P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^x}(x^2 - 1)^n\, dx$$

$$= \frac{1}{2^n n!}\left[\frac{d^{n-1}}{dx^{x-1}}(x^2 - 1)^n\right]_{-1}^{1}$$

Differentiating $\frac{d^{n-1}}{dx^{x-1}}(x^2 - 1)^n = \frac{d^{n-1}}{dx^{x-1}}\left[(x+1)^n (x-1)^n\right]$ using Leibnitz's rule we get

$$\frac{d^{n-1}}{dx^{x-1}}(x^2 - 1)^n =$$

$$\left[\frac{d^{n-1}}{d^{x-1}}(x+1)^n\right](x-1)^n + nc_1\frac{d^{n-2}}{dx^{x-2}}(x+1)^n \frac{d}{dx}(x-1)^n + \ldots\ldots\ldots\ldots +$$

$$(x+1)^n \frac{d^{n-1}}{d^{x-1}}(x-1)^n$$

$$= n\,!(x+1)(x-1)^n + \ldots\ldots\ldots\ldots + (x+1)^n n\,!(x-1)$$

$= \quad 0$ , When x = -1, or +1

Hence $\int_{-1}^{1} P_n(x) = 0, \quad n \neq 0$

**3.4.6 Example :-** Show that $\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,\mathbf{dx} = \dfrac{2n}{4n^2-1}$

**Solution :-** From the recurrance relation (3.2.1) we have

$$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$$

Multiplying both sides of this equation by $P_{n-1}$ (x) and then integrating w.r.t. x. from -1 to +1 we get

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{n+1}{2n+1}\int_{-1}^{1} P_{n+1}(x)\,P_{n-1}(x)\,dx + \frac{n}{2n+1}\int_{-1}^{1}\left[P_{n-1}(x)\right]^2 dx$$

$$= 0 + \frac{n}{2n+1}\int_{-1}^{1}\left[P_{n-1}(x)\right]^2 dx$$

From the orthogonality relation for legendre Polynomials we have

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0, \qquad m \neq n$$

$$= \frac{2}{2n+1}\ ; \text{if} \quad m = r$$

Hence $\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = 0 + \dfrac{n}{2n+1}\left[\dfrac{2}{2(n-1)+1}\right]$

$$= \frac{2n}{4n^2-1}$$

Hence the result.

**3.4.7. Example :-** If m < n, show that

(i) $$\int_{-1}^{1} x^m P_n(x)dx = 0$$

(ii) $$\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+2}(n!)^2}{(2n+1)!}$$

**Solution :-** (i) By Rodrigues formula for Legendre Polynomials, we have

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{1}{2^n \angle n}\int_{-1}^{1} x^m \frac{d^n}{dx^n}\left(x^2-1\right)^n dx$$

$$= \frac{1}{2^n n!}\int_{-1}^{1} x^m D^n (x^2-1)^n dx$$

$$= \frac{1}{2^n n!}\left\{\left[x^m D^{n-1}(x^2-1)^n\right]_{-1}^{1} - m\int_{-1}^{1} x^{m-1} D^{n-1}(x^2-1)^n dx\right\}$$

$$= \frac{(-1)m}{2^n n!}\int_{-1}^{1} x^{m-1} D^{n-1}(x^2-1)^n dx$$

(Since the first term vanishes )

Integrating the RHS again w.r.t. x and substituting the limits we get

$$\int_{-1}^{1} x^m P_n(dx) = \frac{(-1)^2 m(m-1)}{2^n n!}\int_{-1}^{1} x^{m-2} D^{n-2}(x^2-1)^n dx$$

Continuing this process m terms in succession we get

$$\int_{-1}^{1} x^m P_n(x)\,dx = \frac{(-1)^m m(m-1)......3.2.1}{2^n n!} \cdot \int_{-1}^{1} x^{m-m} D^{n-m}(x^2-1)^n dx$$

$$= \frac{(-1)^m m!}{2^n n!} \int_{-1}^{1} D^{n-m}(x^2-1)^n dx$$

$$= \frac{(-1)^m m!}{2^n n!} \left[ D^{n-m-1}(x^2-1)^n \right]_{-1}^{1}$$

$$= 0, \quad \text{Since } m < n$$

(ii) In this case, since m = n

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{(-1)^n n!}{2^n n!} \int_{-1}^{1} x^{n-n} D^{n-n}(x^2-1)^n dx$$

$$= \frac{(-1)^n}{2^n} \int_{-1}^{1} (x^2-1)^n dx$$

Put x = Sin $\theta$ Then dx = Cos $\theta\ d\theta$

Then $$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{1}{2^{n-1}} \int_0^{\frac{\Pi}{2}} \cos^{2n+1}\theta\ d\theta$$

$$= \frac{1}{2^{n-1}} \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma\left(\frac{2n+3}{2}\right)}$$

$$= \frac{n!}{2^n} \frac{2^{n+1}}{(2n+1)(2n-1)\ldots\ldots\ldots 3.1}$$

$$= 2(n!) \frac{2n(2n-2)\ldots\ldots. 4.2}{(2n+1)(2n)(2n-1)(2n-2)\ldots\ldots. 4.3.2.1}$$

$$= 2(n!) \frac{(2.n)[2(n-1)]\ldots\ldots(2.2)(2.1)}{(2n+1)!}$$

$$= 2(n!) \frac{2^n n!}{(2n+1)!}$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

Hence the Result

**SAQS :**

i) Show that $\int_{-1}^{1} P_n(x)\,dx = 0, \quad$ n is even

$$= \frac{(n-1)!(-1)^{\frac{n-1}{2}}}{2^n\left(\frac{n+1}{2}\right)!\left(\frac{n-2}{1}\right)!}, \text{ n is odd}$$

(ii) Evaluate $\int_{-1}^{1} x^3\, P_4\,(x)\;dx$ (Ans : 0)

(iii) Show that $\int_{-1}^{1} P_n\,(x)\,dx = 2, \quad$ if n = 0

$= 0 \quad$ if $n \geq 1$

(iv) Prove that $\int_{-1}^{1} x^2\,P_n^2\,(x)\,dx = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$

(v) Using Rodrigues formula, Prove that $P_{n+1}^1(x) - P_{n-1}^1(x) = (2n+1)P_n(x)$

## 3.5 Summary :

In this lesson, we have established a few recurrence relations for the Legendre Polynomials $P_n(x)$ and derived the Rodrigues for $P_n(x)$ which gives a representation for the Legendre Polynomials.

## 3.6 Model Examination Questions :

1. Prove that $\int_{-1}^{1} (1-x^2)\,P_m^1(x)\;P_n^1(x)\,dx = 0$, if $\quad m \neq n$

$$= \frac{2m(m+1)}{(2m+1)}, \text{ if } \quad m = n$$

2. Prove that $xP_n^1(x) = nP_n(x) + (2n-3)P_{n-2}(x) + (2n-7)P_{n-4}(x) + \ldots\ldots\ldots\ldots$

Hence show that $\int_{-1}^{1} x\, P_n(x)\, P_n^1(x)\, dx = \dfrac{2n}{2n+1}$

3. Prove that $\int_{-1}^{1} \left[P_n^1(x)\right]^2 dx = \dfrac{n}{n+1}$

4. Prove that the function $y = \dfrac{d^n}{dx^n}(x^2-1)^n$ satisfies the Legendre's differential equation $(1-x^2)\, y'' - 2xy' + n(n+1)y = 0$

5. If f is contineous in [ -1,1 ] $f^{(n)}(x)$ denotes the n - th derivation of f and

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

Show that $\int_{-1}^{1} f(x)\, P_n(x)\, dx = \dfrac{(-1)^n}{2^n n!} \int_{-1}^{1} f^{(n)}(x)(x^2-1)^n dx$

6. Show that when $|z| < 1$, $\int_{-1}^{1} (1-2zx+x^2)^{\frac{-1}{2}} P_n(x) dx = \dfrac{2z^n}{(2n+1)}$

7. Show that

$$P_n^1(x) = (2n-1)P_{n-1}(x) + (2n-5)p_{n-3}(x) + (2n-9)P_{n-5} + \ldots\ldots\ldots$$

Also show that the last term of the series is $3P_1$ or $P_0$ according as n is even or odd.

8. Prove that $\sum_{k=0}^{l} (2k+1)P_k^1(x) = \dfrac{l+l}{x-1}\{P_{l+1}(x) - P_l(x)$

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# LESSON 4 : LEGENDRE FUNCTIONS - 4

## 4. ASSOCIATED LEGENDRE FUNCTIONS AND LEGENDRE FUNCTIONS OF THE SECOND KIND.

### 4.1 INTRODUCTION :

Legendre polynomials have several important applications in physics and these applications depend on the number of special properties which Legendre polynomials possess. These polynomials may be introduced either through a solution of a Legendre differential equation or through a generating function. Laplaces equation $\nabla^2 u = 0$ in spherical polar coordinates can be written as :

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2}$$

The solutions of this equation, obtained by the method of seperation of variables, is known as the Associated Legendre's equation. Solutions of the associated Legendre's equation, denoted by $P^m_n(x)$ and $Q^m_n(x)$ are called Associated Legendre functions of the first and second kind.

### 4.2 Associated Legendre functions :

The differential equation, $(1-x^2)y'' - 2xy' + [n(n+1) - \frac{m^2}{1-x^2}]\, y = 0$ ..... (4.1)

n,m are integers is known as the Associated Legendre equation.

**Theorem :** If u is a solution of the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \text{..... (4.2)}$$

then $(1-x^2)^{m/2}\frac{d^m u}{dx^m}$ is a solution of the associated Legendre equation (4.1)

**Proof :** Since u is a solution of the equation (4.2), we have

$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} + n(n+1)u = 0 \qquad \text{...... (4.3)}$$

Differentiating equation (4.3) m times w.r.t. x using Leibnitz's rule, we get

$$(1-x^2)\frac{d^{m+2}u}{dx^{m+2}} + mc_1(-2x)\frac{d^{m+1}u}{dx^{m+1}} + mc_2(-2x) - 2\left\{\frac{d^{m+1}u}{dx^{m+1}} + m\frac{d^m u}{dx^m}\right\} + n(n+1)\frac{d^m u}{dx^m} = 0$$

i.e., $(1-x^2)\dfrac{d^{m+2}u}{dx^{m+2}} - 2(m+1)x\dfrac{d^{m+1}u}{dx^{m+1}} + [n(n+1)-m(m+1)]\dfrac{d^m u}{dx^m} = 0$ .... (4.4)

Put $\dfrac{d^m u}{dx^m} = \upsilon$. Then equation (4.4) reduces to

$$(1-x^2)\frac{d^2\upsilon}{dx^2} - 2(m+1)x\frac{d\upsilon}{dx} + [n(n+1)-m(m+1)]\upsilon = 0 \quad \text{....} \quad (4.5)$$

Now let $Z = (1-x^2)^{m/2}\upsilon$, so that $(1-x^2)^{-m/2}Z = \upsilon$ and $\dfrac{d\upsilon}{dx} = (1-x^2)^{-m/2}\dfrac{dZ}{dx} + m(1-x^2)^{-(m/2)-1}xZ$

and $\dfrac{d^2\upsilon}{dx^2} = (1-x^2)^{-m/2}\dfrac{d^2Z}{dx^2} + 2m(1-x^2)^{-(m/2)-1}.x\dfrac{dZ}{dx} + mZ(1-x^2)^{-(m/2)-1}.$

$$+[m(m+2)x^2Z](1-x^2)^{-(m/2)-2}.$$

Substituting these values in equation (4.5) we get

$$(1-x^2)\ \{(1-x^2)^{-m/2}\frac{d^2Z}{dx^2} + 2m(1-x^2)^{-(m/2)-1}.x\frac{dZ}{dx} + mZ(1-x^2)^{-(m/2)-1}.$$

$$.+[m(m+2)x^2Z](1-x^2)^{-(m/2)-2}.\} - 2(m+1)x\left\{(1-x^2)^{-m/2}\frac{dZ}{dx} + mxZ(1-x^2)^{-(m/2)-1}\right\}$$

$$+\{n(n+1)-m(m+1)\}(1-x^2)^{-m/2}Z = 0$$

or $(1-x^2)^{-m/2}[(1-x^2)\dfrac{d^2Z}{dx^2} - 2x\dfrac{dZ}{dx} + \{n(n+1) - \left(\dfrac{m^2}{1-x^2}\right)\}Z] = 0$

i.e., $(1-x^2)\dfrac{d^2Z}{dx^2} - 2x\dfrac{dZ}{dx} + [n(n+1) - \left(\dfrac{m^2}{1-x^2}\right)]z = 0$

This implies that $Z = (1-x^2)^{m/2}\upsilon = (1-x^2)^{m/2}\dfrac{d^m u}{dx^m}$ forms a solution of the associated Legendre

equation, $(1-x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + [n(n+1) - \left(\dfrac{m^2}{1-x^2}\right)]y = 0$

**Definition :** The associated Legendre function of the first kind, $P^m_n(x)$ is defined by

$P^m_n(x) = (1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x)$, $m \geq 0$. $P^m_n(x)$ thus defined, satisfy the associated Legendre equation (4.1)

The Rodrignes formula for $P^m_n(x)$ is defined by $P^m_n(x) = (1-x^2)^{m/2}\dfrac{1}{2^n n!}\dfrac{d^{m+n}}{dx^{m+n}}(x^2-1)^n$.

which is defined for values of m such that $m + n \geq 0$ i.e., $m \geq -n$.

Using the definition of the associated Legendre function of the first kind and the methods described to derive the various properties in lessons 2 and 3, we can derive the following results analogously :

1. $P^m_n(x) = 0$, if $m > n$ and $P^0_n(x) = P_n(x)$

2. $\displaystyle\int_{-1}^{1} P^m_n(x)\, P^m_{n'}(x)dx = \frac{2(n'+m)!}{(2n'+1)(n'-m)}\,\delta_{nn'}$ **where** $\delta_{nn'}$ **is the** Kronecker delta

3. $(2n+1)\, x\, P^m_n(x) = (n+m)\, P^m_{n-1}(x) - (n-m+1)\, P^m_{n+1}(x)$

4. $\sqrt{1-x^2}\, P^m_n(x) = \dfrac{1}{2n+1}\left[P^{m+1}_{n+1}(x) - P^{m-1}_{n-1}(x)\right]$

5. $P_n^{m+1}(x) - \dfrac{2mx}{\sqrt{1-x^2}} P_n^m(x) + [n(n+1)-m(m-1)]P_n^{m-1}(x) = 0$

## 4.3 Legendre function of the second kind

The Legendre function of the second kind denoted by $Q_n(x)$ is defined by

$$Q_n(x) = \frac{n!}{1.3.5......(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + ......\right]$$

$$= \frac{(2.4.6......2n)n!}{1.2.3...2n(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + ......\right]$$

$$= \frac{2^n(n!)^2}{(2n+1)!}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + ......\right]$$

This forms a solution for the Legendre's equation in descending powers of x, all the powers of x being negative.

### 4.3.1 Recurrence relations for $Q_n(x)$

We shall now derive a few recurrence relations satisfied by the $Q_n(x)$

1. $Q'_{n+1}(x) - Q'_{n-1}(x) = (2n+1)\, Q_n(x)$

We know that $Q_n(x) = \dfrac{2^n (n!)^2}{(2n+1)!}\left[x^{-n-1} + \dfrac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + ......\right]$

$$= \frac{2^n (n!)^2}{(2n+1)!}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + ......\right]$$

$$= \frac{2^n n!}{(2n+1)!}\left[(n!)x^{-n-1} + \frac{(n+2)!}{2(2n+3)} x^{-n-3} + \frac{(n+4)!}{2.4(2n+3)(2n+5)} x^{-n-5} + ......\right]$$

$$= \frac{2^n (n!)}{(2n+1)!}\left[(n!)\, x^{-(n+1)} + \frac{(n+2)!}{2(2n+3)} x^{-(n+2+1)} + \frac{(n+4)!}{2.4(2n+3)(2n+5)} x^{-(n+4+1)} + ......\right]$$

$$= \frac{2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2.4.2r(2n+3)........(2n+2r-1)}$$

$$= \frac{2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r r!(2n+3)........(2n+2r-1)} \quad .... \quad (4.6)$$

Differentiating equation (4.6) w.r.t. x we get

$$Q'_n(x) = -\frac{2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+2)}}{2^r r!(2n+3)........(2n+2r-1)} \quad ..... \quad (4.7)$$

Replacing n by (n-1) we obtain

$$Q'_{n-1}(x) = -\frac{2^{n-1}(n-1)!}{(2n-1)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+2)}}{2^r r!(2n+1)........(2n+2r-1)}$$

$$= -\frac{2^n (n!)}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r r!(2n+1)........(2n+2r-1)} \quad ...... \quad (4.8)$$

Similarly by replacing n by (n+1) in equation (4.7) and on simplification we get

$$Q^1{}_{n+1}(x) = -\frac{2^n n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!x^{-(n+2r+3)}}{2^r r!(2n+1)........(2n+2r+3)} \quad ... \quad (4.9)$$

From equations (4.6) and (4.8) we have

$$Q^1{}_{n-1}(x) + (2n+1)Q_n(x) = \frac{-2^n n!}{(2n)!}$$

$$\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^r(r!)(2n+1)...(2n+2r-1)} + \frac{(2n+1)2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^r(r!)(2n+3)...(2n+2r-1)}$$

$$= \frac{2^n n!}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)....(2n+2r+1)} - \sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)...(2n+2r-1)}\right]$$

$$= \frac{2^n n!}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)....(2n+2r+1)}\{(2n+1) - (2n+2r+1)\}\right]$$

$$= \frac{2^n n!}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}(-2r)}{2^r r!(2n+1).....(2n+2r+1)}\right]$$

$$= \frac{-2^n n!}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1).....(2n+2r+1)}\right]$$

$$= \frac{-2^n n!}{(2n)!}\left[0 + \sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1).....(2n+2r+1)}\right]$$

Let s = r-1 then the RHS reduces to

$$= \frac{-2^n n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!x^{-(n+2r+3)}}{2^r r!(2n+1).....(2n+2r+3)} = Q^1{}_{n+1}(x) \text{ (by equation 4.9)}$$

Hence the result.

**(2)** **$(2n+1)x\, Q'_n(x) - (n+1)\, Q'_{n-1}(x) = n\, Q'_{n+1}(x)$**

**Solution :** From equations (4.7) and (4.8) we have $(2n+1)x\, Q'_n(x) - (n+1)\, Q'_{n-1}(x)$

$$= -(2n+1)x \frac{2^n n!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+2)}}{2^r r!(2n+3).......(2n+2r+1)}$$

$$= -(n+1)(-1)\frac{2^n n!}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^r r!(2n+1).....(2n+2r+1)}$$

$$= \frac{-2^n n!}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r+1)!\, x^{-(n+2r+1)}(2n+1)}{2^r r!(2n+1).......(2n+2r+1)} +$$

$$\frac{2^n n!}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}(n+1)(2n+2r+1)}{2^r r!(2n+1).......(2n+2r+1)}$$

$$= \frac{-2^n n!}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}(2nr)}{2^r r!(2n+1).......(2n+2r+1)}$$

$$= \frac{-2^n n! n}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1).......(2n+2r+1)}$$

$$= \frac{-2^n n! n}{(2n)!} \left[ \sum_{r=0}^{\infty} \frac{(n+2r)!\, x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1).......(2n+2r+1)} + 0 \right]$$

$$= \frac{-n2^n n!}{(2n)!} \sum_{r=0}^{\infty} \frac{(n+2r+2)!\, x^{-(n+2r+3)}}{2^r r!(2n+1).......(2n+2r+3)}$$

$$= n.\ Q'_{n+1}(x).$$

**(3)** **$(n+1)\, Q_{n+1}(x) + n\, Q_{n-1}(x) = (2n+1)\, xQ_n(x)$**

**Proof :** Consider $n\, Q_{n-1}(x) - (2n+1)\, x\, Q_n(x)$ as LHS we have from equation (4.6)

$$\text{LHS} = \frac{n\,2^{n-1}(n-1)!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^r r!(2n+1)\ldots\ldots(2n+2r-1)}$$

$$-\frac{(2n+1)x2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r r!(2n+1)\ldots\ldots(2n+2r+1)}$$

$$= \frac{2^{n-1}n!(2n)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\;x^{-(n+2r)}(2n+2r+1)}{2^r r!(2n+3)\ldots\ldots.(2n+2r-1)}$$

$$\frac{(2n+1)\,x\,2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!x^{-(n+2r)}}{2^r r!(2n+3)\ldots\ldots.(2n+2r+1)}$$

$$= \frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\;x^{-(n+2r)}}{2^r r!(2n+3)\ldots\ldots.(2n+2r-1)}\,X$$

$$X\left[n(2n+2r+1)-(2n+1)(n+2r)\right]$$

$$= \frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\;x^{-(n+2r)}(-2r)(n+1)}{2^r r!(2n+3)\ldots\ldots.(2n+2r+1)}$$

$$= \frac{2^n n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{-(n+2r-1)!\;(n+1)\;x^{-(n+2r)}}{2^{r-1}(r-1)!(2n+3)\ldots\ldots.(2n+2r+1)}$$

$$= \frac{-(n+1)2^n n!}{(2n+1)!}\left[0+\sum_{r=1}^{\infty}\frac{(n+2r-1)!x^{-(n+2r)}}{2^{r-1}(r-1)!(2n+3)\ldots\ldots.(2n+2r+1)}\right]$$

$$= \frac{-(n+1)\,2^n(n!)(2n+2)}{(2n+2)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r+1)!x^{-(n+2r+2)}}{2^r r!(2n+1)\ldots..(2n+2r+3)}\right]$$

$$= \frac{-(n+1)2^{n+1}(n+1)!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!x^{-(n+2r+2)}}{2^r r!(2n+5)\ldots\ldots.(2n+2r+3)}$$

$=$ **- (n+1) $Q_{n+1}(x)$, by equation (4.6)**

**Hence** $\quad (n+1)\, Q_{n+1}(x) + n\, Q_{n-1}(x) = (2n+1)\, x\, Q_n(x)$

(4) $\quad (2n+1)(1-x^2)\, Q^1{}_n(x) = n(n+1)\, [Q_{n-1}(x) - Q_{n+1}]$

**Proof :** We know that $Q_n(x)$ is a solution of the Legendre's differential equation,

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)\,y = 0$$

Hence $\frac{d}{dx}\left[(1-x^2)\; Q^1 n(x)\right] = -n(n+1)\, Q_n(x)$ ..... (4.10)

Integrating both sides of equation (4.10) w.r.t. x from x to $\infty$ we get

$$\left[(1-x)^2 Q_n{}^1(x)\right]_x^\infty = -(n+1)\int_x^\infty Q_n(t)dt$$

i.e., $\left[Q_n{}^1(x) - x^2 Q_n{}^1(x)\right]_x^\infty = -(n+1)\int_x^\infty Q_n(t)dt$

**Since** $Q^1 n(x)_{x=\infty} = 0$ we get

$$-(1-x^2)Q_n{}^1(x) = -(n+1)\int_x^\infty Q_n(t)dt \qquad \text{..... (4.10(a))}$$

By recurrance relation (1) we have, $Q^1 n{+}1(x) - Q^1 n{-}1(x) = (2n+1)Q_n(x)$

Integrating this relation w.r.t. x between the limits x to $\infty$

$$\left[Q_{n+1}(x) - Q_{n-1}(x)\right]_x^\infty = \int_x^\infty (2n+1)Q_n(t)dt \qquad \text{....... (4.10(b))}$$

Since $\left[Q_{n+1}(x)\right]_{x=\infty} = \left[Q_{n-1}(x)\right]_{x=\infty} = 0$, we have from equations (4.10a) & (4.10b)

$$(1-x^2)\; Q^1 n(x) = \frac{-n(n+1)}{2n+1}\left[Q_{n+1}(x) - Q_{n-1}(x)\right]$$

or $(2n+1)(1-x^2)\, Q^1{}_n(x) = n(n+1)\, [Q_{n-1}(x) - Q_{n+1}]$

Hence the result.

**5.** $\quad \mathbf{x\, Q'_n(x) - Q'_{n-1}(x) = n\, Q_n(x)}$

From recurrance relation (3) we have $(n+1)Q_{n+1}(x) - (2n+1)\, x\, Q_n(x) + nQ_{n-1}(x) = 0$

Differentiating this equation w.r.t. x we obtain

$$(n+1)Q'_{n+1}(x) - (2n+1)\,\{\, Q_n(x) + xQ'_n(x)\} + n\, Q'_{n-1}(x) = 0 \quad \ldots\ldots \quad (4.11)$$

From recurrance relation (1), we have

$$Q'_{n+1}(x) = Q'_{n-1}(x) + (2n+1)\, Q_n(x)$$

Substituing for $Q'_{n+1}(x)$ from this equation in equation (4.11) we get

$$(n+1)\,\{\, Q'_{n-1}(x) + (2n+1)\, Q_n(x)\,\} - (2n+1)\,[\, xQ'_n(x) + Q_n(x)] + n\, Q'_{n-1}(x) = 0$$

i.e.. $\quad (2n+1)\, Q'_{n-1}(x) - (2n+1)\, x\, Q'_n(x) + (2n+1)\, n\, Q_n(x) = 0$

or $\quad Q'_{n-1}(x) - x\, Q'_n(x) + n\, Q_n(x) = 0$

**6.** $\quad \mathbf{Q'_n(x) - x\, Q'_{n-1}(x) = n\, Q_{n-1}(x)}$

<u>**Proof :**</u> From recurrance relation (1) we have $Q'_{n+1}(x) - Q'_{n-1}(x) = (2n+1)\, Q_n(x)$

$$\Rightarrow \quad x\, Q'_{n+1}(x) - x\, Q'_{n-1}(x) = (2n+1)\, x\, Q_n(x) \quad \ldots \quad (4.12)$$

From recurrance relation (3) we have

$$(n+1)\, Q_{n+1}(x) + n\, Q_{n-1}(x) - (2n+1)\, x\, Q_n(x) = 0 \quad \ldots. \quad (4.13)$$

Using equation (4.12) in equation (4.13) we obtain

$$(n+1)\, Q_{n+1}(x) + n\, Q_{n-1}(x) - [x\, Q'_{n+1}(x) - x\, Q'_{n-1}(x)] = 0 \quad \ldots. \quad (4.14)$$

From recurrance relation (5) we have

$$n\, Q_n(x) = x\, Q'_n(x) - Q'_{n-1}(x)$$

$$\Rightarrow \quad (n+1)\, Q_{n+1}(x) = x\, Q'_{n+1}(x) - Q'_n(x) \quad \ldots. \quad (4.15)$$

using equation (4.15) in equation (4.14) we obtain

$$x\,Q'_{n+1}(x) - Q'_n(x) + n\,Q_{n-1}(x) - x\,Q'_{n+1}(x) + x\,Q'_{n-1}(x) = 0$$

$$\text{i.e., } Q'_n(x) - x\,Q'_{n-1}(x) = n\,Q_{n-1}(x).$$

**SAQs :**

1. Show that $(x^2-1)\,Q'_n(x) = nx\,Q_n(x) - n\,Q_{n-1}(x)$ ( **Hint** : Use recurrance relation (5) )
2. Show that $(x^2-1)\,Q'_n(x) = (n+1)\,Q_{n+1}x - (n+1)\,x\,Q_n(x)$
   ( **Hint :** Use recurrance relations (5) and (6) )

**Theorem :** If $P_n(x)$ is a solution of the Legendre equation $(1-x^2)\,y'' - 2xy' + n(n+1)y = 0$ and $Q_n(x) = C\,P_n(x)\int\frac{dx}{(1-x^2)P^2{}_n(x)}$, then its complete solution is given by $C_1P_n(x) + C_2Q_n(x)$, where $C_1$, $C_2$ and C are constants.

**Proof :**

Let $y = u(x)\,P_n(x)$ be the complete solution of the Legendre equation $(1-x^2)y''-2xy'+n(n+1)y=0$

$$\text{Then } \frac{dy}{dx} = u(x)\frac{d}{dx}P_n(x) + P_n(x)\frac{d}{dx}u(x)$$

$$\frac{d^2y}{dx^2} = u(x)\frac{d^2}{dx^2}(P_n(x)) + 2\frac{d}{dx}(P_n(x))\frac{du}{dx} + P_n(x)\frac{d^2u}{dx^2}$$

Substituting these values in the Legendre equation we get

$$(1-x^2)\left\{u(x)\frac{d^2}{dx^2}P_n(x) + 2\frac{d}{dx}P_n(x)\frac{du}{dx} + P_n(x)\frac{d^2u}{dx^2}\right\}$$

$$-2x\left\{u(x)\frac{d}{dx}P_n(x) + P_n(x)\frac{du}{dx}\right\} + n(n+1)\,u(x)\,P_n(x) = 0$$

i.e.,
$$(1-x^2)\left\{P_n(x)\frac{d^2u}{dx^2} + 2\frac{d}{dx}P_n(x)\frac{du}{dx}\right\} + (1-x^2)\,u(x)\frac{d^2}{dx^2}P_n(x)$$

$$-2xu\frac{d}{dx}P_n(x) - 2xP_n(x)\frac{du}{dx} + n(n+1)u(x)P_n(x) = 0$$

(or) $$(1-x^2)\left\{P_n(x)\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{d}{dx}P_n(x)\right\}+u[(1-x^2)\frac{d^2}{dx^2}P_n(x)$$

$$-2x\frac{d}{dx}P_n(x)+n(n+1)P_n(x)]-2xP_n(x)\frac{du}{dx}=0$$

i.e., $$(1-x^2)\left\{P_n(x)\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{d}{dx}P_n(x)\right\}-2xP_n(x)\frac{du}{dx}=0$$

Since $P_n(x)$ is a solution of Legendre equation ; Dividing this equation throughout by $(1-x^2)\,P_n(x)\,du/dx$ and rearranging we get $\dfrac{1}{du/dx}\dfrac{d^2u}{dx^2}+\dfrac{2}{P_n(x)}\dfrac{dP_n(x)}{dx}-\dfrac{2x}{(1-x)^2}=0$

on integration we get, $\log\dfrac{du}{dx}+2\log P_n(x)+\log(1-x^2)=\log k$

(or) $$\log\left\{\frac{du}{dx}P^2_n(x)\;(1-x^2)\right\}=\log k$$

$$\Rightarrow\frac{du}{dx}P^2_n(x)\;(1-x^2)=k,\Rightarrow\frac{du}{dx}=\frac{k}{(1-x^2)P^2_n(x)}$$

on integration we get $u(x)=K\displaystyle\int\frac{dx.}{(1-x^2)P^2_n(x)}+C_1$ where K and $C_1$ are arbitrary constants

Hence the complete solution of the Legendre's equation is

$$y=u\,P_n(x)=\left[K\int\frac{dx}{(1-x^2)P^2_n(x)}+C_1\right]P_n(x)$$

$$=C_1P_n(x)+K\,P_n(x)\int\frac{dx}{(1-x^2)P^2_n(x)}$$

$$=C_1P_n(x)+\frac{K}{C}C\,P_n(x)\int\frac{dx}{(1-x^2)P^2_n(x)}$$

$$= C_1 P_n(x) + C_2 Q_n(x), \text{ where } C_2 = \frac{K}{C}.$$

**Hence the Proof.**

## 4.4 Christoffel's second summation formula .

From the recurrance relations of $P_n(x)$ and $Q_n(x)$ we have

$$(2r+1)\, x\, P_r(x) = (r+1)\, P_{r+1}(x) + r\, P_{r-1}(x)$$
$$(2r+1)\, y\, Q_r(x) = (r+1)\, Q_{r+1}(y) + r\, Q_{r-1}(y)$$

Multiplying the first equation by $Q_r(y)$ and the second equation by $P_r(x)$ and then subtracting one from the other we obtain,

$$(2r+1)(x-y)\, P_r(x)\, Q_r(y) = (r+1)\,[\, P_{r+1}(x)\; Q_r(y) - Q_{r+1}(y)\, P_r(x)\,] - r\,[\, P_r(x) \cdot Q_{r-1}(y) - P_{r-1}(x) Q_r(y)]$$

Putting r = 1, 2, 3, ......., n in succession in this equation we get the following n equations,

$$3(x-y)\, P_1(x)\, Q_1(y) = 2\,[\, P_2(x)\; Q_1(y) - P_1(x)\, Q_2(y)\,] - 1\,[\, P_1(x) \cdot Q_0(y) - P_0(x)\, Q_1(y)]$$

$$5(x-y)\, P_2(x)\, Q_2(y) = 3\,[\, P_3(x)\; Q_2(y) - P_2(x)\, Q_3(y)\,] - 2\,[\, P_2(x) \cdot Q_1(y) - P_1(x)\, Q_2(y)]$$

" " " "

" " " "

$$(2n-1)(x-y)\, P_{n-1}(x)\, Q_{n-1}(y) = n\,[\, P_n(x)\; Q_{n-1}(y) - P_{n-1}(x)\, Q_n(y)\,] - (n-1)\,[\, P_{n-1}(x) \cdot Q_{n-2}(y) - P_{n-2}(x) Q_{n-1}(y)]$$

$$(2n+1)(x-y)\, P_n(x)\, Q_n(y) = (n+1)\,[\, P_{n+1}(x)\; Q_n(y) - P_n(x)\, Q_{n+1}(y)\,] - n\,[\, P_n(x) \cdot Q_{n-1}(y) - P_{n-1}(x) Q_n(y)]$$

Adding these n equations we get,

$$\sum_{r=1}^{n} (2r+1)(x-y)\, P_r(x)\, Q_r(y) = (n+1)\,[\, P_{n+1}(x)\; Q_n(y) - P_n(x)\, Q_{n+1}(y)\,] - [\, P_1(x) \cdot Q_0(y) - P_0(x) Q_1(y)]$$

$$= (n+1)\,[\, P_{n+1}(x)\; Q_n(y) - P_n(x)\, Q_{n+1}(y)\,] - [\, x\, Q_0(y) - \{\, y\, Q_0(y) - 1\}\,],$$

since $P_1(x) = x$ and $P_0(x) = 1$, $Q_1(y) = y$, $Q_0(y) = 1$

$$= -1 + (n+1)\ [P_{n+1}(x)\ Q_n(y) - P_n(x)\ Q_{n+1}(y)] - (x-y)\ P_0(x)P_0(y), \text{ since } P_0(x) = 1$$

i.e., $$\sum_{r=0}^{n} (2r+1)(y-x)\ P_r(x)\ Q_r(y) = 1-(n+1)\ [P_{n+1}(x)\ Q_n(y) - P_n(x)\ Q_{n+1}(y)]$$

or $$\sum_{r=0}^{n} (2r+1)\ P_r(x)\ Q_r(y) = \frac{1}{x-y}[\ 1-(n+1)\ \{P_{n+1}(x)\ Q_n(y) - P_n(x)\ Q_{n+1}(y)\}\ ]$$

This equation is known as the Christoffel's second summation formulas.

**Examples :**

**4.4.1** Example : Prove that $(x^2-1)\ [Q_n(x)\ P'_n(x) - P_n(x)\ Q'_n(x)] = C$

**Solution :** We have the Legendre's differential equation, $(1-x^2)\ y'' - 2xy' + n(n+1)y = 0$.

We know that $P_n(x)$ and $Q_n(x)$ are solutions of this equation. Hence.

$$(1-x^2)\ P_n'' - 2x\ P'_n(x) + n(n+1)\ P_n(x) = 0. \quad \ldots \quad (4.15)$$

and $$(1-x^2)\ Q_n'' - 2x\ Q'_n(x) + n(n+1)\ Q_n(x) = 0. \quad \ldots \quad (4.16)$$

Multiplying equation (4.15) by $Q_n(x)$ and equation (4.16) by $P_n(x)$ and subtracting one from the other, we get,

$$(1-x^2)\ [\ P_n''(x)\ Q_n(x) - Q''_n(x)\ P_n(x)\ ] - 2x\ [\ P'_n(x)\ Q_n(x) - Q'_n(x)\ P_n(x)\ ] = 0$$

or $$(1-x^2)\ \frac{d}{dx}\ [\ P'_n(x)\ Q_n(x) - Q'_n(x)\ P_n(x)\ ] - 2x\ [\ P'_n(x)\ Q_n(x) - Q'_n(x)\ P_n(x)\ ] = 0$$

i.e., $$\frac{d}{dx}\ \{\ (1-x^2)\ [P'_n(x)\ Q_n(x) - Q'_n(x)\ P_n(x)\ ]\ \} = 0$$

Integrating, we get $(1-x^2)\ [\ Q_n(x)\ P'_n(x) - P_n(x)\ Q'_n(x)\ ] = \text{Constant (C)}$

$$\text{or } (x^2-1)\ [\ Q_n(x)\ P'_n(x) - P_n(x)\ Q'_n(x)\ ] = C$$

**4.4.2** **Example :** Prove that

(i) $n\ [\ Q_n(x)\ P_{n-1}(x) - P_{n-1}(x)\ Q_n(x)\ ] = (n-1)\ [\ Q_{n-1}(x)\ P_{n-2}(x) - Q_{n-2}(x)\ P_{n-1}(x)\ ]$

(ii) $n\ [\ Q_n(x)\ P_{n-1}(x) - Q_{n-1}(x)\ P_n(x)\ ] = -1$

**Solution :** (i) We have from the recurrance relations,

$$(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$$
$$(2n+1)\, x\, Q_n(x) = (n+1)Q_{n+1}(x) + n\, Q_{n-1}(x)$$

Replacing n by (n-1) in both these equations,

$$(2n-1)\, x\, P_{n-1}(x) = n\, P_n(x) + (n-1)\, P_{n-2}(x) \qquad \ldots. \qquad (4.17)$$
$$(2n-1)\, x\, Q_{n-1}(x) = nQ_n(x) + (n-1)Q_{n-2}(x) \qquad \ldots. \qquad (4.18)$$

Multiplying equation (4.17) by $Q_{n-1}(x)$ and (4.18) by $P_{n-1}(x)$ and subtracting one from the other,

$$0 = n\,(\, P_n(x)\, Q_{n-1}(x) - P_{n-1}(x)\, Q_n(x)\,) + (n-1)\,[\, P_{n-2}(x)\, Q_{n-1}(x) - P_{n-1}(x)\, Q_{n-2}(x)\,]$$

or $$n\,(\, Q_n(x)\, P_{n-1}(x) - Q_{n-1}(x)\, P_n(x)\,) = (n-1)\,[\, Q_{n-1}(x)\, P_{n-2}(x) - Q_{n-2}(x)\, P_{n-1}(x)\,] \qquad \ldots. \qquad (4.19)$$

(ii) Let $t_n = n\,[\, Q_n(x)\, P_{n-1}(x) - Q_{n-1}(x)\, P_n(x)\,]$

The equation (4.19) may be written as $t_n = t_{n-1}$ and this gives $t_{n-1} = t_{n-2} = \ldots\ldots = t_3 = t_2 = t_1$

Thus we have $t_n = t_1$ which implies from Eq. (4.19) that

$$n\,[\, Q_n(x)\, P_{n-1}(x) - Q_{n-1}(x)\, P_n(x)\,] = Q_1(x)\, P_0(x) - Q_0(x)\, P_1(x)$$

or $$n\,[\, Q_n(x)\, P_{n-1}(x) - Q_{n-1}(x)\, P_n(x)\,] = Q_1(x) - x\, Q_0(x) \qquad \ldots.. \qquad (4.20)$$

(Since $P_0 x) = 1$ and $P_1(x) = x$)

But we know that $Q_1(x) - x\, Q_0(x) = \frac{x}{2} \log \frac{x+1}{x+1} - 1 - x\, \frac{1}{2} \log \frac{x+1}{x-1} = -1$

Hence from equation (4.20) we get,

$$n\,[\, Q_n(x)\, P_{n-1}(x) - Q_{n-1}(x)\, P_n(x)\,] = -1$$

or $$P_n(x)\, Q_{n-1}(x) - Q_n(x)\, P_{n-1}(x) = \frac{1}{n}$$

**4.4.3** **Example :** Prove that $Q_2(x) = \frac{1}{2} P_2(x) \log\left(\frac{x+1}{x-1}\right) - \frac{3x}{2}$

**Solution :** From recurrance relation (3) from $Q_n(x)$.
We have $(n+1) Q_{n+1}(x) = (2n+1) x Q_n(x) - n Q_{n-1}(x)$

Replacing n by 1 in this equation,

$$2 Q_2(x) = 3x Q_1(x) - Q_0(x)$$

$$= 3x\left[\frac{x}{2} \log\frac{x+1}{x-1} - 1\right] - \frac{1}{2} \log\frac{x+1}{x-1}$$

$$= \frac{3x^2 - 1}{2} \log\frac{x+1}{x-1} - 3$$

$$= P_2(x) \log\frac{x+1}{x-1} - 3x$$

or $$Q_2(x) = \frac{1}{2} P_2(x) \log\frac{x+1}{x-1} - \frac{3}{2}x$$

**SAQs :**

i) Prove that $P_n(x) Q_{n-2}(x) - Q_n(x) P_{n-2}(x) = \frac{(2n-1)x}{n(n-1)}$

ii) Show that $\frac{d^{n+1}}{dx^{n+1}} Q_n(x)s = -\frac{(-2)^n n!}{(x^2-1)^{n+1}}$

iii) Show that $P^m_n(x) = 0$, if $m > n$

## 4.5 Summary :

Solutions of the Associated Legendre's equation $(1-x^2)y^{11} - 2xy^1 + [n(n+1) - \frac{m^2}{1-x^2}]y = 0$, n and m are integers are known as Associated Legendre's functions. Associated Legendre's functions are widely used in wave mechanics, such as solving the Schrodinger's equation.

In this lesson, we have also established a few recurrance relations for the Legendre function of the second kind, denoted by $Q_n(x)$ and derived the Christoffel's second summation formula.

## 4.6 Model Examination Questions :

1. Define Associated Legendre's Polynomials and prove their orthogonality relation.

2. Show that $P^{-m}_n(x) = (-1)^m \dfrac{(n-m)!}{(n+m)!} P^m{}_n(x)$

3. Prove that $\dfrac{1}{y-x} = \sum\limits_{m=0}^{\infty} (2m+1)P_m(x)Q_m(y)$

4. Show that $Q_m(y) = \dfrac{1}{2}\int\limits_{-1}^{1} \dfrac{P_m(x)}{(y-x)} dx,\ y>1$

5. Using the definition of $Q_n(x)$, obtain the value of $Q_0(x)$ and $Q_1(x)$

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# LESSON - 5
# BESSEL FUNCTIONS - 1

## 5.1 Introduction :

The Bessel functions, denoted by $J_n(x)$ were first introduced by Bessel in 1824 whil discussing a problem in dynamical astronomy. They arise also in the Bernoulli's investigation c oscillations of hanging chains and in the Euler's theory of Vibrations of Circular membrane. Thes functions satisfy the Bessel differential equation :

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + (1 - \frac{n^2}{x^2})y = 0$$

which arises in boundary value problems of mathematical physics. The solutions of the Bessel differential equation are known as Bessel functions of order n.

## 5.2 Bessel differential equation and its solution :

The differential equation $x^2y^{11} + xy^1 + (x^2-n^2)\,y = 0$ ... (5.1)
is called the Bessel equation of order n, n being a non-negative constant. We shall now solve equation (5.1) to obtain a series solution, using the Frobenius method.

We shall assume the solution of equation (5.1) in the form of a series,

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad C_0 \neq 0,$$

Then
$$y^1 = \sum_{m=0}^{\infty} C_m (k+m)\, x^{k+m-1} \quad \text{and}$$

$$y^{11} = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1)\, x^{k+m-2}$$

Substituting for y, $y^1$ and $y^{11}$ in equation (5.1) we get,

$$\rightarrow x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1)\, x^{k+m-2} + x \sum_{m=0}^{\infty} C_m (k+m)\, x^{k+m-1}$$

$$+ (x^2 - n^2) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $\sum_{m=0}^{\infty} C_m[(k+m)(k+m-1)+(k+m)-n^2]x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$

i.e., $\sum_{m=0}^{\infty} C_m(k+m+n)(k+m-n)\, x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$ ... (5.2)

Equating to zero the coefficient of $x^k$, the smallest power of x, we get the indicial equation,

$$C_0 (k+n) (k-n) = 0 \Rightarrow (k+n) (k-n) = 0, \text{ since } C_0 \neq 0.$$

Hence k = n, and k = -n are the roots of the indicial equation. Now equating the coefficient of $x^{k+1}$ to zero we get,

$$C_1 (k+n+1) (k-n+1) = 0, \text{ so that } C_1 = 0, \text{ for } k = n \text{ and } k = -n.$$

Again equating to zero the coefficient of $x^{k+m}$ we obtain $C_m (k+m+n) (k+m-n) + C_{m-2} = 0$

$$\text{or } C_m = \frac{1}{(k+m+n)(n-k-m)} C_{m-2} \quad \dots \quad (5.3)$$

Putting m = 3, 5, 7, ...... in succession in equation (5.3) and using the fact that $C_1=0$, we find that $C_1 = C_3 = C_5 = \dots\dots = 0$.

Putting m = 2, 4, 6, ..... in succession in equation (5.3) we obtain

$$C_2 = \frac{1}{(k+2+n)(n-k-2)} C_0,$$

$$C_2 = \frac{1}{(k+4+n)(n-k-4)} C_2 = \frac{1}{(k+4+n)(n-k-4)} \frac{1}{(k+n+2)(n-k-2)} C_0, \text{ and so on}$$

Substituting these values of $C_i$'s in the assumed series solution y = $\sum C_m x^{k+m}$ we get

$$y = C_0 x^k \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(k+4+n)(n-k-4)} + \dots\right] \quad \dots \quad (5.4)$$

Replacing **k** by n and $C_0$ by 'a' in equation (5.4)

We obtain $$y_1 = a\,x^n \left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4.8.(1+n)(2+n} + \ldots\right] \quad \ldots. \quad (5.5)$$

Replacing **k** by -n and $C_0$ by 'b' in equation (5.4)

We get $$y_2 = b\,x^{-n} \left[1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4.8.(1-n)(2-n)} + \ldots\right] \quad \ldots. \quad (5.6)$$

We now consider a particular solution of equation (5.1) by fixing the constant $a = \dfrac{1}{2^n \Gamma(n+1)}$ in equation (5.5). In this case the obtained solution is called the Bessel function of the first kind and of order n. We denote this solution by $J_n(x)$. Thus

$$y_1 = J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4.8.(n+1)(n+2)} + \ldots\right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} \quad \ldots.. \quad (5.7)$$

If we now subtitute $\dfrac{1}{2^n \Gamma(n+1)}$ for the constant b in equation (5.6) we obtain another solution for equation (5.1) :

$$y_2 = J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\;\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \quad \ldots.. \quad (5.8)$$

We know that $\Gamma(x) \to \infty$, as $x \to 0$ or a negative integer and $\Gamma(x)$ is finite other wise. Suppose that n is not an integer. Since r is always an integer, the factor $\Gamma(-n + r + 1)$ in equation (5.8) is always finite and non-zero.

For $2r < n$, equation (5.8) shows that $J_{-n}(x)$ contains negative powers of x. But from equation (5.6) we observe that $J_n(x)$ does not contain negative powers of x. Also, for x = 0, $J_n(x)$ is finite where as $J_{-n}(x)$ is infinite. Hence one of them can not be expressed as a constant multiple of the others. As such we can conclude that $J_n(x)$ and $J_{-n}(x)$ are two independent solutions of the Bessel equation (5.1) when n is not an integer.

Hence the general solution of the Bessel's differential equation can be expressed as $y = C_1 J_n(x) + C_2 J_{-n}(x)$, where $C_1$ and $C_2$ are arbitary constants. The solutions of Bessel differential equation are called the Bessel functions.

## 5.3 Bessel Function of the second kind and of order n

Suppose that n is an integer. Then we observe that $y_2$ given by equatin (5.8) fails to give a solution for positive values of n and $y_1$ expressed by equation (5.7) fails to give a solution for negative values of n. This suggests us to obtain one another independent solution of the Bessel equation by assuming that $y = \phi(x) J_n(x)$ as the second solution, when n is an integer

Let $y = \phi(x) J_n(x)$

Then $y^1 = \phi^1(x) J_n(x) + \phi(x) J^1_n(x)$ and $y^{11} = \phi^{11}(x) J_n(x) + 2\phi^1(x) J_n^{\,1}(x) + \phi(x) J^{11}_n(x)$

Substituting these values of y, $y^1$ and $y^{11}$ in the Bessel's differential equation (5.1), we get,

$$x^2 \left( \phi^{11}(x) J_n(x) + 2\phi^1(x) J^1_n(x) + \phi(x) J^{11}_n(x) \right)$$
$$+ x\left(\phi^1(x) J_n(x) + \phi(x) J^1_n(x)\right) + (x^2-n^2)\phi(x) J_n(x) = 0$$

or
$$\phi(x) \{ x^2 J^{11}_n(x) + x J^1_n(x) + (x^2-n^2) J_n(x)\}$$
$$+ x^2 \phi^{11}(x) J_n(x) + 2x^2 \phi^1(x) J^1_n(x) + x\phi^1(x) J_n(x)=0$$

Since $J_n(x)$ is a solution for the Bessel's differential equation, the sum in the braces is zero. Hence the above equation reduces to $x^2 \phi^{11}(x) J_n(x) + 2x^2 \phi^1(x) J^1_n(x) + x\phi^1(x) J_n(x)=0$

Dividing both sides of this equation with $x^2 \phi^1(x) J_n(x)$. We get,

$$\frac{\phi^{11}x}{\phi 1(x)} + \frac{2J^1n(x)}{J_n(x)} + \frac{1}{x} = 0.$$

$$\text{or } \frac{d}{dx}(\log\phi^1(x)) + 2\frac{d}{dx}(\log J_n(x)) + \frac{d}{dx}\log(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left[x\log\phi^1(x).J_n{}^2(x)\right] = 0$$

On integrating this equation w.r.t. x we get $\log(x\phi^1(x)\ J^2{}_n(x)) = \log C_2$

or $\quad x\phi^{1}(x)\, J_n^2(x) = \log C_2 \quad \Rightarrow \quad \phi^{1}(x) = \dfrac{C_2}{J_n^2(x).x} \quad \cdots \quad (5.9)$

Integrating equation (5.9) w.r.t. x we get

$$\phi(x) = \int \frac{C_2}{x.J_n^2(x)}\, dx + C_1$$

Hence the assumed second solution of the Bessel's differential equation is

$$y = \phi(x) J_n(x) = \left[\int \frac{C_2}{J_n^2(x).x} dx + C_1\right] J_n(x)$$

$$= C_1 J_n(x) + C_2 J_n(x) \int \frac{1}{x\, J_n^2(x)}\, dx$$

$$= C_1 J_n(x) + C_2 Y_n(x) \text{ where } Y_n(x) = \int \frac{dx}{x\, J_n^2(x)}$$

The function $Y_n(x)$ given above, is known as the Bessel function of the second kind and of order n. This is called the Neumann's function. This is a second independent solution of the Bessel's differential equation.

**Bessel's equation or order zero :**

The equation $xy^{11} + y^{1} + xy = 0$ is known as the Bessels differential equation of order zero. One solution of this equation is given by $y = J_0(x)$ where

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{r!\Gamma(r+1)} = 1 - \frac{x^2}{2^2} + \frac{x4}{2^2.4^2} - \ldots\ldots$$

The second solution denoted by $Y_0(x)$ is called the Bessel function of the second kind of order zero and is obtained from $Y_n(x)$ by putting n = 0. Hence the complete solution of the Bessel's equation of order zero is given by, $y = C_1 J_0(x) + C_2 Y_0(x)$; $C_1$ and $C_2$ are arbitrary constants

## 5.4 Recurrance relations for $J_n(x)$ :

We shall now derive a few recurrance relations for $J_n(x)$ and discuss some of their applications

1. Prove that $\frac{d}{dx} x^n J_n(x) = x^n J_{n-1}(x)$

**Proof :** We have by definition of $J_n(x)$, $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$

Then $\frac{d}{dx} x^n J_n(x) = \frac{d}{dx}\{x^n \sum_{r=0}^{\infty} (-1)^r . \frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\}$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \frac{1}{2^{2r+n}} \frac{d}{dx} x^{2(r+n)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2(r+n)}{r!\Gamma(n+r+1)2^{2r+n}} x^{2r+2n-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r) x^n}{r!(n+r)\Gamma(n+r)} \frac{x^{2r+n-1}}{2^{2r+n}}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} . \frac{1}{\Gamma(n+r)}\left(\frac{x}{2}\right)^{2r+n-1}$$

$$= x^n J_{n-1}(x)$$

2. Prove that $\frac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x)$

**Proof :** We have $\frac{d}{dx}\left\{x^{-n} J_n(x)\right\} = \frac{d}{dx} \sum_{r=0}^{\infty} \frac{(-1)^r x^{-n}}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \cdot \frac{1}{2^n} \frac{d}{dx} \left(\frac{x}{2}\right)^{2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2r)\, x^{2r-1}}{r!\Gamma(n+r+1).2^{2r+n}}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^r}{\angle r-1} \frac{x^{2r-1}}{\Gamma(n+r+1)} \frac{1}{2^{2r+n-1}};$$

Now put r = m + 1. Then the summation shifts to m and m varies between 0 and $\infty$

$$\frac{d}{dx}\left\{-x^{-n} J_n(x)\right\} = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m!\Gamma(n+m+2)} \left(\frac{x}{2}\right)^{2m+n+1} .x^{-n}$$

$$= -x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+n+2)} \left(\frac{x}{2}\right)^{2m+n+1}$$

$$= -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+1+r+1)} \left(\frac{x}{2}\right)^{2r+n+1}$$

$$= -x^{-n} J_{n+1}(x)$$, by the definition of $J_{n+1}(x)$

**3. Prove that** $J^1 n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$ (or) $x J^1 n(x) = -n J_n(x) + x J_{n-1}(x)$

**Proof :** From recurrance relation (1) we have $\frac{d}{dx} x^n J_n(x) = x^n J_{n-1}(x)$

i.e., $n x^{n-1} J_n(x) + x^n J^1{}_n(x) = x^n J_{n-1}(x)$

dividing throughout by $x^{n-1}$ we get, $n J_n(x) + x J^1{}_n(x) = x J_{n-1}(x)$

or $x J^1{}_n(x) = -n J_n(x) + x J_{n-1}(x)$

**4. Prove that** $J^1{}_n(x) = n J_n(x) - x J_{n+1}(x)$

**Proof :** From recurrance relation (2) we have $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$

i.e., $-n x^{-n-1} J_n(x) + x^{-n} J^1 n(x) = -x^{-n} J_{n+1}(x)$

Dividing this equation throughout by $x^{-n}$ we get,

$$-n x^{-1} J_n(x) + J^1 n(x) = -J_{n+1}(x)$$

or $$J^1 n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

5. **Prove that** $2J^1 n(x) = J_{n-1}(x) - J_{n+1}(x)$

**Proof :**

From recurrance relation (3) we have

$$J^1 n(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

From recurrance relation (4) we have

$$J^1 n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

Adding these equations we obtain

$$2J^1 n(x) = J_{n-1}(x) - J_{n+1}(x)$$

## 5.5 Examples :

**5.5.1 Example :** Show that :

(i) $J_{-n}(x) = (-1)^n J_n(x)$ for any (positive or negative) integer n.

(ii) $J_n(-x) = (-1)^n J_n(x)$ for any integer n.

**Solution :** (i) Let n be a positive integer. we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \quad \text{....} \quad (5.10)$$

Since n is an integer, for r = 0, 1, 2, ........, (n-1) it follows that $\Gamma(-n+r+1) \to \infty$ and

$$\frac{1}{\Gamma(-n+r+1)} \to 0.$$

Hence the summation on r in equation (5.10) must be from n to $\infty$, so that

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

Let m = r - n so that r = m + n; m = 0 when r = n, and m $\to \infty$ when r $\to \infty$. Hence

$$J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^{m+n} \frac{1}{(m+n)!\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2(m+n)-n}$$

$$= \sum_{m=0}^{\infty} (-1)^m (-1)^n \frac{1}{\Gamma(m+n+1)\ m!} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m\ 1}{m!\ \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$= (-1)^n J_n(x)$$

Suppose that n is a negative integer. Let p be a positive integer so that n = -p. Then by case (i) :

$$J_{-p}(x) = (-1)^p J_p(x), \text{ so that } J_p(x) = (-1)^{-p} J_{-p}(x).$$

But since p = -n, we have $J_{-n}(x) = (-1)^n J_n(x)$

(ii) Suppose that n is a positive integer

we know that $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Hence $J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{-x}{2}\right)^{n+2r}$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n J_n(x) \qquad \text{since } (-1)^{2r} = 1$$

Let n be a negative integer, say n = -m, where m is a postive integer

Then $J_n(x) = J_{-m}(x) = (-1)^m J_m(x)$, by (i)

Replacing x by -x we get

$$\begin{aligned} J_n(-x) &= (-1)^m J_m(-x) \\ &= (-1)^m (-1)^m J_m(x), \quad \text{(By (ii))} \\ &= (-1)^m J_{-m}(x) = (-1)^{2m} (-1)^m J_{-m}(x) \\ &= (-1)^{-m} J_{-m}(x) = (-1)^n J_n(x) \end{aligned}$$

Hence $J_n(-x) = (-1)^n J_n(x)$ for any integer n.

**5.5.2** **Example :** Show that (i) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cos x$

(ii) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \sin x$

**Solution :**

**(i) we have by definition :**

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2.(n+1)(n+2)} + \dots\right] \quad \dots \quad (5.11)$$

**Putting n =** $-\dfrac{1}{2}$ **in equation (5.11) and simplifying we get,**

$$J_{-\frac{1}{2}}(x) = \frac{x^{-1/2}}{2^{-1/2}\,\Gamma(\frac{1}{2})}\left[1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots\right] = \sqrt{\frac{2}{\pi x}} \cos x \text{, since } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

**(ii) Putting n =** $\dfrac{1}{2}$ **in equation (5.11)**

$$J_{\frac{1}{2}}(x) = \frac{x^{1/2}}{2^{1/2}\,\Gamma(1+\frac{1}{2})}\left[1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} - \dots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \frac{1}{x}\left[x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{\angle 3} + \frac{x^5}{5!} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

**5.5.3. Example :** Show that $J_2(x) - J_0(x) = 2J''_0(x)$

**Solution :** By recurrance relation (5) we have $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ .... (5.12)

Differentiating equation (5.12) throughout w.r.t. x we get

$$2J''_n(x) = J'_{n-1}(x) - J'_{n+1}(x) \quad \ldots \quad (5.13)$$

Replacing n by (n-1) in equation (5.12) and n by (n+1) again in equation (5.12) we get

$$2J'_{n-1}(x) = J_{n-2}(x) - J_n(x)$$
$$2J'_{n+1}(x) = J_n(x) - J_{n+2}(x)$$

Substuting in equation (5.13) we get

$$2J''_n(x) = \frac{1}{2}(J_{n-2}(x) - J_n(x)) - \frac{1}{2}(J_n(x) - J_{n+2}(x))$$

or $$4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x) \quad \ldots\ldots \quad (5.14)$$

Putting n = 0 in equation (5.14) we get

$$4J''_0(x) = J_{-2}(x) - 2J_0(x) + J_2(x)$$
$$= (-1)^2 J_2(x) - 2J_0(x) + J_2(x) \text{ , since } J_{-n}(x) = (-1)^n J_n(x)$$
$$\text{or } 2J''_0(x) = J_2(x) - J_0(x)$$

**5.5.4 Example :** If n > -1, show that $\int_0^x x^{n+1} J_n(x)\,dx = x^{n+1} J_{n+1}(x)$

**Solution :** By recurrance relation (1) we have

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \quad \ldots \quad (5.15)$$

Replacing n by (n+1) in equation (5.15) we get

$$\frac{d}{dx}\left[x^{n+1} J_{n+1}(x)\right] = x^{n+1} J_n(x) \quad \ldots \quad (5.16)$$

Integrating equation (5.16) w.r.t. x from 0 to x we get

$$\left[x^{n+1} J_{n+1}(x)\right]_0^x = \int_0^x x^{n+1} J_n(x)\, dx$$

or
$$x^{n+1} J_{n+1}(x) = \int_0^x x^{n+1} J_n(x)\, dx$$

**5.5.5. Example :** Show that for n > 1, $\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x)$

**Solution :** By recurrance relation (2) we have

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \qquad \ldots\ldots \quad (5.17)$$

Integrating equation (5.17) w.r.t. x from 0 to x we get

$$\left(x^{-n} J_n(x)\right)_0^x = -\int_0^x x^{-n} J_{n+1}(x)\, dx$$

or
$$x^{-n} J_n(x) - \left(\underset{x \to 0}{\text{Lt}} \frac{J_n(x)}{x^n}\right) = -\int_0^x x^{-n} J_{n+1}(x)\, dx$$

But
$$\underset{x \to 0}{\text{Lt}} \frac{J_n(x)}{x^n} = \underset{x \to 0}{\text{Lt}} \frac{1}{x^n} \cdot \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{2.2(n+1)} + \ldots\ldots\right] = \frac{1}{2^n \Gamma(n+1)}$$

Hence
$$x^{-n} J_n(x) - \frac{1}{2^n \Gamma(n+1)} = -\int_0^x x^{-n} J_{n+1}(x)\, dx$$

or
$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \Gamma(n+1)} - x^{-n} J_n(x)$$

**5.5.6. Example :** Prove that $J_n(x)\, J^1_{-n}(x) - J^1_n(x)\, J_{-n}(x) = \frac{-2\sin n\pi}{\pi x}$

**Solution :** We know that Jn(x) and J-n(x) are the solutions of the Bassel's equation $x^2y^{11} + xy^1 + (x^2-n^2)y = 0$

Hence
$$x^2 J_n^{11}(x) + x J_n^1(x) + (x^2-n^2) J_n(x) = 0 \qquad \ldots \quad (5.18)$$

and $$x^2 J_{-n}''(x) + x J_{-n}'(x) + (x^2-n^2) J_{-n}(x) = 0 \qquad \ldots (5.19)$$

Multiplying equation (5.18) by $J_{-n}(x)$ and equation (5.19) by $J_n(x)$ and subtracting one from the other we get,

$$x^2 [ J_n''(x) J_{-n}(x) - J_n(x) J_{-n}''(x) ] + x [ J_n'(x) J_{-n}(x) - J_n(x) J_{-n}'(x) ] = 0$$

or $$J_n''(x) J_{-n}(x) - J_n(x) J_{-n}''(x) + \frac{1}{x}[ J_n'(x) J_{-n}(x) - J_n(x) J_{-n}'(x) ] = 0 \qquad \ldots (5.20)$$

Now let $J_n'(x) J_{-n}(x) - J_n(x) J_{-n}'(x) = u(x)$

Then $$\begin{aligned} u'(x) &= J_n''(x) J_{-n}(x) - J_n(x) J_{-n}''(x) - J_n'(x) J_{-n}(x) - J_n(x) J_{-n}'(x) \\ &= J_n''(x) J_{-n}(x) - J_n(x) J_{-n}''(x) \end{aligned}$$

Substituting the values of u(x) and u'(x) in equation (5.20) we get,

$$u^1(x) + \frac{1}{x} u(x) \Rightarrow \frac{du}{u} + \frac{dx}{x} = 0$$

Integrating w.r.t. x we get, $\log u + \log x = \log C \Rightarrow u(x) = \frac{c}{x}$

$$\text{or } J'_n(x) J_{-n}(x) - J_n(x) J'_{-n}(x) = \frac{c}{x} \qquad \ldots (5.21)$$

i.e., $$\frac{1}{2^n \Gamma(n+1)}\left[ n x^{n-1} - \frac{(n+2)\, x^{n+1}}{4(n+1)} + \frac{(n+4)\, x^{n+3}}{4.8.(n+1)(n+2)} + \ldots \right] X$$

$$\frac{1}{2^{-n}\Gamma(-n+1)}\left[ x^{-n} - \frac{x^{2-n}}{4(1-n)} + \frac{x^{4-n}}{4.8.(1-n)(2-n)} - \ldots \right]$$

$$-\frac{1}{2^n \Gamma(n+1)}\left[ x^n - \frac{x^{n+2}}{4(n+1)} + \frac{x^{n+4}}{4.8.(n+1)(n+2)} + \ldots \right] X$$

$$\frac{(-1)}{2^{-n}\Gamma(-n+1)}\left[ -n x^{-n-1} - \frac{(2-n)]\, x^{1-n}}{4(1-n)} + \frac{(4-n)\, x^{3-n}}{4.8.(1-n)(2-n)} - \ldots \right]$$

$$= \frac{c}{x} \qquad \ldots (5.21)$$

Comparing the coefficient of 1/x on both sides of equation (5.21) we find that

$$\frac{n}{2^n\Gamma(n+1)2^{-n}\Gamma(-n+1)}+\frac{n}{2^{-n}\Gamma(-n+1)2^n\Gamma(n+1)}=C$$

$$\Rightarrow C=\frac{2n}{n\Gamma(n)\Gamma(1-n)}=\left(\frac{2}{\pi/\sin n\pi}\right)=\frac{2\sin n\pi}{\pi}\text{; since } n\,\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

Hence from equation (5.21) we get

$$J'_n(x)\,J_{-n}(x) - J_n(x)\,J'_{-n}(x)=\frac{2\sin n\pi}{\pi x}$$

or $$J_n(x)\,J'_{-n}(x) - J'_n(x)\,J_{-n}(x)=\frac{-2\sin n\pi}{\pi x}$$

**5.5.7 Example :**

Show that $\frac{d}{dx}\left[J^2{}_n(x)+J^2{}_{n+1}(x)\right]=2\left[\frac{n}{x}J^2{}_n(x)-\frac{n+1}{x}J^2{}_{n+1}(x)\right]$

**Solution :** From recurrance relation (3) we have $J^1{}_n(x)=-\frac{n}{x}J_n(x)+J_{n-1}(x)$

Replacing n by (n+1) in this recurrance relation we get $J^1{}_{n+1}(x)=\frac{-(n+1)}{x}J_{n+1}(x)+J_n(x)$ .... (5.22)

Also, we have $\frac{d}{dx}\left[J^2{}_n(x)+J^2{}_{n+1}(x)\right]=2\,J_n(x)J^1{}_n(x)\;+2\,J_{n+1}(x)\times J^1{}_{n+1}(x)$ .... (5.23)

From recurrance relation (4) we have $J^1{}_n(x)\;=\;\frac{n}{x}J_n(x)-J_{n+1}(x)$ .... (5.24)

Using equation (5.22) and (5.24) in equation (5.23)

we obtain $\frac{d}{dx}\left[J^2{}_n(x)+J^2{}_{n+1}(x)\right]=2\,J_n(x)\left[\frac{n}{x}J_n(x)-J_{n+1}(x)\right]+$

$$2\,J_{n+1}(x)\left[-\frac{n+1}{x}J_{n+1}(x)+J_n(x)\right]$$

$$= 2\left[\frac{n}{x} J^2{}_n(x) - \frac{(n+1)}{x} J^2{}_{n+1}(x)\right]$$

**SAQs :**

1. Show that $\int_0^{\pi/2} \sqrt{\pi x}.J_{1/2}(2x)\ dx = 1$

2. Establish the recurrance relation : $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$

3. Show that $\frac{d}{dx} J_0(x) = -\ J_1(x)$ (Hint : Use recurrance relation (2) )

4. Show that all roots of $J_n(x)$ are real.

## 5.6 Summary :

The Bessel differential equation of order n i.e., $x^2y^{11} + xy^1 + (x^2-n^2)\,y = 0$ was solved for non integral n and its general solution was expressed as a linear combination of two independent solutions $J_n(x)$ and $J_{-n}(x)$ as $y = C_1J_n(x) + C_2J_{-n}(x)$. When n is an integer, a second independent solution of the Bessel equation was obtained by assuming $y = \phi\ (x)\ J_n(x)$ and this solution was expressed as a linear combination of $J_n(x)$ and $Y_n(x) = \int \frac{dx}{x\,J^2{}_n(x)}$. The function $Y_n(x)$ is called the Neumann's function, is the Bessel function of the second kind and of order n. A few recurrance relations were established in terms of the Bessel functions.

## 5.7 Model Examination questions :

1. Show that $J_n(x) = \dfrac{x^n \int_0^1 (1-t^2)^{n-1/2} \cos xt\ dt}{2^{n-1}\Gamma(1/2)\Gamma(n+1/2)}$.

2. Show that $J^2{}_0(x) + 2\,(J^2{}_1(x) + J^2{}_2(x) + ....) = 1$. Hence deduce that $|J_0(x)| \le 1$

3. Prove that $\int_a^b J_0(x)\,J_1(x)\,dx = \frac{1}{2}\left[J^2{}_0(a) - J^2{}_0(b)\right]$

4. Determine the values of a and b for which $\frac{d}{dx}\left(J_n(x)\right) = a\, J_{n-1}(x) + b\, J_{n+1}(x)$

(Ans : a = 1/2 , b = - 1/2)

5. Evaluate $\int x^3 J_3(x)\, dx$ (Ans : $-x^3 J_2(x) - 5x^2 J_1(x) - 15\, x\, J_0(x) + 15 \int J_0(x)\, dx$)

6. Show that :

(i) $\frac{d}{dx}\left\{x J_1(x)\right\} = x\, J_0(x)$

(ii) $J_2(x) - J_0(x) = 2\, J^{11}{}_0(x)$

(iii) $J_2(x) = J^{11}{}_0(x) - 1/x\, J^1{}_0(x)$

7. Show that $J_n(x) = 0$ has no repeated roots except at x = 0.

8. Evaluate $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$. (Ans : $J_3(x) = \frac{(8-x^2)}{x^2} J_1(x) - \frac{4}{x} J_0(x)$).

9. Show that $\int_0^x x^2\, J_0(x)\, J_1(x)\, dx = \frac{1}{2} x^2 J^2{}_1(x)$.

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# LESSON 6 : BESSEL FUNCTIONS -2

## 6.1 Introduction

In the earlier lesson we have solved the Bessel's differential equation of order n, n being a non-negative constant and obtained a series solution using the Frobenius method. When n is not an integer, we obtained two linearly independent solutions $J_n(x)$ and $J_{-n}(x)$. However, when n is an integer, we learnt that $J_n(x)$ and $Y_n(x)$ form two linearly independent solutions of the Bessel equation. We have also established a few recurrance relations involving the Bessel functions and their first order derivatives. In the discussion that follows in this lesson we shall define the generating function for the Bessel function, establish the orthogonality property of Bessel functions and obtain the Fourier - Bessel series expansion for f(x)

## 6.2 Generating function for $J_n(x)$

**Theorem :-** For any positive integer n, $J_n(x)$ is the co-efficient of $z^n$ in the expansion of $\exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}$

**Proof :-** Consider $\exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\} = e^{\frac{xz}{2}} \cdot e^{-\left(\frac{x}{2}\right)z^{-1}}$

$$= \left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+\ldots\ldots+\left(\frac{x}{2}\right)^n\frac{z^n}{n!}+\ldots\ldots\ldots\right]\times$$

$$\left[1-\left(\frac{x}{2}\right)z^{-1}+\left(\frac{x}{2}\right)^2\frac{z^{-2}}{2!}+\ldots\ldots+(-1)^n\left(\frac{x}{2}\right)^n\frac{z^{-n}}{n!}+\ldots\ldots\right] \quad ..(6.1)$$

The coefficent of $z^n$ in in the product on the RHS of equation (6.1) is obtained by multplying the coefficient of $z^n, z^{n+1}, z^{n+2}$ .......... in the first bracket with those of the coefficients of $z^0, z^{-1}, z^{-2}$ ................. in the second bracket respectively. Hence the co-efficient of $z^n$ is

$$\left(\frac{x}{2}\right)^n \frac{1}{n\,!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \left(\frac{x}{2}\right)^{n+4} \frac{1}{(n+2)!\;2!} + \ldots\ldots\ldots\ldots$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!(n+r)\,!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum \frac{(-1)^r}{r\,!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

The Co efficient of $z^{-n}$ in the product on the RHS of equation (6.1) is obtained by multplying the coefficient of $z^{-n}$, $z^{-n-1}$ .......... in the second bracket with those of the coefficients of $z^0$, $z, z^2$ ........... in the first bracket respectively. Thus the coefficient of $z^{-n}$ is

$$\left(\frac{x}{2}\right)^n \frac{(-1)^n}{n\,!} + \left(\frac{x}{2}\right)^{n+1} \frac{(-1)^{n+1}}{(n+1)!} \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^{n+2} \frac{(-1)^{n+2}}{(n+2)!\;2!} \left(\frac{x}{2}\right)^2 + \ldots\ldots\ldots\ldots$$

$$= (-1)^n \left\{ \left(\frac{x}{2}\right)^n \frac{1}{n!} - \left(\frac{x}{2}\right)^{n+2} \frac{1}{(n+1)!} + \ldots\ldots\ldots\ldots \right\}$$

$$= (-1)^n J_n(x)$$

$$\text{or } J_n(x) = (-1)^n \text{ co efficient of } z^{-n}$$

In particular the coefficient of $z^0$ in the product on the RHS of equation (6.1) is obtained by multiplying the coefficients of $z^0, z^1, z^2$ ....... in the first bracket with the co-efficients $z^0, z^{-1}, z^{-2}$, .............. in the second bracket respectively. Hence the co-efficient of $z^0$ is

$$= 1 - \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^4 \left(\frac{1}{\angle 2}\right)^2 - \left(\frac{x}{2}\right)^6 \left(\frac{1}{\angle 3}\right)^2 + \ldots\ldots\ldots\ldots$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \ldots\ldots\ldots\ldots\ldots\ldots = J_0(x)$$

We observe here that the coefficient of

$z^0, (z - z^{-1}), (z^2 + z^{-2}) \ldots\ldots\ldots\ldots, (z^n + (-1)^n z^n), \ldots\ldots\ldots$

are $J_0(x)\ J_1(x), \ldots\ldots\ldots\ldots\ldots\ldots, J_n(x)$ respectively.

Hence equation (6.1) gives

$$e^{\frac{x}{2}(z - \frac{1}{z})} = J_0(x) + (z - z^{-1})J_1(x) + (z^2 + z^{-2})\, J_2(x) + \ldots\ldots\ldots\ldots +$$

$$\left[z^n + (-1)^n z^{-n}\right] J_n(x) + \ldots\ldots\ldots$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

Hence exp $\left\{\frac{x}{2}(z - \frac{1}{z})\right\}$ is called the generating function of $J_n(x)$.

## 6.3. Expansion of Trigonometric functions interms of Bessel functions :

**Result :-** Prove that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) + \ldots\ldots\ldots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) + \ldots\ldots\ldots\ldots\ldots$$

**Proof** : From the generating function of $J_n(x)$ we have

$$e^{\frac{x}{2}(z - \frac{1}{z})} = J_0(x) + (z - z^{-1})J_1(x) + (z^2 + \frac{1}{z^2})J_2(x) + \ldots\ldots\ldots\ldots$$

Put $z$ = $e^{i\theta}$ Then

$$e^{\frac{x}{2}(z - \frac{1}{z})} = e^{\frac{x}{2}(e^{i\theta} - e^{-i\theta})}$$

$$= J_0(x) + (e^{i\theta} - e^{-i\theta})\, J_1(x) + (e^{2i\theta} + e^{-2i\theta})J_2(x) + \ldots\ldots\ldots\ldots$$

or $e^{ix\sin\theta} = J_0(x) + (2i\sin\theta)\, J_1(x) + (2\cos 2\theta)\, J_2(x) + \ldots\ldots\ldots\ldots\ldots$

i.e. $\cos(x\sin\theta) + i\sin(x\sin\theta)$

$$= \{J_0(x) + 2(\cos 2\theta)\, J_2(x) + 2(\cos 4\theta)\, J_4(x) + \ldots\} +$$

$$i\left[2(\sin\theta)J_1(x) + 2(\sin 3\theta)J_3(x) + \ldots\ldots\right]$$

Equating the real and imaginary parts on both the sides of this equation we obtain.

$$\cos(x\sin\theta) = J_0(x)+2(\cos 2\theta)J_2(x) + 2(\cos 4\theta)J_4(x)+...... \qquad (6.2)$$

and $$\sin(x\sin\theta) = (2\sin\theta)J_1(x)+(2\sin 3\theta)J_3(x)+........ \qquad (6.3)$$

Putting $\theta = \frac{\pi}{2}$ in both sides of the equations we get

(6.2) and (6.3)

$$\cos x = J_0(x)-2J_2(x)+2J_4(x)-.......$$

and $$\sin x = 2J_1(x)-2J_3(x)+2J_5(x)-.....$$

## 6.4 Orthogonality of Bessel Functions

We shall now establish the orthogonality property of Bessel Functions through the following theorem.

**Theorem** :- If $\lambda_i$ and $\lambda_j$ are the roots of the equation $J_n(\lambda a)=0$, then

$$\int_0^a xJ_n(\lambda_i x)\,J_n(\lambda_j x)dx = 0, \quad \text{if } \lambda_i \neq \lambda_j \quad = \frac{a}{2}J^2_{n+1}(\lambda_i a) \quad \lambda_i = \lambda_j$$

**Proof** :- Suppose that $i \neq j$ and $\lambda_i$ and $\lambda_j$ be unequal roots of the equation $J_n(\lambda a)=0$

Let $u(x) = J_n(\lambda_i x)$ and $v(x) = J_n(\lambda_j x)$ ---- (6.4)

Then we can verify that 4(x) and v(x) satisfy the Bessels' equation :

$$x^2y^{11} + xy^1 + ((\lambda^2x^2 - x^2)y = 0 \qquad \text{(Since they are Bessel functions)}$$

i.e., $$x^2u^{11}+xu^1+(\lambda^2_i x^2-n^2)u = 0 \qquad (6.5)$$

and $$x^2\upsilon^{11}+x\upsilon^1+(\lambda^2_j x^2-n^2)\upsilon = 0 \qquad (6.6)$$

Multiplying equation (6.5) by $\upsilon$ and equation (6.6) by u and subtracting one from the other we get

$$x^2(\upsilon u^{11}-u\upsilon^{11}) + x(\upsilon u^1-u\upsilon^1) + x^2(\lambda^2_i - \lambda^2_j)u\upsilon = 0$$

or $$x(\upsilon u^{11}-u\upsilon^{11}) + (\upsilon u^1-u\upsilon^1) = x(\lambda^2_j - \lambda^2_i)u\upsilon$$

i.e. $x\frac{d}{dx}(\upsilon u^1 - u\upsilon^1) + (\upsilon u^1 - u\upsilon^1) = x(\lambda_j^2 - \lambda_i^2)u\upsilon$

or $\frac{d}{dx}\left\{x(\upsilon u^1 - u\upsilon^1)\right\} = x(\lambda_j^2 - \lambda_i^2)u\upsilon$ --- (6.7)

Integrating equation (6.7) w.r.t. x between 0 and a we get

$$\left[x(\upsilon u^1 - u\upsilon^1)\right]_0^a = (\lambda_j^2 - \lambda_i^2)\int_0^a x\, u\upsilon\, dx$$

Substituting for u and $\upsilon$ from equation (6.4) we get

$$(\lambda_j^2 - \lambda_i^2)\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx$$

$$= \left\{x\left[J_n(\lambda_j x)J_n^1(\lambda_i x) - J_n(\lambda_i x)J_n^1(\lambda_j x)\right]\right\}_0^a$$

$$= a\left[J_n(\lambda_j a)J_n^1(\lambda_i a) - J_n(\lambda_i a)J_n^1(\lambda_j a)\right]$$

$= 0$; Since $J_n(\lambda a) = 0$, for $\lambda = \lambda_i$ and $\lambda = \lambda_j$

Since $\lambda_i \neq \lambda_j$, this implies that

$$\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = 0, \quad i \neq J \quad \text{---(6.8)}$$

**Case 2** : Suppose that j = i, i.e. the roots of $J_n(\lambda a) = 0$ are equal. Then multiplying equation (6.5) with $2u^1$ we get

$$2x^2u^1u^{11} + 2xu^{1^2} + 2(\lambda_i^2x^2 - n^2)uu^1 = 0$$

or $$\frac{d}{dx}\left[x^2u^{1^2} - n^2u^2 + \lambda_i^2x^2u^2\right] - 2\lambda_i^2\, x\, u^2 = 0$$

i.e. $2\lambda_i^2\, x\, u^2 = \frac{d}{dx}\left[x^2u^{1^2} - n^2u^2 + \lambda_i^2x^2u^2\right]$ --- (6.9)

Integrating equation (6.9) w.r.t.x. from 0 to a we get

$$2\lambda_i^2 \int_0^a x\,u^2 dx = \left[x^2 u^{1^2} - n^2u^2 + \lambda_i^2 x^2 u^2\right]_0^a$$

Since u (x) = $J_n(\lambda_i x)$ and $J_n(\lambda_i a) = 0$ this Equation reduces to

$$2\lambda_i^2 \int_0^a x\, J_n^2(\lambda_i x)\,dx = a^2\left[\left\{J_n^1(\lambda_i x)\right\}^2\right]_{x=a} \quad \ldots \quad \textbf{(6.10)}$$

Replacing x by $(\lambda_i x)$ in the recurrance relation

$$\frac{d}{dx}J_n(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \text{ we get}$$

$$\frac{d}{d(\lambda_i x)}J_n(\lambda_i x) = \left(\frac{n}{\lambda_i x}\right)J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$

or $$\frac{1}{\lambda_i}J_n^1(\lambda_i x) = \left(\frac{n}{\lambda_i x}\right)J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$

i.e., $$J_n^1(\lambda_i x) = \left(\frac{n}{x}\right)J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)$$

Hence $$\left[\left\{J_n^1(\lambda_i x)\right\}^2\right]_{x=a} = \left[\left\{\frac{n}{x}J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\right\}^2\right]_{x=a}$$

$$= \left\{0 - \lambda_i J_{n+1}(\lambda_i a)\right\}^2, \text{ since } J_n(\lambda_i a) = 0$$

$$= \lambda_i^2 J_{n+1}^2(\lambda_i a) \quad \ldots. (6.11)$$

Using equation (6.11) in equation (6.10) we get,

$$2\lambda_i^2 \int_0^a x\, J_n^2(\lambda_i x)\,dx = a^2\lambda_i^2 J_{n+1}^2(\lambda_i a)$$

or $$\int_0^a x\, J_n^2(\lambda_i x)\,dx = \frac{a^2}{2}J_{n+1}^2(\lambda_i a) \quad \ldots. (6.12)$$

combining the equations (6.8) and (6.12) we have

$$\int_0^a x\, J_n(\lambda_i x)\; J_n(\lambda_j x)\, dx = 0, \text{ if } \lambda_i \neq \lambda_j \qquad = \frac{a^2}{2} J^2_{n+1}(\lambda_i a) \text{ if } \lambda_i = \lambda_j$$

Note : This result can also be stated alternatively as :

$$\int_0^a x\, J_n(\lambda_i x)\; J_n(\lambda_j x)dx = \frac{a^2}{2} J^2_{n+1}(\lambda_i a)\; \delta_{ij}$$

$$\text{when } \delta_{ij} = 0, \text{ if } i \neq j$$

$$= 1, \text{ if } i = j$$

**6.5 Fourier - Bessel expansion for f(x)**

**Theorem** :- If f(x) is defined for all x in the intervel [ 0, a ] and has an expansion of the form

$$f(x) = \sum_{i=0}^{\infty} c_i J_n(\lambda_i x) \qquad \text{--- (6.13)}$$

where $\lambda_i$ are the roots of the equation $J_n(\lambda_i x) = 0$, then for each i, $c_i$ is given by

$$c_i = \frac{2\int_0^a x\, f(x)\, J_n(\lambda_i x)dx}{a^2 J^2_{n+1}(\lambda_i a)}$$

**Proof** :- Consider equation (6.13) and multiply both sides by $x\, J_n(\lambda_j x)$ we get

$$x\, f(x)\, J_n(\lambda_j x) = \sum_{i=0}^{\infty} c_i\, x\, J_n(\lambda_i x)\, J_n(\lambda_j x)$$

Integrating both sides of this equation w.r.t.x. from 0 to a we get

$$\int_0^a x\, f(x) J_n(\lambda_j x)\, dx = \sum_{i=0}^{\infty} c_i \int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx$$

$$= c_j \frac{a^2}{2} J^2_{n+1}(\lambda_j a)$$

Replacing j by i we obtain

$$\int_0^a x\, f(x) J_n(\lambda_i x)\, dx = c_i \frac{a^2}{2} J^2_{n+1}(\lambda_i a)$$

$$\Rightarrow \qquad c_i = \frac{2\int_0^a x\, f(x)\, J_n(\lambda_i x)\, dx}{a^2 J_{n+1}^2(\lambda_i a)}$$

Because of its similarity to a Fourier series, the series of the type (6.13) is called a Fourier - Bessel series.

## 6.6 Examples :

**6.6.1 Example** : Using the generating function for $J_n(x)$, show that $J_n(-x)=(-1)^n J_n(x)$

**Solution :-** we have from the generating function for $J_n(x)$

$$\sum_{n=-\infty}^{\infty} z^n J_n(x) = e^{\frac{x}{2}(z-\frac{1}{z})}$$

Replacing x by -x in both sides of this equation, we get

$$\sum_{n=-\infty}^{\infty} z^n J_n(-x) = e^{-\frac{x}{2}(z-\frac{1}{z})}$$

$$= e^{\frac{x}{2}(-z+\frac{1}{z})}$$

$$= e^{\frac{x}{2}(-z-\frac{1}{-z})}$$

$$= \sum_{n=-\infty}^{\infty} (-z)^n J_n(x)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n (z)^n J_n(x) \qquad \ldots \qquad (6.14)$$

Equating the co-efficient of $Z^n$ on both sides of the equation (6.14) we get

$$J_n(-x) = (-1)^n J_n(x)$$

**6.6.2 Example : Show that**

(i) $$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta \quad \text{for any integer n}$$

(ii) $$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\, d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\, d\theta .$$

**Solution** :- We have from equation (6.2) and (6.3)

$$\cos(x\sin\theta) = J_0(x) + 2(\cos 2\theta) J_2(x) + 2(\cos 4\theta) J_4(x) + \ldots\ldots \quad \text{---} \quad \mathbf{(6.2)}$$

$$\sin(x\sin\theta) = 2(\sin\theta) J_1(x) + 2(\sin 3\theta) J_3(x) + \ldots\ldots \quad \text{---} \quad \mathbf{(6.3)}$$

Multiplying both sides of equation (6.2) with $\cos 2m\theta$ and integrating w.r.t. $\theta$ from 0 to $\pi$ we get

$$\int_0^{\pi} \cos(x\sin\theta)\cos 2m\theta\, d\theta = J_0(x)\int_0^{\pi}\cos 2m\theta\, d\theta + 2J_2(x)\int_0^{\pi}\cos 2\theta \cos 2m\,\theta\, d\theta$$

$$+ \ldots\ldots\ldots + 2J_{2m}(x)\int_0^{\pi}\cos^2(2m\theta)\, d\theta + \ldots\ldots$$

$$= 0 + 0 + \ldots\ldots + J_{2m}(x)\int_0^{\pi}(1+\cos 4m\theta)\, d\theta$$

$$+ 0 + \ldots\ldots\ldots$$

$$= \pi J_{2m}(x)$$

Similarly we can show that

$$\int_0^{\pi}\cos(x\sin\theta)\cos(2m+1)\theta\, d\theta = 0$$

Multiplying both sides of equation (6.3) by sin (2m+1) $\theta$ and integrating w.r.t. $\theta$ from 0 to $\pi$ we get

$$\int_0^\pi \sin(x\sin\theta)\sin(2m+1)\theta\, d\theta = 2J_1(x)\int_0^\pi \sin\theta \sin(2m+1)\theta\, d\theta +$$

$$2J_3(x)\int_0^\pi \sin 3\theta \sin(2m+1)\theta\, d\theta + \ldots\ldots$$

$$+2J_{2m+1}(x)\int_0^\pi \sin^2(2m+1)\theta\, d\theta + \ldots$$

$$= 0+0+\ldots\ldots+J_{2m+1}(x)\int_0^\pi [1-\cos(2m+1)\theta]d\theta + 0 + \ldots$$

$$= \pi J_{2m+1}(x)$$

In a similar manner we can show that

$$\int_0^\pi \sin(x\sin\theta)\sin 2m\theta\, d\theta = 0$$

Hence, $\int_0^\pi \cos(2m\theta - x\sin\theta)d\theta =$

$$\int_0^\pi \cos 2m\theta \cos(x\sin\theta)d\theta + \int_0^\pi \sin 2m\theta \sin(x\sin\theta)\, d\theta = \pi J_{2m}(x)$$

and $$\int_0^\pi [\cos(2m+1)\theta - x\sin\theta]d\theta = \int_0^\pi [\cos(2m+1)\theta \cos(x\sin\theta]d\theta +$$

$$\int_0^\pi [\sin(2m+1)\theta \sin(x\sin\theta]d\theta$$

$$= \pi J_{2m+1}(x)$$

Hence for all positive integral n, we have

$$\int_0^\pi \cos(n\theta - x\sin\theta)d\theta = \pi J_n(x)$$

suppose that n is a negative integer. Then we can write n = -m where m is a positive integer. Now

$$\int_0^\pi \cos(n\theta - x\sin\theta)d\theta = \int_0^\pi [\cos(-m\theta - x\sin\theta)]d\theta$$

$$= -\int_{\pi}^{0} \cos(-m(\pi-\phi) - \sin(\pi-\phi)\, d\phi$$

if $\theta = (\pi - \phi)$

$$= \int_{0}^{\pi} \cos[-m\pi + (m\phi - x\sin\phi)]\, d\phi$$

$$= \int_{0}^{\pi} \{\cos m\pi . \cos(m\phi - x\sin\phi) + \sin m\pi \ \sin(m\phi - x\sin\phi)\} d\phi$$

$$= (-1)^m \int_{0}^{\pi} \cos(m\phi - x\sin\phi)\, d\phi$$

$$= (-1)^m \pi\, J_m(x) = \pi\, J_{-m}(x) = \pi\, J_n(x)$$

Hence, for all integral values of n we have $\int_{0}^{\pi} \cos(m\theta - x\sin\theta) d\theta = \pi\, J_n(x)$

(ii) We have $\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta\, J_2(x) + ....$ **(6.2)**

$$\int_{0}^{\pi} \cos(x\sin\theta)\, d\theta = J_0(x) \int_{0}^{\pi} d\theta + 2\, J_2(x) \int_{0}^{\pi} \cos 2\theta\, d\theta + 2\, J_4(x) \int_{0}^{\pi} \cos 4\theta\, d\theta + ............$$

we know that $\int_{0}^{\pi} \cos p\theta\, d\theta = 0$, if p is an even integer.

Hence $\int_{0}^{\pi} \cos(x \sin\theta)\, d\theta = \pi\, J_0(x)$

$$\text{or} \quad J_0(x) = \frac{1}{\pi} \int_{0}^{\pi} \cos(x\sin\theta)\, d\theta \qquad (6.15)$$

Replacing $\theta$ by $\left(\frac{\pi}{2} - \theta\right)$ in equation (6.2) we get

$$\cos(x\cos\theta) = J_0(x) - 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) + ..........$$

Integrating both sides of this equation w.r.t. $\theta$ from 0 to $\Pi$ we get

$$\int_0^\pi \cos(x\cos\theta)\,d\theta = J_0(x)\int_0^\pi d\theta - 2J_2(x)\int_0^\pi \cos 2\theta\,d\theta + \ldots\ldots\ldots$$

$$= \pi J_0(x)$$

or $$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)\,d\theta \qquad \ldots \quad \textbf{(6.16)}$$

From equations (6.15) and (6.16) the result (ii) follows

**6.6.3. Example** :- Expand the function f(x) =1, 0 < x < a in a series of the form $\Sigma c_i J_0(\lambda_i x)$, where $\lambda_i$ are the roots of the equation $J_0(\lambda a)=0$,

**Solution :** The Fourier - Bessel expansion for f(x) =1, is given by

$$1 = f(x) = \sum_{i=0}^{\infty} c_i J_0(\lambda_i x) \text{ where } J_0(\lambda_i a) = 0$$

Here $c_i$ is given by $$c_i = 2\int_0^a \frac{xf(x)}{a^2}\frac{J_0(\lambda_i x)\,dx}{J_1^2(\lambda_i a)}$$

$$= 2\int_0^a \frac{x J_0(\lambda_i x)\,dx}{a^2 J_1^2(\lambda_i a)} \text{ ; since f(x) =1,} \qquad \text{--- (6.17)}$$

Let $\lambda_i x = t$ So that $dx = \frac{1}{\lambda_i}dt$ Then

$$\int_0^a x J_0(\lambda_i x)\,dx = \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} t J_0(t)\,dt$$

$$= \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} \frac{d}{dx}\left(t J_1(t)\right)dt \qquad \text{by reccurance relation (1)}$$

$$= \frac{1}{\lambda_i^2}\left[t.J_1(t)\right]_0^{a\lambda_i}$$

$$= \frac{1}{\lambda_i^2}\left[a\lambda_i J_1(\lambda_i a) - 0\right], \quad \text{Since } J_1(0) = 0,$$

$$= \left(\frac{a}{\lambda_i}\right) J_1(\lambda_i a) \qquad \text{---} \qquad (6.18)$$

Using equation (6.18) in equation (6.17) we get

$$c_i = \frac{\frac{2}{a^2}\left[\frac{a}{\lambda_i}\right] J_1(\lambda_i a)}{J_1^2(a\lambda_i)} = \frac{2}{a\lambda_i J_1(a\lambda_i)}$$

consequently

$$1 = \frac{2}{a}\sum_{i=0}^{\infty}\frac{J_0(\lambda_i x)}{\lambda_i J_1(\lambda_i a)}$$

**SAQS :**

(i) Show that $\int_0^{\infty}\frac{J_n(x)}{x}dx = \frac{1}{n}$

(ii) Expand x in a series of the form $\sum_{r=1}^{\infty} c_r J_1(\lambda_r x)$

in the interval $0 \le x \le 1$, where $\lambda_r$ are the roots of the equation $J_1(\lambda_1 a) = 0$

(iii) show that $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \ldots\ldots\ldots$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r r!)^2}$$

## 6.7 Summary :

The generating function for the Bessel Function $J_n(x)$ was defined and the trigonometric functions Cos x and Sin x were expanded interms of Bessel functions $J_0(x)$, $J_1(x)$ .......... The orthogonality property of Bessel functions was established and a Fourier - Bessel series expansion for f(x) was obtained.

**6.8 Model Examination questions :**

1. Prove that $J_n(x) = \dfrac{1}{\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)}\left(\dfrac{x}{2}\right)^n \int_0^{\pi} \cos(x\sin\phi)\cos 2n\phi\ d\phi$

2. if a > o, Prove that $\int_0^x e^{-ax} J_0(bx)dx = \dfrac{1}{\sqrt{a^2+b^2}}$

3 Using the generating function for $J_n(x)$, show that

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x)\, J_{n-r}(x)$$

4. If n is non - negative, show that $\int_0^x J_n(bx)\,dx = \dfrac{1}{b}$

5. Expand $x^2$ in a series of th form $\sum_{r=1}^{\infty} C_r J_0(\lambda_r x)$

in the interval [ 0 , a ] where $\lambda_r$ are the roots of the equation $J_0(\lambda a) = 0$

( Ans :- $x^2 = \dfrac{2}{a}\sum_{r=1}^{\infty} \dfrac{\left[(\lambda_r a)^2 - 4\right]}{\lambda_r^2}\dfrac{J_0(\lambda_r x)}{J_0(\lambda_r a)}$ )

6. If $\alpha$ and $\beta$ are the roots of the equation $J_0(x) = 0$, show that

$$\int_0^1 x\, J_0(\alpha x) J_0(\beta x)\ \text{dx} = \frac{1}{2} J_1^2(\alpha)\ \delta_{\alpha\beta}$$

7. If $\alpha$ and $\beta$ are the roots of the equation $J_0(x) = 0$, then Prove that

$$\int_0^1 x\, J_n(\alpha x) J_n(\beta x)\ \text{dx} = 0. \qquad \alpha \neq \beta$$

8. Prove that $\sum_{n=1}^{\infty} \dfrac{2}{\alpha_n}\dfrac{J_0(\alpha_n x)}{J_1(\alpha_n)} = 1.$

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# LESSON - 7 : TOTAL DIFFERENTIAL EQUATIONS

## 7.1 INTRODUCTION :

An ordinary differential equation of first order and of degree one involving three variables is expressed in the form : $P+Q\frac{dy}{dx}+R\frac{dZ}{dx}=0$, where P,Q,R are functions of x, y, z and x is the independent variable. In terms of differentials, this equation can be written as

$$P\,dx + Q\,dy + R\,dZ = 0 \tag{7.1}$$

Equation (7.1) is called a total differential equation. It can be integrated directly if there exists a function u (x,y,z) whose total differential du is equal to the L.H.S. of (7.1). In other cases, equation (7.1) may not be integrable. We shall now find the condition which P, Q, R must satisfy in order that equation (7.1) may be integrable.

## 7.2 Condition for the integrability of the total differential equation P dx + Q dy + R dZ=0

**Theorem :**
A necessary and sufficient condition for the differential equation P dx + Q dy + R dz=0 to be integrable is that :

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

**Proof :** Suppose that the condition is necessary : consider the differential equation P dx + Q dy + R dz = 0, where P, Q, R are functions of x, y and z that admit first order partial derivatives. Assume that equation (7.1) has an integral: u (x, y, z) = C ... **(7.2)**
Then the total differential du must be equal to P dx + Q dy + R dz or a multiple of it by a factor $\lambda$ (x,y,z). From equation (7.2) we have :

$$du = \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz \quad \ldots \quad (7.3)$$

Since u (x, y, z) = C is the integral of equation (7.1), P, Q, R must be proportional to $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$

i.e., $$\frac{\partial u/\partial x}{P}=\frac{\partial u/\partial y}{Q}=\frac{\partial u/\partial z}{R}=\lambda(x,y,z) \text{ (or) } \frac{\partial u}{\partial x}=\lambda P,\ \frac{\partial u}{\partial y}=\lambda Q,\ \frac{\partial u}{\partial z}=\lambda R. \quad \ldots(7.4)$$

From the first two equations of (7.4) we have

$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}(\lambda Q) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$$

Hence $\frac{\partial}{\partial x}(\lambda P) = \frac{\partial}{\partial x}(\lambda Q)$.

i.e., $\lambda \frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda \frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$

$$\Rightarrow \lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y} \quad \ldots\ldots \quad \textbf{(7.5)}$$

Similarly we obtain :

$$\lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \quad \ldots\ldots \quad \textbf{(7.6)}$$

and

$$\lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x} \quad \ldots\ldots \quad \textbf{(7.7)}$$

Multiplying equations 7.5, 7.6 and 7.7 by R, P and Q respectively and adding, we obtain

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad \ldots\ldots \quad \textbf{(7.8)}$$

Equation (7.8) is the necessary condition for the integrability of equation (7.1)

Conversely now suppose that the coefficients P, Q and R of equation (7.1) satisfy the equation (7.8). We have to now prove that an integral of equation (7.1) can be found. This is equivalent to showing that, if we take $P_1 = \mu P$, $Q_1 = \mu Q$ and $R_1 = \mu R$, where $\mu$ is a function of x, y, z then the same condition (7.8) is satisfied by $P_1$, $Q_1$, $R_1$.

We may now, without loss of generality, regard P dx + Q dy as an exact differential, for if that is not exact differential we can make it exact by multiplying equation (7.1) by the integrating factor $\mu(x, y, z)$. Then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ..... **(7.9)**

Let $V = \int P\,dx + Q\,dy$ ..... **(7.10)**

Then $\dfrac{\partial V}{\partial x} dx + \dfrac{\partial V}{\partial y} dy = P\,dx + Q\,dy$

and hence $P = \dfrac{\partial V}{\partial x}, Q = \dfrac{\partial V}{\partial y}$ ..... **(7.11)**

From equation (7.4).we have

$$\frac{\partial P}{\partial z} = \frac{\partial^2 V}{\partial z \partial x}, \frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z \partial y}$$

using equations (7.9) and (7.11) in equation.(7.8) we obtain

$$\frac{\partial V}{\partial x}\left[\frac{\partial^2 V}{\partial z \partial y} - \frac{\partial R}{\partial y}\right] + \frac{\partial V}{\partial y}\left[\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right] = 0$$

$$\Rightarrow \quad \frac{\partial V}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial V}{\partial z} - R\right] - \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\left[-r + \frac{\partial V}{\partial z}\right] = 0$$

or $$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}(\dfrac{\partial V}{\partial z} - R) \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}(\dfrac{\partial V}{\partial z} - R) \end{vmatrix} = 0$$

This suggests that a relation between V and $\left(\dfrac{\partial V}{\partial z} - R\right)$ exists which is independent of x and y.

Consequently $\left(\dfrac{\partial V}{\partial z} - R\right)$ can be expressed as a function of z and v alone. Hence we can take

$$\left(\frac{\partial V}{\partial z} - R\right) = \phi(z, V) \quad ..... \quad \textbf{(7.12)}$$

using equation (7.11) and (7.12) we get

$$P\,dx + Q\,dy + R\,dz = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + (\frac{\partial V}{\partial z} - \phi)\,dz$$

$$= (\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + (\frac{\partial V}{\partial z} dz) - \phi\,dz$$

$$= dV - \phi\,dz$$

Thus equation (7.1) may be written as $dV - \phi\,dz = 0$, which is an equation in two variables. Its integral is of the form F (V,z) = 0. Hence the condition (7.8) is sufficient.

7)

Thus equation (7.8) is a necessary and sufficient condition for the equation (7.1) to have an integral

**Geometrical Interpretation :**

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and P, Q, R, are functions of x, y, and z, then the total differential equation P dx + Q dy + R dz = 0 can be written as $\vec{F}.\vec{dr} = 0$ ..... **(7.13)**

If equation (7.13) is not exact, we can find an integrating factor such that $\lambda\ \vec{F}.\vec{dr} = 0$ is exact.

This equation has a solution of the Form F (x, y, z) = C ..... **(7.14)**

Equation (7.14) represents a family of single parameter surfaces

## 7.3 Examples :

**7.3.1. Example :** Verify the conditions of integrability for the equation.

$$(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)\,dy + (x - y)(x + y - 2z)\,dz = 0$$

**Solution :** Comparing the given equation with the standard form P dx + Q dy + R dz = 0, we have

$$P = y^2 - z^2 - 2x(y - z)$$
$$Q = z^2 - x^2 - 2y(z - x)$$
$$R = x^2 - y^2 - 2z(x - y)$$

Hence $\dfrac{\partial P}{\partial y} = 2y - 2x$ ; $\dfrac{\partial P}{\partial z} = 2x - 2z$

$$\frac{\partial Q}{\partial x} = 2y - 2x\ ; \qquad \frac{\partial Q}{\partial z} = 2z - 2y$$

$$\frac{\partial R}{\partial x} = 2x - 2z; \qquad \frac{\partial R}{\partial y} = 2z - 2y$$

Since $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$, $\dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}$ and $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

We observe that $P\left(\dfrac{\partial Q}{\partial z} - \dfrac{\partial R}{\partial y}\right) + Q\left(\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}\right) + R\left(\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}\right) = 0$

Hence the condition for integrability is satisfied.

### 7.3.2. Example :

Verify the condition of integrability for the equation $(\cos x+e^x y)dx + (e^x+ze^y)dy + e^y dz = 0$

**Solution :** Comparing the given equation with the equation Pdx + Qdy + Rdz = 0, we have

$$P = \cos x + e^x y$$

$$Q = e^x + z\, e^y$$

$$R = e^y$$

Then $\dfrac{\partial P}{\partial y} = e^x$ , $\dfrac{\partial P}{\partial z} = 0$.

$\dfrac{\partial Q}{\partial z} = e^y$ , $\dfrac{\partial Q}{\partial x} = e^x$

$\dfrac{\partial R}{\partial x} = 0$ , $\dfrac{\partial R}{\partial y} = e^y$

Substituting these values in the condition for integrability, we have

$$(\cos x - e^x y)(e^y - e^y) + (e^x + Ze^y)(0-0) + e^y(e^x - e^x) = 0$$

Hence the condition for integrability is satisfied and the given equation is integrable.

### 7.3.3. Example : Show that condition of integrability is satisfied by the equation :

$$(z+z^2)\cos x \frac{dx}{dt} - (z+z^2)\frac{dy}{dt} + (1-z^2)(y-\sin x)\frac{dz}{dt} = 0$$ and find its solution.

**Solution :** We shall rewrite the given equation as

$$(z+z^2)\cos x\, dx - (z+z^2)dy + (1-z^2)(y-\sin x)dz = 0$$

i.e., $z(1+z)\cos x\, dx - z(1-z)dy + (1+z)(1-z)(y-\sin x)dz = 0$

or $z\cos x\, dx - zdy + (1-z)(y-\sin x)dz = 0$ ..... **(7.15)**

on comparing this equation (7.15) with the equation P dx + Q dy + R dz = 0 we have

$P = z\cos x$, $Q = -z$ and $R = (1-z)(y-\sin x)$

Then $\dfrac{\partial P}{\partial y} = 0$, $\dfrac{\partial P}{\partial z} = \cos x$

$\dfrac{\partial Q}{\partial z} = -1$, $\dfrac{\partial Q}{\partial x} = 0$

$$\frac{\partial R}{\partial x} = -(1-z)\cos x, \quad \frac{\partial R}{\partial y} = (1-z)$$

Substituting these values in the condition for integrability we have
$z \cos x \; (-1-(1-z)) - z((z-1)\cos x - \cos x) + (1-z)(y - \sin x)(0-0) = 0$, which is satisfied.

We shall now write equation (7.15) in the form $\dfrac{\cos x dx - dy}{\sin x - y} - \left(\dfrac{1}{z} - 1\right) dz = 0$ ... **(7.16)**

So that the complete solution can be determined. Integrating equation (7.16) we obtain

$$\log(\sin x - y) - \log z + z = \log C$$

i.e., $\log(\sin x - y) - \log z - \log C = -z$

or $\sin x - y = C Z e^{-Z}$

This is the required solution.

**7.3.4. Example :** Determine the curves represented by the solution of the equation :
$y\,dx + (z-y)\,dy + x\,dz = 0$ ... **(717)**

that lie in the plane $2x - y - z = 1$

**Solution :** Here $P = y$, $Q = (z-y)$ and $R = x$ and we can see that the condition for integrability is not satisfied. Hence the given equation cannot be integrated directly.

Differentiating the equation $2x - y - z = 1$ we get $2\,dx - dy - dz = 0$ ... **(7.18)**

Multiplying equation (7.18) by x and adding the obtained one to equation (7.17) we obtain

$$(2x + y)\,dx + (z - y - x)\,dy = 0 \quad \text{...} \quad \textbf{(7.19)}$$

From the equation of the given plane $2x - y - z = 1$ we have $z = 2x - y - 1$. Hence substituting for z in equation (7.19) we get

$$(2x + y)\,dx + (2x - y - 1 - y - x)\,dy = 0$$

or $(2x + y)\,dx + (x - 2y - 1)\,dy = 0$

i.e., $2x\,dx + (y\,dx + x\,dy) - 2y\,dy - dy = 0$ on integrating this equation we get,

$$x^2 + xy - y^2 - y = C.$$

Thus, we find that the solutions of the given differential equation are the sections of the surface :

$x^2 + xy - y^2 - y = C$ by the plane $2x - y - z = 1$

## 7.4 Solution by inspection :

Sometimes we may rearrange the terms of the given equation and / or divide the equation by a suitable function of x, y, z so that the obtained equation may contain different parts which are exact differentials. In such cases, we proceed to find the solution directly and we do not need to explore or verify the condition of integrability. However, in many cases, the following list will help us to re-write the given equations in the exact differential form :

1. $$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

2. $$\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

3. $$\frac{-y\,dx + x\,dy}{xy} = d\left(\log\frac{y}{x}\right)$$

4. $$\frac{x\,dy + y\,dx}{xy} = d(\log xy)$$

5. $$\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left(\frac{1}{2}\log(x^2 + y^2)\right)$$

6. $$\frac{2xydy - y^2dx}{x^2} = d\left(y^2/x\right)$$

7. $x\,dy + y\,dx = d(xy)$
8. $y^2\,dx + 2xy\,dy = d(y^2x)$
9. $2(x\,dx + y\,dy) = d(x^2 + y^2)$
10. $xy\,dz + xz\,dy + yz\,dx = d(xyz)$

## 7.5. Examples :

**7.5.1.** Solve $(x^2y - y^3 - y^2z)\,dx + (xy^2 - x^2z - x^3)\,dy + (xy^2 + x^2y)\,dz = 0$

**Solution :** Dividing the given equation throughout by $x^2y^2$ we get,

$$\left(\frac{1}{y} - \frac{y}{x^2} - \frac{z}{x^2}\right)dx + \left(\frac{1}{x} - \frac{z}{y^2} - \frac{x}{y^2}\right)dy + \left(\frac{1}{x} + \frac{1}{y}\right)dz = 0$$

Rearranging the like terms we obtain $\dfrac{y\,dx - x\,dy}{y^2} + \dfrac{x\,dy - y\,dx}{x^2} + \dfrac{x\,dz - z\,dx}{x^2} + \dfrac{y\,dz - z\,dy}{y^2} = 0$

If this equation is expressed in exact differential form, we get $d\left(\frac{x}{y}\right)+d\left(\frac{y}{x}\right)+d\left(\frac{z}{x}\right)+d\left(\frac{z}{y}\right)=0$

on Integration we get $\frac{x}{y}+\frac{y}{x}+\frac{z}{x}+\frac{z}{y}=C$

$\Rightarrow \quad x^2 + y^2 + z(x+y) = Cxy$, C is a constant

**7.5.2** **Example :** Solve $(y - z)(y + z - zx)\,dx + (z-x)(z + x - 2y)\,dy + (x-y)(x + y - 2z)\,dz = 0$

**Solution :** The given equation is in the standard form of the exact differential $P\,dx + Q\,dy + R\,dz = 0$, where

$$P = (y - z)(y + z - 2x)$$
$$Q = (z - x)(z + x - 2y)$$
$$R = (x - y)(x + y - 2z)$$

Then $\frac{\partial P}{\partial y} = 2y - 2x$ ; $\quad \frac{\partial P}{\partial z} = 2x - 2z$ ;

$\frac{\partial Q}{\partial x} = 2y - 2x$ ; $\quad \frac{\partial Q}{\partial z} = 2z - 2y$ ;

$\frac{\partial R}{\partial y} = 2z - 2y$ ; $\quad \frac{\partial R}{\partial x} = 2x - 2z$ ;

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$, the given equation is exact.

We now re-write the given equation as,

$$(y^2\,dx + 2xy\,dy) - (z^2\,dx + 2zx\,dz) + (z^2\,dy + 2xy\,dz) - (x^2\,dy + 2xy\,dx) + (x^2\,dz + 2zx\,dx) - (y^2\,dz + 2zy\,dy) = 0$$

(or) $d(y^2x) - d(z^2x) + d(z^2y) - d(x^2y) + d(x^2z) - d(y^2z) = 0$, on integration we get

$$y^2x - z^2x + z^2y - x^2y + x^2z - y^2z = C, \text{ C is a constant}$$

**7.5.3** **Example :** Solve $x^2\,dx^2 + y^2dy^2 - z^2\,dz^2 + 2xy\,dx\,dy = 0$

**Solution :** The given equation is not in the standard form of a total differential equation so we convert it into the standard form by rewriting it as $(x\,dx + y\,dy)^2 - (z\,dz)^2 = 0$.

(or) $(x\,dx + y\,dy + z\,dz)(x\,dx + y\,dy - z\,dz) = 0$

Hence $x\,dx + y\,dy + z\,dz = 0$ (or) $x\,dx + y\,dy - z\,dz = 0$

Integrating these equations we obtain

$$\frac{1}{2}(x^2+y^2+z^2)=C_1, \qquad \frac{1}{2}(x^2+y^2-z^2)=C_2\ ;\ C_1, C_2 \text{ are constants}$$

i.e., $x^2+y^2+z^2=C_1$

$x^2+y^2-z^2=C_2$, where $C_1 = 2\,C_1$ and $C_2 = 2\,C_2$

## 7.6 Homogeneous equations in x, y and z

If in a total differential equation P, Q, R are homogeneous functions of x, y and z then such an equation will always be integrable. We adopt the following systematic procedure to solve such equations.

Step 1 : Compare the give equation with the standard form $P\,dx + Q\,dy + R\,dz=0$ and identify P, Q, R

Step 2 : Verify the condition for integrability :

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

Step 3 : Calculate $D = Px+ Qy + Rz$. If this is non-zero, then $\dfrac{1}{Px+Qy+Rz}$ is taken as the integrating factor of the given equation. We multiply the given equation by 1/D. We find the total differential of D i.e, d(D). Then add and subtract d(D) to and from the numerator of the equation, obtained after multiplying the given equation with 1/D. Several terms in the resulting equations will then be exact differentials. We then integrate this equation. While re-grouping the terms here, we keep in view the list of exact differential forms given in section. 7.4

Step 4 : If $D = Px + Qy + Rz = 0$, then the procedure outlined in step 3 is not applicable. Then we adopt the following procedure.:

Put $x = zu$, and $y = z\upsilon$, so that $dx = z\,du + u\,dz$, $dy = z\,d\upsilon + \upsilon\,dz$

We substitute these values in the given equation.

**Case : 1 :** If the coefficient of dz is equal to zero, then we get an equation in the two variables u and $\upsilon$ only. By regrouping the terms properly, we integrate the equation.

**Case : 2 :** If the coefficient of dz $\neq 0$, then we shall be able to seperate z from u and $\upsilon$. The resulting equation will be of the form : $\dfrac{f_1(u,\upsilon)du+f_2(u,\upsilon)d\upsilon}{f(u,\upsilon)}+\dfrac{dz}{z}=0.$

Denote f(u, $\upsilon$) by G and find d(G). Add and subtract d(G) to and from the numerator. Write the equation in proper form and then integrate. After integration, u and $\upsilon$ are replaced by x/z and y/z respectively to get the desired solution in x,y and z.

We shall now illustrate this procedure with a few examples.

## 7.7 Examples :

**7.7.1. Example 1 :** Solve $(yz+z^2)\,dx + xz\,dy + xy\,dz = 0$ .... **(7.20)**

**Solution :**

Comparing the given equation with the standard form of the equation P dx + Q dy + R dz = 0, we find $P = yz + z^2$, Q = - xz and R = xy,

$$\frac{\partial P}{\partial y} = z, \quad ; \quad \frac{\partial P}{\partial z} = y + 2z,$$

$$\frac{\partial Q}{\partial x} = -z, \quad ; \quad \frac{\partial Q}{\partial z} = -x$$

$$\frac{\partial R}{\partial x} = y, \quad ; \quad \frac{\partial R}{\partial y} = x$$

observe that the condition for integrability is satisfied . Let x = uz and y = $\upsilon z$

Then $dx = u\,dz + z\,du$; $dy = \upsilon\,dz + z\,d\upsilon$.

Substituting these values in equation (7.20), we get

$$(\upsilon z^2 + z^2)(u\,dz + z\,du) - uz^2(\upsilon\,dz + z\,d\upsilon) + \upsilon u z^2\,dz = 0$$

or $$(\upsilon + 1) z^3\,du - u z^3\,d\upsilon + (\upsilon+1) u z^2\,dz = 0 \quad ..... \quad \textbf{(7.21)}$$

Dividing equation (7.21) throughout by $(\upsilon+1)\,u z^3$

We get $\dfrac{du}{u} - \dfrac{d\upsilon}{\upsilon+1} + \dfrac{dz}{z} = 0$ on Integration we get $\log u - \log(\upsilon + 1) + \log z = \log C$

or $uz = C(\upsilon+1)$, C is a constant.

But u = x/z, and $\upsilon$ = y/z. Hence we have (x/z) z = C (y/z+1)

i.e., xz = C(y + z), which is the required solution

**7.7.2 Example :** Solve $(y^2 + yz)\, dx + (xz + z^2)\, dy + (x^2 - xy)\, dz = 0$

**Solution :** Here we have $P = y^2 + yz$, $Q = xz + z^2$ and $R = x^2 - xy$, observe that the conditions of integrability aare satisfied by these P, Q and R (Verify).

Since the given equation is homogeneous, put $x = zu$, $y = zv$ so that
$dx = z\, du + u\, dz$ and $dy = z\, dv + v\, dz$

Substituting these values in the given equation, we get

$$(v^2 + v)\, z^3\, du + (u + 1)\, z^3\, du + [\, u (v^2 + v) + v\, (u + 1) + v^2 - uv\, ]\, z^2\, dz = 0$$

or $$z^3 [\, (v^2 + v)\, du + (u + 1)\, dv\, ] + (u + i)(v^2 + v)]\, z^2\, dz = 0$$

Dividing this equation throughtout by $z^3 (u + 1)(v^2 + v)$ we get,

$$\frac{du}{u+1} + \frac{dv}{v^2 + v} + \frac{dz}{z} = 0$$

or $$\frac{du}{u+1} + \left(\frac{1}{v} - \frac{1}{v+1}\right) dv + \frac{dz}{z} = 0$$

on Integration we obtain $\log (u + 1) + \log v - \log (v + 1) + \log z = \log C$, C is a constant

i.e., $$(u + 1)\, v\, z = C\, (v + 1) \qquad \ldots\ldots \qquad (7.22)$$

Substituting $u = x/z$ and $v = y/z$ in equation (7.22) and simplifying we get $(x + z)\, y = C\, (y + z)$. which is the required solution.

**7.7.3. Example :** Solve $z^2dx + (z^2 - 2yz)dy + (2y^2 - yz - xz)dz = 0$

**Solution :** Here $P = z^2$, $Q = z^2 - 2\, yz$, $R = 2y^2 - yz - xz$ and the condition for integrability is satisfied (verify). Since the given equation is homogeneous, put

$x = zu$, $y = zv$ so that $dx = z\, du + u\, dz$ ana $y = z\, dv + v\, dz$

Substituting these values in the given equation, we get,

$$z^2(z\, du + u\, dz) + z^2 (1 - 2v)(z\, dv + v\, dz) + z^2(2v^2 - v - u)\, dz = 0$$

or $$z\, du + (1 - 2v)\, z\, dv + (u + v - 2v^2 + 2v^2 - v - u)\, dz = 0$$

i.e., $z\, du + (1 - 2v)\, z\, dv = 0 \Rightarrow du + (1 - 2v)\, dv = 0$ on integration we get $u + v - v^2 = C$, C is a constant upon substituting for u and $\upsilon$ in this equation we get $xz + yz - y^2 = C z^2$, which is the desired solution.

## 7.8 General solution of P dx + Q dy + R dz = 0 by considering one of the variables as constant.

The given equation is set into the standard form $P\,dx + Q\,dy + R\,dz = 0$ and we verify the condition for integrability. If it is satisfied, then we consider one of the variables, say z as constant so that $dz = 0$. We then integrate the reduced equation $P\,dx + Q\,dy = 0$ and take the arbitrary constant of integration by $\phi(z)$. Now differentiate the integral just obtained w.r.t. x, y and z and compare it with the given differential equation to determine $\phi(z)$.

We shall now exemplify this procedure by two illustrations.

## 7.9 Examples :

**7.9.1 Example :** Solve $(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0$ ... **(7.23)**

**Solution :** The given equation is in the standard form with $P = y^2 + yz$, $Q = z^2 + zx$ and $R = y^2 - xy$. We find here that the condition for integrability is satisfied. If z is considered as constant, then equation (7.23) reduces to :

$$\frac{dx}{z(z+x)} + \frac{dy}{y(y+z)} = 0$$

Noting that z is a constant and then integrating, we get ,

$$\frac{1}{z}\int \frac{dx}{x+z} + \frac{1}{z}\int\left(\frac{1}{y} - \frac{1}{y+z}\right)dy = \text{a constant say } \phi(z)$$

or $\log(x+z) + \log y - \log(y+z) = \phi^{\bullet}(z)$

i.e., $\dfrac{y(x+z)}{y+z} = \phi(z) \Rightarrow y(x+z) - (y+z)\,\phi(z) = 0$ ... **(7.24)**

Differentiating this equation (7.24) we obtain :

$$y(dz + dx) + (x+z)\,dy - [\,(y+z)\,\phi^{1}(z)\,dz + (dy + dz)\,\phi(z)\,] = 0$$

or $y\,dx + [\,x + z - \phi(z)\,]\,dy + [\,y - (y+z)\,dz\,\phi^{1}(z) - \phi(z)\,]\,dz = 0$ ... **(7.25)**

Comparing equation (7.25) with the equation (7.23) we get,

$$\frac{y^2+yz}{y} = \frac{z^2+zx}{z+x-\phi(z)} = \frac{y^2-xy}{y-(y+z)\phi^1(z)-\phi(z)}$$

Since the relation $\frac{y^2+yz}{y} = \frac{z^2+zx}{z+x-\phi(z)}$ reduces to equation (7.24), it does not provide us with any (new) useful information. Hence we take $\frac{y^2+yz}{y} = \frac{y^2-xy}{y-(y+z)\phi^1(z)-\phi(z)}$

Then $y^2 - xy = (y+z)[y-(y+z)\phi'(z) - \phi(z)]$

$= y^2 + yz - (y+z)^2 \phi'(z) - (y+z)\phi(z)$

$= y^2 + yz - (y+z)^2 \phi'(z) - y(x+z)$, (in virtue of equation 7.24)

$= y^2 - xy - (y+z)^2 y^2 \phi'(z)$

Hence $(y+z)^2 \phi'(z) = 0 \quad \Rightarrow \phi'(z) \quad \Rightarrow \phi(z) = C$

From equation (7.24) the required integral is : $y(x+z) = (y+z)C$.

**7.9.2. Example :** Solve $3x^2\,dx + 3y^2\,dy - (x^3 + y^3 + e^{2z})\,dz = 0$ .... **(7.26)**

**Solution :** The given equation (7.26) is in the standard form with $P = 3x^2$, $Q = 3y^2$ and $R = -(x^3+y^3+e^{2z})$. We can see that the condition of integrability is satisfied. We now treat **z** as constant, so that $dz = 0$. Then equation (7.26) reduces to $3x^2\,dx + 3y^2dy = 0$. On integrating this equation we obtain :

$x^3+y^3 = \phi(z)$, where $\phi(z)$ is a constant of integration .... **(7.27)**

Differentiating equation (7.27)

$$3x^2\,dx + 3y^2\,dy - \phi'(z)\,dz = 0 \quad \text{....} \quad \textbf{(7.28)}$$

Comparing equation (7.28) with equation (7.26) we get

$$x^3 + y^3 + e^{2z} = \phi'(z) \quad \text{....} \quad \textbf{(7.29)}$$

In virtue of equation (7.27) equation (7.29) become

$$\phi(z) + e^{2z} = \phi'(z) \text{ or } \phi'(z) - \phi(z) = e^{2z} \quad \text{....} \quad \textbf{(7.30)}$$

Equation (7.30) is a linear differential equation. Its solution can be seen to be $\phi(z) = e^{2z} + C$

Substituting this solution in equation (7.27) we get

$$x^3 + y^3 = e^{2z} + C e^z$$, which is the required solution.

## 7.10 Method of finding solution using Auxiliary equations

Consider the given equation in the standard form $P\,dx + Q\,dy + R\,dz = 0$ .... **(7.31)**

Assume that the condition for integrability is satisfied,

$$\text{Then } P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad \text{.... } \mathbf{(7.32)}$$

Comparing equation (7.31) with (7.32) we have

$$\frac{dx}{\dfrac{\partial Q}{\partial z} - \dfrac{\partial R}{\partial y}} = \frac{dy}{\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}} = \frac{dz}{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}} \quad \text{.... } \mathbf{(7.33)}$$

Equation (7.33) are called the auxiliary equations. Let $u = C_1$, and $\upsilon = C_2$ be the integrals obtained by solving these equations. With these values we formulate the equation $A\,du + B\,d\upsilon = 0$ ..... **(7.34)**

Comparing equations (7.31) and (7.34) we find the values of A and B. Putting these values of A and B in equation (7.34) and integrating it, we obtain a relation and we substitute the values of u and $\upsilon$ in that relation. We then obtain the required solution.

**Note :** This method fails in case the given equation is exact, for, in this case,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

## 7.11 Examples :

**7.11.1 Example :** Solve $3x^2\,dx + 3y^2\,dy - (x^3 + y^3 + e^{2z})\,dz = 0$ .... **(7.34)**

**Solution :** Here $P = 3x^2$, $Q = 3y^2$ and $R = -(x^3 + y^3 + e^{2z})$.
From the condition for integrability we have,

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= 3x^2\,(0 + 3y^2) + 3y^2\,(-3x^2 - 0) - (x^3 + y^3 + e^{2z})\,(0 - 0) \quad = \quad 0$$

$$\Rightarrow 9x^2y^2 - 9x^2y^2 = 0 \quad \text{.... } \mathbf{(7.35)}$$

comparing equation (7.34) with equation (7.35) we get ,

$$\frac{dx}{3y^2} = \frac{dy}{-3x^2} = \frac{dz}{0} \quad \Rightarrow \quad 3x^2\,dx + 3\,y^2\,dy = 0$$

Hence $\quad x^3 + y^3 = \text{const} = u \text{ (say)}$

Also $\quad dz = 0 \Rightarrow z = \text{const} = \upsilon \text{ (say)}$

Substituting these results in the equation A du + B d$\upsilon$ = 0,

We get $\quad A(3x^2dx + 3y^2dy) + B\,dz = 0$ .... **(7.36)**

Comparing equation (7.34) and (7.36) we have

$$3Ax^2 = 3x^2,\ 3Ay^2 = 3y^2 \text{ and } B = x^3 + y^3 + e^{2z}$$

Hence $A = 1$, $B = -(u + e^{2\upsilon})$

Consequently $A\,du + B\,d\upsilon = 0 \Rightarrow du - (u + e^{2\upsilon})\,d\upsilon = 0$ or $\frac{du}{d\upsilon} - u = e^{2\upsilon}$, which is a linear equation. Its integrating factor = $e^{-\int d\upsilon} = e^{-\upsilon}$.

Hence the solution is $ue^{-\upsilon} = \int e^{-\upsilon}e^{+2\upsilon} + C$

or $ue^{-\upsilon} = e^{\upsilon} + C \Rightarrow u = e^{2\upsilon} + Ce^{\upsilon}$

or $x^3 + y^3 = e^{2z} + C\,e^{z}$, which is the required solution.

**7.11.2 Example :** Solve $x\,z^3\,dx - z\,dy + 2y\,dz = 0$ .... **(7.37)**

**Solution :**

Comparing equation (7.37) with P dx + Q dy + R dz = 0, we have $P = x\,z^3$, $Q = -z$, and $R = 2y$.

Hence equation (7.37) is integrable if the condition for integrability is satisifed,

i.e., $\quad xz^3\,(-1-2) + (-z)\,(0-3xz^2) + zy\,(0) = 0$ .... **(7.38)**

Comparing equations (7.37) and (7.38) we get the auxiliary equations :

$$\frac{dx}{-3} = \frac{dy}{-3xz^2} = \frac{dz}{0} \quad \text{or} \quad \frac{dx}{1} = \frac{dy}{xz^2} = \frac{dz}{0}$$

From the last ratio we have $dz = 0$ or $z = u$, (say)

From the first two ratios, $x z^2 dx = dy$ (or) $2x u^2 dx - 2 dy = 0$

Integrating, $x^2 u^2 - 2y = \upsilon$ , (say) (or) $x^2 z^2 - 2y = \upsilon$

Therefore, $A\, du + B\, d\upsilon = 0$ becomes $A\, dz + B\,(2 xz^2 dx - 2 dy + 2 zx^2 dz) = 0$

(or) $2B xz^2 dx - 2B dy + (A + 2Bx^2 z) dz = 0$ .... **(7.39)**

Comparing equations (7.37) and (7.39) we have, $xz^3 = 2 B xz^2$, $-z = -2B$, $2y = A+2x^2zB$

Therefore, $B = \frac{1}{2}z = \frac{1}{2}u$ and $A = 2y - 2 Bx^2z$

$= 2y - x^2 z^2 = -\upsilon$

Hence $A\, du + B\, d\upsilon = 0$ gives $-\upsilon\, du + 1/2\, u\, d\upsilon = 0$ (or) $-2\left(\frac{du}{u}\right) + \frac{d\upsilon}{\upsilon} = 0$

Integrating, $-2 \log u + \log \upsilon = \log C$ (or) $\upsilon = C u^2$

i.e., $x^2 z^2 - 2y = Cz^2$., which is the required solution.

**SAQs :**

I. Verify that the conditions of integrability is satisfied by the following total differential equations.

(i) $(yz + z^2) dx - xz\, dy + xy\, dz = 0$

(ii) $2(y + z) dx - (x + z) dy + (2y - x + z) dz = 0$

II. Solve the following total differential equations :

(i) $\frac{-yz}{x^2 + y^2} dx - \frac{xz}{x^2 + y^2} dy - \tan^{-1}(y/x)dz = 0$ (Ans : $z \tan^{-1}(y/x) = C$ )

(ii) $(y\, dx + x\, dy)(a - z) + xy\, dz = 0$ (Ans : $xy = C(a-x)$ )

III. Solve the following equations by the method of inspection :

(i) $z(y\, dx - x\, dy) = y^2 dz$ (Ans : $x - Cy = y \log z$)

(ii) $x^2 dx^2 + y^2 dy^2 - z^2 dz^2 + 2xy\, dx\, dy = 0$ (Ans : $x^2 + y^2 + z^2 = C_1$

$x^2 + y^2 - z^2 = C_2$

IV. Solve the following homogeneous total differential equations :

(i) $(2xz - yz)\, dx + (2yz - xy)\, dy - (x^2 - xy + y^2)\, dz = 0$ (Ans: $x^2+y^2-xy = Cz$)

(ii) $z(y + z)\, dx - z(t-x)\, dy + y(x - t)\, dz + y(y + z)\, dt = 0$ (Ans: $xz + yt = c(y+z)$ )

V. Solve the following equation using the auxiliary equations method :

$z(z - y)\, dx + (z + x) z\, dy + x(x + y)\, dz = 0$ (Ans : $z(x + y) = C(x + y)$ )

## 7.12 Summary :

A necessary and sufficient condition for the integrability of the total differential equation was established. A systematic procedure to solve the homogeneous equations in x, y and z was laid and a simple method of finding solution using auxiliary equation was presented.

## 7.13 Model Examination Questions :

1. Show that the equation $3y\, dx + (z-3y)\, dy + x\, dz = 0$ is not integrable. Prove that the projection on the xy plane of the curves that satisfy the equation and lie in the plane $2x + y - z = a$ are the rectangular hyperbolas $x^2 + 3xy - y^2 - ay = C$.

2. (i) Find f(y) so that the total differential equation $\frac{xy + z}{x} dx - z\, dy + f(y)\, dz = 0$ is integrable

(ii) Hence solve it (Ans : (i) $f(y) = k(y+1)$ (ii) $z^k = c(y+1)$ ).

3. Solve the following equations by the method of inspection :

(i) $2yz\, dx + zx\, dy - xy(1+z)\, dz = 0$ (Ans : $x^2y = Cz\, e^z$)

(ii) $(2x^3y + 1)\, dx + x^4\, dy + x^2 \tan z\, dz = 0$ (Ans : $x^2y - 1/x + \log \sec z = C$)

(iii) $3x^2(y + z) + (z^2 + x^3)\, dy + (2yz + x^3)\, dz = 0$ (Ans: $x^3(y + z) + yz^2 = C$)

4. Solve the following total differential equations :

(i) $z(z - y)\, dx + z(z + x)\, dy + x(x + y)\, dz = 0$ (Ans : $z(x + y) = C(x + z)$)

(ii) $(x^2y - y^3 - y^2z)\, dx + (xy^2 - x^3 - x^2z)\, dy + (x^2y^2 + x^2y)\, dz = 0$.

(Ans: $x^2 + y^2 + z(x+y) = C.xy$)

(iii) $(2xyz + y^2z + yz^2)\, dx + (x^2z + 2xy + 2xyz + xz^2)\, dy + (x^2y+xy^2+2xyz)\, dz = 0$

(Ans : $xyz(x + y + z) = C$)

5. Solve the following equation using the auxiliary equations method

$(y^2 + yz)\, dx + (xz + z^2)\, dy + (y^2-xy)\, dz = 0$ (Ans : $(x + z)\, y = C(y + z)$ )

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# LESSON - 8 PARTIAL DIFFERENTIAL EQUATIONS

## 8.1 INTRODUCTION :

Partial differential equations arise in a variety of problems in science and Engineering when the number of independent variables involved in the problem under discussion is greater than or equal to two. In such cases, any dependent variable is likely to be a function of more than one variable so that it possess partial derivatives w.r.t. several variables, instead of ordinary derivatives w.r.t. a single variable. Usually the independent variables are scalars - for example pressure, temperature, density or vectors like velocity, force etc. The task of a mathematician is

(i) to formulate the partial differential equation from the given physical problem and
(ii) to solve the mathematical problem, if a solution exists uniquely, by enforcing the initial and / or boundary conditions.

We shall now consider the formulation of a partial differential equation and learn a few methods of solving a first order partial differential equation of the linear and non-linear type.

## 8.2 Formation of Partial Differential Equations :

It has already been stated that a partial differential equation is one which involves partial derivatives and which contains more than one independent variable. The order of a partial differential equation is the order of the highest derivative occuring in it. For example,

$$x\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=2z \qquad \text{.... (8.1)}$$

$$\frac{\partial^2 z}{\partial t^2}=C^2\frac{\partial^2 z}{\partial x^2} \qquad \text{.... (8.2)}$$

$$\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial z^2}=0 \qquad \text{.... (8.3)}$$

are partial differential equations. Equation (8.1) is the first order and the dependent variable z is considered as a function of two independent variables x and y. Equation (8.2) is an equation of second order involving two independent variables x and t and the depedent variable z being a function of x and t. Equation (8.3) is also of second order but it involves three independent variables.

In the discussion that follows, z is taken as dependent variable and x, y as independent variables, so that z = f(x,y). We shall adopt the following notation :

$$\frac{\partial z}{\partial x}=p, \quad \frac{\partial z}{\partial y}=q, \quad \frac{\partial^2 z}{\partial x^2}=r, \quad \frac{\partial^2 z}{\partial y^2}=t, \quad \frac{\partial^2 z}{\partial x\partial y}=s.$$

We shall now form the Partial differential equation by (i) the elimination of arbitrary constants (ii) elimination of arbitrary functions from the given relation involving three or more variables.

**8.2.1 By elimination of arbitrary constants :**

Let us consider the equation $x^2 + y^2 + (z-c)^2 = a^2$ , a, c are arbitrary constants ... **(8.4)**

This represents the set of all spheres whose centres lie on the z-axis. Differentiating (8.4) w.r.t. x

we obtain $x + p\,(z-c) = 0,\ p = \dfrac{\partial z}{\partial x}$. ... **(8.5)**

By differentiating (8.4) w.r.t. y, we find that

$$y + q\,(z - c) = 0; \qquad q = \frac{\partial z}{\partial y} \quad \dots \quad \textbf{(8.6)}$$

By eliminating the arbitrary constant c from (8.5) and (8.6) we get $y\,p - x\,q = 0$ ... **(8.7)**

Equation (8.7) is a first order partial differential equation representing the set of all spheres with centres on z-axis. Thus by starting with an equation F (x, y, z, a, b) = 0 and eliminating in a systematic way the arbitrary constants a and b, we obtained an equation f (x, y, z, p, q) = 0 we shall now generalise this equation.

Consider the equation F(x, y, z, a, b) = 0 .... **(8.7)**

Where a and b are arbitrary constants. By regarding z as a function of the independent variables x and y and differentiating (8.7) w.r.t. x and y partially we get,

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \quad \dots \quad \textbf{(8.8)}$$

$$\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \quad \dots \quad \textbf{(8.9)}$$

Eliminating a and b from equations (8.7), (8.8) and (8.9) we obtain a first order partial differentia equation of the form f(x, y, z, p, q) = 0 .... **(8.10)**

We call equation (8.7), the primitive or complete solution of the equation (8.10).

In general, if the number of constants to be eliminated is just equal to the number of independent variables, the partial differential equation obtained after the elimination of arbitrary constants will be of first order. However, if the number of constants to be eliminated is more than the number of independent variables contained in the given equation, the resulting partial differential equation will be of order greater than one.

**8.2.2. Example :** Form the partial differential equation by eliminating the arbitrary constants from the equation $(x-a)^2 + (y-b)^2 + z^2 = 1$ .... **(11)**

**Solution :** Differentiating equation (11) partially w.r.t. x we get,

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad (x-a) + zp = 0 \qquad \text{.... } \mathbf{(12)}$$

Differentiating equation (11) partially w.r.t. x we get

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \quad \Rightarrow \quad (y-b) + zq = 0 \qquad \text{.... } \mathbf{(13)}$$

eliminating a and b from equation (11), (12) and (13) we obtain :

$$x - a = -zp$$
$$y - b = -zq$$

and $(-zp)^2 + (-zq)^2 + z^2 = 1$

or $z^2(p^2 + q^2 + 1) = 1$

This is the required partial differential equations.

**8.2.3. Example :** Form the partial differential equation by eliminating the arbitrary constants 'a' and 'b' from $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2z$ ..... **(14)**

**Solution :** Differentiating equation (14) partially w.r.t. x and later w.r.t. y we obtain :

$$\frac{2x}{a^2} = 2\frac{\partial z}{\partial x} \Rightarrow \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}.$$

$$\frac{2y}{b^2} = 2\frac{\partial z}{\partial y} \Rightarrow \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

substituting these values in equation (14) we get $xp + yq = 2z$, which is the desired partial differential equation.

### 8.2.4 Eliminating arbitrary functions :

We now formulate a partial differential equation by the elimination of arbitrary functions $\phi$ from the equation $\phi(u,\upsilon) = 0$, where $u$ and $\upsilon$ are known functions of x,y and z.

Consider the equation $\phi(u,\upsilon) = 0$ ..... **(15)**

we treat x and y as independent variables x and z as a dependent variable.

So that $\dfrac{\partial z}{\partial x} = p,\quad \dfrac{\partial z}{\partial y} = q,\quad \dfrac{\partial x}{\partial y} = 0,\quad \dfrac{\partial y}{\partial x} = 0$

Differentiating equation (15) partially w.r.t. x and y respectively, we obtain

$$\frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right\}+\frac{\partial \phi}{\partial \upsilon}\left\{\frac{\partial \upsilon}{\partial x}+\frac{\partial \upsilon}{\partial z}P\right\}=0 \quad \ldots \quad \textbf{(16)}$$

$$\frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right\}+\frac{\partial \phi}{\partial \upsilon}\left\{\frac{\partial \upsilon}{\partial y}+\frac{\partial \upsilon}{\partial z}q\right\}=0 \quad \ldots \quad \textbf{(17)}$$

Eliminating $\dfrac{\partial \phi}{\partial u}$ and $\dfrac{\partial \phi}{\partial \upsilon}$ from equations (16) and (17) we get the determinant equation.

$$\begin{vmatrix} \dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}p & \dfrac{\partial \upsilon}{\partial x}+\dfrac{\partial \upsilon}{\partial z}p \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}q & \dfrac{\partial \upsilon}{\partial y}+\dfrac{\partial \upsilon}{\partial z}q \end{vmatrix} = 0$$

$$\text{i.e.,}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial \upsilon}{\partial y}+q\frac{\partial \upsilon}{\partial z}\right)-\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)\left(\frac{\partial \upsilon}{\partial x}+p\frac{\partial \upsilon}{\partial z}\right)=0 \quad \ldots \quad \textbf{(18)}$$

Equation (18) on simplication gives a partial differential equation of the form P p + Q q = R...**(19)**

Where

$$P = \frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial \upsilon}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial \upsilon}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial x}$$

Equation (19) is a first order partial differential equation in p and q which is also in the standard form f (x, y, z, p, q) = 0.

**8.2.5 <u>Example :</u>** Form the partial differential equation by eliminating arbitrary functions from the equation xyz = $\phi$ ( x+ y + z)

**<u>Solution :</u>** The given equation is xyz = $\phi$ ( x + y + z ) .... **(20)**

Differentiating equation (20) partially w.r.t. x and later w.r.t. y we get,

$$yz + xyp = (1+p)\ \phi 1\ (x+y+z) \qquad \ldots. \qquad (21)$$

$$zx + xyq = (1+q)\ \phi 1\ (x+y+z) \qquad \ldots. \qquad (22)$$

Eliminating $\phi 1\ (x+y+z)$ from equations (21) and (22) we obtain :

$$\frac{xyp+yz}{xyq+zx} = \frac{1+p}{1+q}.$$

Simplifying this equation we get, $p\,x(y-z) + q\,y(z-x) = z(x-y)$

Which is the desired partial differential equation.

**8.2.6 Example :** Form the partial differential equation by eliminating the arbitrary functions from the equation $z = f(x+it) + g(x-it)$.

**Solution :** Let $z = f(x+it) + g(x-it)$ .... **(23)**

Differentiating z partially w.r.t. x and t we get,

$$\frac{\partial z}{\partial x} = f^{1}(x+it) + g^{1}(x-it) \qquad \ldots. \qquad (24)$$

$$\frac{\partial z}{\partial t} = i\,f^{1}(x+it) - i\,g^{1}(x-it) \qquad \ldots. \qquad (25)$$

Again differentiating equation (24) w.r.t. x and (25) w.r.t. t partially we get

$$\frac{\partial^2 z}{\partial x^2} = f^{11}(x+it) + g^{11}(x-it) \qquad \ldots. \qquad (26)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f^{11}(x+it) + i^2 g^{11}(x-it) = -\frac{\partial^2 z}{\partial x^2}$$, in virtue of equation (26)

Hence the desired partial differential equation is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$ which is of second order.

**8.2.7 Example :** Form the p rtial differential equation by eliminating the arbitrary function from the equation $F(x+y+z,\ x^2,\ y^2 - z^2) = 0$

**Solution :** Let $x+y+z = u$ and $x^2 + y^2 - z^2 = \upsilon$ .... **(27)**

Then the given equation becomes $F(u, \upsilon) = 0$ .... **(28)**

Differentiating equation (28) partially w.r.t. x we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial \upsilon}\left(\frac{\partial \upsilon}{\partial x}+p\frac{\partial \upsilon}{\partial z}\right)=0 \quad \ldots\ldots \quad \mathbf{(29)}$$

From equation (27) we have

$$\left.\begin{array}{lll} \frac{\partial u}{\partial x}=1, & \frac{\partial u}{\partial z}=1, & \frac{\partial u}{\partial y}=1 \\ \frac{\partial \upsilon}{\partial x}=2x, & \frac{\partial \upsilon}{\partial y}=2y, & \frac{\partial \upsilon}{\partial z}=-2z \end{array}\right\} \quad \ldots\ldots \quad \mathbf{(30)}$$

Substituting these values in equation (29) we get, $\frac{\partial F}{\partial u}(1+p)+\frac{\partial F}{\partial \upsilon}(2x-2pz)=0 \ldots$ **(31)**

Differentiating equation (28) partially w.r.t. y we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial \upsilon}\left(\frac{\partial \upsilon}{\partial y}+q\frac{\partial \upsilon}{\partial z}\right)=0 \quad \ldots\ldots \quad \mathbf{(32)}$$

Substituting the results of equations (30) in equation (32) we get

$$\frac{\partial F}{\partial u}(1+q)+\frac{\partial F}{\partial \upsilon}(2y-2zq)=0 \quad \ldots\ldots \quad \mathbf{(33)}$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial \upsilon}$ from equations (31) and (33) we get,

$$\begin{vmatrix} 1+p & 2x-2pz \\ 1+q & 2y-2pz \end{vmatrix} = 0$$

$$\Rightarrow \quad (1+p)(2y-2qz)-(1+q)(2x-2pz)=0$$

on simplifying this equation, we obtain, $(y+z)p-(x+z)q=x-y$. This is the desired partial differential equation which is of first order.

## 8.3 Linear partial differential equation of first order - Lagrange's equation :

A differential equation involving only first order partial derivatives p and q is called first order equation. If the degrees of p and q are one, it is called a linear partial differential equation of first order, otherwise it is non-linear.

A linear partial equation of first order of the form $Pp+Qq=R$ ... **(34)**

Where P, Q, R are functions of x, y, z and $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$ is known as Lagranges equation.

In Section (8.2.4) we have shown that by the elimination of arbitrary function $\phi$ (u, $\upsilon$ )=0 where u and $\upsilon$ are functions of x, y, z we obtained a partial differential equation of the form.

$$Pp + Qq = R \quad \text{.....} \quad \textbf{(19)}$$

Where

$$P = \frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial \upsilon}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial x}$$

Hence $\phi$ (u, $\upsilon$ ) = 0, is the general solution of equation (34), $\phi$ being any arbitrary function :

Now suppose u = a, and $\upsilon$ = b, where a and b are constants, so that

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$$

$$\frac{\partial \upsilon}{\partial x}dx + \frac{\partial \upsilon}{\partial y}dy + \frac{\partial \upsilon}{\partial z}dz = 0$$

By Applying the method of cross-multiplication to these equations, we obtain :

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial \upsilon}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial x}}$$

or $\quad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$, by virtue of equation (19)

The solution of these equations are u = a and $\upsilon$ = b. Hence $\phi(u,\upsilon) = 0$ is the general solution of Pp + Qq = R where u = a and $\upsilon$=b are the solutions of $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

**Remark :** We adopt the following working procedure to solve the equation Pp + Qq = R..

**Step 1** : Form the subsidiary equation $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

**Step 2** : Obtain any two solutions u = a and $\upsilon$ = b, where u and $\upsilon$ are functions of x, y, z and a, b are constants

**Step 3** : Write the general solution as $\phi$ (u, $\upsilon$ ) = 0 or u = f( $\upsilon$ ) or $\upsilon$ = f(u) where the function f is arbitrary.

**8.4. Examples :**

**8.4.1 Example :** Solve $(p - q) z = z^2 + (x + y)^2$

**Solution :** The given equation can be written as $p z - q z = z^2 + (x + y)^2$

Here $P = z$, $Q = -z$ and $R = z^2 + (x+y)^2$.

Hence the Lagranges subsidiary equations are $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2+(x+y)^2}$

Considering the first two ratios : $dx = -dy$ (or) $dx + dy = 0$

Integrating we get $x + y = C_1$, where $C_1$ is a constant,

considering the first and the last ratios, $dx = \dfrac{Z\,dz}{z^2 + C_1^2}$ ; since $x + y = C_1$

Integrating this equation we get $x = \dfrac{1}{2}\log(z^2 + C_1^2) + C_2$ where $C_2$ is a constant.

Hence the general solution is : $\phi\left[x + y,\ x - \dfrac{1}{2}\log(z^2 + (x+y)^2)\right] = 0$

**8.4.2 Example :** Solve $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$

**Solution :** Here $P = x^2 - yz$, $Q = y^2 - zx$ and $R = z^2 - xy$

Hence the subsidiary equation for the given equation is : $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$

Then $\dfrac{dx - dy}{(x-y)(x+y+z)} = \dfrac{dy - dz}{(y-z)(x+y+z)} = \dfrac{dz - dx}{(z-x)(x+y+z)}$

considering the first and the second ratios we get ,

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} \Rightarrow \log(x - y) = \log(y - z) + \log C_1$$

or $$\log\left(\frac{x-y}{y-z}\right) = \log C_1 \Rightarrow \frac{x-y}{y-z} = C_1$$

Similarly considering the last two ratios and simplifying we get $\dfrac{z-x}{y-z} = C_2$

Hence the general solution can be expressed as : $\phi\left(\dfrac{x-y}{y-z}, \dfrac{z-x}{y-z}\right) = 0$

**8.4.3. Example :** Solve the equation $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2x\, z$

**Solution :** Here $P = x^2 - y^2 - z^2$, $Q = 2xy$, $R = 2x\, z$. Hence the subsidiary equations are :

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \dots \quad (35)$$

From the subsidiary equation (35) each fraction is equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} \quad \dots \quad (36)$$

From the second and third ratios of equation (35) we have,

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log C_1 \quad \text{or} \quad y/z = C_1 \quad \dots \quad (37)$$

Also, $\dfrac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)} = \dfrac{dz}{2xz}$ which on integration gives $\dfrac{x^2 + y^2 + z^2}{z} = C_2$ .....(38)

From equations (37) and (38), the general solution can be written as $\phi\left( y/z, \dfrac{x^2 + y^2 + z^2}{z} \right) = 0$

or $\quad x^2 + y^2 + z^2 = z\, f(y/z)$

## 8.5 Integral surfaces passing through a given curve :

In the preceding section we have discussed the method of obtaining a general solution of the equation $Pp + Qq = R$. We shall now discuss the method of utilising the general solution for obtaining the integral surface that passes through a given curve.

Let $Pp + Qq = R$ be the given equation. Assume that the subsidiary equation give the independent solutions :

$$\phi_1(x, y, z) = C_1 \text{ and } \phi_2(x, y, z) = C_2 \quad \dots \quad (39)$$

Suppose we wish to find the integral surface passing through the curve whose equation is paramatric form is given by :

$$x = x(t), \qquad y = y(t) \qquad \text{and} \qquad z = z(t) \quad \dots \quad (40)$$

where t is the parameter.
Then equation (39) can be expressed as :

$$\left.\begin{aligned} \phi_1[\,x(t), y(t), z(t)\,] &= C_1 \\ \phi_2[\,x(t), y(t), z(t)\,] &= C_2 \end{aligned}\right\} \quad \dots \quad (41)$$

By eliminating the parameter t from equations (41), we obtain a relation involving $C_1$ and $C_2$. We then express the general solution with the help of equations (36) in the form $F(\phi_1, \phi_2) = 0$.

## 8.6 Examples :

### 8.6.1 Example :

Find the integral surface of the differential equation $(x - y) y^2 p + (y - x) x^2 q = (x^2 + y^2) z$ passing through the curve $xz = a^3$, $y = 0$.

**Solution :** Comparing the given equation with the standard form we get
$(x - y) y^2 = P$, $(y - x) x^2 = Q$, and $(x^2 + y^2) = R$

Then the Lagranges subsidiary equations become,

$$\frac{dx}{y^2(x-y)} = \frac{dy}{(y-x)x^2} = \frac{dz}{z(y^2-x^2)} \qquad \ldots \quad (42)$$

Considering the first and the second ratios we get,

$$x^2\,dx = -y^2\,dy \quad \Rightarrow \quad x^3 + y^3 = C_1 \qquad \ldots \quad (43)$$

By choosing 1, -1, 0 as the multipliers, each ratio of equation (42) is equal to

$$\frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$$

or $$\frac{dx - dy}{x - y} = \frac{dz}{z} \quad \Rightarrow \quad x - y = C_2 z$$

$$\Rightarrow \quad \frac{x-y}{z} = C_2 \qquad \ldots \quad (44)$$

By hypothesis the given curve is $xz = a^3$, $y = 0$ .... **(45)**

Using equation (45) in equations (43) and (44) we get,

$$x^3 = C_1, \text{ and } x = C_2 z. \text{ so that } x^4 = C_1 C_2 z \qquad \ldots \quad (46)$$

Using the values of $C_1$ and $C_2$ from equations (43) and (44) in equation (46) we get,

$$x^4 = \left(x^3 + y^3\right)\left(\frac{x-y}{z}\right)z$$

or $x^4 = (x^3 + y^3)(x - y)$. This is the equation of the required surface.

**8.6.2. Example :**

Obtain the integral surface of the equation $x(y^2+z)\,p - y(x^2+z)\,q = (x^2-y^2)z$. which contains the straight line $x + y = 0$, $z = 1$.

**Solution :** The auxiliary equations for the given PDE become

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \qquad \dots \quad \textbf{(47)}$$

From the first two ratios and the third ratios we have $\dfrac{x\,dx + y\,dy}{z(x^2-y^2)} = \dfrac{dz}{z(x^2-y^2)}$

Integrating this equation we get $\dfrac{1}{2}(x^2+y^2) = z + \dfrac{1}{2}C_2$

$$\Rightarrow \quad x^2 - y^2 - 2z = C_2 \qquad \dots \quad \textbf{(48)}$$

Similarly by multiplying each of the ratios of eq (47) with xyz, summing them and integrating we get,

$$xyz = C_1 \qquad \dots \quad \textbf{(49)}$$

The parametric representation of the line given by $x + y = 0$, $z = 1$ can be taken as $x = t, y = -t, z = 1$
Substituting these values in equtions (48) and (49) we get

$$-t^2 = C_1 \text{ and } 2t^2 - 2 = C_2 \qquad \dots\dots \quad \textbf{(50a)}$$

Eliminating t from the equations (50a) we have the relation $2C_1 + C_2 + 2 = 0$ ..... **(50b)**

Hence the desired integral surface is written using equations (48) (49) and (50b) as :

$$x^2 + y^2 + 2xyz - 2z + 2 = 0$$

## 8.7 Surfaces orthogonal to a given system of surfaces

Suppose we are given a one-parameter (C) family of surfaces characterised by the equation :

$$f(x, y, z) = C \qquad \dots \quad \textbf{(51)}$$

Suppose we wish to find a system of surfaces which cut each of the given surfaces (51) at right angles. Then the direction ratios of the normal at (x, y, z) to the surfaces (51) which pass through that point are $\left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right)$

Let the surface $z = \phi(x, y)$ .... **(52)**

Cut each surface of (51) at right angles. Then the normal at (x,y,z) to the surface (52) has the direction ratios $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$, i.e., ( p, q, -1).

Since the normals at (x, y, z) to the surface (51) and to (52) are at right angles, we have

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} = 0 \text{ (or) } p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \qquad \text{.... } \mathbf{(53)}$$

This is of the form Pp + Qq = R

Conversely, we can easily verify that any solution of equation (53) is orthogonal to every surface represented by equation (51).

**8.7.1. Example :** Find the surface which intersects the surfaces of the system $z(x + y) = C(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1$, $z = 1$.

**Solution :** The given system of surfaces is $f(x, y, z) = \frac{z(x+y)}{3z+1} = C$

Therefore, $\frac{\partial f}{\partial x} = \frac{z}{3z+1}$, $\frac{\partial f}{\partial y} = \frac{z}{3z+1}$

$$\frac{\partial f}{\partial z} = (x+y)\frac{3z+1-3z}{(3z+1)^2} = \frac{x+y}{(3z+1)^2}.$$

The required orthogonal surface is the solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \quad \text{i.e.,} \quad \frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $z(3z + 1)p + z(3z + 1)q = x + y$ ..... **(54)**

Lagranges auxiliary equations for the equation (54) are

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \qquad \text{..... } \mathbf{(55)}$$

From the first two ratios we get $dx - dy = 0$, so that $x - y = C_1$

Also, choosing x, y and $-z(3z + 1)$ as the multipliers, each of the ratios of equation (55) is equal to

$$\frac{x\,dx + y\,dy - z(3z+1)\,dz}{0}$$

Hence $x\,dx + y\,dy - 3z^2\,dz - z\,dz = 0$

Integrating, we get $x^2 + y^2 - 2z^3 - z^2 = C_2$

Hence any surface that is orthogonal to the given system has the equation of the form $x^2 + y^2 - 2z^3 - z^2 = \psi\,(x-y)$.

To obtain the desired surface passing through the circle $x^2 + y^2 = 1, z = 1$ we have to choose $\psi\,(x-y) = -2$. Hence the required particular surface is $x^2 + y^2 - 2z^3 - z^2 = -2$

## 8.8 Summary :

In this lesson, we have discussed the method of formulating a partial differential equation by elimination of arbitrary constants and also by elimination of arbitrary functions. We have solved the first order linear partial differential equation of the form $Pp + Qq = R$ - known as Lagranges equation - by forming the subsidiary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. The general solution is expressed as $\phi\,(u, v) = 0$. We have also discussed the method of utilizing the general solution towards obtaining the integral surface that passes through a given curve.

## SAQs :

1. Eliminate the arbitrary constants indicated in brackets from the following equations and for the partial differential equations

   (i) $z = ax + by + ab$ ; ( a, b) (Ans : $z = px + qy + pq$)
   
   (ii) $az + b = a^2 x + b$ ; ( a, b) (Ans : $pq = 1$)

2. From the partial differential equation by eliminating the arbitrary functions.

   (i) $z = f(y/x)$ (Ans : $px + qy = 0$)
   
   (ii) $z = x + y + f(x, y)$ (Ans : $px - qy = x - y$ )

3. Solve the following equations by Lagranges method

   (i) $p + q = 1$ (Ans : $\phi\,(x - y, x - z) = 0$)
   
   (ii) $xp + yq = z$ (Ans : $\phi\,(x/z, y/z) = 0$)

## 8.9 Model Examination papers :

1. Form the Partial Differential equations by eliminating the arbitrary constants / arbitrary functions

(1) $z = ax + a^2 y^2 + b$ (Ans : $q = 2y\, p^2$)

(2) $z = a e^{pt} \sin px$ (Ans : $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$)

(3) $z = ax + by + a^2 + b^2$ (Ans : $z = px + qy + p^2 + q^2$)

(4) $z = xy + f(x^2 + y^2)$ (Ans : $py - qx = y^2 - x^2$)

(5) $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$ (Ans : $p - q = (y - x)/z$ )

(6) $z = y^2 - 2f(1/x + \log y)$ (Ans : $px^2 + qy = 2y^2$)

2. Solve the following first order partial differential equations :

1. $x^2 p + y^2 q = z^2$ (Ans : $\phi(1/x - 1/y,\ 1/y - 1/z) = 0$

2. $p \tan x + q \tan y = \tan z$ (Ans : $\sin z/\sin y = \phi(\sin x / \sin y)$)

3. $(y^2 z / x)\, p + x zq = y^2$ (Ans : $\phi(x^3 - y^3, x^2 - z^2) = 0$)

4. $z(z^2 + xy)(px - qy) = x^4$, (Ans : $\phi(xy, x^4 - z^4 - 2xy\, z^2) = 0$)

5. $p - 2q = 3x^2 \sin(2x + y)$ (Ans : $\phi(2x+y, x^3 \sin(y+2x) - z) = 0$)

6. $x(y^2 - z^2)\, p + y(z^2 - x^2)\, q = z(x^2 - y^2)$ (Ans : $\phi(x^2+y^2+z^2, xyz) = 0$)

7. $(y - zx)\, p - (x + yz)\, q = x^2 + y^2$ (Ans : $\phi(x^2 - y^2 + z^2, xy - z) = 0$)

8. $yp - xq = 2x - 3y$ (Ans : $\phi(x^2+y^2, 3x+2y+z) = 0$)

9. $(y+zx)p - (x + yz)\, q = x^2 - y^2$ . (Ans : $\phi(x^2+y^2-z^2, xy + z) = 0$)

10. $(y + z)p + (z+x)q = x + y$ . (Ans : $\phi[(x-y)(y-z), (x-y).(x+y+z)^{1/2}] = 0$ )

11. $(1+y)\, p + (1+x)\, q = z$ (Ans . $\phi[(1+x)^2 - (1+y)^2\ 1/2(x+y+z)] = 0$ )

12. $x^2 p + y^2 q = (x + y)\, z$ (Ans : ( $\phi[(xy/z), (x-y)/z] = 0$)

13. $(x^2 - y^2 - yz)\, p + (x^2 - y^2 - zx)\, q = z(x - y)$ (Ans : $\phi[z - x + y, (x^2 - y^2)/z] = 0$)

14. Find the surface which is orthogonal to the one-parameter system $z = cxy(x^2+y^2)$ and which passes through the curve $x^2 - y^2 = a^2$, $z = 0$.

(Ans : $(x^2+y^2+4z^2)(x^2-y^2)^2 = a^4(x^2+y^2)$)

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## LESSON - 9 NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE

### 9.1 INTRODUCTION

We have already learnt that, equations of the type $f(x, y, z, p, q) = 0$ where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ are called partial differential equations of first order. If the degrees of p and q are one, it is termed as linear partial differential equation of first order, otherwise non-linear. The complete solution of such non-linear equation contains only two arbitrary constants and a particular integral can be obtained by giving particular values to the constants.

The general method of solving such first order non-linear partial differential equations is called the Charpit's method. Before taking up this method we shall know some special methods of solving them by classifying these equations into various standard forms.

### 9.2 Equations in the standard form I :

In this section, we shall cc sider equations in the standard form $F(p, q) = 0$ i.e., equations containing p and q only.

An integral of the equation $F(p, q) = 0$ .... (1)

is given by $z = ax + by + c$, .... (2)

Where a and b are conneted by the relation $f(a, b) = 0$ ..... (3)

Since $p = \frac{\partial z}{\partial x} = a$, and $q = \frac{\partial z}{\partial y} = b$, they can be substituted in equation (3) to get $F(p,q) = 0$.

If we solve equation (3) for b, we can express b as a function of a, say $b = \phi(a)$. Substituting this value of b in equation (2) we obtain the complete integral of (1) as.

$$z = ax + \phi(a)\, y + c \qquad ..... \quad (4)$$

The general integral is obtained by taking $c = \psi(a)$ in equation (4), where is an arbitrary function, and eliminating a between

$$z = ax + \phi(a)\, y + \psi(a)$$

and $$0 = x + y\phi'(a) + \psi'(a).$$

## 9.3 Examples :

**9.3.1. Example :** Solve $x^2 p^2 + y^2 q^2 = z^2$

**Solution :** We shall reduce the given equation to the standard form by expressing it as :

$$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \ldots\ldots \quad (5)$$

We now set $\frac{dx}{x} = du$, $\frac{dy}{y} = d\upsilon$ and $\frac{dz}{z} = d\omega$, so that $u = \log x$, $\upsilon = \log y$ and $\omega = \log z$.

Then equation (5) becomes $\left(\frac{\partial \omega}{\partial u}\right)^2 + \left(\frac{\partial \omega}{\partial \upsilon}\right)^2 = 1$ or $P^2 + Q^2 = 1$, Where $P = \frac{\partial \omega}{\partial u}$; $Q = \frac{\partial \omega}{\partial \upsilon}$

The complete solution (integral) is : $\omega = au + b\upsilon + C$ ...... **(6)**

Where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$

Hence equation (6) become $\omega = au + \sqrt{1-a^2}\,\upsilon + C$

or $\log z = a \log x + \sqrt{1-a^2} \log y + C$

**9.3.2. Example :** Solve $p^2 - q^2 = 4$

**Solution :** The given equation is of the form $f(p,q) = 0$. Hence its complete integral is given by $z = ax + by + C$, where $a^2 - b^2 = 4$.

Therefore, The complete integral is $z = ax + \sqrt{a^2-4}\, y + C$

To obtain the general integral, put $C = \psi(a)$. Then $z = ax + \sqrt{a^2-4}\, y + \psi(a)$ ...... **(7)**

Differentiating partially w.r.t. a we get $0 = x + \frac{a}{\sqrt{a^2-4}} y + \psi'(a)$ ...... **(8)**

By eliminating a from equations (7) and (8) we obtain the general integral.

**9.3.3 Example :** Find the complete integral of $(y-x)(qy - px) = (p-q)^2$

**Solution :** The given equation is not in the standard form 1. However, we can reduce it to the standard form through the following substitution : Put $u = x + y$ and $\upsilon = xy$

$$\text{Then } p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial \upsilon}\frac{\partial \upsilon}{\partial x} = \frac{\partial z}{\partial u} + y\frac{\partial z}{\partial \upsilon} = P + yQ$$

Where $P = \frac{\partial z}{\partial u}$, $Q = \frac{\partial z}{\partial \upsilon}$

Similarly $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial \upsilon}\frac{\partial \upsilon}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}x = P + xQ$

Then $p - q = (y - x) Q$ and

$$qy - px = py + xy\,Q - Px - xy.Qy = P(y - x)$$

Substituting the values of (p-q) and qy - px in the given equation we get $(y-x)^2P = (y-x)^2Q^2 \Rightarrow P=Q^2$
The complete integral is $z = au + b\,\upsilon + C$ where $a = b^2$ i.e., $z = b^2(x+y) + bxy + C$

## 9.4 Equations in the standard form - II

We shall now consider equations which do not contain x and y explicitly - i.e., equations of the form $f(z, p, q) = 0$. To solve such equations, we put $t = x + ay$, where a is any arbitrary constant

Assume that $z = z(t)$. Then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}.\frac{\partial t}{\partial x} = \frac{\partial z}{\partial t} = \frac{dz}{dt}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}.\frac{\partial t}{\partial y} = a\frac{\partial z}{\partial t} = a\frac{dz}{dt}$$

Then $f(z, p, q) = 0$ becomes $f(z, \frac{dz}{dt}, a\frac{dz}{dt}) = 0$. which is an ordinary differential equation.

Solving for $\frac{\partial z}{\partial t}$ we get $\frac{\partial z}{\partial t} = \phi(z, a)$

or $\frac{dz}{dt} = \phi(z, a) \Rightarrow \frac{dz}{\phi(z,a)} = dt$, which can be easily integrated.

Hence $\int \frac{\partial z}{\phi(z,a)} = \int dt$ or $F(z, a) = t + b$, where b is a constant of integration.

i.e., $F(z, a) = x + ay + b$ which is a complete integral.

## 9.5 Examples :

**9.5.1. Example :** Solve $z^2 (p^2z^2 + q^2) = 1$

**Solution :** Given that $z^2(p^2z^2+q^2)=1$. Let $q = ap$.

Then $z^2(p^2z^2+q^2) = p^2z^2(z^2+a^2) = 1$ or $p = \pm\dfrac{1}{z\sqrt{a^2+z^2}}$

Also $dz = p\,dx + q\,dy = p\,dx + ap\,dy = p\,(dx + a\,dy)$ (or) $dz = \pm\dfrac{dx + a\,dy}{z\sqrt{a^2+z^2}}$

or $dx + a\,dy = \pm\dfrac{1}{2}\sqrt{a^2+z^2}\,(2z)\,dz$

Integrating, $x + ay + b = \pm\dfrac{1}{2}\dfrac{(a^2+z^2)^{3/2}}{(3/2)}$

or $9(x+ay+b)^2 = (a^2+z^2)^3$, which is the complete integral.

**9.5.2 Example :** Solve $9(p^2z+q^2)=4$

**Solution :** Let $u = x + ay$, so that

$$p \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = a\frac{\partial z}{\partial u} = a\frac{dz}{du}$$

Substituting these values in the given equation, we get,

$$9\left[\left(\frac{dz}{du}\right)^2 z + a^2\left(\frac{dz}{du}\right)^2\right] = 4$$

$$9(z+a^2)\left(\frac{dz}{du}\right)^2 = 4$$

$$\text{i.e.,}\quad \frac{3}{2}\sqrt{(z+a^2)}\,dz = du$$

Integrating both sides of this equation we get $(z+a^2)^{3/2} = u + C$, where C is a constant.

Hence the required complete integral is $(z+a^2)^{3/2} = x + ay + C$.

**9.5.3. Example :** Solve $z^2 (p^2 + q^2 + 1) = C^2$

**Solution :** Let $u = x + ay$, so that $p = \frac{dz}{du}$, $q = a\frac{dz}{du}$

Then the given equation reduces to $z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = C^2$

$$\text{or} \qquad z^2(1+a^2)\left(\frac{\partial z}{\partial u}\right)^2 = C^2 - z^2$$

$$\text{i.e.,} \qquad \sqrt{(1+a^2)}\frac{dz}{du} = \pm\sqrt{\frac{C^2-z^2}{z}}$$

$$\text{or} \qquad \frac{z}{\sqrt{C^2-z^2}}dz = \pm\frac{du}{\sqrt{1+a^2}}$$

Integrating both sides we get $\sqrt{C^2 - z^2} = \pm\frac{u}{\sqrt{1+a^2}} + K$

$$\text{or} \qquad \sqrt{C^2-z^2} = \pm\frac{x+ay}{\sqrt{1+a^2}} + K.$$

This is the required complete integral.

## 9.6 Equations in standard form III

In this section we consider equations in which z is absent and the terms containing x and p can be seperated from those containing y and q.

i.e., Equations in the standard form $f_1(x,p) = f_2(y,q)$

Let $f_1(x,p) = f_2(y,q) = a$, say

Then solving for p and q we get

$$p = \phi(x,a) \text{ and } q = \psi(y,a).$$

But $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy = \phi(x,a)\,dx + \psi(y,a)\,dy.$

on integrating, $z = \int \phi\ (x, a)\ dx + \int \psi\ (y, a)\ dy + b$

This is the desired complete solution involving two constants a and b.

## 9.7 Examples :

**9.7.1 Example :** Solve $p^2 + q^2 = x + y$

**Solution :** The given equation in the standard form is $p^2 - x = y - q^2 = a$ (say)

Then $p^2 = a + x \Rightarrow p = \sqrt{a+x}$

$$q^2 = y - a \Rightarrow q = \sqrt{y-a}$$

Then $dz = p\ dx + q\ dy = \sqrt{a+x}\ dx + \sqrt{y-a}\ dy$

On integration we get $z = \frac{2}{3}(a + x)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b$

This is the complete integral of the given equation.

**9.7.2. Example :** Solve $p^2\ y\ (1 + x^2) = q\ x^2$

**Solution :** The given equation is seperable. It can be expressed in the standard form as

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \text{ (say)}$$

Then, $p = \sqrt{\frac{a}{1+x^2}}\ x$ and $q = ay$

Substituting these values of p,q in $dz = pdx + qdy$ we get

$$dz = \sqrt{\frac{a}{1+x^2}}\ x\ dx + ay\ dy$$

Integrating of this equation,

$$z = \sqrt{a(1+x^2)} + \frac{a}{2}y^2 + b,$$

Where a, b are arbitrary constants. This is the complete integral of the given equation.

**9.7.3 Example :** Solve $z(p^2 - z^2) = x - y$ .... **(9)**

**Solution :** To express the given equation in the standard form we shall re-write it as

$$\left(\sqrt{z}\,\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\,\frac{\partial z}{\partial y}\right)^2 = x - y \qquad ..... \quad \textbf{(10)}$$

Put $\sqrt{z}\, dz = du$, $u = u(x, y)$

Then $\frac{2}{3} z^{3/2} = u$, apart from additive constant

$$\frac{\partial u}{\partial x} = \frac{2}{3} \cdot \frac{3}{2} z^{1/2} \frac{\partial z}{\partial x} = \sqrt{z}\,\frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{2}{3} \cdot \frac{3}{2} z^{1/2} \frac{\partial z}{\partial y} = \sqrt{z}\,\frac{\partial z}{\partial y}$$

Equation (10) can be written as,

$$\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 = x - y$$

i.e., $$\left(\frac{\partial u}{\partial x}\right)^2 - x = \left(\frac{\partial u}{\partial y}\right)^2 - y$$

Let $\left(\frac{\partial u}{\partial x}\right)^2 - x = a = \left(\frac{\partial u}{\partial y}\right)^2 - y$ ; 'a' is a constant.

$$\left(\frac{\partial u}{\partial x}\right)^2 = a + x, \quad \left(\frac{\partial u}{\partial x}\right) = \sqrt{x+a} \quad \left(\frac{\partial u}{\partial y}\right) = \sqrt{a+y}$$

Since $u = u(x,y)$, $\quad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \sqrt{x+a}\, dx + \sqrt{y+a}\, dy$

Integrating, $u = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$; b is a constant.

$$\frac{2}{3} z^{3/2} = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + b$$

or $\quad z^{3/2} = (x+a)^{3/2} + (y+a)^{3/2} + \frac{3b}{2} = (x+a)^{3/2} + (y+a)^{3/2} + C$, where $C = 3b/2$

## 9.8 Equations in the standard form IV : Clairauts type

Suppose the given first order P.D.E. equation is of the clairauts type, in the form $z = px + qy + f(p,q)$

We now show that the complete integral of this type of equations can be expressed as :

$$z = ax + by + f(a,b) \qquad \text{.....} \qquad \textbf{(12)}$$

Where a, b are arbitrary constants.

Differentiating equation (12) partially w.r.t. a and b we get,

$$x \frac{\partial f}{\partial a} = 0 \text{ and } y \frac{\partial f}{\partial b} = 0 \qquad \text{......} \qquad \textbf{(13,14)}$$

Eliminating a and b from equations (12) (13) (14) we get the singular integral,

Let $b = \phi(a)$. Then equation (12) becomes,

$$z = ax + \phi(a)\, y + f[\, a, \phi(a)\, ] \qquad \text{......} \qquad \textbf{(15)}$$

Differentiating partially w.r.t. a we get

$$0 = x + y\, \phi^{1}(a) + f^{1}(\, a, \phi(a)) \qquad \text{.....} \qquad \textbf{(16)}$$

Eliminating a between equations (15) and (16) we get the required general integral.

## 9.9 Examples :

**9.9.1. Example :** Solve $z = px + qy + pq$

**Solution :** The given equation is in the standard form of $z = px + qy + f(p, q)$. Hence its complete integral shall be $z = ax + by + ab$, where a, b are arbitrary constants ...... **(17)**

Differentiating equation (17) partially w.r.t. a and b and equating the obtained expressions to zero, we get $x + b = 0$ and $y + a = 0$ ..... **(18,19)**

Eliminating a and b from equations (17) (18) & (19)

We obtain $b = -a$ and $a = -y$ so that, $z = -xy - xy + xy = -xxy$, which is the singular integral. To obtain the general integral, put $b = f(a)$. Then ,

$$z = ax + f(a)\, y + a\, f(a) \qquad \text{.....} \qquad \textbf{(20)}$$

Differentiating equation (20) partially w.r.t. a and equating the obtained expression to zero, we have

$$x + y f^{1}(a) + f(a) + a f^{1}(a) = 0 \qquad ..... \qquad \textbf{(21)}$$

The general integral is now obtained by eliminating 'a' between equations (20) and (21).

**9.9.2.** **Example :** Solve $4 xyz = pq + 2p x^2y + 2q xy^2$.

**Solution :** The given equation is not in the standard form. To convert it to clairauts form,

put $x = \sqrt{u}$ and $y = \sqrt{\upsilon}$. Then,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = 2\sqrt{u}\ \frac{\partial z}{\partial u}$$

Similarly $q = 2\sqrt{\upsilon}\ \dfrac{\partial z}{\partial \upsilon}$

Substituting, for x, y, p and q in the given equation.

We get $\quad z = u\dfrac{\partial u}{\partial u} + \upsilon\dfrac{\partial z}{\partial \upsilon} + \dfrac{\partial z}{\partial u}\cdot\dfrac{\partial z}{\partial \upsilon}$

$$\text{or} \qquad z = u P + \upsilon Q + PQ \qquad .... \qquad \textbf{(22)}$$

This equation is in the standard form IV.

The complete solution is given by $z = au + b\upsilon + ab$

$$\text{i.e.,} \qquad z = ax^2 + by^2 + ab \qquad .... \qquad \textbf{(23)}$$

Differentiating this equation (23) partially w.r.t. a and b and equating the obtained expressions to zero we get, $x^2 + b = 0$ and $y^2 + a = 0$ .... **(24,25)**

Eliminating 'a' and 'b' from equations (23) (24) and (25) we obtain the singular integral

To find the general integral, let $b = f(a)$, where f is arbitrary. Then $z = ax^2 + f(a)\, y^2 + af(a)$. ..... **(26)**

Differentiating equation (26) partially w.r.t. a and eliminating a between the obtained equation and equation (26), we get the general integral.

## 9.10 General method of solving a non-linear equation of first order : Charpit's method :

We shall now discuss the Charpit's method of solving a first order non-linear partial differential equation. This general method is applied when the given the equation can not be reduced to any one of the four standard forms discussed earlier.

Let the given equation be $f(x, y, z, p, q) = 0$ .... **(27)**

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \quad \text{.... (28)}$$

To solve equation (27), we consider an auxiliary function $\phi(x, y, z, p, q) = 0$ .... **(29)**

Differentiating equation (27), first w.r.t. x and then w.r.t. y we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0 \quad \text{.... (30)}$$

and
$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial y} = 0 \quad \text{.... (31)}$$

Similarly differentiating equation (29) first w.r.t. x and later w.r.t. y we obtain

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}p + \frac{\partial \phi}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}.\frac{\partial q}{\partial x} = 0 \quad \text{.... (32)}$$

and
$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}q + \frac{\partial \phi}{\partial p}.\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}.\frac{\partial q}{\partial y} = 0 \quad \text{.... (33)}$$

Eliminating $\frac{\partial p}{\partial x}$ from equations (30) and (32) we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \quad \text{(34)}$$

Eliminating $\frac{\partial q}{\partial y}$ from equations (31) and (33) we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \quad \text{(35)}$$

We now add equations (34) and (35), use the relation $\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$ and simplify the equation by rearranging the terms, we get

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}+$$

$$\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}=0$$

or $$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z}$$

$$+\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q}=0 \quad \ldots \quad (36)$$

We note that equation (36) is a linear Lagrange equation $\phi$ as the dependent variable and x, y, z, p, q as in dependent variables. Hence we write the subsidiary equations.

$$\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{\partial \phi}{0} \quad \ldots \quad (37)$$

These equations (37) are known as the Charpits equations. Once an integral $\phi$ (x, y, z, p, q) has been found, the problem reduces to solving for p, q and then integrating equation (28) by the already known methods. We note here that not all of the equations (37) need be used, but we choose the simplest of the integrals so as to solve for p, q easily.

## 9.11 <u>Examples :</u>

### 9.11.1 <u>Example :</u> Solve $(p^2 + q^2)\, y = q\, z$

<u>**Solution :**</u> Let $f(x, y, z, p, q) = (p^2 + q^2)\, y - qz = 0$ ... (38)

Then $\frac{\partial f}{\partial p} = 2py$, $\frac{\partial f}{\partial q} = 2\, qy - z$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = p^2 + q^2$, $\frac{\partial f}{\partial z} = -q$

Substituting these values in the Charpits subsidiary equations (37) we get

$$\frac{dx}{-2py}=\frac{dy}{z-2py}=\frac{dz}{-2p^2y+qz-2q^2y}=\frac{dp}{-pq}=\frac{dq}{p^2}$$

Taking the last of the two ratios we have $p\, dp + q\, dq = 0 \Rightarrow p^2 + q^2 = C^2$ ... (39)

We now use equation (39) in equation (38) to solve for p and q. Then $C^2y = qz$ so that $q = C^2y / z$

Substituting this value of q in equation (39) we obtain p = $\frac{C}{z}\sqrt{z^2 - C^2y^2}$

Hence dz = p dx + q dy = $\frac{C}{z}\sqrt{z^2 - C^2y^2}\,dx + \frac{C^2y}{z}dy$

or $\quad z\,dz - C^2y\,dy = C\sqrt{z^2 - C^2y^2}\,dx$

or $$\frac{\frac{1}{2}d(z^2 - C^2y^2)}{\sqrt{z^2 - C^2y^2}} = C\,dx$$

Integrating both sides we obtain $\sqrt{z^2 - C^2y^2} = Cx + d$

or $\quad z^2 = (Cx + d)^2 + C^2y^2 \qquad \ldots\ldots \quad \textbf{(40)}$

This is the required complete integral.

To obtain the general integral, we replace d by $\phi$ (C) in equation (40) so that

$$z^2 - C^2y^2 = [Cx + \phi(C)]^2 \qquad \ldots\ldots \quad \textbf{(41)}$$

Differentiating partially w.r.t. C we get

$$-2Cy^2 = 2[Cx + \phi(C)][x + \phi'(C)] \qquad \ldots\ldots \quad \textbf{(42)}$$

Eliminating C from equations (41) and (42) we obtain the general integral.

**9.11.2 Example :** Solve 2 zx - px$^2$ - 2q xy + pq = 0

**Solution :** Given that f (x, y, z, p, q) = 2 zx - px$^2$ - 2qxy + pq = 0 ..... **(43)**

The Charpits subsidiary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{p^2x + 2xqy - 2pq}$$

From the second ratio we have dq = 0 $\Rightarrow$ q = a

Substituting q = a in equation (43) we get p = $\frac{2x(z - ay)}{x^2 - a}$.

Hence $dz = p\,dx + q\,dy = \dfrac{2x(z-ay)}{x^2-a}\,dx + a\,dy$ (or) $\dfrac{dz - a\,dy}{z - ay} = \dfrac{2x\ dx}{x^2-a}$

Integrating, $\log(z-ay) = \log(x^2-a) + \log b$ or $z - ay = b(x^2-a^2)$ which is the complete integral.

**9.11.3. Example :** Find the complete integral of the equation $p\,xy + pq + qy = yz$

**Solution :** Here $f(x, y, z, p, q) = pxy + pq + qy - yz = 0$ ... **(44)**

Then the Charpits auxiliary equations are :

$$\frac{dp}{0} = \frac{dq}{(px+q)+qy}\ \frac{q\,dz}{-p(xy+q)-q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}$$

From the first ratio we have $dp = 0 \Rightarrow p = a$

Substituting $p = a$ in equation (44) we have $a\,xy + aq + qy = yz$ (or) $(a+y)q = y(z-ax) \Rightarrow q = \dfrac{y(z-ax)}{a+y}$

Now $dz = p\,dx + q\,dy = a\,dx + \dfrac{y(z-ax)}{a+y}\,dy$ or $\dfrac{dz - a\,dx}{z-ax} = \dfrac{y\,dy}{a+y} = \left(1 - \dfrac{a}{a+y}\right)dy$

Integrating $\log(z-ax) = y - a\log(a+y) + \log b$

i.e., $\log(z-ax) + \log(a+y)^a = \log b + y$ (or) $(z-ax)(a+y)^a = b\,e^y$ which is the complete integral

**SAQs :**

I. Find Complete integral of the following equations :

1. $p^2 - q^2 = 1$ (Ans : $z = ax + \sqrt{a^2-1}\ y + C$)
2. $pq = k$ (Ans : $z = ax + (k/a)\,y + C$ )

II. Find the complete integral and the singular integral of the following equations.

1. $z = px + qy + pq$ (Ans : S.I. : $z = xy$
   C.I. : $z = ax + by + ab$ )
2. $z = px + qy + p^2 + q^2$ (Ans : S.I. : $4z + x^2 + y^2 = 0$
   C.I.: $z = ax + by + a^2 + b^2$)

III. Find the complete and singular integral of the following equations :

1. $p^3 + q^3 = 27\,z$ (Ans : S.I. : $z = 0$
C.I. :

2. $q^2 = z^2p^2\,(1-p^2)$ (Ans : S.I. : $z = 0$
C.I. : $z^2 - a^2 = (x+ay+b^2)$

IV. Find the complete integrals of the following equations using Charpits method.

1. $q = px + q^2$

$$\text{(Ans : } z = -\frac{x^2}{4} \pm \left[\frac{x}{2}\sqrt{x^2+4a} + 2a\,\log\{x+\sqrt{x^2+4a}\} + ay + b\right]$$

2. $px + qy + pq = 0$ (Ans : $az = -\frac{1}{2}(y + ax)\,(dy + a\,dx)$

## 9.12 Summary :

The integral $\phi$ (x, y, z, a, b) = 0 which has as many arbitrary constants as there are independent variables is known as the complete integral of the first order partial differential equation f(x, y, z, p, q) = 0. A particular integral of this equation is obtained by giving particular values to the constants a, b in the complete integral. By eliminating a, b between the equations :

$\phi$ (x, y, z, a, b) = 0, $\dfrac{\partial \phi}{\partial a} = 0$, and $\dfrac{\partial \phi}{\partial b} = 0$, we obtain a relation between x, y and z and this gives the singular integral of the considered differential equation. In some cases, the singular integral may be obtained from the complete integral by giving particular values to the involved constants.

If in the complete integral of the given partial differential equation $\phi$ (x, y, z, a, b) = 0, one of the constants, say b, is a function of the other. i.e., b = F(a). Then we have $\phi$ (x,y, z, a, F(a)) = 0 This equation represents one of the families of the surfaces given by the system $\phi$ (x, y, z, a, b) = 0 The general integral of the given equation is obtained by eliminating 'a' between

$\phi$ [(x, y, z, a, f(a)]=0 and $\dfrac{\partial \phi}{\partial a}$=0. Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of general integral satisfies the given differential equation.

In this lesson we have considered first order non-linear partial differential equations, in which the degree of p and q is other than one. Initially we began our discussion (study) with four standard forms to which many equations can be reduced and for which a complete integral can be obtained either by inspection or by other simple methods. Finally we discussed the general method of solving the given equation with two independent variables, known as the Charpit's method, to solve equations which can not be reduced to any one of the standard forms.

## 9.13 Model Examination Questions :

I. Find the complete integral of the equations :

(1) $(x^2+y^2)(p^2+q^2) = 1$ (Ans : $z = a \log r + \sqrt{1-a^2}\,\theta + C$

$r = \sqrt{x^2-y^2}$, $\theta = \tan^{-1}(y/x)$)

(2) $p^2 + q^2 = 1$ (Ans : $z = ax + \sqrt{1-a^2}\,y + C$ )

(3) $p^2 + p = q^2$ (Ans : $z = ax + \sqrt{a+a^2}\,y + C$)

(4) $q = 3p^2$ (Ans : $z = ax = 3a^2y + C$)

II. Find the complete and singular integrals of the following equations :

1. $z = px + qy - p^2q$ (Ans : C.I. : $z = ax + by - a^2b$
   S.I. : $z^2 = x^2 y$ )

2. $z = px + qy - 2\sqrt{pq}$ (Ans : C.I. : $z = ax + by - 2\sqrt{ab}$
   S.I. : $xy = 1$ )

3. $z^2(p^2z^2+q^2) = 1$ (Ans : C.I. : $9(x+ay+b)^2 = (z^2+a^2)^3$
   S.I. : No singular integral)

III. Prove that the complete integral of $z = px + qy + \sqrt{p^2+q^2+1}$ represents all planes at unit distance from the origin.

1. $z = pq$ (Ans : $(x + ay + b)^2 = 4az$ )
2. $p(1+q^2) = q(z-a)$ (Ans : $4a\,(z-a) = 4 + (x+ay+b)^2)$
3. $z = p^2 - q^2$ (Ans : $az = (x+ay+b)^2)$
4. $z^2\ p^2 + q^2 = p^2q$ (Ans : $z = a \tan (x+ay+b))$

V. Find the complete integrals of the following equations :

1. $pq = ay$ (Ans : $2\,az = a^2x^2 + y^2 + b)$
2. $p^2y\,(1+x^2) = q\,x^2$ (Ans : $z^2 = ax^2 \pm \sqrt{a-1}\,y^2+b)$
3. $x^2p^2 = y\,q^2$ (Ans : $z = a \log x + 2a\,\sqrt{y} +b)$
4. $q = xy\,p^2$ (Ans : $(2z - ay^2 - 2b)^2 = 16\,ax$ )
5. $p^2 = q + x$ (Ans : $3z = 3ay + 2(a+x)^{3/2} + 3b)$

VI Apply the Charpits method to find the complete integral of the following equations :

1. $p = (\,z + qy\,)^2$ (Ans : $yz = ax + 2\,\sqrt{ay} + b)$
2. $(p^2 + q^2)\,y = qz$ (Ans : $z^2 - a^2y^2 = (ax+b)^2)$
3. $yz\,p^2 = q$ (Ans : $z^2 = 2ax + a^2y^2 + b$ )
4. $q = 3p^2$ (Ans : $z = ax + 3a^2y +b)$

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# LESSION - 10 PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

## 10.1 INTRODUCTION :

So far we discussed various methods of solving first order partial differential equations. In the discussion that follows, we shall confine ourselves to the linear partial differential equations of second order. A study of such equations is of great practical interest since they arise frequently in several problems of mathematical physics. For example, the equation $\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{k}\frac{\partial \phi}{\partial z}$ called the one dimensional diffusion equation, is applicationally interesting. Before taking up the method of solving such equations, we shall first consider those equations that can be reduced to linear equations and thereafter discuss the method of solving equations by Lagranges method and finally consider the canonical forms.

## 10.2 Given equation reducible to a linear equation :

A partial differential equation is said to be of second order if it contains atleast one partial derivative of second order and none of the order greater than two. The most general form of a second order partial differential equation is given by F (x, y, z, p, q, r, s, t) = 0, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}$

## 10.3 Example : Solve $t - qx = x^2$

**Solution :** The given equation is $\frac{\partial q}{\partial y} - xq = x^2$. This equation is linear in q and y, if x is treated as constant. The integrating factor is given by $e^{-x\int dy} = e^{-xy}$.

Hence the solution of the given equation is :

$$q\,e^{-xy} = \int x^2 e^{-xy}\,dy + \phi(x) = -x\,e^{-xy} + \phi(x)$$

i.e., $$\frac{\partial z}{\partial y} = -x + e^{xy}\phi(x)$$

Now integrating w.r.t. y, treating x as a constant.

We obtain $z = -xy + \dfrac{e^{xy}}{x}\phi(x) + f(x)$ which is the required solution.

**10.2.2. Example :** Solve yt - q = xy

**Solution :** The given equation can be written as $\dfrac{\partial q}{\partial y} - \dfrac{q}{y} = x$, which is a linear equation of first order.

The integrating factor = $e^{-\int \frac{1}{y} dy} = \dfrac{1}{y}$. Hence the solution of the given equation is

$$q.\frac{1}{y} = \int \frac{x}{y}\, dy + \phi(x) \quad \text{(or)} \quad q = y \,.\, x \log y + y\,\phi(x)$$

$$\text{i.e.,} \quad \frac{\partial z}{\partial y} = x\, y \log y + y\,\phi(x)$$

Integrating again w.r.t. y, treating x as constant, we obtain $z = (x \log x)\dfrac{y^2}{2} - \dfrac{1}{4}xy^2 + \dfrac{y^2}{2}\phi(x) + f(x)$

This is the required solution.

**10.2.3. Example :** x s + q = 4x + 2y + 2

**Solution :** The given equation can be re-written as $x\dfrac{\partial q}{\partial x} + q = 4x + 2y + 2$.

$$\text{or} \quad \frac{\partial q}{\partial x} + \frac{1}{x}q = 4 + \frac{2y}{x} + \frac{2}{x} \quad \ldots \quad \textbf{(1)}$$

which is a linear equation. The integrating factor = $e^{\int \frac{dx}{x}} = e^{\log x} = x$

The solution as now obtained by multiplying the equation (1) with the integrating factor and integrating both sides with respect to x.

$$\text{i.e.,} \quad x.q = \int \left(4 + \frac{2y}{x} + \frac{2}{x}\right) x\, dx + f(y) = 2x^2 + 2xy + 2x + f(y)$$

$$\text{or} \quad \frac{\partial z}{\partial y} = 2x + 2y + 2 + \frac{1}{x} f(y)$$

Integrating w.r.t. y we get $z = 2\,xy + y^2 + 2y + \frac{1}{x}\int f(y)dy + F(x)$

or $\quad xz = 2x^2y + xy^2 + 2xy + \phi\,(y) + F(x)$ ; where $\int f(y)dy = \phi\,(y)$

## 10.3 Equations solved by Lagrange's method :

In this method, the given equation is first integrated w.r.t. one of the variables and is put in the standard form of $Pp + Qq = R$. The Lagrange's subsidiary equations are then written and solved to obtain a linear equation. The required solution is then obtained by solving the linear equation, as has been done in section (10.2) we shall illustrate the method with a few examples :

**10.3.1. Example :** Solve $p + r + s = 1$

**Solution :** The given equation expressed as $\frac{\partial z}{\partial x} + \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 1$

Integrating this equation w.r.t. x, we have $z + p + q = x + f(y)$ (or) $p + q = x - z + f(y)$

The Lagrange's subsidiary equations are $\frac{\partial x}{1} = \frac{\partial y}{1} = \frac{dz}{x - z + f(y)}$

From the first two ratios, we have $dx = dy$ on integration we get $x - y = C_1$

Considering the last two ratios we get $\frac{dz}{dy} + z = x + f(y)$ or $\frac{dz}{dy} + z = C_1 + y + f(y)$

This is a linear equation, for which the Integrating Factor is $e^{\int dy} = e^y$.

$$
\begin{aligned}
\text{Therefore, } z\,e^y &= \int [C_1 + y + f(y)]\,ey\,dy + C_2 \\
&= C_1\,e^y + \int [y + f(y)]\,e^y\,dy + C_2 \\
&= C_1\,e^y + \phi\,(y) + C_2 \\
\text{or} \quad z &= C_1 + e^{-y}\,\phi\,(y) + C_2\,e^{-y} \\
&= x - y + e^{-y}\,\phi\,(y) + e^{-y}\,\phi\,(x-y), \text{ which is the required solution.}
\end{aligned}
$$

**10.3.2. Example :** Solve $s - t = \frac{x}{y^2}$.

**Solution :** The given equation can be written as $\frac{\partial p}{\partial y} - \frac{\partial q}{\partial y} = \frac{x}{y^2}$

Integrating w.r.t. y we get $p - q = -\frac{x}{y} + \phi\,(x)$.

Then the Lagranges subsidiary equations are $\frac{\partial x}{1} = \frac{\partial y}{-1} = \frac{dz}{-x/y+\phi(x)}$

Taking the first two ratios we get dx + dy = 0 i.e., x + y = a, 'a' is a constant.

Considering the first and the last ratios we have dz $= -\frac{x}{y}\,dx + \phi(x)\,dx$

$$= -\frac{x}{a-x}\,dx + \phi\,(x)\,dx$$

$$= \left(1 - \frac{a}{a-x}\right) dx + \phi\,(x)\,dx$$

Integrating we get, $z = x + a \log(a-x) + \phi\,(x) + b$

or $z = a \log(a-x) + \psi\,(x) + b\,;\ (x) = x + \phi\,(x)$

$= (x+y)\log y + \psi\,(x) + f(x+y)$. This is the required solution.

**10.3.3. Example :** Determine the surfaçe satisfying r + s = 0 and touching the curve $z = 4x^2$ along its section by the plane y = 2x + 1

**Solution :** The given equation is $\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 0.$

Integrating w.r.t. x we get p + q = f(y).

The Lagranges subsidiary equations are $\frac{\partial x}{1} = \frac{\partial y}{1} = \frac{dz}{f(y)}$.

From the first two ratios we get dy = dx $\Rightarrow$ $y - x = C_1$

From that last two ratios we have dz = f(y) dy $\Rightarrow$ $z = \phi\,(y) + C_2$

Hence the solution $z = \phi(y) + F\,(y-x)$ ..... (1)

Now $p = \frac{\partial z}{\partial x} = -F^1\,(y-x)$

$q = \frac{\partial z}{\partial y} = \phi^1(y) + F^1\,(y-x)$

From $z = 4x^2 + y^2$ we have p = 8x and q = 2y ..... (2)

If (1) touches (2) along its section by the plane y = 2x + 1, then the values of p and q for any point n y = 2x + 1 must be equal i.e., $-F^1\,(y-x) = 8x$ ..... (3)

and $\phi^1(y) + F^1(y - x) = 2y$ ..... **(4)**

$$y = 2x + 1 \qquad ..... \quad (5)$$

from (3) and (5) we have $-F^1(y - x) = 8(y - x - 1)$

or $\quad -F(y - x) = 8\left[\frac{1}{2}(y - x)^2 - (y - x)\right] + a_1$

i.e., $\quad F(y - x) = -4(y - x)^2 + 8(y - x) + C$

From (3) and (4) we have $\quad \phi^1(y) = 8x + 2y$

$$\phi^1(y) = \frac{8}{2}(y - 1) + 2y = 6y - 4$$

$$\phi(y) = 3y^2 - 4y + b_1$$

The required surface is now obtained by substituting $\phi(y)$ in equation (1)

i.e., $\quad z = 3y^2 - 4y + b_1 - 4(y - x)^2 + 8(y - x) + C$

or $\quad z = 4x^2 + y^2 - 8xy + 8x - 4y - C$ ..... **(6)**

Equating the values of z from equations (2) and (6) we get C = -2.

The required solution is $z = 4x^2 + y^2 - 8xy + 8x - 4y + 2 = 0$.

## 10.4. <u>Differential Cononical Forms :</u>

We shall now consider equations of the type $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ ( 7 ) , where R, S, T are continuous functions of x and y possessing continuous partial derivatives of n-th order. By changing the independent variables to u, $\upsilon$ by the transformation $u = u(x.y)$ , $\upsilon = \upsilon(x.y)$ we shall show that the given equation (7) can be transformed into one of the three cononical forms which are easily integrable. Now ,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial \upsilon}.\frac{\partial \upsilon}{\partial x}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial \upsilon}.\frac{\partial \upsilon}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}.\frac{\partial}{\partial u} + \frac{\partial \upsilon}{\partial x}.\frac{\partial}{\partial \upsilon}\right)\left(\frac{\partial u}{\partial x}.\frac{\partial z}{\partial u} + \frac{\partial \upsilon}{\partial x}.\frac{\partial z}{\partial \upsilon}\right)$$

$$= \frac{\partial^2 z}{\partial \upsilon^2}\left(\frac{\partial \upsilon}{\partial x}\right)^2 + 2\,\frac{\partial^2 z}{\partial u \partial \upsilon}\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial x} + \frac{\partial^2 z}{\partial \upsilon^2}\left(\frac{\partial \upsilon}{\partial x}\right)^2$$

$$\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial \upsilon}\frac{\partial^2 \upsilon}{\partial x^2}$$

Similarly $\quad s \quad = \quad \dfrac{\partial^2 z}{\partial u^2}\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 z}{\partial u \partial \upsilon}\left(\dfrac{\partial u}{\partial x}\dfrac{\partial \upsilon}{\partial y} + \dfrac{\partial u}{\partial y}\dfrac{\partial \upsilon}{\partial x}\right)$

$$+ \frac{\partial^2 z}{\partial \upsilon^2}\frac{\partial \upsilon}{\partial x}\frac{\partial \upsilon}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 \upsilon}{\partial y \partial x} + \frac{\partial z}{\partial \upsilon}\frac{\partial^2 \upsilon}{\partial x \partial y}$$

and $\quad t \quad = \quad \dfrac{\partial^2 z}{\partial u^2}\left(\dfrac{\partial u}{\partial y}\right)^2 + 2\dfrac{\partial^2 z}{\partial u \partial \upsilon}\dfrac{\partial u}{\partial y}\dfrac{\partial \upsilon}{\partial y}$

$$+ \frac{\partial^2 z}{\partial \upsilon^2}\left(\frac{\partial \upsilon}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial \upsilon}\frac{\partial^2 \upsilon}{\partial y^2}$$

Substituting these values of p, q, r, s, t in equation (7) and simplifying we get.

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial \upsilon} + C\frac{\partial^2 z}{\partial \upsilon^2} + F\left(u, \upsilon, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial \upsilon}\right) = 0 \qquad \ldots \quad \textbf{(8)}$$

where $A = R\left(\dfrac{\partial u}{\partial x}\right)^2 + S\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + T\left(\dfrac{\partial u}{\partial y}\right)^2 \qquad \ldots. \quad \textbf{(8(a))}$

$$B = R\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial y}$$

$$C = R\left(\frac{\partial \upsilon}{\partial x}\right)^2 + S\frac{\partial \upsilon}{\partial x}\frac{\partial \upsilon}{\partial y} + T\left(\frac{\partial \upsilon}{\partial y}\right)^2$$

and $F\left(u, \upsilon, z, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial \upsilon}\right)$ is the transformed form of f (x, y, z, p, q)

We now determine u and $\upsilon$ so that the equation (8) reduces to the most simple form. The method of evaluation of the desired values of u and $\upsilon$ become easy when the discriminant $S^2$-4RT of the quadratic equation $R\lambda^2 + S\lambda + T = 0$..... **(9)** is either positive, negative or zero every where.

We shall now discuss these three cases seperately.

**Case 1 :** $S^2 - 4RT > 0$

In this case the roots $\lambda_1$ and $\lambda_2$ of equation (9) are real and distinct.

We choose u and $\upsilon$ such that $\dfrac{\partial u}{\partial x} = \lambda_1 \dfrac{\partial u}{\partial y}$ ... **(10)**

and $\dfrac{\partial \upsilon}{\partial x} = \lambda_2 \dfrac{\partial \upsilon}{\partial y}$

Therefore $A = (R\lambda_1^2 + S\lambda_1 + T)\left(\dfrac{\partial u}{\partial y}\right)^2 = 0$; where $\lambda_1$ is a root of $R\lambda_1^2 + S\lambda_1 + T = 0$

Similarly $C = 0$. The Lagranges auxiliary equation for the equation (10) are $\dfrac{dx}{1} = \dfrac{dy}{-\lambda_1} = \dfrac{du}{0}$

From the third ratio we have $du = 0$; $u = C_1$, $C_1$ is a constant.

From the first and second ratios we have $\dfrac{dy}{dx} + \lambda_1 = 0$ ... **(11)**

Let $f_1(x,y) = C_2$ be the solution of equation (11).

Therefore, the solution of equation (10) can be taken as $u = f_1(x,y)$.

Similarly if $f_2(x,y) = \text{const}$ is a solution of $\dfrac{dy}{dx} + \lambda_2 = 0$, then the solution of $\dfrac{\partial \upsilon}{\partial x} = \lambda_2 \dfrac{d\upsilon}{dy}$ can be taken as $\upsilon = f_2(x,y)$.

It can be shown that $AC - B^2 = \dfrac{1}{4}(4RT - S^2)\left(\dfrac{\partial u}{\partial x}\dfrac{\partial \upsilon}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial \upsilon}{\partial x}\right)^2$,

$$A = 0 = C, \text{ and } B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial \upsilon}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial \upsilon}{\partial x}\right)^2$$

Also, since $S^2 - 4RT > 0$, it follows that $B^2 > 0$. Hence we may divide both sides of equation (8a) by it with $A = 0$, $C = 0$ and using the above facts, equation (7) takes the form $\dfrac{\partial^2 z}{\partial u \partial \upsilon} = \phi\left(u, \upsilon, z, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial \upsilon}\right)$. This is the canonical form of equation (7).

## 10.5 Examples :

**10.5.1 Example :** Reduce the equation $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form

**Solution :** The given equation can be written as $r + x^2 t = 0$ .... **(12)**

Comparing equation (12) with the equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have R=1, S = 0 and T = $x^2$.

Hence $R\lambda^2 + S\lambda + T = 0$, for our case become $\lambda^2 + x^2 = 0$, giving $\lambda_1 = i\,x$, and $\lambda_2 = -i\,x$

Also, $\frac{dy}{dx} + \lambda_1 = 0$ become $\frac{dy}{dx} + i\,x = 0$

$\frac{dy}{dx} + \lambda_2 = 0$ become $\frac{dy}{dx} - i\,x = 0$

Integrating these equations we get $y + \frac{ix^2}{2} = C_1$, $y - \frac{ix^2}{2} = C_2$. Where $C_1$, $C_2$ are constants.

To reduce equation (12) to cononical form, we change the variables x, y to u, $\upsilon$ respectively by taking $u = y + \frac{ix^2}{2}$ and $\upsilon = y - \frac{ix^2}{2}$.

If $u = \alpha + i\beta$ (say) and $\upsilon = \alpha - i\beta$ (say) then we have $\alpha = y$ and $\beta = \frac{1}{2}x^2$. Then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left(\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right)$$

$$= \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$$

Substituting these values in the given equation, we have

$$\left(\frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2}\right) + x^2\left(\frac{\partial^2 z}{\partial \alpha^2}\right) = 0$$

i.e.,
$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2}\frac{\partial z}{\partial \beta}$$

or
$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \alpha},$$ which is the required cononical form.

**10.5.2 Example :** Reduce the equation $(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n} \frac{\partial^2 z}{\partial y^2} = n\, y^{2n-1} \frac{\partial z}{\partial y}$ **(13)**

to canonical form and find its general solution.

**Solution :** The given equation can be written as $(n-1)^2 r - y^{2n} t - n\, y^{2n-1} q = 0$ **(14)**

Comparing equation (14) with the standard form $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ we have, $R = (n-1)^2$, $S = 0$, and $T = -y^{2n}$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes $(n-1)^2 \lambda - y^{2n} = 0$, so that $\lambda = \pm\frac{1}{(n-1)} y^n$

Hence $\lambda_1 = (n-1)^{-1} y^n$ and $\lambda_2 = -(n-1)^{-1} y^n$ are the two distinct real roots.

$$\frac{dy}{dx} + \lambda_1 = 0 \text{ becomes } \frac{dy}{dx} + \frac{1}{(n-1)} y^n = 0 \text{ and}$$

$$\frac{dy}{dx} + \lambda_2 = 0 \text{ becomes } \frac{dy}{dx} - \frac{1}{(n-1)} y^n = 0$$

Integrating these equations we get $x - y^{-n+1} = C_1$, and $x + y^{-n+1} = C_2$, $C_1$, $C_2$ are constants

To reduce the given equation (13) to canonical form, we change the independent variables x, y to u $\upsilon$ by taking $u = x - y^{-n+1} = C_1$, $\upsilon = x + y^{-n+1}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}$$

$$q = \frac{\partial z}{\partial y} = (n-1)\, y^{-n} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial \upsilon}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial u}{\partial \upsilon}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial \upsilon} + \frac{\partial^2 z}{\partial \upsilon^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[(n-1)\, y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right)\right]$$

$$= -n\,(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right) + (n-1)\, y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right)$$

$$= -n\,(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right) + n(n-1)\, y^{-n}$$

$$\left(\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial \upsilon}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right)\frac{\partial \upsilon}{\partial y}\right)$$

$$= -n\,(n-1)^{-n-1}\left(\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right) + (n-1)^2\, y^{-2n}\left(\frac{\partial^2 z}{\partial u^2}\right.\right.$$

$$\left. - 2\frac{\partial^2 z}{\partial u \partial \upsilon} + \frac{\partial^2 z}{\partial \upsilon^2}\right)$$

Substituting these values of q, r, t in equation (13) and simplifying, we obtain $\frac{\partial^2 z}{\partial u \partial \upsilon} = 0$, which is the canonical form of the given equation.

Integrating this equation again w.r.t. $\upsilon$ we get $\frac{\partial z}{\partial u} = F(u)$.

On integrating this equation again w.r.t. u we obtain $z = \int F(u)\, du + G(\upsilon) = H(u) + G(\upsilon)$ (Say)

Hence the solution of the equation (13) is $z = H(x - y^{-n+1}) + G(x + y^{-n+1})$

**Case 2 :** $S^2 - 4RT = 0$

In this case, the two roots of the equation (9) i.e., $R\lambda^2 + S\lambda + T = 0$, would be real and equal. We take u, as in case1 i.e., $\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y}$ giving u = f(x,y). We consider $\upsilon$ to be a function of x,y which is independent of u. Then, as in case 1 we have A = 0, $B^2 = 0$ since $S^2$-4RT = 0 and hence B = 0.

Substituting A = 0, B = 0 in equation (8) and dividing the equation throughout by C ($\neq 0$) we get

$$\frac{\partial^2 z}{\partial \upsilon^2} = \phi\left(u, \upsilon, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial \upsilon}\right)$$

This is the second canonical form of equation (7)

**10.5.3. Example :** Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

**Solution :** The given equation is r + 2s + t = 0 comparing this equation with equation (7) we get R = 1, S = 2 and T = 1 so that $R\lambda^2 + S\lambda + T = 0$ gives $\lambda^2 + 2\lambda + 1 = 0$.

Hence $\lambda$ = -1, -1 and $\frac{\partial y}{\partial x} + \lambda = 0$ become $\frac{\partial y}{\partial x} - 1 = 0$.

Integrating we get x - y = $c_1$, $c_1$ is a constant. To reduce the given equation to the canonical form, we transform x, y to u, $\upsilon$ by taking u = x - y, $\upsilon$ = x + y.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}, \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial \upsilon}$$

$$q = \frac{\partial z}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \upsilon} - \frac{\partial}{\partial u}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial \upsilon}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}\right) = \frac{\partial^2 z}{\partial u^2} + 2\left(\frac{\partial^2 z}{\partial u \partial \upsilon}\right) + \frac{\partial^2 z}{\partial \upsilon^2}$$

Similarly, $$t = \frac{\partial^2 z}{\partial \upsilon^2} - 2\left(\frac{\partial^2 z}{\partial u \partial \upsilon}\right) + \frac{\partial^2 z}{\partial u^2}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial \upsilon^2}$$

Substituting the values of r, s, t in the given equation and simplifying, we get $\frac{\partial^2 z}{\partial \upsilon^2} = 0$ which is the required cononical form.

Integrating w.r.t. $\upsilon$ we get $\frac{\partial z}{\partial \upsilon}$ = F(u) and on further integration w.r.t. $\upsilon$ we get

$z \quad \upsilon\ F(u) + G(u)$, $F(u)$, $G(u)$ are arbitrary functions of u

Hence the required solution is Z = $(x+y)\ F(x-y) + G(x-y)$

**10.5.4 Example :-** Reduce the equation

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial v}$$

to canonical form and solve it. 15)

**Solution :-** The given equation is

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0 \qquad \text{---(16)}$$

comparing equation (16) with equation (7) we get $R = y^2$, $S = -2xy$, $T = x^2$ so that

$R\lambda^2 + S\lambda + T = 0$ gives $y^2\lambda^2 - 2xy\lambda + x^2 = 0$ or $(y\lambda - x)^2 = 0$

Therefore, $\lambda = \frac{x}{y}, \frac{x}{y}$ and $\frac{dy}{dx} + \lambda = 0$ becomes $\frac{dy}{dx} + \frac{x}{y} = 0$

i.e. 2y dy + 2x dx = 0, Integrating we get $x^2 + y^2 = c_1$

To reduce equation (15) to the canomical form we transform x, y to u, v, by taking $u = x^2 + y^2$

and choose $\upsilon = x^2 - y^2$ ; which is independent of u

$$\text{Then p} = \frac{\partial z}{\partial x} = 2x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}\right);\ q = 2y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial \upsilon}\right)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[2x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial \upsilon}\right)\right]$$

$$= 2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial \upsilon}\right)+4x^2\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u \partial \upsilon}+\frac{\partial^2 z}{\partial \upsilon^2}\right)$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial \upsilon}\right)\right)$$

$$= 2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial \upsilon}\right) + [2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial \upsilon}\right)]$$

$$= 2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial \upsilon}\right)+4y^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u \partial \upsilon}+\frac{\partial^2 z}{\partial \upsilon^2}\right)$$

$$s = 2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial \upsilon}\right)$$

$$= 4xy\left(\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial \upsilon^2}\right)$$

Subsituting these values in equation (16) and simplifying we get $\frac{\partial^2 z}{\partial y^2}=0$, which is the required canonical form. Integrating this equation w.r.t. $\upsilon$, we get $\frac{\partial z}{\partial \upsilon}$ = F(u) on further integration w.r.t. $\upsilon$ we get z = $\upsilon$ F(u) + G(u). Hence the required solution of equation (15) is given by

$$z = (x^2-y^2)\, F(x^2+y^2) + G(x^2+y^2)$$

**Case 3 :** $S^2 - 4RT < 0$

In this case the roots of the equation $R\lambda^2 + S\lambda + T = 0$ are complex and we have a similar discussion as in case (1). Hence proceeding as in Case (1), we find that equation (7) reduces to the form

$$\frac{\partial^2 z}{\partial u \partial \upsilon} = \phi\left(u, \upsilon, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial \upsilon}\right) \quad ..... \textbf{(17)}$$ but the variables u and $\upsilon$ in this case are complex conjugates. To obtain a real canonical form, we apply transformation

$u = \alpha + i\beta$ and $\upsilon = \alpha - i\beta$, so that $\alpha = \frac{1}{2}(u+\upsilon)$ and $\beta = \frac{1}{2}(u-\upsilon)$ .. **(18)**

Now $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial \alpha}\cdot\dfrac{\partial \alpha}{\partial u} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial u} = \dfrac{1}{2}\left(\dfrac{\partial z}{\partial \alpha} - i\dfrac{\partial z}{\partial \beta}\right)$

Similarly $\dfrac{\partial z}{\partial \upsilon} = \dfrac{1}{2}\left(\dfrac{\partial z}{\partial \alpha} + i\dfrac{\partial z}{\partial \beta}\right)$

$$\frac{\partial^2 z}{\partial u \partial \upsilon} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial \upsilon}\right) = \frac{1}{4}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right)$$

Substituting these values in equation (17), the transformed canonical form of the given equation become

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

**10.5.5. Example :** Reduce the equation $\dfrac{\partial^2 z}{\partial x^2} + x^2\dfrac{\partial^2 z}{\partial y^2} = 0$ to the canonical form.

**Solution :** The given equation can be expressed as $r + x^2 t = 0$

Comparing this equation with equation (7) we have $R = 1$, $S = 0$, and $T = x^2$, so that $R\lambda^2 + S\lambda + T = 0$ become $\lambda^2 + x^2 = 0$.

Therefore, $\lambda = \pm i x$ and $\lambda_1 = i x$ and $\lambda_2 = -i x$.

$\dfrac{dy}{dx} + \lambda_1 = 0$ become $\dfrac{dy}{dx} + i x = 0$

$\dfrac{dy}{dx} + \lambda_2 = 0$ become $\dfrac{dy}{dx} - i x = 0$

Integrating these equations we get $y + \dfrac{1}{2} i x^2 = C_1$, and $y - \dfrac{1}{2} i x^2 = C_2$

To reduce the given equation to cononical form we transform $x, y$ to $u, \upsilon$ to $\alpha, \beta$ by taking

$u = y + \dfrac{1}{2} i x^2 = \alpha + i\beta$ and $\upsilon = y - \dfrac{1}{2} i x^2 = \alpha - i\beta$

Solving $\alpha, \beta$ we get $\alpha = y$ and $\beta = \dfrac{1}{2} x^2$.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}$$

$$\Rightarrow \frac{\partial}{\partial y} = \frac{\partial}{\partial \alpha}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left(\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right)$$

$$= \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$$

Substituting these values of r and t in the equation $r + x^2t = 0$, we obtain

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{(or)} \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \alpha}$$

This is the required carnonical form of the given equation.

## 10.6 Classification of second order partial differential equations :

Depending on the cononical form, the partial differential equation of type (7) is classified into 3 types.

(i) It is hyperbolic if $s^2 - 4RT > 0$. The one-dimensional wave equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ is hyperbolic with the cornonical form $\frac{\partial^2 z}{\partial u \partial v} = 0$.

(ii) It is parabolic if $S^2 - 4RT = 0$. The one-dimensional diffusion equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$

is parabolic, being already in canonical form.

(iii) It is elliptic if $S^2 - 4RT < 0$. The two-dimensional haramonic equation

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is elliptic and is also in the canonical form.

**SAQ's :**

I. Solve the following equations by inspection or by Lagranges method.

1. $s + t + q = 0$ (Ans : $z e^x - F(x) = \phi(x-y)$ )
2. $p + r x = 9x^2 y^3$ (Ans : $z = x^3 y^3 + f(y) \log x + \phi(y)$)
3. $s = 2(x + y)$ (Ans : $z = x^2 y + xy^2 + f(x) + \phi(y)$)

II. Find a surface satisfying $t = 6x^3 y$ and containing the two lines $y = z = 0$ and $y = 1 = z$

(Ans : $z = x^3 y^3 + y(1 - x^3)$)

III. Reduce the following equations to canonical form and hence solve them.

1. $x^2 (y-1) r - x(y^2-1) s + y(y-1) t + xy p - q = 0$

(Ans : $\frac{\partial^2 z}{\partial u \partial v} = 0$; $z = F(xy) + 1 + (e^y x)$ }

2. $x^2 r - 2xy s + y^2 t - xp + 3yq = 8y / x$ (Ans : $\upsilon \left(\frac{\partial^2 z}{\partial \upsilon^2}\right) + 2\left(\frac{\partial z}{\partial \upsilon}\right) = z$, $z = y / x + x^2 F(xy) + 1 + (xy))$

## 10.7 SUMMARY :

In this lesson we discussed the method of inspection for solving partial differential equations of second order. The constant of integration involved in the resulting solution consists of an arbitrary function of the variable which is considered constant during integration of the given equation. However they can be determined if some geometrical conditions are specified. Then we obtain surfaces which satisfy the given geometrical conditions.

By a suitable change of the independent variables, we have shown that the equation $Rr+Ss+Tt+f(x, y, z, p, q) = 0$ can be transformed into one of the three canonical forms, which are easily integrable. Finally the second order par'ial differential equations were classified by their canonical forms into three types - hyperbolic, parabolic and elliptic according to $S^2-4RT>0, = 0$ or $< 0$ respectively.

## 10.8 Model Examination questions :

1. Solve the following equations :

(i) $xys = 1$ (Ans : $z = \log x + \log y + f(x) + F(y)$)

(ii) $t = \sin xy$ (Ans : $z = -1/x^2 \sin xy + y f(x) + F(x)$)

(iii) $r = 2y^2$. (Ans : $z = x^2y^2 + x f(y) + F(y)$)

(iv) $\log s = x + y$ (Ans : $z = e^{x+y} + f(x) + F(y)$)

(v) $x^2 s = \sin y$ (Ans : $z = f(x) + F(y) + 1/x \cos y$)

(vi) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$ (Ans : $\dfrac{x^2}{y} \log y + axy + f(x) + F(y)$)

(vii) $t - xq = x \cos y + \sin y$ (Ans : $z = f(x) + F(x) e^{xy} - \sin y$)

2. Show that the surface of revolution satisfying the equation $r = 12x^2 + 4y^2$ and touching the plane $z = 0$ is $z = (x^2+y^2)^2$

3. Reduce the following equations to canonical form and solve them.

(i) $(y - 1) r - (y^2 - 1) s + y (y - 1) t + p - q = 2y e^{2x} (1 - y)^2$

(Ans : $\dfrac{\partial^2 z}{\partial u \partial \upsilon} = 2\upsilon$, $z = (x + y) y^2 e^{2x} + F(x+y) + H ye^{2x}$)

(ii) $x (xy - 1) r - (x^2y^2 - 1) s + y (xy - 1) t + (x - 1) p + (y - 1) q = 0$

(Ans : $\dfrac{\partial^2 z}{\partial u \partial \upsilon} = 0$, $z = F(ye^x) + 1 + H(x e^y)$ )

(iii) $xyr - (x^2-y^2) s - xy t + py - qx = 2 (x^2-y^2)$

(Ans : $\dfrac{\partial^2 z}{\partial u \partial \upsilon} = \dfrac{\upsilon^2 - 1}{(\upsilon^2 + 1)^2}$ ; $z = - xy + F(x^2+y^2) + H (y/x)$)

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# LESSON-11 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

## 11.1 INTRODUCTION

A partial differential equation which is linear with respect to the dependent variable and its derivatives and in which the coefficients are merely constants is called a linear partial differential equation with constant coefficients. In the most general form it is expressed as

$$\left(A\frac{\partial^n z}{\partial x^n}+A_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+\dots\dots+A_n\frac{\partial^n z}{\partial y^n}\right)+\left(B_0\frac{\partial^{n-1} z}{\partial x^{n-1}}+\dots\dots+B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)$$

$$+\dots\dots+\left(M_0\frac{\partial z}{\partial x}+M_1\frac{\partial z}{\partial x}\right)+N_z=f(x,y) \qquad \dots \qquad (1)$$

Here $A_0, A_1, \dots\dots, A_n, B_0, \dots\dots, B_{n-1}, \dots\dots, M_0$, M, N, are all constants

If we take $D=\frac{\partial}{\partial x}$, $D^1=\frac{\partial}{\partial y}$, $D^{D^1}=\frac{\partial^2}{\partial x\,\partial y}$ in equation (1) then it gets transformed to

$$(A_0D^n + A_1D^{n-1}D^1 + \dots + A_n D^{1n}) + B_0 D^{n-1} + B_1 D^{n-2}D_1 + \dots + B_{n-1} D^{1n-1}) + \dots$$

$$(M_0D+M_1D^1)+N)\, z = f(x,y) \qquad \dots \qquad (2)$$

or more briefly $F(D,D^1)\, z = f(x,y)$ .... (3)

If all the derivatives appearing in equation (1) are of the same order, then such an equation is called a linear homogeneous partial differential equation with constant coefficients. It is expressed as $(A_0D^n + A_1D^{n-1}D^1 + \dots + A_n D^{1n})\, z = f(x,y)$

If all the derivatives in equation (1) are not of the same order, then we call such equation, a non-homogeneous linear equation with constant coefficients.

We shall now discuss the methods of solving linear partial differential equation.

## 11.2 Solution of linear partial differential equations : The complementary function :

As is the case with ordinary linear differential equations, the complete solution of equation (3) is expressed as the sum of (i) the complementary function and (ii) the particular integral

Let $z = \phi(y + mx)$ be a solution of equation $F(D,D^1)\, z = 0$. Then

$$Dz = \frac{\partial z}{\partial x} = m\phi^1(y + mx)$$

$$D^2z = \frac{\partial^2 z}{\partial x^2} = m^2\phi^{11}(y + mx)$$

" " "

" " "

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n\phi^{(n)}(y + mx)$$

and $$D^1 z = \frac{\partial z}{\partial y} = \phi^1(y + mx)$$

" " "

" " "

$$D^{1n} z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y + mx)$$

Also $$D^r D^{1^s} z = \frac{\partial^{r+s} z}{\partial x^r \partial y^s} = m^r \phi^{r+s}(y + mx)$$

Substituting these results in the equation $F(D,D^1)\, z = 0$ we get,

$$F(D,D^1)z = (A_0 m^n + A_1 m^{n-1} + \dots + A_{n-1}\, m + A_n)\, \phi^n(y+mx) = 0$$

which is satisfied only if $A_0 m^n + A_1 m^{n-1} + \dots + A_{n-1}\, m + A_n = 0$

This equation is known as the auxiliary equation. Note that it can be obtained by writing m for D and 1 for $D^1$ in the equation $F(D,D^1) = 0$.

### 11.2.1 <u>When the auxiliary equation has unequal roots</u>

If the auxiliary equation has 'n' distinct roots $m_1, m_2, \dots, m_n$, then the complementary function of equation (3) is obtained as follows :

Let $F(D,D^1) = (D-m_1 D^1)(D-m_2 D^1) \dots (D-m_r D^1) \dots (D-m_n D^1)\, z = 0$

Then $D - m_r D^1 = \frac{\partial z}{\partial x} - m_r \frac{\partial z}{\partial y} = 0$, r = 1, 2, ....., n (or) $p - m_r q = 0$

The subsidiary equations corresponding to the Lagranges form are $\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}$

Considering the first two ratios we have $dy + m_r dx = 0$ (or) $y + m_r x = C_1$.

From the last relation $dz = 0 \Rightarrow z = C_2$. Hence $z = \phi_r(y + m_r x)$ is the solution of $(D - m_r D^1)z = 0$
Then the complementary function of equation (3) in this case is :

$$z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + ..... + \phi_n(y+m_nx)$$

**Working rule for finding the complementary function (C.F) :**

Consider the given equation $F(D,D^1)\, z = f(x,y)$ Factorize $F(D,D^1)$ into linear factors of the form $(bD - aD^1)$.

Corresponding to each distinct factor $(bD-aD^1)$ the part (or contribution) for C.F is taken as $\phi(by+ax)$

**11.2.2. When the auxiliary equation has (equal) repeated roots :**

Let us consider the equation having two equal roots i.e. $(D-mD^1)^2 z = 0$. Then putting $(D-mD^1)z = u$ in this equation, we have $(D-mD^1)\, u = 0$.

Therefore, $u = \phi_1(y + mx)$

or $(D-mD^1)\, z = \phi_1(y+mx) \Rightarrow p - mq = \phi_1(y+mx)$

Then the subsidiary equations corresponding to this Lagrangian form of equation is

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi_1(y+mx)}$$

From the first two ratios, we have,

$$dy + m\, dx = 0 \Rightarrow y + mx = C_1.$$

From the first and the third ratios we have, or $\quad z = x\,\phi_1(C_1) + C_2.$

Hence the solution becomes $z = x\,\phi_1(y+mx) + \phi_2(y+mx)$

In general, if a factor in the auxiliary equation repeats itself for 'r' times, then $(D-mD^1)^r z=0$ and in this case,

$$z = x\,\phi_1(y+mx) + \phi_2(y+mx) + \ldots\ldots + x^{r-1}\,\phi_r(y+mx)$$

**Working rule for finding the complementary function :**

(i) Factorize $F(D,D^1)$ into linear factors of the form $(bD-aD^1)$. Corresponding to each of the non-repeated factor $(bD-aD^1)$, the part of the C.F is taken as $\phi\,(by+ax)$

(ii) Corresponding to a repeated factor $(bD-aD^1)^m$, the part of the C.F is taken as $\phi_1(by+ax) + x\,\phi_2(by+ax) + \ldots\ldots + x^{m-1}\,\phi_m(by+ax)$

(iii) Corresponding to a non-repeated factor D, the part of C.F is taken as $\phi\,(y)$.

(iv) Corresponding to a repeated factor $D^m$, the part of C.F is taken as $\phi_1(y) + x\,\phi_2(y) + x^2.\phi_3(y) + \ldots. + x^{m-1}\,\phi_m(y)$

(v) Corresponding to a non-repeated factor $D^1$, the part of C.F is taken as $\phi\,(x)$.

(vi) Corresponding to a repeated factor $D^{1m}$, the part of C.F is taken as $\phi_1(x) + y\,\phi_2(x) + y^2.\phi_3(x) + \ldots. + y^{m-1}\,\phi_m(x)$

We shall now solve a few problems by applying these working rules.

## 11.3 Examples :

**11.3.1 Example :** Solve $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^2 z}{\partial x^2\,\partial y} + 2\dfrac{\partial^3 z}{\partial x\,\partial y^2} = 0$

**Solution :**

The auxiliary equation for the given differential equation is $(D^3 - 3D^2 D^1 + 2D\,D^{12})z=0$
From this equation we get $D\,(D-1)\,(D-2)\,z = 0$. Here all the factors are non-repeated. The part of the C.F corresponding to these factors are $\phi_1(y)$, $\phi_2(y+x)$, $\phi_3(y+2x)$. Hence the required solution is $z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x)$.

**11.3.2 Example :** Solve r + t + 2s = 0

**Solution :** We recall that if z is a function of the variables x and y, we employed the following notation. $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x\,\partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Following this notation, the given equation turns out as,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} = 0.$$

If we let $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$, this equation becomes $(D^2+D'^2+2DD')\,z = 0$ or $(D+D')^2 z = 0$

Then the auxiliary equation is $(D+1)^2 = 0$, so that D = -1, -1. Hence the solution (C.F) is

$$z = \phi_1(y-x) + x\,\phi_2(y-x).$$

**11.3.3. Example :** Solve 2r + 5s + 2t = 0

**Solution :** The given equation can be expressed as $(2D^2 + 5\,DD' + 2\,D'^2)z = 0$

Then the auxiliary equation is $2\,m^2 + 5m + 2 = 0$,

i.e., (m+2) (2m+1) = 0. Hence m = -2 or -1/2.

Hence the required solution is $z = \phi_1(y-2x) + \phi_2(y - x/2)$

**11.3.4. Example :** Solve 25r - 40 s + 16 t = 0

**Solution :** The given equation can be written as $(25D^2 - 40DD' + 16\,D'^2)\,z = 0$

The auxiliary equation is $25m^2 - 40m + 16 = 0$ i.e., $(5m-4)^2 = 0$. Hence m = 4/5, 4/5

Hence the required solution is, $z = \phi_1\left(y + \frac{4}{5}x\right) + x\,\phi_2\left(y + \frac{4}{5}x\right)$

or $$z = \phi_1(5y + 4x) + x\,\phi_2(5y+4x)$$

**11.3.5. Example :** Solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

**Solution :** The given equation can be expressed as $(D^4 - D^{14})\, z = 0$.

Its auxiliary equation is $m^4 - 1 = 0$, (or) $(m-1)(m+1)(m^2+1) = 0$

Hence $m = 1, -1 \pm i$. Hence the required equation is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix)$$

**11.3.6. Example :** Solve $\frac{\partial^4 z}{\partial x^3 \partial y} - 4 \frac{\partial^4 z}{\partial x^2 \partial y^2} + 4 \frac{\partial z}{\partial x \partial y^3} = 0$

**Solution :** The given equation can be expressed as $DD^1 (D^2 - 4\, DD^1 + 4\, D^{12}) = 0$

or $DD^1(D-2D^1)^2 = 0$

Hence the auxiliary equation is $mm^1\,(m-2)^2 = 0$.

Then the required solution is given by $z = \phi_1(y) + \phi_2(x) + \phi_3(y+2x) + x\phi_4(y+2x)$

## 11.4. The particular Integral

Before we take up the general method of finding the particular integral (P.I) of the equation $F(D,D^1)\, z = f(x,y)$ we shall discuss a few special cases when f(x,y) bears the following two special forms :

**Special case 1 :** **When f(x,y) is a function of ax + by**

We shall apply a simpler method for finding the particular integral.

Let $f(x,y) = \phi(ax + by)$

Then $D\phi(ax + by) = a\,\phi^1(ax + by)$

$D^2\phi(ax + by) = a^2\,\phi^{11}(ax + by)$ and so on.

Also. $D^1\phi(ax + by) = b\,\phi^1(ax + by)$

$$D^{1 2}\,\phi\,(ax+by) \quad = \quad b^2\,\phi^{11}(ax+by) \text{ and so on.}$$

Where $\phi^1$, $\phi^{11}$, ..... are derivatives of $\phi$ w.r.t. ax + by.

Hence $F(D,D^1)\ \phi\,(ax+by) = F(a,b)\ \phi^{(n)}\,(ax+by)$ if $F(D,D^1)$ is of degree n.

i.e., $$\frac{1}{F(D,D^1)}\phi^{(n)}(ax+by) = \frac{1}{F(a,b)}\phi\ (ax+by) \qquad \text{if } F\ (a,\ b) \neq 0$$

Put ax + by = t. Then $\dfrac{1}{F(D,D^1)}\phi^{(n)}t = \dfrac{1}{F(a,b)}\phi\ (t)$

Integrating n times w.r.t. t we get

$$\frac{1}{F(D,D^1)}\phi\ (t) = \frac{1}{F(a,b)}\int\int \ldots\ldots \int \phi\,(t)\,dt^n, \quad \text{where } t = ax+by$$

### 11.4.1. Working procedure :

To obtain the P.I of the equation $F(D,D^1)\,z = \phi\,(ax+by)$, where $F(D,D^1)$ is a homogeneous function of D, $D^1$ of degree n, we put ax + by = t, integrate $\phi$ (t) w.r.t. t, n times we shall then put 'a' for D and 'b' for $D^1$ in $F(D,D^1)$ to get F(a,b). Then P.I. = $\dfrac{1}{F(a,b)}$ x (nth integral of $\phi$ (t) w.r.t. t) Where t = (ax + by).

### Case 2 : Special case when F (a,b) = 0

When $F(a,b) \neq 0$, we have seen in case (1) that the particular integral can be obtained as :

$$\frac{1}{F(D,D^1)}\phi\ (t) = \frac{1}{F(a,b)}\int\int \ldots\ldots \int \phi\,(t)\,dt^n,$$ with the notation explained therein.

However if F(a,b) = 0, we observe that this method fails. In this case $(bD-aD^1)$ must be a factor of $F(D,D^1)$, so that we can write $F(D,D^1) = (bD-aD^1)\,f(D,D^1)$.

If $(bD - aD^1)z = \phi\ (ax+by)$, the subsidiary equations are : $\dfrac{dx}{b} = \dfrac{dy}{-a} = \dfrac{dz}{\phi_1(ax+by)}$

From the first two ratios we get ax + by = C, C is a constant.

From the first and the last ratios we have $\dfrac{1}{b}\phi\,(ax+by)dx = dz$

$$\text{or} \quad \frac{1}{b}\phi(C)dx = dz$$

Integrating we obtain $\frac{x}{b}\phi(C) = \frac{x}{b}\phi(ax+by) = z$

Hence $F(D,D^1)\, z = (bD - aD^1)\, f(D,D^1)\, z = \phi(ax+by)$

$$\Rightarrow \quad z = \frac{1}{bD - aD^1} \cdot \frac{1}{f(D,D^1)}\phi(ax+by)$$

$$= \frac{1}{f(D,D^1)} \frac{x}{b} \phi(ax+by)$$

$$= \frac{1}{f(D,D^1)} \frac{x}{b} \psi(ax+by),$$

where $\psi(t) = \int \ldots\ldots \int \phi(t)\, dt^n$,

Now differentiating $F(D,D^1) = (bD - aD^1)\, f(D,D^1)$ w.r.t D we get

$$F^1(D,D^1) = b\, f(D,D^1) + (bD - aD^1)\, f^1(D,D^1)$$

So that $F^1(a,b) = b\, f(a,b) + \phi$

$$\text{Hence } z = \frac{1}{F^1(D,D^1)} x\, \psi(ax+by)$$

**Working Procedure :**

(i) When $f(a,b) = 0$, evaluate $\frac{1}{F(D,D^1)}\phi(ax+by)$ by differentiating $F(D,D^1)$ w.r.t. D and multiply the expression by x so that $\frac{1}{F(D,D^1)}\phi(ax+by) = x\frac{1}{F^1(D,D^1)}\phi(ax+by)$

(ii) If $F^1(a,b) = 0$, differentiate $F^1(D,D^1)$ again w.r.t. D and multiply by x so that

$$\frac{1}{F(D,D^1)}\phi(ax+by) = x^2 \frac{1}{F^{11}(D,D^1)}\phi(ax+by) \quad \text{In general}$$

$$\frac{1}{F(D,D^1)}\phi(ax+by) = x^n \frac{1}{F^{(n)}(D,D^1)}\phi(ax+by)$$

**Case 3 :** When f(x,y) is of the form $x^m y^n$, m and n being non-negative integers.

Here we have two alternatives.

(i) If $n < m$, we expand $\dfrac{1}{F(D,D^1)}$ in ascending powers of $D^1/D$.

(ii) If $m < n$ we expand $\dfrac{1}{F(D,D^1)}$ in ascending powers of $D/D^1$.

It may be noted that one might arrive at two different solutions for a considered problem When $\dfrac{1}{F(D,D^1)}$ is expanded in these two possible ways. However, this variation in the solution can be absorbed in the arbitrary functions occurring in the complementary function of the solution.

## 11.5 **Examples :**

**11.5.1 Example :** Solve $(D^2 + 3\,DD^1 + 2\,D^{1^2})z = x + y$

**Solution :** The auxiliary equation of the given equation is $m^2 + 3m + 2 = 0$ i.e., $(m+1)(m+2)=0$ So that $m = -1, -2$.

Hence the complementary function = $\phi_1(y-x) + \phi_2(y-2x)$

Particular integral $= \dfrac{1}{D^2 + 3DD^1 + 2D^{1^2}}(x+y)$

$= \dfrac{1}{1^2 + 3.1.1 + 2.1^2}\iint \upsilon\, d\upsilon^2$ where $\upsilon = x + y$

$= \dfrac{1}{6}\,\dfrac{1}{6}\,\upsilon^3 \quad = \quad \dfrac{1}{36}(x+y)^3.$

Hence the required solution $= \phi_1(y-x) + \phi_2(y-2x) + \dfrac{1}{36}(x+y)^3.$

**11.5.2 Example :** Solve $(D^2 + 2\,DD^1 + D^{1^2})\,Z = e^{2x+3y}$.

**Solution :** The auxiliary equation for the given equation is $m^2 + 2m + 1 = 0$ so that $m = -1, -1$
Hence the complementary function $= \phi_1(y - x) + x\,\phi_2(y - 2x)$

The particular integral $= \dfrac{1}{D^2 + 2DD^1 + D^{1^2}}\, e^{2x+3y}$

$= \dfrac{1}{2^2 + 2.2.3. + 3^2} \int\int e^{\upsilon}\, d\upsilon^2$, where $\upsilon = 2x + 3y$.

$= \dfrac{1}{25}\, e^{2x+3y}$, since the integral of $e^{2x+3y}$ with respect of 2x+3y carried twice gives $e^{2x+3y}$.

Hence the complete solution $= \phi_1(y - x) + x\,\phi_2(y - x) + \dfrac{1}{25}\, e^{2x+3y}$

**11.5.3. Example :** Solve $r - 2s + t = \sin(2x+3y)$

**Solution :**

In terms of D and $D^1$, the given equation can be expressed as $(D\text{-}2\,DD^1 + D^{1^2})\,z = \sin(2x+3y)$
The auxiliary equation is $m^2 - 2m + 1 = 0$.

i.e., $(m-1)^2 = 0 \quad \Rightarrow \quad m = 1; 1$

Hence the complementary function $= \phi_1(y + x) + x\,\phi_2(y + x)$

The particular Integral $= \dfrac{1}{D^2 - 2DD^1 + D^{1^2}}\, Sin(2x + 3y)$

$= \dfrac{1}{2^2 - 2.2.3. + 3^2} \int\int \sin\upsilon\;\, d\upsilon^2$ where $\upsilon = (2x+3y)$

$= \dfrac{1}{1}[-\sin(2x + 3y)]$

Hence the complete integral $= \phi_1(y + x) + x\,\phi_2(y + x) - \sin(2x+3y)$.

**11.5.4. Example :** Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$

**Solution :** The given equation can be written in terms of D and $D^1$ as

$(D^2 + D^{1^2})z = \cos mx \cos ny.$

Its auxiliary eqution $= m^2 + 1 = 0$, so that $m = \pm i$.

The complementary function $= \phi_1(y + i x) + \phi_2(y - i x)$

Particular integral $= \dfrac{1}{D^2 + D^{1^2}} \cos mx \cos ny$

$$= \frac{1}{D^2 + D^{1^2}} \left[ \frac{1}{2} [\cos(mx+ny) + \cos(mx-ny)] \right]$$

$$= \frac{1}{2}\left\{\frac{1}{D^2 + D^{1^2}} \cos(mx+ny)\right\} + \frac{1}{2}\left\{\frac{1}{D^2 + D^{1^2}} \cos(mx-ny)\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{l} \dfrac{1}{m^2+n^2}(-\cos(mx+ny) - \\ \dfrac{1}{m^2+n^2}(-\cos(mx-ny) \end{array}\right\}$$

$$= \frac{-1}{m^2+n^2} \cos mx \cos ny$$

Hence the complete integral $= \phi_1(y + i x) + \phi_2(y - i x) - \dfrac{\cos mx \cos ny}{m^2 + n^2}$

**11.5.5. Example :** Solve $(D^3 - 4D^2D^1 + 4DD^{1^2})z = 4\sin(2x+y)$

**Solution :** The auxiliary equation for the given equation is $m^3 - 4m^2 + 4m = 0$ or $m(m-2)^2 = 0$

Hence m = 0, 2, 2 and the C.F. $= \phi_1(y) + \phi_2(2x+y) + x\phi_3(2x+y)$

Particular integral $= \dfrac{1}{D^3 - 4D^2D^1 + 4DD^{1^2}} . 4\sin(2x+y)$

If $D = 2$, $D^1 = 1$ is substituted in the denominator on the R.H.S, the denominator vanishes. Hence we mutliply the numerator by x and differentiate the denominator w.r.t. D. Then

$$\text{P.I} = x.\frac{1}{3D^2 - 8DD^1 + 4DD^{1^2}}.\ 4\sin(2x+y)$$

The denominator vanishes again if we substitute $D = 2$, and $D^1 = 1$ Hence by repeating the aforesaid procedure we obtain.

$$\text{P.I.} = x^2 . \frac{1}{6D - 8D^1}.\ 4\sin(2x+y)$$

$$= \frac{4x^2}{6.2 - 8.1}\int \sin(2x+y)d(2x+y)$$

$$= -x^2\cos(2x+y)$$

Hence the complete solution $z = \phi_1(y) + \phi_2(2x+y) + x\,\phi_3(y+2x) - x^2\cos(2x+y)$

**11.5.6. Example :** Solve $4r - 4s + t = 16 \log(x+2y)$

**Solution :** Re-writing the given equation interms of D, $D^1$ we get

$$(4D^2 - 4DD^1 + D^{1^2})\,z = 16\log(x+2y)$$

The auxiliary equation is $4D^2 - 4D + 1 = 0$ or $(2D-1)^2 = 0$.

Hence $D = 1/2, 1/2$ and C.F. $= \phi(2y+x) + x\phi_2(2y+x)$

$$\text{Particular integral} = \frac{1}{4D^2 - 4DD^1 + D^{1^2}}\ 16\log(x+2y)$$

$$= 16\frac{1}{\left(2D - D^1\right)^2}\log(x+2y)$$

$$= \frac{16x}{2\left(2D - D^1\right).2}\log(x+2y)$$

$$= 2x^2\log(x+2y)$$

Hence the required general solution : $z = \phi_1(2y+x) + x\phi_2(2y+x) + 2x^2\log(x+2y)$

**11.5.7. Example :** Solve $(D^2 - 2DD^1 + D^{1^2})\, z = 12\, xy$

**Solution :** The auxiliary equation for the given equation is $D^2 - 2D + 1 = 0$, so that $D = 1,1$.
Hence the C.F $= \phi_1(y+x) + x\phi_2(y+x)$.

The particular integral $= \dfrac{1}{D^2 - 2DD^1 + D^{1^2}}\, 12xy$

$$= 12.\ \frac{1}{D^2\left(1 - \frac{D^1}{D}\right)^2}\, xy$$

$$= \frac{12}{D^2}.\left(1 - \frac{D^1}{D}\right)^{-2} xy$$

$$= \frac{12}{D^2}.\left(1 + 2\frac{D^1}{D} + \ldots\ldots\right) xy$$

$$= \frac{12}{D^2}.\left(xy + \frac{2}{D}x\right)$$

$$= \frac{12}{D^2}.\left(xy + 2.\frac{x^2}{2}\right)$$

$$= 12.\left(y\frac{x^3}{6} + \frac{x^4}{12}\right)$$

$$= 2x^3 y + x^4.$$

Hence the general solution $z = \phi_1(y+x) + x\phi_2(y+x) + 2x^3y + x^4$.

**11.5.8. Example :** Solve $(2D^2 - 5\, DD^1 + 2\, D^{1^2}) = 24\,(y - x)$

**Solution :** The auxiliary equation for the given equation is $2m^2 - 5m + 2 = 0$

i.e., $(2m-1)(m-2) = 0$, so that $m = 1/2, 2$

The complementary function $= \phi_1(2y+x) + \phi_2(y+2x)$.

The particular integral $= \dfrac{1}{2D^2 - 5DD^1 + 2D^{1^2}} 24(y - x)$

$$= \frac{1}{2D^2\left(1 - \frac{5D^1}{2D} + \frac{D^{1^2}}{D^2}\right)} 24(y - x)$$

$$= \frac{24}{D^2}\left(1 - \frac{5D^1}{2D} + \frac{D^{1^2}}{D^2}\right)^{-1} (y - x)$$

$$= \frac{12}{D^2}(y - x) + \frac{30}{D^3}(y - x)$$

$$= 12.\left(\frac{x^2 y}{2} - \frac{x^3}{6}\right) + 30\left(\frac{x^3}{6}\right) = 6x^2y + 3x^3.$$

Hence the general solution is $z = \phi_1(2y+x) + \phi_2(y+x) + 6x^2y + 3x^3$.

## 11.6 General method for finding the particular integral :

Let $F(D,D^1)$ be a homogeneous function of D and $D^1$. Now consider the equation.

$$(D - m D^1)\, z = \phi(x,y)$$

The Lagranges subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{\phi(x,y)}$

From the first and second ratios we have $y + mx = C$.

From the first and third ratios we have $dz = \phi(x,y)\, dx$ i.e., $dz = \phi(x, C-mx)\, dx$

Integrating $z = \int \phi(x, C - mx)\, dx$

Hence $z = \dfrac{1}{(D - m D^1)}, \phi(x,y) = \int \phi(x, C - mx)\, dx$, and C is to be replaced by $(y+mx)$ after the integration is carried out, for, the P.I should not contain any arbitrary constant.

Suppose that in the given equation $F(D, D^1)\, z = \phi(x,y)$, $F(D,D^1)$ is of the form :

$$F(D,D^1) = (D-m_1D^1)(D-m_2D^1)......(D-m_kD^1), \text{ then}$$

Particular Integral (P.I) $= \dfrac{1}{(D-m_1D^1)}\dfrac{1}{(D-m_2D^1)}\cdots\cdots\dfrac{1}{(D-m_kD^1)}f(x,y)$

In this case, the P.I is evaluated by a repeated application of the above method.

## 11.7 Examples :

**11.7.1 Example :** Solve $(D^2 + 2DD^1 + D^{1^2})\,z = 2\cos y - x\sin y$

**Solution :** The auxiliary equation for the given equation is $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

Hence the complementary function $= \phi_1(y-x) + x\phi_2(y-x)$

The particular Integral $= \dfrac{1}{(D+D^1)(D+D^1)}(2\cos y - x\sin y)$

$= \dfrac{1}{D+D^1}\int[(2\cos(C+x) - x\sin(C+x)\,dx]$

since $C = y + mx = y - x$.

$= \dfrac{1}{D+D^1}[(2\sin(C+x) + x\cos(C+x) - \sin(C+x)]$

$= \dfrac{1}{D+D^1}[\sin(C+x) + x\cos(C+x)]$

$= \int\sin(C+x) + x\cos(C+x)\,dx$

$= -\cos(C+x) + x[\sin(C+x) - (-\cos(C+x)]$

$= x\sin(C+x)$

$= x\sin y$

The complete solution $z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y$.

**11.7.2. Example:** Solve $(D^2 - DD^1 - 2D^{1^2})\,z = (y-1)e^x$

**Solution :** The auxiliary equation is $m^2 - m - 2 = 0$ i.e., $(m-2)(m+1) = 0$, since $m = 2$ or $-1$

Hence the C.F. $= \phi_1(y+2x) + \phi_2(y-x)$

Particular Integral $= \dfrac{1}{D^2 - DD^1 - D^{1^2}}(y-1)e^x$

$$= \frac{1}{\left(D-2D^1\right)} \frac{1}{D+D^1} (y-1)\, e^x$$

$$= \frac{1}{\left(D-2D^1\right)} \int (x+a-1)\, e^x\, dx, \quad \text{since } y - x = a.$$

$$= \frac{1}{\left(D-2D^1\right)} [(x-1)e^x + (a-1)e^x]$$

$$= \frac{1}{\left(D-2D^1\right)} (x+a-2)\, e^x$$

$$= \frac{1.(y-2)e^x}{(D-2D^1)} \quad \text{since } y = x + a$$

$$= \int (b-2x-2)\, e^x\, dx,$$

$$= (b-2) \int e^x\, dx - 2 \int x e^x\, dx$$

$$= (b-2)e^x - 2(x-1)e^x = (b-2x)e^x$$

$$= y e^x$$

The general solution $z = \phi_1(y+2x) + \phi_2(y-x) + ye^x$

**SAQs :**

I. Find the complementary function for the following linear partial differential equations.

1. $r = a^2 t$ (Ans : $z = \phi_1(y+ax) + \phi_2(y-ax)$ )

2. $(D^3 - 4D^2D^1 + 4DD^{1^2})\, z = 0$ (Ans : $z = \phi_1(y+2x) + x\phi_2(y+2x) + \phi_3(y)$ )

3. $$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 2\frac{\partial^4 z}{\partial x^2 \partial y^2}$$

II. Solve the following equations completely.

1. $(D^2+3DD^1+2{D^1}^2)\, z = x + y$ (Ans : $z = \phi_1(y-x) + \phi_2(y-2x) + 1/36(x+y)^3$)
2. $r - t = x - y$ (Ans : $z = \phi_1(y+x) + \phi_2(y-x) + x/4\,(x-y)^2$)
3. $2r - s - 3t = 5\, e^{x-y}$ (Ans : $z = \phi_1(y-x) + \phi_2(3x+2y) + x\, e^{x-y}$)
4. $(D^2+3DD^1+2{D^1}^2) = 2x + 3y$ $= \phi_1(y-x) + \phi_2(y-2x) - 7/6\, x^2 + 3/2\, x^2 y$)

III. Solve the following equations :

1. $r + s - 6t = y \cos x$ (Ans : $z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y \cos x$)
2. $(D^2-4{D^1}^2)\, z = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$ (Ans : $y = \phi_1(y+2x) + \phi_2(y-2x) +$
   $x \log y + y \log x + 3x$ )

## 11.8 Summary :

A partial differential equation that is linear w.r.t. the dependent variable and its derivatives and in which the coefficients are merely constants is called a linear partial differential equation with constant coefficients . It is expressed in the form $F(D,D^1) = f(x,y)$. The auxiliary equation for this equation is obtained by writing 'm' for D and '1' for $D^1$ in $F(D, D^1) = 0$ and the complementary function is obtained by solving the auxiliary equation. A few rules of working for finding the complementary function are given similarly a few simple methods of obtaining the particular integrals are also discussed before discussing a general method of finding the same. The general solution of the considered equation is expressed as $z = C.F + P.I.$

## 11.9 Model Examination Questions :

I. Solve the following homogeneous equations :

1. $(4D^2+12DD^1+9{D^1}^2)\, z = 0$ (Ans : $z = \phi_1(2y-x) + x\, \phi_2(2y-3x)$ )
2. $(D^3-6D^2D^1+11{D^1}^2-6{D^1}^3)z = 0$ (Ans : $z = \phi_1(x+y) + \phi_2(2x+y) + \phi_3(3x+y)$ )
3. $\dfrac{\partial^4 z}{\partial x^4} + \dfrac{\partial^4 z}{\partial y^4} = 2\dfrac{\partial^2 z}{\partial x^2 \partial y^2}$

II. Solve the following equations :

1. $(D^2-2DD^1+D^{1^2})\, z = \tan(x+y)$ (Ans : $z = \phi_1(y+x)+x\phi_2(y+x)+x^2/2\tan(y+x)$ )

2. $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 12\,(x+y)$ (Ans : $z = \phi_1(y+ix)+\phi_2(y-ix)+(x+y)^3$ )

3. $r - s + t = \sin(2x+3y)$ (Ans : $z = \phi_1(x+y)+x\phi_2(x+y) - \sin(2x+3y)$ )

4. $(D^2-a^2 D^{1^2})\, z = x$ (Ans : $z = \phi_1(y+ax)+\phi_2(y-ax)+x^2/6$ )

5. $r + (a+b)\, s + ab\, t = xy$ (Ans : $z = \phi_1(y-ax)+\phi_2(y-bx)+ 1/6\, x^3y-1/24(a+b)x^4$ )

6. $(D^2+DD^1-6D^{1^2})\, z = y \sin x$ (Ans : $z = \phi_1(y-3x)+\phi_2(y+2x)-(y\sin x+\cos x)$)

7. $(D^3-4D^2D^1+4DD^{1^2})\, z = \cos(y+2x)$ (Ans : $z = \phi_1(y)+\phi_2(y+2x)+\phi_3(y+2x).x + x^2/4 \sin(2x+y)$ )

8. $(D^2+DD^1-6D^{1^2})\, z = x^2 \sin(x+y)$ (Ans : $z = \phi_1(y+2x)+\phi_2(y-3x) +1/4(x^3-13/8)\sin(x+y)$)

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# LESSON -12 NON-HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

## 12.1 INTRODUCTION:

We have discussed several cases for finding the complementary function as well as particular integral to obtain the general solution of a homogeneous linear equations, in the preceding lesson. But there exist linear partial differential equations that are not homogeneous that is $F(D, D^1)$ is not necessary homogeneous when we consider equations of the form $F(D,D^1)\, z = f(x,y)$. In such cases $F(D,D^1)$ may or may not be reducible into linear factors. We shall deal with these cases now.

## 12.2 Complementary function corresponding to linear factors.

Let $F(D,D^1)\, z = f(x,y)$ .... **(1)** be the linear partial differential equation, where $F(D,D^1)$ is not necessarily homogeneous. Suppose that $F(D,D^1)$ can be expressed as product of linear factors.

i.e., $F(D,D^1) = (D - m_1 D^1 - \alpha_1)(D - m_2 D^1 \alpha_2) \ldots (D - m_n D^1 - \alpha_n)$

We shall now obtain the C.F. corresponding to the first factor.

Then $(D - m_1 D^1 - \alpha_1)\, z = 0$ i.e., $p - m_1 q = \alpha_1 z$ ..... **(2)**

This is of the form of Lagranges linear equation, so that the subsidiary equations are :

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\alpha_1 z}$$

Considering the first and the second ratios we have on integration, $y + m_1 x = C_1$.

Considering the first and the last ratios we have $\dfrac{dz}{z} = \alpha_1\, dx$

$$\Rightarrow \log z = \alpha_1 x + \log C_2 \quad \text{or} \quad z = C_2 e^{\alpha_1 x}$$

Hence the complementary function of equation (2) is $z = e^{\alpha x}\, \phi\,(y + m_1 x)$

By generalising this situation, it can be shown that the solution of equation (1) will be

$$z = e^{\alpha_1 x}\, \phi_1 (y + m_1 x) + e^{\alpha_2 x}\, \phi_2 (y + m_2 x) + \ldots + e^{\alpha_n x}\, \phi_n (y + m_n x)$$

In case there are any repeated factors of the type $(D-m\,D_1-\alpha)^r z=0$, then the corresponding solution can be obtained as

$$z=e^{\alpha x}\phi_1(y+mx)+x\,e^{\alpha x}\phi_2(y+mx)+\ldots\ldots+x^r e^{\alpha x}\phi_n(y+mx)$$

**Note :** If $F(D,D^1)$ is expressed as a product of non-repeated factors of the form $(bD-aD^1-C)$, the part of the C.F. is taken as $e^{cx/b}\phi(by+ax)$ for each non-repeated factor. Corresponding tc a repeated factor $(bD-aD^1-C)^m$, the part of the C.F is taken as

$$e^{cx/b}\left[\phi_1(by+ax)+x\,\phi_2(by+ax)+\ldots\ldots+x^{m-1}\phi_m(by+ax)\right]$$

We observe here that if the format of the linear factor change, the corresponding contribution of that factor for the complementary function also varies accordingly. Thus,

(i) Corresponding to a linear factor $(aD^1+C)$, the part of the CF is taken as $e^{-cy/a}\phi(ax)$ In case $(aD^1+C)$ repeats for m times, the C.F. is seen to be $e^{-cy/a}\left[\phi_1(ax)+x\,\phi_2(ax)+\ldots\ldots+x^{m-1}\phi_m(ax)\right]$

(ii) Corresponding to a linear factor $(bD+C)$ the C.F is $e^{cx/b}\phi(by)$

(iii) Corresponding to a non-repeated factor $(bD-aD^1)$ the part of the CF is taken as $\phi(by+ax)$. Corresponding to a repeated factor $(bD-aD^1)^m$, the part of the CF is taken as $\phi_1(by+ax)+x\,\phi_2(by+ax)+\ldots\ldots+x^{m-1}\phi_m(by+ax)$

(iv) Corresponding to a factor $D^1$, the C.F. is taken as $\phi(x)$ corresponding to a factor $D^{1m}$, the C.F. is taken as $\phi_1(x)+y\,\phi_2(x)+y^2\phi_3(x)+\ldots.+y^{m-1}\phi_m(x)$

## 12.3 Particular Integral of non-homogenous Equation $F(D,D^1)\,z=f(x,y)$

In the case of non-homogenous linear equations, the particular integral can be obtained in the same way, as in the case of linear equations with constant coefficients. We shall now enlist a few cases now, for the sake of ready reference.

(1) When $f(x,y) = e^{ax+by}$.

In this case $\dfrac{1}{F(D,D^1)}e^{ax+by}=\dfrac{1}{F(a,b)}e^{ax+by}$, provided that $f(a,b)\neq 0$

Thus we substitute 'a' for D and 'b' for $D^1$

However, if f(a,b) = 0, then $\dfrac{1}{F(D,D^1)} e^{ax+by} = e^{ax+by} \dfrac{1}{F(D+a,D^1+b)}$

(2) When f(x,y) is of the form Sin(ax+by) or Cos (ax+by)

In this case the particular integral is taken as,

$$\frac{1}{F(D^2,DD^1,D^{1^2})}\sin(ax+by) = \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by),$$

Provided that F $(-a^2, -ab, -b^2) \neq 0$

Similarly $\dfrac{1}{F(D^2,DD^1,D^{1^2})}\cos(ax+by) = \dfrac{1}{F(-a^2,-ab,-b^2)}\cos(ax+by)$,

Provided F $(-a^2, -ab, -b^2) \neq 0$

If F $(-a^2, -ab, -b^2) = 0$, we may express $\dfrac{1}{F(D^2,DD^1,D^{1^2})}\sin(ax+by) =$

Imaginary part of $\dfrac{1}{F(D^2,DD^1,D^{1^2})} e^{i(ax+by)}$

Similar expression can be made for f(x,y) = cos (ax+by) also.

(3) When f (x,y) = $x^m y^n$, where m and n are positive integers, then

$$\frac{1}{F(D,D^1)} x^m y^n = [F(D, D^1)]^{-1} x^m y^n.$$

The R.H.S is evaluated by expanding $[F(D, D^1)]^{-1}$ in ascending power of $D^1/D$ or $D/D^1$ or D and/or $D^1$ as the case may be according to the different situations discussed in lesson - 11.

(4) When f (x,y) = $e^{ax+by}$. g(x,y), then

$$\frac{1}{F(D,D^1)} e^{ax+by}.g(x,y) = \frac{1}{F(D+a,D^1+b)} g(x,y)$$

and is simplified by applying an appropriate rule discussed in 1-3.

We shall now apply these results to some problems and solve them.

## 12.4 Examples :

**12.4.1. Example :** Solve $\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=xy$

**Solution :** Transforming the given equation into the standard form we get $(DD^1+D-D^1-1)z=xy$
i.e., $(D^1+1)(D-1)=xy$

Hence the complementary function $z=e^x\phi_1(y)+e^{-x}\phi_2(x)$

Particular Integral $z = \frac{1}{(D-1)(D^1+1)}xy$

$= -(1-D)^{-1}(1+D^1)^{-1}xy$

$= -[(1-D+D^2+\ldots\ldots)(1-D^1+\ldots\ldots)]\,xy$

$= -xy-y+x+1$, (by case 3)

Hence the complete solution $z=e^x\phi_1(y)+e^{-x}\phi_2(x)-xy-y+x+1$.

**12.4.2. Example :** Solve $(D^2+DD^1+D^1-1)z=\text{Sin}(x+2y)$.

**Solution :** The given equation can be expressed as $(D+1)(D+D^1-1)z=\sin(x+2y)$.

Hence the complementary function $z = e\text{-}x\,\phi\,1(y) + ex\,\phi\,2(y\text{-}x)$.

Particular integral $= \frac{1}{(D^2+DD^1+D^1-1)}\sin(x+2y)$

$= \frac{1}{-1^2-1(2)+D^1-1}\sin(x+2y)$

$= \frac{1}{(D^1-4)}\sin(x+2y)$

$= \frac{D^1+4}{D^{12}-16}\sin(x+2y)$

$= (D^1+4).\frac{1}{(-2)^2-16}\sin(x+2y)$

$$= \quad \cdot \frac{-1}{20} \, (D^1+4) \sin (x+2y)$$

$$= \quad -\frac{1}{20} \, [\, 2 \cos (x+2y) + 4 \sin (x+2y) \,]$$

Hence the required general solution : $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y-x) - \frac{1}{10}\cos(x+2y) - \frac{1}{5}\sin(x+2y)$.

**12.4.3. Example :** Solve $(D^2 - D^{1^2} - 3D + 3D^1)z = xy + e^{x+2y}$

**Solution :** The given equation can be expressed as $(D-D^1)(D+D^1-3)z = xy + e^{x+2y}$.

Hence the complementary function = $\phi_1(y+x) + e^{3x}\phi_2(y-x)$.

The particular integral corresponding to xy =

$$P.I_1 = \frac{1}{(D-D^1)(D+D^1-3)} xy$$

$$= \frac{-1}{3D}\left(1-\frac{D^1}{D}\right)^{-1}\left(1-\frac{D+D^1}{3}\right)^{-1} xy$$

$$= \frac{-1}{3D}\left(1+\frac{D^1}{D}+\ldots\ldots\right)\left(1+\frac{D+D^1}{3}+\left(\frac{D+D^1}{3}\right)^2+\ldots.\right) xy$$

$$= \frac{-1}{3D}\left(1+\frac{D^1}{D}+\ldots\ldots\right)\left(1+\frac{D+D^1}{3}+\frac{2DD^1}{9}+\ldots.\right) xy$$

$$= \frac{-1}{3D}\left(1+\frac{D}{3}+\frac{D^1}{3}+\frac{D^1}{D}+\frac{D^1}{3}+\frac{2DD^1}{9}+\ldots\ldots\right) xy$$

$$= \frac{-1}{3D}\left(xy+\frac{y}{3}+\frac{2x}{3}+\frac{1}{D}x+\frac{2}{9}\right)$$

$$= \frac{-1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right)$$

Particular integral corresponding to $e^{x+2y}$.

$$P.I_2 = \frac{1}{(D+D^1-3)(D-D^1)}e^{x+2y} = \frac{1}{(D+D^1-3)(1-2)}e^{x+2y}$$

$$= e^{-x+2y}\frac{1}{D+1+D^1+2-3}.1 = e^{-x+2y}\frac{1}{D^1(1+\frac{D}{D^1})}.1$$

$$= e^{-x+2y}\frac{1}{D^1}.(1+\frac{D}{D^1})^{-1}$$

$$= -y\ e^{-x+2y}$$

Hence the complete solution z = $\phi_1(y+x) + e^{3x}\phi_2(y-x) - y\,e^{x+2y} - 1/54\,(6x^2y+6xy+6x^2+3x^3+4x)$

**12.4.4. Example :** Solve $(D^2-DD^1+D^1-1)\,z = Cos(x+2y) + e^y$

**Solution :** The given equation can be expressed as $(D-1)(D-D^1+1) = \cos(x+2y) + e^y$

The complementary function = $e^x\phi_1(y) + e^{-x}\phi(y+x)$

Particular integral corresponding to cos (x+2y) is

$$P.I_1 = \frac{1}{D^2-DD^1+D^1-1}\cos(x+2y)$$

$$= \frac{1}{-1^2-\{-1(1X2)+D^1-1\}}\cos(x+2y) = \frac{1}{D^1}\cos(x+2y)$$

$$= \frac{1}{2}\sin(x+2y)$$

Particular integral corresponding to $e^y$ =

$$P.I_2 = \frac{1}{(D-D^1+1)(D-1)}e^{0.x+1y} = \frac{1}{(D-D^1+1)}\frac{1}{-1}e^{0.x+1y}$$

$$= -e^y\frac{1}{(D+0)-(D^1+1)+1} = -e^y\frac{1}{D}\left(1-\frac{D^1}{D}\right)^{-1}.1$$

$$= -e^y.x$$

Hence the complete solution z = $e^x\phi_1(y) + e^{-x}\phi(y+x)$ 1/2 sin (x + y) - x $e^y$

**SAQs:**

1. Solve $2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} - 3\dfrac{\partial z}{\partial y} = 5\cos(3x-2y)$ (Ans : $z = \phi_1(x) + e^{3x/2}\phi_2(2y-x)$ $+ 1/10\,[4\cos(3x-2y) + 3\sin(3x-2y)]$ )

2. Solve $(D^2 - D^{1^2} + D + 3D^1 - 2)z = e^{x-y} - x^2 y$
(Ans : $z = e^{-2x}\phi_1(x+y) + e^x\phi_2(y-x) - 1/4\, e^{x-y} + 1/2\,(x^2y + xy + 3x^2/2 + 3y/2 + 3x + 21/4)$}

## 12.5 Special case : When F(D,D$^1$) cannot be resolved into linear factors.

Suppose $F(D,D^1)$ is such that it cannot be resolved or factorised into linear factors in D and $D^1$ i.e., $F(D,D^1)$ is irreducible. Then the techniques for finding the complementary function discussed in Sec.12.2 do not help us. In this case, we have to seek a trial solution, usually taken as $e^{hx+ky}$. We then substitute this solution in the given equation to obtain a relation between h and k. A general solution - which serves as a complementary function to the given problem - is then written using this relationship with some arbitrary constants. This procedure is to be repeated corresponding to each non-linear factor of $F(D,D^1)$ and the solution is then compiled. Following examples illustrate this method.

## 12.6 Examples :

**12.6.1 Example :** Solve$(D^2-D^1)z = \cos(3x-y)$

Since $F(D,D^1) = (D^2-D^1)$ is irreucible, we assume that $z = A\,e^{hx+ky}$ is the complementary function

Then $D^2z = Ah^2 e^{hx+ky}$ and $D^1z = Ak\, e^{hx+ky}$ so that $(D^2-D^1)z = A(h^2-k)\, e^{hx+ky} = 0$

This is held only if $h^2 - k = 0$ and $k = h^2$

Hence a more general solution, which is the C.F. is taken as $z = \sum A\, e^{hx + h^2 y}$

The particular integral $\quad z \quad = \quad \dfrac{1}{D^2 - D^1}.Cos(3x-y)$

$$= \frac{1}{-9-D^1} .Cos(3x-y)$$

$$= -\frac{(D^1-9)}{D^{1^2}-81} \cos(3x-y)$$

$$= -\frac{(D^1-9)}{-1-81} \cos(3x-y)$$

$$= \frac{1}{82}[\sin(3x-y) - 9\cos(3x-y)]$$

Hence the complete solution $z = \sum A\, e^{hx+h^2y} + \frac{1}{82}[\sin(3x-y) - 9\cos(3x-y)]$

**Example 12.6.2.** Solve $\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial z^2} = y + e^{x+z}$

**Solution :** Since x and z are the two independent variables we put $D = \frac{\partial}{\partial x}$ and $D^1 = \frac{\partial}{\partial z}$ to transform the given equation into $(D^2-D^{12}-1)y = e^{x+z}$.

Let the trial solution be $y = A\, e^{hx+kz}$

Then substituting this solution in the given equation $A(h^2-k^2-1)\, e^{hx+kz} = 0$, which holds if $h^2-k^2-1=0$ or $h = \text{Sec}\,\theta$ and $k = \tan\theta$.

Hence the complementary function is taken as

$$y = \sum A\, e^{x\,\text{Sec}\theta + z\tan\theta} \text{ ; A and } \theta \text{ are arbitrary constants}$$

$$\text{P.I.} = \frac{1}{D^2 - D^{1^2} - 1} e^{x+z} = \frac{1}{(1-1-1)} e^{x+z} = -e^{x+z}$$

Therefore, the general solution $y = \sum A\, e^{x\,\text{Sec}\theta + z\tan\theta} - e^{x+z}$

**12.6.3. Example :** Solve $(2D^4 - 3D^2D^1 + D^{12})z = 0$

**Solution :** The given equation can be rewritten as $(2D^2-D^1)(D^2-D^1)z = 0$

The $C.F_1$ corresponding to the factor $(D^2-D^1)$ is $\sum A e^{hx+h^2y}$ (follows from example 12.6.1)

To find the $C.F_2$ corresponding to $(2D^2-D^1)$ consider $(2D^2-D^1)z = 0$.

Let the trial solution be $z = B e^{ux+\upsilon y}$

Then $(2D^2-D^1)z = B(2u^2 - \upsilon)e^{ux+\upsilon y} = 0$ which is held only if $2u^2-\upsilon = 0$, so that $\upsilon = 2u^2$.

Hence the $C.F_2$ can be taken as $z = B e^{ux+2u^2y}$ where B and u are arbitrary constants.

Thus the general solution of the given equation is given by $z = CF_1 + CF_2$

$$= \sum A e^{hx+h^2y} + \sum B e^{ux+2u^2y}$$

## 12.7 Equations reducible to linear form with constant coefficients :

A partial differential equation of the form $F(xD, yD^1) = f(x,y)$ having variable coefficients can sometimes be reduced to a linear equation with constant coefficient by suitable subsitution.

For instance let $x = e^u$ and $y = e^{\upsilon}$ so that $u = \log x$, and $\upsilon = \log y$.

Denoting $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial \upsilon}$ respectively by D and $D^1$, it can be shown that

$$\frac{\partial}{\partial y} = D^1,\quad y^2\frac{\partial}{\partial y^2} = D^1(D^1-1), \text{ and so on.}$$

In general
$$x^m y^n \frac{\partial^{m+n}}{\partial x^m \partial y^n} = x^m \frac{\partial^m}{\partial x^m} \cdot y^n \frac{\partial^n}{\partial y^n}$$
$$= D(D-1)\ldots..(D-m+1)\,D^1(D^1-1)\ldots.(D^1-n+1)$$

These subsitution when affected to the given equation, we see that if reduces to an equation having constant coefficients, which can be solved by the methods discussed so far. The following examples illustrate the procedure.

**12.8 Examples :**

**12.8.1 Example :** Solve $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = xy$

**Solution :** Let $x = e^u$ and $y = e^{\upsilon}$ so that $u = \log x$ and $\upsilon = \log y$ .

$$x \frac{\partial}{\partial x} = D, \qquad x^2 \frac{\partial^2}{\partial x^2} = D(D-1)$$

$$y \frac{\partial}{\partial y} = D^1, \qquad y^2 \frac{\partial^2}{\partial y^2} = D^1(D^1-1), \qquad D = \frac{\partial}{\partial u}, \quad D^1 = \frac{\partial}{\partial \upsilon}$$

with these subsitutions, the given equation is transformed to $[D(D-1) - D^1(D^1-1)]z = e^{u+\upsilon}$.

i.e., $(D-D^1)(D+D^1-1)z = e^{u+\upsilon}$, which is a linear equation with constant co-efficients

The complementary function is given by :

$$\begin{aligned} z &= \phi_1(u+\upsilon) + e^u \phi_2(u-\upsilon) \\ &= \phi_1(\log x + \log y) + x\phi_2(\log x - \log y) \\ &= f_1(x.y) + x f_2(y/x). \end{aligned}$$

$$\text{Particular Integral} = \frac{1}{(D-D^1)(D+D^1-1)} e^{u+\upsilon}$$

$$= \frac{1}{D-D^1} \cdot \frac{1}{1+1-1} e^{u+\upsilon} = \frac{1}{D-D^1} e^{u+\upsilon}$$

$$= u\, e^{u+\upsilon} = xy \log x .$$

The complete solution $z = f_1(xy) + x f_2(y/x) + xy \log x$.

**12.8.2. Example :** Solve $x^2 r - y^2 t + xp - yq = \log x$

**Solution :** The given equation can be expressed as

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \log x$$

Let $x = e^u$, and $y = e^{\upsilon}$ so that $u = \log x$, $\upsilon = \log y$.

Therefore, $x\dfrac{\partial}{\partial x} = D$, $\quad y\dfrac{\partial}{\partial y} = D^1$, $\quad x^2\dfrac{\partial^2}{\partial x^2} = D(D-1)$ and $y^2\dfrac{\partial^2 z}{\partial y^2} = D^1(D^1-1)$

Substituting these expressions in the given equation we get $[D(D-1)-D^1(D^1-1)+D-D^1]z = u$,
i.e., $(D^2-D^1)^2 z = u$ (or) $(D-D^1)(D+D^1)z = u$

Complementary function $= \phi_1(u+\upsilon) + \phi_2(\upsilon - u)$

$= \phi_1(\log x + \log y) + \phi_2(\log y - \log x)$

$= \phi_1(\log x.y) + \phi_2(\log y/x)$.

Particular integral $= \dfrac{1}{\left(D^2 - D^1\right)^2}u = \dfrac{1}{D^2}\left(1 - \dfrac{D^{1^2}}{D^2}\right)^{-1}u$

$= \dfrac{1}{D^2}u = \dfrac{u^2}{6} = \dfrac{1}{6}(\log x)^2$

Complete solution $z = \phi_1(\log xy) + \phi_2(\log(y/x)) + 1/6(\log x)^2$.

**12.8.3. Example :** Solve $x^2\dfrac{\partial^2}{\partial x^2} - 4xy\dfrac{\partial^2 z}{\partial x \partial y} + 4y^2\dfrac{\partial^2 z}{\partial y^2} + 6y\dfrac{\partial z}{\partial y} = x^3y^4$.

**Solution :** Putting $x = e^u$, $y = e^{\upsilon}$, $D = \dfrac{\partial}{\partial u}$, $D^1 = \dfrac{\partial}{\partial \upsilon}$

So that $x\dfrac{\partial}{\partial x} = D$, $\quad y\dfrac{\partial}{\partial y} = D^1$, $\quad x^2\dfrac{\partial^2}{\partial x^2} = D(D-1)$, $\quad xy\dfrac{\partial^2}{\partial x \partial y} = DD^1$, $\quad y^2\dfrac{\partial^2}{\partial y^2} = D^1(D^1-1)$

in the given equation, we obtain $[D(D-1) - 4DD^1 + 4D^1(D^1-1) + 6D^1]z = e^{3u}.e^{4\upsilon}$,

i.e., $(D-2D^1)(D-2D^1-1)z = e^{3u+4\upsilon}$,

Complementary function $= \phi_1(\upsilon+2u) + e^u\phi_2(\upsilon+2u)$

$= \phi_1(\log y + 2\log x) + x\phi_2(\log y + 2\log x)$

$= \phi_1(\log x^2y) + x\phi_2(\log x^2y)$

$= f_1(x^2y) + x f_2(x^2y)$

Particular Integral $= \dfrac{1}{(D-2D^1)(D-2D^1+1)} e^{3u+4\upsilon}$

$= \dfrac{1}{(3-8)(3-8-1)} e^{3u+4\upsilon}$

$= \dfrac{1}{30} e^{3u+4\upsilon} = \dfrac{1}{30} x^3 y^4$

Hence the general solution $z = f_1(x^2y) + x\, f_2\,(x^2y) + \dfrac{1}{30}\, x^3 y^4$

**SAQs :**

1. Solve $s + p - q = z + xy$ (Ans : $z = e^x \phi_1(y) + e^{-y} \phi_2(x) + 1 + x - y - xy$)

2. Solve $(D^2 - D^1)\, z = 2y - x$. (Ans : $z = \Sigma\; A\; e^{hx+h^2x+x^2y}$)

3. Solve $r - s + p = 1$ (Ans : $z = \phi_1(y) + e^{-x} \phi_2(y + x) + x$)

4. Solve $yt - q = xy$. (Ans : $z = f_1(x) + y^2 f_2(x) + 1/2\, xy^2 \log y$)

5. Solve $(x^2D^2 - y^2 D^{12})\, z = x^2y$. (Ans : $z = x\phi_1(y/x) + \phi_2(xy) + 1/2\, x^2y$)

## 12.9 Summary :

In this lesson, we discussed some methods of finding the complementary function of a non-homogeneous linear partial differential equation $F(D,D^1)z = f(x,y)$ where $F(D,D^1)$ is not homogeneous function of D and $D^1$. Since $F(D,D^1)$ is not always resolvable into linear factors. If $F(D,D^1)$ is expressed as product of linear factors of the form $(bD-aD^1-C)$, the part of complementary function is taken in a specified form depending on the form (structure) of the non-repeated or repeated factor. When $F(D,D^1)$ cannot be resolved into linear factors, we apply a trial method with a trial solution, corresponding to each non-linear factor of $F(D,D^1)$, we get a general solution of the form $z = \Sigma\; A\; e^{hx+kh^2t}$ where A and h are arbitrary constants and this is the complementary function. The methods of finding particular integrals of non-homogeneous equations are similar to those of the linear equations with constant coefficients and they are dealt with here under four broad cases.

## 12.10 Model Examination questions :

I. Solve the following non-homogeneous equations :

1. $(D-D^1-1)(D-D^1-2)z = \sin(2x+3y)$ (Ans : $z = e^x \phi_1(x+y) + e^{2x}\phi_2(x+y) + 1/20 \sin(2x+3y)$)

2. $(D+D^1-1)(D+2D^1-3)z = 4+3x+6y$ (Ans : $z = e^x\phi_1(x+y) + e^{3x}\phi_2(y-2x)+x+2y+6$)

3. $(D^2-D^1-1)z = x^2 y$ (Ans : $z = \sum A e^{hx+(h^2-1)y} + x^2 - 2y - x^2y + 4$)

4. $(D^2-D^{12})z = \cos(x-3y)$ (Ans : $z = \sum A e^{k^2x+ky}$ $+1/82 [\sin(x-3y) + 9\cos(x-3y)]$

II. Solve the following equations by reducing them into linear equations :

1. $x^2\left(\dfrac{\partial^2 z}{\partial x^2}\right) - y^2\left(\dfrac{\partial^2 z}{\partial y^2}\right) = xy$ (Ans : $z = \phi_1(y/x) + x^n\phi_2(y/x^2) + (x^2+y^2/2-x)$)

2. $(x^2 D^2 + 2xy DD^1 + y^2 D^{12})z = 0$ (Ans : $z = \phi(y/x) + x\,\phi_2(y/x)$)

3. $x^3 r - 3xy\, s + 2y^2 t + px + 2\, qy = x + 2y$ (Ans : $z = \phi_1(xy) + \phi_2(x^2y) + x + y$)

4. $(x^2D^2 - y^2 D^{12} - y\, D^1 + xD)z = 0$ (Ans : $z = \phi_1(xy) + \phi_2(y/x)$)

5. $$\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3}\frac{\partial z}{\partial x} = \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3}\frac{\partial z}{\partial y}$$

(Ans : $z = \phi_1(x^2+y^2) + \phi_2(y^2-x^2)$)

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# LESSON-13 NON-LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

## 13.1 INTRODUCTION :

We have already learnt that a partial differential equation is said to be of second order if it contains atleast one partial derivative of second order and none of order higher than two. The most general form of such equation is given by F (x, y, z, p, q, r, s, t) = 0. We have learnt a few methods of solving linear equations with constant coefficients in the earlier lesson. In this lesson we shall derive a second order non-linear partial differential equation Rr + Ss + Tt = V - called the uniform non-linear equation and also the equation $Rr+Ss+Tt+u(rt-s^2) = V$, called the non-uniform non linear equation. We shall learn how to obtain the integrals of such equations and also know the monge's method of solving such non-linear equations of order two.

## 13.2 Derivation of a second order non-linear partial differential equation ;

A second order partial differential equation can be derived by the elimination of an arbitrary function from a partial differential equation of first order. Let us consider two known functions :

$$u = u(x, y, z, p, q) \quad \ldots \quad (1)$$

$$\upsilon = \upsilon(x, y, z, p, q) \quad \ldots \quad (2)$$

which are connected by the relation $u = \phi(\upsilon)$ ..... (3) we now differentiate equation (3) partially w.r.t. x and y, so that

$$\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} + r\frac{\partial u}{\partial p} + s\frac{\partial u}{\partial q} = \phi^1(\upsilon)\left[\frac{\partial \upsilon}{\partial x} + p\frac{\partial \upsilon}{\partial z} + r\frac{\partial \upsilon}{\partial p} + s\frac{\partial \upsilon}{\partial q}\right] \quad \ldots. \quad (4)$$

and
$$\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} + s\frac{\partial u}{\partial p} + t\frac{\partial u}{\partial q} = \phi^1(\upsilon)\left[\frac{\partial \upsilon}{\partial y} + q\frac{\partial \upsilon}{\partial z} + s\frac{\partial \upsilon}{\partial p} + t\frac{\partial \upsilon}{\partial q}\right] \quad \ldots. \quad (5)$$

Eliminating $\phi^1(\upsilon)$ between equations (4) and (5), we find that terms in rs and st cancel out leaving an equation of the form. $Rr + Ss + Tt + U(rt - s^2) = V$ .... (6)

Where R, S, T, U and V involve p,q and the partial derivatives of u and $\upsilon$ w.r.t. x, y, z, p, q

In particular if $\upsilon$ or u happens to be function of x, y, z only and not of p or q, then

$$u = \left(\frac{\partial u}{\partial p}\frac{\partial \upsilon}{\partial q} - \frac{\partial \upsilon}{\partial p}\frac{\partial u}{\partial q}\right) \quad \text{will vanish} \quad \ldots. \quad (7)$$

In such case, the desired equation reduces to $Rr + Ss + Tt = V$ .... (8)

This is a non-linear equation, since the coefficients R, S, T, and V are functions of p, q as well as x,y, z. This form of equation is known as the uniform non-linear equation, where as equation of Type (6) is known as non-uniform non-linear equation.

We note here that an integral of the equation of either type (6) or type (8) will be of the form (3), which is itself a partial differential equation of first order. It it is possible to determine one or two such first integrals, known as intermediate integrals, from which equations (6) or (8) can be derived, then the values of p and q may be determined. By substituting the values of p and q in the identity $dz = p\,dx + q\,dy$, the complete integral of the equations of the types (6) or (8) may be obtained.

In the discussion that follows, we shall develop a procedure for obtaining such 'intermediate Integrals'. One such method that can be applied to solve a majority of the equations is known as 'Monge's method'. We shall now discuss it with reference to equations or type (8).

## 13.3. Monge's method of integrating Rr + Ss + Tt = V

As has been outlined in the earlier section, we shall first determine the intermediate integrals.

We know that $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy$ ... (1)

and $dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = s\,dx + t\,dy$ ... (2)

These equations give $r = \frac{dp - s\,dy}{dx}$ and $t = \frac{dq - s\,dx}{dy}$ .... (3)

Eliminating r, t from the equation $Rr + Ss + Tt = V$ with the help of equations (3) we get

$$R\left(\frac{dp - s\,dy}{dx}\right) + Ss + T\left(\frac{dq - s\,dx}{dy}\right) = V$$

or $(R\,dp\,dy + T\,dq\,dx - V\,dy\,dx) - S\,(R dy^2 - S\,dy\,dx + T\,dx^2) = 0$ ..... (4)

Any relation between x, y, z, p, q which satisfies equation (4) must also satisfy the two equations :

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad ..... (5)$$

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \quad ..... (6)$$

Equations (5) (6) are called the Monge's subsidiary equations and these simultaneous equations give us the desired intermediate integrals.

**Case 1 :** Suppose the quadratic equation (6) can be factored into two linear equations in dx and dy of the form : $dy - m_1\, dx = 0$ and $dy - m_2 dx = 0$ ... (7)

From the equation $dy - m_1\, dx = 0$, combined with (5) and if necessary with $dz = p\, dx + q\, dy$ will give us two integrals of the type $u_1 = u_1(x, y, z, p, q)$ and $\upsilon = \upsilon_1(x, y, z)$. Then $u_1 = f(\upsilon_1)$ will be a first integral of the given equation. It is called the Intermediate integral.

By considering the second equation $dy - m_2\, dx = 0$ and adopting the similar procedure, the other Intermediate integral, $u_2 = f(\upsilon_2)$ can be obtained. From the two intermediate integrals, we can obtain the values of p and q interms of x and y. By subsituting these values in the equation $dz = p\, dx + q\, dy$ and integrating it, we can obtain the complete integral of the given equation $Rr + Ss + Tt = V$.

**Case 2 :** Suppose the equation (6) is a perfect square. Then $m_1 = m_2$ and we get only one intermediate integral of the form $Pp + Qq = R$. We solve this equation by applying the Lagrange's method.

We shall now solve a few equations that can be identified by these two cases :

## 13.4 Case 1 : Type 1 :

Here we consider equations of the form $Rr + Ss + Tt = V$ which lead to two factors and the second intermediate integral can be obtained from the first one by inspection.

**13.4.1 Example :** Solve $t - r \sec^4 y = 2q \tan y$.

**Solution :** Given that $-r \sec^4 y + t = 2q \tan y$ .... (1)

Comparing equation (1) with the equation $Rr + Ss + Tt = V$, we find that $R = -\sec^4 y$, $S = 0$, $T = 1$, and $V = 2q \tan y$.

Hence Monge's subsidiary equations are :

$$R\, dp\, dy + T\, dq\, dx - V\, dx\, dy = 0 \quad .... \quad (2)$$
$$R\, dy^2 - S\, dx\, dy + T\, dx^2 = 0 \quad .... \quad (3)$$

Substituting the values of R, S, T and V in equation (2) and (3) we get

$$-\operatorname{Sec}^4 y\, dp\, dy + dq\, dx - 2q \tan y\, dx\, dy = 0 \quad .... \quad (4)$$
$$-\operatorname{Sec}^4 y\, dy^2 + dx^2 = 0 \quad .... \quad (5)$$

Equation (5) can be factorised as $(dx - \sec^2y\, dy)(dx + \sec^2y\, dy) = 0$

so that, $\quad dx - \sec^2y\, dy = 0 \quad .... \quad (6)$

and $\quad dx + \sec^2y\, dy = 0 \quad .... \quad (7)$

From equation (6) we have $dx = \sec^2y\, dy$. Substituting this value of dx in equation (4) we get

$$- dp + \cos^2y\, dq - 2q \sin y \cos y\, dy = 0$$

(or) $$dp - (\cos^2y\, dq - 2q \sin y \cos y\, dy) = 0$$

i.e., $$dp - d(q \cos^2y) = 0$$

Integrating we get, $$p - q \cos^2y = C_1 \quad .... \quad (8)$$

Integrating equation (6) we get $x - \tan y = C_2 \quad .... \quad (9)$

From equations (8) and (9), one integral of equation (1) can be taken as

$p - q \cos^2y = f(x-\tan y) \quad .... \quad (10)$

From equations (7) and (4), the other integral of equation (1) follows :

$p + q \cos^2y = g(x + \tan y) \quad .... \quad (11)$

Solving equations (10) and (11) for p and q we find that $p = \frac{1}{2}(f+g)$, $q = \frac{1}{2}(g-f)\sec^2y \quad .... (12)$

Also $$dz = p\, dx + q\, dy$$

$$= \frac{1}{2}(f+g)\, dx + \frac{1}{2}(g-f)\sec^2y\, dy$$

$$= \frac{1}{2} f(x - \tan y)(dx - \sec^2y\, dy) + \frac{1}{2} g(x + \tan y)(dx + \sec^2y\, dy)$$

Integrating, $$z = F(x-\tan y) + G(x + \tan y)$$

**13.4.2. Example :** Solve $pt - qs = q^3$.

**Solution :** Here $R = 0$, $S = -q$, $T = p$ and $V = q^3$. Then the Monge's subsidiary equations will be

$$q\, dy\, dx + p\, dx^2 = 0 \quad .... \quad (1)$$

and $$p\, dq\, dx - q^3\, dy\, dx = 0 \quad .... \quad (2)$$

From equation (1), $dx = 0$ or $q\, dy = - p\, dx$.

Hence $x = C_1$ and $dz = p\, dx + q\, dy = 0$ i.e., $z = C_2$. Where $C_1$, $C_2$ are arbitrary constants.

If $z = C_2$ and $q\, dy = - p\, dx$ then equation (2) reduces to $p\, dq + q^2p\, dx = 0$ or $\frac{dq}{q^2} + dx = 0$

Integrating we get $\frac{-1}{q} + x = C_3 = \phi(z)$ say (or) $\frac{\partial z}{\partial y} = q = \frac{1}{x - \phi(z)}$

Integrating w.r.t. y we obtain $y = xz - \int \phi(z)\,dz + F(x)$

Hence the required solution is $y = xz + \int f(z) F(x)$

**Case 1 : Type 2 :**

Suppose that we are able to obtain two factors as in Type 1 but the inspection method fails to provide us with another integral of a given differential equation. In this case we start with only one integral by putting it in the form Pp + Qq = R and apply the Lagranges method. We now illustrate this technique with a few examples.

**13.4.3 Example :** Solve $q(1+q)r - (p+q+2pq)s + p(1+p)t = 0$ ... (13)

**Solution :** Comparing equation (13) with the equation Rr + Ss + Tt = V, we find that $R = q(1+q)$, $S = -(p+q+2pq)$, $T = p(1+p)$, $V = 0$.

The Monge's subsidiary equations are :

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad \ldots \quad (14)$$

and $$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \quad \ldots \quad (15)$$

Substituting the expressions for R, S, T and V in equations (14) and (15) we obtain,

$$(q+q^2)\,dp\,dy + (p+p^2)\,dq\,dx = 0 \quad \ldots \quad (16)$$

and $$(q+q^2)dy^2 + (p+q+2pq)\,dx\,dy + (p+p^2)\,dx^2 = 0 \quad \ldots \quad (17)$$

Equation (17) can be factorised into $(q\,dy + p\,dx)\,[(1+q)dy + (p+1)dx] = 0$

So that $q\,dy + p\,dx = 0 \Rightarrow q\,dy = -p\,dx$ ... (18)

and $(1+q)\,dy + (1+p)\,dx = 0$ ... (19)

In view of equation (18) equation (16) may be written as $(1+q)\,dp\,(q dy) - (1+p)\,dq\,(-p dx) = 0$

We now divide this equation by q dy ( or its equivalent -p dx) so that,

$$(1+q)\,dp - (1+p)\,dq = 0 \quad \text{or} \quad \frac{dp}{1+p} - \frac{dq}{1+q} = 0$$

Integrating, $\log(1+p) - \log(1+q) = \log C_1$. (or) $\frac{1+p}{1+q} = C_1$ ... (20)

Also $dz = p\,dx + q\,dy = 0$, in virtue of equation (18). Hence $z = C_2$. ... (21)

From equation (20) and (21) we find that one integral of equation (13) is $\frac{1+p}{1+q}$=f(z) or 1+p=(1+q)f(z) or p - f(z) q = f(z) - 1.

This equation is of the form Pp + Qq = R. Hence the Lagranges auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = -\frac{dz}{f(z)-1} = \frac{dx+dy+dz}{0} \qquad \ldots \quad (22)$$

Hence $dx + dy + dz = 0 \Rightarrow x + y + z = C_3$. ... (23)

From the first and third ratios of equation (22) we get $dx - [\,f(z)-1]^{-1}\,dz = 0$

Integrating we get $x + F(z) = C_4$. ... (24)
from equation (23) and (24) we have x + F(z) = G(x+y+z), which is the required general solution.

**13.4.4. Example :** Solve $y^2r - 2ys + t = p + 6y$

**Solution :** Since $r = \frac{dp - s\,dy}{dx}$, $t = \frac{dq - s\,dx}{dy}$

the given equation is $y^2\left(\frac{dp - s\,dy}{dx}\right) - 2ys + \frac{dq - s\,dx}{dy} = p + 6y$

(or) $y^2\,dp\,dy + dq\,dx - (p+6y)\,dy\,dx - s^2(y^2dy^2 + 2y\,dy\,dx + dx^2) = 0$ (25)

Hence Monge's subsidiary equations are :

$y^2\,dy\,dp + dq\,dx - (p+6y)\,dy\,dx = 0$ ... (26)

and $y\,dy + dx = 0$ ... (27)

Integrating equation (27), $y^2 + 2x = C_1$

Substituting the value of y dy from equation (27) in (26) and dividing with by-dx, we have

$y\,dp - dq + (p+6y)\,dy = 0$

or $(y\,dp + p\,dy) - dq + 6y\,dy = 0$

Integrating, we have $py - q + 3y^2 = C_2$.

Hence the intermediate integral is $py - q + 3y^2 = f(y^2 + 2x)$

The Lagranges subsidiary equations are $\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{-3y^2 + f(y^2+2x)}$

The first two ratios give $y^2 + 2x = C_1$. Using this in the last ratio, we have $\frac{dy}{-1} = \frac{dz}{-3y^2 + f(C_1)}$

or $\quad dz = [\, 3y^2 - f(C_1)\,]\, dy.$

Integrating, we get $\quad z = y^3 - y\, f(C_1) + C_3.$

Hence the solution $\quad z = y^3 - y\, f(y^2+2x) + \phi\, (y^2+2x)$

## 13.5 Case 2 : Type 3 :

If the Monge's subsidiary equation has R.H.S. as a perfect square, it gives only one distinct factor. In this case we obtain only one intermediate integral, which is expressed in the form Pp+Qq=R

**13.5.5. Example :** Solve $y^2r + 2xy\, s + x^2\, t + px + qy = 0$

**Solution :** Given that $y^2r + 2xy\, s + x^2\, t = -(\, px + qy)$ ... (28)

Comparing equation (28) with $Rr + Ss + Tt = V$, we get $R = y^2$, $S = 2xy$, $T = x^2$, $V = -(px+qy)$

Mange's Subsidiary equations are :

$$R\, dp\, dy + T\, dq\, dx + V\, dx\, dy = 0 \quad \ldots \quad (5)$$
$$R\, dy^2 - S\, dx\, dy + T\, dx^2 = 0 \quad \ldots \quad (6)$$

Substituting the expressions for R, S, T, V in equations (29) and (30) we get the Monge's equations of Equation (28) :

$$y^2\, dp\, dy + x^2\, dq\, dx + (\, px + qy\,)\, dx\, dy = 0 \quad \ldots \quad (29)$$
$$y^2\, dy^2 - 2xy\, dx\, dy + x^2\, dx^2 = 0 \quad \ldots \quad (30)$$

From equation (30) we have $(x\, dx - y\, dy)^2 = 0$ so that, $x\, dx - y\, dy = 0$ (or) $x\, dx = y\, dy$ (31)

Hence equation (29) may be written as $y\, dp\, (y\, dy) + x\, dq\, (x\, dx) + p\, dy\, (x\, dx) + q\, dx\, (y\, dy) = 0$

Dividing throught by x dx or its equivalent y dy,

we get, $y\, dp + x\, dq + p\, dy + q\, dx = 0$ (or) $(y\, dp + p\, dy) + (\, x\, dq + q\, dx) = 0$

Integrating we obtain $\quad y\, p + x\, q = C_1.$ .... (32)

Integrating equation (31), $\frac{1}{2}x^2 - \frac{1}{2}y^2 = C_2/2$.

(or) $x^2 - y^2 = C_2$. ..... (33)

From equation (32) and (33), one of the integral of (28) is $y\,p + x\,q = F(x^2 - y^2)$

Since this is of the form $Pp + Qq = R$, its Lagrange's auxiliary equations are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{F(x^2 - y^2)}$

From the first two ratios we have $x\,dx - y\,dy = 0$ so that $x^2 - y^2 = C_2$.

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Taking the last two factors and using equation (33) we obtain

$$\frac{dy}{\sqrt{y^2 + C_2}} = \frac{dz}{F(C_2)} \quad \text{or} \quad dx - F(C_2)\frac{dy}{\sqrt{y^2 + C_2}} = 0.$$

Integrating, $z - F(C_2)\log[\,y + \sqrt{y^2 + C_2}\,] = C_3$.

i.e., $z - F(x^2-y^2)\log[\,y + \sqrt{y^2 + x^2 - y^2}\,] = C_3$.

or $z - F(x^2-y^2)\log(x+y) = C_3$.

Hence the general solution is $z - F(x^2-y^2)\log(x + y) = G(x^2-y^2)$.

**13.5.2. <u>Example :</u>** Solve $y^2r - 2ys + t = p + 6y$ .... (33)

**<u>Solution :</u>** The Monge's subsidiary equations for the equation (33) are :

$y^2\,dp\,dy + dq\,dx - (p+6y)\,dx\,dy = 0$ .... (34)

$y^2\,dy^2 + 2y\,dy\,dx + dx^2 = 0$ (or) $(y\,dy + dx)^2 = 0$ .... (35)

From equation (35), $y\,dy + dx = 0$ or $dx = -y\,dy$ .... (36)

Substituting for dx from (36) in equation (34) we obtain :

$y^2\,dp\,dy + dq(-y\,dy) - (p+6y)\,dy(-y\,dy) = 0$

i.e., $y\,dp - dq + (p+6y)\,dy = 0$ (or) $(y\,dp + p\,dy) - dq + 6y\,dy = 0$

Integrating, $yp - q + 3y^2 = C_1$. ... (37)

Integrating equation (36), $x + \frac{1}{2} y^2 = C_2/2$ or $y^2 + 2x = C_2$. ... (38)

From equation (37) and (38) we can write one integral of equation (33) as $y p - q + 3y^2 = F(y^2+2x)$ or $yp - q = F(y^2 + 2x) - 3y^2$ which is in the standard form $Pp + Qq = R$. Hence the Lagranges auxiliary equations become $\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{F(y^2+2x)-3y^2}$ ... (39)

From the first two ratios, we find $y\,dy + dx = 0$ (or) $y^2 + 2x = C_2$. ... (38)

From the last two ratios, using equation (38) we obtain : $dz + [\,F(C_2) - 3y^2\,]\,dy = 0$

Integrating. $z + y\,F(C_2) - y^3 = C_3$. (or ) $z + y\,F(y^2 + 2x) - y^3 = C_3$.

Hence tne required solution is $z + y\,F(y^2 + 2x) - y^3 = G(y^2 + 2x)$

**13.5.3. Example :** Obtain the integral of $q^2r - 2pqs + p^2t = 0$ in the form of $y + x\,f(z) = F(z)$

**Solution :** For the given equation, the Monge's Subsidiary equations are

$$q^2\,dp\,dy + p^2\,dp\,dx = 0 \quad \dots \quad (40)$$

and $$q^2\,dy^2 + 2pq\,dx\,dy + p^2dx^2 = 0$$

or $$(q\,dy + p\,dx)^2 = 0 \quad \text{i.e.,} \quad q\,dy = -p\,dx \quad \dots \quad (41)$$

Hence, in view of (41), equation (40) becomes $q\,dp\,(q dy) - p\,dq\,(-p\,dx) = 0$ dividing throughout by q dy or its equivalent p dx, we find $q\,dp - p\,dq = 0$ or $\frac{dp}{p} = \frac{dq}{q}$

Integrating $\frac{p}{q} = C_1$.

From equation (41) $dz = p\,dx + q\,dy = 0$, so $z = C_2$.

Hence one integral of the given equation is $\frac{p}{q} = f(z)$. (or) $p - f(z)\,q = 0$. This is of the form $Pp+Qq = R$. Hence Lagranges auxiliary equations for this equations are $\frac{dx}{y} = \frac{dy}{-f(z)} = \frac{dz}{0}$. Hence $dz = 0$, so $z = C_2$. From the first two ratios $\frac{dx}{1} = \frac{dy}{-f(z)}$ (or) $dy = -f(C_2)\,dx$

Integrating $y + x\,f(C_2) = C_3$ (or) $y + x\,f(z) = C_3$.

Hence the required integral is $y + x\,f(z) = F(z)$.

## 13.6 Case 3 : Type 4 :

Suppose that the Monge's subsidiary equation $R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0$ has the R.H.S. such that it is neither a perfect square nor it gives two factors. In such case, we cancel factors such as dx, dy, p, .... etc and an integral of the given equation is obtained. This integral is then integrated by a known method (lesson -10). We shall now illustrate this case with a few examples.

**13.6.1. Example :** Solve $z(qs - pt) = pq^2$.

**Solution :** Given that $z(qs - pt) = pq^2$. ... (42)

The Monge's subsidiary equations for equation (42) are $-zp\,dq\,dx - pq^2\,dx\,dy = 0$ ... (43)

and $-zq\,dx\,dy - zp\,dx^2 = 0$ ... (44)

Dividing equation (44) by $-z\,dx$ and (43) by $-p\,dx$ we obtain :

$$z\,dq + q^2\,dy = 0$$

$$q\,dy + p\,dx = 0 \quad \Rightarrow \quad dz = 0 \quad \Rightarrow z = C_1.$$

Hence $C_1\,dq + q^2\,dy = 0$ (or) $\dfrac{dq}{q^2} + \dfrac{dy}{C_1} = 0$

Integrating, $-\dfrac{1}{q} + \dfrac{y}{C_1} = C_2$ (or) $-\dfrac{1}{q} + \dfrac{y}{z} = C_2$

Hence one integral of equation (42) is $-\dfrac{1}{q} + \dfrac{y}{z} = f(z)$ (or) $\dfrac{dy}{dz} - \dfrac{y}{z} = -f(z)$

This equation is linear in y and z if x is treated as constant. Its integrating factor is

$$e^{-\int \frac{1}{z} dz} = e^{-\log z} = \frac{1}{z}$$

Hence its solution is $y/z = -\int \frac{1}{z} f(z)dz + G(x)$ (or) $y z^{-1} = F(z) + G(x)$

i.e., $y = z\,F(z) + z\,G(x)$

**13.6.2. Example :** Solve $(q + 1)s = (p+1)\,t$

**Solution :** From the given equation we find that $R = 0$, $S = (q+1)$, $T = -(p+1)$ and $V = 0$.

Hence Monge's subsidiary equations are

$$- (p + 1)\, dq\, dx = 0$$

and $\quad - (q+1)\, dx\, dy - (p+1)\, dx^2 = 0$

or $\quad dq = 0$ and $\quad (q + 1)\, dy - (p+1)\, dx = 0$

i.e., $\quad dx + dy + p\, dx + q\, dy = 0$

or $\quad dx + dy + dz = 0$, since $p\, dx + q\, dy = dz$

Integrating we get, $\quad x + y + z = C_1$. $\quad$ Also $dq = 0 \quad \Rightarrow \quad q = C_2$.

Hence an integral of the given equation is $q = f(x+y+z)$ (or) $\dfrac{\partial z}{\partial y} = f(x+y+z)$.

Integrating this equation partially w.r.t. y. we obtain,

$z = F(x+y+z) + G(x)$, where F, G are arbitrary functions.

**13.6.3. Example :** Solve $( x - y )\,( xr - xs - ys - yt) = (x+y)\,(p-q)$

**Solution :** We first subsitute $r = \dfrac{dp - s\, dy}{dx}$ and $t = \dfrac{dp - s\, dx}{dy}$ in the given equation. to obtain the following Monge's subsidiary equations.

$$x ( x - y )\, dp\, dy + y ( x- y )\, dq\, dx - (x + y)( p - q)\, dx\, dy = 0 \quad \ldots \quad (45)$$

and $\quad x dy^2 + ( x + y)\, dx\, dy + y\, dx^2 = 0 \quad \ldots \quad (46)$

From equation (46) we obtain, $x\, dy + y\, dx = 0 \Rightarrow xy = C_1$. $\quad \ldots \quad (47)$

and $\quad dx + dy = 0 \quad \ldots \quad (48)$

In view of equation (47), equation (45) can be written as

$$- y (x-y)\, dp\, dx + y (x-y)\, dq\, dx - (p-q)(-y\, dx^2 + y\, dx\, dy) = 0$$

i.e., $\quad (x -y) ( dp -dq) - (p-q) (dx-dy) = 0$

or $\quad \dfrac{dp - dq}{p - q} = \dfrac{dx - dy}{x - y}$

Integrating we get $p - q = C_2 (x - y)$

Hence one integral of the given equation is $p - q = ( x- y )\, f (xy)$, which in the Lagranges form.

Hence the subsidiary equations are $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{(x-y)f(x,y)} = \dfrac{f(xy)(y\ dx + x\ dy + dz)}{0}$

From the first two ratios we have $x + y = C_3$.

From the last ratio, we get $dz = f(xy)\, d(xy)$ (or) $z = F_1(xy) + \text{const.}$

Hence the complete integral is $z = F_1(xy) + F_2(x + y)$

**SAQs :** Solve the following equations using the Monge's method :

**Type 1:**

(1) $r - t\cos^2 x + p\tan x = 0$ (Ans : $F(y-\sin x) + G(y+\sin x)$]

(2) $r = a^2 t$. (Ans : $z = F(y-ax) + G(y+ax)$)

**Type 2:**

(1) $2x^2 r - 5xys + 2y^2 t + 2(px + qy) = 0$ (Ans : $z + F(x^2 y) = G(xy^2)$ )

(2) $(x - y)(xr - sx - ys + yt) = (x + y)(p - q)$ (Ans : $F(xy) + z = G(x+y)$)

**Type 3:**

(1) $(1+q^2) r - 2(1+p+q+pq)s + (1+p)^2 t = 0$ (Ans : $y + xF(x+y+z) = G(x+y+z)$)

(2) $x^2 r + 2xys + y^2(t) = 0$ (Ans : $z + F(y/x) = y\, G(y/z)$)

**Type 4 :**

(1) $pq = x(ps - qr)$ (Ans : $F(z) = \log x + G(y)$ )

(2) $pt - qs = q^3$. (Ans : $y = xz - F(z) + G(x)$ )

## 13.7 Summary

In this lesson we have derived a second order non-linear partial differential equation $Rr+Ss+Tt=V$, called the uniform non-linear equation and also $Rr+ Ss+Tt+U(rt-s^2) = V$, called the non-uniform non-linear equation. The integrals of these equations are of the form $u = \phi(\upsilon)$, which is itself a partial differential equation of first order. We have developed a procedure for obtaining the intermediate integrals for those equations of the type $Rr + Ss + Tt = V$, using the Monge's method by writing the subsidiary equations. The solution for the considered equation is obtained using the Lagránge's method.

## 13.8 Model Examination Questions :

Solve the following equations using Monge's method :

1. $r + ka^2t - 2as = 0$ (Ans : $z = F[y + a(1+b)x] + G[y+a(1-b)x]$ )

2. $rx^2 - 3s\,xy + 2ty^2 + px + 2qy = x + 2y$ (Ans : $y = F(x+z) + G(z)$ )

3. $x^2r - y^2t - 2xp + 2z = 0$ (Ans : $yz = (xy)^{3/2} F(y/x) + G(xy)$)

4. $(r - s)y + (s - t)x + q - p = 0$ (Ans : $z = F(x+y) + G(x^2-y^2)$ )

5. $x^2r - y^2t = xy$ (Ans : $z = xy \log x + x F(y/x) + G(xy)$ )

6. $x^2r - 2xs + t + q = 0$ (Ans : $F(y+\log x) + x\,G(y+\log x)$ )

7. $q^2r - 2pqs + p^2t = pq^2$ (Ans : $y = e^{y} F(z) + G(x)$ )

8. $(e^x-1)(qr-ps) = pq\,e^x$ (Ans : $x = F(z) + G(y) + e^x$)

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## LESSON-14 NON - UNIFORM NON - LINEAR EQUATIONS OF SECOND ORDER - MONGE'S METHOD

### 14.1 Introduction :

In lesson 13, a second order partial differential equation was derived by eliminating an arbitary function from a partial differential equation of first order. By considering two known functions u, v of x,y, z, p and q which are connected by a relation u = $\phi$ ($\upsilon$), we derived an equation of the form $Rr + Ss + Tt + U(rt - s^2) = V$ known as a non- uniform non -linear equation

In particular if u or v happens to be a function of x,y, z only (and not of p or q), we formed that the derived equation will be reduced to $Rr + ss + Tt$ =V, known as uniform non-linear equation . In the last lesson, we have discussed about four types of solving such equations - categorised into two broad cases- depending on whether $m_1 = m_2$ or $m_1 \neq m_2$.

In this lesson we shall discuss the Monge's method of finding a solution to the non - uniform non-linear equation $Rr + Ss + Tt + U(rt - s^2) = V$ through a classification into three types. we shall first derive the Monge's subsidiary equations and obtain the intermediate integrals.

### 14.2 Monges subsidiary equations and the intermediate integrals.

Consider the equation $Rr + Ss + Tt + U(rt - s^2) = V$ ...(1) where R,S,T, U and V are functions of x,y, z, p and q we know that $\frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = rdx + sdy$

$$\text{or } r = \frac{dp - sdy}{dx}$$

similarly $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy = sdx + t\,dy$$

$$\text{or } \frac{dq - sdx}{dy} = \text{t}$$

substituting these values of r and t in equation (1) we have

$$R\left(\frac{dp - sdy}{dx}\right) + Sr + T\left(\frac{dq - sdx}{dy}\right) + U\left\{\frac{dp - sdy}{dx}\frac{dq - sdx}{dy} - s^2\right\} = \text{V},$$

i.e

$$(R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy) - s(R\,dy^2 - S\,dx\,dy + T\,dx^2 + U\,dp\,dx + V\,dq\,dy) = 0$$

Hence the the Monges subsidiary equations are

$$R\,dp\,dy + T\,dq\,dx + U\,dp\,dq - V\,dx\,dy = 0 \qquad \text{--- (2)}$$

and $(R\,dy^2 - S\,dx\,dy + T\,dx^2 + U\,dp\,dx + V\,dq\,dy) = 0$ --- (3)

Here we notice that the presence of the factor U dp dx + U dq dy in equation (3) stalls the process of factorising it. So we shall now try to factorize

$$(R\,dy^2 - s\,dxdy + T\,dx^2 + U\,dp\,dx + U\,dq\,dy$$
$$+ \lambda\ (R\ dp\,dy + T\,dq\,dx + U\,dp\,dx - V\,dq\,dy) = 0 \qquad \text{--- (4)}$$

for some multiplier $\lambda$ to be determined later Equation (4) can be expressed after rearrangement of terms as

$$R\,dy^2 + T\,dx^2 - (S + \lambda V)\,dx + dy + U\,dp\,dy - V\,dq\,dy + \lambda\ R\,dp\,dy + \lambda\ T\,dq\,dx + \lambda\ U\,dp\,dq = 0 \qquad \text{--- (5)}$$

Let us suppose that the factors of (5) are

$(R\,dy + m\,T\,dx + kU\,dp)\ (dy + \frac{1}{m}dx + \frac{\lambda}{k}dq) = 0$ where k and m are constants --- (6)

comparing the co-efficients in equations (5) and (6) we get $\frac{R}{m} + mT = -(S + \lambda V)$

$$k = m \qquad \text{and} \qquad \frac{R\lambda}{k} = U \qquad \text{---} \quad (7)$$

The last two relations in equations (7) give $m = \frac{R\lambda}{U}$

which when substituted in the first give $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ --- (8)

called the $\lambda$ equation. In general $\lambda$ will be a function of x,y, z, p and q. If $\lambda_1$ and $\lambda_2$ are assumed to be the two values of $\lambda$ satisfying the quadratic equation (8), then the factors corresponding to these values will be.

$$(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp)(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq) = 0$$

or $(U\,dy+\lambda_1 T\,dx+\lambda_1 U\,dp)\,(U\,dx+\lambda_1 R\,dy+\lambda_1 U\,dq) = 0.$

as $m = k = \dfrac{R\lambda_1}{U}$ --- (9)

similarly corresponding to $\lambda_2$ we obtain

$(U\,dy+\lambda_2 T\,dx+\lambda_2 U\,dq)\,(U\,dx+\lambda_2 R\,dy+\lambda_2 U\,dq) = 0$ --- (10)

To get intermediate integrals in the form $u = f(\upsilon)$ one of the factors from equation (9) is to be combined with a one from equation (10). In a similar way the left over pairs will give rise to another intermediate integral. However, in order to obtain proper solutions, we combine the factors in the following manner :

$$U\,dy+\lambda_1 T\,dx+\lambda_1 U\,dp = 0 \quad \}$$

and $$U\,dx+\lambda_2 R\,dy+\lambda_2 U\,dq = 0 \quad \} \qquad \text{---} \quad (11)$$

$$U\,dy+\lambda_2 T\,dx+\lambda_2 U\,dp = 0$$

and $$U\,dx+\lambda_1 R\,dy+\lambda_1 U\,dq = 0 \qquad \text{---} \quad (12)$$

If the total differential equations (11) and (12) are integrable, we get intermediate integrals, from which p and q can be determined. These p and q when substituted in dz = p dx + q dy and after integration give the desired general solution.

we shall now exemplify the Monge's method of solving the equation (1) by considering three types (1-3) : we begin with examples where in the $\lambda$- equation has equal roots. In this case, we proceed with only one intermediate integral.

### 14.3 Examples of Type 1 : $\lambda$- equation having equal roots

**14.3.1. Example :** Solve $2r + te^x - (rt - s^2) = 2e^x$

<u>**Solution:-**</u> comparing the given equation with the standard form

$$Rr + Ss + Tt + U(rt - s^2) = V$$

we have R =2, S = 0, T = $e^x$, U = -1, and V = 2 $e^x$

Hence the $\lambda$-equation $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$

yields $\lambda^2(2e^x - 2e^x) + \lambda(0) + 1 = 0$, which can't be solved.

Here $r = \frac{dp - s\,dy}{dx}$, $t = \frac{dq - s\,dx}{dy}$

The Monge's subsidiary equations are

$$2\,dp\,dy + e^x\,dq\,dx - dp\,dq - 2\,e^x\,dx\,dy = 0 \quad \text{---} \quad (A)$$

and $dy^2 + e^x dx^2 - dp\,dx - dq\,dy = 0$

From eqaution (A) we have $(2\,dy - dq)(dp - e^x\,dx) = 0$

$\Rightarrow 2\,dy - dq = 0 \quad \Rightarrow 2y - q = a$, where a is a constant

Similarly $dx - e^x dx = 0 \quad \Rightarrow p - e^x = b$, where b is a constant

$q = 2y - a$ and $p = b + e^x$

substituting these values of p and q in $dz = p\,dx + q\,dy$

we get $dz = (b + e^x)\,dx + (2y - a)\,dy$

Integrating we get $z = e^x + dx + y^2 - ay + c$

which is the required solution.

**14.3.2. Example : Solve $2s + (rt - s^2) = 1$,**

**Solution : Comparing the given equation with the equation**

$Rr + Ss + Tt + U(rt - s^2) =$ V, we have R = 0, S = 2, T = 0, U = 1, and v = 1 --- (13)

Hence the $\lambda$-equation becomes $\lambda^2 + 2\lambda + 1 = 0$, so that $\lambda_1 = \lambda_2 = -1$ and we have equal values of $\lambda$. Hence we have only one intermediate integral given by

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0,$$

and $$U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0,$$

i.e., $dy - dp = 0$ and $dx - dq = 0$ (In virtue of equ 13)

Integrating, y - p = $C_1$ and x - q = $C_2$

$$\text{Hence } dz = p\,dx + q\,dy = (y-c_1)dx + (x-c_2)dy$$

$$= (y\,dx + x\,dy) - c_1\,dx - c_2\,dy$$

$$= d(xy) - c_1\,dx - c_2\,dy$$

Integrating, $z = xy - c_1x - c_2y + c_3$, which is the desired solution.

**14.4 Examples of Type .2 :** $\lambda$ - equation with different roots

**14.4.1. Example :** Solve $r + 3s + t + (rt - s^2) = 1,$

**Solution** :

Comparing the given equation with the standard form, we find that R = 1, S = 3, T = 1, U=1,V = 1,

Hence the $\lambda$-equation $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$

become $2\lambda^2 + 3\lambda + 1 = 0$ which gives $\lambda_1 = -1$ and $\lambda_2 = -1/2$ (distinct roots)

The first system of integrals are given by

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0 \quad \Rightarrow dy - dx - dp = 0$$

$$U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0 \quad \Rightarrow dy - 2dx + dq = 0$$

Integrating y - x - p = constant and y - 2x + q = constant

Hence the first intermediate integral is given by y - x - p $= f_1(y - 2x + q) = f_1(\alpha)$, say

similarly the second intermediate integral can be seen to be

$$2y - x - p = f_2(x - q - y) = f_2(\beta), \text{ say}$$

From the relations $\alpha = y - 2x + q$ and $\beta = x - y - q$

$-\beta - \alpha = x$ and $y = f_2(\beta) - f_1(\alpha)$

so that $dx = -d\beta - d\alpha$ and dy = $f_2^1(\beta) - f_1^1(\alpha)\,d\alpha$

Then from the first intermediate integral we have $p = y - x - f_1(\alpha)$

and from the second intermediate integral q $= x - y - \beta$

Substituting these values of p and q in dz = pdx + qdy

we obtain $dz = \left[y - x - f_1(\alpha)\right] dx + (x - y - \beta)dy$

or $\left[-(x-y)(dx-dy) - f_1(\alpha)\{-d\beta - d\alpha\} - \beta\left\{f_2(\beta)d\beta - f_1^1(\alpha)d\alpha\right\}\right]$

Integrating, z $= -\frac{1}{2}(x-y)^2 + \int f_1(\alpha)d\alpha - \int \beta\, f_2^1(\beta)d\beta + \beta f_1(\alpha)$

$$= -\frac{1}{2}(x-y)^2 + F_1(\alpha) + F_2(\beta) - \beta\, f_2(\beta) + \beta\, f_1(\alpha)$$

which is the required solution.

**14.4.2 Example :** Solve $r + 4s + t + rt - s^2 = 2$

**Solution** :

Comparing the given equation with the standard form

$Rr + Ss + Tt + U(rt - s^2)$ = V we have R = 1, S = 4, T = 1, U =1, and V = 2 ---(14)

The $\lambda$ -equation, with these values substituted become $3\lambda^2 + 4\lambda + 1 = 0$

so that $\lambda_1 = -1$; $\lambda_2 = -1/3$ --- (15)

The two intermediate integrals for the given equation are given by the following sets :

$$U\,dy + \lambda_1 T\,dx + \lambda_1 U\,dp = 0$$

$$U\,dx + \lambda_2 R\,dy + \lambda_2 U\,dq = 0 \qquad \text{---} \quad (16)$$

and

$$U\,dy + \lambda_2 T\,dx + \lambda_2 U\,dp = 0$$

$$U\,dx + \lambda_1 R\,dy + \lambda_1 U\,dq = 0 \qquad \text{---} \quad (17)$$

Using the results of (14) and (15) in Eq's (16) and(17) we get

| | | | | | |
|---|---|---|---|---|---|
| | $dy - dx - dp = 0$ | or | $dp + dx - dy = 0$ | | |
| | $dx - 1/3\, dy - 1/3 dq = 0$ | or | $dq + dy - 3dx = 0$ | --- | (16 a) |
| and | $dy - 1/3\, dx - 1/3\, dp = 0$ | or | $dp + dx - 3\, dy = 0$ | | |
| | $dx - dy - dq = 0$ | or | $dq + dy - dx = 0$ | --- | (17a) |

Integrating equations 16(a) we get $p + x - y = c_1$ , $q + y - 3x = c_2$

Integrating equations 17(a) we get $p + x - 3y = c_3$ , $q + y - x = c_4$

Thus the two integrals of the given equations are

$p + x - y = f\,(q+y - 3x)$ and $p + x - 3y = F\,(q + y - x)$ --- (18)

Now let $q + y - 3x = \alpha$ and $q + y - x = \beta$ --- (19)

Then $p + x - y = f(\alpha)$ and $p + x - 3y = F(\beta)$ --- (20)

Solving equations (19) for x we get $x = \frac{1}{2}(\beta - \alpha)$ --- (21)

similarly solving equations (20) for y we have $y = \frac{1}{2}[f(\alpha) - F(\beta)]$ --- (22)

From eq (15) we get $p = y - x + f(\alpha)$

From eq (19) we have $q = x - y + \beta$

Also from eq (21) $dx = \frac{1}{2}(d\beta - d\alpha)$

and from eq (22) $dy = \frac{1}{2}\left[f^1(\alpha)d\alpha - F^1(\beta)d\beta\right]$ --- (23)

Hence , $dz = p\, dx + q\, dy$

$$= [y - x + f(\alpha)]\, dx + (x - y + \beta)\, dy$$

$$= ydx + xdy - xdx - ydy + f(\alpha)\, dx + \beta\, dy$$

$$= d(xy) - xdx - ydy + f(\alpha)\frac{1}{2}(d\beta - d\alpha) + \beta\frac{1}{2}\left[f^1(\alpha)d\alpha - f^1(\beta)d\beta\right] \text{ by eq(23)}$$

$$= d(xy) - xdx - ydy + \frac{1}{2}\left[f(\alpha)\, d\beta + \beta\, f^1(\alpha)\, d\alpha\right] - \frac{1}{2}\left[f(\alpha)\, d\alpha + \beta\, F^1(\beta)\, d\beta\right]$$

or $2\,dz = 2d(xy) - 2x\,dx - 2y\,dy + d[\beta f(\alpha)] - f(\alpha)\,d\alpha - \beta F^{1}(\beta)\,d\beta$

Integrating $2z = 2xy - x^2 - y^2 + \beta f(\alpha) - \int f(\alpha)\,d\alpha - \beta F^{1}(\beta)\,d\beta$

or $2\,dz = 2xy - x^2 - y^2 + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)\,d\alpha + \int F(\beta)\,d\beta$

Let $\int f(\alpha)\,d\alpha = \phi(\alpha)$ and $\int F(\beta)\,d\beta = \psi(\beta)$ ... (25)

so that $f(\alpha) = \phi^{1}(\alpha)$ and $F(\beta) = \psi^{1}(\beta)$ ... (26)

using (25) and (26) in equations (21), (22) and (24)

we have $2x = \beta - \alpha$ ; $2y = \phi^{1}(\alpha) - \psi^{1}(\beta)$

and $2z = 2xy - x^2 - y^2 + \beta[\phi^{1}(\alpha) - \psi^{1}(\beta)] - \phi(\alpha) + \psi(\beta)$

This is the required solution in the parametric form with $\alpha$, $\beta$ as parameters and $\phi$, $\psi$ as arbitrary functions.

## 14.5 Examples of Type 3 :

In case we fail to obtain a solution for an equation of the form $Rr+Ss+Tt+U(rt-s^2)=V$ with the earlier methods discussed in 14.3 and 14.4, we proceed with the method of type 3. Here, in place of the usual $u_1 = f(\upsilon_1)$, we obtain an integral of the given equation in the linear form $u_1 = m\upsilon_1 + n$. we then integrate this form by Lagranges method to obtain the general solution involving two arbitrary constants m, n. This method, thus gives the desired solution in more general form, for problems which were considered in Types 1 and 2. We shall now illustrate the procedure with a few examples.

**14.5.1 Example :** Solve $qrx + (x+y)s + pyt + xy(rt - s^2) = 1 - pq$

**Solution :-** comparing the given equation with the standard form of the equation

$Rr + Ss + Tt + U(rt - s^2) = V$ we find that R = qx, S = x + y, T = p y,

U = x y and V = 1- pq with these values of R, S, T, U and V the $\lambda$-equation becomes

$$[xy(1-pq) + qx(py)]\lambda^2 + xy\,(x+y)\lambda + x^2y^2 = 0$$

or $\lambda^2 + (x+y)\lambda + +xy = 0 \Rightarrow (\lambda + x)(\lambda + y) = 0,$

so that $\lambda_1 = -x$ and $\lambda_2 = -y$

we have one intermediate integral given by $Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$

$$Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

i.e, xy dy - x py dx - x $^2$ y dp = 0 $\Rightarrow$-dy + (p dx + x dp) = 0

xy dy - yq x dy - y $^2$ x dq = 0 $\Rightarrow$-dx + (q dy + y dq) = 0

Integrating, -y + px = $c_1$ and -x + qy = $c_2$

Hence one intermediate integral may be taken as

px - y = m ( qy - x) + n or

or px - mqy = y - mx + n, m,n are artibitary constants.

The Lagranges auxiliary equations for this equations are $\dfrac{dx}{x} = \dfrac{dy}{-my} = \dfrac{dz}{y - mx + n}$ --- (27)

From the first two ratio's, we have $m\dfrac{dx}{x} + \dfrac{dy}{y} = 0$

Integrating m log x + log y = log $c_1$ or $x^m y = c_1$ --- (28)

choosing $m, \dfrac{1}{m}, 1$ as multipliers in equation (27), each of the fraction $= \dfrac{1}{n}\left(m\,dx + \dfrac{dy}{m} + dz\right)$

--- (29)

From the first fraction of equation (27) and (29) we have $\dfrac{dx}{x} = \dfrac{\left(m\,dx + \dfrac{1}{m}dy + dz\right)}{n}$

or $$dz + m\,dx + \frac{1}{m}dy - n\frac{dx}{\mathrm{x}} = 0$$

Integrating, $z + mx + \dfrac{y}{m} - n\log \mathrm{x} = c_2$ --- (30)

From equations (28) and (30), the required general solution is $z + mx + \dfrac{y}{m} - n\log \mathrm{x} = \phi(x^m y)$,

$\phi$ is an arbitrary function.

**14.5.2 Example :** Solve $(rt - s^2) - s(\sin x + \sin y) = \sin x \sin y$

**Solution :** Here R = 0, $S = -s(\sin x + \sin y)$ T = 0, U = 1, and $V = \sin x \sin y$

Hence the $\lambda$ -equation becomes

$$\lambda^2 (\sin x \sin y) - \lambda (\sin x + \sin y) + 1 = 0$$

so that $\lambda_1 = \operatorname{cosec} x$ and $\lambda_2 = \operatorname{cosec} y$

Then one of the intermediate integrals will be $\sin x \, dy + dp = 0$ and $\sin y \, dx + dq = 0$

Since these are not integrable, we consider the other intermediate integrals :

$$\sin y \, dy + dp = 0 \quad \text{and} \quad \sin x \, dx + dq = 0$$

Integrating p - cos y = a, and q - cos x = b, where a, b are arbitrary constants.

Hence we can write p - cos y = f (q-cos x) which cannot be further integrated unless we know f. So let us suppose that the arbitrary function f is linear, i.e., p - cos y = m (q - cosx) + n where m,n are constants.

Lagranges subsidiary equations for this equations will be

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\cos y - m\cos x + n}$$

From the first two ratios we have $y + mx = c_1$ or $y = (c_1 - mx)$

From the first and last ratio's $\dfrac{dx}{1} = \dfrac{dz}{\cos(c_1 - mx) - m\cos x + n}$

or $dz = \left[\cos(c_1 - mx) - m\cos x + n\right]dx$

Integrating, $z = -\dfrac{1}{m}\sin(c_1 - mx) - m\sin x + nx + c_2$

or $mz + \sin y + m^2 \sin x - mnx = mc_2$, which is the required solution.

**SAQS :** **Solve the following equations by the Monges method**

1. $r+t-(rt-s^2)=1$

(Ans : $2z=c_1x+2c_2y+x^2+y^2+c_3$)

2. $3r+4s+t+rt-s^2=1$

(Ans : $2z=2c_1x+2c_2y-x^2-3y^2+4xy+c_3$)

3. $3s+rt-s^2=2$

(Ans : $z=xy+\beta[\phi''(\alpha)-\psi'(\beta)-\phi(\alpha)+\psi(\beta)]$

4. $s^2-rt=a^2$.

(Ans : $2ax=\beta-\alpha$, $2ay=\phi'(\alpha)-\psi'(\beta)$, $2az=-2a^2xy+\beta[\phi'(\alpha)-\psi'(\beta)]$
$-\phi(\alpha)+\psi(\beta)]$

5. Solve $2r+te^x-(rt-s^2)=2e^x$

(Ans : $z=e^x+y^2+nx+\phi(y+mx)$)

6. Solve $2s+(rt-s^2)=1$

(Ans : $z=xy-nx+\phi(y+mx)$)

## <u>14.6 Summary</u>

The non-uniform, non -linear partial differential equations $Rr+Ss+Tt+U(rt-s^2)=V$, is solved by applying the Monges method through a classification into 3 types. In the process of obtaining the solution, we derived the Monges subsidiary equations and obtained the intermediate integrals. When two values of $\lambda$ are equal, we got only one intermediate integral, which together with one of the integrals, gave the values of p and q suitable to solve dz = p dx + q dy. If suitable values of p and q could not be obtained from the two intermediate integrals, for integration in dz = p dx + q dy, we take one of the intermediate integrals and one of the integrals from $u_2=a_2$, $v_2=b_2$. Substituting the values of p,q in dz = p dx + q dy and integrating, we get the solution. An integral of more general form is obtained by assuming the arbitrary function occuring in intermediate integral to be linear and integrating by Lagranges method.

## 14.7 Model Examination questions :-

1. Solve $z(1+q^2)r - 2pqzs + z(1+p^2)t + z^2(s^2-rt) + 1 + p^2 + q^2 = 0$

(Ans : $z^2 = 2c_1x + 2c_2y - x^2 - y^2 + c_3$)

2. Solve $5r + 6s + 3t + 2(rt - x^2) + 3 = 0$

(Ans : $4z = 6xy - 3x^2 - 5y^2 - 2c_1x - 2c_2y + c_3$)

3. Solve $2pr + 2qt - 4pq(rt - s^2) = 1$

(Ans : $3z = 3c_2 + 2(x+c_1)^{\frac{3}{2}} + 2(y+c_2)^{\frac{3}{2}}$

4. Solve $7r - 8s - 3t + rt - s^2 = 36$

(Ans : $2x = \beta - \alpha$, $2y = \phi'(\alpha) - \psi'(\beta)$, $2z = 3x^2 - 7y^2 - 10xy + \beta[\phi'(\alpha) - \psi'(\beta)]$ $+ \psi(\beta) - \phi'(\alpha)$ |

5. Solve $qr + (p+x)s + yt + y(rt - s^2) + q = 0$

(Ans : $z = cx - \frac{1}{2}x^2 + F(\frac{c}{y}) + G(c)$)

6. Solve $5r + 6s + 3t + 2(rt - s^2) + 3 = 0$

(Ans : $4z = 6xy - 3x^2 - 5y^2 - 2nx + \phi(y + mx)$ )

7. Solve $ar + bs + ct + d(rt - s^2) = h$, a,b,c,d and h are constants

(Ans : $dz = -\frac{1}{2}cx^2 - \frac{1}{2}ay^2 + m_2xy + c_3x + F\{(m_2 - m_1)y + c_3\} + k$

8. Solve $z(1+q^2)r - 2pqzs + z(1+p^2)t + z^2(rt - s^2) + 1 + q^2 = 0$

(Ans : $z^2 + x^2 + y^2 - 2nx = \phi(y + mx)$)

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Unit - 3 **ANALYTIC FUNCTIONS**

Lesson - 15

# COMPLEX NUMBER SYSTEM

### 15.0 OBJECTIVE OF THE LESSON

After going through this lesson, one should be able to (i) observe that the complex number system is a field, that cannot be ordered (ii) note that the Real number system can be regarded as a subfield of the complex number system (iii) calculate the $n^{\text{th}}$ roots of a non-zero complex number (iv) describe lines and half planes in the complex plane and (v) explain the extended complex plane and the stereographic projection.

### 15.1 INTRODUCTION :

The complex number system $(\mathbb{C})$ is introduced and suitable addition and multiplication are defined on $\mathbb{C}$ to get a field. Absolute value of a complex number is defined and that enables to define a metric on $\mathbb{C}$. Nonzero complex numbers are represented in polar form and that fecilitation to obtain $n^{\text{th}}$ roots of a non-zero complex number. An element, denoted by $\infty$, is adjoined to $\mathbb{C}$ to obtain extended complex number system $\mathbb{C}_\infty$. Finally stereographic projection is discussed and that results in the identification of $\mathbb{C}_\infty$ with the unit sphere in $\mathbb{R}^3$ ( the three dimensional real space).

$\mathbb{R}$ denotes the real number system and for any positive integer $n$, $\mathbb{R}^n$ denotes the set of ordered $n$-tuples of real numbers.

### 15.2 REAL NUMBERS :

The set of all real numbers is denoted by $\mathbb{R}$. We know that $(\mathbb{R}, +, \cdot)$, where + is the usual addition and $\cdot$ is the usual multiplication of reals, is a complete ordered field.

Now, we study the complex number system.

### 15.3 THE FIELD OF COMPLEX NUMBERS :

An ordered pair $(x, y)$, where $x, y$ are real numbers, is called a complex number.

The set of all complex numbers is denoted by $\mathbb{C}$.

We define, binary operations $\oplus$ and $\odot$ on $\mathbb{C}$ as follows.

For any $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$, $(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$(x_1, y_1) \odot (x_2, y_2) = (x_1 \cdot x_2 - y_1 \cdot y_2, x_1 \cdot y_2 + x_2 \cdot y_1)$

Where + and • have their usual meanings.

When there is no confusion, we write $x_1 x_2$ etc.. instead of $x_1 \cdot x_2$ etc. Also, we write + and in stead of $\oplus$ and $\odot$ ready and one has to understand the operation properly.

It is routine to observe that $(\mathbb{C}, \oplus, \odot)$ is a complete field $(0, 0)$ and $(1, 0)$ are the additive and multiplicative identities respectively, in $\mathbb{C}$.

We identify the real number $x$ with the complex number $(x, 0)$. With this identification, we can regard $\mathbb{R}$ as a subset of $\mathbb{C}$.

Denoting $(0, 1)$ by $i$, we observe that any complex number $(x, y)$ can be written as,

$$(x, y) = (x, 0) \oplus (0, y) = (x, 0) \oplus \{(0, 1) \odot (y, 0)\}$$

$$= (x, 0) \oplus \{i \odot (y, 0)\} = x \oplus (i \odot y)$$

$$\simeq x + iy \text{ (with the convention).}$$

We observe that $i^2 = i \odot i = (0, 1) \odot (0, 1)$

$$= (-1, 0) \simeq -1$$

Hence the equation $z^2 + 1 = 0$ has a solution in $\mathbb{C}$ but not in $\mathbb{R}$.

If $z = x + iy \in \mathbb{C}$ then $x$ is called the real part of $z$ and $y$ is called the imaginary part of $z$ and are denoted by Re z and Im z respectively.

If $z = x + iy \in \mathbb{C}$ then $x - iy$ is called the (complex) conjugate of $z$ and is denoted by $\bar{z}$

We observe that $\overline{(\bar{z})} = z$.

If $z = x + iy \in \mathbb{C}$ then $\sqrt{x^2 + y^2}$ is called the absolute value of $z$ and is denoted by $|z|$.

We observe that $|z| = |-z| = |\bar{z}| = |-\bar{z}|$,

$$|z|^2 = z \odot \bar{z},\ \mathrm{Re}\ z \le |z| \text{ and } \mathrm{Im}\ z \le |z|.$$

Further $\mathrm{Re}\,z = \mathrm{Re}\ \bar{z} = (z+\bar{z})/2$ and

$$-\mathrm{Im}\,\bar{z} = \mathrm{Im}\ z = (z-\bar{z})/2i$$

$$|z| = 0 \Leftrightarrow z = 0 \Leftrightarrow \mathrm{Re}\,z = 0 = Im\ z.$$

$|z| \ge 0$ for every $z \in \mathbb{C}$

If $z = x+iy \ne 0$ then $\dfrac{1}{z} = \dfrac{x-iy}{x^2+y^2} = \dfrac{\bar{z}}{|z|^2}$

## 15.3 OBSERVATIONS :

The following can easily be verified. For any $z_1, z_2 \in \mathbb{C}$,

(i) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2,\ \overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$.

(ii) $|z_1 z_2| = |z_1|\,|z_2|;\ |z_1/z_2| = |z_1|/|z_2| \quad (z_2 \ne 0)$.

(iii) $z_1 = \bar{z}_1 \Leftrightarrow \mathrm{Im}\ z_1 = 0 \Leftrightarrow z_1$ is real.

(iv) $|z_1 + z_2| \le |z_1| + |z_2|,\ \big||z_1| - |z_2|\big| \le |z_1 - z_2|$

(v) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$

## 15.4 POLAR REPRESENTATION :

Let $0 \ne z = x+iy \in \mathbb{C}$. This $z$ has polar coordinates $(r, \theta)$ where $x = r\cos\theta,\ y = r\sin\theta$. So $r = |z|$ and $\theta$ is the angle between the positive real axis and the line segment from 0 to $z$. '$\theta$' is called an argument (amplitude) of $z$. We observe that if '$\theta$' is an argument of $z$, then for any integer $k$, $(\theta + 2\pi x)$ is also an argument of $z$.

We denote cis $\theta = \cos\theta + i\sin\theta$

**15.4.1 OBSERVATION :**

If $z_1, \cdots\cdots, z_n$ are non zero complex numbers and $z_j = r_j$ cis $\theta$ $(j=1, \cdots\cdots, n)$ then $z_1 z_2 \cdots\cdots z_n = (r_1 r_2 \cdots r_n)$ cis $(\theta_1 + \theta_2 + \cdots\cdots\cdots + \theta_n)$.

Hence $|z_1\ z_2 \cdots\cdots z_n| = |z_1|\,|z_2| \cdots\cdots |z_n|$ and an argument of $(z_1 z_2 \cdots\cdots z_n)$ is $\sum_{j=1}^{n} \arg z_j$.

Taking $z_1 = z_2 = \cdots\cdots\cdots = z_n = z = r \cos\theta$, we get that

$z^n = r^n \cos n\theta$ for any positive itneger $n$. Also, we observe that $z\left(r^{-1} \operatorname{cis}(-\theta)\right) = 1 \Rightarrow z^{-1} = r^{-1} \operatorname{cis}(-\theta) \Rightarrow z^{-n} = r^{-n} \operatorname{cis}(-n\theta)$. Thus $z^n = r^n \operatorname{cis} n\theta$ holds for any integer $n$.

From this, we get De Moivre's formula.

**15.4.2 $n^{\text{th}}$ ROOTS OF A NON-ZERO COMPLEX NUMBER :**

Let $0 \neq a \in \mathbb{C}$ and $n$ be a positive integer. The problem is to find complex numbers $z$ such that $z^n = a = |a| \operatorname{cis}\theta$ where $\theta$ is an $\arg(a)$. So $z^n = |a| \operatorname{cis}(2k\pi + \theta)$ where $k$ is any integer $\Rightarrow z = |a|^{1/n} \operatorname{cis}\left(\dfrac{2k\ \pi + \theta}{n}\right)$ and we observe that $\left(\dfrac{2k\pi + \theta}{n}\right)$ are distinct for $k = 0, 1, \cdots\cdots (n-1)$.

Thus $\left\{|a|^{1/n} \operatorname{cis}\left(\dfrac{2k\pi + d}{n}\right) : k = 0, 1, \cdots\cdots\cdots, (n-1)\right\}$ are the (distinct) solutions of $z^n = a$. Hence there are $n$ (distinct) $n^{\text{th}}$ roots for any non-zero complex number 'a' ($n$ being any positive integer).

**15.4.3 EXAMPLE :**

Find the solutions $z^4 = 1 + i$.

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left[\cos\left(2r\pi + \frac{\pi}{4}\right) + i\ \sin\left(2r\pi + \frac{\pi}{4}\right)\right] \quad (r \text{ being any integer})$$

$$\therefore \quad z = 2^{\frac{1}{8}}\left[\cos\frac{(8r+1)\pi}{16} + i\sin\frac{(2r+1)\pi}{16}\right] \quad (r=0,1,2,3)$$

### 15.4.4 SELF ASSESSMENT QUESTION

Find the solutions of $z^2 = -9$.

### 15.4.5 SAQ :

Find all the solutions of $z^4 + z^2 + 1 = 0$.

### 15.4.6 SAQ :

Find the square roots of $-15-8i$.

### 15.4.7 SAQ :

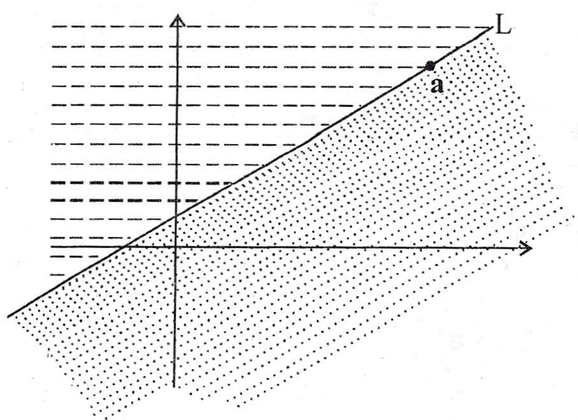
$n$ is an integer $\geq 2$ and $z = \text{cis}\, 2\pi/n$. Show tht $1+z+z^2+\cdots\cdots+z^{n-1}=0$

### 15.4.8 SAQ :

Find the fourth roots of -1.

## 15.5 LINES AND HALF PLANES IN THE COMPLEX PLANE :

Any straight line $L$ in the complex plane $\mathbb{C}$ is uniquely determined by a point on $L$ and a direction vector.




Hence if '$a$' is a point on $L$ and $b$ is its direction vector then

$$L = \{z \in \mathbb{C} : z = a + tb,\ t \in \mathbb{R}\}.$$

Since $b$ is not the null vector,

$$z \in L \Rightarrow \frac{z-a}{b} \text{ is real. Hence}$$

$L$ divides $\mathbb{C}$ into two half planes given by

$$H_a = \left\{ z \in \mathbb{C} : \operatorname{Im}\left(\frac{z-a}{b}\right) > 0 \right\}$$

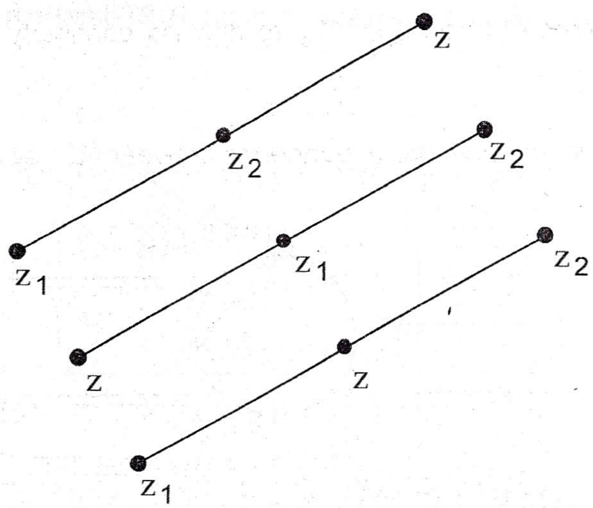
and

$$K_a = \left\{ z \in \mathbb{C} : \operatorname{Im}\left(\frac{z-a}{b} < 0\right) \right\}.$$

**15.5.1 EXAMPLE :**

Find the equation of the line passing through the complex numbers $z_1$ and $z_2$.

**Solution :** Let $L$ be the line joining $z_1$ and $z_2$ and $z$ be any point on $L$. It follows that the complex numbers $(z - z_1)$ and $(z_1 - z_2)$ have the either the same arguments or their arguments differ by $\pi$ depending on the position of $z$ with respect to $z_1$ and $z_2$.




Hence

$$\arg\left(\frac{z - z_1}{z_1 - z_2}\right) = 0 \text{ or } \pi \Rightarrow$$

$$\operatorname{Im}\left(\frac{z-z_1}{z_1-z_2}\right)=0 \Rightarrow \frac{z_1-z_2}{z_1-z_2} \text{ is real} \Rightarrow$$

$$\frac{z-z_1}{z_1-z_2}=\overline{\left(\frac{z-z_1}{z_1-z_2}\right)}$$

$$=\frac{\bar{z}-\bar{z}_1}{\bar{z}_1-\bar{z}_2} \Rightarrow$$

$$(z-z_1)(\bar{z}_1-\bar{z}_2)=(\bar{z}-\bar{z}_1)(z_1-z_2) \Rightarrow$$

$$z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)=z_1(\bar{z}_1-\bar{z}_2)-\bar{z}_1(z_1-z_2)$$

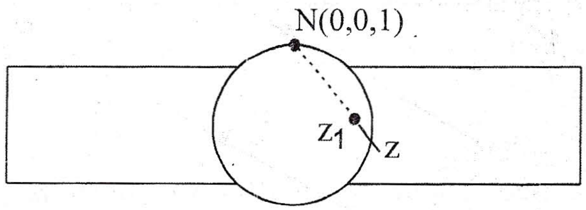
$\Rightarrow z\bar{a}-\bar{z}a=c$ where $a=z_1-z_2,\ c=2i\operatorname{Im}\{z_1(\bar{z}_1-\bar{z}_2)\}$.

## 15.6 THE EXTENDED COMPLEX PLANE AND ITS SPHERICAL REPRESENTATION

We know that $\lim\limits_{z\to 0} 1/|z|$ exists and $=\infty$. To discuss such situations, we introduce extended complex plane by introducing $\infty$ in $\mathbb{C}$. We denote $\mathbb{C}_\infty=\mathbb{C}\cup\{\infty\}$.

We introduce a distance function on $\mathbb{C}_\infty$ to discuss continuity properties of extended complex valued functions.

We exhibit one - to - one and onto correspondence between $\mathbb{C}_\infty$ and the unit sphere in $\mathbb{R}^3$.




Denote $S=\{(x_1, x_2, x_3)\in\mathbb{R}^3 : x_1^2+x_2^2+x_3^2=1\}$

Let $N=(0,0,1)$. We identity $\mathbb{C}$ with $\{(x_1,x_2,0): x_1, x_2\in\mathbb{R}\}$. Now for each $z\in\mathbb{C}$ consider the straight line in $\mathbb{R}^3$ through $z$ and $(0,0,1)$. This intersects the sphere in exactly one point say $Z(\neq 0,0,1)$. We associate to $(0,0,1)$ the point $\infty$ of $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty$ is represented as the unit spheres.

Denote $z = x+iy$ and $Z=(x_1, x_2, x_3)$ be the corresponding point on $S$.

The line through $N(0, 0, 1)$ and $z=x+iy=(x, y, 0)$ is given by $\{(1-t)x,(1-t)y, t : t \in \mathbb{R}\}$.

The coordinates of $Z$ are obtained by finding the points of intersection of this line with the unit sphere. So $t \in \mathbb{R}$ should be such that

$$(1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1 \Rightarrow (1-t)^2 |z|^2 + t^2 = 1$$

$t=1$ gives $N$. So, for $t \neq 1$,

$$|z|^2 = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t} \Rightarrow \frac{(1+t)-(1-t)}{(1+t)-(1-t)} = \frac{|z|^2-1}{|z|^2+1}$$

$$\Rightarrow t = \left(|z|^2-1\right)/\left(|z|^2+1\right)$$

$$\Rightarrow x_1 = (1-t)x = \frac{2}{|z|^2+1}x = \frac{z+\bar{z}}{|z|^2+1}.$$

$$x_2 = (1-t)y = \frac{2y}{|z|^2+1} = \frac{1}{i}\left\{\frac{z-\bar{z}}{|z|^2+1}\right\} = -i\left\{\frac{z-\bar{z}}{|z|^2+1}\right\},$$

$$x_3 = t = \frac{|z|^2-1}{|z|^2+1}.$$

Conversely, if the point $Z=(x_1, x_2, x_3)$ $(\neq N(0, 0, 1))$ is given then the point $z$ in the complex plane is obtained from the relation $(x_1, x_2, x_3) = ((1-t)x, (1-t)y, t)$

$$\Rightarrow t = x_3 \text{ and so } (1-t)y = x_2 \Rightarrow y = x_2/1-x_3 \text{ and } (1-t)x = x_1 \Rightarrow x = x_1/1-x_3$$

Hence $z = x+iy = \dfrac{x_1+ix_2}{1-x_3}$

Thus the correspondence between the points of $\mathbb{C}$ and the points of the unit sphere, except for the point $N$ (which we call as north pole) are obtained. We associate $\infty$.

This correspondence between the points of $\mathbb{C}_\infty$ and $S$ is called the stereographic projection.

**15.6.1 WORKED EXAMPLE :**

For the points $-1, -4-5i$ find the corresponding points on the unit sphere.

**Solution :** Denote $z=-1=x+iy \Rightarrow x=-1,\ y=0$. The corresponding point on the unit sphere is $(x_1, x_2, x_3)$. Where

$$x_1=\frac{2x}{|z|^2+1}=\frac{2(-1)}{1+1}=-1,\ x_2=\frac{2y}{|z|^2+1}=\frac{0}{1+1}=0,\ x_3=\frac{|z|^2-1}{|z|^2+1}=0$$

so the point on the unit sphere is $(-1, 0, 0)$.

Denote $z=-4-5i=x+iy \Rightarrow x=-4,\ y=-5$

The corresponding point on the unit sphere is $(x_1, x_2, x_3)$, where

$$x_1=\frac{2x}{|z|^2+1}=\frac{-8}{41+1}=\frac{-4}{21},\ y_1=\frac{2y}{|z|^2+1}=\frac{-10}{41+1}=\frac{-5}{21},$$

$$z_1=\frac{|z|^2-1}{|z|^2+1}=\frac{41-1}{41+1}=\frac{40}{42}=\frac{20}{21}$$

So, the corresponding point on the unit sphere is $\left(-\frac{4}{21}, \frac{-5}{21}, \frac{20}{21}\right)$.

**15.6.2 SAQ :**

For the point $3+2i$ in $\mathbb{C}$, find the correspoding point on the unit sphere.

**15.6.3 WORKED EXAMPLE :**

For the point $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ on the unit sphere find the corresponding point on $\mathbb{C}$.

**Solution :** Denote $X:(x_1, x_2, x_3) = \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}\right)$. The corresponding points in $\mathbb{C}$ is

$$z = \frac{x_1 + i x_2}{1 - x_3} = \frac{\frac{1}{2} - i\frac{1}{2}}{1 - \frac{1}{\sqrt{2}}} = \frac{(1-i)}{\sqrt{2}\left(\sqrt{2}-1\right)}.$$

**15.6.4 SAQ :**

For the point $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ on the unit sphere, find the corresponding point on $\mathbb{C}$.

**15.6.5 DISTANCE FUNCTION ON $\mathbb{C}_\infty$ :**

Let $z, z'$ be points in $\mathbb{C}$ and $Z = (x_1, x_2, x_3)$, $Z' = (x_1', x_2', x_3')$ be the corresponding point in $\mathbb{R}^3$. We define the distance $d(z, z')$ between $z$ and $z'$ in $\mathbb{C}_\infty$ to be the distance between the corresponding points $Z$ and $Z'$ in $\mathbb{R}^3$. i.e.

$$\left[d(z, z')\right]^2 = \left[(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\right]$$

$$= \left(x_1^2 + x_2^2 + x_3^2\right) + \left(x_1'^2 + x_2'^2 + x_3'^2\right) - 2\left(x_1 x_1' + x_2 x_2' + x_3 x_3'\right)$$

$$= 2\left[1 - \left(x_1 x_1' + x_2 x_2' + x_3 x_3'\right)\right]$$

(since $Z, Z'$, lie on the unit sphere)

$$= 2\left[1 - \frac{(z+\bar{z})}{1+|z|^2} \cdot \frac{z' + \bar{z}'}{1+|z'|^2} - (-i)^2 \frac{(z-\bar{z})}{1+|z|^2} \frac{(z' - \bar{z}')}{1+|z'|^2} - \left(\frac{|z|^2 - 1}{|z|^2 + 1}\right)\left(\frac{|z'|^2 - 1}{|z'|^2 + 1}\right)\right]$$

$$= \frac{2}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} \left\{\left(1+|z|^2\right)\left(1+|z'|^2\right) - (z+\bar{z})\left(z'+\bar{z}'\right)\right.$$

$$\left. + (z-\bar{z})\left(z'-\bar{z}'\right) - \left(|z|^2 - 1\right)\left(|z'|^2 - 1\right)\right\}$$

$$= \frac{2}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} \left\{\left(1+|z|^2+|z'|^2+|z|^2\,|z'|^2\right) - \left(z\,z' + z\,\overline{z'} + \bar{z}\,z' + \bar{z}\,\overline{z'}\right)\right.$$

$$\left. + \left(z\,z' - z\,\overline{z'} - \bar{z}\,z' + \bar{z}\,\overline{z'}\right) - \left(|z|^2\,|z'|^2 - |z|^2 - |z'|^2 + 11\right)\right.$$

$$= \frac{2}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} \left[2\left(|z|^2 + |z'|^2 - z\,\overline{z'} - \bar{z}\,z'\right)\right]$$

$$= \frac{4}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} \left[\,z\,\bar{z} + z'\,\overline{z'} - z\,\overline{z'} - \bar{z}\,z'\right]$$

$$= \frac{4}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} \left[\left(z - z'\right)\left(\bar{z} - \overline{z'}\right)\right]$$

$$= \frac{4}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} \left[\left(z - z'\right)\left(\overline{z - z'}\right)\right]$$

$$= \frac{4}{\left(1+|z|^2\right)\left(1+|z'|^2\right)} |z - z'|^2$$

So $$d(z, z') = \frac{2|z - z'|}{\left[\left(1+|z|^2\right)\left(1+|z'|^2\right)\right]^{1/2}}$$

### 15.6.6 OBSERVATION :

For $z' \neq 0$

$$d(z, z') = \frac{2|z - z'|}{\left[\left(1+|z|^2\right)\left(1+|z'|^2\right)\right]^{1/2}} = \frac{2\left|z/z' - 1\right|}{\left(1+|z|^2\right)^{1/2}\left(\dfrac{1}{|z'|^2} + 1\right)^{1/2}}$$

$$\Rightarrow \frac{2|0-1|}{\left(1+|z|^2\right)^{1/2}(0+1)} = \frac{2}{\left(1+|z|^2\right)^{1/2}} \quad \text{as } |z'| \to \infty$$

$$\Rightarrow d(z,\infty) = \frac{2}{\left(1+|z|^2\right)^{1/2}}$$

## 15.7 MODEL EXAMINATION QUESTIONS :

**15.7.1 :** For any complex numbers $z_1, z_2$ show that

(i) $|z_1 + z_2| \le |z_1| + |z_2|$

(ii) $\big||z_1| - |z_2|\big| \le |z_1 - z_2| \le |z_1| + |z_2|$

**15.7.2 :** $z_1 = 2+3i$ and $z_2 = 3-2i$; evaluate

$$\left|\frac{2z_1 + z_2 + 4 - 2i}{z_1 - 2z_2 + 7 + 3i}\right|^2$$

**15.7.3 :** Prove that

(i) $|z_1 \, z_2| \ge (z_1 \bar{z}_2 + \bar{z}_1 z_2)/2$

(ii) $|z_1|^2 + |z_2|^2 \ge (z_1 \bar{z}_2 + \bar{z}_1 z_2)$

**15.7.4 :** Find the square roots of $\sqrt{3} + 3i$

**15.7.5 :** Express $\sqrt{6} - \sqrt{2}\, i$ in polar coordinates.

## 15.8 SOLUTIONS TO SAQ'S AND M.E.QS (HINTS)

**15.4.4 :** $z = x+iy \Rightarrow x^2 - y^2 = -9,\ 2xy = 0 \Rightarrow x=0$ or $y=0$ observe that

$y \ne 0 \Rightarrow x = 0 \Rightarrow y = \pm 3i \Rightarrow z = \pm 3i$

**15.4.5 :** Roots of $z^4 + z^2 + 1 = 0$ are the roots of $z^6 - 1 = 0$ omitting 1 and -1; $z^6 = 1 \Leftrightarrow z =$ cis $k\pi/3$ $(k = 0, 1, \cdots\cdots, 5)$ $k = 0 \Rightarrow z = 1$ and $k = 3 \Rightarrow z = -1$. So the roots are

(i) cis $\pi/3$ (i) cis $2\pi/3$ (i) cis $\frac{4\pi}{3} = \text{cis}\left(-\frac{2\pi}{3}\right)$, (i) cis $\frac{5\pi}{3} = \text{cis}\left(-\pi/3\right)$

**15.4.6 :** $z = x + iy$ and $z^2 = -15 - 8i \Rightarrow x^2 - y^2 = -15$ and $2xy = -8 \Rightarrow y = -\frac{4}{x}$.

So $x^2 + \frac{16}{x^2} = -15 \Rightarrow x^4 + 15x^2 - 16 = 0$

$$\Rightarrow \left(x^2+16\right)\left(x^2-1\right) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow y = \mp 4$$

so roots are $1 - 4i,\ -1 + 4i$

**15.4.7 :** $z \neq 1 \Rightarrow (z-1) \neq 0,\ z^n - 1 = (Cis\ 2\pi) - 1 = 0$. So

$$0 = \left(z^n - 1\right) = (z-1)\left(z^{n-1} + z^{n-2} + \cdots\cdots + z + 1\right)$$

$$\Rightarrow 1 + z + z^2 + \cdots\cdots + z^{n-1} = 0$$

**15.4.8 :** $z^4 = -1 = \text{cis}\,\pi = \text{cis}(2r+1)\pi \qquad (r = 0,1,2,3)$

$$\Rightarrow z = Cis\,\frac{(2r+1)\pi}{4} \quad (r = 0,1,2,3)$$

**15.6.2 :** $z = 3 + 2i \Rightarrow x = 3,\ y = 2.\ \ |z|^2 = 9 + 4 = 13.$

$$x_1 = \frac{6}{13+1} = \frac{3}{7},\ y_1 = \frac{4}{14} = \frac{2}{7},\ z_1 = \frac{(13-1)}{(13+1)} = \frac{6}{7}$$

So the point is $\left(\frac{3}{7}, \frac{2}{7}, \frac{6}{7}\right)$

**15.6.4 :** $z = \dfrac{(x_1 + ix_2)}{1 - x_3} = \dfrac{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}{1 + \frac{1}{2}} = \dfrac{2(1-i)}{3\sqrt{2}} = \dfrac{\sqrt{2}}{3}(1-i)$

**15.7.1 : (i)** $|z_1 + z_2|^2 = (z_1 + z_2)\left(\bar{z}_1 + \bar{z}_2\right)$

$$= |z_1|^2 + |z_2|^2 + \left(z_1\,\bar{z}_2 + \overline{\bar{z}_1\,z_2}\right)$$

$$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}\left(z_1\,\bar{z}_2\right)$$

$$\le |z_1|^2 + |z_2|^2 + 2|z_1\,\bar{z}_2|$$

$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1 + z_2| \le |z_1| + |z_2|$$

**(ii)** $|z_1| = |z_1 - z_2 + z_2|$

$$\le |z_1 - z_2| + |z_2|$$

$$\Rightarrow |z_1| - |z_2| \le |z_1 - z_2|$$

Further $-\{|z_1| - |z_2|\} = |z_2 - z_1| = |z_1 - z_2|$

So $\max\left[|z_1| - |z_2|, -\{|z_1| - |z_2|\}\right] \le |z_1 - z_2|$

$$\Rightarrow \big||z_1| - |z_2|\big| \le |z_1 - z_2|$$

**15.7.2 :** $\left|\dfrac{2z_1 + z_2 + 4 - 2i}{z_1 - 2z_2 + 7 + 3i}\right|^2 = \left|\dfrac{2(2+3i) + (3-2i) + 4 - 2i}{(2+3i) - 2(3-2i) + 7 + 3i}\right|$

$$= \left|\frac{11 + 2i}{3 + 10i}\right|^2 = \frac{|11+2i|^2}{|3+10i|^2} = \frac{|2|+4}{9+100} = \frac{125}{109}$$

**15.7.3 :** $z_1\,\bar{z}_2 + \bar{z}_1\,z_2 \le 2\,\mathrm{Re}(z_1\,\bar{z}_2) \le 2|z_1||z_2| \le |z_1|^2 + |z_2|^2$

**15.7.4 :** $z^2 = \sqrt{3} + 3i$ and $z = x + iy \Rightarrow x^2 - y^2 = \sqrt{3}$ and $2xy = 3 \Rightarrow y = 3/2x$

$$\Rightarrow x^2 - 9/4x^2 = \sqrt{3} \Rightarrow x^4 - \sqrt{3}x^2 - 9/4 = 0$$

$$\Rightarrow x^2 = \frac{\sqrt{3} \pm \sqrt{3+9}}{2} \Rightarrow x^2 = \frac{3\sqrt{3}}{2} \Rightarrow x = \pm\frac{3^{3/4}}{2^{1/2}}$$

So roots are $\pm \dfrac{3^{1/4}}{2^{1/2}}\left(\sqrt{3}+1\right)$

**15.7.5 :** $\sqrt{6}-\sqrt{2}i = \sqrt{8}\ (\cos\theta + i\ \sin\theta)$

$$\Rightarrow \cos\theta = \sqrt{3}/2,\ \sin\theta = -1/2,\ \theta = -\pi/6$$

$$\therefore \sqrt{6}-\sqrt{2}i = \sqrt{8}\left[\cos\left(2k\pi - \frac{\pi}{6}\right) + i\ \sin\left(2k\pi - \frac{\pi}{6}\right)\right]$$

k being any integer.

**REFERENCE BOOKS :**


1. J. B. Connay : Functions of One Complex Variable - Second Edition - Springer International Student Edition.
2. Ruel V. Churchil; Jamesward Brown : Complex Variables and Applications - Mc-Graw Hill International Editions - Fifth Edition.



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Lesson - 16

# POWER SERIES

## 16.0 OBJECTIVE OF THIS LESSON

After going through this lesson, one should be able to (i) define a power series about a point in $\mathbb{C}$ (ii) define the radius of convergence of a power series about a point (iii) calculate the radius of convergence of a power series and (iv) derive the formula for the radius of convergence of a power series (about a point in $\mathbb{C}$).

## 16.1 INTRODUCTION :

Power series is an important topic in complex Analysis. We observe that to each power series, a circle is associated, called the circle of convergence, such that the given power series converges absolutely inside the circle of convergence, does not converge outside the circle. On the circle, the behaviour of the power siries is zig zag. There are power series such that (i) the series converges at every point on the circle of convergence (ii) converges at no point on the circle of convergence and (iii) converges at some points and does not converge at some other points on the circle of convergence. This concept is useful to introduce 'Analytic functions'.

## 16.2 CONCEPTS OF CONVERGENCE ETC.

We first consider the following elementary concepts :

**16.2.1 DEFINITIONS :** A sequence $\{z_n; n=1, 2, \cdots\cdots, \text{ and } O\}$ of complex numbers is said to converge to a complex number z if and only if (iff) to each $\in > 0$ there exists a positive integer $N$ (depending on $\in$) such that $|z_n - z| < \in$ for all $n \geq N$. (It can be shown that such a $z$ is unique, if exists).

We say that $z$ is the limit of the sequence $\{z_n\}$ and we write this as $\lim\limits_{n\to\infty} z_n = z$ (we read this as $\lim\limits_{n\to\infty} z_n$ exists and $= z$).

$\{z_n;\ n=1, 2, \cdots\cdots\}$ be a sequence of complex numbers and $s_n = z_1 + \cdots\cdots + z_n$ for $n = 1, 2, \cdots\cdots$ If the sequence $\{s_n\}$ (of complex numbers) converges to (a complex number) $z$ and we write this as $\sum\limits_{n=1}^{\infty} z_n = z$ ($s_n$ is called the $n^{\text{th}}$ partial sum of the series $\sum\limits_{n=1}^{\infty} z_n$ )

(we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{r=1}^{n} a_r = z$).

A series $\sum_{n=1}^{\infty} z_n$, of complex numbers, is said to converge absolutely iff $\sum_{n=1}^{\infty} |z_n|$ converges (in $\mathbb{R}$ or to some real number; in fact to a non-negative real).

**16.2.2 THEOREM :**

Absolute convergence implies convergence. (i.e. if $\sum_{n=1}^{\infty} z_n$ converges absolutely then $\sum_{n=1}^{\infty} z_n$ converges; $z_n$ s being complex numbers).

**Proof :** Let $\sum_{n=1}^{\infty} z_n$ , $z_n$ being complex numbers, converge absolutely. Hence, by definition, $\sum_{n=1}^{\infty} |z_n|$ converges (in $\mathbb{R}$). Let $\sum_{n=1}^{\infty} |z_n| = \lambda$ (say). Now, to each $\in > 0$ there corresponds a positive integer $N$ such that

$$\left|\sum_{r=1}^{n} |z_r| - \lambda\right| < \in \text{ whenever } n \geq N$$

i.e. $$\left|\sum_{r=1}^{n} |z_r| - \sum_{r=1}^{\infty} |z_r|\right| < \in \text{ whenever } n \geq N.$$

$$\Rightarrow \left|\sum_{r=n+1}^{\infty} |z_r|\right| = \sum_{r=n+1}^{\infty} |z_r| < \in \text{ whenever } n \geq N \text{ --------- (i)}$$

Denote $s_n = z_1 + \cdots\cdots + z_n$ ($n$ being any +ve integer).

For $m > n \geq N$, consider

$$|s_m - s_n| = \left|\sum_{r=1}^{m} z_r - \sum_{r=1}^{n} z_r\right| = \left|\sum_{r=n+1}^{m} z_r\right|$$

$$\leq \sum_{r=n+1}^{m} |z_r| \leq \sum_{r=n+1}^{\infty} |z_r| < \in \text{ (by (i))}$$

Hence $\{s_n\}$ is a Cauchy sequence in the complete metric space $\mathbb{C}$ with the usual metric).

Hence $\{s_n\}$ is a convergent sequence (in $\mathbb{C}$). So there exists a $z$ in $\mathbb{C}$ such that $s_n \to z$ as $n \to \infty$; this implies $\sum_{n=1}^{\infty} z_n$ converges and has the sum $z$.

**16.2.3 REMARK :**

The converse of 16.2.2 is false in view of $z_n = (-1)^{n+1}/n$ ($n$ being +ve integer). From Real Analysis, we have $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges; but $\sum_{n=1}^{\infty} \left|(-1)^{n+1}/n\right| = \sum_{n=1}^{\infty} 1/n$ diverges.

For, further development we recollect the following concepts of Real Analysis.

**16.2.4 DEFINITIONS :**

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence real numbers. Then

$$\underline{\lim}_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \left[\sup\{a_n, a_{n+1}, \cdots\cdots\}\right]$$

and

$$\overline{\lim}_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left[\inf\{a_n, a_{n+1}, \cdots\cdots\}\right]$$

(If $b_n = \sup\{a_n, a_{n+1}, \cdots\}$ for $n = 1, 2, \cdots$ then $\{b_n\}$ is a decreasing sequences of reals. If it is bounded below then $\lim_{n\to\infty} b_n = g.l.b$ of $\{b_n\}$; otherwise this is $-\infty$; Thus $\underline{\lim}\ a_n$ exists. So is the case with $\overline{\lim}\ a_n$).

**16.3 POWER SERIES :**

We now study about power series in $\mathbb{C}$.

**16.3.1 DEFINITION :** Let $\{a_n; n = 0, 1, 2, \cdots\}$ and $z_0$ be complex numbers. A series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ $(z \in \mathbb{C})$ is called a (complex) power series about $z_0$.

**16.3.2 WORKED EXAMPLE :** Show that the power series $\sum_{n=0}^{\infty} z^n$ $(z \in \mathbb{C})$ about '0' (origin) converges iff $|z|<1$.

(This series is known as geometric series)

**Solution :** Let $s_n$ be the $(n+1)^{\text{th}}$ partial sum of the series $\sum_{n=0}^{\infty} z^n \ (z \in \mathbb{C})$. So

$s_n = 1+z+z^2+\cdots+z^n = \left(1-z^{n+1}\right)/(1-z)$ for $z \neq 1$.

We know that, if $|z|<1,\ z^{n+1} \to 0$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} s_n = 1/(1-z)$$

Hence $\sum_{n=0}^{\infty} z^n$ converges to $1/(1-z)$ if $|z|<1$.

If $|z|>1,\ |z|^{n+1} \to \infty$ as $n \to \infty \Rightarrow z^{n+1} \not\to 0$ as $n \to \infty$ and if $|z|=1$, $|z|^{n+1}=1$ and so $z^{n+1} \not\to 0$ as $n \to \infty$.

Hence $\sum_{n=0}^{\infty} z^n$ does not converge if $|z| \geq 1$.

So $\sum_{n=0}^{\infty} z^n$ converges $\left(\text{to } \dfrac{1}{1-z}\right)$ iff $|z|<1$.

**16.3.3 THEOREM :** Consider the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where $a_n$'s and $z_0$ are complex numbers and $z \in \mathbb{C}$. Denote $\Lambda = \limsup_{n\to\infty} |a_n|^{1/n}$ and define

$$R = \begin{cases} 1/\Lambda & \text{if } 0<\Lambda<+\infty \\ 0 & \text{if } \Lambda=+\infty \\ +\infty & \text{if } \Lambda = 0 \end{cases}$$

Then, we have the following :

(a) The power series converges absolutely in

$$B(z_0;\ R) = \{z : |z-z_0| < R\};$$

$(=\mathbb{C} \text{ if } R=+\infty \text{ and } \phi \text{ if } R=0)$

(b) If $0 \le R < +\infty$, the series does not converge for $|z-z_0| > R$ (infact the terms of the series become unbounded and hence the series diverges)

(c) If $r$ is such that $0 < r < R (\Rightarrow R > 0)$ then the series converges uniformly on $\bar{B}(z_0, r) = \{z \in \mathbb{C} : |z-z_0| \le r\}$

Further $R$ is unique satisfying (a) and (b).

**Proof :** Under the given conditions, if $R=0$, Clearly $B(z_0, R) = \phi$. Let $R > 0$. Select $z$ such that $|z-z_0| < R$, choose a real $r$ with $|z-z_0| < r < R$ (such a selection is possible). Since $\frac{1}{r} > \frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$, there exists a positive integer $N$ such that $|a_n|^{1/n} < 1/r$ for $n \ge N$. Since $|z-z_0|/r < 1$, we have that $\sum (|z-z_0|/r)^n$ converges. Hence, by (i), follows that $\sum |a_n (z-z_0)^n|$ converges $\Rightarrow \sum a_n (z-z_0)^n$ converges absolutely whenever $|z-z_0| < r$ and this is true for all $r < R \Rightarrow \sum a_n (z-z_0)^n$ converges absolutely for all $z \in B(z_0; R)$. This proves (a).

Let $0 \le R < +\infty$ and $z \in \mathbb{C}$ be such that $|z-z_0| < R$ select real $r$ such that $|z-z_0| > r > R \Rightarrow \frac{1}{r} < \frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$ (with the convention $1/R = +\infty$ when $R=0$) $\Rightarrow$ $|a_n|^{1/n} > \frac{1}{r}$ for infinitily many $n$; $\Rightarrow |a_n (z-z_0)^n| > (|z-z_0|r)^n$ for infinitely many $n$.

Since $|z-z_0|/r > 1$, $|a_n (z-z_0)^n| \to +\infty$ as $n \to \infty$

Hence the terms of the series become unbounded and so $\sum a_n (z-z_0)^n$ diverges for all $z \in \mathbb{C}$ with $|z-z_0| > r > R$. This is true for all $r > R \Rightarrow \sum a_n (z-z_0)^n$ does not converge (in fact diverge) for all $z \in \mathbb{C}$ satisfying $|z-z_0| > R$.

This proves (b).

Let $R > 0$ and $r$ be such that $0 < r < R$. Let $\rho$ be such that $0 < \rho < R$. As in the proof of (a) there exists a positive integer, say $N_0$, such that

$$|a_n| < \frac{1}{\rho^n} \text{ for all } n \geq \mathbb{N}_0$$

Let $z \in \mathbb{C}$ be such that $|z - z_0| \leq r$

$$|a_n (z - z_0)^n| \leq \left(\frac{|z - z_0|}{\rho}\right)^n \leq \left(\frac{r}{\rho}\right)^n \text{ for } n \geq N_0 \qquad \text{(by (ii))}$$

Since $r/\rho < 1$, by Weierstrass M-test follows that the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges uniformly on $\overline{B}(z_0, r) = \{z \in \mathbb{C} / |z - z_0| \leq r\}$. This proves (c).

Since $\lim \sup$ is unique follows that such $R$ is unique.

This completes the proof of the Theorem.

**16.3.4 DEFINITION :** Given the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $a_n^s$ and $z_0$ are (fixed) complex number and $z \in \mathbb{C}$. Denote $R = \dfrac{1}{\lim\limits_{n\to\infty} \sup |a_n|^{\frac{1}{n}}}$ (in the extended real number system)

(that mean we take it as $+\infty$ or 0 according as $\lim\limits_{n\to\infty}\sup |a_n|^{1/n}$ is 0 or $+\infty$). Then $R$ is called the radius of convergence of the power series and the circh $\{z \in \mathbb{C} : |z - z_0| = R\}$ is called the (i) circle of convergence of the power series.

**16.3.5 REMARK :** In view of 16.3.4, the theorem 16.3.3 can be re-stated as follows. Every power series converges absolutely inside its circle of convergence, does not converge outside the circle of convergence but converges uniformly on the closed discs inside the circular disc of convergence (The behaviour of the series on the circle of convergence cannot be specified) (see example given under)

We, now give an elegant formula for the radius of convergence of a particular class of power series.

**16.3.6 THEOREM :** Consider the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ $(z \in \mathbb{C})$, $a_n{}^s$ and $z_0$ are (fixed) complex numbers. If $\lim_{n\to\infty} \left| a_n / a_{n+1} \right|$ exists and is positive and finite then the radius of convergence, $R$, of the power series $= \lim_{n\to\infty} \left| a_n / a_{n+1} \right|$

**Proof :** Under the given hypothesis, dentoe $\eta = \lim_{n\to\infty} \left| a_n / a_{n+1} \right|$.

Suppose $R < \alpha$. Since $z \in \mathbb{C}$ and $r > 0$ such that

$$R < |z - z_0| < r < \eta = \lim_{n\to\infty} \left| a_n / a_{n+1} \right|$$

By definition, the exists a positive integer $N$ such that

$$\left| a_n / a_{n+1} \right| > r \text{ for } n \geq N \Rightarrow |a_n| r^n > |a_{n+1}| r^{n+1} \text{ for } n \geq N \text{ ------- (1)}$$

Denote $\lambda = |a_N| r^N$

Now $|a_{N+1}| \, r^{N+1} < |a_N| r^N = \lambda$

$|a_{N+2}| r^{N+2} < |a_{N+1}| r^{N+1} < \lambda$ and so on. (by (i))

$\Rightarrow |a_n| r^n \leq \lambda$ for $n \geq N$ ------------- (ii)

Now, $\left| a_n (z-z_0)^n \right| = |a_n| r^n \left( \frac{|z-z_0|}{r} \right)^n \leq \lambda \left( \frac{|z-z_0|}{r} \right)^n$ for $n \geq N$ (by (i))

Since $\frac{|z-z_0|}{r} < 1$ follows that $\sum_{n=0}^{\infty} \left( \frac{|z-z_0|}{r} \right)^n$ converges and hence, by comparison test, follows that $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges (absoutely) and this is a contradiction (by a known Theorem) since $|z-z_0| > R$.

Hence our supposition is false and $\Rightarrow \alpha \le R$.

Now, suppose that $\eta < R$. Select $z \in \mathbb{C}$ and $s > 0$ such that $\eta < s < |z-z_0| < R$.

In this case also there exists a +ve integer $N_0$, such that $|a_n/a_{n+1}| < s$ for $n \ge N_0$

$\Rightarrow |a_n| s^n \ge |a_{N_0}| s^{N_0}$ for all $n \ge i$

$$\Rightarrow |a_n (z-z_0)^n| = |a_n| s^n \left(\frac{|z-z_0|}{s}\right)^n$$

$$\ge \left(|a_{N_0}| s^{N_0}\right)\left(\frac{|z-z_0|}{s}\right)^n \text{ for } n \ge N_0 \text{ ------- (iii)}$$

Since $\frac{|z-z_0|}{s} > 1$ follows that $\sum_{n=0}^{\infty}\left(\frac{|z-z_0|}{s}\right)^n$ diverges; hence by comparison test, follows that (by (iii)).

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$ diverges and this contradicts the fact that $|z-z_0| < R$ (by a ) known theorem; i.e. Theorem 16.3.6)

$\Rightarrow \eta < R$ is false $\Rightarrow \eta = R$.

This completes the proof of the Theorem.

**16.3.7 EXAMPLE :**

Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{z^n}{n} (z \in \mathbb{C})$

**Solution :** In the usual rotation, $a_n = \frac{1}{n} (n = 1, 2, \cdots)$ and hence $\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1$ as $n \to \infty$. Hence the radius of convergence of the series is '1'.

**16.3.8 EXAMPLE :**

Find the radius of converges of $\sum_{n=0}^{\infty} z^n / \lfloor n \ (z \in \mathbb{C})$.

**Solution :** In the usual notation, $a_n = 1/\lfloor n \ (n = 0, \cdots)$

$$|a_n|^{1/n} = \left(\frac{1}{\lfloor n}\right)^{1/n} \to 0 \text{ as } n \to \infty.$$

Hence the radius of convergence of the series is '$+\infty$'.

(That is the series converges over $\mathbb{C}$, in fact it is $e^z$ )

**16.3.9 EXAMPLE :**

Find the radius of convergence of

(i) $\sum_{n=0}^{\infty} z^n$ (ii) $\sum_{n=1}^{\infty} z^n / n^2$

**Solution :** (1) In the usual notation $a_n = 1$ for $n = 0, \cdots$ s $\left|\frac{a_n}{a_{n+1}}\right| = 1 \to 1$ as $n \to \infty$. Hence the radius of convergence of the series is 1.

(2) In the usual notation, $a_n = 1/n^2 \ (n = 1, 2, \cdots)$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2 \to 1 \text{ as } n \to \infty.$$

Hence the radius of convergence of the powerseries is 1.

**16.3.10 EXAMPLE :**

Find the radius of convergence of $\sum_{n=1}^{\infty} n^n z^n \ (z \in \mathbb{C})$.

**Solution :** In the usual notation, $a_n = n^n \ (n = 1, 2, \cdots)$

$\Rightarrow |a_n|^{1/n} = n \to +\infty$ as $n \to \infty$. Hence, the radius of convergence of the power series is $R = 1 \Big/ \overline{\lim_{n\to\infty}} |a_n|^{1/n} = 0$ ($\Rightarrow$ The series convergence only at $z = 0$).

**16.3.11 OBSERVATIONS :**

(i) The series $\sum_{n=0}^{\infty} z^n \ (z \in \mathbb{C})$ has the circle of convergence the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. This series converges at no point on the circle of convergence; on the circle of convergence, for any $z$, $|z| = 1 \Rightarrow |z|^n \to 1$ as $n \to \infty \Rightarrow z^n \not\to 0$ as $n \to \infty$.

(ii) The series $\sum_{n=1}^{\infty} z^n/n \ (z \in \mathbb{C})$ has the circle of convergence, the unit circle. $z = -1$ lies on the circle of convergence and $\sum_{n=1}^{\infty} (-1)^n/n$ converges. $z = 1$ lies on the circle on convergence and $\sum_{n=1}^{\infty} 1/n$ diverges. Thus, on the circle of convergence, the series converges at some points and does not converge at some other points.

(iii) The series $\sum_{n=1}^{\infty} z^n/n^2 \ (z \in \mathbb{C})$ has the circle of convergence the unit circle. At every point of the circle of convergence $|z^n/n^2| = 1/n^2$ and $\sum 1/n^2$ converges. So, the given series converges (absolutely) on the circle of convergence.

**16.4 SELF ASSESSMENT PROBLEMS :**

**16.4.1 SEQ :** Test the convergence of

(i) $\sum_{n=1}^{\infty} \left( e^{in^2}/n^3 \right)$ (ii) $\sum_{n=1}^{\infty} (\cos in)/2^n$

**16.4.2 SEQ :** For what values of $z$, the following series converges :

(i) $\sum_{n=1}^{\infty} n^p z^n$ (ii) $\sum_{n=1}^{\infty} p^{n^2} z^n \ (p \neq 0)$ (iii) $\sum_{n=0}^{\infty} n! z^n$

**16.4.3 SEQ :** Find the radius of convergence of

(i) $\sum_{n=1}^{\infty} n^n z^n$ (ii) $\sum_{n=1}^{\infty} \frac{n}{2^n} z^n$ (iii) $\sum_{n=1}^{\infty} 2^n z^n$

**16.4.4 SEQ :** If $R$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$, determine the radius of convergence of

(i) $\sum_{n=0}^{\infty} \left(3^n - 1\right) a_n z^n$ (ii) $\sum_{n=0}^{\infty} \frac{a_n}{\lfloor n} z^n$ (iii) $\sum_{n=0}^{\infty} n^n a_n z^n$

## 16.5 MODEL EXAMINATION QUESTIONS :

**16.5.1 :** Define the radius of convergence of a power series. If $R$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$ exists $s = \ell$ then show that $R = \ell$.

**16.5.2 :** Prove that every power series converges absolutely inside its circle of convergence and does not converge outside the circle of convergece.

**16.5.3 :** Give examples of power series that converges (i) at every pt. on the circle of convergence (ii) at no point on the circle of convergence (iii) at some pts and not at all points on the circle of convergence.

**16.5.4 :** Find the radius of convergence of

(i) $\sum_{n=0}^{\infty} 2^{-n} z^n$ (ii) $\sum_{n=0}^{\infty} z^{\lfloor n}$

**16.5.5 :** If $R$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, find the radius of convergence of

(i) $\sum_{n=0}^{\infty} \left(sc^n - 1\right) a_n z^n$ ($sc$ being a +ve integer) (ii) $\sum_{n=1}^{\infty} \frac{a_n}{\left(\lfloor n\right)^2} z^n$

## 16.6 HINTS TO SAQ AND MEQ'S

**16.4.1 : (i)** $\left| \frac{e^{in^2}}{n^3} \right| = \frac{1}{n^3} \Rightarrow$ The series converges absolutely

**(ii)** $\cos in = \cos hn = \dfrac{e^n + e^{-n}}{2} \geq \dfrac{e^n}{2} \Rightarrow$

$\dfrac{\cos in}{2^n} \geq \dfrac{1}{2}\left(\dfrac{e}{2}\right)^n$. since $\dfrac{e}{2} \geq 1$ follows that the series does not converge (In fact the series is a real series and it diverges to $+\infty$)

**16.4.2 (i) :** $\left|n^p\right|^{1/n} = \left(n^{1/n}\right)^p \to 1$ as $n \to \infty$.

So the series converges absolutely for $|z| < 1$.

**(ii) :** $\left|p^{n^2}\right|^{1/n} = \left|p^n\right| \to 0$ or 1 or $+\infty$ according as $|p| < 1$ or 1 or >1. Hence the series converges absolutely for all $z \in \mathbb{C}$ when $|p| < 1$, converges (absolutely) for $|z| < 1$ when $|p| = 1$, converges only at $z = 0$ when $|p| > 1$.

**(iii) :** $\left|\dfrac{n!}{(n+1)!}\right| = \dfrac{1}{n+1} \to 0$ as $n \to \infty$. So the radius of convergence of the power series is 0. Hence the series converges only at $z = 0$.

**16.4.3 : (i)** $\left|n^n\right|^{1/n} = n \to +\infty$ as $n \to \infty$. So $R$ =0.

**(ii)** $\left|\dfrac{n}{2^n}\right|^{\frac{1}{n}} = \dfrac{n^{1/n}}{2} \to \dfrac{1}{2}$ as $n \to \infty$. So $R = 2$.

**(iii)** $\left|2^n\right|^{1/n} = 2 \to 2$ as $n \to \infty$. so $R = 1/2$

**16.4.4 (i) :** $\overline{\lim} \left|a_n\right|^{1/n} = 1/R$ (with the usual convention)

$\left(3^n - 1\right) \sim 3^n$ as $n \to \infty$. So $\overline{\lim} \left|\left(3^n - 1\right)a_n\right|^{1/q} = \lim\left(3^n - 1\right)^{1/n} \; \overline{\lim}\left|a_n\right|^{1/n}$

$$= \lim \left(3^n\right)^{1/n} \cdot \frac{1}{R} = \frac{3}{R}$$

So the radius of converges is $\frac{R}{3}$.

**(ii)** $\overline{\lim} \left|\frac{a_n}{\lfloor n}\right|^{1/n} = \left(\overline{\lim}\, |a_n|^{1/n}\right) \lim \left(\frac{1}{\lfloor n}\right)^{1/n}$

$$= \frac{1}{R} \lim \left(\frac{1}{\lfloor n+1} \Big/ \frac{1}{\lfloor n}\right) = \frac{1}{R} \lim \left(\frac{1}{n+1}\right) = 0.$$

The radius of convergence is $+\infty$.

**(iii)** $= \overline{\lim} \left|n^n a_n\right|^{1/n} = (\lim n)\, \overline{\lim}\, |a_n|^{1/n} = (\infty)\left(\frac{1}{R}\right) = +\infty$

So, the radius convergence of the series is '0'

**16.5.1 :** See (16.3.4) and (16.3.6)

**16.5.2 :** See (16.3.3).

**16.5.3 :** See (16.3.11)

**16.5.4 : (i)** $\left|2^{-n}\right|^{1/n} = 2^{-1} = 1/2 \Rightarrow R = 2$.

**(ii)** $a_n = \begin{cases} 1 & \text{if } n = \lfloor k \text{ for some non -ve integer k} \\ 0 & \text{other wise} \end{cases}$

Hence $\overline{\lim}\, |a_n|^{1/n} = 1 \Rightarrow R = 1$.

**16.5.5 :** $\left(k^n - 1\right) \sim k^n$ as $n \to \infty$. So

$$\overline{\lim} \left\{\left|\left(k^n - 1\right)a_n\right|\right\}^{1/n} = \lim\left(k^n - 1\right)^{1/n} \overline{\lim}\, |a_n|^{1/2}$$

$$= \lim\left(k^n\right) \cdot \frac{1}{R} = \frac{k}{R}$$

So the radius of convergence is $R/k$.

**(ii)** $$\overline{\lim}\left|\frac{a_n}{\left(\lfloor n\right)^2}\right|^{1/n} = \overline{\lim}\left|a_n\right|^{1/n} \lim\left|\frac{1}{\left(\lfloor n\right)^2}\right|^{1/n}$$

$$=\frac{1}{R}\cdot \lim\left|\frac{\left(\lfloor n\right)^2}{\left(\lfloor n+1\right)^2}\right| = \frac{1}{R}\lim\frac{1}{(n+1)^2} = 0$$

The radius of convergence is $+\infty$.

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## Lesson - 17

# ANALYTIC FUNCTIONS

### 17.0 OBJECTIVE OF THIS LESSON :

After going through this lesson one should be able to (i) define an analytic function (ii) give examples of analytic functions (iii) prove important fundamental properties of analytic functions and (iv) discuss about cauchy-Riemann equations and derive useful consequences.

### 17.1 INTRODUCTION :

In complex analysis, the theory of Analytic Functions plays a very important role to develop the subject. First, the concept of differentiability is considered and otherwords analytic functions are considered. It is observed that analyticity is a stronger condition than differentiability. Also, it is established that the truth of Cauchy-Riemann equations. Mainly a necessary condition but not a sufficient condition for a (complex) function to be analytic (at a point) with some extra condition the analyticity of the function can be obtained. Finally we observe that the real and imaginary parts of an analytic function are harmonic functions; with this we calculate either the imaginary part or the real part of an analytic function when the other is known.

### 17.2 DIFFERENTIABLE FUNCTION

#### 17.2.1 DEFINITION :

$G$ be a a nonempty, open subset of $\mathbb{C}$ and $f: G \to \mathbb{C}$. Let $z_0 \in G$. $f$ is said to be differentiable at $z_0$ iff

$$\lim_{\substack{z \to 0 \\ (z \in G - \{z_0\})}} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

(We know that the limit is unique). The limit is called the derivative of $f$ and $z_0$ and is denoted by $f'(z_0)$.

If $f$ is differentiable at every point of $G$ then we say that $f$ is differentiable on $G$.

#### 17.2.2 THEOREM :

$G$ is a nonempty subset of $\mathbb{C}$. $z_0 \in G$. $f: G \to \mathbb{C}$ is differentiable at $z_0$ then $f$ is continuous at $z_0$.

**Proof :** Under the given hypothesis, since $f$ is differentiable at $z_0$, $f'(z_0)$ exists and is finite. So, for $z \in G-\{z_0\}$, $f(z)-f(z_0) = \left\{\dfrac{f(z)-f(z_0)}{z-z_0}\right\}(z-z_0)$

$$\to f'(z_0)(0) = 0 \text{ as } z \to z_0$$

$\Rightarrow \lim\limits_{z \to z_0} f(z)$ exists $z = f(z_0)$. Thus $f$ is continuuous at $z_0$.

**17.2.3 REMARK :**

The converge of Theorem (16.3.2) is false. For, consider $f(z)=|z|$ for every $z \in \mathbb{C}$. It is trivial that $f$ is continuous on $\mathbb{C}$, in particular, it is continuous at $z_0 = 0$. But, for $z \neq 0$ and $z$ is real, say $x$, consider.

$$\frac{f(z)-f(0)}{z-0} = \frac{f(x)-f(0)}{x-0} = \frac{|x|}{x}$$

$\dfrac{|x|}{x} \to 1$ as $x \to 0+0$ and $\dfrac{|x|}{x} \to -1$ as $x \to 0-0$. So $\dfrac{f(z)-f(0)}{z-0}$ does not tend to a limit as $z \to 0$. Hence $f$ is not differentiable at '0'.

Thus differentiability at a point implies continuity at that point; but continuity at a point need not imply differentiability at that point.

**17.3 ANALYTIC FUNCTION :**

**17.3.1 DEFINITION :** $G$ is a nonempty open subset of $\mathbb{C}$ and $f: G \to \mathbb{C}$. $f$ is said to be analytic on $G$ iff $f$ is continuously differentiable on $G$ (i.e. $f'$ exists on $G$ and $f': G \to \mathbb{C}$ is continuous).

**17.3.2 NOTE :**

It will be shown that $f$ is differentiable on $G \Rightarrow f$ is infinitely differentiable on $G$ ($\Rightarrow T$ has continuity of $f'$ on $G$ is reduntant).

**17.3.3 SAQ :**

$G$ is a nonempty, open subset of $\mathbb{C}$ and $f, g: G \to \mathbb{C}$ are analytic. Then, show that,

(i) for any complex numbers $\lambda, \mu, (\lambda f + \mu g)$ is analytic on $G$ and $(\lambda f + \mu g)'(z) = \lambda f'(z) + \mu g'(z)$ for all $z \in G$.

(ii) $(f \cdot g)$ is analytic on $G$ and, for all $z \in G$, $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(iii) If $G_1$ is the set of pts in $G$ where $g \neq 0$ (observe that $G_1$ is open in $\mathbb{C}$), then $f/g$ is analytic on $G_1$ and, for any $z \in G_1$.

$$\left(f/g\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$$

**17.3.4 OBSERVATIONS :**

(i) Any constant function on a nonempty open set $G$ in $\mathbb{C}$ is analytic on $G$.

(ii) The identity function on a non-empty open set $G$ is analytic on $G$.

(iii) implies that

(iv) Any complex, polynomial function is analytic on $\mathbb{C} - A$, where $A$ is the set of zeros of the denominator.

**17.3.5 OBSERVATION :**

A complex function $f$ defined on a nonempty, open set $G \subseteq \mathbb{C}$ is differentiable at $z_0 \in G$ iff

$$\lim_{\substack{z \to z_0 \\ (z \in G - \{z_0\})}} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists } z = f'(z_0)$$

$\Rightarrow$ given $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \text{ whenever } z \in G \text{ with } |z - z_0| < \delta$$

$$\Rightarrow f(z) - f(z_0) = (z - z_0)\left[f'(z_0) + \eta(z)\right]$$

for $z \in G$ with $\eta(z) \to 0$ as $z \to z_0$.

**17.3.6 THEOREM :**

(Chai Rule) $G_1, G_2$ are nonempty, open subsets of $\mathbb{C}$; $f_1$ is analytic on $G_1$; $f_1(G_1) \subseteq G_2$; $f_2$ is analytic on $G_2$. Then $(f_2 \circ f_1)$ is analytic on $G_1$ and $(f_2 \circ f_1)'(z_0) = f_2'(f_1(z_0))\, f_1'(z_0)$ for all $z_0 \in G_1$.

**Proof :** Under the given hypothesis; let $z_0 \in G_1$. Now $f_1$ is differentiable at $z_0$. So, for all $z \in G_1$,

$$\left.\begin{array}{l} f_1(z)-f_1(z_0)=(z-z_0)\left[f_1'(z_0)+\eta(z)\right], \\ \text{where } \eta(z)\to 0 \text{ as } z\to z_0 \end{array}\right\}\to \text{(i)}$$

Since $f_2$ is differentiable at $W_0 = f_1(z_0)$ for all $W \in G_2$,

$$\left.\begin{array}{l} f(W)-f_2(W_0)=(W-W_0)\left[f_2'(W_0)+g(W)\right], \\ \text{where } g(W)\to 0 \text{ as } W\to W_0 \end{array}\right\}\to \text{(ii)}.$$

Now $z \in G_1 \Rightarrow f_1(z) \in f_1(G_1) \subseteq G_2$. Taking $W = f_1(z)$ in (ii), we get that

$$f_2(f_1(z))-f_2(f_1(z_0))=\{f_1(z)-f_1(z_0)\}\left[f_2'(W_0)+g(W)\right]$$

$$=(z-z_0)\left[f_1'(z_0)+\eta(z)\right]\left[f_2'(f_1(z_0))+g(W_1)\right]$$

$$=(z-z_0)\left[f_1'(z_0)+\eta(z)\right]\left[f_2'(f_1(z_0))+g(f_1(z))\right]$$

$$=(z-z_0)\left[f_1'(z_0)+y(z)\right]\left[f_2'(f_1(z_0))+y(f_1(z))\right] \text{------- (iii)}$$

Since $f_1$ is differentiable at $z_0$ follows that $f_1$ is continuous at $z_0$. Hence $z \to z_0 \Rightarrow f_1(z) \to f_1(z_0) = W_0$. Hence $\eta(z) \to 0$ and $g(f_1(z)) \to 0$. Hence (iii) gives

$$(f_2 \circ f_1)(z)-(f_2 \circ f_1)(z_0)=(z-z_0)\left[f_1'(z_0)f_2'(f_1(z_0))+K(z)\right],$$

where $K(z)=f_2'(f_1(z_0))\eta(z)+f_1'(z_0)g(f_1(z))+\eta(z)g(f_1(z))\to 0$ as $z \to z_0$

Hence $(f_2 \circ f_1)$ is differentiable at $z_0$ and

$$(f_2 \circ f_1)'(z_0)=f_2'(f_1(z_0))f_1'(z_0).$$

Since $f_1'$ is continuous at $z_0$ and $f_2'$ is continuous at $W_0$ follows that $(f_2 \circ f_1)'$ is continuous at $z_0$ and this is true for all $z_0 \in G_1$. Hence $(f_2 \circ f_1)$ is analytic on $G_1$.

**17.3.7 DEFINITION :**

A (complex valued) function $f$ defined on $A \subseteq \mathbb{C}$ is said to be analytic on $A$ iff $f$ is analytic on an open set $G \supseteq A$ ( $\Rightarrow f$ is analytic at a point $z_0 \in \mathbb{C}$ iff $f$ is analytic) ($\Rightarrow f$ is differentiable in a nbd of $z_0$ ).

We now prove that every power series represents an analytic function inside its circle of convergence. The converse of this result will be provide in a latter unit. This result has significance only when the radius of convergence of the power series is positive.

**17.3.8 THEOREM :**

Let the complex power sereis $\sum_{n=0}^{\infty} a_n (z-z_0)^n \ (z \in \mathbb{C})$ has radius of convergence $R > 0$.

Then

(a) for each integer $K \geq 1$, the series

$$\sum_{n=K}^{\infty} n(n-1)\cdots(n-K+1) a_K (z-z_0)^{n-K} \ (z \in \mathbb{C})$$

has the same radius of convergence $R$.

(b) The function $f$ defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, z \in B(z_0, R) = \{z \in \mathbb{C} : |z-z_0| < K\}$$

is infinitely differentiable on $B(z_0, R)$.

Further the $K^{\text{th}}$ derivative of $f$, $(K = 1, 2, \cdots)$, if

$$f^{(K)}(z) = \sum_{n=K}^{\infty} n(n-1)\cdots(n-K+1)(z-z_0)^{n-K} \ \forall z \in B(z_0, K)$$

($\Rightarrow f$ is analytic on $B(z_0, R)$ and term by term differentiation is valid)

(c) $a_n = \frac{1}{\lfloor n} f^{(n)}(z_0)$ for all $n \in \{0, 1, 2, \cdots\}$.

**Proof :** Under the given hypothesis, we prove (a) for $K = 1$ and the cases $K = 2, \cdots$ follow from the p receedings ones. Without loss of generality we can assume that $z_0 = 0$. Since $R$ is the rdius of convergence of the given power sereis, we have

$$\frac{1}{R} = \lim\sup |a_n|^{1/n} \text{ ------------- (i)}$$

Let $R'$ be the radius of convergence of the derived series

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \, (z \in \mathbb{C}). \text{ So}$$

$$\frac{1}{R'} = \lim\sup |(n+1) a_{n+1}|^{1/n}$$

$$= \lim\sup |a_{n+1}|^{1/n} \text{ ------------ (ii)}$$

(since $(n+1)^{1/n} \to 1$ as $n \to \infty$)

So by (ii), $R'$ is the radius of convergence of $\sum_{n=0}^{\infty} a_{n+1} z^n \, (z \in \mathbb{C})$ as well.

For any $z \in \mathbb{C}$,

$$z \sum_{n=0}^{\infty} a_{n+1} z^n + a_0 = \sum_{n=0}^{\infty} a_n z^n$$

So $|z| < R' \Rightarrow$

$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \le \sum_{n=0}^{\infty} |a_n z^n| \le |a_0| + \sum_{n=0}^{\infty} |a_n z^n|$$

$$= |a_0| + |z| \sum_{n=0}^{\infty} |a_{n+1} z^n| < +\infty \text{ (by (iii))}$$

Hence follows that $R' \le R$.

At $z = 0$ both the series converge. Let $0 < |z| < R$. Now

$$\sum_{n=0}^{\infty}\left|a_{n+1}z^n\right|=\frac{1}{|z|}\sum_{n=0}^{\infty}\left|a_{n+1}z^{n+1}\right|=\frac{1}{|z|}\left\{\sum_{n=0}^{\infty}\left|a_n z^n\right|-\left|a_0\right|\right\}$$

$<+\infty$ (by (i)).

Hence $R \le R'$ and so $R'=R$. This proves (a)

Now, we have, for $z \in B(0;R)$

$$f(z)=\sum_{n=0}^{\infty} a_n z^n ;$$

define $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1};\ s_n(z) = \sum_{K=0}^{n} a_K z^K$,

$$R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$$ ($n$ being any nonnegative integer).

Let $W \in B(0;R)$ and $r \in \mathbb{R}$ be such that $|W|<r<R$. Now $W \in B(0;r)$. Select $\delta_1>0$ such that $B(W;\delta_1) \subseteq B(0,r)$. (This is possible since $W$ is an interier point of $B(0;r)$).

Let $z \in (W,\delta_1)\setminus\{W\}$. Now

$$\frac{f(z)-f(W)}{z-W}-g(W)=\left[\frac{s_n(z)-s_n(W)}{z-W}-s_n'(W)\right]+\left[s_n'(W)-g(W)\right]$$

$$+\left[\frac{R_n(z)-R_n(W)}{z-W}\right] \quad \text{---------- (iii)}$$

But $$\left|\frac{R_n(z)-R_n(W)}{z-W}\right|=\left|\sum_{K=n+1}^{\infty} a_K\left(\frac{z^K-W^K}{z-W}\right)\right|$$

$$\le \sum_{K=n+1}^{\infty}\left|a_K\right|\left|\frac{z^K-W^K}{z-W}\right|$$

$$= \sum_{K=n+1}^{\infty} |a_K| \left| z^{K-1} + z^{K-2} W + \cdots\cdots + zW^{K-2} + W^{K-1} \right|$$

$$\leq \sum_{K=n+1}^{\infty} |a_K| \left\{ |z|^{K-1} + |z|^{K-2} |W| + \cdots + |z||W|^{K-2} + |W|^{K-1} \right\}$$

$$\leq \sum_{K=n+1}^{\infty} |a_K| K r^{K-1} \left( \because |z| \,\&\, |W| < r \right) \text{ -------- (iv)}$$

Since $r < R$, follows that $\sum_{K=1}^{\infty} K|a_K|r^{K-1} < \frac{\in}{3}$ wheneever $n \geq N_1$.

So (iv) $\Rightarrow$

$$\left| \frac{R_n(z) - R_n(W)}{z - W} \right| < \frac{\in}{3} \text{ for all } n \geq N_1 \text{ ---------- (v)}$$

Since $\lim_{n \to \infty} s_n'(W) = g(W)$, there exists a positive integer $N_2$ such that

$$\left| s_n'(W) - g(W) \right| < \frac{\in}{3} \text{ for all } n \geq N_2 \text{ --------- (vi)}$$

Since $s_n'(W)$ exists there exists a $\delta_2 > 0$ such that

$$\left| \frac{s_n(z) - s_n(W)}{z - W} - s_n'(W) \right| < \frac{\in}{3} \text{ for } 0 < |z - W| < \delta_2 \text{ --------- (vii)}$$

Let $N = \max\{N_1, N_2\}$ and $\delta = \min\{\delta_1, \delta_2\}$

Now for $n \geq N$ and $0 < |z - W| < \delta$,

$$(\text{i}) - (\text{vii}) \Rightarrow \left| \frac{f(z) - f(W)}{z - W} - g(W) \right| < \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in$$

$\Rightarrow f'(W) = g(W)$ and (b) follows by induction on $K$.

Clearly $f(a) = a_0$ and by (b), $f^{(K)}(a) = \lfloor K \; a_K \; (K = 1, 2, 3, \cdots)$

$\Rightarrow a_n = f^{(n)}(a)/\underline{|n}\ (n = 0, 1, \cdots)$. This proves (c) and the Theorem.

**17.3.9 COROLLARY :**

Let $R \in (0, +\infty)$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n \ (z \in \mathbb{C})$

and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $z \in B(z_0; R)$. Then $f$ is analytic on $B(z_0; R)$.

**Proof :** By Theorem 16.3.8 follows that $f$ has derivatives of all orders on $B(z_0; R)$. Hence follows that $f$ is analytic on $B(z_0; R)$.

**17.3.10 OBSERVATION :**

In the previous lesson we observed that $\sum_{n=0}^{\infty} z^n/\underline{|n}$ converges for every $z \in \mathbb{C}$. We denote the sum by $\exp(z)$. By corollary (16.3.9) follows that the exponential function is analytic on $\mathbb{C}$. Now, we consider the following.

**17.4 DEFINITION :**

A nonempty, open, connected subset of the complex plane is called a region in $\mathbb{C}$.

**17.4.1 THEOREM :**

Let $G$ be a region in $\mathbb{C}$ and $f : h \to \mathbb{C}$ be such that $f(z) = u(x, y) + i\upsilon(x, y)$ for all $z = x + iy \in G$. If $f$ is analytic on $G$ then $u$ and $\text{v}$ have first order partial derivatives at any point $(x, y)$ where $z = x + iy \in G$ and further

$$\frac{\partial u}{\partial x} = \frac{\partial \text{v}}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial \text{v}}{\partial x} \text{ at } (x, y)$$

**Proof :** Let $f, u, \text{v}$ and $G$ be as in the statement. Let $z \in G$ be such that $f$ is differentiable at $z$ So

$$f'(z) = \lim_{h \to \infty} \frac{f(z+h) - f(z)}{h} \qquad (0 \neq h \text{ and } z + h \in G)$$

Suppose $h$ is real and $z = x + iy$ . So

$$f'(z) = \lim_{h \to 0} \frac{f(x+h+iy) - f(x+iy)}{h}$$

$$= \lim_{h \to 0} \frac{\{u(x+h,y) - u(x,y)\} + i\{\upsilon(x+h,y) - \upsilon(x,y)\}}{h}$$

$$= \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h \to 0} \frac{\upsilon(x+h,y) - \upsilon(x,y)}{h}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial \upsilon}{\partial x}\right)_{(x,y)} \quad \text{---------- (i)}$$

Suppose $h$ is purely imaginary, say $h = i\,h'$ where $h'$ is real. So

$$f'(z) = \lim_{h' \to 0} \frac{f(z+h') - f(z)}{i\,h'}$$

$$= \lim_{h' \to 0} \frac{\{u(x,y+h') - u(x,y)\} + i\{\upsilon(x,y+h') - \upsilon(x,y)\}}{i\,h'}$$

$$= \lim_{h' \to 0} \frac{\upsilon(x,y+h') - \upsilon(x,y)}{h'} - i \lim_{h' \to 0} \frac{u(x,y+h') - u(x,y)}{h'}$$

$$= \left(\frac{\partial \upsilon}{\partial y} - i\frac{\partial u}{\partial y}\right)_{(x,y)} \quad \text{---------- (ii)}$$

(i) & (ii) $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial \upsilon}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial \upsilon}{\partial x} \text{ at } (x, y)$$

**17.4.2 NOTE :** The relations in Theorem (16.4.1) are called the Cauchy Riemann (C.R) equations.

**17.4.3 OBSERVATION :**

The truth of C.R. equations at a point $(x_0, y_0)$ is a necessary condition put not a sufficient condition for a function $f(z) = u(x,y) + i\,\upsilon(x,y)$ to be diferentiable at the point $z_0 = x_0 + i\,y_0$.

This we conclude by the following.

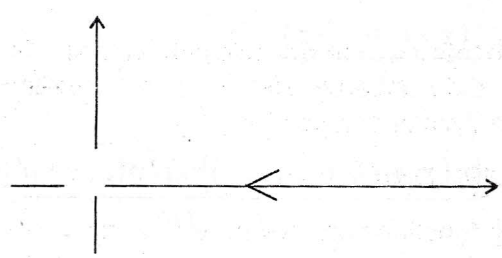
### 17.4.4 COUNTER EXAMPLE :

Let $f(z)=\sqrt{|x\,y|}$ for $\forall\, z=x+iy\in\mathbb{C}$. We now show that $f$ the C.R. equations are satisfied at $(0,\,0)$ but $f$ is not differentiable at $0=0+i0$.

Write $f=u+i\,\text{v}\Rightarrow u(x,y)=\sqrt{|x\,y|}$ and $u(x,y)=0$ for all $(x,y)\in\mathbb{R}^2$. Now for any $x\neq 0$,

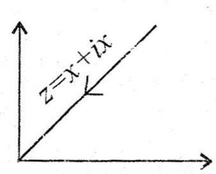
$$\frac{u(x,0)-u(\text{v},0)}{x-0}=\frac{0-0}{x}=0\Rightarrow\left(\frac{\partial u}{\partial x}\right)_{(0,0)}\ \text{exists}\ x=0.$$

III$^{rly}$ $\frac{\partial u}{\partial y}=0=\frac{\partial \upsilon}{\partial x}=\frac{\partial \upsilon}{\partial y}$ at $(\upsilon,0)$. Thus C.R. equations are satisfied at $(0,0)$.



Now, for $z\neq 0,\ z=x(\Rightarrow y=0)$

$$\frac{f(z)-f(\text{v})}{z-0}=\frac{\sqrt{|x\cdot 0|}-0}{x}=0\to 0\ \text{as}\ z=x\to 0 \quad \text{-------- (i)}$$




For $z=x+ix,\ x>0$

$$\frac{f(z)-f(u)}{z-0}=\frac{\sqrt{|x|^2}-0}{x+ix}=\frac{1}{1+i}\to\frac{1}{1+i}\neq 0\ \text{as}\ x\to 0+ \quad \text{------- (ii)}$$

(i) and (ii) $\Rightarrow f$ is not differentiable at '0'.

### 17.4.5 EXAMPLE :

Show that $f$ defined on $\mathbb{C}$ by $f(z)=|z|^2$ is differentiable only at $z=0$ (origin).

**Solution :** Given that $f(z)=|z|^2 \ \forall z=x+iy\in\mathbb{C}$. Denote $f=u+i\text{v} \Rightarrow u(x,y)=|z|^2=x^2+y^2$ and $\upsilon(x,y)=0 \ \forall (x,y)\in\mathbb{R}^2$. Now, if $(x,y)\neq(0,0)$ atlest one of $\frac{\partial u}{\partial x}=2x, \frac{\partial u}{\partial y}=2y$ differ from the corresponding $\frac{\partial u}{\partial y}=0, -\frac{\partial u}{\partial x}=0$. Hence $f$ is not differentiable at any $z\neq 0$. When $z_0=\text{v}$, consider for $z\neq 0$

$$\frac{f(z)-f(0)}{z-0}=\frac{|z|^2-0}{z-0}=\frac{z\bar{z}}{z}=\bar{z}\to 0 \text{ as } z\to 0 \Rightarrow$$

$f$ is differentiable at '0' and $f'(0)=0$.

**17.4.6 OBSERVATION :**

The above function is such that it is differentiable only at origin. So there is no neighbourhood of origin where the function is differentiable. Hence, it is an example of a function which is differentiable at a point but not analytic at that point.

**17.4.7 SELF ASSESMENT QUESTIONS :**

**17.4.7.1 :** Show that the following functions are nowhere analytic.

(i) $(a)\ f(z)=\bar{z} \quad (b)\ f(z)=z-\bar{z} \ \forall\ z\in\mathbb{C}$

(ii) $f(z)=2x+ixy^2 \forall z=x+iy\in\mathbb{C}$

(iii) $f(z)=(\exp x)(\exp -iy)=e^{x-iy}=e^{\bar{z}} \ \forall\ z=x+iy\in\mathbb{C}$

(iv) $f(z)=xy+iy \forall\ z=x+iy\in\mathbb{C}$

(v) $f(z)=e^y\cdot e^{ix} \forall z=x+iy\in\mathbb{C}$

**17.4.7.2 SAQ :** Show that $f$ defined on $\mathbb{C}$ by $f(0)=0$ and $f(z)=(\bar{z})^2/2$ for $z\neq 0$, the CR equations are satisfied at $(0,0)$ but $f$ is not differentiable at $(0,0)$.

We now prove a sufficient condition that ensures analycity of a function.

**17.4.8 THEOREM :**

$G$ is a region in $\mathbb{C}$ and $u$, v are real valued functions with continuous partial derivatiives on

$G$ (regarded as a subset of $\mathbb{R}^2$). If Cauchy-Riemann equations are satisfied on $G\left(\subseteq \mathbb{R}^2\right)$ then $f = u + i\mathrm{v}$ is analytic (differentiable) on $G(\subseteq \mathbb{C})$.

**Proof :** Let $G, u,$ v and $f$ be as in the statement of the Theorem. Let $z_0 = x_0 + iy_0 \in G$. Since $G$ is open there exists an $r > 0$ such that $B(z_0, \gamma) = \{z \in \mathbb{C} : |z - z_0| < r\} \subseteq G.$ Let $0 \neq h = s + it \in B(0; r) \Rightarrow (z_0 + h) \in G$. Now

$$u(x_0 + s,\ y_0 + t) - u(x_0,\ y_0) = \left[u(x_0 + s,\ y_0 + t) - u(x_0,\ y_0 + t)\right]$$

$$+\left[u(x_0, y_0 + t) - u(x_0, y_0)\right] \text{ ------- (i)}$$

By mean value Theorem for the derivative of a function of one variable of reals, there exist $s_1, t_1$ with $|s_1| < |s|$ and $|t_1| < |t|$ with

$$\left.\begin{array}{l} u(x_0 + s, y_0 + t) - u(x_0, y_0 + t) = s\ u_x(x_0 + s_1,\ y_0 + t) \\ \text{and} \\ u(x_0, y_0 + t) - u(x_0, y_0) = t\ u_y(x_0,\ y_0 + t_1) \end{array}\right\} \text{ -------- (ii)}$$

(where $u_x, u_y, \cdots\cdots$ have their usual meanings).

For any $(s,t) \in R^2 \setminus \{(0,0)\}$ with $s + it \in B(0,r)$ define

$$\phi(s,t) = \left[u(x_0 + s,\ y_0 + t) - u(x_0, y_0)\right] - \left[su_x(x_0, y_0) + \left(-u_y(x_0, y_0)\right)\right]$$

So, by (i) and (ii),

$$\frac{\phi(s,t)}{s+it} = \frac{s}{s+it}\left[u_x(x_0 + s_1,\ y_0 + t) - u_x(x_0, y_0)\right] + \frac{t}{s+it}\left[u_y(x_0, y_0 + t_1) - u_y(x_0, y_0)\right] \rightarrow \text{ (iii)}$$

Since $|s_1| < |s| \leq |s + it|$ and $|t_1| < |t| \leq |s + it|$ and $|t_1| < |t| \leq |s + it|$ and $u_x,\ u_y$ are continuous at $(x_0,\ y_0)$ follows that (from (iii)).

$$\lim_{s+it \to \infty} \frac{\phi(s,\ t)}{s + it} \text{ exists } s = 0 \text{ (by (ii)) --------- (iv)}$$

So,

$$u(x_0+s,\ y_0+t)-u(x_0,\ y_0)=s\,u_x(x_0,y_0)+t u_y(x_0,y_0)+\phi(s,t)$$

where $\phi$ satisfies (iv), similarly

$$u(x_0+s,\ y_0+t)-\mathrm{v}(x_0,\ y_0)=s\ u_x(x_0,y_0)+t\ u_y(x_0,\ y_0)+\psi(s,\ t) \text{ ------ (5)}$$

where $\psi$ is such that

$$\lim_{s+it}\frac{\psi(s,\ t)}{s+it} \text{ exists} = 0 \text{ -------- (vi)}$$

Now

$$\frac{f(z_0+h)-f(z_0)}{h}=\frac{\{u(x_0+s,\ y_0+t)-u(x_0,y_0)\}+i\{\upsilon(x_0+s,\ y_0+t)-\upsilon(x_0-y_0)\}}{s+it}$$

$$=\frac{su_x(x_0,y_0)+tu_y(x_0,y_0)+\phi(s,t)+i\{s\upsilon_x(x_0,y_0)+t\upsilon_y((x_0,y_0)+\upsilon(s,t))\}}{s+it}$$

$$=\frac{s\{u_x(x_0,y_0)+i\upsilon_x(x_0,\ y_0)\}+it\{\upsilon_y(x_0,\ y_0)-iu_y(x_0,y_0)\}+\phi(s,t)+i\phi(s,t)}{s+it}$$

$$=\frac{s\{u_x(x_0,\ y_0)+i\upsilon_x(x_0,y_0)\}+it\{u_x(x_0,y_0)+i\upsilon_x(x_0,\ y_0)\}+\phi(s,\ t)+i\psi(s,t)}{s+it}$$

(Since C-R equations are satisfied)

$$=u_x(x_0,y_0)+i\upsilon_x(x_0,\ y_0)+\frac{\phi(s,t)+i\psi(s,t)}{s+it}$$

Since $\dfrac{\phi(s,t)+i\psi(s,t)}{s+it}\to 0$ as $s+it\to 0$ (by (iv) and (vi)), by 16.3.6.1 of the previous lesson follows that $f$ is differentiable at $z_0$ and $f'(z_0)=u_x(x_0,y_0)+i\upsilon_x(x_0,\ y_0)$.

Since $u_x$ and $\upsilon_x$ are cotinuous at $(x_0,y_0)$ follows that $f'$ is continuous at $z_0$. This is true for all $z\in G$. Hence $f$ is analytic on $G$. This completes the proof of the Theorem.

(By C-R equations, observe that $f'(z_0)$ is also $=\upsilon_y(x_0,y_0)-iu_y(x_0,y_0)$)

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**17.4.9 COROLLARY :**

$G$ is a region in $\mathbb{C}$ and $f: G \to \mathbb{C}$ is analytic such that $f' = 0$ on $G$. Thus $f$ is $G$ constant.

**Proof :** Let $f$ and $G$ be as in the statement. Let $f = u + i\text{v}$. Now, by Theorem 16.4.8 follows that $f'(z) = u_x(x,y) + i\upsilon_x(x,y) = \upsilon_y(x,y) - iu_y(x,y)$ for every $z = x + iy \in G$. Since $f' = 0$ on $G$ follows that $u_x = u_y = \upsilon_x = \upsilon_y = 0$ on the connected set $G\left(\subseteq \mathbb{R}^2\right)$. Hence $u$ and $\upsilon$ are independent of $x$ $y$

$\Rightarrow u$ is a constant and $\upsilon$ is a constant on $G\left(\subseteq \mathbb{R}^2\right)$

$\Rightarrow f$ is a constant on $G$. This proves the result.

**17.4.10 DEFINITION :**

A complex valued function $f$ defined on $\mathbb{C}$, i.e. $f: \mathbb{C} \to \mathbb{C}$ is said to be an entire function if and only $f$ is analytic (differentiable) on $\mathbb{C}$.

**17.4.11 SELF ASSESSMENT QUESTIONS :**

Prove that the following are entire functions.

(i) $f(z) = e^{iz} \ \forall \ z \in \mathbb{C}$

(ii) $f(z) = \sin z \ \forall \ z \in \mathbb{C}$

(iii) $f(z) = 4x - y + i(4y + x) \forall z = x + iy \in \mathbb{C}$

(iv) $f(z) = \left(z^3 + z\right)e^z \forall z \in \mathbb{C}$

## 17.5 HARMONIC FUNCTIONS :

**17.5.1 DEFINITION :**

A function $\phi$ defined on a non-empty set $G \subseteq \mathbb{R}^2$ is said to be harmonic iff $\phi$ has continuous, second order partial derivatives and $\nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ on $G$ (This equation is called Laplace (potential) equation).

**17.5.2 EXAMPLE :**

$f = u + i\upsilon$ is analytic on a nonempty, open set $G \subseteq \mathbb{C}$. Then $u$ and $\upsilon$ are harmonic on $G\left(\subseteq \mathbb{R}^2\right)$.

**Solution :** Let $f = u + i\text{v}$ be analytic on $\phi \neq G$ (open) $\subseteq \mathbb{C} \Rightarrow f$ has derivatives of all orders on $G \Rightarrow u$ and $\upsilon$ have partial derivatives of all orders on $G\left(\subseteq \mathbb{R}^2\right)$. Since $f$ is analytic on $G$, $u$ and $\upsilon$ satisfy C-R equations on $G$. So $\dfrac{\partial u}{\partial x} = \dfrac{\partial \upsilon}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial \text{v}}{\partial x}$ on $G$.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \upsilon}{\partial x \partial y} - \frac{\partial^2 \upsilon}{\partial y \partial x} = \frac{\partial^2 \upsilon}{\partial x \partial y} - \frac{\partial^2 \upsilon}{\partial x \partial y} = 0 \text{ on } G.$$

Thus $u$ is harmonic on $G$. Similarly $\upsilon$ is harmonic on $G$.

This means $f$ is analytic on $G \Rightarrow Re\, f$ and $\operatorname{Im} f$ are harmonic on $G \subseteq \mathbb{R}^2$.

**17.5.3 DEFINITION :**

If $f = u + i\text{v}$ is analytic on $\phi \neq G \subseteq \mathbb{C}$ then $\upsilon$ is called a harmonic conjugate of $u$ on $G$ (Note that $-u$ is a harmonic conjugate of $\upsilon$ since if $\upsilon - iu$).

**17.5.4 THEOREM :**

Let $G$ be a nonempty open disc in $\mathbb{R}^2$ or $\mathbb{R}^2$. If $u : h \to \mathbb{R}$ is harmonic then $u$ has a harmonic conjugate on $G$.

**Proof :** Let $G = B(0; R)$ $(0 < R \leq +\infty)$ and $u : G \to \mathbb{R}$ be harmonic. Let $u : G \to \mathbb{R}$ be defined by

$$\upsilon(x, y) = \int_0^y u_x(x, t)\, dt + \phi(x) \;\forall\; (x, y) \in G.$$

(where $\phi$ is specified in due course).

Now follows that

$$u_x(x, y) = \int_0^y u_{xx}(x, t)\, dt + \phi'(x)$$

$$= - \int_0^y u_{yy}(x,t)\,dt + \phi'(x) \quad (\because \nabla^2 u = 0 \text{ on } G)$$

$$= -u_y(x,y) + u_y(x,0) + \phi'(x)$$

We select $\phi$ such that $\phi'(x) = -u_y(x,0)$. So

$$u(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(s,0)\,ds$$

( $\Rightarrow \upsilon_y = u_x$ and $\upsilon_x = -u_y$ on $G$)

**17.5.5 EXAMPLE :**

Show that $u(x,y) = e^x \cos y \ \forall \ (x,y) \in \mathbb{R}^2$ is harmonic (on $\mathbb{R}^2$). Find an analytic function $f$ whose real part is $u$.

**Solution :** Given that $u(x,y) = e^x \cos y \forall (x,y) \in \mathbb{R}^2$

So $u_x(x,y) = e^x \cos y \Rightarrow u_{xx}(x,y) = e^x \cos y$,

$u_y(x,y) = -e^x \sin y \Rightarrow u_{yy}(x,y) = -e^x \cos x$

So $\nabla^2 u = e^2 \cos y - e^x \cos y = 0$ on $\mathbb{R}^2$

A harmonic conjugate $\upsilon$ of $u$ on $\mathbb{R}^2$ is

$$\upsilon(x,y) = \int_0^y e^x \cos t \, dt - \int_0^x 0\, ds$$

$= e^x \sin y + c,$ where $c$ is a real constant.

$$\therefore f(z)\ e^x \cos y + i\left\{e^x \sin y + c\right\}$$

$$= e^x \cdot e^{iy} + ic = e^{x+iy} + c = e^z + c \forall \ z \in \mathbb{C}$$

### 17.5.6 SELF ASSESSMENT QUESTIONS :

Show that the following are harmonic and find their harmonic conjugates.

(i) $u(x,y)=2x(1-y)$ $\forall (x,y)\in \mathbb{R}^2$

(ii) $u(x,y)=2x-x^3+3xy^2$ $\forall (x,y)\in \mathbb{R}^2$

(iii) $u(x,y)=\sinh x \sin y$ $\forall (x,y)\in \mathbb{R}^2$

(iv) $u(x,y)=y/x^2+y^2$ $\forall (x,y)\in \mathbb{R}^2-\{0\}$

### 17.6 MODEL EXAMINATION QUESTIONS :

**17.6.1 :** Define an analytic function.

**17.6.2 :** State and prove chain-rule

**17.6.3 :** Show that every power series represents an analytic function inside its circle of convergence.

**17.6.4 :** State and derive C-R equations.

**17.6.5 :** Show that the truth of C-R equations is a necessary condition but not a sufficient conditin for a complex function to be analytic at a point.

**17.6.6 :** State and establish a set of sufficient conditions for a complex function to be analytic at a point.

**17.6.7 :** If $f$ is analytic on a region $G$ in $\mathbb{C}$ and $f'=0$ on $G$ show that $f$ is a constant on $G$.

**17.6.8 :** Define a harmonic function.

**17.6.9 :** If $f$ is analytic on a region $G$, show that the $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are harmonic on $G(\subseteq \mathbb{R}^2)$.

**17.6.10 :** $u$ is harmonic on a non-empty, open set $G$ in $\mathbb{R}^2$. Show that $u$ has a harmonic conjugate on $G$.

### 17.7 HINTS TO SAQS AND MEQS :

**17.3.3 :** (i) $z\in G$. Select $0\neq h\in\mathbb{C}\ni(z+h)\in G$. Now

$$\frac{(\lambda f+\mu g)(z+h)-(\lambda f+\mu g)(z)}{h}=\lambda\frac{f(z+h)-f(z)}{h}+\mu\frac{g(z+h)-g(z)}{h}$$

$$\to \lambda f'(2)+\mu g'(z)$$

(ii) $$\frac{(f.g)(z+h)-(f\cdot g)(z)}{h}=\frac{f(z+h)-f(z)}{h}\cdot g(z+h)$$

$$+\frac{g(z+h)-g(z)}{h}\ f(z)\to f'(z)g(z)+g'(z)f(z)$$

as $h\to 0$ ($\because$ differentiability $\Rightarrow$ continuity)

(iii) For any $z\in G$,

$$\left(\frac{f}{g}\right)'(z)=\left(f\cdot\frac{1}{g}\right)'(z)=f'(z)\cdot\frac{1}{g(z)}+f(z)\left[\frac{-g'(z)}{\{g(z)\}^2}\right]$$

**17.4.7.1 :**

(1) (a) $f=u+iu\Rightarrow u(x,y)=x$ and $\upsilon(x,y)=-y\,\forall(x,y)\in\mathbb{R}^2$

$\dfrac{\partial u}{\partial x}=1\neq-1=\dfrac{\partial\upsilon}{\partial y}\forall(x,y)\in\mathbb{R}^2\Rightarrow f$ is no where analytic in $\mathbb{C}$.

(i) (b) $f=u+i\,\mathrm{v};\ u(x,y)=0,\ \upsilon(x,y)=2y, u_x=0\neq 2=\mathrm{v}_y$.

(ii) $u(x,y)=2x,\ \upsilon(x,y)=xy^2.\ u_x=2=2xy=\upsilon_y\Rightarrow xy=1(i)$

$u_y=0=-y^2\Rightarrow y=0\to(\text{ii})$. (i) & (ii) C.R. equas are satisfied at no $(x,y)\in\mathbb{R}^2$.

(iii) $u(x,y)=e^x\cos y,\ \mathrm{v}(x,y)=-e^x\sin y\forall(x,y)\in\mathbb{R}^2$

$u_x=\upsilon_y\Rightarrow\cos y=0, u_y=-\upsilon_x\Rightarrow\sin y=0\Rightarrow$ C.R. equations are satisfied at no $(x,y)\in\mathbb{R}^2$.

(iv) $u_x=y=1=\upsilon_y;\ u_y=x=-0$ C - R equations are satisfied only at $(0,1)\Rightarrow f$ can be differentiable only at $0\Rightarrow f$ is now where analytic.

(v) $u(x,y)=e^y\cos x,\ \upsilon(x,y)=e^y\sin x;\ u_x=\upsilon_y\Rightarrow\sin x=0;$

$u_y=-\upsilon_x\Rightarrow\cos x=0\Rightarrow$ now where satisfied.

**17.4.7.2 :**

$u_x(0,0)=1=\upsilon_y(0,0), u_y(0,0)=0=-\text{v}_x(0,0)$. For $z\neq 0$

$$\{f(z)-f(0)\}/(z-0)=\left(\frac{\bar{z}}{z}\right)^3 \to 1 \text{ as } z=x\to 0;$$

$$\to\left(\frac{1-i}{1+i}\right)^3=(-i)^3=i \text{ as } z=x+ix\to 0$$

**17.4.11 :**

(i) $f=u+i\text{v}\Rightarrow u(x,y)=e^{-y}\cos x, \upsilon(x,y)=e^{-y}\sin x \forall (x,y)\in\mathbb{R}^2$

$u_x=-e^{-y}\sin x=\upsilon_y,\ u_y=-e^{-y}\cos x=-\upsilon_x$ and these are all continuous in any nbd of $(x,y)\Rightarrow f$ is entire

(ii) $u(x,y)=\sin x\cos xy,\ \upsilon(x,y)=\cos x\sin hy \forall (x,y)\in\mathbb{R}^2$

(iii) $u(x,y)=4x-y,\ \upsilon(x,y)=4y+x;\ u_x=4=\upsilon_y,\ u_y=-1=-\upsilon_x\Rightarrow$

(iv) $f_1(z)=z^3,\ f_2(z)=2$ and $f_3(z)=e^{-z}\forall z\in\mathbb{C}$ are entire (prove)

$\Rightarrow f=f_1+f_2+f_3$ is entire.

**17.5.6 :**

(i) $\nabla^2 u=0+0=0\Rightarrow u$ is harmonic.

$$\upsilon(x,y)=\int_0^y u_x(x,t)\,dt-\int_0^x u_y(s,0)\,ds$$

$$=\int_0^y[2(1-y)]_{(0,t)}\,dt+\int_0^x(-2x)_{(s,0)}\,ds$$

$$=\int_0^y 2(1-t)\,dt-\int_0^x 2s\,ds=2y-y^2-x^2+c,$$ $c$ is a real constant.

(ii) $u_x(x,y)=2-3x^2+3y^2,\ u_{xx}(x,y)=-6x$

$u_y(x,y)=6yx,\ u_{yy}(x,y)=6x\Rightarrow\nabla^2 u=0$

$$u(x,y)=\int_0^y\left(2-3x^2+3t^2\right)dt-\int_0^x 0\,ds = 2y-3x^2y+y^3+c \text{ (real constant)}$$

(iii) $u_x(x,y)=\cos hx\sin y,\ u_{xx}(x,y)=\sin hx\sin y$

$u_y(x,y)=\sin hx\cos y,\ u_{yy}(x,y)=-\sin hx\sin y$.

$\nabla^2 u=0$

$$u(x,y)=\int_0^y \cos hx\sin t\,dt-\int_0^x \sin hs\,ds=-\cos hx\cos y+c$$

(iv) $u_x(x,y)=\cdots\cdots\cdots\nabla^2 u=0$ etc.

$z=x+iy\Rightarrow y=\mathrm{Re}(-iz),\ x^2+y^2=\left|z^2\right|$

$$\frac{y}{x^2+y^2}=\mathrm{Re}\left(\frac{-i_z}{|z|^2}\right)\Rightarrow \mathrm{Im}\left(\frac{-iz}{|z|^2}\right)=-\frac{x}{x^2+y^2}+c \text{ (real constant,}$$

**17.6**

**17.6.1 :** See definition (17.3.1)

**17.6.2 :** See theorem (17.3.6)

**17.6.3 :** See corollary (17.3.9)

**17.6.4 :** See theorem (17.4.1)

**17.6.5 :** See theorem (17.4.1) and counter example (17.4.4)

**17.6.6 :** See Theorem (17.4.8)

**17.6.7 :** See corollary (17.4.9)

**17.6.8 :** See definition (17.5.1)

**17.6.9 :** See example (17.5.2)

**17.6.10 :** See theorem (17.5.4)

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Lesson - 18

# STANDARD FUNCTIONS & ANALYTIC FUNCTIONS AS MAPPINGS

## 18.0 OBJECTIVE OF THIS LESSON :

After going through this lesson one should be able to (i) observe that the complex exponential function has the same properties as that of the real exponential function and so are the complex trigonometric functions (ii) define a branch of complex logarithmic function (iii) mention the properties of certain analytic functions and (iv) prove that any analytic mapping $f$ defined on a region preserves angles, both in magnitude and direction at any $z_0$ of $G$ where $f'(z_0) \neq 0$.

## 18.1 INTRODUCTION :

The complex exponential function has an extra property, namely that it is periodic with period $2\pi i$. This leads to the fact that the complex logarithmic function is a many vlaued one. This leads to the fact that the complex logarthmic function is a many valued one. There by, we consider various branches of this function and observe that any two differ by a multiple of $2\pi i$. Further the mpping properties of certain analytic functions lead to the fact that for any pair of open, connected sets $G$, $\Omega$ in $\mathbb{C}$ there is an analytic function on $G$ such that $f(G) = \Omega$.

## 18.2 COMPLEX EXPONENTIAL FUNCTION :

We have already introduced the complex exponential function in the previous lesson (see 17.3.10). Now, we consider the following.

### 18.2.1 THEOREM :

Let $f(z) = \sum\limits_{n=0}^{\infty} z^n / \underline{|n} = e^z\,(\exp z)$ for $\forall z \in \mathbb{C}$. Then

(i) $f$ is entire and $f' = f$ on $\mathbb{C}$.

(ii) $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$ for all $z_1, z_2 \in \mathbb{C}$.

(iii) $e^z \neq 0$ for all $z \in \mathbb{C}$ and $e^{-z} = 1/e^z \;\forall z \in \mathbb{C}$.

(iv) $\overline{e^z} = e^{\overline{z}}$ for all $z \in \mathbb{C}$;

and $\left|e^{z}\right| = e^{\operatorname{Re}(z)}$ for all $z \in \mathbb{C}$ and

(v) $\left|e^{i\lambda}\right| = 1$ for any $\lambda \in \mathbb{R}$.

**Proof :** Given that $f(z) = e^{z} = \sum_{n=0}^{\infty} z^{n} / \underline{\lfloor n}$ $\forall z \in \mathbb{C}$.

Clearly by 16.3.8, $f$ is entire and by Theorem (17.3.8) (c),

$$f'(z) = \left(e^{z}\right)' = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{\underline{\lfloor n}} = \sum_{n=1}^{\infty} z^{n-1} / \underline{\lfloor n-1} = \sum_{n=0}^{\infty} z^{n} / \underline{\lfloor n} \quad \forall z \in \mathbb{C}$$

Thus $f' = f$ on $\mathbb{C}$. This proves (i).

Let $z_1 \in \mathbb{C}$. Define $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = e^{z} \cdot e^{z_1 - z}$ for all $z \in \mathbb{C}$. Now, follows that for any $z \in \mathbb{C}$, $g'(z)$ exists = $e^{z} \cdot e^{z_1 - z_2} + e^{z}\left(-e^{z_1 - z}\right) = e^{z} \cdot e^{z_1 - z} - e^{z} \cdot e^{z_1 - z} = 0$.

Since $\mathbb{C}$ is connected, by a known result (17.4.9) follows that $g$ is a constant (on $\mathbb{C}$). So $e^{z} \cdot e^{z_1 - z} = g(z) = g(0) = e^{z_1}\ \forall\ z \in \mathbb{C}\left(\because e^{0} = 1\right) \Rightarrow e^{z_1} \cdot e^{z_2} = e^{z_1}\, e^{(z_2 + z_1) - z_1} = e^{z_2 + z_1}$

$= e^{z_1 + z_2} \forall z_1, z_2 \in \mathbb{C}$. This proves (ii).

Since $e^{z}$ is defined for all $z \in \mathbb{C}$, follows from (ii) that, for any $z \in \mathbb{C}$, $e^{z} \cdot e^{-z} = e^{0} = 1 \neq 0$.

$\Rightarrow e^{z} \neq 0$ for all $z \in \mathbb{C}$. This proves (iii)

Since all the coefficients in $\sum_{n=0}^{\infty} \frac{1}{\underline{\lfloor n}} z^{n}$ all real, we get that, for any $z \in \mathbb{C}$.

$$\overline{e^{z}} = \overline{\sum_{n=0}^{\infty} \frac{1}{\underline{\lfloor n}} z^{n}} = \sum_{n=0}^{\infty} \frac{1}{\underline{\lfloor n}} \overline{(z_n)} = \sum_{n=0}^{\infty} \frac{1}{\underline{\lfloor n}} (\overline{z})^{n} = e^{\overline{z}}$$

This proves (iv).

For any $z \in \mathbb{C}$, $\left|e^{z}\right|^{2} = e^{z} \cdot \overline{e^{z}} = e^{z} \cdot e^{\overline{z}} = e^{z + \overline{z}}$

$$= e^{2\operatorname{Re} z} = \left(e^{\operatorname{Re} z}\right)^2$$

$$\Rightarrow \left|e^z\right| = e^{\operatorname{Re} z}$$

Also, for any $\lambda \in \mathbb{R}$, $\operatorname{Re}(i\lambda) = 0 \Rightarrow \left|e^{i\lambda}\right| = e^0 = 1$.

This proves (v) and that the result follows.

Now, we consider the following.

**18.2.2 DEFINITIONS :**

For any $z \in \mathbb{C}$, we define

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{\lfloor(2n+1)} = z - \frac{z^3}{\lfloor 3} + \frac{z^5}{\lfloor 5} - + \cdots$$

and $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{\lfloor 2n} = 1 - \frac{z^2}{\lfloor 2} + \frac{z^4}{\lfloor 4} - + \cdots$$

(observe that each of the series on the RHS has radius of convergence $+\infty$)

Since each of the power series, considered above, has radius of convergence $+\infty$ follows that the complex sine and cosine functions are entire. It is easy to prove the following :

**18.2.3 SELF ASSESSMENT QUESTIONS :**

For any $z \in \mathbb{C}$

(i) $(\cos z)' = -\sin z$;

(ii) $(\sin z)' = +\cos z$

(iii) $\cos z = \left(e^{iz} + e^{-iz}\right)/2$

(iv) $\sin z = \left(e^{iz} - e^{-iz}\right)/2i$

(v) $\sin^2 z + \cos^2 z = 1$

(vi) $e^{iz} = \cos z + i \sin z$

Before we consider complex logarithmic function, we introduce the following.

## 18.3 PERIODIC FUNCTIONS :

### 18.3.1 DEFINITION :

A (complex) function $f$ defined on $\mathbb{C}$ is said to be a periodic function iff there is a complex number $\mathscr{G}$ such that $f(z+\mathscr{G})=f(z)$ for all $z\in\mathbb{C}$. $\mathscr{G}$ is called a period of $f$.

We observe that this property is enjoyed by complex exponential function but not by real one.

### 18.3.2 OBSERVATION :

Since $\exp(2\pi i)=e^{2\pi i}=1$ follows that $\exp(z+2\pi i)=\exp(z)$ for all $z\in\mathbb{C}$. Hence, the complex exponential function is a periodic function with a period $2\pi i$.

Further if $\mathscr{G}\in\mathbb{C}$ be such that $\exp(z+\mathscr{G})=\exp(z)\ \forall z\in\mathbb{C}$ such that $\exp(\mathscr{G})=1\Rightarrow \exp\mathscr{G}=|\exp\mathscr{G}|=1 \Leftrightarrow e^{\mathrm{Re}\,\mathscr{G}}=0\Leftrightarrow \mathrm{Re}\,\mathscr{G}=0$. So $\mathscr{G}=i\lambda$ for some $\lambda\in\mathbb{R}$ and $1\Leftrightarrow\lambda$ is a multiple of $2\pi$. So $\lambda=2\pi k$, where $k$ is any integer. Thus $\{2\pi ki : k \text{ is an integer}\}$ sequence of periods of complex exponential function.

We now consider complex logarithmic function.

## 18.4 COMPLEX LOGARITHMIC FUNCTION :

### 18.4.1 DEFINITION :

Let $G$ be a region (non-empty, open, connected set) in $\mathbb{C}$. Let $f:G\to\mathbb{C}$ be a continuous function such that $z=\exp f(z)$ for all $z\in G$. Then $f$ is called a branch of the logarithm on $G$.

(since $\exp\neq 0$ follows that $0\notin G$)

### 18.4.2 THEOREM :

Let $G$ be a region in $\mathbb{C}$ and $f$ be a branch of the logarithm on $G$, $H$ continuous function $g:G\to\mathbb{C}$ is also a branch of the logarithm if and only if there is an integer $K$ such that $g=f+2\pi iK$ (that means any two branches of the logarithm differ by a multiple of $2\pi i$).

**Proof :** Let $f$ be a branch of the logarithm on the region $G$. Let $k$ be an integer. Defining : $g:G\to\mathbb{C}$ by $g(z)=f(z)+2\pi ik$ for $\forall z\in G$. Clearly $g$ is continuous on $G$. Further, for any $z\in G$, $\exp g(z)=\exp(f(z)+2\pi ik)=\exp f(z)\ \exp(2\pi ik)=\exp f(z)$

(since $\exp((2\pi iK)=1)=z$, by hypothesis. Hence $y$ is also a branch of the logarithm on $G$.

Conversly assume that $f$ and $g$ are branches of the lograrithm on $G$. So, for any $z \in G,\ z = \exp g(z) = \exp f(z) \Rightarrow$ for any $z \in G,\ z = \exp g(z) = \exp f(z) \Rightarrow$ for any $z \in G$, there is an integer $k_z$ such that $g(z) = f(z) + 2\pi i k_z$. Since the function $h : G \to \mathbb{C}$ defined by $h(z) = [g(z) - f(z)]/2\pi i$, for every $z \in G$, is continuous and is integer valued on the connected set $G$ follows that $h$ is a constant. Thus $k_z$ is the same for all $z \in G$. Denote $k_z = k$. So there exists an integer $k$ such that $g(z) = f(z) + 2\pi i k$ for all $z \in G$. This proves the result.

**18.4.3 REMARK :**

From Theorem (18.4.2) follows that if one branch of the logarithm on a connected, open set is known then all the other branches are determined. In fact if $f$ is a branch of the logarthm on $G$ then $\{f + 2\pi i k : k \text{ is an integer}\}$ is the set of all branches of the logarithm on $G$, where $G$ is a non-empty, open, connected subset of $\mathbb{C}$.

Now, we show the existence of one branch of the logarithm on a non-empty, open, connected set $G$, where $0 \notin G$.

**18.4.4 RESULT :**

There is a branch of the logarithm on a non-empty, open, connected sub set $G$ of $\mathbb{C}$ with $0 \notin G$.

**Proof :** Let $G = \mathbb{C} - \{r \in \mathbb{R} / r \le 0\}$ (The complex plane with a slit (cut along the non positive real axis). Clearly $G$ is a non-empt, open, conncted subset of $\mathbb{C}$. Any $z \in G$ can uniquely be written as $z = r e^{i\theta}$ where $r = |z|$ and $\theta \in (-\pi, \pi)$. Define $f : G \to \mathbb{C}$ by $f(z) = \ln r + i\theta \forall z = r e^{i\theta} \in G$ $(r = |z|, -\pi < \theta < \pi)$. Let $\{z_n ; n = 1, 2, \cdots\}$ be a sequence in $G$ such that $z_n \to z \in G$ as $n \to \infty$. Let $z_n = r_n e^{i\theta_n}$ and $z = r e^{i\theta}$ where $|z_n| = r_n (n = 1, 2, \cdots\cdots), r = |z|, \theta_n\ \theta \in (-\pi, \pi))$. Since $||z_n| - |z|| (= |r_n - r|) \le |z_n - z|$ follows that $r_n \to r$ as $n \to \infty \Rightarrow \ln r_n \to \ln r$ as $n \to \infty$. Since $\gamma_n$ cis $d_n = z_n \to z = r \operatorname{cis}\theta$ as $n \to \infty$. So $f(z_n) = \ln r_n + i d_n \to \ln r + i\theta = f(z)$ as $n \to \infty$. Thus $f$ is continuous on $G$. For any $z = re^{i\theta} \in G$,

$$\exp(f(z)) = \exp(\ln r + id) = \exp(\ln r) \cdot \exp(i\theta)$$

$$= r e^{i\theta} = z$$

Thus $f$ is a branch of the logarithm on $G$.

**18.4.5 DEFINITION :**

Let $G = \mathbb{C} - \{r \in \mathbb{R} / r \le 0\}$. Any $z \in G$ can uniquely be written as $z = re^{i\theta}$ where $r = |z|$ and $\theta \in (-\pi, \pi)$. Define $f : G \to \mathbb{C}$ by $f(z) = \ln r + i\theta$ for every $z = re^{i\theta} \in G$. Then $f$ is called the principal branch of the logarithm and is denoted by log.

(Any branch is given by $\{\log + 2\pi ik : k \text{ is an integer}\}$. This set is denoted by Log).

Now, we discuss about the analyticity of a branch of iogarithm on a nonempty, open, connected subset $G$ of $\mathbb{C}$ with $0 \notin G$.

**18.4.6 THEOREM :**

Let $G$ and $H$ be non-empty, open subsets of $\mathbb{C}$. Let $g : G \to \mathbb{C}$ and $h : H \to \mathbb{C}$ be continuous such that $g(G) \subseteq H$ $h \circ g = I$, the identity function on $G$. If $h$ is differentiable on $H$ and $h'(g(z)) \ne 0$ for all $z \in G$ then $g$ is differentiable on $G$ and $g'(z) = 1/h'(g(z))$ for every $z \in G$ ($\Rightarrow h$ is analytic on $H$ $\Rightarrow g$ is analytic on $G$).

**Proof :** Let $G, H, g$ and $h$ be as in the statement of the theorem. Let $z_0 \in G$. Since $G$ is open, there exists a $0 \ne h_0 \in \mathbb{C}$ such that $z_0 + h_0 \in G$. Now $h(g(z_0)) = z_0$ and $h(g(z_0 + h_0)) = z_0 + h_0$. Since $h_0 \ne 0$ follows that $g(z_0) \ne g(z_0 T h_0)$. Further

$$1 = \frac{(z_0 + h_0) - z_0}{h_0} = \frac{h(g(z_0 + h_0)) - h(g(z_0))}{g(z_0 + h_0) - g(z_0)} \cdot \frac{g(z_0 + h_0) - g(z_0)}{h_0} \text{ -------- (i)}$$

Since $g$ is continuous on $G$ follows that $g(z_0 + h_0) \to g(z_0)$ as $h_0 \to 0$ and so, by hypothesis,

$$\lim_{h_0 \to 0} \frac{h(g(z_0 + h_0)) - h(g(z_0))}{g(z_0 + h_0) - g(z_0)} \text{ exists } f = h'|g(z_0)| \ne 0 \text{ ------ (ii)}$$

(i) & (ii) imply that

$$\lim_{h_0 \to 0} \frac{g(z_0 + h_0) - g(z_0)}{h_0} \text{ exists } f = 1/h'(g(z_0))$$

This is true for all $z_0 \in G$. Hence follows that $g$ is differentiable on $G$ and

$$g'(z) = \frac{1}{h'(g(z))} \quad \forall\, z \in G.$$

This completes the proof of the Theorem.

**18.4.7 COROLLARY :**

Any branch of the logarithm on a non-empty, open connected subset $G$ of $\mathbb{C}-\{0\}$ is analytic and its derivative for any $z \in G$ is $1/z$.

**Proof :** Let $g$ be any branch of the logarithm on a non-empty open connected subset of $\mathbb{C}-\{0\}$. By definition $g$ is continuous and $\exp(g(z)) = z \forall z \in G$. Since exponential function is entire and $(\exp)' \neq 0$ on $\mathbb{C}$ follows that (by the previous Theorem) that $g$ is analytic on $G$ and for any $z \in G,\ g'(z) = 1/(\exp)'(g(z)) = 1/\exp g(z) = 1/z$. Thus the result follows.

**18.4.8 EXAMPLE :**

If $z_1, z_2 \in \mathbb{C}$, (i) $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$;

(ii) $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

**Solution :** Let $z_1, z_2 \in \mathbb{C}$. We know that

$$\exp\{i(z_1+z_2)\} = \exp(iz_1)\exp(iz_2)$$

$$= (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$$

$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i[\cos z_1 \sin z_2 + \sin z_1 \cos z_2] \text{ ---- (i)}$$

Replacing $z_1, z_2$ by $-z_1, -z_2$ and we get

$$\exp\{-i(z_1+z_2)\} = \{\cos(-z_1)\cos(-z_2) - \sin(-z_1)\sin(-z_2)\} +$$

$$i\{\cos(-z_1)\sin(-z_2) + i\sin(-z_1)\cos(-z_2)\}$$

$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) -$$

$$i(\cos z_1 \sin z_2 + \sin z_1 \cos z_2) \text{ ----------- (ii)}$$

(i) + (ii) $\Rightarrow$

$$2\cos\left(z_1+z_2\right)=2\left(\cos z_1\cos z_2-\sin z_1\sin z_2\right)\Rightarrow$$

$$\cos\left(z_1+z_2\right)=\cos z_1\cos z_2-\sin z_1\sin z_2$$

(i) - (ii) $\Rightarrow$

$$2i\sin\left(z_1+z_2\right)=2i\left(\cos z_1\sin z_2+\sin z_1\cos z_2\right)\Rightarrow$$

$$\sin\left(z_1+z_2\right)=\sin z_1\cos z_2+\cos z_1\sin z_2$$

**18.4.9 DEFINITION :**

$G$ is a region in $\mathbb{C}$ with $0\notin G$. $f$ is a branch of the logarithm on $G$ and $b\in\mathbb{C}$. Define $g:G\to\mathbb{C}$ by $g(z)=\exp(b\,f(z))\,\forall\, z\in G$. If $b$ is an integer then $g(z)=z^b\,\forall\, z\in G$. If we write $g(z)=z^b\,(z\in G)$ we understand it as $\exp(b\log z)\,\forall\, z\in G,$ $\log$ being the principal branch.

**18.5 ANALYTIC FUNCTIONS AS MAPPINGS :**

We first consider the following :

**18.5.1 DEFINITIONS :**

(i) Let $[a, b]$ be a bounded closed interval of reals and $G$ be a region in $\mathbb{C}$. Any continuous map $r:[a, b]\to G$ is called a path in $G$.

(ii) A function $r:[a, b]\to\mathbb{C}$ is said to be diffrentiable on $[a, b]$ iff for every $t$, in $[a, b]$,

$\lim\limits_{h\to\infty}\dfrac{r(t+h)-r(t)}{h}$ exists; the limit is denoted by $r'(t)$ and it is called the derivative of $r$ and $t$ (at $t=a,$ we take right limit and at $t=b$ we take left derivative).

(Observe that, if $r=r_1+ir_2,$ where $r_1=\operatorname{Re}r$ and $R_2=\operatorname{Im}r$, then $r'(t)$ exiss iff (if and only) $r_1'(t)$ and $r_2'(t)$ exist for any $t\in[a, b]$ and in this case $r'(t)=r_1'(t)+ir_2'(t)$.

(iii) A path $r$ in the region $G$ (i.e. $r:[a, b]\to G$ is a continuous map is said to be a smooth path in $G$ iff $r$ is continuously differentiable on $[a, b]$ (i.e. $r'(t)$ exists for all $t\in[a, b]$ and $r':[a, b]\to\mathbb{C}$ is continuous).

(iv) $r$ is said to be piecewise smooth in G iff there exists a partition $P(a=t_0<t_1<\cdots<t_n=b)$ of $[a, b]$ such that $r$ is smooth on each sub interval $[t_{j-1}, t_j]$ of ([$a$, $b$]) for $j=1,\cdots n$.

### 18.5.2 OBSERVATION :

Let $r:[a,b]\to G$ be a smooth path in $G$ and $r'(t_0)\neq 0$ for some $t_0\in(a,b)$. Then $r$ has a tangent line at the point $z_0=r(t_0)$. This line passes through the point $z_0$ in the direction of the vector $r'(t_0)$ (or the scope of the line is $\tan(\arg r'(t_0))$.

### 18.5.3 DEFINITION :

Let $[a, b]$ be a bounded, closed interval of reals and $G$ be a region in $\mathbb{C}$. Let $r_1, r_2 :[a, b]\to G$ be smooth paths such that for some $t_1, t_2\in[a, b]$, $r_1(t_1)=r_2(t_2)=z_0$ (say) and $r_1'(t_1)\neq 0, r_2'(t_2)\neq 0$. Then the angle between $r_1$ and $r_2$ at $z_0$ is defined to be $\arg r_2'(t_2)-\arg r_1'(t)$.

### 18.5.4 DEFINITION :

Let $G$ be a region in $\mathbb{C}$ and $f:G\to\mathbb{C}$. Let $z_0\in G$. $f$ is said have angle preserving property at $z_0$ iff the angle between two curves through $z_0$ in $G$ is same as the angle between the curves $f(r_1), f(r_2)$ (throught $f(z_0)$).

### 18.5.5 DEFINITION :

Let $G$ be a region in $\mathbb{C}$. A mapping $f:G\to\mathbb{C}$ is called a conformal mapping at $z_0\in G$ iff $f$ preserves angles at $z_0$ and $\underset{z\to z_0}{\mathrm{Lt}}\dfrac{|f(z)-f(z_0)|}{|z-z_0|}$ exists.

### 18.5.6 THEOREM :

Let $G$ be a region in $\mathbb{C}$ and $f:G\to\mathbb{C}$ be an analytic function. Then $f$ is conformal ($\Rightarrow$ preserves angles) at each point $z_0\in G$ where $f'(z_0)\neq 0$.

**Proof :** Let $G$, $z_0$ and $f$ be as in the statement of the Theorem. Suppose $r$ is a smoth path in $G$. Now follows that $\sigma=$ for is also a smooth path in $\mathbb{C}$ and $\sigma'(t)=f'(r(t))\,r'(t)$ for all $t\in$ Domain$(r)$. Let $z_0=r(t_0)$ and $r'(t_0)\neq 0$ and $f'(z_0)\neq 0$. Now follows that $\sigma'(t_0)\neq 0$ and

$\arg\sigma'(t_0)=\arg f'(z_0)+\arg r'(t_0)$

$$\Rightarrow \arg\sigma'(t_0)-\arg r'(t_0)=\arg f'(z_0) \text{ ---------------- (i)}$$

Let $r_1, r_2$ be smooth paths in $G$ with $r_1(t_1)=r_2(t_2)=z_0$ where $t_1\in \mathrm{Dom}(r_1)$ and $t_2\in \mathrm{Dom}(r_2)$. Further $r_1'(t_1)\neq 0\neq r_2'(t_2)$. Denote $\sigma_1=f\circ r_1$, $\sigma_2=f\circ r_2$. Also suppose that $r_1$ and $r_2$ are not tangent to each other at $z_0$; that is suppose that $r_1'(t_1)\neq r_2'(t)$. Now (i) gives that

$$\arg r_2'(t_2)-\arg r_1'(t_1)=\arg\sigma_2'(t_2)-\arg\sigma_1'(t_1) \text{ ------------ (ii)}$$

$$\left(\because \arg\sigma_2'(t_2)-\arg r_2'(t_2)=\arg f'(z_0)=\arg\sigma_1'(t_1)-\arg r_1'(t_1)\right)$$

(ii) says that given any two paths through $z_0$; $f$ maps these paths onto two paths through $f(z_0)$ and, when $f'(z_0)\neq 0$, the angles between the corresponding curves are preserved both in magnitude and direction.

Since $f$ is analytic on $G$, follows that

$$\lim_{z\to z_0}\frac{|f(z)-f(z_0)|}{|z-z_0|} \text{ exists.}$$

Thus $f$ is conformal at the point $z_0$ where $f'(z_0)\neq 0$. This completes the proof of the Theorem.

**18.5.7 EXAMPLE :**

Discuss the transformation $w=f(z)=z^2 \forall z\in\mathbb{C}$

**Solution :** Given that $f:\mathbb{C}\to\mathbb{C}$ is such that $f(z)=z^2\ \forall\ z\in\mathbb{C}$. $f'(z)$ exists $f'=2z\forall z\in\mathbb{C}$. So $f$ is entire and $f'(z)\neq 0$ for all $z\neq 0$. So $f$ is conformal at all $z\neq 0$.

Write $z=x+iy$ and $w=u+i\mathrm{v}$. So

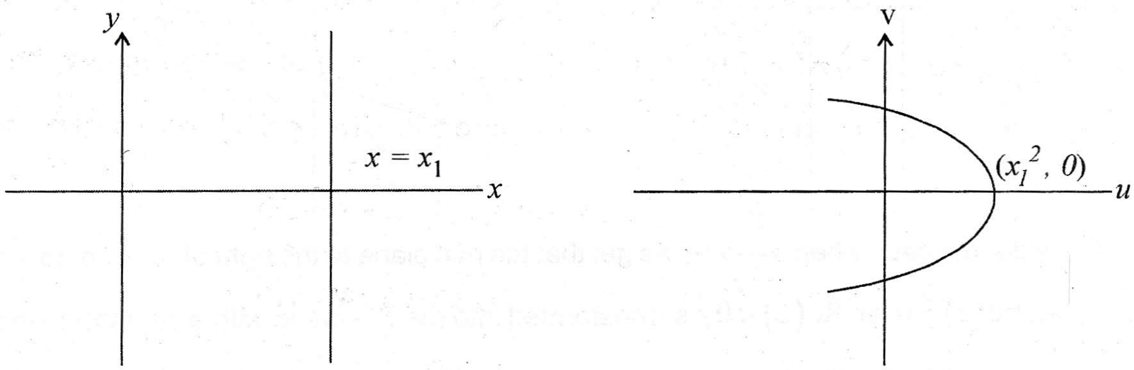
$$u+i\mathrm{v}=(x+iy)^2=(x^2-y^2)+i(2xy)\Rightarrow u=x^2-y^2,\ \mathrm{v}=2xy\,\forall (x,y)\in\mathbb{R}^2$$

(i) Consider $x=x_1\,(>0)$. It is transformed into the curve given by

$$u=x_1^2-y^2 \text{ and } \mathrm{v}=2x_1y.$$

Eliminating $y$ we get that

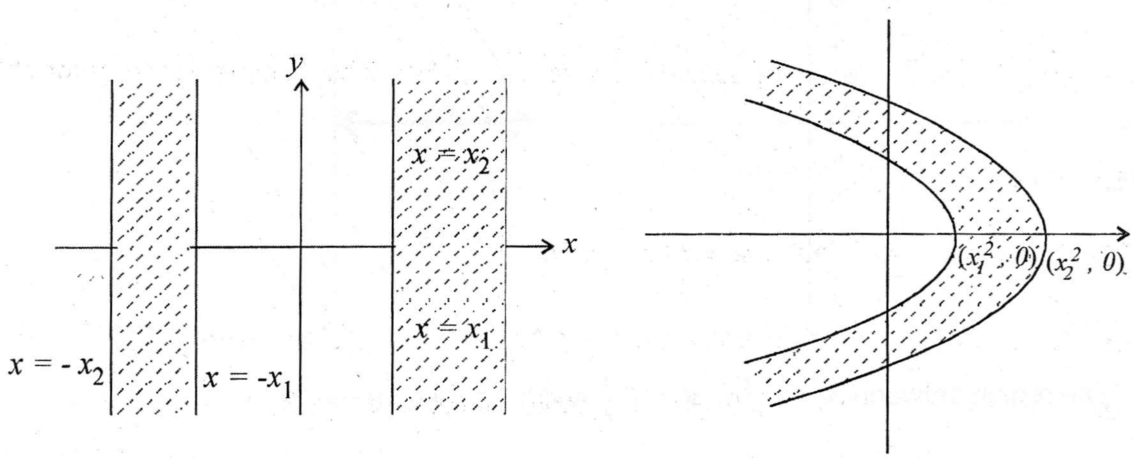
$$u = x_1^2 - \left(\upsilon/2x_1\right)^2 \Rightarrow \upsilon^2 = -4x_1^2\left(u - x_1^2\right)$$



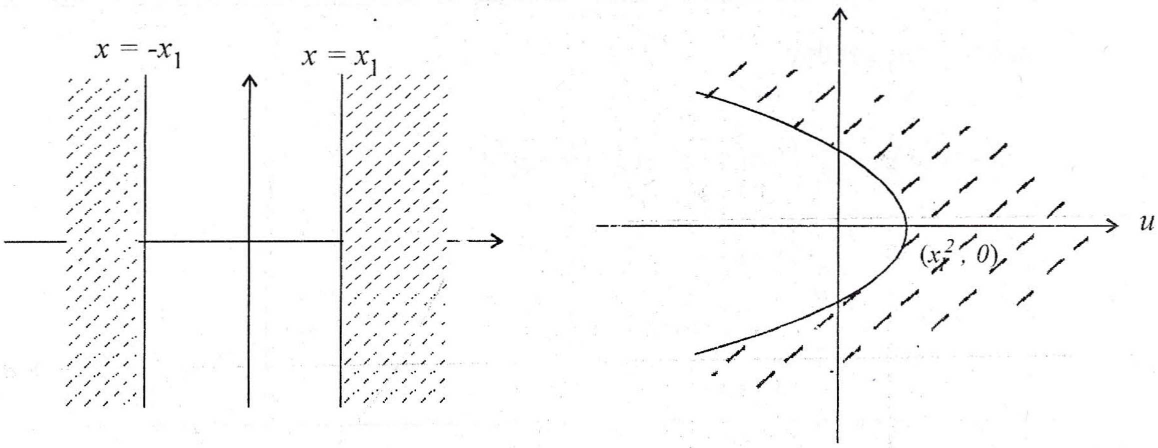

Thus the line $x = x_1$ is transformed into the parabola $\upsilon^2 = -4x_1^2\left(u - x_1^2\right)$ by means of $f$.

Also $x = -x_1$ is transofrmed into the same parabola.

Let $0 < x_1 < x_2$; the region between $x = x_1$ and $x = x_2$ or between $x = -x_1$ and $x = -x_2$ is transformed into the region between $\upsilon^2 = -4x_1\left(u - x_1^2\right)$ and $\upsilon^2 = -4x_2\left(u - x_2^2\right)$.

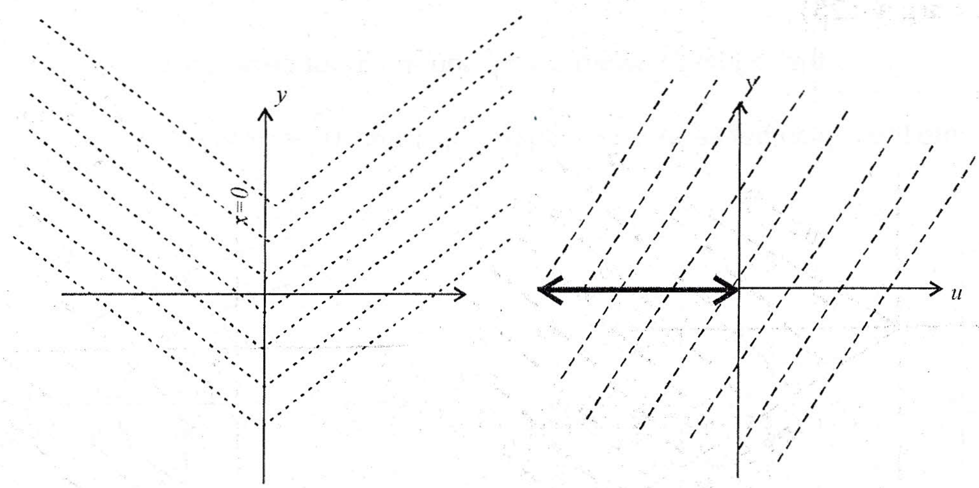



Consider the case when $x_2 \to \infty$; we get that the half plane to the right of $x = x_1$ or the half plane to the left of $x = -x_1$ is mapped into the region which is exterior to $v^2 = -4x_1\left(u - x_1^2\right)$.




Consider the case when $x_1 \to 0$; we get that the half plane to the right of $x = 0$ or to the left of $x = 0$ (i.e. $\mathrm{Re}(z) > 0$ or $\mathrm{Re}(z) < 0$) is transformed into the $W$ - plane with a cut (sht) along the -ve real axis $(u \le 0)$ in the $W$ -plane.

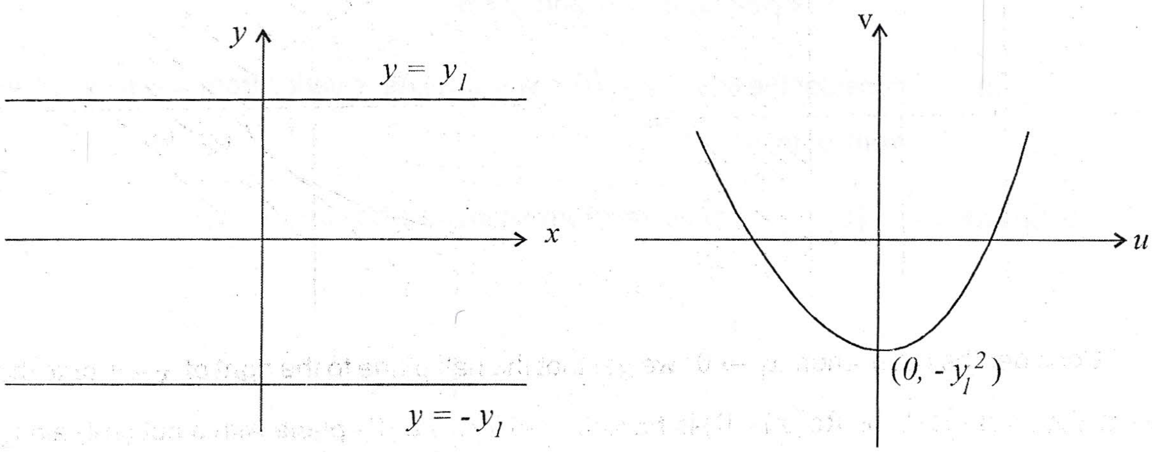
$x_1 \to 0 \Rightarrow u = 0$ and $u \le 0$.




The region between $v^2 = -4x_1\left(u - x_1^2\right)$ and $u^2 = -4x_2\left(u - x_2^2\right)$

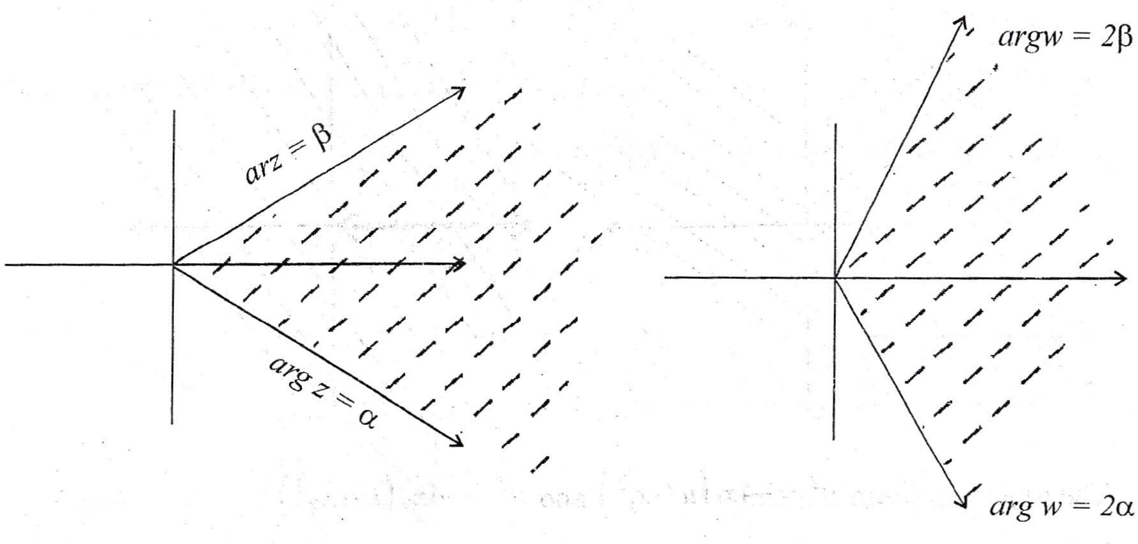
Similarly $y = y_1 (> 0)$ or $y = -y_1$ is transformed into the parabola $v^2 = 4y_1^2\left(u + y_1^2\right)$.

(ii) (a) Consider the circle with a origin as centre and $r(>0)$ as radius; i.e. $|z| = r \Rightarrow r e^{i\theta}, 0 \le \theta < \pi$. Now $w = z^2 = r^2 e^{2i\theta} \Rightarrow |w| = r^2, 0 \le \theta < \pi \Rightarrow 0 \le 2\theta < 2\pi$.




So $|z| = r$ transformed into $|w| = r^2$ described with.

(b) The sector $\{z \in \mathbb{C} : \alpha < \arg z < \beta\}$ $(\beta - \alpha < \pi)$ is transformed into the sector $\{w \in \mathbb{C} : 2\alpha < \arg w < 2\beta\}$.




**18.5.8 EXAMPLE :**

Discuss the transformation $w = \exp(z) = e^z \forall z \in \mathbb{C}$.

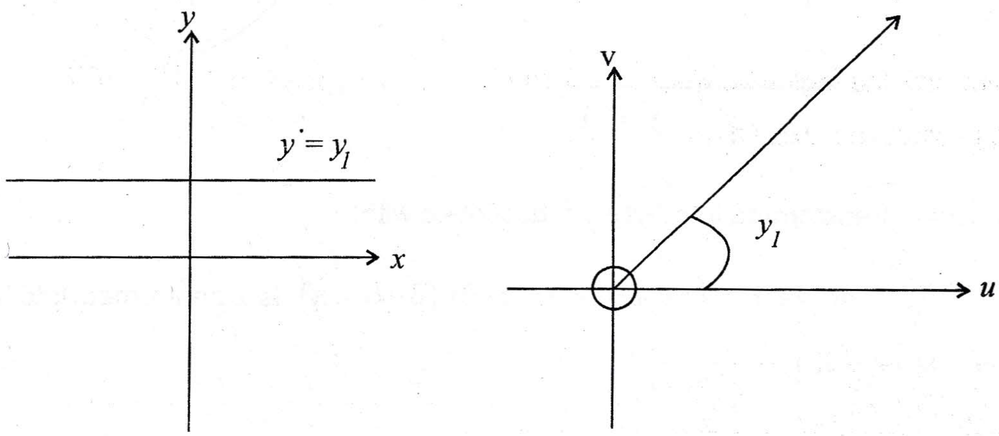
**Solution :** Given that $f : \mathbb{C} \to \mathbb{C}$ is such that $f(z) = e^z \ \forall z \in \mathbb{C}$. $f'(z)$ exists $f = e^z \forall z \in \mathbb{C}$. Sc $f$ is entire and $f'(z) \neq 0 \forall z \in \mathbb{C}$. So $f$ is conformal on $\mathbb{C}$.

(1) Let $z = x + iy$ and $w = \rho e^{ie}$

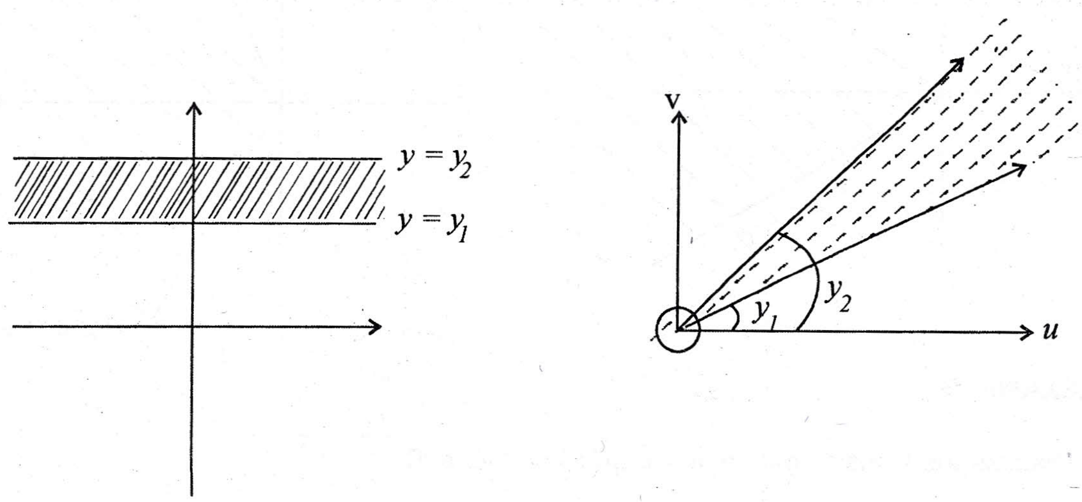
$\therefore e^{x+iy} = \rho e^{i\phi} \Rightarrow e^x = \rho$ and $y = \phi$

(i) consider the line $y = y_1 \left(0 < y_1 < 2\pi\right)$. As $x$ varies from $-\infty$ to $\infty$, $e^x$ varies from $0$ to $\infty$.

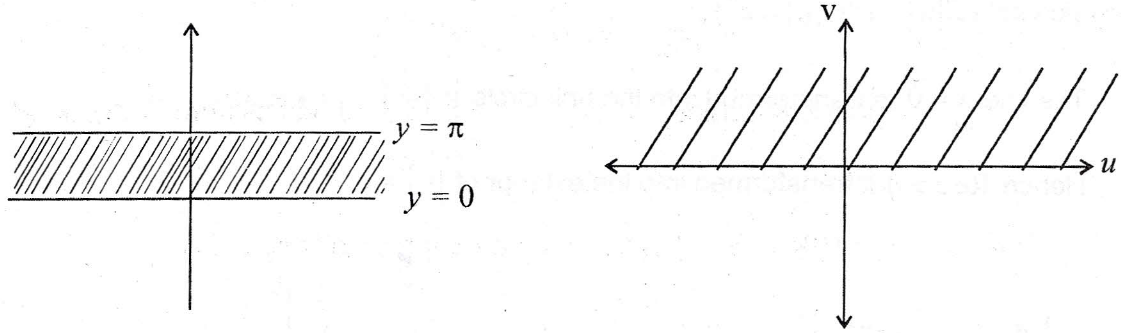
So the line $y = y_1 \left(0 < y_1 < 2\pi\right)$ is transformed into the half ray $\phi = y_1$;



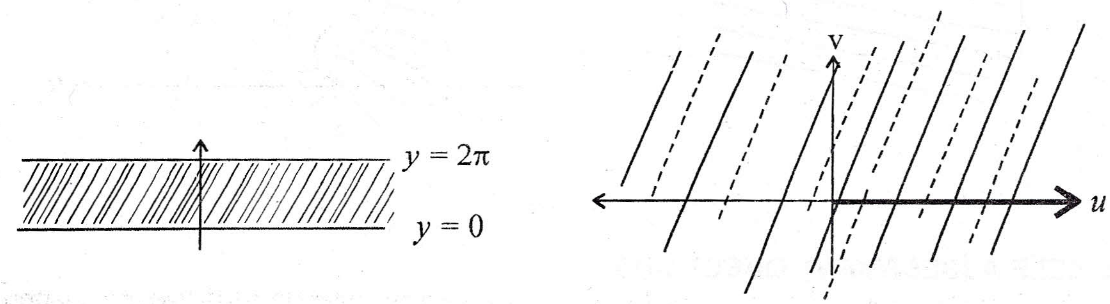

So the region $y_1 < I\left(z\right) = y < y_2 \ \left(y_2 - y_1 < \pi\right)$ is transformed into the Wedge shaped region in the $w$ plane bounded by the radial lines $\phi = y_1$, $\phi = y_2$;



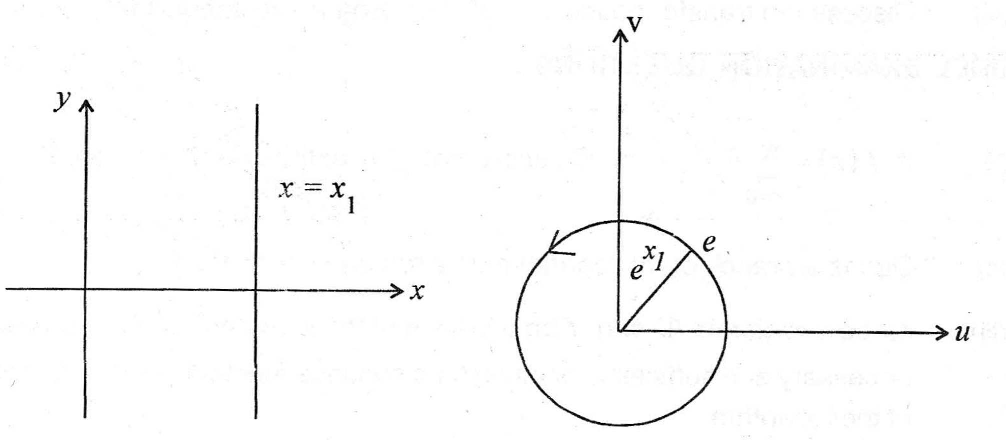

The region between $y = 0$ and $y = \pi$ is mapped into the upper half $w$-plane.




Similarly, the region between $y = 0$ and $y = 2\pi$ is mapped into the $W$-plane with a cut (slit) along the positive real axis ($0$ to $\infty$).



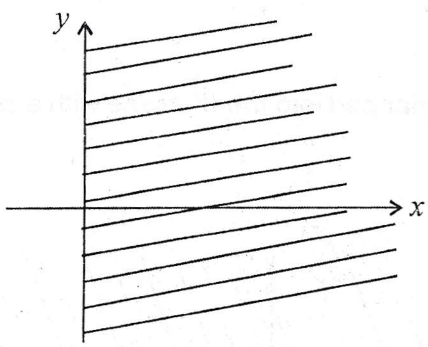

(2) Consider the line $x = x_1 \Rightarrow \rho = e^{x_1} \Rightarrow |w| = \rho = e^{x_1}$. Thus the line $x = x_1$ is transformed into the circle $|w| = e^{x_1}$.

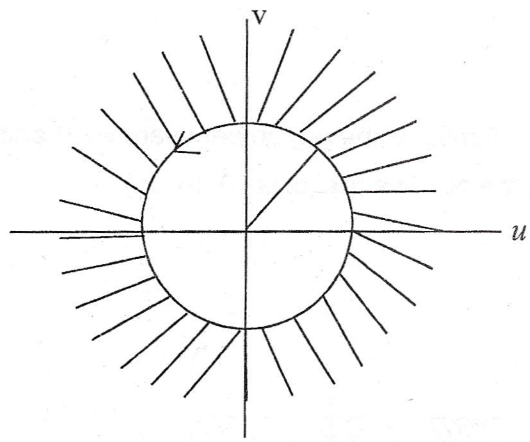


Similarly, the region between $x = x_1$ & $x = x_2$ is transformed into the region between the concentric circles $|w| = e^{x_1}, |w| = e^{x_2}$.

The line $x = 0$ is transformed into the unit circle $|w| = 1$ $x_1 \to \infty \Rightarrow e^{x_1} \to \infty$.

Hence $\operatorname{Re} z > 0$ it transformed into the exterior of $|w| = 1$.




### 18.5.9 SELF ASSESSMENT QUESTIONS :

(i) Discuss the transformation $w = \sin z\,(z \in \mathbb{C})$

(ii) Discuss the transformation $w = \cos z\ (z \in \mathbb{C})$

(iii) Discuss the transformation $w = z^n$ ($n$ being a +ve integer) for $\forall z \in \mathbb{C}$.

## 18.6 MODEL EXAMINATION QUESTIONS :

(i) If $f(z) = \sum_{n=0}^{\infty} z^n / \underline{|n}$ $\forall\ z \in \mathbb{C}$, show that $f$ is entire and $f' = f$ on $\mathbb{C}$.

(ii) Define a branch of the logarithm on a region $G$ with $0 \notin G$.

(iii) $G$ be a region in $\mathbb{C}$ and $f$ be a branch of the logarithm on $G$. State and prove a necessary and sufficient condition for a continus function $g : G \to \mathbb{C}$ to be a branch of the logarithm.

(iv) Show that there is a branch of the logarithm on a region $G$ in $\mathbb{C}$ when $0 \notin G$.

(v) Define the principal branch of the logarithm.

(vi) Show that any branch of the logarithm on a region $G \subseteq \mathbb{C}-\{0\}$ is analytic and its derivative is $\frac{1}{2}$ for $\forall z \in G$.

(vii) Explain the term "$f$ is conformal at $z_0 \in \mathrm{Dom}\, f$".

(viii) $f$ is analytic on a region $G$ and $f'(z_0) \neq 0$ for some $z_0 \in G$ show that $f$ is conformal at $z_0$.

(ix) Discuss the transformation $z \to z^2 \forall z \in \mathbb{C}$

(x) Discuss the transformation $z \to \exp(z) \forall z \in \mathbb{C}$.

## 18.7 HINTS TO SAQ'S AND MEQ'S :

**18.2.3.1 :**

(i) $$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{\lfloor 2n} = \cos z \,\forall\, z \in \mathbb{C}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n\, 2n \frac{z^{2n-1}}{\lfloor 2n} = (\cos z)' \Rightarrow (\cos z)' = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{\lfloor 2n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{\lfloor 2n+1}$$

(ii) $$\sin z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1} \Big/ \lfloor 2n+1 \;\forall\, z \in \mathbb{C} \Rightarrow$$

$$(\sin z)' = \sum_{n=0}^{\infty} (-1)^n (2n+1) z^{2n} \Big/ \lfloor 2n+1 = \sum_{n=0}^{\infty} \left(-1^n\right) z^{2n} \Big/ \lfloor 2n = \cos z$$

(iii) $$\exp(iz)+\exp(-iz) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\lfloor n} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{\lfloor n}$$

$$= 2\sum_{n=0}^{\infty} \frac{(iz)^{2n}}{\lfloor 2n} = 2\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{\lfloor 2n} = 2\cos z$$

$$\Rightarrow \cos z = \{\exp(iz)+\exp(-iz)\}/2 \ \forall \ z \in \mathbb{C}.$$

(iv) $$\exp(iz)-\exp(-iz) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\lfloor n} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{\lfloor n}$$

$$= 2\sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{\lfloor 2n+1} = 2i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{\lfloor 2n+1} = 2i \sin z$$

(v) $$\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= \frac{1}{4}\left[\left(e^{iz}+e^{-iz}\right)^2 - \left(e^{iz}-e^{-iz}\right)^2\right]$$

$$= \frac{1}{4}\left[4e^{iz}\cdot e^{-iz}\right] = 1$$

(vi) $$\cos z + i \sin z = \left[\frac{\exp(iz)+\exp(-iz)}{2}\right] + i\left[\frac{\exp(iz)-\exp(-iz)}{2i}\right]$$

$$= \exp\left(i^2 z\right)$$

**18.5.9 :**

(i) $\dfrac{dw}{dz} = \cos z = 0 \Leftrightarrow z = (2n+1)\dfrac{\pi}{2}$, $n$ being an integer.

So the mapping is conformal on $\mathbb{C} - \left\{(2n+1)\frac{\pi}{2};\ n \in I(\text{integer})\right\}$

Write $w = u + i\,\mathrm{v}$, $z = x + iy \cdot y = \lambda \Rightarrow \dfrac{u^2}{\cos h^2 \lambda} + \dfrac{\mathrm{v}^2}{\sin h^2 \lambda} = 1$

$$x = a \Rightarrow \frac{u^2}{\sin^2 a} - \frac{u^2}{\cos^2 a} = 1$$

(ii) $\dfrac{dw}{dz} = -\sin z \neq 0$ iff $z \neq n\pi\,(n \in I)$

So the mapping is conformal on $\mathbb{C}-\{n\pi : n \in I\}$.

Put $z = x+iy,\ w = u+i\text{v}$. So

$u+i\text{v} = \cos(x+iy) = \cos x \cos hy - i\sin x \sin hy$

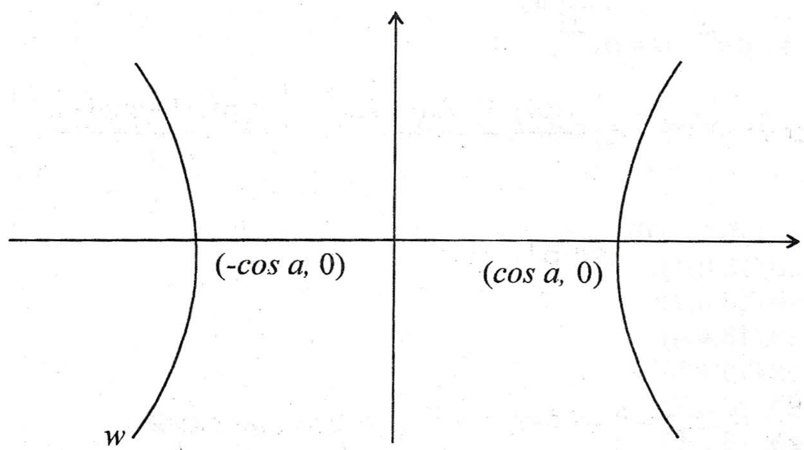
$u(x,y) = \cos x \cos hy,\ u(x,y) = \sin x \sin \text{h}\, y \,\forall (x,y) \in \mathbb{R}^2$

$$x = a\,(0 < a < \pi/2) \Rightarrow$$

$$\frac{u^2}{\cos^2 x} - \frac{\text{v}^2}{\sin^2 x} = \cos h^2 y - \sin h^2 y = 1$$

As $y$ varies from $-\infty$ to $\infty$, $u > 0$ and $\text{v} > 0$.

So $x = a$ is transformed in to the branch of the hyperbola which is to the right of the imaginary $W$-axis. Similarly $x = (\pi - a)$ is transformed into the other branch.



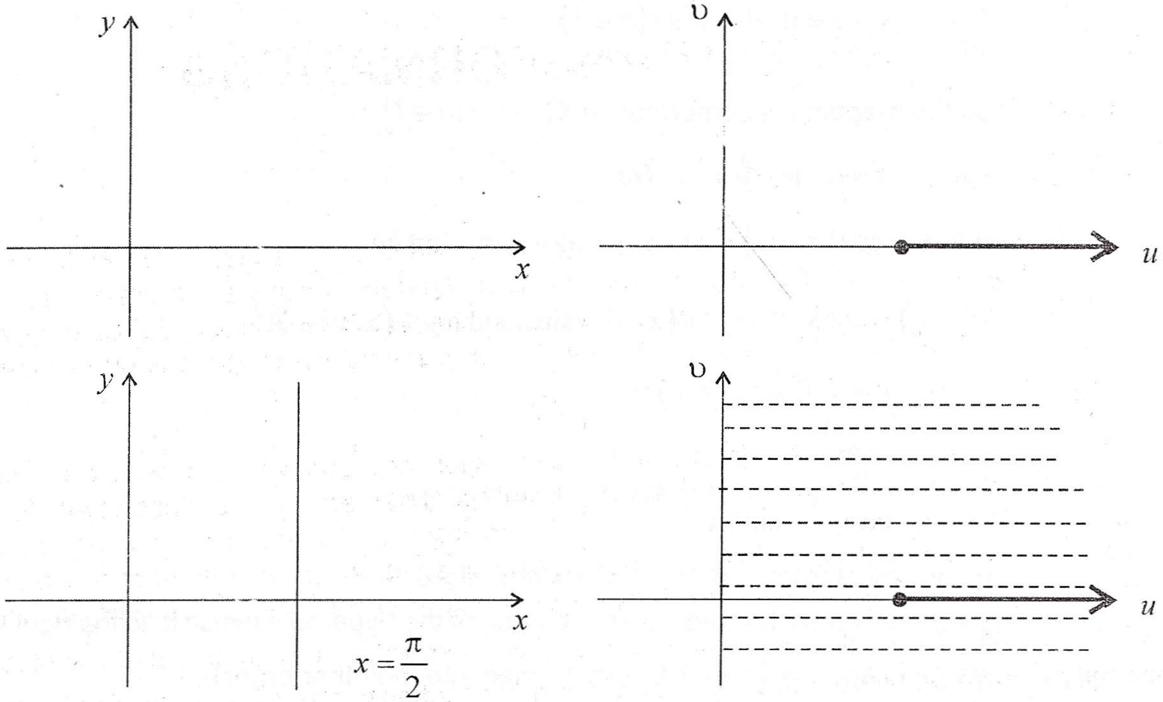

$x = 0 \Rightarrow u = \cos \text{h}\, y,\ \text{v} = 0$

$y$ varies from $-\infty$ to $0$, $u$ varies from $\infty$ to 1.

$y$ varies from 0 to $\infty$, $u$ varies from 1 to $\infty$.

$\Rightarrow 0 < \text{Re}\, z < \pi/2$ is transformed into the right half $W$-plane with a slit (cut) along the +ve $W$- axis from 1 to $\infty$ (taken twice (iii) $n = 1,$ the identity transformation.

$n \geq 2,\ \dfrac{dw}{dz} = nz^{n-1} = 0 \Leftrightarrow z = 0$. So the transformation is conformal on $\mathbb{C}-\{0\}$




$$z = r\rho e^{id}, \; w = \rho e^{i\phi}$$

$$\rho\, e^{i\phi} = r^n e^{ni\theta} \Rightarrow \rho = r^n, \; \phi = n\theta$$

**18.60 MEQs**

(i) See (18.2.1) (i)
(ii) See (18.4.1)
(iii) See (18.4.2)
(iv) See (18.4.4)
(v) See (18.4.5)
(vi) See (18.4.7)
(vii) See (18.5.5)
(viii) See (18.5.6)
(ix) See (18 5.7)
(x) See (18.5.8)

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## Lesson - 19

# MOBIUS TRANSFORMATIONS

### 19.0 OBJECTIVE OF THIS LESSON :

After going through this lesson one should be able to (i) realize a linear fractional transformation (ii) identify a Mobius transformation (iii) understand a translation, dilation, rotation and inversion (iv) derive certain useful properties of a Mobius transformation (v) define cross ratio and observe its properties and (vi) state and derive the principles of symmetry and orientation.

### 19.1 INTRODUCTION :

First we introduce a linear fractional transformation and consider a special and useful transformation known as Mobius transformation. It will be observed that the class of all Mobius transformations form a group under the composition of mappings. Trnaslation, diletion, rotation and inversion are special types of Mobius transformations. It will be observed that a Mobius transformation is a composition of the above types of transformations. We introduce cross ratio of four complex numbers in $\mathbb{C}_\infty$ and observe that the cross ratio is invariant under a Mobius transformation. Also we prove that a Mobius transformation takes circles onto circles. Further it will be observed that symmetric points w.r.t a circle are transformed into symmetric points w.r.t. the transformed circle under a Mobius transformation.

### 19.2 MOBIUS TRANSFORMATION :

#### 19.2.1 DEFINITION :

Let $a,b,c,d$ be complex numbers such that at least one of $c,d$ is not zero. A mapping $S$ defined by $S(z)=\dfrac{az+b}{cz+d}$ for all $z\in\mathbb{C}$ with $cz+d\neq 0$ is called a linear fractional transformation.

If further $ad-bc\neq 0$ then $S$ is called a Mobius transformation.

#### 19.2.2 EXTENSION :

Let $S$ be a Mobius transformation, given by $S(z)=\dfrac{az+b}{cz+d}$ for all $z\in\mathbb{C}$ with $cz+d\neq 0$; $a,b,c,d\in\mathbb{C}$.

Define $S(\infty)=a/c$ and $S\left(-d/c\right)=\infty$ (we observe that, $ad-bc\neq 0 \Rightarrow$ atleast one of $a,c$ is not zero and at least one of $b,d$ is not zero). Now $S:\mathbb{C}_\infty\to\mathbb{C}_\infty$.

### 19.2.3 EXAMPLE :

Show that the set of all Mobius transformations from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, under the composition of mappings is a group.

**Solution :** We know that the class of mappings from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ is a group under the composition of mappings. Let $\mathcal{M}$ be the class of all Mobius transformations (from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$). Let $S, T \in \mathcal{M}$ defined by $S(z) = \dfrac{az+b}{cz+d}$ and $T(z) = \dfrac{\alpha z+\beta}{\gamma z+\delta}$ (with $ac - bd \neq 0$ and $\alpha\delta - \beta\gamma \neq 0$).

For any $z \in \mathbb{C}$

$$(S \circ T)(z) = S(T(z)) = \frac{a\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)+b}{c\left(\dfrac{\alpha z+\beta}{\gamma z+\delta}\right)+d}$$

$$= \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+d\delta)}$$

Since $(a\alpha+b\gamma)(c\beta+d\delta)-(a\beta+b\delta)(c\alpha+d\gamma) =$

$$(ac\alpha\beta + ad\alpha\delta + bc\beta\alpha + bd\gamma\delta) -$$

$$(ac\alpha\beta + ad\beta\gamma + bc\alpha\delta + bd\gamma\delta) =$$

$$(ad-bc)(\alpha\delta-\beta\gamma) \neq 0 \Rightarrow S \circ T \in \mu_0 .$$

Let $S_1(z) = \dfrac{dz-b}{-cz+a}\ \forall z \in \mathbb{C}_\infty$. Clearly $S_1 \in \mathcal{M}$

and $$(S \circ S_1)(z) = \frac{a\left(\dfrac{dz-b}{-cz+a}\right)+b}{c\left(\dfrac{dz-b}{-cz+a}\right)+d} = \frac{(ad-bc)z}{(ad-bc)} = z\ \forall z \in \mathbb{C}_\infty$$

$\text{III}^{\text{rly}}$ $(S_1 \circ S) = I$ (the identity transformation on $\mathbb{C}_\infty$).

Hence $S^{-1}$ exists $f = S_1 \in \mathcal{M}$.

This $\mathcal{M}$ is a group under the composition of mappings.

**19.2.4 OBSERVATION :**

From (19.2.3) we conclude that $S$ is 1 - 1 and onto from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$.

**19.2.5 SELF ASSESSMENT QUESTION (S.A.Q.) :**

$S_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, $S_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$ $\forall z \in \mathbb{C}_\infty$ be Mobius transformations. Show that $S_1 = S_2$ if and only if there is a $0 \neq \lambda \in \mathbb{C}$ such that $a_1 = \lambda a_2,\ b_1 = \lambda b_2,\ c_1 = \lambda c_2,\ d_1 = \lambda d_2$.

**19.2.6 DEFINITION :**

A transformation $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is called

(i) a translation iff $S(z) = z + b \ \forall\ z \in \mathbb{C}_\infty \ (b \in \mathbb{C})$

(ii) a rotation iff $S(z) = e^{i\theta} z \ \forall z \in \mathbb{C}_\infty \ (\theta \in \mathbb{R})$

(iii) a dilation iff $S(z) = az \ \forall\ z \in \mathbb{C}_\infty \ (0 \neq a \in \mathbb{C})$ and

(iv) an inversion iff $S(z) = 1/z \ \forall\ z \in \mathbb{C}_\infty$.

**19.2.7 OBSERVATION :**

All the transformations in the definition (19.2.6) are Mobius transformations (in the usual notation, we can show that $ad - bc \neq 0$ in all the ones).

**19.2.8 THEOREM :**

Any Mobius transformation is the composition of translations, dilations and the inversion (some of them may not be present depending upon the nature of the transformation).

**Proof :** Let $f(z) = \dfrac{az+b}{cz+d} \forall z \in \mathbb{C}$, where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0 \ (\Rightarrow S \in \mathcal{M})$.

If $c = 0$ then $d \neq 0 \Rightarrow S(z) = \dfrac{az+b}{d} = \left(\dfrac{a}{d}\right) z + \dfrac{b}{d} \forall z \in \mathbb{C}_\infty$

Define $S_1(z) = \left(\dfrac{a}{d}\right) z$ and $S_2(z) = z + \dfrac{b}{d} \ \forall\ z \in \mathbb{C}_\infty$. Now

$$(S_2 \circ S_1)(z) = S_2(S_1(z)) = \left(\frac{a}{d}\right)z + \frac{b}{d} = S(z) \Rightarrow S_2 \circ S_1 = S.$$

Thus $S$ is a composition of translation and dilation.

If $c \neq 0$, define $S_1(z) = z + \frac{a}{c}$, $S_2(z) = \left(\frac{bc-ad}{c^2}\right)z$,

$S_3(z) = \frac{1}{z}$ and $S_4(z) = z + \frac{d}{c}$ $\forall\, z \in \mathbb{C}_\infty$. Now

$$(S_1 \circ S_2 \circ S_3 \circ S_4)(z) = (S_1 \circ S_2 \circ S_3)\left(z + \frac{d}{c}\right) = (S_1 \circ S_2)\left(\frac{1}{z + d/c}\right)$$

$$= S_1\left(\frac{bc-ad}{c^2} \frac{1}{z + \frac{d}{c}}\right) = \frac{bc-ad}{c(cz+d)} + \frac{a}{c} = \frac{a(cz+d)+(bc-ad)}{c(cz+d)} = \frac{c(az+b)}{c(cz+d)} = S(z)$$

$$\forall\, z \in \mathbb{C}_\infty \Rightarrow S_1 \circ S_2 \circ S_3 \circ S_4 = S.$$

Thus $S$ is a composition of translation, inversion, dilation and again a translation.

This completes the proof of the Theorem.

**19.2.9 DEFINITION :**

Let $f$ be a selfmap on the non-empty sets (i.e. $f : S \to S$) $\mathscr{E} \in S$ is said to be a fixed point of $S$ iff $S(\mathscr{E}) = \mathscr{E}$.

**19.2.10 EXAMPLE :**

A Mobius transformation has atmost two fixed points, if it is not the identity transformation.

**Solution :** Let $S(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ and $z \in \mathbb{C}_\infty$. So $S$ is a Mobius transformation. Now, for any $z \in \mathbb{C}$, $S(z) = z \Leftrightarrow \frac{az+b}{cz+d} = z \Leftrightarrow$

$$az+b=z(cz+d)\Leftrightarrow(cz^2)+(d-a)z-b=0.$$

This is an equation in $z$, of degree $\leq z$. If $S\neq I$. So it has atmost two roots $\Rightarrow$ there are atmost two fixed points to a Mobius transformation.

**19.3 THE CROSS RATIO :**

**19.3.1 Theorem :** A Mobius transformation is uniquely determined by its action on any three points in $\mathbb{C}_\infty$.

**Proof :** Let $S,T$ be Mobius transformations and $z_j\in\mathbb{C}(j=1,2,3)$ be such that $S(z_j)=T(z_j)$ for $j=1,2,3$. Now follows that $T^{-1}\text{o }S$ in a Mobius transformation such that $(S^{-1}\text{o }T)(z_j)=S^{-1}(T(z_j))=z_j\,(j=1,2,3)$, by hypothesis. So $S^{-1}\text{ o }T$ has three fixed points. By 19.2.1 follows that $S^{-}\text{ o }T=I\Rightarrow T=S$. This proves the Theorem.

**19.3.2 THEOREM :**

$z_2,z_3,z_4$ are three distinct points in $\mathbb{C}_\infty$. There is a (unique) Mobius transformation $S$ such that $S(z_2)=1,S(z_3)=0,\ S(z_4)=\infty$.

**Proof :** Let $z_2,z_3,z_4$ be distinct points in $\mathbb{C}_\infty$. Consider $S$ defined on $\mathbb{C}_\infty$ by

$$S(z)=\left(\frac{z-z_3}{z-z_4}\right)\Big/\left(\frac{z_2-z_3}{z_2-z_4}\right)\ \forall\ z\in\mathbb{C}_\infty;\ \text{if}\ \ z_2,z_3,z_4\in\mathbb{C};$$

$$S(z)=\frac{z-z_3}{z-z_4}\ \forall\ z\in\mathbb{C}_\infty,\ \text{if}\ z_2=\infty,$$

$$S(z)=\frac{z_2-z_4}{z-z_4}\ \forall\ z\in\mathbb{C}_\infty,\ \ \text{if}\ \ z_3=\infty,$$

$$S(z)=\frac{z-z_3}{z_2-z_3}\ \forall\ z\in\mathbb{C}_\infty,\ \text{if}\ \ z_4=\infty.$$

In all the cases, $S(z_2)=1,\ S(z_3)=0$ and $S(z_4)=\infty$.

Thus $S$ is the required transformation.

**19.3.3 DEFINITION :**

Let $z_1, z_2, z_3, z_4$ are in $\mathbb{C}_\infty$ and $z_2, z_3, z_4$ are distinct. Then the cross ratio of $z_1, z_2, z_3, z_4$ denoted by $(z_1, z_2, z_3, z_4)$, is the image of $z_1$ under the (unique, Mobius) transformation that maps $z_2, z_3, z_4$ respectively to $1, 0, \infty$.

**19.3.4 OBSERVATIONS :**

(i) $(z_2, z_2, z_3, z_4) = 1; (z_3, z_2, z_3, z_4) = 0; (z_4, z_2, z_3, z_4) = \infty$.

(ii) If $T$ is the (Mobius) transformation and $w_2, w_3, w_4 \in \mathbb{C}_\infty$ with $Tw_2 = 1, Tw_3 = 0, Tw_4 = \infty$, then for any $z \in \mathbb{C}_\infty, (z, w_2, w_3, w_4) = Tz$.

So $(z, 1, 0, \infty) = I(z) = z$, since the identity transformation maps $1, 0, \infty$ on $1, 0, \infty$ respectively.

**19.3.5 SAQ :**

Evaluate

(i) $(1, -1, \infty, 0)$ (ii) $(0, -1, \infty, i)$ (iii) $(0, 1, i, 3)$.

**19.3.6 THEOREM :**

If $z_1, z_2, z_3, z_4$ are distinct points in $\mathbb{C}_\infty$ and $z_1 \in \mathbb{C}_\infty$ then under any Mobius transformation $T$,

$$(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4).$$

(Cross ratio is preserved under a Mobius transformation)

**Proof :** Let $z_2, z_3, z_4, z_1$ and $T$ be as in the statement

Let $S$ be the Mobius transformtion that takes $z_2, z_3, z_4$ to $1, 0, \infty$ respy. (i.e. $Tz_2 = 1, Tz_3 = 0, Tz_4 = \infty$). So $(z_1, z_2, z_3, z_4) = Sz_1$. Denote $M = S o T^{-1}$.

Clearly $M \in \mathcal{M}$ and

$$M(Tz_2) = S(T^{-1}(Tz_2)) = S(z_2) = 1,$$

$$M(Tz_3) = S(T^{-1}(Tz_3)) = S(z_3) = 0,$$

$$M(Tz_4) = S\left(T^{-1}(Tz_4)\right) = S(z_4) = \infty.$$

Hence, by definition,

$$(Tz_1, Tz_2, Tz_3, Tz_4) = M(Tz_1) = \left(S\,o\,T^{-1}\right)(Tz_1)$$

$$= S\left(T^{-1}(Tz_1)\right) = S(z_1) = (z_1, z_2, z_3, z_4).$$

Then the result follows.

Now we prove a result that generalizes Theorem (19.3.2).

**19.3.7 THEOREM :**

Let $\{z_2, z_3, z_4\}$ and $\{w_2, w_3, w_4\}$ be triads of distinct points of $\mathbb{C}_\infty$. Then threre is a uniqe, Mobius transformation $T$ such that $T(z_j) = w_j; (j = 1, 2, 3)$.

**Proof :** Let $z_2, z_3, z_4$ be distinct and $w_2, w_3, w_4$ be distinct numbers in $\mathbb{C}_\infty$. Let $Sz = (z_1, z_2, z_3, z_4)$ and $Mz = (z, w_1, w_2, w_3, w_4)$ $\forall\ z \in \mathbb{C}_\infty$ ($S$ and $M$ exist by definition). Denote $T = M^{-1}\,o\,S$. Clearly $T$ is a Mobius transformation. $T(z_2) = M^{-1}(Sz_z) = M^{-1}(1) = W_2$, $T(z_3) = M^{-1}(Sz_3)$ $= M^{-1}(0) = w_3$ and $T(z_4) = M^{-1}(Sz_4) = M^{-1}(\infty) = w_4$.

Hence $T$ is a tranformation of the required type. By Theorem (19.3.1), uniqueness follows. Thus the result follows.

**19.3.8 REMARK :**

A line in $\mathbb{C}$ can be regarded as a circle in $\mathbb{C}_\infty$ passing through $\infty$. We know that any three points in the plane determine a circle.

We now, present a necessary and sufficient condition for four points in $\mathbb{C}_\infty$ to be concyclic.

**19.3.9 THEOREM :**

$z_1, z_2, z_3, z_4$ are four distinct points in $\mathbb{C}_\infty$. Then these four points lie on a circle if and only if (the cross ratio) $(z_1, z_2, z_3, z_4)$ is real.

**Proof :** Let $z_1, z_2, z_3, z_4$ be four distinct points in $\mathbb{C}_\infty$. Let $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by $Sz = (z, z_2, z_3, z_4)$ for $\forall z \in \mathbb{C}_\infty$ ($\Rightarrow S$ carries $z_2, z_3, z_4$ to $1, 0, \infty$ respy). Now

$S^{-1}(\mathbb{R})=\left[z \in \mathbb{C}_{\infty}:(z, z_1, z_2, z_3) \in \mathbb{R}\right]$. Hence follows the result if the image of $\mathbb{R}_{\infty}$ under a Mobius transformation is a circle.

Let $S(z)=\dfrac{az+b}{cz+b}$ with $a,b,c,d \in \mathbb{C}$ & $ad-bc \neq 0$.

If $z=x \in \mathbb{R}$ and $w=S^{-1}(x)$ then $S(w)=x \Rightarrow S(w)=\overline{f(w)}$

$$\therefore \frac{aw+b}{cw+d}=\overline{\left(\frac{aw+b}{cw+d}\right)}=\frac{\bar{a}\,\bar{w}+\bar{b}}{\bar{c}\,\bar{w}+\bar{d}} \Rightarrow$$

$$(a\bar{c}-\bar{a}\,c)\,|w|^2+(a\bar{d}-\bar{b}\,c)w+(b\bar{c}-\bar{a}\,d)\bar{w}+(b\bar{d}-\bar{b}\,d)=0 \text{ ------- (i)}$$

If $a\bar{c}$ is real then $a\bar{c}=\overline{a\bar{c}}=\bar{a}\,c \Rightarrow a\bar{c}-\bar{a}\,c=0$

So (i) becomes

$$(a\bar{d}-\bar{b}c)w+(b\bar{c}-\bar{a}d)\bar{w}+(b\bar{d}-\bar{b}d)=0$$

$\Rightarrow \alpha w-\bar{\alpha}\,\bar{w}+\beta=0$ where $\alpha=a\bar{d}-\bar{b}\,c,\ \beta=b\bar{d}-\bar{b}\,d$,

$$\Rightarrow 2i\,\mathrm{Im}\,(\alpha\, w)+\beta=0 \Rightarrow 0$$

$$\mathrm{Im}(\alpha w)+\frac{1}{2i}\beta=0 \Rightarrow \mathrm{Im}(\alpha w)-i\left(\frac{\beta}{2}\right)=0$$

$$\Rightarrow \mathrm{Im}\left(\alpha w-\frac{i\beta}{2}\right)=0\left(\because \frac{i\beta}{2} \text{ is real}\right) \text{ ------------ (ii)}$$

$\Rightarrow w$ lies on the line determined by (ii) for fixed $\alpha$ and $\beta$.

Suppose $a\bar{c}$ is not real. $\Rightarrow a\bar{c}-\bar{a}\,c \neq 0$. So (i) becomes

$$|w|^2+\left(\frac{a\bar{d}-\bar{b}c}{a\bar{c}-\bar{a}\,c}\right)w+\left(\frac{b\bar{c}-\bar{a}\,d}{a\bar{c}-\bar{a}\,c}\right)\bar{w}+\frac{b\bar{d}-\bar{b}d}{a\bar{c}-\bar{a}\,c}=0$$

$$\text{i.e. } |w|^2+\left(\frac{b\bar{c}-\bar{a}\,d}{a\bar{c}-\bar{a}\,c}\right)w+\left(\frac{b\bar{c}-\bar{a}\,d}{a\bar{c}-\bar{a}\,c}\right)\bar{w}+\left(\frac{b\bar{d}-\bar{b}\,d}{a\bar{c}-\bar{a}\,c}\right)=0$$

$$\Rightarrow (w-r)\overline{(w-r)} = |r|^2 - \left(\frac{b\bar{d}-\bar{b}\,d}{a\bar{c}-\bar{a}\,c}\right) \text{ ---------- (iii)}$$

where $r = \left(\frac{b\bar{c}-\bar{a}\,d}{a\bar{c}-\bar{a}\,c}\right)$

Now $|r|^2 - \left(\frac{b\bar{d}-\bar{b}\,d}{a\bar{c}-\bar{a}\,c}\right) = \frac{(b\bar{c}-\bar{a}\,d)(\bar{b}\,c-a\bar{d})-(b\bar{d}-\bar{b}\,d)(\bar{a}\,c-a\bar{c})}{|a\bar{c}-\bar{a}\,c|^2}$

$$= \frac{\left(|b|^2\,|c|^2 - ab\,\bar{c}\,\bar{d} - \bar{a}\,\bar{b}\,c\,d + |a|^2\,|d|^2\right) - \left(\bar{a}\,b\,c\,\bar{d} - \bar{a}\,\bar{b}\,c\,d - a\bar{b}\,\bar{c}\,\bar{d} + a\,\bar{b}\,\bar{c}\,d\right)}{|a\bar{c}-\bar{a}\,c|^2}$$

$$= \frac{|b|^2\,|c|^2 - \left(bc\,\overline{ad} + \overline{bc}\,ad\right) + |a|^2\,|d|^2}{|a\bar{c}-\bar{a}\,c|^2}$$

$$= \frac{|bc-ad|^2}{|a\bar{c}-\bar{a}\,c|^2}.$$

$\therefore$ (iii) becomes $|w-r|^2 = \lambda^2 \Rightarrow |w-r| = \lambda$ where

$\lambda = \left|\frac{bc-ad}{a\bar{c}-\bar{a}\,c}\right| > 0$ and this represents a circle.

This completes the proof of the Theorem.

**19.3.10 THEOREM :**

A Mobius transformation takes circles onto circles.

**Proof :** Let $S$ be a Mobius transformation and $\Gamma$ be a circle in $\mathbb{C}_\infty$. Let $z_2, z_3, z_4$ be distinct points on $\Gamma$ and $Sz_j = w_j\,(j = 2,3,4)$. $w_2, w_3, w_4$ determine a circle, say $\Gamma'$. For any $z \in \mathbb{C}_\infty$, by Theorem (19.3.6),

$$(z,\, z_2, z_3, z_4) = (Sz,\, w_2, w_3, w_4) \text{ ------- (i)}$$

If $z \in \Gamma$, then by Theorem (19.3.9), follows that the L.H.S. of (i) is a real number and so is

R.H.S. Since $w_2, w_3, w_4 \in \Gamma'$, by Theorem (19.3.9), follows that $Sz \in \Gamma'$. This is true for all $z \in \Gamma \Rightarrow S(\Gamma) = \Gamma'$ and this completes the proof of the Theorem.

**19.3.11 THEOREM :**

$\Gamma$ and $\Gamma'$ are circles in $\mathbb{C}_\infty$. Then, there is a Mobiun transformation $R$ such that $R(\Gamma) = \Gamma'$. Further $R$ takes any three points on $\Gamma$ onto any three points on $\Gamma'$. If $z_j\,(j=1,2,3)$ are distinct points on $\Gamma$ and $w_j\,(j=1,2,3)$ are distinct points on $\Gamma'$ with $R(z_j) = w_j\,(j=1,2,3)$ then such $R$ is unique.

**Proof :** Let $\Gamma$ and $\Gamma'$ be circles in $\mathbb{C}_\infty$. Let $z_j \in \Gamma$ and $W_j \in \Gamma'(j=1,2,3)$ be distinct. Let $Sz = (z, z_2, z_3, z_4)$ and $Tz = (z, w_2, w_3, w_4)\,\forall\, z \in \mathbb{C}_\infty$. Denote $R = T^{-1} \circ S$. Clearly $R$ is Mobius transformation. Now $R(z_2) = T^{-1} \circ S$. Clearly $R$ is a Mobius transformation. Now $R(z_2) = T^{-1}(S(z_2)) = T^{-1}(1) = w_2 \cdot R(z_3) = T^{-1}(S(z_3)) = T^{-1}(0) = w_3$ and $R(z_4) = T^{-1}(S(z_4))$

$= T^{-1}(\infty) = w_4$. Now by (19.3.10) follows that $R(\Gamma) = \Gamma'$.

This proves the result except the uniqueness. Let $R_1$ be also a Mobius transformation such that $R_1(z_j) = w_j\,(j=1,2,3)$. Now consider the transformation $\left(R_1^{-1} \circ R\right)$. Clearly it is a Mobius transformation. Further $\left(R_1^{-1} \circ R\right)$. Clearly it is a Mobius transformation. Further $\left(R_1^{-1} \circ R\right)(z_j) = R_1^{-1}(R(z_j)) = R_1^{-1}(w_j) = z_j\,(j=1,2,3)$. Hence $\left(R_1^{-1} \circ R\right)$ has three fixed points $z_j\,(j=1,2,3)$. This, by a known result, implies that $R_1^{-1} \circ R = I$ (the identity transformation on $\mathbb{C}_\infty$) $\Rightarrow R_1 = R$. This completes the proof of the Theorem. We can use Theorem 19.3.1).

**19.4 PRINCIPLE OF SYMMETRY :**

We observed that a Mobius transformation maps circles onto circles. Now, the further problem is about the inside and outside of these circles. For discussing about this, we need the following :

**19.4.1 DEFINITION :**

Let $\Gamma$ be a circle in $\mathbb{C}_\infty$ and $z_2, z_3, z_4$ be distinct points on $\Gamma$. The points $z, z^*$ in $\mathbb{C}_\infty$ are said to be symmetric points with respect to $\Gamma$ iff

$$\left(z^*, z_2, z_3, z_4\right) = \overline{\left(z, z_2, z_3, z_4\right)}$$

**19.4.2 OBSERVATION :**

(i) The definition of symmetry does not depend on the choice of points $z_2, z_3, z_4$ on $\Gamma$.

(ii) $z \in \mathbb{C}_\infty$ is symmetric to itself iff $z \in \Gamma$; for $z \in \mathbb{C}_\infty$ is symmetric to $z \Leftrightarrow \left(z, z_2, z_3, z_4\right)$

$= \overline{\left(z, z_2, z_3, z_4\right)}$

$\Leftrightarrow \left(z, z_2, z_3, z_4\right)$ is real $\Leftrightarrow z \in \Gamma$ (by 19.3.9).

Now we discuss the geometric significance of symmetry.

**19.4.3 PROBLEM :**

If $\Gamma$ is a straight line, then show that $z, z^* \in \mathbb{C}$ are symmetric with respect to $\Gamma$ then $z, z^*$ are equidistant from $\Gamma$ and the line through $z, z^*$ is perpendicular to $\Gamma$ (when $z$ is not on $\Gamma$).

**Solution :** Let $\Gamma$ be a straight line in $\mathbb{C}_\infty$. So, it can be regarded as a circle in $\mathbb{C}_\infty$ through $\infty$. Let $z, z^* \in \mathbb{C}$ and are not on $\Gamma$. Let $z_2, z_3 \in \Gamma$. Now $z$ and $z^*$ are symmetric with respect to $\Gamma$ if and only if

$$\left(z^*, z_2, z_3, \infty\right) = \overline{\left(z, z_2, z_3, \infty\right)}$$

$$\Leftrightarrow \frac{z^* - z_3}{z_2 - z_3} = \overline{\left(\frac{z - z_3}{z_2 - z_3}\right)} = \frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3} \text{ --------- (i)}$$

$$\Leftrightarrow \left|\frac{z^* - z_3}{z_2 - z_3}\right| = \left|\frac{\bar{z} - \bar{z}_3}{\bar{z}_2 - \bar{z}_3}\right|$$

$$\Leftrightarrow \left|z^* - z_3\right| = \left|\bar{z} - \bar{z}_3\right| \quad \left(\because \left|z_2 - z_3\right| = \left|\overline{z_2 - z_3}\right| = \left|\bar{z}_2 - \bar{z}_3\right|\right)$$

$$= \left|z - z_3\right|$$

Since $z_3$ is an arbitrary point on $\Gamma$, follows that $z$ and $z^*$ are equidistant from each point on $\Gamma$.

Also (i) $\Rightarrow \mathrm{Im}\left(\dfrac{z^{*}-z_3}{z_2-z_3}\right)=\mathrm{Im}\overline{\left(\dfrac{z-z_3}{z_2-z_3}\right)}=\mathrm{Im}\left(\dfrac{z-z_3}{z_2-z_3}\right)$

$\Rightarrow z$ and $z^{*}$ lie in different half planes determined by $\Gamma$. Thus the line through $z, z^{*}$ is perpendicular to $\Gamma$.

**19.4.4 PROBLEM :**

If $\Gamma$ is a circle, given by $|w-z_0|=R$ where $z_0 \in \mathbb{C}$ and $R>0$ then $z, z^{*} \in \mathbb{C}$ are symmetric w.r.t. $\Gamma \Rightarrow z, z_0, z^{*}$ are collinear and $|z-z_0|\,|z^{*}-z_0|=R^2$.

**Solution :** Let $\Gamma : |w-z_0|=R\left(z_0 \in \mathbb{C},\ R>0\right)$ and $z, z^{*} \in \mathbb{C}$ are symmetric w.r.t. $\Gamma$. Let $z_2, z_3, z_4$ be points on $\Gamma$. Since $z, z^{*}$ are symmetric points w.r.t. $\Gamma$, we have

$$\left(z^{*}, z_2, z_3, z_4\right)=\overline{\left(z, z_2, z_3, z_4\right)}$$

$$=\overline{\left(z-z_0, z_2-z_0, z_3-z_0, z_4-z_0\right)} \text{ (by 19.3.6)}$$

$$=\left(\bar{z}-\bar{z}_0\ \overline{z_2-z_0},\ \overline{z_3-z_0},\ \overline{z_4-z_0}\right)$$

$$=\left(\bar{z}-\bar{z}_0, \frac{R^2}{z_2-z_0}, \frac{R^2}{z_3-z_0}, \frac{R^2}{z_4-z_0}\right)$$

$$\left(\because z_2,\ z_3,\ z_4 \in \Gamma \Rightarrow |z_2-z_0|^2=|z_3-z_0|^2=|z_4-z_0|^2=R^2\right)$$

$$=\left(\frac{R^2}{\bar{z}-\bar{z}_0}, z_2-z_0, z_3-z_0, z_4-z_0\right) \text{ (the mapping } 2 \rightarrow R^2/2\text{ )}$$

$$=\left(\frac{R^2}{\bar{z}-\bar{z}_0}+z_0, z_2, z_3, z_4\right)\left(z \rightarrow z+z_0\right) \text{ ----------- (i)}$$

Since a Mobius transformation is one - to - one, it follows that

$$z^* = \frac{R^2}{\bar{z} - \bar{z}_0} + z_0 = z_0 + R^2 \left(\bar{z} - \bar{z}_0\right)^{-1}$$

$$\Rightarrow z^* - z_0 = R^2 \left(\bar{z} - \bar{z}_0\right)^{-1} \Rightarrow \left(z^* - z_0\right)\left(\bar{z} - \bar{z}_0\right) = R^2$$

$$\Rightarrow \frac{z^* - z_0}{z - z_0} = \frac{R^2}{\left(z - z_0\right)\left(\bar{z} - \bar{z}_0\right)} = \frac{R^2}{\left(z - z_0\right)\left(\overline{z - z_0}\right)} = \frac{R^2}{\left|z - z_0\right|^2} > 0$$

$\Rightarrow z^* = z_0 + t\left(z - z_0\right)(t > 0) \Rightarrow z^*$ lies on the line joining $z$ and $z_0 \Rightarrow z, z_0, z^*$ are collinear and $\left|z - z_0\right|\left|z^* - z_0\right| = R^2$.

**19.4.5 THEOREM (SYMMETRIC PRINCIPLE) :**

$\Gamma_1$ and $\Gamma_2$ are circles in $\mathbb{C}_\infty$ and $T$ be a Mobius transformation that maps $\Gamma_1$ and $\Gamma_2$. Then any pair of points symmetric with respect to $\Gamma_1$ are mapped by $T$ to a pair of points symmetirc with respect to $\Gamma_2$.

**Proof :** Let $T, \Gamma_1, \Gamma_2$ be as in the statement of the Theorem. Let $z, z^* \in \mathbb{C}$ be symmetric points with respect to $T$. Let $z_2, z_3, z_4$ be distinct points on $\Gamma$. Since $T$ is one-to-one follows that $Tz_2, Tz_3, Tz_4$ are distinct points on $\Gamma_2$. By Theorem (19.3.6), since $T$ is a Mobius transformation, follows that

$$\left(Tz^*, Tz_2, Tz_3, Tz_4\right) = \left(z^*, z_2, z_3, z_4\right)$$

$$= \overline{\left(z, z_2, z_3, z_4\right)} \text{ (by definition)}$$

$$= \overline{\left(Tz, Tz_2, Tz_3, Tz_4\right)}$$

Hence, by definition, follows that $Tz, Tz^*$ are symmetric points with respect to $\Gamma_2$. This completes the proof of the Theorem.

**19.5 ORIENTATION FOR CIRCLES IN $\mathbb{C}_\infty$ :**

We, now, discuss orientation for circles in $\mathbb{C}_\infty$. This leads to distinguish between the interior (inside) and exterior (outside) of a circle.

**19.5.1 DEFINITION :**

An orientation of a circle $\Gamma$ (in $\mathbb{C}_\infty$) is an ordered triple $(z_1, z_2, z_3)$ of points (complex members) such that each $z_j\,(j=1,2,3)$ is in $\Gamma$.

**19.5.2 OBSERVATION :**

An ordered triple $(z_1, z_2, z_3)$ of points on a circle $\Gamma$ gives a direction to $\Gamma$; on $\Gamma$.

We move from $z_1$ to $z_2$ and then $z_2$ to $z_3$ (and from $z_3$ to $z_1$).

Only two points $z_1, z_2$ are given on $\Gamma$, moving from $z_1$ to $z_2$ leads to amibiguity.

**19.5.3 OBSERVATIONS :**

(i) Let $\Gamma = \mathbb{R}$ and $z_2, z_3, z_4 \in \Gamma = \mathbb{R}$. Let $T$ be the Mobius transformation specified by $Tz = (z,\ z_2, z_3, z_4) = \dfrac{az+b}{cz+d}, a,b,c,d \in \mathbb{C}, \forall z \in \mathbb{C}_\infty$. Since $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$, by Theorem (19.3.9) follows that $a,b,c,d$ be taken as reals. So

$$Tz = \frac{az+b}{cz+d} = \frac{(az+b)\overline{(cz+d)}}{(cz+d)\overline{(cz+d)}} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}$$

$$= \frac{\left(ac|z|^2 + bd + a\,d\,z + b\,c\,\bar{z}\right)}{|cz+d|^2}$$

$$(\because z = \operatorname{Re} z + i \operatorname{Im} z \Rightarrow \bar{z} = \operatorname{Re} z - i \operatorname{Im} z)$$

$$\therefore \operatorname{Im}(z, z_2, z_3, z_4) = \operatorname{Im} Tz = \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} \quad \forall\ z \in \mathbb{C}.$$

$\therefore\ ad-bc > 0 \Rightarrow \{z \in \mathbb{C} : \operatorname{Im}(z, z_2, z_3, z_4) > 0\}$ is the upper half plane w.r.t. $\Gamma (\because \Rightarrow \operatorname{Im} z > 0)$

$ad-bc < 0 \Rightarrow \{z \in \mathbb{C} : \operatorname{Im}(z, z_2, z_3, z_4) < 0\}$ is the upper half plane w.r.t. $\Gamma (\because \Rightarrow \operatorname{Im} z > 0)$

Thus $\{z \in \mathbb{C} : \operatorname{Im}(z, z_2, z_3, z_4) > 0\}$ is either the upper half plane or lower half plane w.r.t. $\Gamma$

according as $ad-bc>$ or $<0$ (Clearly $ad-bc=\det T$).

(ii) Let $\Gamma$ be a circle in $\mathbb{C}$ and $z_2, z_3, z_4 \in \Gamma$. For any Mobius transformation $T$, by Theorem (19.3.6),

$$\{z \in C_\infty : \operatorname{Im}(z, z_2, z_3, z_4) > 0\} = \{z \in \mathbb{C}_\infty : \operatorname{Im}(Tz, Tz_1, Tz_2, Tz_3, Tz_4) > 0\}$$

$$= T^{-1}\{z \in \mathbb{C}_\infty : \operatorname{Im}(Tz, Tz_2, Tz_3, Tz_4) > 0\} \text{ -------- (i)}$$

If $T$ is such that $T(\Gamma) = \mathbb{R}_\infty$, then (i) $\Rightarrow$

$\{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) > 0\}$ is either $T^{-1}$ (uper half plane) or $T^{-1}$ (lower half plane).

**19.5.4 DEFINITION :**

$\Gamma$ is a circle in $\mathbb{C}_\infty$ and $\{z_2, z_3, z_4\}$ is an orientation of $\Gamma$. Then $\{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) > 0\}$ is called the right side of $\Gamma$ w.r.t. $(z_2, z_3, z_4)$ and

$\{z \in \mathbb{C}_\infty : \operatorname{Im}(z, z_2, z_3, z_4) < 0\}$ is called the left side of $\Gamma$ w.r.t. $(z_2, z_3, z_4)$.

We now prove the following.

**19.5.5 THEOREM (ORIENTATION PRINCIPLE) :**

Let $\Gamma_i\,(j = 1, 2)$ be circles in $\mathbb{C}_\infty$ and $T$ be a Mobius transformation such that $T(\Gamma_j) = \Gamma_{j'}$ where $j, j' \in \{1, 2\}$, $j \neq j'$. Let $\{z_2, z_3, z_4\}$ be an orientation for $T_j$. Then $T$ takes the right side / left side of $\Gamma_j$. Onto the right side / left side of $\Gamma_j$ respectively with respect to the orientation $(Tz_2, Tz_3, Tz_4)$.

**Proof :** Under the given hypothesis, for any $z \in \mathbb{C}_\infty$ by Theorem (19.3.6), $(z, z_2, z_3, z_4) = (Tz, Tz_2, Tz_3, Tz_4)$. Hence $\operatorname{Im}(z, z_2, z_3, z_4) > 0 \Leftrightarrow \operatorname{Im}(Tz, Tz_2, Tz_3, Tz_4) > 0$. So $z$ lies on the right side / left side of $\Gamma_j$ if and only if $Tz$ lies on the right side / left side of $\Gamma_{j'}$ respectively, with respective to the orientation $(Tz_2, Tz_3, Tz_4)$. This completes the proof of the Theorem.

**19.5.6 OBSERVATION :**

Consider the orientation $(1, 0, \infty)$ of $\mathbb{R}$. For any $z \in \mathbb{C}$, by the definition of cross ratio.

$(z,1,0,\infty) = I(z) = z$ ($I$ being the identity transformation on $\mathbb{C}_\infty$). So the right side of $\mathbb{R}$ w.r.t. to the orientation $(1,0,\infty)$ is the upper half plane (in $\mathbb{C}$).

**19.5.7 EXAMPLE :**

Find an analytic function $f : G \to \mathbb{C}$ where $G = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ such that $f(G) = D = \{w \in \mathbb{C} : |w| < 1\}$

**Solution :** We will find a Mobius transformation that taxes the imaginary axis in the $z$-plane onto the unit circle in $w$-plane. We consider the orientation such that $G$ is mapped onto $D$. Consider the orientation $(-i,0,i)$ of the imaginary in the $z$-plane axis. Then $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is on the right side of this axis.

For any $z \in \mathbb{C}$, $(z,-i,0,i) = \dfrac{(z-0)(-i-i)}{(z-i)(-i-0)} = \dfrac{2z}{z-i}$

$$= \frac{2z}{z-i} \cdot \frac{\overline{z-i}}{\overline{z-i}} = \frac{(2z)(\bar{z}+i)}{|(z-i)|^2}$$

$$= \frac{2}{|z-i|^2}\left(|z|^2 + iz\right)$$

So, $\{z \in \mathbb{C} : \operatorname{Im}(z,-i,0,i) > 0\} = \{z \in \mathbb{C} : \operatorname{Im}(iz) > 0\}$

$$\left(\because \operatorname{Im}\left(\frac{2|z|^2}{|z-i|^2}\right) > 0\right)$$

$$= \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$$

$(\because \operatorname{Im}(iz) = \operatorname{Re}(z))$.

Consider the orientation $(-i,-1,i)$ for the unit circle $\Gamma$ in the $w$-plane. $D$ lies on the right side of $\Gamma$.

Now, for any $z \in \mathbb{C}, (z,-i,-1,i) = \dfrac{(z+1)(-i-1)}{(z-i)(-i+1)}$

$$= \frac{2i}{i-1} \cdot \frac{z+1}{z-1}$$

So, if $S(z) = \frac{2z}{z-i}$ and $T(z) = \frac{2i}{i-1}\left(\frac{z+1}{z-i}\right)$ then $R = T^{-1} \text{o} S$ is a Mobius transformation that maps $G$ onto $D$.

## 19.6 MODEL EXAMINATION QUESTIONS :

(i) Define a Mobius transformation (See 19.2.1).

(ii) Show that the set of all Mobius transformations from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$, under the composition of mappings is a group (see 19.2.3).

(iii) Define (a) translation, rotation, dilation, inversion (see 19.2.6)

(iv) Show that any Mobius transformation is the composition of translations, dilations and inversions (see 19.2.8).

(v) Show that a Mobius transformation has atmost two fixed points, if it is not the identity transformation (see 19.2.10)

(vi) Show that a Mobius transformation is uniquely determined by its action on any three points in $\mathbb{C}_\infty$ (see 19.3.1).

(vii) Define cross ratio. Show that it is invariant under any Mobius transformation (see 19.3.3 and 19.3.5).

(viii) State and prove a necessary and sufficient condition for four distinct points in $\mathbb{C}_\infty$ to lie on a circle (see 19.3.9).

(ix) Show that a Mobius transformation takes circle onto circles. (see 19.3.10).

(x) Show that there is a Mobius transformation that takes a circle onto a circle (see 19.3.11).

(xi) Define the concept "$z, z^*$ in $\mathbb{C}_\infty$ are symmetric with respect to a circle in $\mathbb{C}_\infty$" (see 19.4.1).

(xii) State and prove symmetric principle (see 19.4.5)

(xiii) State and prove orientation principle (see 19.5.5)

(xiv) Find a Mobius transformation that maps right half plane into the interior of the unit disc (see 19.5.7).

**19.7 HINTS TO SAQ :**

**19.2.5 :** $S_1 = S_2 \Rightarrow S_1(0) = S_2(0) \Rightarrow \frac{b_1}{d_1} = \frac{b_2}{d_2} \Rightarrow \frac{b_1}{b_2} = \frac{d_1}{d_2}$ -------- (i)

$$S_1\left(-\frac{b_1}{a_1}\right) = S_2\left(-\frac{b_1}{a_1}\right) \Rightarrow 0 = a_2 b_1 - b_2 a_1 \Rightarrow \frac{b_1}{b_2} = \frac{a_1}{a_2} \text{ -------- (ii)}$$

$$S_1\left(-\frac{d_1}{c_1}\right) = S_2\left(-\frac{d_1}{c_1}\right) \Rightarrow c_2 d_1 - c_1 d_2 = 0 \Rightarrow \frac{d_1}{d_2} = \frac{c_1}{c_2} \text{ ------- (iii)}$$

(i), (ii) & (iii) $\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$. The converse is trivial.

**19.3.5 :** (i) $(1, -1, \infty, 0) = \frac{-1-0}{1-0} = -1$

$$\left(\because S(z) = \frac{z_2 - z_4}{z - z_4} \text{ if } z_3 = \infty\right)$$

(ii) $(0, -1, \infty, i) = \frac{-1-i}{0-i} = \frac{1+i}{i} = 1 + \frac{1}{i} = 1 - i$

(iii) $(0, 1, i, 3) = \frac{(0-i)(1-3)}{(0-3)(1-i)} = \frac{2i}{3(i-1)} = \frac{2i(i+1)}{3(i^2-1)}$

$$= \frac{2(i^2 + i)}{3(-2)} = -\frac{1}{3}(-1+i) = \frac{1}{3}(1-i).$$

**REFERENCE BOOKS :**


1. J. B. Connay : Functions of One Complex Variable - Second Edition - Springer International Student Edition.

2. Ruel V. Churchil; Jamesward Brown : Complex Variables and Applications - Mc-Graw Hill International Editions - Fifth Edition.



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Lesson - 20

# BASIC RESULTS IN COMPLEX INTEGRATION

## 20.0 INTRODUCTION

We use the results of complex integration to prove the basic results of complex analysis in the simplest cases : Cauchy Integral Formula

and

Cauchy's Theorem.

using these, we prove that an analytic functionis infinitely differeniable and it has a primitive in a sufficiently small neighbourhood of each point.

We prove Morera's theorem.

We present, finally, a proof of Goursat's theorem.

These two results clarify the notion of differentiability in the complex domain.

## 20.1 PRELIMINARY RESULTS :

### 20.1.1 DEFINITION :

Suppose $f, \varphi : [a, b] \to \mathbb{C}$.

We say that Riemann-Stiltjes Integral of $f$ with respect to $\varphi$ exists if there is a complex number $\alpha$ with the following property :

Given $\varepsilon > 0$ it is possible to choose a $\delta(\varepsilon) > 0$ such that for any partition

$$P = a = t_0 < t_1 < \cdots < t_n = b$$

and each choice

$$e_0 \in [t_0, t_1], \cdots, e_{n-1} \in [t_{n-1}, t_n]$$

We have $\left| \sum_{j=0}^{n-1} f(e_j)\left[\varphi(t_{j+1}) - \varphi(t_j)\right] - \alpha \right| < \varepsilon$

if $\|P\| < \delta(\varepsilon)$.

(observe that $\alpha$ is unique).

**Notation :** If for a pair of functions $f, \varphi$ above the Riemann-Stieltjes Integral of $f$ with respect to $\varphi$ exists, we denote it variously by

$$I(f,\varphi),\ \int_{\varphi} f\, d\varphi,\ \int f,\ \int_a^b f\, d\varphi,\ \int_a^b f(t)\, d\varphi(t).$$

We state two simple results. The proofs are routine and are left as exercises. In both the integrator $\varphi$ is

$$\varphi(t) = t$$

and $f$ is assumed continuous.

**20.1.2 PROPOSITION :**

(i) Suppose $f(t) = A(t) + iB(t)$

where $A(t)$, $B(t)$ are real. Then

$$\int_a^b f(t)\,dt = \int_a^b A(t)\,dt + i\int_a^b B(t)\,dt.$$

(ii) Suppose $f'$ exists and is continuous. Then

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

**20.1.3 REMARK :**

The integrals on the right hand side in (a) are the usual Riemann Integrals. It is easy to prove and known that $A, B$ are continuous if $f$ is continuous and $A, B$ are differentiable if $f$ is differentiable. The result (b) may be treated as a simple generalisation of the fundamental theorem of calculus. We do not have the mean value theorem of differential calculus for complex valied functions. The result (b) above may be used in its place when we need only inequality.

It is clear that the integrals on the left exist is part of the assertion.

**20.1.4 PROPOSITION :**

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt, \text{ if } a < b.$$

$$\leq \left| \int_a^b |f(t)|\, dt \right|.$$

**20.1.5 DEFINITION :**

Suppose $\varphi:[a, b] \to \mathbb{C}$ is a function. We say that

$\varphi$ is smooth or $\varphi$ is $C'$

if the derivative $\varphi'$ of $\varphi$ exists on $[a, b]$ and $\varphi'$ is a continuous function.

**20.1.6 REMARK :**

It is easy to see that if $f(t) = A(t) + i\, B(t)$ then $f$ is smooth if and only if $A$, $B$ the real valued functions are smooth.

**20.1.7 THEOREM :**

Suppose $f, \varphi:[a, b] \to \mathbb{C}$ are such that $f$ is continuous and $\varphi$ is smooth. Then $I(f, \varphi)$ exists and

$$I(f, \phi) = \int_a^b f(t)\, \phi'(t)\, dt.$$

**20.1.8 REMARK :**

The function $f\varphi'$ is continuous on $[a, b]$ and so by proposition (20.1.2) the integral on the right exists.

**Proof :** Let $\varepsilon > 0$.

We have assumed that $\varphi'$ is continuous. Therefore $|\varphi'|$ is continuous and

$$\int_a^b |\varphi'(t)|\, dt$$

exists and equals some $M \in \mathbb{R}$.

We have assumed that $f$ is continuous. Since $[a, b]$ is compact, $f$ is uniformly continuous on $[a, b]$. Therefore we can find a $\delta(\varepsilon) > 0$ such that for all $s, t$ in $[a, b]$

$$|f(s)=f(t)|<\frac{\varepsilon}{1+M} \text{ if } |s-t|<\delta(\varepsilon) \text{ --------- (1)}$$

Now we let

$P: a=t_0<t_1<\cdots<t_n=b$ be a parition with

$\|P\|<\delta(\varepsilon)$, and let

$$\tau_0\in[t_0,t_1],\cdots,\tau_{n-1}\in[t_{n-1},t_n]$$

For arbitrary $s,t$ in $[a, b]$, $s<t$ we have

$$\int_s^t \varphi'(t)\,dt=\varphi(t)-\varphi(s);$$

$$\left|\sum_{k=0}^{n-1} f(\tau_R)[\varphi(t_{k+1})-\varphi(t_k)]-\int_a^b f(t)\varphi'(t)\,dt\right|$$

$$=\left|\sum_{k=0}^{n-1}\left\{f(\tau_k)[\varphi(t_{k+1})-\varphi(t_k)]-\int_{t_k}^{t_{k+1}} f(t)\varphi'(t)\,dt\right\}\right|$$

$$=\left|\sum_{k=\infty}^{n-1}\int_{r_k}^{t_{k+1}}[f(\tau_k)-f(t)]\varphi'(t)\,dt\right| \text{ by (2)}$$

$$\leq\sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}}[f(\tau_k)-f(t)]\varphi'(t)\,dt\right| \text{ by triangle inequality,}$$

$$\leq\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}|f(\tau_k)-f(t)||\varphi'(t)|\,dt \text{ by Proposition 2}$$

$$<\frac{\varepsilon}{1+M}\int_a^b|\varphi'(t)|\,dt \text{ by (1)}$$

$$\leq\varepsilon\cdot\frac{M}{1+M}\leq\varepsilon$$

This implies the Theorem.

**20.1.9 DEFINITION :**

If $\phi:[a, b] \to \mathbb{C}$ is a map, by Trace of $\varphi$, we mean the set

$$\{\phi(t) : t \in [a, b]\}$$

The trace of $\phi$ is denoted by $\{\phi\}$.

**20.2 MAIN RESULTS :**

Before we state the next theorem we introduce some notation. This is to keep the ideals clear.

Suppose $f:[a,b]\times[c,d]\to\mathbb{C}$ is a map. We write $f(x,y)$ for $f$. It suggests that in the usual terminology, $f$ is a function of two variables, $x$ is the first variable and $y$ is the second variable. If the partial derivative of $f$ with respect to the first variable $x$ exists, we denote it by

$$D_1 f \text{ and } D_2 f$$

is similarly defined. Thus, if we use the usual notation for partial derivatives, we have

$$D_1 f = \frac{\partial}{\partial x} f, \; D_2 f = \frac{\partial}{\partial y} f.$$

**20.2.1 THEOREM (LEIBNIZ'S RULES : REAL CASE) :**

Suppose we denote $[a, b] \times [c, d]$ by $E$, and $f : E \to C$ is a map.

(a) If $f$ is continuous on $E$, then for each $y$ in $[c, d]$

$$F(y) = \int_a^b f(t, y)\, dt$$

exists and is a continuous function on $[c, d]$.

(b) If $D_2 f = \frac{\partial}{\partial y} f(x,y)$ exists and is continuous on $E$, then $F'$ exists on $[c,d]$.

$$F'(y) = \int_a^b D_2 f(t,y)\, dt \quad \text{---------- (1)}$$

and is continuous.

**Proof : (a)** We shall directly prove that $F$ is uniformly continuous on $[c,d]$. Let $\varepsilon > 0$.

Since $E$ is compact and $f$ is continuous on $E$, we can find a $\delta(\varepsilon) > 0$ such that

$$|f(x,y) - f(x',y')| < \frac{\varepsilon}{(b-a)}$$

if $\left[(x-x')^2 + (y-y')^2\right]^{1/2} < \delta(E)$. So

if $|y-y'| < \delta(\varepsilon)$, then

$$|f(t,y) - f(t,y')| < \frac{\varepsilon}{(b-a)}$$

We have

$$|F(y) - F(y')| = \left|\int_a^b [F(t,y) - f(t,y')]\,dt\right|$$

$$\leq \int_a^b |f(t,y) - f(t,y')|\,dt \text{ by proof 2,}$$

$$< \frac{\varepsilon}{(b--a)} \cdot (b-a) = \varepsilon. \text{ If } |y-y'| < \frac{\delta}{\varepsilon}.$$

(b) Let $\varepsilon > 0$ and $\alpha_0$ be in $[c,d]$. $D_2 f$ is continuous on $E$, a compact set. Therefore there is a $\delta(\varepsilon) > 0$ such that

$$|D_2 f(x,y) - D_2 f(x',y')| < \frac{\varepsilon}{b-a}$$

if $\left[(x-x')^2 + (y-y')^2\right]^{1/2} > \delta(\varepsilon)$.

For $\alpha$ in $[c, d]$ we have

$$F(\alpha)-F(\alpha_0)=\int_a^b [f(t,\alpha)-f(t,\alpha_0)]dt$$

$$=\int_a^b \left\{\int_{\alpha_0}^{\alpha} D_2 f(t,s)ds\right\}dt$$

by proposition 1. (b).

Therefore

$$\left|F(\alpha)-F(\alpha_0)-(\alpha-\alpha_0)\int_a^b D_2 f(t,\alpha_0)dt\right|$$

$$=\left|\int_a^b \left\{\int_{\alpha_0}^{\alpha} [D_2 f(t,s)-D_2 f(t,\alpha_0)]ds\right\}dt\right|$$

$$\leq \int_a^b \left|\int_{\alpha_0}^{\alpha} [D_2 f(t,s)-D_2 f(t,\alpha_0)]ds\right|dt$$

by a known result.

$$\leq \int_a^b \left|\int_{\alpha_0}^{\alpha} |D_2 f(t,s)-D_2 f(t,\alpha_0)|ds\right|dt$$

by a proposition 2.

$<\varepsilon|\alpha-\alpha_0|$ if $|\alpha-\alpha_0|<\delta(\varepsilon)$.

This proves (1) and that in turn prove (b).

**20.22 REMARK :**

In the theorem we may replace $[c,d]$ by $[c,d]\times[c',d']$ or by an open set in $\mathbb{R}^n$. The derivative $\frac{\partial f}{\partial y}$ may be replaced by

$$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$$

in case $U \subset \mathbb{R}^n$ we assume this is done.

**20.2.3 THEOREM (LEBNIZ RULE : COMPLEX CASE) :**

Suppose $U$ is an open set in $\mathbb{C}$ and

$$f : [a,b] \times U \to \mathbb{C}$$

is a function with the following properties.

2) For each fixed $t_0$ in $[a,b]$, $f(t_0, z)$ defined on $U$ is analytic in $U$.

3) We denote the derivative of $f(t_0, z)$ with respect to $z$ by

$$\frac{\partial f}{\partial z}(t_0, z).$$

The function $\frac{\partial f}{\partial z}(t,z)$ defined on $[a,\ b] \times U$ is continuous. Then

1) For each $z$ in $U$ is the integral

$$\int_a^b f(t,z)\,dt$$

exists. The function $g : U \to \mathbb{C}$ defined by

$$g(z) = \int_a^b f(t,z)\,dt \text{ is continuous.}$$

2) $g$ is differentiable with respect to $z$ and

$$\frac{dg}{dz}(z) = g'(z) = \int_a^b \frac{\partial f}{\partial z}(t,z)\,dt.$$

3) $g$ is analytic in $U$, that is $g'(z)$ is continuous.

**Proof :** We derive the theorem from Leibniz Rule : Real Case.

## 20.3 SOME MORE RESULTS :

**20.3.1 DEFINITION :** Suppose $z_0 \in \mathbb{C}$, $\rho > 0$,

$$c(z_0,\rho)=\{w \in \mathbb{C}:|w-z_0|=\rho\} \text{ and}$$

$$\varphi :[0, 2\pi] \to C(z_0,\rho) \text{ is given by}$$

$$\varphi(t)=z_0+\rho e^{it}.$$

Suppose $f$ is a continuous function from $C(z_0,\rho)$ to $\mathbb{C}$. Then we write

$$\int_{|w-z_0|=\rho} f(w)dw, \quad \int_{C(z_0,\rho)} f(w)dw \text{ for } I(f,\phi).$$

**20.3.2 PROPOSITION :**

If $|z|<1$ then

$$\int_{|w|=1} \frac{dw}{w-z} = 2\pi i$$

**Proof :** By definition $\displaystyle\int_{|w|=1} \frac{dw}{w-z} = \int_0^{2\pi} \frac{1}{e^{is}-z} i\, e^{is}\, ds$.

So the assertion is

$$\int_0^{2\pi} \frac{e^{is}}{e^{is}-z} ds = 2\pi.$$

We note that when $z=0$, the result is clear. Assume $|z|<1$ and set

$$\phi(s,t)=\frac{e^{is}}{e^{is}-tz}, \; s \in [0, 2\pi], \; t \in [0, 1].$$

$|e^{is}|=1$, $|tz|=t|z|<t<1$, Therefore $\phi$ is continuous it is differentiable as a function of $t$ and

$$\frac{\partial \phi}{\partial t} = \frac{ze^{is}}{\left(e^{is} - tz\right)^2}.$$

Suppose we define

$$\varphi_1(s) = \frac{iz}{\left(e^{is} - tz\right)} \text{ for } s \in [0, 2\pi]$$

Then we notice that

$$\frac{d\varphi_1}{ds} = iz \times \frac{(-1)\, i e^{is}}{\left(e^{is} - tz\right)^2}.$$

$$= \frac{ze^{is}}{\left(e^{is} - z\right)^2}$$

Let us set

$$g(t) = \int_0^{2\pi} \varphi(s,t)\, ds \cdot$$

Then we have

$$g(1) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - z} ds \text{ and}$$

$$g(0) = \int_0^{2\pi} \frac{e^{is}}{e^{is}} ds = 2\pi.$$

So, the proposition states that $g(1) = g(0)$. We shall prove that $g'(t)$ exists and is zero on $[0,1]$. Then it follows that $g(1) = g(0)$, and the result gets proved.

By Leibniz's Rule

$$g'(t) = \int_0^{2\pi} \frac{ze^{is}}{\left(e^{is} - tz\right)^2} ds$$

$$= \int_0^{2\pi} \frac{d\varphi_1}{ds} \cdot ds = \varphi_1(2\pi) - \varphi_1(0)$$

$$= \left(\frac{iz}{1-tz}\right) - \frac{iz}{(1-tz)} = 0.$$

Therefore the result.

**20.3.3 THEOREM :**

(Cauchy Integral Formula for Circle). Suppose $f$ is an analytic function in an open set $U \subseteq \mathbb{C}$ and for some $r > 0$

$$\overline{B(0,r)} \subset U.$$

Then for any $z$ in $B(0,r)$ i.e. for $|z| < r$

$$2\pi i f(z) = \int_{|w|=r} \frac{f(w)}{w-z} dw$$

**Proof :** By the definition

$$\int_{|w|=r} \frac{f(w)}{w-z} dw = \int_0^{2\pi} \frac{f(re^{it})}{(re^{it} - z)} i\, re^{it}\, dt.$$

We write

$$w = z + (w-z),\ f_0(t) = f\left(z + \left(re^{it} - z\right)\right)$$

and define for $(s,t)$ in $[0,1] \times [0, 2\pi]$

$$g(s,t) = \frac{f\left(z + s\left(re^{it} - z\right)\right)}{\left(re^{it} - z\right)} ire^{it}$$

$re^{it} - z$ is never zero for $t \in [0, 2\pi]$, $z \in \mathbb{C}$, $|z| \neq r$. So $g(s,t)$ is a continuous unction on $[0,1] \times [0, 2\pi]$. We define

$$h(s) = \int_{0}^{2\pi} g(s,t)\,dt \cdot$$

We find that $h(1) = \int\limits_{|w|=r} \frac{f(w)}{(w-z)} dw$ and

$$h(0) = \int\limits_{|w|=r} \frac{f(z)}{(w-z)} dw = f(z) \int\limits_{|w|=\cdot} \frac{dw}{w-z}$$

$= f(z) \cdot 2\pi i$ by proposition 2.

If we prove $h(1) = h(0)$ our Theorem becomes proved.

By Leibniz's rule $h(s)$ is a continuous function. We shall prove that $h'(s)$ exists and is zero on $(0, 1]$. We have by Leibniz's Rule

$h'(s) = \int_{0}^{2\pi} \frac{\partial}{\partial s}[g(s,t)]\,dt$ and by chain rule for $f\left(z+s\left(re^{it}-z\right)\right)$ we have

$$\frac{\partial}{\partial s} g(s,t) = \frac{f'\left(z+s\left(re^{it}-z\right)\right)\left(re^{it}-z\right) \times ire^{it}}{\left(re^{it}-z\right)}$$

$$= f'\left(z+s\left(re^{it}-z\right)\right) \times ir\,e^{it}$$

Here $f'$ denotes the derivative of $f$ as a function on $U$.

Now consider

$$f\left(z+s\left(re^{it}-z\right)\right)$$

as a function of $t$ and differentiate with respect to $t$. Then

$\frac{d}{dt} f\left(z+s\left(re^{it}-z\right)\right) = f'\left(z+s\left(re^{it}-z\right)\right) s \cdot r \cdot i e^{it}$, by chain rule

$$= s\,\frac{\partial g}{\partial s}(s,t).$$

Thus for $s \in (0, 1]$

$$h'(s) = \int_0^{2\pi} \frac{\partial g}{\partial s}(s,t)\,dt$$

$$= \frac{1}{s}\int_0^{2\pi} \frac{d}{dt}\left[f\left(z+s\left(re^{it}-z\right)\right)\right]dt$$

$$= \frac{1}{s}\left[f\left(z+s\left(re^{it}-z\right)\right)\right]_0^{2\pi}$$

$$= \frac{1}{s}\left[f\left(z+s(r-z)\right)-f\left(z+s(r-z)\right)\right]$$

= 0.

It follows that $h'$ being continuous on $[0, 1]$ by Leibniz's Rule $h'(0) = 0$. Thus $h$ is constant and so

$h(1) = h(0)$ and the Theorem is proved.

It is possible to follow the method of proof in Proposition 3 and prove, using the above formula, that $f(z)$ is infinitely differentiable, we shall prove a stronger result. This is the basic result for studying analytic functions in sufficiently small neighbourhoods of the points.

**20.3.4 THEOREM :**

(1) Suppose $f$ is analytic in $U$ and $B(a,r) \subset U$. Then for $z$ in $B(a,r)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

The power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence $\geq r$.

For any $\rho$ with $0<\rho<r$

$$a_n = \frac{1}{2\pi i}\int_{|w-a|=\rho}\frac{f(w)}{(w-a)^{n+1}}dw\cdot$$

(2) $f$ is infinitely differentiable and

$$a_n = \frac{f^{(n)}(a)}{n!}$$

(3)

**Proof :** We have $\overline{B(a,\rho)}\subset B(a,r)\subset U$. Therefore by the exercise

$$f(z)=\frac{1}{2\pi i}\int_{|w-a|=\rho}\frac{f(w)}{w-z}dw.$$

In the above formula, we confine $z$ to $B(a,\rho)$. Then we have

$$|z-a|<|w-a|=\rho;$$

if we set $|z-a|=\sigma$, then $0<\frac{\sigma}{\rho}<1$. We may write

$$\frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}=\frac{1}{(w-a)}\cdot\frac{1}{\left(1-\frac{z-a}{w-a}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{(z-a)^n}{(w-a)^{n+1}} \text{ since } \left|\frac{z-a}{w-a}\right|=\frac{\sigma}{\rho}<1.$$

The series converges uniformly as a function of $w$, on the set

$$|w|=\rho.$$

Now we use the following result :

Suppose $g_n:[a,b]\to\mathbb{C}$ is a sequence of continuous functions such that

$$\sum g_n$$

converges uniformly to a function $g$ on $[a, b]$. Then $g$ is a continuous function and we have

$$\sum_{n=0}^{\infty} \int_a^b g_n(t)\,dt = \int_a^b g(t)\,dt$$

Consider

$$\int_{|w-a|=\rho} \frac{f(w)}{w-z}\,dw = \int_0^{2\pi} \frac{f\left(a+\rho e^{i\theta}\right)}{\left(re^{i\theta}-z\right)} ire^{i\theta}\,d\theta .$$

Let us denote

$\sup\limits_{|w|=\rho} |f(w)|$ by $M(\rho)$. Then

$$\left| f\left(a+\rho e^{i\theta}\right) i\ e^{i\theta} \frac{(z-a)^n}{(w-a)^{n+1}} \right| \le M(\rho)\left(\frac{\sigma}{\rho}\right)^n .$$

By M-test the series of functions of $\theta \in [0, 2\pi]$

$$\sum_{n=0}^{\infty} f\left(a+\rho e^{i\theta}\right) \frac{i\rho e^{i\theta}(z-a)^n}{\left(\rho e^{i\theta}-a\right)^{n+1}}$$

converges uniformly to the function of $\theta$

$$\frac{f\left(a+\rho e^{i\theta}\right) i\ \rho e^{i\theta}}{\left(re^{i\theta}-z\right)}$$

Therefore $\displaystyle\int_0^{2\pi} \frac{f\left(a+\rho e^{i\theta}\right) i\rho e^{i\theta}}{\left(re^{i\theta}-z\right)}\,d\theta$

$$= \sum_{n=0}^{\infty} \int_0^{2\pi} f\left(a+\rho e^{i\theta}\right) i\rho e^{i\theta} \frac{(z-a)^n}{\left(\rho e^{i\theta}-a\right)} d\theta$$

By the definition

$$\int_0^{2\pi} f\left(a+\rho e^{i\theta}\right) \cdot i\rho e^{i\theta} \cdot \frac{(z-a)^n}{\left(\rho e^{i\theta}-a\right)^{n+1}} d\theta$$

$$= \int_{|w-a|=\rho} (z-a)^n \frac{f(w)}{(w-a)^{n+1}} dw$$

$$=(z-a)^n \int_{|w-a|=\rho} \frac{f(w)}{(w-a)^{n+1}} dw$$

Therefore we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z-a| < \rho$$

From the theorems on convergent power series it follows that

(1) $f(z)$ is infinitely differentiable in $B(a, \rho)$ for $\rho < r$ and

(2) $a_n = \dfrac{f^{(n)}(a)}{n!}$

Thus $f(z) = \sum_{n=0}^{\infty} \dfrac{f^n(a)}{n!}(z-a)^n$;

the right hand side converges if there is a $\rho$ such that

$|z-a| = \sigma < \rho < r$.

Consequently the above series converges in $B(a,r)$. That is the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n \text{ is } \geq r \text{ if } B(a,r) \subset U.$$

The theorem is proved.

**20.3.5 DEFINITION :**

Suppose $U$ is an open set in $\mathbb{C}$; $F, f, : U \to \mathbb{C}$ are two functions. We say that $F$ is a primitive of $f$ if $F$ analytic in $U$ and

$$F' = f \cdot$$

**20.3.6 THEOREM :**

If $f : B(a,r) \to \mathbb{C}$ is analytic then $f$ has a primitive in $B(a,r)$.

**Proof :** By Theorem 4 we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

The power series converges in $B(a,r)$. We have for $z$ in $B(a,r)$

$$\left|\frac{f^{(n)}(a)}{n!(n+1)}(z-a)^{n+1}\right| \le \left|\frac{f^{(n)}(a)}{n!}(z-a)^n\right| \cdot r.$$

Therefore if $0 \le |z-a| = \rho < r$ then by comparison test

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!(n+1)}(z-a)^{n+1}$$

converges in $B(a,r)$. If we set for $|z-a| < r$

$$F(z) = \sum \frac{f^n(a)}{(n+1)!}(z-a)^{n+1}$$

the results on power series give

(1) $F$ is analytic and

(2) $F'(z) = f(z)$.

The theorem is proved.

The next Theorem is the other basic result of complex analysis. It is called Cauchy's theorem for the disc. Here we derive the result as a consequence of Cauchy Integral Formula. If Cauchy's Theorem is granted, Cauchy Integral Formula may be derived from it.

**20.3.7 THEOREM (CAUCHY'S THEOREM FOR DISC) :**

Suppose $f$ is analytic in $B(a,r)$, $F$ a primitive of $f$

$$\varphi : [a,b] \to B(a,r)$$

is a smooth function. Then

$$\int_a^b f\,d\varphi = F(\varphi(b)) - F(\varphi(a)).$$

So, if path $\varphi(a) = \varphi(b)$

$$\int_\varphi f = 0$$

**Proof :** Consider the function

$$g(t) = F(\varphi(t))$$

By chain rule

$$g'(t) = F'(\varphi(t))\varphi'(t)$$

where $F'$ is the derivative of $F$ as a function on $B(a,r)$. So

$$g'(t) = f(\varphi(t))\varphi'(t).$$

By definition

$$\int_a^b f\,d\varphi = \int_a^b f(\varphi(t))\varphi'(t)dt = \int_a^b g'(t)dt = g(b) - g(a)$$

**20.3.8 DEFINITION OF POLYGONAL PATH :**

Suppose $t_1 < t_2 < \cdots < t_n$ is a sequence of real numbers and $z_1, z_2, \cdots, z_n$ is a sequence of complex numbers. By a polygonal path we mean a function $\varphi : [t_1, t_n] \to \mathbb{C}$ given by

$$\varphi(t) = z_j + \frac{(t - t_j)}{(t_{j+1} - t_j)}(z_{j+1} - z_j),\ t \in [t_j, t_{j+1}]$$

$$j = 0, 1, \cdots, n-1.$$

The sequences $(t_1, \cdots, t_n)$ and $(z_1, \cdots, z_n)$ define the polygonal path uniquely. So it may be denoted in many ways. For example

$$((t_1, \cdots, t_n); (z_1, \cdots, z_n)),\ (\underline{t},\ \underline{z}),$$

$$[z_1(t_1), \cdots, z_n(t_n)],\ [z_1, z_2, \cdots, z_n]$$

For $1<j<n-1$ the right hand derivtive of $\varphi$ at $t_j$ is $\dfrac{z_{j+1}-z_j}{t_{j+1}-t_j}$

$\dfrac{z_{j+1}-z_j}{t_{j+1}-t_j}$, the left hand derivative at $t_j$ is $\dfrac{z_j-z_{j-1}}{t_j-t_{j-1}}$ every other point of $(t_j,t_{j+1})\varphi$ has derivative and its value is

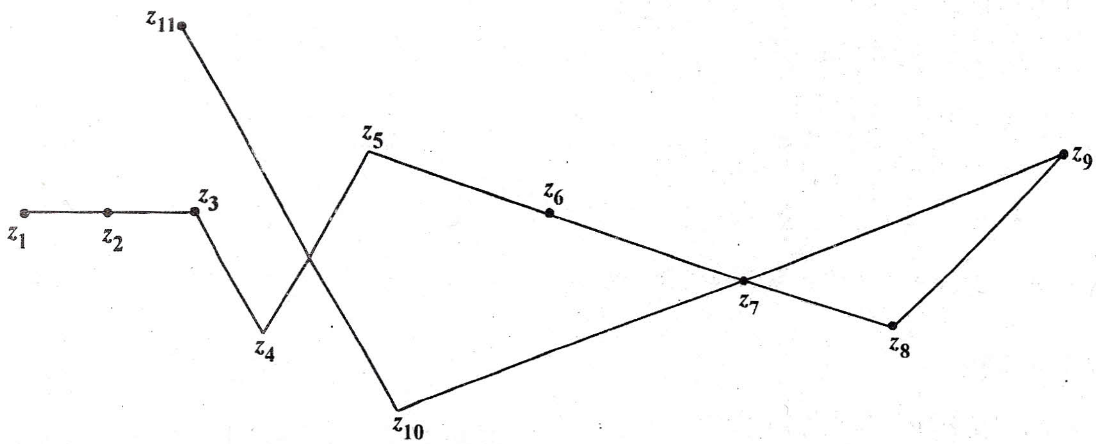
$$\frac{(z_{j+1}-z_j)}{(t_{j+1}-t_j)}.$$

Thus $\varphi$ is a piecewise differential map. The map $\varphi$ on $[t_j,t_{j+1}]$ is

$$\varphi(t)=z_j-\frac{(z_{j+1}-z_j)}{(t_{j+1}-t_j)}t_j+t\frac{(z_{j+1}-z_j)}{(t_{j+1}-t_j)}$$

$$=\frac{t_{j+1}z_j-t_jz_{j+1}}{(t_{j+1}-t_j)}+t\frac{(z_{j+1}-z_j)}{(t_{j+1}-t_j)}$$

Therefore $\varphi(t)$ is of the form $\alpha+\beta t$; that is it is what is called an affine map of simples $\varphi(t)$ is a polynomial of degree one. When $\alpha=\theta$ it is a linear map.




The image of $\varphi$ is illustrated in the figure above. Suppose we define

$$\varphi_j=\varphi/[t_j,t_{j+1}]\quad j=1,\cdots,n-1.$$

That is $\varphi_j : [t_j, t_{j+1}] \to \mathbb{C}$ given by

$$\varphi_j(t) = z_j + \frac{(t - t_j)}{(t_{j+1} - t_j)}(z_{j+1} - z_j).$$

Now suppose $f$ is a continuous on the trace of $\varphi$. Then we have

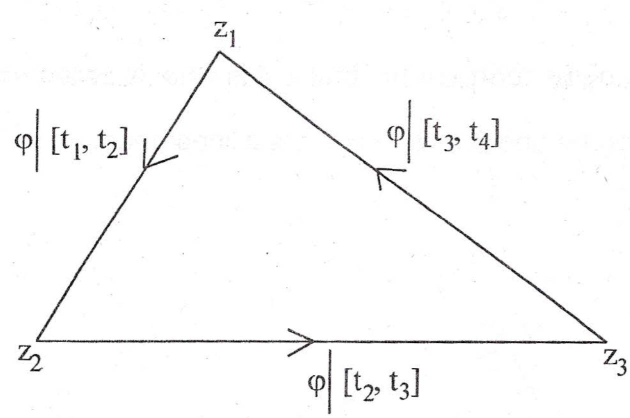
$$\int_{\gamma(\varphi)} f = \int_{\gamma(\varphi_1)} f \cdots + \int_{\gamma(\varphi_{n-1})} f$$

$$= \int_{z_1}^{z_2} f + \cdots + \int_{z_{n-1}}^{z_n} f.$$

**20.3.9 DEFINITION :**

By a Triangular path we mean a polygonal Path $\varphi : [z_1(t_1), z_2(t_2), z_3(t_3), z_1(t_4)]$.

That is we are given three complex numbers $z_1, z_2, z_3$ and three intervals $[t_1, t_2], [t_2, t_3], [t_3, t_4]$.




It is to be noted that the first number and the last number of the sequence $z_1, z_2, z_3, z_4 = z_1$ are equal.

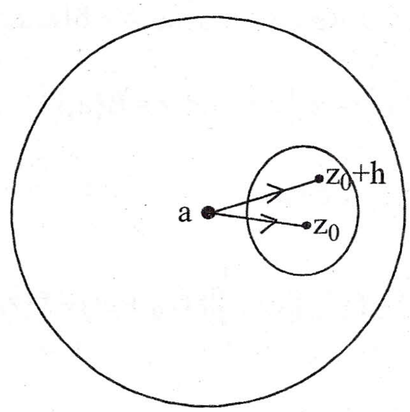
**20.3.10 THEOREM :**

Suppose $f : B(a,r) \to \mathbb{C}$ is a continuous function such that the integral of $f$ along every triangular path $\gamma(\varphi)$ with trance $\varphi = \{\varphi\} \subset B(a,r)$ is zero. Then $f$ is analytic in $B(a,r)$.

**Proof :** We shall prove that $f$ has a primitive in $B(a,r)$, say $F(z)$ Then $F'(z)$ being equal to $f(z)$ is continuous. It folows that $F(z)$ is analytic. By Theorem it follows that $F$ is infinitely differentiable. Therefore $f$ is infinitely differentiable. So $f$ is analytic.

Let us set for $z \in B(a,r)$

$$F(z) = \int_a^z f(w)\,dw.$$




Let $z_0 \in B(a,r), |z_0| = r_0$ and $|h| < r - r_0$. Then $|z_0 + h| < |z_0| + |h| = r$. Then since $B(a,r)$ is a convex set the segments $[a, z_0], [a, z_0 + h], [z_0, z_0 + h]$ are all contained in $B(a,r)$.

We have $F(z_0 + h) - F(z_0)$

$$= \int_a^{z_0+h} f(w)\,dw - \int_a^{z_0} f(w)\,dw$$

$$= \int_a^{z_0+h} f(w)\,dw + \int_{z_0}^{a} f(w)\,dw + \int_{z_0+h}^{z_0} f(w)\,dw - \int_{z_0}^{z_0+h} f(w)\,dw$$ by proposition 6.

$$= \int_{[a, z_0+h, z_0, a]} f + \int_{z_0}^{z_0+h} f$$ by remark after proposition 5

$$= \int_{z_0}^{z_0+h} f = \int_0^1 f(z_0+th)h\,dt$$

Therefore for $0<|h|<r-r_0$,

$$\left|\frac{F(z_0+h)-F(z_0)}{h}-f(z_0)\right| = \left|\int_0^1 [f(z_0+th)-f(z_0)]dt\right| \le \int_0^1 |f(z_0+h)-f(z)|dt$$

$f$ is continuous. Given $\varepsilon>0$ we can choose $\delta=\delta(\varepsilon,x_0)>0$ s.t

$|f(z)-f(z_0)|<\varepsilon$ if $|z-z_0|<\delta$ and $z\in B(a,r)$.

If $h$ is s.t. $0<|h|<\min\{r,\delta\}$ we have

$$|f(z_0+th)-f(z_0)|<\varepsilon,\ \int_0^1 |f(z_0+th)-f(z_0)|dt \le \varepsilon$$

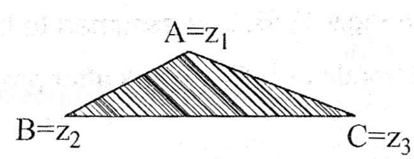
and so $$\left|\frac{F(z_0+h)-F(z_0)}{h}-f(z_0)\right|<\varepsilon \text{ if } 0<h<\min\{r,\delta\}.$$

Thus $F$ is differentiable and $F'(z_0)=f(z)$. The theorem is proved.

**20.3.11 COROLLARY :**

Suppose $U$ is an open set, $f:U\to\mathbb{C}$ a continuous function. If $f$ is such that $\int_\gamma f=0$ for each triangular path $\gamma\subset U$ with the property that interior of the triangle of the path is also contained in $U$ then $f$ is analytic.

**Proof :** The problem is to define interior of the triangle of a triangular path it is indicated by the figure. One way to define it is




$$[\lambda_1 z_1+\lambda_2 z_2+\lambda_3 z_3/\lambda_1+\lambda_2+\lambda_3=1,\ \lambda_1,\lambda_2,\lambda_3\ge 0].$$

The proof is left as an exercise.

**20.3.12 DEFINITION :**

$$A, B, C \in \mathbb{C}.\ \Delta ABC = \{\lambda A + \mu B + \upsilon C / \lambda, \mu, \upsilon \geq 0,\ \lambda + \mu + \upsilon = 1\}$$

**20.3.13 REMARK :**

The corollary is called Morera's Theorem. The main argument is the proof of the previous theorem.

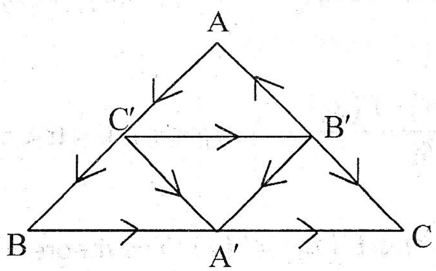
The next result is called Goursat's Theorem.

**20.3.14 THEOREM (GOURSAT'S THEOREM) :**

Suppose $U \subseteq \mathbb{C}$ is an open set $f : U \to \mathbb{C}$ is a differentiable function and the triangle $ABC$ with its interior is a subset of $U$. Then

$$\int_{[A,B,C,A]} f = 0;\ \int_{[A,B]} f + \int_{[B,C]} f + \int_{[C,A]} f = \int_A^B f + \int_B^C f + \int_C^A f = 0$$

**Proof :**




Let $A', B', C'$ be the mid points of the sides of the triangle $ABC$ that are opposite the vertices $A, B, C$.

Then by remark after proposition 5

$$\int_{[A,B,C,A]} = \left( \int_A^{C'} + \int_{C'}^{B} + \int_B^{A'} + \int_{A'}^{C} + \int_C^{B'} + \int_{B'}^{A} \right) f$$

Since the interior of the triangle $A, B, C$ is assumed to be a subset of $U$, the segments $[A', B'], [B', C'], [C', A']$ are contained in $U$. By remark after proposition 5.

$$\therefore \int_{[A,B,C,A]} f = \int_{[B',A,C',B']} f + \int_{[C',B,A',C']} f + \int_{[A',C,B',A']} f - \int_{C'}^{B'} f - \int_{A'}^{C'} f - \int_{B'}^{A'} f$$

By proposition 5

$$-\int_{C'}^{B'} f - \int_{A'}^{C'} f - \int_{B'}^{A'} f = \int_{[A',B',C',A']} f$$

Therefore

$$\int_{[A,B,C,A]} f = \int_{[A,B',C',A]} f + \int_{[B,A',C',B]} f + \int_{[C,A',C',C]} f + \int_{[A',B',C',A']} f$$

We consider the numbers

$$\left|\int_{[A,C',B',A]} f\right|, \left|\int_{[B,A',C',B]} f\right|, \left|\int_{[C,B',A',C]} f\right|, \left|\int_{[A',B',C',A']} f\right|.$$

We choose one of the which is greater than or equal to the other three. We call the corresponding triangle $A_1\ B_1\ C_1$. We have

$$\Delta A_1\ B_1\ C_1 \subset \Delta ABC \subset U$$

$$A_1B_1 = \frac{1}{2}AB,\ B_1C_1 = \frac{1}{2}BC,\ C_1A_1 = \frac{1}{2}CA$$

and $$\left|\int_{[A,B,C,A]} f\right| \leq 4\left|\int_{[A_1,B_1,C_1,A_1]} f\right|$$

We repeat the construction with the triangular path $[A_1, B_1, C_1, A_1]$ and obtain $[A_2, B_2, C_2, A_2]$. By induction we obtain triangles $A_n\ B_n\ C_n$ s.t.

$$\Delta A_{n+1}\ B_{n+1}\ C_{n+1} \subset \Delta A_n\ B_n\ C_n$$

$$A_{n+1}\ B_{n+1} = \frac{1}{2}A_n, B_n,\ B_{n+1}\ C_{n+1} = \frac{1}{2}B_nC_n, C_{n+1}\ A_{n+1} = \frac{1}{2}C_n\ A_n$$

$$\left|\int_{[A_n,B_n,C_n,A_n]} f\right| \leq 4\left|\int_{[A_{n+1},B_{n+1},C_{n+1},A_{n+1}]} f\right|$$

$$\left| \int_{[A,B,C,A]} f \right| \le 4^n \left| \int_{[A_n,B_n,C_n,A_n]} f \right|$$

$$d(\Delta A_n B_n C_n) = \frac{1}{2^n} d(\Delta ABC).$$

$\Delta_n = \Delta A_n\, B_n\, C_n$ are closed subsets of $\mathbb{C}$, $\Delta_{n+1} \subseteq \Delta_n$ and $d(\Delta_n) \to 0$ as $n \to \infty$ by Cantor Intersection Theorem there is a unique $z_0 \in \mathbb{C}$ in $\bigcap \Delta_n$.

Since $\Delta_n \subset \Delta = \Delta\, ABC$, we obtain $z_0 \in ABC$; $z_0 \in U$ an open set. Let $\varepsilon > 0$. Then we can choose a $\delta = \delta(\varepsilon, f, z_0) > 0$ such that

1) $B(z_0, \delta) \subset U$

2) $|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon |z - z_0|$ for $z$ is in $B(z_0, \delta)$

We define $r(z)$ on $U$ by

$$r(z) = f(z) - f(z_0) - f'(z_0)(z - z_0)$$

Then $r(z)$ is a continuous function on $U$; $f(z_0)(z - z_0)$, $\dfrac{f'(z_0)(z - z_0)^2}{2}$ are primitives of $f(z_0)$ and $f'(z_0)(z - z_0)$. Therefore for each $n$.

$$\int_{[A_n B_n C_n A_n]} f = \int_{[A_n B_n C_n A_n]} r$$

since $f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z)$ and

$$\int_{[A_n,B_n,C_n,A_n]} f(z_0) = 0 \qquad \int_{[A_n,B_n,C_n,A_n]} f'(z_0)(z - z_0) = 0$$

Suppose we choose a positive integer $n_0$ such that

$$d(\Delta_{n_0}) = \frac{1}{2^{n_0}} d(\Delta) < \delta$$

Then since $z_0 \in \Delta_n$, we have $\Delta_{n_0} \subset B(z_0, \delta)$ and

$$\left| \int_{[A_{n_0}, B_{n_0}, C_{n_0}, A_{n_0}]} r(z) \right| \leq \int_{[A_{n_0} B_{n_0} C_{n_0} A_{n_0}]} |r(z)| \leq \varepsilon \int_{[A_{n_0}, B_{n_0}, C_{n_0}, A_{n_0}]} |z - z_0|$$

$$\leq \varepsilon \cdot \frac{1}{2^{n_0}} d(\Delta) \frac{1}{2^{n_0}} (AB + BC + CA)$$

Thus $\left| \int_{[A_{n_0}, B_{n_0}, C_{n_0}, A_{n_0}]} f \right| \leq \varepsilon \cdot d(\Delta)(AB + BC + CA) \frac{1}{4^{n_0}}$

Thus $\left| \int_{[A,B,C,A]} f \right| \leq d(\Delta)(AB + BC + CA)\varepsilon$.

This proves the theorem.

**20.3.15 REMARK :**

We have called a function $f$ on an open set $U \subset \mathbb{C}$ analytic if $f$ is differentiable and the derivative is continous Goursat's theorem says if $f$ is differentiable then the integral of $f$ along a triangular path in $U$ is zero if the path and its interior are contained in $U$. Then Morerals Theorem implies $f$ is analytic. So we may say

**20.3.16 THEOREM (GOURSAT - MORERA) :**

If $U \subseteq \mathbb{C}$ is an open set, $f : U \to \mathbb{C}$ is a differentiable functions then $f$ is an analytic function.

**20.4 SAQ'S & SOLUTIONS :**

**SAQ 1 :** Suppose $\varphi : [a, b] \to \mathbb{C}$ is given by

$\varphi(t) = \alpha$ for all $t$.

Then prove that $I(f, \varphi)$ exists and evaluate it.

**Solution :** Let $P : a = t_0 < t_1 < \cdots < t_n = b$

be any partition and $\tau_0 \in [t_0, t_1], \cdots, \tau_{n-1} \in [t_{n-1}, t_n]$. Then

$$\sum f(\tau_j)\left[\varphi(t_j+1)-\varphi(t_j)\right]=0$$

This implies that $I(f, \varphi)$ exists and is zero.

**SAQ 2 :** Suppose $\varphi:[0,1] \to \mathbb{C}$ is defined by

$$\varphi(t)=\begin{cases} 0 & t \in \left[0, \frac{1}{2}\right] \\ 1 & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Then prove that if $f$ is continuous on $[0, 1]$, $I(f, \varphi)$ exists and evaluate it.

**Solution :** Let

$$P: a = t_0 < t_1 < \cdots < t_n = b$$

be any partition and $\tau_0, \cdots \tau_{n-1}$ be in $[t_0, t_1], \cdots, [t_{n-1}, t_n]$.

There is a unique $j$ s.t.

$$0 \le j \le n-1 \text{ and } t_j \le \frac{1}{2} \le t_j.$$

We have $\varphi(t_{k+1}) - \varphi(t_k) = 0$ if $k \ne j$.

Let $\varepsilon > 0$. Since $f$ is continuous at $\frac{1}{2}$ we can find a $\delta(\varepsilon) > 0$ such that

$$f\left(\frac{1}{2}\right)-\varepsilon < f(t) < f\left(\frac{1}{2}\right)+\varepsilon$$

for $\frac{1}{2}-\delta(\varepsilon) < t < \frac{1}{2}+\delta(\varepsilon)$. Therefore if $\|P\| < \delta/\varepsilon$ then for $\tau_j$ in $\left[t_j, t_{j+1}\right]$ we have

$$\left|\tau_j - \frac{1}{2}\right| \le \left(t_{j+1} - t_j\right) < \delta(\varepsilon)$$

and so $\left|f(\tau_j)\left[\varphi(t_{j+1})-\varphi(t_j)\right]\right|-f\left(\frac{1}{2}\right)\left[\varphi(t_{j+1})-\varphi(t_j)\right]<\varepsilon$.

By our choice of $j$ we have

$$\varphi(t_j+1)-\varphi(t_j)=1.$$

This proves that

$$I(f,\varphi) \text{ exists and } I(f,\varphi)=f\left(\frac{1}{2}\right).$$

**SAQ 3 :** Evaluate the integral

$$\int_{w=2}\frac{dw}{w-1}$$

**Solution :** Let us write $\alpha$ for the value of the integral. First we shall write down the integral to be evaluated. By our conventions the path is

$$\varphi(t)=2e^{it} \qquad t\in[0,2\pi]$$

Since $\varphi(t)$ is smooth by                of lesson 1 we have

$$\int_{|w|=2}\frac{dw}{w-1}=\int_0^{2\pi}\frac{2ie^{it}}{\left(2e^{it}-1\right)}dt=2i\int_0^{2\pi}\frac{e^{it}}{\left(2e^{it}-1\right)}dt.$$

We have

$$\frac{e^{it}}{\left(2e^{it}-1\right)}=\frac{e^{it}\left(2e^{-it}-1\right)}{\left(2e^{it}-1\right)\left(2e^{-it}-1\right)}=\frac{2-e^{it}}{\left|2e^{it}-1\right|^2}-\frac{2-e^{it}}{5-4\cos t}$$

$$=\frac{2-\cos t}{5-4\cos t}+i\frac{\sin t}{5-4\cos t}$$

Since $\frac{\sin t}{5-4\cos t}=\frac{1}{4}\frac{d}{dt}\log(5-4\cos t)$ and $5-4\cos\theta=5-4\cos 2\pi$

We have $\int_0^{2\pi} \frac{\sin t}{5-4\cos t} dt = 0$.

Thus $\alpha = 2i \int_0^{2\pi} \frac{2-\cos t}{5-4\cos t} dt$.

We have $\frac{2-\cos t}{5-4\cos t} = \frac{1}{4}\left(\frac{8-4\cos t}{5-4\cos t}\right) = \frac{1}{4}\left(\frac{3}{5-4\cos t}+1\right)$

$$= \frac{1}{4} + \frac{3}{4}\frac{1}{5-4\cos t}$$

$$\therefore \alpha = 2i \int_0^{2\pi} \frac{dt}{4} + \frac{6}{4} i \int_0^{2\pi} \frac{dt}{5-4\cos t} = \pi i + \frac{3}{2} i \int_0^{2\pi} \frac{dt}{5-4\cos t} \quad \text{---------- (1)}$$

We have $\int_0^{2\pi} \frac{dt}{5-4\cos t} = \left( \int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi} \right) \frac{dt}{5-4\cos t}$

$$= I_1 + I_2 + I_3 + I_4$$

Setting $t = \pi - s$, we have $dt = -ds$, $\cos t = -\cos s$ and so

$$\int_{\pi/2}^{\pi} \frac{dt}{5-4\cos t} = \int_0^{\pi/2} \frac{ds}{5+4\cos s}.$$

Setting $t = \pi + s$, we have $dt = ds$, $\cos t = -\cos s$ and so

$$\int_{\pi}^{\pi+\frac{\pi}{2}} \frac{dt}{5-4\cos t} = \int_0^{\pi/2} \frac{ds}{5+\cos s}$$

Setting $t = 2\pi - s$, we have $dt = -ds$, $\cos t = \cos s$ and so

$$\int_{\frac{3\pi}{2}}^{2\pi} \frac{dt}{5-4\cos t} = \int_0^{\pi/2} \frac{ds}{5-4\cos s}$$

Therefore $\int_0^{2\pi} \frac{dt}{5-4\cos t} = 2\int_0^{\pi/2} \frac{dt}{5-4\cos t} + 2\int_0^{\pi/2} \frac{dt}{5+4\cos t}$ ------------- (2)

We substitute $\tan\frac{t}{2} = s$. Then

$$\sec^2\frac{t}{2}\frac{dt}{2} = ds$$

We have

$$5-4\cos t = 5-4\frac{(1-s)^2}{1+s^2} = \frac{9+s^2}{1+s^2}$$

and $\int_0^{\pi/2} \frac{dt}{5-4\cos t} = 2\int_0^1 \frac{ds}{1+9s^2}$

$$\int_0^{\pi/2} \frac{dt}{5+4\cos t} = 2\int_0^1 \frac{ds}{9+s^2}.$$

Setting $3s = t$ we obtain

$$\int_0^1 \frac{ds}{1+9s^2} = \frac{1}{3}\int_0^3 \frac{dt}{1+t^2} = \frac{1}{3}\tan^{-1}(3);$$

setting $s = 3t$, we obtain

$$\int_0^1 \frac{ds}{9+s^2} = \frac{1}{3}\int_0^{\frac{1}{3}} \frac{dt}{1+t^2} = \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right).$$

$$\therefore \int_0^{\pi/2} \frac{dt}{5-4\cos t} + \int_0^{\pi/2} \frac{dt}{5+4\cos t} = \frac{2}{3}\left(\tan^{-1}(3)+\tan^{-1}\left(\frac{1}{3}\right)\right) = \frac{2}{3}\frac{\pi}{2}$$

$\therefore$ By (2) $$\int_0^{2\pi} \frac{dt}{5-4\cos t} = 2\times\frac{2}{3}\times\frac{\pi}{2} = \frac{2}{3}\pi$$

and by (1) $\alpha = \pi i + \left(\frac{3}{2}i\right)\left(\frac{2}{3}\pi\right) = \pi i + \pi i = 2\pi i$.

## 20.5 MODEL EXAMINATION QUESTIONS :

1. Evaluate $\int_{|w|=1} dw$

2. Evaluate $\int_{|w|=1} w\, dw$

3. Suppose $\varphi : [0,1] \to \mathbb{C}$ is given by

$$\varphi(t) = \begin{cases} 0 & 0 \le t < \frac{1}{2} \\ 1 & \frac{1}{2} \le t \le 1 \end{cases}$$

and $f(t) = t$. Then evaluate $I(f,\varphi)$.

4. Let $U = \{z \in \mathbb{C} : z \ne 0\}$ and $f(z) = \frac{1}{z}$ in $U$. Then show that $f$ has no primitive in $U$.

5. (Abel's Formula) Suppse $\sum a_n z^n$ has radius of convergence 1 and $\sum a_n$ converges to a limit $\alpha$. Then show that

$$\lim_{z\to 1^-}\left(\sum a_n z^n\right) = \sum_n \lim_{z\to 1^-}\left(a_n z^n\right) = \alpha.$$

6. Expand $\sqrt{z}$ in power series around $z = 1$.

7. Suppose $\varphi(t) = z_1 + t(z_2 - z_1)$ $t \in [0,1]$

and $f$ is a continuous function on $\mathbb{C}$. Then show that

$$\int_{\varphi} f = (z_2 - z_1) \int_0^1 f(\varphi(t))\, dt$$

8. State and prove Cauchy's Integral formula for the circle.
9. State and prove Morera's Theorem
10. State and prove Goursat's Theorem
11. Prove that every entire function has a primitive.

**REFERENCE BOOKS :**


1. J. B. Connay : Functions of One Complex Variable - Second Edition - Springer International Studer :dition.
2. Ruel V. Churchil; Jamesward Brown : Complex Variables and Applications - Mc-Graw Hill International Editions - Fifth Edition.



**Lesson Writer :**

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Lesson - 21

# INTEGRATION RESULT ON ANALYTIC FUNCTIONS IN $\mathbb{C}$ AND IN A NEIGHBOURHOOD OF A POINT

## 21.0 INTRODUCTION

Suppose $U$ is an open set in $\mathbb{C}$

$$\overline{B(a,r)} \subset U \quad (a \in \mathbb{C} \text{ and } r > 0)$$

and $f$ is analytic function on $U$. Then $f$ a continuous function on $\overline{B(a,r)}$ a compact set. So the function $|f|$ attains its minimum as well as its maximum on the set $\overline{B(a,r)}$.

We consider the following questions.

1. Where does $|f|$ attain its maximum and where does $|f|$ attain its minimum ? In the interior $B(a,r)$ or on the boundary of $\overline{B(a,r)}$ in $U$?

2. Suppose $f$ attains a value $\alpha$. How many times does $f$ attain the value $\alpha$?

3. Suppose $f$ attains a value $\alpha$. Is there a $\delta > 0$

   such that $f$ attains all the values of $B(\alpha;\delta)$? i.e. there a $\delta > 0$ such that

$$B(\alpha,\delta) \subset f(U) \text{ ?}$$

## 21.1 RESULTS :

### 21.1.1 PROPOSITION (CAUCHY'S ESTIMATE) :

Suppose $f$ is analytic in $B(a,r)$ and for some $M > 0$

$$|f(z)| \le M \text{ for all } z \text{ in } B(a,r). \text{ Then}$$

$$\left|f^{(n)}(a)\right| \le n!\,\frac{M}{r^n} \text{ for } n = 0,1,2,\cdots.$$

**Proof :** For each $\rho$ in $(0,r)$ we have

$$\frac{\left|f^{(n)}(a)\right|}{n!} \le \frac{1}{2\pi} \int_{|w-a|=\rho} \left|\frac{f(w)}{(w-a)^{n+1}}\right| |dw|$$

$$\le \frac{1}{2\pi} \frac{M}{\rho^{n+1}} 2\pi\,\rho$$

$$= \frac{M}{\rho^n}$$

Thus $\dfrac{\left|f^{(n)}(a)\right|}{n!}$ is a lower bound for the numbers $\left\{\dfrac{M}{\rho^n} : 0 < \rho < r\right\}$

Therefore it is less than or equal to the greatest lower bound of the set i.e.

$$\frac{\left|f^{(n)}(a)\right|}{n!} \le \frac{M}{r^n}.$$

**21.1.2 DEFINITION :**

An analytic function on $\mathbb{C}$ is called an entire function.

**21.1.3 THEOREM (LIOUVILLE'S THEOREM) :**

Suppose $f$ is a bounded entire function. Then

$$f(z) = f(0)$$

for all $z$ i.e. $f$ is a constant function.

**Proof :** $f$ is a bounded. That means there is an $M \ge 0$ such that

$$|f(z)| \le M \text{ for } z \in \mathbb{C}.$$

By Cauchy's estimate we have

$$\left|f^{(n)}(0)\right| \le n! \frac{M}{r^n}$$

for all $r > 0$ and $n = 1, 2, \cdots$. This implies that

$$f^{(n)}(0)=0$$

Since we have

$$f(z)=\sum \frac{f^{(n)}(0)}{n!} z^n$$

for every $z$ in $\mathbb{C}$, we obtain

$$f(z)=f(0) \text{ for all } z \text{ in } \mathbb{C}.$$

**21.1.4 THEOREM (FUNDAMENTAL THEOREM OF ALGEBRA) :**

Suppose

$$P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

is a polynomial of degree $n \geq 1$ . Then it has a zero, i.e. there is atleast one $\alpha$ in $\mathbb{C}$ such that $P(\alpha)=0$.

**Proof :** Suppose $P(z)$ does not vanish at any $z$ in $\mathbb{C}$. Then

$f(z)=\dfrac{1}{P(z)}$ $(z \in \mathbb{C})$ is an entire function. We claim $f$ is bounded.

Let $K>2$ and set

$$\rho=nK\left(1+|a_{n-1}|+\cdots+|a_0|\right)$$

For $z \neq 0$ we have

$$P(z)=z^n\left(1+\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right)$$

for $|z| \geq \rho$ we have

$$\left|\frac{a_j}{z^j}\right| \leq \frac{1}{(nK)^j} \leq \frac{1}{nK}$$

for $j=0,1,2,\cdots,n-1$. Therefore,

$$\left|\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right| \le \frac{1}{nK}\cdot n = \frac{1}{K};$$

$$|P(z)| = |z^n|\left|1-\left(-\left(\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right)\right)\right|$$

$$\ge |z^n|\left|1-\frac{1}{K}\right|$$

$$\ge \frac{1}{2}\rho^n.$$

So

$$|f(z)| \le \frac{2}{\rho^n} \text{ for } |z| \ge \rho.$$

$f(z)$ is bounded on the compact set $\overline{B(0,e)}$. Therefore $f$ is bounded. This implies that $P$ is a constant function. This not possible since $n \ge 1$. Thus we have a contradiction and the Theorem is proved.

**21.1.5 COROLLARY :**

Suppose $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ then there are complex numbers $\alpha_1, \cdots, \alpha_r$ and natural numbers $n_1, \cdots, n_r$ such that

$$P(z) = (z-\alpha_1)^{n_1} \cdots (z-\alpha_r)^{n_r}.$$

**REMARK :**

We may take $\alpha_1, \cdots, \alpha_r$ to be different from each other. The consideration of degree gives

$$n = n_1 + n_2 + \cdots + n_r.$$

**Proof :** By the above theorem there is an $\alpha$ in $\mathbb{C}$ such that $P(\alpha) = 0$.

Then by algebra we have

$$P(z) = (z-\alpha)P_1(z) \text{ where}$$

$$P_1(z) = z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0$$

By induction we may assume that

$$P_1(z) = (z-\beta_1)^{m_1} \cdots (z-\beta_s)^{m_s},$$

where $\beta_1, \beta_2, \cdots\cdots, \beta_s$ are s complex numbers and $m_1, m_2, \cdots\cdots, m_s$ are positive integers. Then $p(z) = (z-\alpha)(z-\beta_1)^{m_1} \cdots\cdots (z-\beta_s)^{m_s}$.

If $\alpha = \beta_j$ then

$$P(z) = (z-\beta_1)^{m_1} \cdots (z-\beta_j)^{m_j+1} \cdots (z-\beta_s)^{m_s}$$

The results is proved.

**21.1.6 THEOREM :**

Suppose $f$ is analytic on $B(a,r)$ and $f$ is not the constant function $f(a)$ on $B(a,r)$. Then there is a positive integer $m$ and an analytic function $g(z)$ on $B(a,r)$ such that

$$f(z) - f(a) = (z-a)^m g(z) \text{ where } g(a) \neq 0.$$

**Proof :** By Theorem 20.3.4 of lesson 20 we have

$$f(z) - f(a) = \sum_{n=1}^{\infty} a_n (z-a)^n.$$

The power series on the right converges for $z$ in $B(a,r)$. If all the $a_n$ are zero then for each $z$ in $B(a,r)$.

$$\sum_{n=1}^{\infty} a_n (z-a)^n = 0;$$

this implies $f$ is equal to the constant function $f(a)$ on $B(a,r)$ contrary to our hypothesis. Therefore there is atleast one $n$ such that $a_n \neq 0$. Let $m$ be the least positive integer such that

$a_m \neq 0$ :

if $m > 1 : a_1 = \cdots = a_{m-1} = 0$ and $a_m \neq 0$

We have $f(z)-f(a)=\sum_{n=m}^{\infty} a_n (z-a)^n$

$$=(z-a)^m \sum_{k=0}^{\infty} a_{m+k} (z-a)^k$$

We claim that the power series $\sum_{k=0}^{\infty} a_{m+k} (z-a)^k$ converges in $B(a,r)$. For $0<|z-a|=r_0<r$ we have

$$\sum \left|a_{m+k} (z-a)^k\right| = \frac{1}{r_0^{\ m}} \sum \left|a_n (z-a)^n\right|$$

since the series $\sum_{n=1}^{\infty} a_n (z-a)^n$ converges absolutely for $|z-a|=r_0$. Therefore

$\sum_{k=0}^{\infty} a_{m+k} (z-a)^k$ converges in $B(a,r)$. If we set $g(z)=\sum_{k=0}^{\infty} a_{m+k} (z-a)^k$

we have $g(a)=a_m \neq 0$ and

$f(z)=(z-a)^m g(z)$ the theorem is proved.

**21.1.7 THEOREM :** Suppose $U$ is a connected open set in (1), $f$ is an analytic function in $U$ and

$$z(f)=\{z \in U/f(z)=0\}$$

1) $f$ is the zero function on $U$.

2) There is an $a$ in $U$ such that $f^{(n)}(a)=0$ for $n=0,1,2,\cdots$

Here $f^{(0)}(a)$ is $f(a)$.

3) $z(f)$ has a limit point $z_0$ belonging to $U$.

**Proof :** $(1) \Rightarrow (2)$ is clear. We may take $a$ to be any point in $U$.

$(2) \Rightarrow (3)$ suppose $r>0$ is such that $B(a,r) \subset U$. Then for $z$ in $B(a,r)$ we have

$$f(z)=\sum\frac{f^{(n)}(a)}{n!}(z-a)^n=0$$

Therefore $z_0=a$ is a limit point of $z(f)$.

$(3)\Rightarrow(1)$.

Let $r_0>0$ be such that $B(z_0,r_0)\subset U$. Then we have $f(z_0)=0$ and

$$f(z)=f(z)-f(z_0)=\sum_{n=1}^{\infty}a_n(z-z_0)^n$$

If there is an $n$ such that $a_n\neq 0$ by Theorem (21.1.6) we have

$$f(z)=(z-z_0)^m g(z)$$

where $g(z)$ is analytic and $g(z_0)\neq 0$. Then there is some $s>0$ such that $B(z_0,s)\subset B(z_0,r_0)$ and

$$g(z)\neq 0 \text{ for } z\in B(z_0,s)$$

This implies that.

$$z(f)\cap B(z_0,s)=\{z_0\}$$

which contradicts our hypothesis $z_0$ is a limit point of $z(f)$. Therefore we must have $a_n=0$ for $n=1,2\cdots$ This implies that

$$f(z)=0 \text{ for } z\in B(z_0,r_0)$$

We define $V=\{z\in U/f(w)=0 \text{ for } w\in B(z,s) \text{ for some } s=s(w)>0\}$

we have proved above that V is a non-empty subset of $U$.

**Claim :** V is an open set

Let $z_1\in V$. Then there is an $r_1>0$ such that

$$f(w)=0 \text{ for } w\in B(z_1,r_1).$$

Let $w_1$ be such that

$|w_1 - z_1| = s_1 < r_1$. Then

$B(w_1, r_1 - s_1) \subset B(z_1, r_1)$

This implies $f$ is zero on $B(w_1, r_1 - s_1)$ and so $w_1 \in V$. That is $B(z_1, r_1) \subset V$.

So V is open.

We claim that V is closed in $U$.

Suppose $a \in U$ is a limit point of V. Suppose $B(a, r) \subset U$.

Then $f(a) = 0$ and

$$f(z) = f(z) - f(a) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n .$$

If there is an $n$ such that $f^{(n)}(a) \neq 0$, then by Theorem (21.1.6), we have

$$f(z) = f(z) - f(a) = (z-a)^m g(z)$$

where $g(a) \neq 0$. This implies that there is an $r_0 > 0$ such that

$g(z) \neq 0$ in $B(a, r_0)$;

and so $f(z) \neq 0$ for $0 < |z - a| < r_0$.

Since $f = 0$ on V and $a$ is a limit point of V, this is not possible. This contradiction proves that

$f^{(n)}(a) = 0$ for $n = 1, 2, \cdots$

This implies that $f(z) = 0$ for $z \in B(a, r)$ and so $a \in V$.

We have assumed that $U$ is a connected set. Therefore either $V = U$ or $V = \phi$. Since we have proved that V is non-empty we have

$V = U$. The theorem is proved.

**21.1.8 REMARK :**

Suppose $U$ is a connected open set, $f$ is analytic on $U$ and $f$ is not a constant function. Then

$$z(f)=\{z\in U/f(z)=0\}$$

has not limit points in $U$. Therefore if $f(a)=0$, there is an $r>0$ such that

$$f(z)\neq 0 \text{ if } 0<|z-a|<r.$$

**21.1.9 EXAMPLE :**

Consider $U=\mathbb{C}\setminus\{0\}$ and $f=\sin\frac{1}{z}$ on $U$.

The set

$$z(f)=\left\{z/z=\frac{1}{i\pi n}\right\}$$

is the set of zeros of $f$. This set has a limit point, namely 0. However $a\notin U$.

The next result says that $|f|$ cannot attain maximum on $U$, if $U$ is connected.

**21.1.10 THEOREM 5 MAXIMUM MODULUS THEOREM :**

Suppose $U$ is an open set in $\mathbb{C}$ and $f$ is analytic in $U$. If there is a $z_0\in U$ such that

$$|f(z)|\leq|f(z_0)|$$

and $U$ is connected, then $f(z)=f(z_0)$ for all $z$ in $U$.

**Proof :** $|f(z)|\leq|f(z_0)|$ for $z\in B(z_0,r_0)$

Choose any $\rho$ such that $0<\rho<r_0$. By Cauchy Integral formula we have

$$f(z_0)=\frac{1}{2\pi i}\int_{|z-z_0|=\rho}\frac{f(w)}{w-z_0}\,dw$$

Therefore

$$|f(z_0)| = \left|\frac{1}{2\pi i}\int_{|w-z_0|=\rho} \frac{f(w)}{(w-z_0)}\,dw\right|$$

$$\leq \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f\left(z_0+\rho e^{i\theta}\right)}{\rho e^{i\theta}}\, i\rho e^{i\theta}\, d\theta\right|$$

$$= \frac{1}{2\pi}\left|\int_0^{2\pi} f\left(z_0+\rho e^{i\theta}\right) d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0+\rho e^{i\theta}\right)\right| d\theta .$$

That is $2\pi|f(z_0)| \leq \int_0^{2\pi} \left|f\left(z_0+\rho e^{i\theta}\right)\right| d\theta$;

$$\int_0^{2\pi}\left[|f(z_0)| - \left|f\left(z_0+\rho e^{i\theta}\right)\right|\right] d\theta \leq \theta$$

By our hypothesis

$|f(z)| \leq |f(z_0)|$ for all z in $U$. Therefore $|f(z_0)| - \left|f\left(z_0+\rho e^{i\theta}\right)\right| \geq 0$. By the properties of the Riemann Integral,

$$\int_0^{2\pi} |f(z_0)| - \left|f\left(z_0+\rho e^{i\theta}\right)\right| d\theta \geq 0$$

Therefore $\int_0^{2\pi}\left\{|f(z_0)| - \left|f\left(z_0+\rho e^{i\theta}\right)\right|\right\} d\theta = 0$.

The integrand is a non-negative continuous function. By properties of Riemann Integral we obtain

$$|f(z_0)| = \left|f\left(z_0+\rho e^{i\theta}\right)\right| \text{ for any } \rho s \neq \theta,\ 0<\rho<r_0,\ \theta\in[0,2\pi]$$

Therefore $|f(z)| = |f(z_0)|$ for $z \in B(z_0, r_0)$.

This implies $f(z) = f(z_0)$ for $z \in B(z_0, r_0)$.

Now consider the function

$$g(z) = f(z) - f(z_0)$$

on $U$ $g(z)$ is analytic on $U$. We have

$$g(z) = 0 \text{ for } z \in B(z_0, r_0)$$

By a known result

$$f(z) = f(z_0) \text{ on } U$$

**SAQ 2 :** If $f$ is analytic and

$$|f(z)| = |f(a)| \text{ on } B(a, r)$$

Prove that $f(z) = f(a)$ for $z$ in $B(a, r)$.

**21.1.11 EXAMPLE :**

Suppose

$$U = B(0,1) \cup B(2,1)$$

Define $f$ on $U$ by

$$f(z) = z \text{ for } z \in B(0,1)$$

$$f(z) = 1 \text{ for } z \in B(2,1)$$

Here $U$ is not connected and $f$ is not a constant function.

## 21.2 FURTHER RESULTS :

### 21.2.1 DEFINITION :

Suppose $U$ is an open set, $f$ is an analytic function on $U$, $a \in U$ and $\alpha \in \mathbb{C}$. We say that

"$\alpha$ is a value of $f$ at a with multiplicity $m$" or

"$f$ takes the value $\alpha$ at a with multiplicity $m$"

if $f(a)=\alpha$ and $f(z)-f(a)=(z-a)^m \cdot g(z)$

where $g$ is an analytic function on $U$ and $g(a) \neq 0$.

**21.2.2 DEFINITION :**

We say that $\alpha$ is a simple value of $f$ if

(1) $f$ takes the value $\alpha$ at some $a$ in $U$ and

(2) At any $z_0 \in U$, $f(z_0)=\alpha$ implies $f$ takes the value $\alpha$ at $z_0$ with multiplicity one.

**21.2.3 PROPOSITION :**

Suppose $U$ is a connected open set, $f$ is analytic on $U$ and not a constant. Then $f$ takes the value $f(a)$ with multiplicity $\geq 2$ if and only if $f'(a)=0$.

**Proof :** Suppose $f$ takes the value $f(a)$ with multiplicity $m \geq 2$. Then $f(z)-f(a)=(z-a)^m g(z)$

where $g(z)$ is anlaytic in $U$. Then

$$f'(z)=(f(z)-f(a))'=m(z-a)^{m-1}g(z)+(z-a)^m g'(z)$$

Since $m \geq 2$, $(m-1) \geq 1$ and so $f'(a)=0$.

Suppose $a \in U$ and $f'(a)=0$. Since $f$ is not a constant we have by Theorem

$$f(z)-f(a)=(z-a)^m g(z)$$

where $g(z)$ is an analytic function on $U$ such that $g$ then

$$f'(z)=m(z-a)^{m-1}g(z)+(z-a)^m g(z)$$

Therefore $f'(a)=0$ implies $(m-1) \geq 1$ or $m \geq 2$.

**21.2.4 PROPOSITION :**

Suppose that $f$ is analytic on $B(a,r)$ and $f$ is not the constant function $f(a)$ on $B(a,r)$ Then there is an $r_0 \leq r$, $0 < r_0$ such that $f$ assumes only simple values on

$$\{z : 0 < |z-a| < r_0\} = B(a,r_0) \setminus \{a\}.$$

**Proof :** Consider the function $f'(z)$ on $B(a,r)$. If $f'(z)$ is a constnat on $B(a,r)$ then

$$f'(z) = f'(a), \text{ for } z \in B(a,r).$$

Since $f$ is not a constant function $f'(a)$ cannot be zero. Therefore $f'(z) \neq 0$ for any $z$ in $B(a,r)$. By the previous proposition every value of $f$ is a simple value.

Suppose $f'(z)$ is not equal to a constant function. Then by Remark after theorem there is an $r_0 > 0$ such that $0 < r_0 \leq r$ and

$$f'(z) \neq 0 \text{ for } 0 < |z-a| < r_0$$

By proposition (21.2.2) this implies that every value of $f$ on $B(a,r_0) \setminus \{a\}$ is a simple value. The theorem is proved.

We shall now prove that if $U$ is a conected open set and $f$ is an analytic function on $U$ and not a constant function then $f$ is an open map.

We shall state and prove the main step in the proof as a Theorem.

**21.2.5 THEOREM :**

Suppose $f$ is analytic in $B(a,r)$ and $f$ is not the constant function $f(a)$. Then there is a $\delta > 0$ such that

$$B(f(a), \delta) \subset f(B(a,r))$$

**Proof :** Clearly there is an analytic function $g$ on $B(a,r)$ such that

(i) $f(z) - f(a) = (z-a)^m g(z), \quad m \in \mathbb{N}$ and

(ii) $g(a) \neq 0$

Since $g$ is continuous we can find an $r_0 > 0$ such that

$$g(z) \neq 0 \text{ for } |z-a| \leq 2r_0.$$

The main ideas behind the proof of thc theorem are the following two facts

(a) Since $g(z)$ is never zero on $\bar{B}(a, 2r_0)$, $\dfrac{g'(z)}{g(z)}$ is an analytic function on $B(a, 2r_0$

therefore

$$\int_{|z-a|=r} \frac{g'(z)}{g(z)} dz = 0 \text{ for } r_1 \le 2r_0$$

(b) $$\int_{|w-a|=r_1} \frac{f'(w)}{f(w)-f(a)} = 2\pi i m \text{ for } r_1 \le 2r_0$$

We have $f(z)-f(a)=(z-a)^m g(z)$;

$$f'(z) = (f(z)-f(a))' = m(z-a)^{m-1} g(z)+(z-a)^m g'(z)$$

$$\frac{f'(z)}{f(z)-f(a)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

Therefore $$\int_{|w-a|=r_1} \frac{f'(w)}{f(w)-f(a)} dw = \int_{|w-a|=r_1} \frac{m}{(w-0)} dw + \int_{|w-a|=r_1} \frac{g'(w)}{g(w)} dw$$

$$= 2\pi i m$$

Let us set $S(a,r_0) = \{z \in \mathbb{C} / |z-a| = r_0\}$, and

$$2\Delta = d(0, f(S(a,r_0)))$$

$S(a,r_0)$ is a compact set and $f$ being continuous $f(S(a,r_0))$ is a compact set. Therefore there is a $w_0 \in S(a,r_0)$ such that $2\Delta = |f(w_0)|$. We have assumed that $f(w) \ne 0$ on $S(a,r_0)$ therefore $\Delta > 0$. Let us choose any $\alpha$ in C with $|\alpha| < \Delta$. Then $f(w)-\alpha \ne 0$ for $w \in S(a,r_0)$ because

$$|f(w)-\alpha| \ge \big||f(w)|-|\alpha|\big|$$

$$= |f(w)|-|\alpha| \text{ since } |\alpha| < \Delta < 2\Delta \le |f(w)|$$

$$> 2\Delta - \Delta$$

$$= \Delta$$

Therefore $\eta(\alpha)=\int\limits_{|w-a|=r_0}\frac{f'(w)}{f(w)-\alpha}dw$ is defined. We have

$$\eta(\alpha)-\eta(0)=\int\limits_{|w-a|=r_0} f'(w)\left\{\frac{\alpha}{f(w)\cdot(f(w)-\alpha)}\right\}dw$$

We haave seen above that for $|w-a|=r_0$ we have

$|f(w)-\alpha|>\Delta$ and $|f(w)|\geq 2\Delta$.

Therefore $|\eta(\alpha)-\eta(0)|\leq \int\limits_0^{2\pi}|f'(w)|\frac{|\alpha|}{2\Delta^2}r_0\,d\theta$

$\leq\frac{2\pi r_0}{2\Delta^2}M(f')\,|\alpha|$, where $M(f)=\sup\limits_{|w-a|=r_0}|f(w)|$

We know that $n(0) = 2\pi im,\ m\geq 1$. Therefore if we set

$$\delta=\min\left\{\Delta,\frac{\Delta^2}{(1+M(f'))(r_0+1)}\right\}$$

and choose any $\alpha$ with $|\alpha-f(a)|<\delta$ then

$\eta(\alpha)\neq 0$.

This implies that $f(z)-\alpha$ vanishes in $B(a,r_0)$ by (1) above. If $f(z)-\alpha$ is never zero in $\overline{B(a,r_0)}$, $\frac{f'(z)}{f(z)-\alpha}$ is analytic in $\overline{B(a,r_0)}$ and so

$$\eta(\alpha)=\int\limits_{|w-a|=r_0}\frac{f'(w)}{f(w)-\alpha}dw=0$$

The result is proved.

**21.2.6 THEOREM :**

Suppose $U$ is a connected open set and

$$f : U \to \mathbb{C}$$

is an analytic function which is not a constant function. Then $f$ is an open map.

**Proof :** Let $a \in U$. Then there is an $r > 0$ such that

$$B(a, r) \subset U.$$

By Theorem $f$ is not equal to $f(a)$ on $B(a,r)$. Therefore we can apply the previous theorem and conclude that there is a $\delta > 0$ such that

$$B(f(a), \delta) \subset f(B(a,r)) \subset f(U).$$

Thus $f$ is an open map.

**21.2.7 THEOREM :**

Suppose $f$ is analytic in $(a, r)$ and $f$ is not a constant function. Then there is an $r_1 > 0$, a $\delta > 0$ such that

1) an $B(a, r_1) \setminus \{a\}$ every value of $f$ is a simple value

2) $f(B(a, r_1)) \supset B(f(a), \delta)$.

**Proof :** By proposition there is an $r_1$ such that $0 < r_1 < r$ every value of $f$ on $B(a, r_1) \setminus \{a\}$ is a simple value. By Theorem above there is a $\delta > 0$ such that

$$f(B(a, r)) \supset B(f(a), \delta)$$

The theorem is proved.

**21.3 SOLUTIONS TO SAQ'S :**

**SAQ 1 :** Suppose that $f$ is continuous on $\overline{B}(a, r) = \{z : |z - a| \le r\}$.

Then $\displaystyle \mathop{Lt}_{\substack{\rho \to r \\ \rho < r}} \int_{|z-a|=\rho} f(t)\,dt = \int_{|z-a|=\rho} f(t)\,dt$

**Solution :** Let $\varepsilon > 0$. Since $\bar{B}(a,r)$ is compact and $f$ is continuous, $f$ is uniformly continuous on $\bar{B}(a,r)$. Therefore we can choose a $\delta = \delta(\varepsilon) > 0$ s.t. for all $z_1 w$ in $B(a,r)$

$$|f(z) - f(w)| < \varepsilon \text{ if } |z - w| < \delta.$$

We have

$$\left| \int_{|z-a|=\rho} f(z)dz - \int_{|z-a|=r} f(z)dz \right|$$

$$\left| \int_0^{2\pi} f\left(a + \rho e^{i\theta}\right) ie^{i\theta} d\theta - \int_0^{2\pi} f\left(a + re^{i\theta}\right) ie^{i\theta} dw \right|$$

$$\leq \int_0^{2\pi} \left| f\left(a + \rho e^{i\theta}\right) - f\left(a + re^{i\theta}\right) \right| d\theta < \int_0^{2\pi} \varepsilon = 2\pi\varepsilon$$

Since $\left| \left(r + \rho e^{i\theta}\right) - \left(a + re^{i\theta}\right) \right| = r\rho 2\delta$.

This proves assertion.

**SAQ 2 :** Suppose that $U$ is a connected open set and $f(z)$ is an analytic function such that $|f(z)| = c$ a constant for $z \in U$. Then $f(z)$ is a constant.

**Solution :** Let $f(z) = u(x, y) + i\upsilon(x, y)$. Then $|f(z)| = c$ implies

$$u^2 + v^2 - c^2$$

Differentiating with respect to $x$ first and with $y$ next, we obtain

$$2u u_x + 2v v_x = 0$$

$$2u u_y + 2v v_y = 0$$

We obtain for $x + iy$ in $U$

$$\left(u_x v_y - u_y v_x\right) \cdot u = 0, \left(u_x v_y - u_y v_x\right) v = 0$$

If $u_x v_y - u_y v_x \neq 0$ for some $z_0 = x_0 + iy_0 \in U$, then, since we have assumed $f'(z)$ is continuous, $u_x, u_y, v_x, v_y$ are continuous and so $u_x v_y - u_y v_x \neq 0$ in some neighbourhood $B(z_0, \bar{0})$ $(x_0, y_0)$. In that neighbourhood $u = 0,\ v = 0$. This implies that $u_x = u_y = v_x = v_y = 0$ in $B(z_0, \delta)$; this contradicts our assumption

$$\left(u_x v_y - u_y v_x\right)(x_0 y_0) \neq 0.$$

Therefore we must have

$$u_x v_y - u_y v_x \equiv 0 \text{ in } U.$$

By Cauchy Riemann equation this implies

$$u_x^2 + u_y^2 = 0,\ v_x^2 + v_y^2 = 0,$$

This gives $f'(z) \equiv 0$ in $U$.

Suppose $B(a, r) \subset U$.

Consider $g(t) = f\left(a + t(z - a)\right)$.

We have $g'(t) = f'\left(a + t(z - a)\right)(z - a)$

$= 0$

and $$g(1) - g(0) = \int_0^1 g'(t)\,dt = 0$$

Therefore $f(z) = g(1) = g(0) = f(a)$

This implies that $f(z) - f(a)$ vanishes on $B(a, r)$. By Theorem ........ we obtain $f(z) \equiv f(a)$.

**SAQ 3 :** Suppose $U$ is an open set in $\mathbb{C}$, $f : U \to \mathbb{C}$ is an analytic function, $a \in U$, $m_1$, $m_2$ are non-negative integers $g_1(z)$, $g_2(z)$ are analytic function on $U$ such that

$$f(z) = (z - a)^{m_1} g_1(z),\ \ g_1(a) \neq 0$$

$$f(z) = (z-a)^{m_2} g_2(z),\ g_2(a) \neq 0$$

Then $m_1 = m_2$ and $g_1 = g_2$.

**Proof :** Suppose we have proved $m_1 = m_2$. Then we obtain

$$(z-a)^{m_1}\left(g_1(z) - g_2(z)\right) = 0 \text{ for } z \in U.$$

If $z \neq a$, $(z-a) \neq 0$ and so we obtain

$$g_1(z) - g_2(z) = 0 \qquad z \in U \setminus \{a\}$$

Since $g_1, g_2$ are continuous on $U$ we obtain $g_1(a) = g_2(a)$ and therefore

$$g_1 = g_2$$

So we need only prove $m_1 = m_2$, suppose if possible $m_1 \neq m_2$ either $m_1 < m_2$ or $m_1 > m_2$, we shall show $m_1 < m_2$ is impossible.

$$(z-a)^{m_1}\left(g_1(z) - (z-a)^{m_2-m_1} g_2(z)\right) = 0$$

This implies that $g_1(z) = (z-a)^{m_2-m_1} g_2(z)$ for all $z$ in $U$, $z \neq a$. Again by continuity we obtain

$$g_1(z) = (z-a)^{m_2-m_1} g_2(z) \text{ for all } z \text{ in } U.$$

This implies $g_1(a) = 0$ since we have assumed $(m_2 - m_1) > 0$. This contradicts the hypothesis $g_1(a) \neq 0$. So $m_1 < m_2$ is impossible. Similarly $m_1 > m_2$ is impossible. Thus

$m_1 = m_2$ and the Theorem is proved.

## 21.4 MODEL EXAMINATION QUESTIONS :

1) If $U$ is a connected open set, $f$ is analytic on $U$ and on some non-empty open set $V \subset U$, $f = i$, then

$$f = i \text{ on } U$$

2) Suppose $U$ is an open set in $\mathbb{C}$ and $f$ is an analytic function. Show that if $U$ is connected either $f(U)$ is a single point or an open subset of $\mathbb{C}$.

3) Find all entire functions $f$ such that

$f(x) = e^x$ for $x \geq 0$

4) Suppose $f$ is an entire function such that

$|f(z)| \leq M|z|^n$ for $z \in \mathbb{C}$.

Then show that $f$ is a polynomial of degree $\leq n$.

5) $f$ is analytic in an open connected set $U$ and $a \in U$ is such that

$|f(a)| \leq |f(z)|$ for all $z \in U$

Then either $f(a) = 0$ or $f$ is a constant.

6) State and prove Liovville's Theorem.

7) State and prove Fundamental Theorem of Algebra.

8) State and prove Maximum Modulus Principle.

9) State and prove open mapping theorem for analytic functions.

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Lesson - 22

# RIEMANN - STIELYIES INTEGRAL

## 22.0 INTRODUCTION

Suppose $f,\varphi:[a,b]\to\mathbb{C}$ are functions. Then we have defined $I(f,\varphi)$ in Lesson 20.

In this lesson we prove that $I(f,\varphi)$ exists when $\varphi$ is a function of bounded variation and $f$ is a continuous function.

It is proved that $\varphi$ is of bounded variation when $\varphi$ is a monotonic function and when $\varphi$ is a piecewise smooth function.

We prove that when $f$ is a continuous function and $\varphi$ is a continuous function of bounded variation, given $\varepsilon>0$ we can find a polygonal path, with vertices on $\{\varphi\}$, say $\Gamma(\varepsilon)$ such that

$$\left|\int_{\varphi} f - \int_{\Gamma} f\right| < \varepsilon.$$

## 22.1 PRELIMINARY RESULTS :

### 22.1.1 DEFINITION :

Suppose $\varphi:[a,b]\to\mathbb{C}$ is a function. We say that $\varphi$ is of bounded variation if there is some real number $K$ with the following property : For each partition

$$P: a=t_0<t_1<\cdots\cdots<t_n=b$$

$$|\varphi(t_1)-\varphi(t_0)|+|\varphi(t_2)-\varphi(t_1)|+\cdots+|\varphi(t_n)-\varphi(t_{n-1})|\le K.$$

### 22.1.2 REMARKS :

(i) A function of bounded variation is a bounded function. Let

$P: a=t_0<t=t_1<t_2=b$. Then

$$|\varphi(t)-\varphi(t_0)|+|\varphi(t_2)-\varphi(t)|\le K$$

So $\quad |\varphi(t)-\varphi(a)|\le K$

or $\quad |\varphi(t)| \leq |\varphi(a)| + K$

(ii) Suppose $\varphi[a,b] \to \mathbb{C}$ is of bounded variation. Let us denote by $\varphi_t$ the restriction of $\varphi$ to $[a,t]$, $t \in [a,b]$.

For $a \leq t \leq b$, $\varphi_t : [a,t] \to \mathbb{C}$ is defined by

$\varphi_t(s) = \varphi(s)$ for $s \in [a,t]$.

Then $\varphi_t$ is of bounded variation.

Suppose $P : a = t_0 < \cdots < t_m = t < b$ is a partition of $[a,t]$. Then let a partition $P'$ of $[a,b]$ be defined by

$$P' : a = t_0 < \cdots < t_m < t_{m+1} = b$$

Then $\quad |\varphi_t(t_1) - \varphi_t(t_0)| + \cdots + |\varphi_t(t_m) - \varphi_t(t_{m-1})|$

$$= |\varphi(t_1) - \varphi(t_0)| + \cdots + |\varphi(t_m) - \varphi(t_{m-1})|$$

$$\leq |\varphi(t_1) - \varphi(t_0)| + \cdots + |\varphi(t_m) - \varphi(t_{m-1})| + |\varphi(t) - \varphi(t_m)|$$

$$\leq K.$$

(iii) We have for any two complex numbers $z, w$

$$|\operatorname{Re} z - \operatorname{Re} w| \leq |z - w| \leq |\operatorname{Re} z - \operatorname{Re} w| + |\operatorname{Im} z - \operatorname{Im} w|$$

$$|\operatorname{Im} z - \operatorname{Im} w| \leq |z - w| \leq |\operatorname{Re} z - \operatorname{Re} w| + |\operatorname{Im} z - \operatorname{Im} w|$$

It follows that $\varphi$ is of bounded variation if and only if $\operatorname{Re}\varphi$ and $\operatorname{Im}\varphi$ are of bounded variation.

### 22.1.3 DEFINITION :

Let $\varphi : [a,b] \to \mathbb{C}$ be of bounded variation. Then the numbers

$$v(\varphi, P)$$

for all possible P of $[a,b]$ from a bounded set in $\mathbb{R}$. Its least upper bound is called

The variation of $\varphi$ on $[a,b]$ and is denoted by $V(\varphi)$.

**22.1.4 REMARK :**

It is clear that for each $t \in [a,b]$

$$V(\varphi_t) \leq V(\varphi)$$

and $V(\varphi_t)$ is a non-negative, non-decreasing function of $t$ we denote $V(\varphi_t)$ by

$$|\varphi|(t) \text{ or } V(\varphi)(t)$$

**22.1.5 PROPOSITION :**

Suppose $f:[a,b] \to \mathbb{C}$ is such that

$$M(f) = \sup_{t \in [a,b]} |f(t)| < +\infty$$

and $\varphi$ is a function of bounded variation s.t. $I(f,\varphi)$ exists. Then

$$|I(f,\varphi)| \leq M(f) \cdot V(\varphi).$$

**Proof :** Let $\varepsilon > 0$. Then we can select a $\delta = \delta(\varepsilon) > 0$ s.t. for all partitions

$$P: a = t_0 < \cdots < t_n = b$$

and each sequence $\tau_0, \cdots, \tau_{n-1}$, $\tau_j \in [t_j, t_{j+1}]$ we have

$$\left|\sum f(\tau_j)\left[\varphi(t_{j+1}) - \varphi(t_j)\right] - I(t,\varphi)\right| < \varepsilon \text{ if } \|P\| < i(\varepsilon)$$

So $|I(f,\varphi)| \leq \sum |f(\tau_j)| |\varphi(t_{j+1}) - \varphi(t_j)| + \varepsilon \leq M(f) \cdot V(\varphi) + \varepsilon$.

This implies the proposition.

**22.1.6 PROPOSITION :**

Suppose $\varphi:[a,b] \to \mathbb{C}$ is smooth then $\varphi$ is of bounded variation and

$$V(\varphi) = \int_a^b |\varphi'(t)| dt$$

**Proof :** Let

$$P : a = t_0 < t_1 < \cdots < t_n = b$$

be a partition of $[a,b]$. Since $\varphi$ is smooth we have

$$\int_{t_k}^{t_{k+1}} \varphi'(t)\,dt = \varphi(t_{k+1}) - \varphi(t_k)$$

So we have

$$V(P,\varphi) = \sum_{k=0}^{n-1} \left|\varphi(t_{k+1}) - \varphi(t_k)\right| = \sum_{k=0}^{n-1} \left| \int_{t_k}^{t_{k+1}} \phi'(t)\,dt \right|$$

$$\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left|\varphi'(t)\right| dt$$

$$= \int_a^b \left|\varphi'(t)\right| dt.$$

Thus we obtain $\varphi$ is bounded variation and further

$$V(\varphi) \leq \int_a^b \left|\varphi'(t)\right| dt$$

We shall now prove $\int_a^b \left|\varphi'(t)\right| dt \leq V(\varphi)$. For this let $\varepsilon > 0$ $\varphi'$ on $[a,b]$ is given to be continuous, and $[a,b]$ is compact. Therefore there is a $\delta = \delta(\varepsilon) > 0$ such that for all $s,t$ in $[a,b]$.

$$\left|\varphi'(s) - \varphi'(t)\right| < \varepsilon \text{ if } |s-t| < \delta.$$

Let $P$ be any partition of $[a,b]$. We have

$$\int_a^b \left|\varphi'(t)\right| dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left|\varphi'(t)\right| dt.$$

By the mean value theorem of integral calculus there is a $\tau_k \in [t_k, t_{k+1}]$ such that

$$\int_{t_k}^{t_{k+1}} |\varphi'(t)|dt = (t_{k+1}-t_k)|\varphi'(\tau_k)| = \int_{t_k}^{t_{k+1}} |\varphi'(z_k)|dt$$

$$= \left|\int_{t_k}^{t_{k+1}} \varphi'(\tau_k)dt\right|$$

Therefore

$$\int_a^b |\varphi'(t)|dt = \sum_{k=0}^{n-1}\left|\int_{t_k}^{t_{k+1}} \varphi'(\tau_k)dt\right|$$

By a known result we have

$$\varphi(t_{k+1}) - \varphi(t_k) = \int_{t_k}^{t_{k+1}} \varphi'(t)dt$$

So, we have

$$\left||\varphi(t_{k+1}) - \varphi(t_k)| - \left|\int_{t_k}^{t_{k+1}} \varphi'(\tau_k)dt\right|\right|$$

$$= \left|\left|\int_{t_k}^{t_{k+1}} \varphi'(t)dt\right| - \left|\int_{t_k}^{t_{k+1}} \varphi'(\tau_k)dt\right|\right|$$

$$\leq \left|\int_{t_k}^{t_{k+1}} [\varphi'(t) - \varphi'(\tau_k)]dt\right|$$

$$\leq \int_{t_k}^{t_{k+1}} |\varphi'(t) - \varphi'(\tau_k)|dt$$

$$\leq \int_{t_k}^{t_{k+1}} \varepsilon dt \text{ if } t_{k+1} - t_k < \delta$$

$$= \varepsilon(t_{k+1} - t_k) \text{ if } \|P\| < \delta.$$

Therefore

$$\int_{t_k}^{t_{k+1}} |\varphi'(t)|\,dt = \left|\int_{t_k}^{t_{k+1}} \varphi'(\tau_k)\,dt\right|$$

$$= \left|\int_{t_k}^{t_{k+1}} \varphi'(\tau_k)\,dt\right| - |\varphi(t_{k+1}) - \varphi(t_k)| + |\varphi(t_{k+1}) - \varphi(t_k)|$$

$$\le \left|\left|\int \varphi'(\tau_k)\,dt\right| - |\varphi(t_{k+1}) - \varphi(t_k)|\right| + |\varphi(t_{k+1}) - \varphi(t_k)|$$

$$\le \varepsilon(t_{k+1} - t_k) + |\varphi(t_{k+1}) - \varphi(t_k)|$$

Therefore $\int_a^b |\varphi'(t)|\,dt \le \varepsilon + V(P,\varphi)$

$$\le \varepsilon + V(\varphi).$$

This being true for all $\varepsilon > 0$ we obtain

$$\int_a^b |\varphi'(t)|\,dt \le V(\varphi)$$

Thus $V(\varphi) = \int_a^b |\varphi'(t)|\,dt$.

### 22.1.7 COROLLARY :

Suppose $\varphi:[a,b] \to \mathbb{C}$ is a piecewise smooth, and

$$P: a = t_0 < t_1 < \cdots < t_n = b$$

is such that

$$\varphi_j = \varphi/[t_j, t_{j+1}] \text{ is smooth.}$$

Then

$$V(\varphi)=\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left|\varphi_j'(t)\right|dt$$

**Proof** is left as an exercise.

**22.1.8 REMARK :**

With the notation above we know that

$\varphi_0', \varphi_1', \cdots, \varphi_{n-1}'$ are continuous functions on $[t_0, t_1], \cdots, [t_{n-1}, t_n]$

Suppose we define

$$\varphi^{(1)}(t)=\varphi'(t) \text{ if } t\in(t_j, t_{j+1}) \text{ some } j$$

$$\varphi^{(1)}(t_0)=\varphi'(a),\ \varphi^{(1)}(t_n)=\varphi'(b)$$

$$\varphi^{(1)}(t_j)=\frac{1}{2}\left(\varphi_{j-1}'(t_j)+\varphi_j'(t_j)\right) \text{ for } j\neq 0, n$$

Then we know that

$$\int_a^b\left|\varphi^{(1)}(t)\right|dt=\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\left|\varphi'(t)\right|dt$$

We may conveniently write this as

$$V(\varphi)=\int_a^b\left|\varphi'(t)\right|dt$$

**22.1.9 THEOREM :**

Suppose $f, \varphi:[a,b]\to\mathbb{C}$ are such that $f$ is continuous and $\varphi$ is of bounded variation.

Then $I(f,\varphi)$ exists.

**Proof :** Let $P', P$ be partitions of $[a,b]$ and $P'$ be refinement

$$P: a=t_0<t_1<\cdots<t_n=b;\ P': a=s_0<s_1<\cdots<s_m=b$$

$$t_j=s_{k_j}<s_{k_j+1}<\cdots<s_{k_{j+1}-1}<s_{k_{j+1}}=t_{j+1}$$

$j = 0,1,2,\cdots,n-1$

Let $\tau_0 \in [t_0,t_1],\cdots,\tau_{n-1} \in [t_{n-1},t_n],\sigma_0 \in [s_0,s_1],\sigma_1 \in [s_1,s_2],\cdots,\sigma_{m-1} \in [s_{m-1},s_m]$ be choosen arbitrarily. Then

$$f(\tau_j)[\varphi(t_{j+1})-\varphi(t_j)]-\sum_{r=k_j}^{k_{j+1}-1} f(\sigma_r)[\varphi(s_{r+1})-\varphi(s_r)]$$

$$=\sum_{r=k_j}^{k_{j+1}-1}[f(\tau_j)-f(\sigma_N)][\phi(s_{r+1})-\varphi(s_r)]$$

$= A_r$ say.

Now let $\varepsilon > 0$ be given. Since $f$ is continuous and $[a,b]$ is compact we can find a $\delta(\varepsilon)>0$ such that for all $s,t$ in $[a,b]$

$$|f(s)-f(t)|<\varepsilon \text{ if } |s-t|<\delta(\varepsilon).$$

Suppose now that $\|P\|<\delta(\varepsilon)$. Then

$$|A_j| \le \sum_{r=k_j}^{k_{j+1}-1}|f(\tau_j)-f(\sigma_r)|\,|\varphi(s_{r+1})-\varphi(s_r)|$$

$$\le \sum_{r=k_j}^{k_{j+1}-1}\varepsilon|\varphi(s_{r+1})-\varphi(s_n)|$$

Thus $|A_0+\cdots+A_{n-1}| \le |A_0|+\cdots+|A_{n-1}|$

$$\le \varepsilon\sum_{r=0}^{m-1}|\varphi(s_{r+1})-\varphi(s_r)|$$

$$\le \varepsilon V(\varphi).$$

Thus for an intermediary sum $(f,\varphi,P,\tau)$ and another $(f,\varphi,P',\sigma)$ we have

$$|(f,\varphi,P,\tau)-(f,\varphi,P',\sigma)| \le \varepsilon V(\varphi)$$

if $P'$ is a refinement of $P$ and $\|P\|<\delta(\varepsilon)$

Suppose $P_1$, $P_2$ are partitions of $[a,b]$, $P=P_1 \cup P_2$ be the common refinement consisting of points in either $P_1$ or $P_2$ arranged in increasing order. $(f,\varphi,P_1,\tau)$, $(f,\varphi,P_2,\sigma)$, $(f,\varphi,P,u)$ be arbitrary intermediary sums.

Then

$$|(f,\varphi,P_1,\tau)-(f,\varphi,P_2,\sigma)|=|(f,\varphi,P_1,\tau)-(f,\varphi,P,u)+(f,\varphi,P,u)-(f,\varphi,P_2,\sigma)|$$

$$\leq|(f,\varphi,P_1,\tau)-(f,\varphi,P,u)|+|(f,\varphi,P,u)-(f,\varphi,P_2,\sigma)|$$

$$\leq \varepsilon V(\varphi)+\varepsilon V(\varphi) \text{ if } \|P_1\|<\delta(\varepsilon), \|P_2\|<\delta(\varepsilon).$$

Let us denote the set of complex numbers corresponding to intermediary sums of $(f,\varphi)$ for $P$ with $\|P\|<\delta$ by

$\mathscr{S}(\delta)$:

$$\mathscr{S}(\delta)=\{(f,\varphi,P,\tau): P \text{ is a partition of } [a,b] \text{ with } \|P\|<\delta\}$$

What we have proved is that if $\alpha,\beta \in \mathscr{S}(\delta(\varepsilon))$ then

$$|\alpha-\beta| \leq \varepsilon V(\varphi).$$

It follows that the diameter $d\left(\overline{\mathscr{S}(\delta(\varepsilon))}\right)$ of the set $\overline{\mathscr{S}(\delta(\varepsilon))}$ is less than $\varepsilon \cdot V(\varphi)$.

For $\varepsilon$ we choose successively $1, \frac{1}{2}, \cdots, \frac{1}{m}, \cdots$

$\Delta(1), \Delta(2), \cdots, \Delta(m), \cdots$ are defined by

$$\Delta(1)=\frac{1}{2}\delta(1),$$

$$\Delta(m+1)=\frac{1}{2}\min\left\{\Delta(m), \delta\left(\frac{1}{m+1}\right)\right\}.$$

Then we have

$$\overline{\mathscr{S}(\Delta(m+1))} \subset \overline{\mathscr{S}(\Delta(m))}$$

$$d\left(\overline{\mathscr{S}(\Delta(m))}\right) \le 2 \cdot \frac{1}{m} \cdot V(\varphi)$$

The sets $\overline{\mathscr{S}(\Delta(m))}$ are subsets of $\mathbb{C}$, a complete metric space. By Cantor's Theorem, there is a unique complex number $\alpha$ such that

$$\alpha = \bigcap_{m=1}^{\infty} \overline{\mathscr{S}(\Delta(m))}$$

Since $\alpha \in \overline{\mathscr{S}(\Delta(m))}$ for any $\beta$ in $\mathscr{S}(\Delta(m)) \subset \overline{\mathscr{S}(\Delta(m))}$ we have

$$|\alpha - \beta| \le d\left(\overline{\mathscr{S}(\Delta(m))}\right) \le 2\frac{1}{m} V(\varphi).$$

Inother words if $P$ is any partition of $[a,b]$ with $\|P\| < \Delta(m)$ for any intermediary sum of $P$ say $(f, \varphi, P, \tau)$ we have

$$\left|\sum_{j=0}^{n-1} f(\tau_j)\left[\varphi(t_{j+1}) - \varphi(t_j)\right] - \alpha\right| < \frac{2}{m} V(\varphi)$$

This proves that $I(f, \varphi)$ exists and it is the number $\alpha$. (Note that $\alpha$ is unique).

**22.1.10 OBSERVATION :**

Suppose $U$ is an open set $f : U \to \mathbb{C}$ is continuous

$\varphi_1 : [a,b] \to U,\ \varphi_2 : [b,c] \to U$ are of bounded variation and $\varphi_1(b) = \varphi_2(b)$.

Then $\varphi : [a,c] \to U$ defined by

$\varphi(t) = \varphi_1(t) \quad \text{if } t \in [a,b]$

$\varphi(t) = \varphi_2(t) \quad \text{if } t \in [b,c]$ is of bounded variation, $\int f\, d\varphi$ exists and

$$\int_{\varphi} f = \int f\, d\varphi = \int_{\varphi_1} f + \int_{\varphi_2} f$$

**22.1.11 PROPOSITION :**

Suppose $f:[a,b]\to\mathbb{C}$ is continuous and $\varphi:[a,b]\to\mathbb{C}$ is of bounded variation. Then

$$|I(f,\varphi)|\le I(|f|,|\varphi|)$$

is $$\left|\int_a^b f(t)\,d\varphi(t)\right| \le \int_a^b |\varphi(t)|\,d|\varphi|(t)$$

(we denote

$$I(f,|\varphi|)=\int_a^b f\,d|\varphi|(t) = \int_{|\varphi|} f = \int f d|\varphi|$$

by $$\int_a^b f(t)|d\varphi(t)|\,)$$

**Proof :** Let $\varepsilon>0$. Then we can select $\delta_1(\varepsilon),\ \delta_2(\varepsilon)>0$ such that for any partition

$$P: a<t_0<\cdots<t_n=b$$

and each $\tau_0\in[t_0,t_1],\cdots,\tau_{n-1}\in[t_{n-1},t_n]$ we have

$$\left|\sum f(\tau_j)\left[\varphi(t_{j+1})-\varphi(t_j)\right]-I(f,\varphi)\right|<\varepsilon \text{ if } \|P\|<\delta_1(\varepsilon)$$

$$\left|\sum |f(\tau_j)|\left[|\varphi|(t_{j+1})-|\varphi|(t_j)\right]-I(|f|,|\varphi|)\right|<\varepsilon \text{ if } \|P\|<\delta_2(\varepsilon)$$

Therefore if we choose $\delta\le\min\{\delta_1(\varepsilon),\delta_2(\varepsilon)\}$ and $P$ with $\|P\|<\delta$ we have

$$|I(f,\varphi)|\le\sum|f(\tau_j)||\varphi(t_{j+1})-\varphi(t_j)|+\varepsilon$$

$$\le\sum|f(\tau_j)|\left(|\varphi|(t_{j+1})-|\varphi|(t_j)\right)+\varepsilon \qquad \because |\varphi(t_{j+1})-\varphi(t_j)|\le V(\varphi/[t_j,t_{j+1}])$$

$$\le I(|f|,|\varphi|)+\varepsilon+\varepsilon \qquad = \left(|\varphi|(t_{j+1})-|\varphi|(b_j)\right)$$

This implies the result.

**22.1.12 THEOREM :**

Suppose that $U$ is an open set

$\varphi:[a,b]\to\overline{B(\alpha,r)}\subset B(\alpha,r')\subset U$ is a rectifiable path,

$f:U\to\mathbb{C}$ is a continuous function and $\varepsilon,\delta>0$ are given.

Then there is a partition

$$P: a=t_0<t_1<\cdots<t_n=b$$

such that

1) $\|P\|<\delta$ and

2) if we denote the polygonal path $[\varphi(t_0),\varphi(t_1),\cdots,\varphi(t_n)]$ by $\Gamma$ we have

$$\left|\int_{\gamma(\varphi)} f-\int_{\Gamma} f\right|<\varepsilon.$$

**22.1.13 REMARKS :**

We have assumed that $\{\varphi\}\subset\overline{B(\alpha,r)}$. The set $\overline{B(\alpha,r)}$ is convex, therefore $\Delta$ in $[0,1]$

$$\varphi(t_j)+s(\varphi(t_{j+1})-\varphi(t_j))\in\overline{B(\alpha,r)}:$$

$$\left|\varphi(t_j)+s(\varphi(t_{j+1})-\varphi(t_j))-\alpha\right|$$

$$=\left|(1-s)[\varphi(t_j)-\alpha]+s[\varphi(t_{j+1})-\alpha]\right|$$

$$\le\left|(1-s)(\varphi(t_j)-\alpha)\right|+\left|s(\varphi(t_{j+1})-\alpha)\right|$$

$$<(1-s)r+s\ r=r.$$

The polygonal path $\Gamma$ is defined by $\Gamma_0,\Gamma_1,\cdots,\Gamma_{n-1}$ :

$$\Gamma_j(t)=\varphi(t_j)+\frac{(t-t_j)}{(t_{j+1}-t_j)}(\varphi(t_{j+1})-\varphi(t_j)):t\in[t_j,t_{j+1}]$$

Let us set $z_j = \varphi(t_j), \quad j = 0,1,2,\cdots,n-1$

Then $\Gamma_j(k) = \dfrac{(t_{j+1}-t)}{(t_{j+1}-t_j)} z_j + \dfrac{(t-t_j)}{(t_{j+1}-t_j)} z_{j+1}$

$= (1-s)z_j + s z_{j+1}$ where $s = \dfrac{(t-t_j)}{(t_{j+1}-t_j)}$.

This implies that $\{\Gamma\} \subset \overline{B(\alpha,r)}$. Thus $\int\limits_{\Gamma} f$ is defined.

**Proof of 22.1.12 :** We know that $I(f,\varphi)$ exists. Therefore there is a $\delta_1 = \delta(\varepsilon) > 0$ such that for any partition

$$P : a = t_0 < t_1 < \cdots < t_n = b$$

and all choices $\tau_0 \in [t_0, t_1], \cdots, \tau_j \in [t_j, t_{j+1}], \cdots, \tau_{n-1} \in [t_{n-1}, t_n]$ we have

$$\left|\sum f(\varphi(\tau_j))[\varphi(t_{j+1}) - \varphi(t_j)] - I(f,\varphi)\right| < \varepsilon \text{ if } \|P\| < \delta_1$$

$f$ is a continuous function on $U$ and $\overline{B(\alpha,r)} \subset U$ is a compact set. Therefore we can find $\Delta = \Delta(\varepsilon) > 0$ such that for all $z, w$ in $\overline{B(\alpha,r)}$

$$|f(z) - f(w)| < \varepsilon \text{ if } |z - w| < \Delta$$

$\varphi$ is a continuous function on $[a,b]$. Therefore we can choose $\delta_2 = \delta_2(\Delta) > 0$ such that for all s,t in $[a,b]$

$$|\varphi(s) - \varphi(t)| < \Delta \text{ if } |s - t| < \delta_2.$$

Let us set $\delta_0 = \min\{\delta, \delta_1, \delta_2\}$

Let $P$ above be such that $\|P\| < \delta_0$. Then $\|P\| < \delta$.

We are going to look at

$$A_j = f(\varphi(\tau_j))\left[\varphi(t_{j+1}) - \varphi(t_j)\right] - \int_{\Gamma_j} f$$

We note that

$$\Gamma_j(t) = \varphi(t_j) + \frac{(t - t_j)}{(t_{j+1} - t_j)}\left[\varphi(t_{j+1}) - \varphi(t_j)\right]$$ is smooth on $[t_j, t_{j+1}]$ and

$$\frac{d\Gamma_j}{dt} = \frac{\varphi(t_{j+1}) - \varphi(t_j)}{(t_{j+1} - t_j)} = \frac{z_{j+1} - z_j}{(t_{j+1} - t_j)}.$$

Therefore

$$\int_{\Gamma_j} f = \frac{z_{j+1} - z_j}{(t_{j+1} - t_j)} \int_{t_j}^{t_{j+1}} f(\Gamma_j(t))\,dt.$$

We have $f(\varphi(\tau_j))\left[\varphi(t_{j+1}) - \varphi(t_j)\right]$

$$= \frac{(z_{j+1} - z_j)}{(t_{j+1} - t_j)} \int_{t_j}^{t_{j+1}} f(\varphi(\tau_j))\,dt.$$

Therefore

$$|A_j| = \left| f(\varphi(\tau_j))(z_{j+1} - z_j) - \int_{\Gamma_j} f \right|$$

$$= \frac{|z_{j+1} - z_j|}{(t_{j+1} - t_j)} \left| \int_{t_j}^{t_{j+1}} \left[ f(\varphi(\tau_j)) - f(\Gamma_j(t)) \right] dt \right|$$

$$\leq \frac{|z_{j+1} - z_j|}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} \left| f(\varphi(\tau_j)) - f(\Gamma_j(t)) \right| dt$$

$$\left|\varphi(\tau_j)-\varphi(\Gamma_j(t))\right|$$

$$=\left|\varphi(\tau_j)-\frac{(t_{j+1}-t)\varphi(t_j)+(t-t_j)\varphi(t_{j+1})}{t_{j+1}-t_j}\right|$$

$$=\left|\frac{(t_{j+1}-t)\left[\varphi(\tau_j)-\varphi(t_j)\right]+(t-t_j)\left[\varphi(\tau_j)-\varphi(t_{j+1})\right]}{t_{j+1}-t}\right|$$

$\tau_j \in \left[t_j, t_{j+1}\right]$, $t_{j+1}-t_j \le \|P\| < \delta_0 < \delta_2$ and so

$$\left|\varphi(\tau_j)-\varphi(t_j)\right| < \Delta, \left|\varphi(\tau_j)-\varphi(t_{j+1})\right| < \Delta$$

Therefore by triangle inequality we have

$$\left|\varphi(\tau_j)-\varphi(\Gamma_j(t))\right| < \Delta,$$

and so $\left|f(\varphi(\tau_j))-f(\Gamma_j(t))\right| < \varepsilon;$

thus $\displaystyle\int_{t_j}^{t_{j+1}} \left|f(\varphi(\tau_j))-f(\Gamma_j(t))\right| dt < \varepsilon(t_{j+1}-t_j)$

It follows that $\displaystyle\left|f(\varphi(\tau_j))\left[\varphi(t_{j+1})-\varphi(t_j)\right]-\int_{\Gamma_j} f\right|$

$$< \varepsilon\left|\varphi(t_{j+1})-\varphi(t_j)\right|;$$

hence $\displaystyle\left|\sum_{j=0}^{n-1} f(\varphi(\tau_j))\left[\varphi(t_{j+1})-\varphi(t_j)\right]-\int_{\Gamma} f\right| \le \varepsilon V(\varphi)$

Since $\|P\| < \delta_0 < \delta_1$ we have

$$\left|I(f,\varphi)-\sum f(\varphi(\tau_j))\left[\varphi(t_{j+1})-\varphi(t_j)\right]\right|<\varepsilon$$

Therefore we have

$$\left|\int_{\gamma(\varphi)} f-\int_{\Gamma} f\right|<\varepsilon(1+V(\varphi)).$$

This is clearly implies the theorem. We state a theorem without proof.

**22.1.14 THEOREM :**

Suppose $U$ is an open subset of $\mathbb{C}$,

$\varphi:[a,b]\to U$ is a rectifiable path, $f:U\to\mathbb{C}$ a continuous function.

Then given any pair $\varepsilon,\delta>0$ we can find a partition

$$P: a=t_0<t_1<\cdots<t_n=b$$

such that

1) $\|P\|<\delta$

2) the polygonal path $\Gamma:[\varphi(t_0),\varphi(t_1),\cdots,\varphi(t_n)]$ is such that $\{\Gamma\}\subset U$ and

3) $\left|\int_{\gamma} f-\int_{\Gamma} f\right|<\varepsilon.$

**22.1.15 THEOREM :**

Suppose $U$ is an open set, $f:U\to\mathbb{C}$ is a continuous function with primitive $F$ on $U$ and $\varphi:[a,b]\to U$ is a rectifiable path. Then

$$\int_{\gamma(\varphi)} F=F(\varphi(b))-F(\varphi(a))$$

(This is an analogue of the fundamental theorem, of the integral calculus : If $F:[a,b]\to\mathbb{R}$ is differentiable, $F'=f$ and $f$ is Riemann Integrable then

$$\int_a^b f(t)\,dt=F(b)-F(a))$$

**Proof :** Suppose $\varphi$ is smooth. Then the function $g$ defined by

$g(t)=F(\varphi(t))$ is a smooth function with

$$g'(t)=F'(\varphi(t))\varphi'(t)$$

Therefore by Theorem 1 of Lesson 1

$$g(b)-g(a)=\int_a^b F'(\phi(t))\phi'(t)\,dt=\int_{\gamma(\varphi)} f.$$

Suppose next $\varphi$ is piecewise smooth and the partition

$$P: a=t_0<t_1<\cdots<t_n=b$$

be a partition such that $\varphi_j$ the restriction of $\varphi$ to $\left[t_j, t_{j+1}\right]$ is smooth. Then by the result we have proved

$$\int_{\gamma(\varphi_j)} f=F(\varphi(t_{j+1}))-F(\varphi(t_j))$$

By proposition we have

$$\int_{\gamma(\varphi)} f=\sum_{j=0}^{n-1}\int_{\gamma(\varphi_j)} f=\sum_{j=0}^{n-1}\left[F(\varphi(t_{j+1}))-F(\varphi(t_j))\right]$$

$$=F(\varphi(b))-F(\varphi(a)).$$

Now let $\varphi$ be any rectifiable curve. Let $\varepsilon>0$. Then by Theorem there is a polygonal path $\Gamma$ in $U$ such that

$$\Gamma(a)=\varphi(a), \Gamma(b)=\varphi(b)$$

and $$\left|\int_\varphi f-\int_\Gamma f\right|<\varepsilon.$$

A polygonal path is piecewise differentiable path. Therefore

$$\int_\Gamma f=F(\varphi(b))-F(\varphi(a))$$

Since $\varepsilon$ is arbitrary we obtain

$$\int_{\varphi} f = F(\varphi(b)) - F(\varphi(a))$$

## 22.2 MODEL EXAMINATION QUESTIONS :

1. Show that $\varphi(t) = i$ for $t \in [0,1]$ is a function of bounded variation.

2. Show that if $\varphi:[0,1] \to \mathbb{R}$ is monotonic then $\varphi$ is of bounded variation.

3. Show that if $\varphi:[a,b] \to \mathbb{R}$ is a function of bounded variation then

$$\varphi = \varphi_1 - \varphi_2$$

where $\varphi_1, \varphi_2 : [a,b] \to \mathbb{R}$ are monotonic functions.

[Hint : Take $\varphi_1(t) = V(\varphi)(t), \varphi_2(t) = V(\varphi(t)) - \varphi(t)$ ]

4. Let $\varphi:[0,1] \to \mathbb{C}$ be such that for all s,t in $[0,1]$

$$|\varphi(s) - \varphi(t)| \le M|s - t|$$

for some $M > 0$. Then show that $\varphi$ is of bounded variation.

5. Let $\alpha = a + ib$ and $\beta = c + id$. Then evaluate

$$\int_{[\alpha,\beta]} |z|^2 \, dz$$

6. If $\varphi:[a,b] \to \mathbb{C}$ is smooth show that $\varphi$ is of bounded variation and

$$V(\varphi) = \int_a^b |\varphi'(t)| dt$$

7. If $\varphi:[a,b] \to U$ is piecewise smooth and $F$ is an analytic function in $U$ such that

$$F'(z) = f(z)$$

then $$\int_{\varphi} f = F(\varphi(b)) - F(\varphi(a))$$

8. Suppose $U$ is an open set, $\gamma, \sigma$ are paths in $U$ :

$$\gamma : [a, b] \to U, \ \sigma : [c, d] \to U \text{ and such that}$$

$$\sigma(t) = \gamma\left(a + \left(\frac{b-a}{d-c}\right)(t-c)\right).$$

If $\gamma$ is rectifiable and $f$ is continuous on $U$, then show

$$\int_{\sigma} f = \int_{\gamma} f$$

**REFERENCE BOOKS :**

1. J. B. Connay : Functions of One Complex Variable - Second Edition - Springer International Student Edition.

2. Ruel V. Churchil; Jamesward Brown : Complex Variables and Applications - Mc-Graw Hill International Editions - Fifth Edition.

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Lesson - 23

# CAUCHYS INTEGRAL FORMULA

## 23.0 INTRODUCTION

In lesson 20 we have proved two important results.

1) Cauchy's Integral Formula
2) Cauchy's Theorem

In the first the path is a circle. In the second the open set is a disc.

In this lesson we extend the results to more general paths and more general open sets. The key concept is the index of a closed path $\gamma$ with respect to a point $z$ or the winding number $\gamma$ around $z$. In these considerations $\gamma$ is assumed to be rectifiable.

The basic result is that when $\{\gamma\} \subset U$ and

$$\eta(\gamma, z) = 0 \qquad \forall\, z \notin U$$

we have Cauchy Integral formula for $\gamma$. For such a $\gamma$ we write $\gamma \approx 0(U)$.

We extend the results of Lesson 20. We prove that if $U$ is such that $\gamma \approx 0(U)$ for each closed rectifiable path $\gamma$ in $U$, then every analytic function $f$ on $U$ has a primitive.

Then for such an open set and every analytic function $f$ on $U$ that never assumes the value 0, there is an analytic function $\operatorname{Log} f$ on $U$ s.t.

$$e^{\operatorname{Log} f} = f.$$

## 23.1 SOME RESULTS :

### 23.1.1 THEOREM :

Suppose $\gamma : [a, b] \to \mathbb{C}$

is a closed rectifiable path and $z \in \mathbb{C}$ does not belong to the trace $\{\gamma\}$ of $\gamma$. Then

$$\int \frac{d\gamma}{\gamma - z} = 2\pi i\, k \text{ for some } k \in \mathbb{Z}.$$

**Proof :** We have assumed that $z \notin \{\gamma\}$. This means $\gamma(t) - z \neq 0$ for $t \in [a,b]$. Therefore $\frac{1}{\gamma(t)-z}$ is a continuous function on $[a,b]$. We have assumed that $\gamma$ is a rectifiable path. Therefore

$$\int \frac{d\gamma}{\gamma - z}$$

is defined. Let us denote $\int \frac{d\gamma}{\gamma - z}$ by $2\pi i\, \eta(\gamma, z)$

$$\eta(\gamma, z) = \frac{1}{2\pi i} \int \frac{d\gamma}{\gamma - z}.$$

We divide the proof into three steps.

1) $\gamma$ smooth, 2) $\gamma$ piecewise smooth 3) $\gamma$ rectifiable

We deal here with smooth case. The other two are beyond the scope of the study.

**1) $\gamma$ smooth :** In this case we have proved that

$$\int \frac{d\gamma}{\gamma - z} = \int_a^b \frac{\gamma'(t)}{\gamma(t) - z} dt$$

By our assumption $\gamma'(t)$ is continuous and so $\frac{\gamma'(t)}{\gamma(t)-z}$ is a continuous function on $[a,b]$.

Let us set

$$g(s) = \int_a^s \frac{\gamma'(t)}{\gamma(t) - z}\, dt$$

Then we know from Riemann Integration that $g(s)$ is differentiable and

$$g'(s) = \frac{\gamma'(s)}{\gamma(s) - z}$$

Now consider the function

$$h(s) = e^{-g(s)}(\gamma(s) - a)$$

We have

$$h'(s) = -e^{-g(s)}\, g'(s)(\gamma(s)-a)+e^{-g(s)}\,\gamma'(s)$$

$$= e^{-g(s)}(-\gamma'(s)+\gamma'(s)) = 0$$

It follows that $h$ is constant. Thus

$$h(a) = e^{-g(a)}(\gamma(a)-z)$$

$$= \gamma(a)-z \text{ since } g(a)=0$$

$$h(b) = e^{-g(b)}(\gamma(b)-z)$$

$$= e^{-g(b)}(\gamma(a)-z) \text{ since } \gamma \text{ is closed.}$$

Since $\gamma(a)-z \neq 0$ it follows that

$$e^{-g(b)} = 1$$

That is $g(b) \in 2\pi i\mathbb{N}$.

By definition $g(b) = \int_a^b \frac{\gamma'(t)}{\gamma(t)-z} = \int \frac{d\gamma}{\gamma - z}$

Hence $\int \frac{d\gamma}{\gamma - z} = 2\pi i k$ for some $k \in \mathbb{Z}$.

**23.1.2 PROPOSITION :**

Suppose $\gamma:[a,b] \to \mathbb{C}$ is a rectifiable path and

$$U = \mathbb{C} \setminus \{\gamma\}.$$

Then $U$ is an open set and $f : U \to \mathbb{C}$ defined by $f(z) = \int_a^b \frac{d\gamma(t)}{\gamma(t)-z}$ is analytic function on $U$. Its derivative is

$$f'(z) = \int_a^b \frac{d\gamma(t)}{(\gamma(t)-z)^2}$$

**Proof :** Let us choose some point $z_0$ in $U$. Then the function of $t$

$\gamma(t)-z_0$ is a continuous function on $[a,b]$. Since $z_0 \notin \{\gamma\}$

$\gamma(t)-z_0 \neq 0$ for any $t$ in $[a,b]$.

Therefore $\dfrac{1}{\gamma(t)-z_0}$ is a continuous function on $[a,b]$ and so

$I\left(\dfrac{1}{\gamma-z_0}, \gamma\right) = \displaystyle\int_a^b \dfrac{d\gamma(t)}{\gamma(t)-z_0}$ exists. Let us set for $z \in U$,

$$g(z) = \int_a^b \frac{d\gamma(t)}{\gamma(t)-z}.$$

We now define $\Delta : U \to \mathbb{R}$ by

$$2\Delta(z) = d(z,\{\gamma\}) = g.\ell.b\{|z-\gamma(t)| : t \in [a,b]\}$$

The function $|z-\gamma(t)|$ is a continuous function on $[a,b]$; it attains its minimum; so there is a $t_0 \in [a,b]$ such that

$$2\Delta(z) = |z-\gamma(t_0)|.$$

Since $z-\gamma(t_0) \neq 0$ it follows that

$$2\Delta(z) > 0.$$

We note that

$$2\Delta(z) \leq |z-\gamma(t)| \text{ and } \frac{1}{|\gamma(t)-z|} \leq \frac{1}{2\Delta(z)}.$$

We claim that

$$B(z, 2\Delta(z)) \subset U$$

That is if $|w-z| < 2\Delta(z)$, then $|w-\gamma(t)| \neq 0$ for $t$ in $[a,b]$:

We have

$$2\Delta(z) \le |\gamma(t) - z| \le |\gamma(t) - w| + |w - z|$$

$$\le |\gamma(t) - w| + \Delta(z);$$

this implies $|\gamma(t) - w| \ge \Delta(z)$, $2\Delta(w) \ge \Delta(z)$ and

$$\frac{1}{|\gamma(t) - w|} \le \frac{1}{\Delta(z)}.$$

Let $h$ be a complex number satisfying

$$0 < |h| < \Delta(z).$$

Then $\dfrac{f(z+h) - f(z)}{h} = \displaystyle\int_a^b \frac{d\gamma(t)}{h} \left\{ \frac{1}{(\gamma(t) - (z+h))} - \frac{1}{\gamma(t) - z} \right\}$

$$= \int_a^b d\gamma(t) \frac{1}{(\gamma(t) - (z+h))(\gamma(t) - z)}$$

and $\dfrac{f(z+h) - f(z)}{h} - \displaystyle\int_a^b d\gamma(t) \frac{1}{(\gamma(t) - z)^2}$

$$= \int_a^b d\gamma(t) \left\{ \frac{h}{(\gamma(t) - (z+h))(\gamma(t) - z)^2} \right\}$$

Therefore $\left| \dfrac{f(z+h) - f(z)}{h} - \displaystyle\int_a^b \frac{d\gamma(t)}{(\gamma(t) - z)^2} \right| \le \displaystyle\int_a^b \left| \frac{h}{(\gamma(t) - (z+h))(\gamma(t) - z)^2} \right| |d\gamma(t)|$

$$\le |h| \frac{1}{\Delta(z)(2\Delta(z))^2} V(\gamma).$$

$$= |h| \frac{V(\gamma)}{4\Delta^3(z)}$$

This proves our proposition.

**23.1.3 COROLLARY :**

Suppose $\gamma$ is a closed rectifiable path then $\eta(r, z)$ is constant on each connected component of $U = \mathbb{C} \setminus \{\gamma\}$.

**Proof :** $\eta(r, z)$ is a continuous function on $U$. Let V be a connected subset of $U$. Then the image of V is a connected subset of $\mathbb{C}$. Since $\eta(r, z)$ takes only integer values and the only subsets of $\mathbb{Z}$ which are connected subbsets of $\mathbb{C}$ are subsets having only one point the corollary follows.

**23.1.4 COROLLARY :**

Suppose $|\gamma| \leq M$ and $|z| > M$ then

$$\eta(\gamma, z) = 0$$

**Proof :** The set $U = \{z \in \mathbb{C} : |z| > M\}$ is in the same connected component of $\mathbb{C} \setminus \{\gamma\}$.

We have

$$|\eta(\gamma, z)| \leq \frac{\text{Var}\ (\gamma)}{|z| - M}$$

The right hand side tends to zero as $z \to \infty$, while the left hand side depends only on $U$.

**23.1.5 DEFINITION :**

Suppose $\gamma$ is a closed rectifiable path in $\mathbb{C}$ and $z \notin \{\gamma\}$. The integer

$\eta(\gamma, z)$ is called The index of $\gamma$ with respect to $z$ or The winding number of $\gamma$ around $z$.

Now we state a Theorem.

**23.1.6 PROPOSITION :**

Suppose $\gamma$ and $\sigma$ are closed rectifiable curves having the same initial points. Then

(a) $\eta(\gamma, a) = -\eta(-\gamma, a)$ for $a \notin \{\gamma\}$

(b) $\eta(\gamma + \sigma, a) = \eta(\gamma, a) + \eta(\sigma, a)$ for $a \notin \{\gamma\} \cup \{\sigma\}$

**23.1.7 PROPOSITION :**

Suppose $\gamma$ is a closed rectifiable curve in $\mathbb{C}$. Then for $z$ in the unbounded component of

$\mathbb{C} \setminus \{\gamma\}$

we have

$$\eta(\gamma, z) = 0$$

**23.1.8 DEFINITION :**

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a map. Then $-\gamma$ is a map

$[-b,-a] \to \mathbb{C}$ defined by

$$-\gamma(t) = \gamma(-t).$$

**23.1.9 DEFINITION :**

Suppose $\gamma$, $\sigma$ are maps from

$$\gamma:[a,b] \to \mathbb{C},\ \sigma:[b,c] \to \mathbb{C}.$$

Then $\gamma + \sigma$ is defined to be a map from $[a,e] \to \mathbb{C}$

given by

$$(\gamma+\sigma)(t) = \gamma(t) \text{ if } a \le t \le b$$

$$(\gamma+\sigma)(t) = \sigma(t) \text{ if } b \le t \le c$$

**23.1.10 THEOREM :**

Suppose $f$ is an analytic function in an open set $U \subseteq \mathbb{C}$ and we define

$F : U \times U \to \mathbb{C}$ by

$$F(z,w) = \frac{f(z) - f(w)}{z - w} \text{ if } z \ne w$$

$$F(z,z) = f'(z).$$

Then (1) $F$ is a continuous function on $U \times U$

(2) For each $w_0 \in U$, the function

$F_{w_0} : U \to \mathbb{C}$ defined by

$F_{w_0}(z) = F(z, w_0)$ is an analytic function in $U$.

**23.2.1 THEOREM (CAUCHY INTEGRAL FORMULA) :**

Suppose $\gamma$ is a rectifiable path in an open set $U \subseteq \mathbb{C}$ such that

$$\eta(\gamma, z) = 0 \text{ for } z \notin U,$$

and $f : U \to \mathbb{C}$ is an analytic function. Then for $z \in U \setminus \{\gamma\}$

$$2\pi i\ \eta(\gamma, z)\ f(z) = \int_\gamma \frac{f(w)}{w-z}\, dw$$

**Proof :** We have proved previously that for each $w \in U$ the function

$$F_w : U \to \mathbb{C}$$

defined by $F_w(z) = F(z, w)$ is analytic in $U$.

Recall that $F(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{if } z \neq w,\ z,\ w \in U \\ f'(z) & \text{if } z \in U \end{cases}$

We define

$$g_1 : U \to \mathbb{C}$$

by $$g_1(z) = \int_\gamma F(z, w)\, dw$$

Bya known Theorem $g_1$ is an analytic function on $U$.

We consider the set

$$W = \{z \in \mathbb{C} / \eta(\gamma, z) = 0\}$$

W is an open set. We define

$$g_2 : w \to \mathbb{C}$$

by $$g_2(z) = \int_\gamma \frac{f(w)}{w-z}\, dw$$

Since $\eta(\gamma,z)$ is defined only for $z\notin\{\gamma\}, w-z\neq 0,\ w\in\{\gamma\}$, we know that $g_2$ is an analytic function on w.

We claim that

$$g_1(z)=g_2(z) \text{ for } z\in U\cap w$$

We have

$$g_1(z)=\int_\gamma F(z,w)\,dw \quad \text{since } z\in U$$

$$=\int_\gamma \frac{f(z)-f(w)}{w-z}\,dw \quad \text{since } z\in w,\ z\notin\{\gamma\} \text{ and so } w\neq z$$

$$=\int_\gamma \frac{f(w)}{w-z}\,dw - \int \frac{f(z)}{w-z}\,dz$$

$$=\int_\gamma \frac{f(w)}{w-z}\,uw - 2\pi i\, f(z)\eta(r,z)$$

$$=\int_\gamma \frac{f(w)}{w-z}\,dw$$

Thus the function

$$g(z)=\begin{cases} g_1(z) & \text{for } z\in U \\ g_2(z) & \text{for } z\in w\end{cases}$$

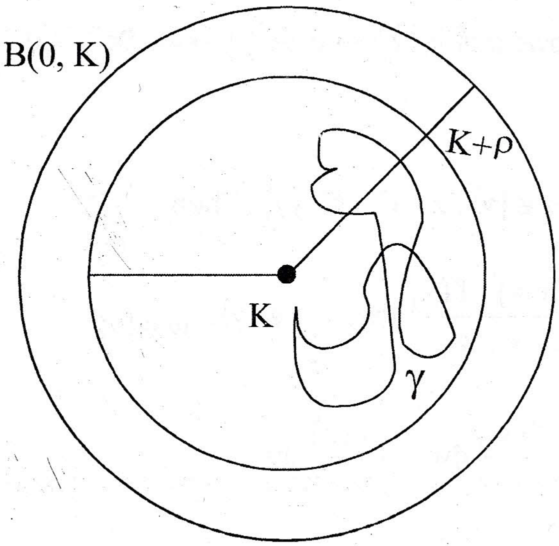
is an analytic function on $U\cup W$.

For $z\notin U$, we are given that $\eta(\gamma,z)=0$. Thus $\mathbb{C}\setminus U\subseteq w$

Therefore $C=U\cup W$ and we have an entire function.

We claim that $g$ is bounded. To see this we note that $\{\gamma\}$, the trace of $\gamma$ is a compact set. It is contained in

Then for $z\in\mathbb{C}\setminus B(0,K+\rho)$ we have




$d(z,\{\gamma\}) \geq \rho$ and therefore

$$|w - z| \geq ||w| - |z|| \geq |K - |z|| \geq \rho$$

and so

$$|g(z)| \leq \left| \int_{\gamma} \frac{f(w)}{w - z} \, dw \right|$$

$$\leq \int_{\gamma} \left| \frac{f(w)}{w - z} \right| d|\gamma|$$

$$\leq \frac{M_{\gamma}(f)}{\rho} V(\gamma)$$

where $M_{\gamma}(f) = \sup\limits_{w \in \{\gamma\}} |f(w)|$

Thus $|g|$ is bounded on $\mathbb{C} \setminus B(0,2K)$. On the compact set $\overline{B(0,2K)}$ it is clearly bounded. Therefore $g$ is bounded on $\mathbb{C}$. By Liuville's Theorem $g$ is constant. Since

$$|g(z)| \leq M_{\gamma}(f) V(\gamma) \frac{1}{\rho}$$

if $|z| \geq K+\rho$ it follows that $|g(z)| \to 0$ as $|z| \to \infty$. So $g=0$.

Thus $g_1 = 0$ in $U$.

Suppose we take $z \notin \{\gamma\}$, $z \in U = \mathbb{C} \setminus \{\gamma\}$. Then

$$g_1(z) = \int_\gamma \frac{f(w)-f(z)}{w-z}\, dw, \quad \because z \notin \{\gamma\},\ w \in \{\gamma\}$$

$$= \int_\gamma \frac{f(w)}{w-z} dw - \int_\gamma \frac{f(z)}{w-z} dw$$

$$= \int_\gamma \frac{f(w)}{w-z} dw - 2\pi i\, \eta(\gamma, z) f(z)$$

The theorem is proved.

**23.2.2 DEFINITION :**

Suppose $U$ is an open set in $\mathbb{C}$ and $\gamma$ is a closed rectifiable path in $U$. We say that

$\gamma$ is homologous to zero in $U$ and write

$\gamma \approx 0$ in $U$ or $\gamma \approx 0(U)$

if $\quad \eta(\gamma, z) = 0 \qquad \forall\, z \notin U$

With this definition we may state the Theorem we have proved as.

**23.2.3 THEOREM :**

If $U$ is an open subset of $\mathbb{C}$, $f$ is an analytic function in $U$, $\gamma$ is a closed rectifiable curve in $U$ such that $\gamma \approx 0(U)$, then

Cauchy Integral Formula is valied for $\gamma$ :

$$2\pi i\, \eta(\gamma, z) f(z) = \int_\gamma \frac{f(w)}{w-z} dw$$

**23.2.4 THEOREM (CAUCHY'S THEOREM SECOND VERSION):**

Suppose $U$ is an open set in $\mathbb{C}$, $f$ is an analytic function in $U$ and $\gamma$ is a closed rectifiable path in $U$ such that

$$\gamma \approx 0\,(U).$$

Then $$\int_\gamma f = 0.$$

**Proof :** Suppose we choose an $a$ in $U$ such that

$a \notin \{\gamma\}$, and set

$$g(z) = (z-a)f(z).$$

By Cauchy integral formula we have

$$0 = g(a) = 2\pi i\ \eta(\gamma, a)\ g(a)$$

$$= \int_\gamma \frac{g(w)}{w-a}\,dw$$

$$= \int_\gamma f(w)\,dw$$

$$= \int_\gamma f$$

## 23.3 COUNTING ZEROS

### 23.3.1 THEOREM :

Suppose $U$ is an open set in $\mathbb{C}$, $f : U \to \mathbb{C}$ is an analytic function. Suppose that

$$\{a_1, a_2, \cdots, a_k\}$$

is the set of all zeros where each $a_j$ is repeated as many times as its multiplicity. If $\gamma : [a,b] \to U$ is a closed rectifiable path which does not pass through any $a_j$: i.e. $a_j \notin \{\gamma\}$ and $\gamma$ is homologous zero in $U$. $\gamma \approx 0$ then

$$\int_\gamma \frac{f'(z)}{f(z)} = 2\pi i \sum_{j=1}^{k} \eta(\gamma, a_j)$$

**Proof :** Suppose that the multiplicity of $a_1$ is $m_1$. Then we know that

$$f(z)=(z-a_1)^{m_1}\cdot g_1(z)$$

where $g_1$ is an analytic function on $U$ and there is an $r_1>0$ such that

$$g_1(z)\neq 0 \text{ in } B(a_1, r_1)$$

We have

$$\frac{f'(z)}{f(z)}=\frac{m_1}{(z-a_1)}+\frac{g_1'(z)}{g(z)} \text{ in } B(a_1, r_1).$$

$$=\frac{1}{(z-a_1)}+\cdots+\frac{1}{(z-a_1)}+\frac{g_1'(z)}{g_1(z)}, \frac{1}{z-a_1} \text{ occurs } m_1 \text{ times.}$$

The only possible zeros of $g_1(z)$ are the $a_j$, $a_j\neq a_1$. Therefore by induction on the number of zeros we obtain

$$\frac{f'(z)}{f(z)}=\frac{1}{(z-a_1)}+\cdots+\frac{1}{(z-a_k)}+\frac{g_1'(z)}{g(z)}$$

Where $g$ is analytic in $U$ and $g(z)\neq 0$ for $z$ in $U$. Thus

$$\frac{g'(z)}{g(z)} \text{ is analytic in } U.$$

We have assumed $\gamma\approx 0$ in $U$. Therefore by Cauchy's Theorem $\int_\gamma \frac{g'(z)}{g(z)}dz=0$

Thus we have $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\sum_{j=1}^{k}\eta(\gamma, a_j)$

The theorem is proved.

**23.3.2 REMARKS :**

In the above theorem we have assumed that the number of zeros of $f$ is finite.

It is not necesary to assume this. In the next proposition we shall prove that if $\gamma \approx 0$ in $U$ the set of $a$ in $U$ such that

$f(a)=0$ and $\eta(\gamma, a) \neq 0$ is finite. Let us write

$z(f)=\{a \in U / f(a)=0\}$.

Then

$\sum_{a \in z(f)} \eta(\gamma, a)$ is a finite sum. The theorem may be stated as

$$\int_{\gamma} \frac{f'(z)}{f(z)}\, dz = 2\pi i \left\{ \sum_{a \in z(f)} \eta(\gamma, a) \right\}$$

**23.3.3 PROPOSITION :**

Suppose $U$ is an open connected set in $\mathbb{C}$, $f: U \to \mathbb{C}$ is analytic and $\gamma$ is a closed rectifiable path in $U$ which is homologous to zero in $U$ and does not have any point $w$ s.t. $f(w)=0,\ w \in \{\gamma\}$

$z(f)=\{a \in U / f(a)=0\}$ and let

$E=\{a \in Z(f) / \eta(\gamma, a) \neq 0\}$.

Then $E$ is a finite set

**Proof :** Suppose $r>0$ is such that

$\{\gamma\} \subset \overline{B(a,r)}$.

Then for $z \notin \overline{B(a,r)}$, $\eta(\gamma, z)=0$ since $z$ belongs to the unbounded component of $\mathbb{C} \setminus \{\gamma\}$. This implies that

$E$ is a bounded set.

Suppose $E$ is infinite. Then by Bolzano-Weierstrass theorem $E$ has a limit point say $z_0$. By Theorem of Lesson 2, $z_0$ cannot be in $U$. So $z_0$ belongs to the complement of $U$. This implies

$\eta(\gamma, z_0)=0$

since we have assumed that $\gamma \approx 0$ in $U$.

We have a sequence $z_n$ in $E$ such that

$$\lim_{n\to\infty} z_n = z_0$$

since $z_0$ is a limit point of $E$. We know that none of the $z_n$ are in $\{\gamma\}$. Therefore $\eta(\gamma, z_n)$ is defined. Since $\eta(\gamma, z)$ is continuous on $\mathbb{C}\setminus\{\gamma\}$

$$\lim_{n\to\infty} \eta(\gamma, z_n) = \eta(\gamma, z_1) = 0$$

Since $z_n \in E$, we have

$$\eta(\gamma, z_n) \neq 0$$

Thus we obtain a sequence of non-zero integers that converges to zero. This is impossible. So, $E$ must be a finite set.

Now suppose $f : U \to \mathbb{C}$ is a function and we define

$$z(f,\alpha) = \{z \in U / f(z) = \alpha\}$$

As an application of the Theorems we have proved above we prove the following result.

**23.3.4 THEOREM :**

Suppose $U$ is a connected open set in $\mathbb{C}$, $f : U \to \mathbb{C}$ is analytic function which is not a constant function. Suppose $\gamma$ is a rectifiable closed path in $U$, such that $\gamma \approx 0$ in $U$, and $\overline{\gamma}$ is defined by $\overline{\gamma} = f \circ \gamma$. Then for any $\alpha$ in $\mathbb{C}$

$$\eta(\overline{\gamma}, \alpha) = \sum_{a \in z(f,\alpha)} \eta(\gamma, a)$$

**Proof :** Let us set $w = \mathbb{C}\setminus\{\alpha\}$ and define $g : w \to \mathbb{C}$ by

$$g(w) = \frac{1}{w-\alpha}.$$

We have

$$\eta(\overline{\gamma}, \alpha) = \int_{\overline{\gamma}} \frac{dw}{w-\alpha}$$

$$= \int_{\bar{\gamma}} (g \circ f) f' \text{ by Theorem.}$$

$$= \sum_{a \in z(t,\alpha)} \eta(\gamma, a) \text{ by Theorem.}$$

**23.3.5 DEFINITION :**

Suppose $U$ is an open subset of $\mathbb{C}$,

$$\gamma_1, \gamma_2, \cdots, \gamma_k$$

are closed rectifiable paths in $U$. Then we say that the formal sum

$\gamma_1 + \gamma_2 + \cdots + \gamma_k$ is homologous to zero in $U$ and write

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k \approx 0 \text{ in } U$$

or $\gamma_1 + \gamma_2 + \cdots + \gamma_k \approx 0(U)$ if for all $z \notin U$

$$\eta(\gamma_1, z) + \cdots + \eta(\gamma_k, z) = 0$$

We define

$$\eta(\gamma_1 + \gamma_2 + \cdots \gamma_k, z) = \eta(\gamma_1, z) + \cdots + \eta(\gamma_k, z).$$

**23.3.6 DEFINITION :**

Suppose $f$ is a continuous function in $U$ and $\gamma_1, \cdots, \gamma_k$ in $U$ are closed rectifiable paths We define

$$I(f, \gamma_1 + \cdots + \gamma_k) = I(f, \gamma_1) + \cdots + I(f, \gamma_k)$$

We write

$$\int_{\gamma_1 + \cdots + \gamma_k} f \quad \text{for} \int_{\gamma_1} f + \cdots + \int_{\gamma k} f.$$

There is an important extension of Cauchy Integral Formula.

**23.3.7 THEOREM (CAUCHY INTEGRAL FORMULA : SECOND FORM) :**

Suppose $U$ is an open set in $\mathbb{C}$,

$f : U \to \mathbb{C}$ is an analytic function and

$\gamma_1, \gamma_2, \cdots, \gamma_m$ are paths in $U$ such that

$\eta(\gamma_1, w)+\cdots+\eta(\gamma_m, w)=0$ for $w \notin U$.

Then for each $a$ in $U$, $a \notin \{\gamma_1\} \cup \cdots \cup \{\gamma_m\}$ we have

$$\left(2\pi i \sum_{k=1}^{m} \eta(\gamma_k, w)\right) f(a) = \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(z)}{z-a} dz$$

**Proof :** The proof is similar to the proof of Cauchy Integral formula. The only thing that is needed is that the set

$$V = \left\{ z \in \mathbb{C} / \sum_{k=1} \eta(\gamma_k, z) = 0 \right\}$$ is an open set. The proof is as follows.

The complement of

$\{\gamma_1\} \cup \cdots \cup \{\gamma_m\}$ is an open set. It is the union of its connected components $\{V_p\}$.

Each $V_p$ is a conected set and so is contained in some connected component of

$$\mathbb{C} \setminus \{\gamma_k\}.$$

It follows that

$$\eta(\gamma_k, z) = \eta(\gamma_k, w) \text{ for } z, w \subseteq V_p,\ k = 1, \cdots, m.$$

Thus we have

$\sum_{k=1}^{m} \eta(\gamma_k, z)$ as a function of $z$ is a constant function on $V_p$. Thus the set

$$\left\{ z : \sum_{k=0}^{m} \eta(\gamma_k, z) = 0 \right\}$$

is a union of some subset of $V_p$'s therefore it is an open set.

The rest of the proof is left to the reader.

**23.3.8 THEROEM :**

The hypotheses kept as in the above theorem, we have

$$f^{(n)}(a)\left(\sum_{k=1}^{m}\eta(\gamma_k,a)\right)=\frac{n!}{2\pi i}\sum_{k=1}^{m}\int_{\gamma_k}\frac{f(z)}{(z-a)^{n+1}}\,dz$$

**Proof :** It is similar to the proof of Theorem.

Next we prove Cauchy's Theorem. This Theorem is, together with Cauchy Integral Formula is the foundation for the subject.

**23.3.9 THEOREM (CAUCHY'S THEOREM. FIRST FORM):**

Suppose $U$ is an open set, $f$ is analytic in $U$, $\gamma_1,\cdots,\gamma_m$ are closed paths in $U$ such that

$$\eta(\gamma_1,z)+\cdots\cdots+\eta(\gamma_m,z)=0 \text{ for } z\notin U$$

Then

$$\sum_{k=1}^{m}\int_{\gamma_k} f(z)\,dz=0$$

**Proof :** Suppose we take an $a$ in $U$, $a\notin\{\gamma_1\}\cup\cdots\cup\{\gamma_m\}$ and set

$$g(z)=(z-a)f(z).$$

By Cauchy Integral Formula we have

$$0=g(a)=g(a)\times 2\pi i\sum_{k=1}^{m}\eta(\gamma_k,a)$$

$$=\sum_{k=1}^{m}\int_{\gamma_k}\frac{g(z)}{(z-a)}dz$$

$$=\sum_{k=1}^{m}\int_{\gamma_k} f(z)\,dz.$$

We have proved in lesson 21 that if $U$ is a connected open set and $f$ is a non-constant analytic function on $U$, then $f$ is an open map. That is if $z_0$ is in $U$, then there is an $r_0>0$ such that every value $\alpha$, $|\alpha-f(z_0)|<r_0$ is attained by $f$. With the generalisation of Cauchy Integral Formula we can say how many times $f$ takes the value near $z_0$.

**23.3.10 THEOREM :**

Suppose $f$ is an analytic function on $B(a,K_1)$ and it is not a constant. Suppose $f$ takes the value $f(a)$ with multiplicity $m$. Then there is a pair of positive numbers $r,K$ such that

1) Every value of $f$ on $B(a,K)$ is a simple value of $f$, with the possible exception of $f(a)$.

2) Every $\alpha$ in $B(f(a),r)$ is attained by $f$ on $B(a,K)$

3) For $\alpha \neq f(a)$ and in $B(f(a),r)$ the set

$\{z \in B(a,K)/f(z)=\alpha\}$ has $m$ elements.

**Proof :** By a known result there is a $K_2 < K_1$ such that every value of $f$, assumed on $B(a,K_2)$, except possibly $f(a)$, is a simple value of $f$.

Since $a$ is not a limit point of the zeros of $f(z)-f(a)$, there is a $K_3 < K_1$ such that

$$f(z) \neq f(a) \quad \text{if } 0<|z-a| \leq K_3$$

Recall that $S(a,K)=\{z \in \mathbb{C}/|z-a|=K\}$. By our choice of $K, f(z)-f(a) \neq 0$ for $z$ in $S(a,K)$. We set

$$r = d(f(a),\, f(S(a,K)))$$

Since $S(a,K)$ is compact the distance r is attained at some poin on $S(a,K)$ and therefore >0.

We are going to prove the r above satisfies the assertion (3) of the Theorem.

We consider the path

$\gamma:[0,\, 2\pi] \to U$ defined by

$\gamma(t) = a + K e^{it}$ and the path $\sigma$ defined by

$\sigma(t) = f(\gamma(t))$

By Theroem we have for $\alpha \notin \{\sigma\}$

$$\int_{\sigma} \frac{dw}{w-\alpha} = \int_{\gamma} \frac{f'(z)}{f(z)-\alpha} dz$$

By our choice $B(f(a),r) \cap \{\sigma\} = \phi$. Therefore $B(f(a),r)$ is contained in some connected component of $\mathbb{C} \setminus \{\sigma\}$. Therefore

$$\eta(\sigma,\alpha) = \eta \mid \sigma, f(a) \quad \forall \alpha \text{ in } B(f(a),r)$$

We have

$$\int_{\gamma} \frac{f'(z)}{f(z)-f(a)} dz = m$$

by a known result. Therefore we have proved the Theorem.

## 23.4 SOLUTIONS TO SAQ :

**SAQ 1 :** Suppose $0 < r < R$, $U = \{z \in \mathbb{C} : r < |z-a| < R\}$.

$\bar{U} = \{z \in \mathbb{C} : r \leq |z-a| \leq R\}$, $f : \bar{U} \to \mathbb{C}$ is continuous and $f$ is analytic on $U$. Then for $z \in U$

$$2\pi i\, f(z) = \int_{|w-a|=R} \frac{f(w)}{w-z} dw - \int_{|w-a|=r} \frac{f(w)}{w-z} dw$$

**Proof :** Let $\delta > 0$ be such that

$$r < r+\delta < R-\delta < R$$

Define $\gamma_2 = \gamma(R,\delta)(t) = (R-\delta)2^{2\pi i t}$

$\gamma_1(t) = \gamma(r,\delta)(t) = (r+\delta)e^{-2\pi i t}$ and choose $\delta > 0$ s.t.

$r+\delta < |z| < R-\delta$,

We have for $|\alpha - a| \leq r+\delta$

$$\eta(\gamma_2,\alpha)=\eta(\gamma_2,0)=1$$

$$\eta(\gamma_1,\alpha)=\eta(\gamma_1,0)=-1;\text{ for } |\alpha-a|>R-\delta$$

$$\eta(\gamma_2,\alpha)=\eta(\gamma_1,\alpha)=0;$$

since $\alpha$ is in the unbounded component of $\mathbb{C}\setminus\gamma_1$ and also of $\mathbb{C}\setminus\gamma_2$.

Therefore Cauchy Integral formula form 2 gives

$$2\pi i\, f(z)=\int_{|z-a|}$$

We note that

$$\int_{\gamma_1}\frac{f(w)}{w-z}\,dw=-\int_{|z-a|=r+\delta}\frac{f(w)}{w-z}\,dz$$

Therefore for each $\delta>0$ s.t. $r+\delta<R-\delta$ we have

$$2\pi i f(z)=\int_{|z-a|=R-\delta}\frac{f(w)}{w-z}\,dw-\int_{|z-a|=r+\delta}\frac{f(w)}{w-z}\,dw$$

By SAQ : Lesson 2 taking limit as $\delta\to 0+$ we obtain

$$2\pi i\, f(z)=\int_{|z-a|=k}\frac{f(w)}{w-z}dw-\int_{(z-a)=r}\frac{f(w)}{w-z}dw$$

## 23.5 MODEL EXAMINATION QUESTIONS :

1) Suppose $U=\{z:\mathbb{C} : z \text{ is not a non-negative real number}\}$

Then, show that if $\gamma$ is a closed rectifiable path in $U$

$$\gamma\approx 0(U)$$

2) Suppose $U=\{z\in\mathbb{C}:z\neq 0\}$.

Then show that $\gamma(t)=e^{2\pi it}$ $\quad t\in[0,1]$

is not homologous to zero in $U$.

3) Suppose $U=\{z: \quad 1<\operatorname{Im} z<2\}$ and $\gamma$ is a closed rectifiable path in $U$. Then show that $\gamma\approx 0(U)$

4) If $f:\mathbb{C}\to\mathbb{C}$ is a continuous function such that $f$ is analytic in the complement of $[-1, +1]$ show that $f$ is an entire function.

5) Let V be an openset in $\mathbb{C}$, $f_n:V\to\mathbb{C}$ an analytic function for each $n\in\mathbb{N}$. If $f_n$ converge uniformly to $f$ in V, show that $f$ is analytic in V.

## 23.6 MODEL EXAMINATION QUESTIONS :

1) State and Prove Cauchy's Integral Formula Second Form

2) Prove that $e^z$ is not one-one.

3) Suppose $U=\{z\in\mathbb{C}:z\neq 0\}$ and

$$\gamma_1(t)=e^{-2\pi it} \qquad t\in[0,1]$$

$$\gamma_2(t)=2e^{2\pi it} \qquad t\in[0,1]$$

Prove that $\gamma_1+\gamma_2\approx 0(U)$

## REFERENCE BOOKS :

1. J. B. Connay : Functions of One Complex Variable - Second Edition - Springer International Student Edition.

2. Ruel V. Churchil; Jamesward Brown : Complex Variables and Applications - Mc-Graw Hill International Editions - Fifth Edition.

**Lesson writer :**

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Lesson - 24

# CAUCHY'S THEOREM : HOMOTOPIC TO LERO PATHS

## 24.0 INTRODUCTION

In this lesson, we try to find out more about paths $\gamma$ for which Cauchy's Results hold and open sets $U$ for which Cauchy's results are valid for every rectifiable closed curve.

Familiarity with topology is assumed in this lesson. The treatment is sketchy.

The basic notion in this lesson is homotopy of maps.

Two theorems are proved.

If $\gamma_0, \gamma_1$ are closed rectifiable paths in $U$ such that $\gamma_0$ is homotopic to $\gamma_1$ in $U$ then for every analytic function $f$ on $U$ we have

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

If $\gamma_0, \gamma_1$ are rectifiable paths in $U$ such that $\gamma_0 \sim \gamma_1$ $(\mathrm{FEP}\ U)$ then also

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

We define three notions of simply connected open set $U$. We prove some relations among them.

The main result all the three notions are the same is beyond the scope of the Lesson.

## 24.1 PRELIMINARIES :

### 24.1.1 DEFINITION :

By a change of parameter we mean a map $\varphi$ from some closed interval $[a,b]$ onto a closed interval $[c,d]$ such that

i) $\varphi(a)=c,\ \varphi(b)=d$

ii) $\varphi$ is strictly increasing :

$\varphi(s)<\varphi(t)$ if $s<t$

and iii) $\varphi$ is continuous

**24.1.2 DEFINITION :**

A path $\sigma:[c,d]\to\mathbb{C}$ is said to be equivalent to a path $\gamma:[a,b]\to\mathbb{C}$ if f there is a change of parameter.

$$\varphi:[a,b]\to[c,d] \text{ such that}$$

$$\sigma=\gamma\cdot\varphi$$

**Now we come to the main concept for this lesson.**

**24.1.3 DEFINITION :**

We say that a closed path $\gamma_0:[a,b]\to U$ is homotopic to a closed path $\gamma_1:[a,b]\to U$ in $U$. if there is a continuous map

$$\Gamma:[0,1]\times[a,b]\to U \text{ such that}$$

$$\Gamma:(0,t)=\gamma_0(t),\ \Gamma(1,t)=\gamma_1(t) \text{ for all } t \text{ in } [a,b]$$

$$\text{and } \Gamma(s,a)=\gamma(s,b) \text{ for all s in } [0,1]$$

**24.1.4 DEFINITION :**

We say that a path $\gamma_0:[a,b]\to U$ is fixed end point homotopic to a path $\gamma_1:[a,b]\to U$ in $U$ if there is a continuous map

$$\Gamma:[0,1]\times[a,b]\to U$$

such that

(1) $\gamma_0(t)=\Gamma(0,t),\ \gamma_1(t)=\Gamma(1,t)$ for $t\in[a,b]$;

(2) $\gamma_0(0)=\gamma_1(0),\ \gamma_0(1)=\gamma_1(1)$;

(3) and for all s in $[0,1]$

$$\gamma_0(0)=\Gamma(s,0)=\gamma_1(0) \text{ and}$$

$$\gamma_0(1)=\Gamma(s,1)=\gamma_1(1)$$

our main concern is with FEP homotopic paths in an open set $U$; of special interest are the closed paths $\gamma$ which are homotopic to the null path $[\gamma(0)]$.

Before we proceed f ther we give som examp'es that make us familiar with the idea of FEP homotopy.

**24.1.5 EXAMPLE :**

Suppose

$$\gamma_0 : [a,b] \to U \text{ and } \gamma_1 : [a,b] \to U$$

define the same curve that is there is a

$$\varphi : [a,b] \to [a,b]$$

such that $\varphi$ is a change of parameter and

$$\gamma_1 = \gamma_0 \circ \varphi$$

Then we claim that $\gamma_0$ is FEP homotopic to $\gamma_1$.

We shall give

$$\Gamma : [0,1] \times [a,b] \to U$$

and leave the details to the reader. We define

$$\Gamma(s,t) = \gamma_0((1-s)t + s\varphi(t))$$

We make only a remark. The interval $[a,b]$ is convex, $t, \varphi(t)$ belong to it and therefore

$$(1-s)t + s\varphi(t)$$

also belongs to $[a,b]$, and this $\gamma_0((1-s)t + s\varphi(t))$ is defined.

The example implies that if paths $\gamma_0$ and $\gamma_1$ define the same curve then they are FEP homotopic to each other

(Notation. If $\gamma_0$ and $\gamma_1$ are paths in $U$ and $\gamma_1$ is FEP homotopic to $\gamma_0$ in $U$ we write $\gamma_0 \sim \gamma_1$ FEP in $U$).

**24.1.6 PROPOSITION :**

"$\sim$ FEP in $U$" is an equivalence relation.

**Proof :** We shall only give the proof that it is transitive. Suppose

$$\gamma_0 \sim \gamma_1 \text{ FEP in } U$$

and $\gamma_1 \sim \gamma_2$ FEP in U;

and $\Gamma_1 : [0,1] \times [a,b] \to U$

$\Gamma_2 : [0,1] \times [a,b] \to U$

are maps that give

$\gamma_0 \sim \gamma_1$ FEP in U

$\gamma_1 \sim \gamma_2$ FEP in U.

Then define

$$\Gamma(s,t) = \begin{cases} \Gamma_1(2s,t) & \text{if } 0 \le s \le \frac{1}{2} \\ \Gamma_2(2s-1,t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

It is easy to verify that $\Gamma$ gives "$\gamma_0 \sim \gamma_2$ in U"

Suppose now that

$\gamma_1 : [a,b] \to U_1, \ \gamma_2 : [b,c] \to U$

are paths such that $\gamma_1(b) = \gamma_2(b)$, we consider

$\gamma : [a,c] \to U$

defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a,b] \\ \gamma_2(t) & \text{if } t \in [b,c] \end{cases}$$

Because we have assumed $\gamma_1(b) = \gamma_2(b)$, $\gamma$ is defined. That $\gamma$ is continuous can be proved easily and left as an exercise for the reader.

In lesson 20 and 22 we have proved that if $\gamma_1, \gamma_2$ are rectifiable, then $\gamma$ is rectifiable and for each $f$ continuous on $[a,c]$ we have

$$\int_{\gamma} f = \int_{\gamma_1} f_1 + \int_{\gamma_2} f_2$$

where $f_1 = f|_{[a,b]}$ and $f_2 = f|_{[b,c]}$

With this in mind we give the following.

We now consider an open set $U$ in $\mathbb{C}$ and $(n+1)^2$ points

$$Z_{jk}$$

for $j = 0, 1, \cdots, n$ and $k = 0, 1, \cdots, n$. For each $(j,k)$ $0 \le j \le n-1$, $0 \le k \le n-1$. We denote the polygonal path

$\left[Z_{jk}, Z_{j+1k}, Z_{jk+1}, z_{jk}\right]$ by $\Gamma_{jk}$. $\Gamma_{jk}$. is a closed path. We assume now that for each $(j,k)$ there is an $a_{jk} \in \mathbb{C}$ and $\rho_{jk} > 0$ such that the four points $Z_{jk}, Z_{jk}, Z_{j+1k}, Z_{j+1\,k+1}$, $Z_{j\,k+1}$ are contained in $B\left(a_{jk}, \rho_{jk}\right)$. Then since B is a convex set we obtain that

$$\left\{\Gamma_{jk}\right\} = \text{the trace of } \Gamma_{jk} \subset B\left(a_{jk}, \rho_{jk}\right) \subset U$$

Suppose we are given a continuous function $f : U \to \mathbb{C}$. Then $\int_{\Gamma_{jk}} f$ is defined and we then look at the sum

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \int_{\Gamma_{jk}} f.$$

We express the integrals along $\Gamma_{jk}$ as sums of integrals of the type

$$\int_{Z_{jk}}^{Z_{j+1k}} f \quad \int_{z_{jk}}^{z_{jk+1}} f; \text{ and } \int_{Z_{j+1k}}^{Z_{jk}} f, \quad \int_{Z_{j\,k+1}}^{Z_{j,k}} f$$

By proposition we have

$$\int_{z_{jk}}^{z_{j+1k}} f + \int_{z_{j+1k}}^{z_{jk}} f = 0 \text{ and } \int_{z_{jk}}^{z_{jk+1}} f + \int_{z_{j\,k+1}}^{z_{jk}} f = 0.$$

Therefore as in the case of squares earlier we obtain

$$\sum_{j=0}^{n-1} \int_{Z_{j0}}^{Z_{j+10}} f + \sum_{k=0}^{n-1} \int_{Z_{n,k}}^{Z_{n\,k+1}} f + \sum_{j=n-1}^{0} \int_{Z_{j+1n}}^{Z_{jn}} f + \sum_{k=n-1}^{0} \int_{Z_{0\,k+1}}^{Z_0} f$$

$$= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \int_{\Gamma_{jk}} f$$

**24.2 MAIN RESULTS :**

**24.2.1 THEOREM (CAUCHY'S THEOREM THIRD VERSION) :**

Suppose $U \subset \mathbb{C}$ is an open set, $f : U \to \mathbb{C}$ is an analytic function. Suppose that

$$\gamma_0, \gamma_1 : [a,b] \to U$$

are rectifiable paths such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. Further suppose that there is a map

$$\Gamma : [0,1] \times [a,b] \to U$$

with the following properties

1) $\Gamma(0,t) = \gamma_0(t)$ $\quad t \in [a,b]$

2) $\Gamma(1,t) = \gamma_1(t)$ $\quad t \in [a,b]$

3) $\gamma_0(a) = \Gamma(s,a) = \gamma_1(a)$ $\quad s \in [0,1]$

4) $\gamma_0(b) = \Gamma(s,b) = \gamma_1(b)$ $\quad s \in [0,1]$

Then $\int_{\gamma_0} f = \int_{\gamma_1} f$

**Proof :** All the necessary work is done in the general explanation given before the statement of the Theorem. We shall make use of it in the proof.

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The set $X=[0,1]\times[a,b]$ is a compact set and $\Gamma$ is assumed to be continuous. Therefore we have

1) $\Gamma(X)$ is a compact subset of $U$

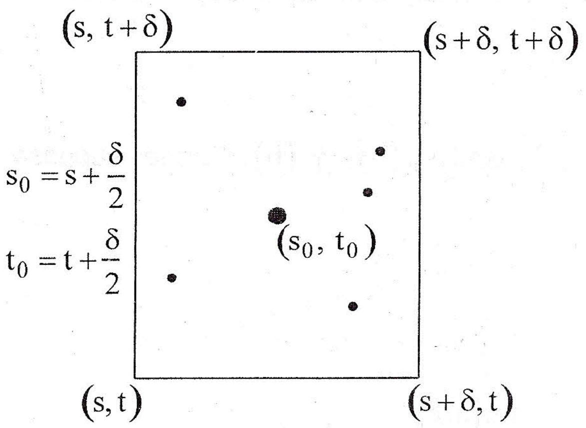
and 2) $\Gamma : X \to U$ is uniformly continuous.

Since $\Gamma(X)$ is compact and $\mathbb{C}\setminus U$ is closed and disjoint from $\Gamma(X)$ we have

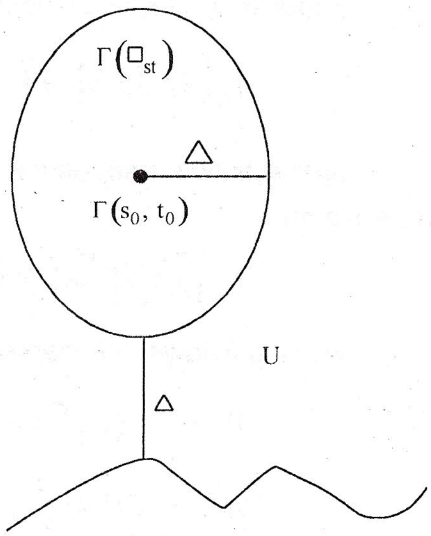
$$d(\Gamma(X), \mathbb{C}\setminus U)=\Delta>0.$$

Since $\Gamma$ is uniformly continuous and $\Delta>0$ we can find a $\delta>0$ such that for all $(s,t)$, $(s',t')$ in $X=[0,1]\times[a,b]$

$$|\Gamma(s,t)-\Gamma(s',t')|<\Delta \text{ if } |s-s'|<\delta, |t-t'|<\delta$$




$(s', t')$ inside the square




Choose a positive integer $n_0$ such that

$$\text{Max}\left\{\frac{1}{n_0}, \frac{b-a}{n_0}\right\}<\delta$$

Define

$$s_j = \frac{j}{n_0},\ t_j = a + \frac{j}{n_0}(b-a) \quad j = 0, 1, \cdots, n-1.$$

We have $s_0 = 0,\ s_{n_0} = 1,\ t_0 = a,\ t_{n_0} = b$. We now define $z_{jk}$ :

$$z_{jk} = \Gamma(s_j, t_k)$$

Let us denote $\frac{s_j + s_{j+1}}{2}$ by $s'_j$, $\frac{t_k + t_{k+1}}{2}$ by $t'_k$ and define

$$z'_{jk} = \Gamma(s'_j, t'_k) : j \le n_0 - 1,\ k \le n_0 - 1.$$

We have

$$\{z_{jk}, z_{j+1k}, z_{j+1\,k+1}, z_{jk+1}\} \subset B(z'_{jk}, \Delta) \subset U.$$

For each $(j, k)$ we denote the polygonal path

$$[z_{jk}, z_{j+1k}, z_{j+1k+1}, z_{jk+1}, z_{jk}] \text{ by } \Gamma_{jk}$$

Since the verticies of $\Gamma_{jk}$ are contained in $B(z'_{jk, \Delta})$ we obtain

$$\{\Gamma_{jk}\} \subset B(z'_{jk}, \Delta) \subset U$$

Therefore $\int_{\Gamma_{jk}} f$ is defined. Therefore by what we have concluded earlier

$$\sum_{j=0}^{n_0-1} \int_{z_{j0}}^{z_{j+10}} f + \sum_{k=0}^{n-1} \int_{z_{n_0 k}}^{z_{n_0 k+1}} f + \sum_{j=n-1}^{0} \int_{z_{j+1\, n_0}}^{z_{j n_0}} f + \sum_{k=n_0-1}^{0} \int_{z_{0k+1}}^{z_{0k}} f$$

$$= \sum_{j=0}^{n_0-1} \sum_{k=0}^{n_0-1} \int_{\Gamma_{jk}} f$$

We have assumed that $f$ is analytic and the $\Gamma_{jk}$ are such that

$$\{\Gamma_{jk}\} \subset B(z'_{jk}, \Delta) \subset U$$

Therefore $\int_{\Gamma_{jk}} f = 0$

We claim that the sums

$$\sum_{j=0}^{n_0-1} \int_{z_{j0}}^{z_{j+1\,0}} f \text{ and } \sum_{j=n_0-1}^{0} \int_{z_{j+1\,n_0}}^{z_{j\,n_0}} f$$

add upto zero. This is because by (3) of our hypothesis we have

$$z_{j0} = \Gamma(s_j, a) = \Gamma(s_j, b) = z_{jn_0}$$

and so

$$\int_{z_{j0}}^{z_{j+1\,0}} f + \int_{z_{j+1\,n_0}}^{z_{jn_0}} f = 0$$

Therefore

$$\sum_{k=0}^{n-1} \int_{z_{n_0k}}^{z_{n_0\,k+1}} f + \sum_{k=n_0-1}^{0} \int_{z_{0\,k+1}}^{z_{0k}} f = 0$$

We note that

$$z_{n_0k} = \Gamma(s_{n_0}, t_k) = \Gamma(1, t_k)$$

$$z_{0k} = \Gamma(s_0, t_k) = \Gamma(0, t_k)$$

Let us denote the restriction of $\gamma_0$ to $[t_k, t_{k+1}]$ by $\gamma_{0k}$ and the restriction of $\gamma_1$ to $[t_k, t_{k+1}]$ by $\gamma_{1k}$. We have

$$|\gamma_0(t) - \gamma_0(t_k)| = |\Gamma(s_0, t) - \Gamma(s_{n_0}, t_{k+1})| < \Delta \text{ since } t \in [t_k, t_{k+1}]\ t_{k+1} - t_k = \frac{b-a}{n_0}.$$

$$|\gamma_1(t) - \gamma_1(t_k)| = |\Gamma(s_{n_0}, (t)) - \Gamma(s_{n_0}, t_{k+1})| < \Delta \text{ since } t \in [t_k, t_{k+1}]\ t_{k+1} - t_k = \frac{b-a}{n_0} < \delta.$$

It follows that

$$\left[z_{0k}, z_{0\,k+1}\right] \cup \left\{\gamma_{0k}\right\} \subset B\left(z_{0k}, \Delta\right) \subset U$$

$$\left[z_{n_0 k}, z_{n_0\,k+1}\right] \cup \left\{\gamma_{1k}\right\} \subset B\left(z_{n_0 k}, \Delta\right) \subset U$$

and therefore

$$\int_{z_{0k}}^{z_{0\,k+1}} f = \int_{\gamma_{0\,k}} f, \quad \int_{z_{n_0 k}}^{z_{n_0\,k+1}} f = \int_{\gamma_{1\,k}} f$$

Therefore our identity is

$$\sum_{j=0}^{n-1} \int_{\gamma_{1j}} f - \sum_{j=0}^{n-1} \int_{\gamma_{0j}} f = 0$$

i.e. $\int\limits_{\gamma_1} f - \int\limits_{\gamma_0} f = 0$.

**24.2.2 THEOREM :**

Suppose $U$ is an open subset of $\mathbb{C}$; $c_0$, $c_1$ are path rectifiable curves such that

$c_0$ is FEP homotopic to $c_1(U)$.

Then

$$\int_{c_0} f = \int_{c_1} f$$

for every analytic function $f$ on $U$.

**Proof :** We shall deduce this Theorem from Theorem 1.

We suppose $c_0, c_1$ are represented by rectifiable paths

$\gamma_0 : [a,b] \to U$ and $\gamma_1 : [a,b] \to U$.

Our hypothesis is that there is a continuous function

$$\Gamma : [0,1] \times [a,b] \to U$$

satisfying the following conditions.

1) For all $t$ in $[a,b]$

$$\gamma_0(t)=\Gamma(0,t),\ \gamma_1(t)=\Gamma(1,t)$$

and,

2) For all s in $[0,1]$

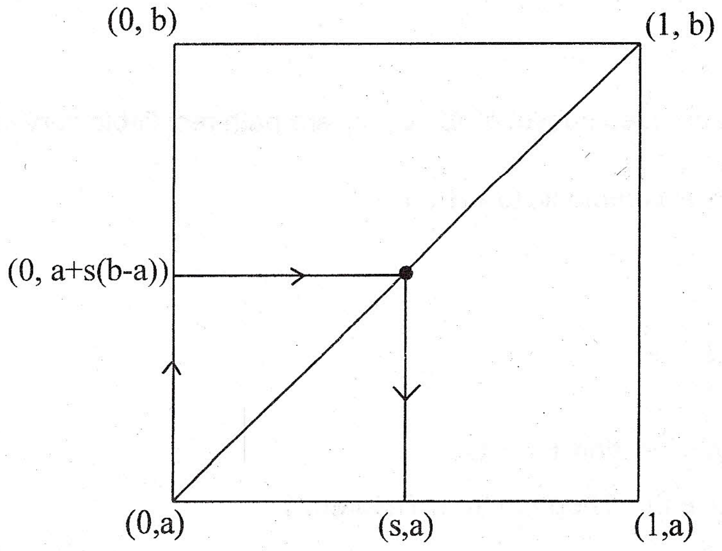
$$\Gamma(s,a)=\gamma_0(a)=\gamma_1(a)$$

$$\Gamma(s,b)=\gamma_0(b)=\gamma_1(b)$$

We shall prove that the closed rectifiable curve $c_1^{-1}\cdot c_0$ is homotopic to $[c_0(a)]$, the null path at $c_0(a)$.

We shall present the idea behind the proof by means of a diagram first.

For s in $[0,1]$ we define a path $\varphi_s:[a,b]\to\mathbb{C}$




$\varphi_s$ is affine on

$\left[a,\ a+\frac{b-a}{3}\right],\ \left[a+\frac{b-a}{3},\ a+2\left(\frac{b-a}{3}\right)\right],\ \left[a+2\left(\frac{b-a}{3}\right),b\right]$. It maps the first interval, second interval, and third interval onto segments $[(0,a),(0,a+s(b-a))]$

$[(0,\ a+s(b-a)),(s,a+s(b-a))]$ and

$[(s,a+s(b-a)),(s,a)]$ respectively. We shall set

$\Gamma_0(s,t)=\Gamma o \varphi_s(t).$

Then we shall prove $\Gamma_0$ gives a homotopy of the closed curve $c_1^{-1} o\ c_0$ with $[c_0(a)]$.

$\varphi_s$ is uniquely determined by the conditions laid down on it. We have

$$\varphi_s(t)=(0,\ a+3s(t-a)) \qquad t\in\left[a,\ a+\left(\frac{b-a}{3}\right)\right]$$

$$\varphi_s(t)=\left(\frac{3s}{b-a}\left(t-a+\frac{(b-a)}{3}\right),\ a+s(b-a)\right) \qquad t\in\left[a+\frac{(b-a)}{3},\ a+2\left(\frac{b-a}{3}\right)\right]$$

$$\varphi_s(t)=\left(s,a+s(b-a)-3s\left(t-\left(a+2\left(\frac{b-a}{3}\right)\right)\right)\right) \qquad t\in\left[a+2\frac{(b-a)}{3},\ a+3\left(\frac{b-a}{3}\right)\right]$$

when s=0 it is clear that

$\varphi_0(t)=(0,a)$

and so $\Gamma o \varphi_0$ is $[c_0(a)]$.

We define

$\Gamma_0(s,t)=\Gamma o \varphi_s(t).$

for $(s,t)$ in $[0,1]\times[a,b]$. We have proved.

$\Gamma_0(0,t)=c_0(a)$ for all $t$ in $[a,b]$.

We look at $\Gamma_0(s,a)$ and $\Gamma_0(s,b)$. By our hypothesis.

$\Gamma(s,a)=\gamma_0(a)$

We have

$\varphi_s(a) = (0, a)$ and

$\varphi_s(b) = (s, a)$ and therefore

$\Gamma_0(s,a) = \Gamma_0(s,b) = \gamma_0(a) = c_0(a)$

Now we consider $\Gamma_0(1,t)$.

It is left as an exercise to the reader to check that the $\Gamma_0(1,t)$ defines the curve $c_1^{-1} o\ c_0$. (See the definition of $\gamma_1^{-1}\ o\ \gamma_0$ ) therefore by Theorem 1.

$$\int_{c_1} f = \int_{c_0} f$$

**24.2.3 COROLLARY :**

If $U$ is an open subset of $\mathbb{C}$, $\gamma$ is a closed rectifiable path in $U$ such that $\gamma \sim 0(U)$, then $\gamma \approx 0(U)$. This corollary is treated as SAQ 1. Its solution is presented after the main text of the lesson.

In $c_0$ 24.3 results in $\mathbb{C}_\infty$ next we are going to make a definition for that we need some preliminaries.

**24.3.1 PRELIMINARIES :**

(1) We take the symbol $\infty$. We define

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$$

we are going to make $\mathbb{C}_\infty$ a topological space.

For $r > 0$ we define

$$B(\infty, r) = \{\infty\} \cup \left\{z \in \mathbb{C} : |z| > \frac{1}{r}\right\}.$$

It is clear that if $r_1 < r_2$ then

$$B(\infty, r) \subset B(\infty, r_2)$$

Denote by B the collection of subsets of $\mathbb{C}_\infty$ consisting of

$\{B(a,r)\}\ a \in \mathbb{C},\ r > 0$

$\{B(\infty,r)\}\ r > 0$

$\varphi$ and $\mathbb{C}_\infty$.

It is checked easily that B is a basis for topology on $\mathbb{C}_\infty$ what is to be checked is that if $U_1, U_2$ are any two elements in B, then their intersection is a union of sets belonging to B. For $B(a,r)$, $B(b,s)$ it is done in topology. The other cases the verification is imeediate.

We describe the above topology by means of opensets. We call $U \subseteq \mathbb{C}_\infty$ an open subset of $\mathbb{C}_\infty$ if either

1) $U \subseteq \mathbb{C}$ and $U$ is an open set in $\mathbb{C}$

or 2) $\infty \in U$ and then

i) there is an $r > 0$ such that $B(\infty,r) \subset U$

and ii) $U \cap \mathbb{C}$ is an open subset of $\mathbb{C}$.

Thus we have a topological space $\mathbb{C}_\infty$. If one is familiar with stereographic projection one can verify that $\mathbb{C}_\infty$ may be identified with Riemann sphere.

We extend the definition of winding number of a closed rectifiable curve around **z** in $\mathbb{C}$ to z in $\mathbb{C}_\infty$.

**24.3.2 DEFINITION :**

Suppose C is a closed rectifiable curve in $\mathbb{C}$. Then we define

$$n(C, \infty) = 0.$$

We have already defined $n(c,z)$ for $z \in \mathbb{C}$.

**24.3.3 PROPOSITION :**

Suppose C is a closed rectifiable curve in $\mathbb{C}$. Then $n(c,) : \mathbb{C}_\infty \setminus \{c\} \to \mathbb{Z}$ is a continuous function.

**Proof :** We have to check the continuity at $\infty$.

Let $\gamma:[a,b]\to\mathbb{C}$ be a closed rectifiable path defining C. Set

$$A=\ell ub\{|\gamma(t)|: t\in[a,b]\}.$$

Then $0\le A<\infty$. Then the set

$\{z\in\mathbb{C}:|z|>A\}$ is contained in the unbounded component of $\mathbb{C}\setminus\{\gamma\}$. Therefore

$$n(c,z)=n(r,z)=0 \text{ for } |z|>A.$$

This implies that

$n(\gamma,z)=0$ on $B\left(\infty,\ \dfrac{1}{A}\right)$ and so $n(c,\ )$ is continuous at $\infty$.

**24.3.4 COROLLARY :**

$n(c,\ )$ is constant on the connected components of $\mathbb{C}_\infty\setminus\{c\}$.

**Proof :** The connected components of $\mathbb{C}_\infty\setminus\{c\}$ are the bounded components of $\mathbb{C}_\infty\setminus\{c\}$ and $\{\infty\}\ U$. The unbounded component of $\mathbb{C}\setminus\{C\}$. So the result is clear.

**24.3.5 DEFINITION :**

We say that $U$ is topologically simply connected iff $\mathbb{C}_\infty\setminus U$ is connected.

**24.3.6 PROPOSITION :**

If $U$ an open set in $\mathbb{C}$ is topologically simply connected, then $U$ is homologically simply connected.

**Proof :** Let us set $E=\mathbb{C}_\infty\setminus U$. By Corollary $n(c,\ )$ is constant on E. Since $\infty\in E$, that constant must be the integer 0. That is

$$n(c,z)=0 \text{ if } z\notin U$$

That is $c\approx 0(U)$

**24.4 SOLUTIONS TO SAQ'S :**

**SAQ 1 :** Suppose $U$ is an open set, $\gamma$ is a closed rectifiable path in $U$ such that $\gamma\sim o(U)$. Then show that

$n(\gamma, z) = 0$ for $z \notin U$.

**Solution :** Suppose we define on $U$

$$f(w) = \frac{1}{w - z}.$$

Then $f$ is analytic in $U$. By Theorem 1

$$\int_{\gamma} f = \int_{[P]} f = 0$$

where $p$ is the origin of $\gamma$. By definition we have

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma(w)} \frac{dw}{w - z} = \frac{1}{2\pi i} \int_{\gamma} f = 0$$

## 24.5 MODEL EXAMINATION QUESTIONS :

1. If $U$ is a convex open set, then show that every closed path $\gamma$ in $U$ is $\sim o(U)$.

2. If $U$ is an open set and $\gamma$ is a rectifiable closed path in $U$ such that $\gamma \sim o(U)$ then show that

$$\gamma \approx o(U)$$

3. Suppose $U = \{z \in \mathbb{C} : z \text{ is not a non positive real number}\}$ and $\gamma$ is a closed path in $U$. Show that

$$\gamma \sim o(U).$$

4. Suppose that $V = \mathbb{C} \setminus \{0\}$ and $\gamma$ is a closed rectifiable curve in V. Then show that

$$\gamma \sim C_m(|\gamma(a)|)$$

where $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$
$m \in \mathbb{Z}$

and $C_m(|\gamma(a)|)$

is $\varphi : [0, 2\pi] \to \mathbb{C}$

given by $\varphi(t) = e^{imt}$.

5. Prove that if $f$ is an analytic function in an open set $U$ and $\gamma$ is a closed rectifiable path in $U$ such that $\gamma \sim o(U)$, then

$$\int_{\gamma} f = 0.$$

6. Let $U = \{z \in \mathbb{C} : z \neq 0\}$ and

$$\gamma(t) = e^{2\pi it} \qquad t \in [\gamma, 1].$$

Show that $\gamma \sim o(U)$ is not true.

**REFERENCE BOOKS :**

1. J. B. Connay : Functions of One Complex Variable - Second Edition - Springer International Student Edition.

2. Ruel V. Churchil; Jamesward Brown : Complex Variables and Applications - Mc-Graw Hill International Editions - Fifth Edition.

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Lesson - 25

# CLASSIFICATION OF SINGULARITIES

## 25.0 INTRODUCTION

In this lesson, we study the notion of an isolated singularity of a function and its' types - namely, removale singularity, pole and essential singularity. Further, we study (i) a necessary and sufficient condition for an isolated singularity to be removable (see Theorem 25.1.3); (ii) a sufficient condition for an isolated singularity to be removable (see Theorem 25.1.4) and the relation between the singular parts of a rational function at its' poles and its partial fractions (See Discussion 25.1.12).

## 25.1 CLASSIFICATION OF SINGULARITIES :

### 25.1.1 DEFINITION :

A function $f$ is said to have an isolated singularity at $z=a$ iff $f$ is defined and analytic in $B'(a;R)=B(a;R)-\{a\}$ but not in $B(a;R)$ for some $R>0$.

### 25.1.1.1 NOTE :

Isolated singularity of $f$ at $z=a$ may occur either because of the fact that $f$ is not defined at $z=a$ or $f$ is defined at $z=a$ but not analytic at $z=a$ in addition to that $f$ is analytic in a deleted neighborhood of $a$. $\frac{\operatorname{Sin} z}{z}$, $\frac{\operatorname{Cos} z}{z}$ and $\frac{1}{z}$ - all have isolated singularity at $z=0$ as each of these functions is analytic at every $z \neq 0$ in $\mathbb{C}$ and is not defined at $z=0$.

### 25.1.2 DEFINITION :

Suppose $f$ has an isolated singularity at $z=a$. The point $z=a$ is called a removable singularity of $f$ if there exists an analytic function $g$ defined in a neighborhood of a $a-B(a;R)$ say such that

$$f(z)=g(z)$$

for all $z$ in $B'(a;R)$.

### 25.1.2.1 NOTE :

In the above definition, we can assume (with out loss of generality) that $f$ is defined and analytic on $B'(a;R)$.

**25.1.2.2 NOTE :**

$f$ has a removable singularity at $z=a$ means that we define $f$ at $z=a$ (if $f$ is not defined at $z=a$) or $f$ is redefined at $z=a$ (if $f$ is defined at $z=a$) so that $f$ is analytic at $z=a$.

Now, the problem before us is - How to determine the removable singularity? The following theorem gives a necessary sufficient condition for a function $f$ to have removable singularity at $z=a$.

**25.1.3 THEOREM :**

Suppose $f$ has an isolated singularity at $z=a$. The point $z=a$ is a removable singularity of $f$ if and only if

$$\lim_{z\to a}(z-a)\,f(z)=0$$

**Proof :** Assume that $f$ has removable singularity at $z=a$. So, there exists an analytic function

$$g:B(a;R)\to\mathbb{C}$$

such that $f(z)=g(z)$ f all $z$ in $B'(a;R)$ for some $R>0$. Now,

$$\lim_{z\to a}(z-a)\,f(z)=0=\lim_{z\to a}(z-a)\,g(z)$$

$$=(a-a)\,g(a)=0$$

Conversely assume that

$$\lim_{z\to a}(z-a)\,f(z)=0$$

Since $f$ has an isolated singularity at $z=a$, $f$ is defined and analytic in a deleted neighborhood of a - $B'(a;R)$ say. Define $g:B(a;R)\to\mathbb{C}$ by

$$g(z)=\begin{cases}(z-a)\,f(z) & \text{if } z\neq a\\ 0 & \text{otherwise} \qquad (\text{i.e. if } z=a)\end{cases}$$

Clearly, $g$ is analytic in $B'(a;R)$ and is continuous at $z=a$ (by our assumption). So, $g$ is continuous in $B(a;R)$.

Suppose we have proved that $g$ is analytic in $B(a;R)$, then, since $g(a)=0$, there exists an analytic function $h$ on $B(a;R)$ such that

$$g(z)=(z-a)h(z)$$

for all $z$ in $B(a;R)$. For any $z$ in $B'(a;R)$,

$$(z-a)f(z)=g(z)=(z-a)h(z)$$

and so $f(z)=h(z)$

Hence $f$ has removable singularity at $z=a$.

Now, the theorem is complete if we prove that $g$ is analytic in $B(a;R)$. We prove this fact using Morera's Theorem.

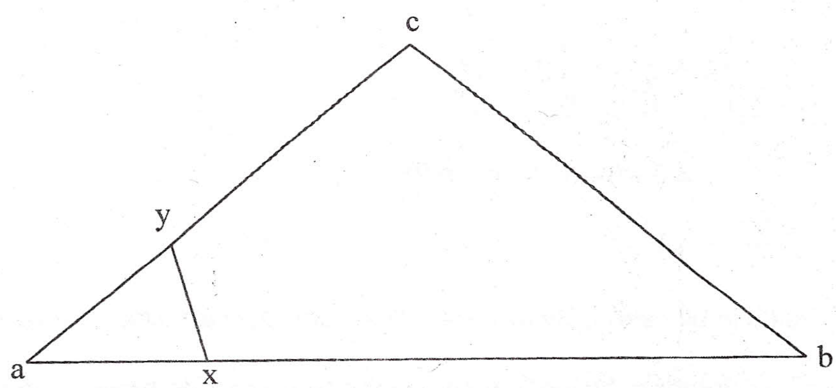
Let $T$ be a triangle in $B(a;R)$. Now, we show that

$$\int_T g=0 \text{ --------- (1)}$$

Let $\Delta$ be the inside of $T$ together with $T$. Now, we have the following cases.

**Case 1 :** $a\notin\Delta$: Clearly $T\sim 0$ (i.e. $T$ is homotopic to zero). By Cauchy's Theorem, (1) holds.

**Case 2 :** Suppose $z=a$ is a vertex of the triangle $T$. Let $T=[a,b.c,a]$ (i.e. $a,b,c$ are vertices of $T$). Choose points $x$ in $[a,b]$ and $y$ in $[a,c]$ and form the triangle $T_1=[a,x,y,a]$ and the polygon $P=[x,b,c,y,x]$. Clearly




$$\int_T g=\int_{T_1} g+\int_P g$$

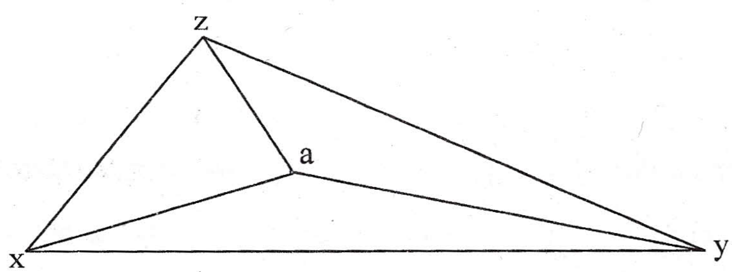
$$=\int_{T_1} g \quad \text{(since } P\sim 0 \text{ and by Cauchy's Theorem)}$$

Let $\in > 0$. Since $g$ is continuous at $z = a$, there exists $\delta > 0$ such that $|z-a| < \delta$ implies $|g(z)| < \frac{\in}{\ell}$ where $\ell$ is the length of $T$. Now, we choose $x$ in $[a,b]$ and $y$ in $[a,c]$ such that the triangle $T_1 = [a,x,y,a]$ lies in the interior of $B(a;\delta)$. Now,

$$\left|\int_T g\right| = \left|\int_{T_1} g\right| \leq \frac{\in}{\ell} \text{ x length of } T_1 < \in \text{ ------------ (2)}$$

since length of $T_1 < \ell$. Thus (2), holds for all $\in > 0$. Hence, (1) holds.

**Case 3 :** Suppose $a$ lies in the interior of $T$. Let $T =$




$[x, y, z, x]$. Consider the triangles -

$$T_1 = [x,y,a,x],\ T_2 = [y,z,a,y],\ T_3 = [z,x,a,z].$$

Clearly,

$$\int_T g = \int_{T_1} g + \int_{T_2} g + \int_{T_3} g$$

$$= 0 + 0 + 0 \text{ (by case 2)}$$

$$= 0.$$

Thus (1) holds in all cases. By Morera's Theorem, $g$ is analytic in $B(a;R)$.

Now, we give another sufficient condition for a function $f$ to have removable singularity at $z = a$. Consider the following.

**25.1.4 THEOREM :**

If $f$ is defined, analytic and bounded in a deleted neighborhood of '$a$' then $f$ has removable singularity at $z = a$.

**Proof :** Suppose $f$ is defined, analytic and bounded in $B'(a;R)$ for some $R>0$. So, there exists $M>0$ such that $|f(z)| \le M$ for all $z$ in $B'(a;R)$. For any $z$ in $B'(a;R)$,

$$|(z-a)\, f(z)| \le |z-a| M.$$

So,

$$\lim_{z\to a} (z-a)\, f(z) = 0$$

Hence, $f$ has removable singularity at $z=a$ (by Theorem 25.1.3), we now, consider the following.

**25.1.5 EXAMPLE :**

Consider the function $f(z) = \sin z/z\,(z \ne 0)$. Since $f(z)$ is not defined at $z=0$ and since $f$ is analytic at each $z \ne 0$ in $\mathbb{C}$, $f$ has an isolated singularity at $z=0$.

Consider

$$\lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} \sin z = 0.$$

By Theorem 25.1.3, $f$ has removable singularity at $z=0$.

Now, we prove the same in another way. We know that for any $z$ in $\mathbb{C}$,

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots.$$

So, for any $z \ne 0$ in $\mathbb{C}$,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Define $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Thus, $h$ has power series expansion about 0. By Theorem 17.3.10, $h$ is analytic in $\mathbb{C}$. Clearly, $f(z) = h(z)$ for all $z \neq 0$ in $\mathbb{C}$. So, $f$ has removable singularity at $z = 0$. Infact, if we define, $f(0) = 1$ then $f$ is analytic at 0 also and hence $f$ becomes an entire function.

**25.1.6 DEFINITION :**

An isolated singularity $z = a$ of $f$ is called a pole of $f$ if and only if $\lim\limits_{z \to \infty} |f(z)| = \infty$.

**25.1.7 DISCUSSION :**

Suppose $f$ has a pole at $z = a$. So,

$$\lim_{z \to a} |f(z)| = \infty .$$

(i.e. given M>0, there exists a positive number $\in > 0$ such that $0 < |z - a| < \in$ implies $|f(z)| > M$) and hence

$$\lim_{z \to a} \frac{1}{f(z)} = 0 .$$

So, the function $q(z)$ defined $q(z) = \frac{1}{f(z)}$ is analytic in a punctured neighborhood $B'(a;R)$ of a and is bounded in $B'(a;R)$. By Theorem 25.1.4, $q(z)$ can be made analytic in $B(a;R)$ i.e. $q(z)$ can be defined at $z = a$ so that $q$ is analytic at $a$. Hence

$$q(a) = \lim_{z \to a} q(z) = \lim_{z \to a} \frac{1}{f(z)} = 0 .$$

(i.e. $q$ has removable singularity at $z = a$). Suppose $q(z)$ has a zero of multiplicity m at $z = a$. So,

$$q(z) = (z - a)^m p(z)$$

for some analytic function $P(z)$ on $B(a;R)$ with $P(a) \neq 0$. Since $P$ is continuous at $a$, there exists $\delta > 0$ such that $P(z) \neq 0$ for any $z$ in $B(a;\delta) \subseteq B(a;R)$. Thus $\frac{1}{P(z)}$ is analytic in

$B(a;\delta)$ and for any $z$ in $B'(a;\delta)$,

$$(z-a)^m f(z) = \frac{1}{P(z)} \; (= h(z) \text{ say})$$

$$\text{i.e. } f(z) = \frac{h(z)}{(z-a)^m}$$

Obviously, $(z-a)^m f(z)$ has removable singularity at $z=a$. Let $n$ be a positive integer such that $(z-a)^n f(z)$ has removable singularity at $z=a$. For $z \neq a$ in $B(a;\delta)$,

$$(z-a)^n f(z) = \frac{(z-a)^n}{(z-a)^m P(z)} = (z-a)^{n-m} h(z)$$

Now, $(z-a)^n f(z)$ removable singularity if and only if $(z-a)^{n-m}$ has removable singularity (since $h$ is analytic in $B(a;\delta)$ and $h(z) \neq 0$ for all $z$ in $B(a;\delta)$) if and only if $n-m \geq 0$ i.e. $m \leq n$.

Hence, if $f$ has a pole at $z=a$ then there exists a smallest positive integer m such that $(z-a)^m f(z)$ has removable singularity at $z=a$. Infact, if f has a pole at $z=a$ then there exists a positive integer m and a nonvanishing analytic function $h$ on a neighborhood $B(a;\delta)$ of a such that

$$f(z) = \frac{h(z)}{(z-a)^m} \quad (z \in B'(a;\delta)).$$

**25.1.8 DEFINITION :**

If $f$ has a pole at $z=a$ and m is the smallest positive integer such that $(z-a)^m f(z)$ has removable singularity at $z=a$, then, we say that $f$ has a pole of order m at $z=a$.

**25.1.9 DISCUSSION :**

Suppose $f$ has a pole of order m at $z=a$. So, m is the smallest positive integer such that $h(z)=(z-a)^m f(z)$ has removable singularity at $z=a$. So, $h(z)$ can be defined at $z=a$. So that $h(z)$ is analytic in a neighborhood of $a$. So, there exists $R>0$ such that $h(z)$ is analytic in

$B(a;R)$. Hence $h(z)$ has power series expansion in $B(a;R)$.

i.e. $$h(z)=\sum_{n=0}^{\infty} b_n (z-a)^n \quad (z\in B(a;R))$$

i.e. $$(z-a)^m f(z)=\sum_{n=0}^{\infty} b_n (z-a)^n (z\in B(a;R)).$$

If $h(a)=b_0=0$, then

$$(z-a)^{m-1} f(z)=\sum_{n=1}^{\infty} b_n (z-a)^{n-1} \qquad (z\in B'(a;R))$$

and hence $(z-a)^{m-1} f(z)$ has removable singularity, a contradiction to that m is the least positive integer with $(z-a)^m f(z)$ has removable singularity at $z=a$. So, $h(a)\neq 0$

i.e. $$\lim_{z\to a} (z-a)^m f(z)=\lim_{z\to a} h(z)=h(a)\neq 0.$$

Since $h$ is continuous at $z=a$, there exists $\delta>0$ such that $\delta<R$ and $h(z)\neq 0$ for any $z$ in $B(a;\delta)$. Now, for any $z$ in $B'(a;\delta)$,

$$(z-a)^m f(z)=h(z)=\sum_{n=0}^{\infty} b_n (z-a)^n$$

and hence $$f(z)=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots+\frac{b_{m-1}}{z-a}+\sum_{n=0}^{\infty} b_{n+m} (z-a)^n$$

$$=R(z)+g(z)$$

where $$R(z)=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots+\frac{b_{m-1}}{z-a}$$ and

$$g(z)=\sum_{n=0}^{\infty} b_{n+m} (z-a)^n.$$

Clearly, $g(z)$ is analytic in $B(a;\delta)$.

**25.1.10 DEFINITION :**

If $f(z)$ has a pole of order m at $z=a$, then we can write

$$f(z)=R(z)+g(z) \quad (z\in B'(a,\delta) \text{ for some } \delta>0)$$

where $R(z)=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots+\frac{b_{m-1}}{z-a}$ and

$$g(z)=\sum_{n=0}^{\infty} b_{n+m}(z-a)^n \text{ is analytic in } B(a;\delta)$$

$R(z)$ is called the principal part of or the singular part of $f$ at $z=a$.

**25.1.10.1 NOTE :**

Let $n$ be a positive integer. Clearly, the function $f$ on $\mathbb{C}-\{a\}$ (where $a\in\mathbb{C}$) defined by

$$f(z)=\frac{1}{(z-a)^n}$$

has a pole of order $n$ at $z=a$.

**25.1.10.2 NOTE :**

If $f$ has a pole of order 1 at $z=a$, then $z=a$ is called a simple pole of $f$ in which case $(z-a)f(z)$ has removable singularity at $z=a$ and

$$\lim_{z\to a}(z-a)\,f(z)\neq 0$$

**25.1.10.11 DEFINITION :**

An isolated singularity of a function $f$ which is neither a removable singularity nor a pole is called an essential singularity of $f$.

**25.1.12 DISCUSSION :**

Consider the rational function $r(z)=\frac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials and

$q(z) \neq 0$. Without loss of generality, we can assume that $p(z)$ and $q(z)$ have no common zeros (i.e. no common factors). The poles of $r(z)$ are precisely the zeros of $q(z)$. Clearly, $r(z)$ has a pole of order m at $z = a$ if and only if $z = a$ is a zero of multiplicity m for $q(z)$. Clearly, any rational function with out poles is a polynomial. Since $q(z)$ is a polynomial it has only a finite number of distinct zeros – $a_1, a_2, \cdots, a_n$ say. Let $S_j(z)$ be the singular part of $r(z)$ at $z = a_j$ $(j = 1, 2, \cdots, n)$. Let $r_1(z) = r(z) - S_1(z)$. Clearly $r_2(z)$ is a rational function with poles precisely $a_2, a_3, \cdots, a_n$. Now $r_2(z) = r_1(z) - S_2(z)$ is a rational function with poles $a_3, a_4, \cdots, a_n$. Continuing this process, we have that $r_n(z) = r_{n-1}(z) - S_n(z)$ is a rational function with no poles. So, $r_n(z)$ is a polynomial $-p(z)$ say.

So, $$p(z) = r_n(z)$$

$$= r_{n-1}(z) - S_n(z) = \cdots\cdots\cdots\cdots$$

$$= r(z) - \sum_{j=1}^{n} S_j(z) = \cdots\cdots\cdots\cdots$$

i.e. $$r(z) = p(z) + \sum_{j=1}^{n} S_j(z)$$

This is nothing but the partial fractions expansion of $r(x)$.

**25.1.13 EXAMPLE :**

Consider the rational function

$$r(z) = \frac{z^4 - 16z^2 + 3z + 61}{(z-3)^2 (z+4)}$$

Now, we determine the poles of $r(z)$ and the singular part of $r(z)$ at each pole. Inview of the discussion 25.1.12, it is enough if we write partial fractions of $r(z)$ from which we can easily write the singular parts of $r(z)$ at each pole.

Let $r(z) = Az+B+\dfrac{C}{z-3}+\dfrac{D}{(z-3)^2}+\dfrac{E}{z+4}$

So, $(Az+B)(z-3)^2(z+4)+C(z-3)(z+4)+D(z+4)+E(z-3)^2$

$\equiv z^4-16z^2+3z+61$

Comparing the coefficients of $z^4$, $z^3$, $z^2$, $z$ and constants both sides, we have

$A=1 \cdots\cdots\cdots (1)$

$-2A+B=0 \cdots\cdots\cdots (2)$

$-2B-15A+C+E=-16 \cdots\cdots\cdots (3)$

$36A-15B+C+D-6E=3 \cdots\cdots\cdots (4)$

$36B-12C+4D+9E=61 \cdots\cdots\cdots (5).$

From (1) and (2), $B=2$. Substituting the values of A and B in (3), we have

$C+E=3 \cdots\cdots\cdots (6)$

So, $C=3-E$. Substituting $A=1$, $B=2$ and $C=3-E$ in (4) and (5), we have

$D-7E=-6 \cdots\cdots\cdots (7)$

$4D+21E=25 \cdots\cdots\cdots (8)$

Solving (7) and (8), we have $D=1$, $E=1$. Hence $C=3-E=2$. Hence

$$r(z)=z+2+\frac{2}{z-3}+\frac{1}{(z-3)^2}+\frac{1}{z+4}$$

Clearly, the poles of $r(z)$ are $3, -4$. The singular parts of $r(z)$ at $z=3$, at $z=-4$ are precisely

$\frac{2}{z-3}+\frac{1}{(z-3)^2}$ and $\frac{1}{z+4}$ respectively.

## 25.2 SHORT ANSWER QUESTIONS :

**25.2.1 :** Write isolated singular point of the function $f(z)$ defined by $f(z)=\frac{\sin z}{z}$

**25.2.2 :** Is the singularity '0' of the function $f(z)=\frac{\sin z}{z}$ removable ?

**25.2.3 :** What is the type of the singularity at $z=0$ for the function $f(z)=\frac{\cos z}{z}$

**25.2.4 :** Define removable singularity.

**25.2.5 :** Define pole.

**25.2.6 :** Define essential singularity.

**25.2.7 :** Define singular part.

**25.2.8 :** Write the singular part of the function

$$f(z)=\frac{e^z-1}{z^3} \text{ at } z=0$$

**25.2.9 :** For the function $\frac{1}{(z-2)^2}$ 2 is a pole of order , $\underline{\quad (a) \quad}$ and the singular part of this function at $z=2$ in $\underline{\quad (b) \quad}$.

**25.2.10 :** Determine the nature of the singularity of the function

$$f(z)=z+3-\frac{1}{z-1}+\frac{2}{(z-1)^2} \text{ at } z=1.$$

**25.2.11 :** Is it possible to define $f(z)=\frac{e^z-1-z}{z^2}$ at $z=0$ so that $f$ is an entire function.

**25.2.12 :** Determine the isolated singular points of the function

$$f(z)=\frac{z^2+1}{z(z-1)}$$ and determine the type of each singularity.

**25.2.13 :** Write down the order of the pole of the function

$$f(z)=\frac{\log(1+z)}{z^2}$$

**25.2.14 :** What is the relation between the singular parts of a rational function and its partial fractions?

**25.3 MODEL EXAMINATION QUESTIONS :**

**25.3.1 :** Suppose $f$ has an isolated singularity at $z=a$. Prove that the point $z=a$ is a removable singularity of $f$ if and only if

$$\lim_{z\to a}(z-a)f(z)=0.$$

Deduce that the function $f(z)$ defined by

$$f(z)=\frac{\cos z-1}{z}$$ has removable singularity at $z=0$.

**25.3.2 :** Define removable singularity. If $f$ is defined, analytic and bounded in a deleted neighborhood of $a$, prove that $f$ has removable singularity at $z=a$. Deduce that the function $f(z)$ defined by

$$f(z)=z\sin\frac{1}{z}$$

has removable singularity at $z=0$.

**25.3.2 :** Define pole. If $f$ has a pole at $z=a$, prove that there exists a positive integer m and a non vanishing analytic function $h$ on a neighborhood $B(a;\delta)$ of a such that

$$f(z)=\frac{h(z)}{(z-a)^m} \qquad (z\in B'(a;\delta)).$$

**25.3.3 :** Let

$$f(z)=\frac{1}{z(z-1)(z-2)}.$$

(i) Determine the isolated singular points and their type.

(ii) Write down the partial fractions expansion of $f(z)$.

(iii) Determine the singular part of $f(z)$ at each of its pole.

## 25.4 EXERCISES :

**25.4.1 :** Each of the following functions $f$ has an isolated singularity at $z=0$. Determine the nature; if it is a removable singularity define $f(0)$ so that $f$ is analytic at $z=0$; if it is a pole find the singular part.

(a) $\dfrac{\sin z}{z}$ (b) $\dfrac{\cos z}{z}$

(c) $\dfrac{\cos z-1}{z}$ (d) $\exp\left(z^{-1}\right)$

(e) $\dfrac{\log(z+1)}{z^2}$ (f) $\dfrac{\cos\left(z^{-1}\right)}{z^{-1}}$

(g) $\dfrac{z^2+2}{z(z-1)}$ (h) $\left(1-e^z\right)^{-1}$

(i) $z\sin\dfrac{1}{z}$ (j) $z^n\sin\dfrac{1}{z}$

**25.4.2 :** Show that $f(z)=\operatorname{Tan} z$ is analytic in $\mathbb{C}$ except for simple poles at $z=\dfrac{\pi}{2}+n\pi$, for each integer $n$. Determine the singular part at each of these poles. (Hint : Let $z=a$ be a simple pole of $f(z)$. So, $\lim\limits_{z\to a}(z-a)\,f(z)=b$ (say) $\neq 0$. So, singular part of $f(z)$ at $z=a$ is $\dfrac{b}{z-a}$ ).

**25.4.3 :** If $f:G\to\mathbb{C}$ is analytic except for poles in $G$, prove that the set of poles of $f$ cannot have a limit point.

**25.4.4 :** Give the partial fractions expansion of

$$f(z)=\frac{z^2+1}{\left(z^2+z+1\right)(z-1)^2}$$

Determine the isolated singular points of $f$ and determine the nature of each isolated singular point and further, if $f$ has removable singularity at $z=a$ then define $f(a)$ so that $f$ is analytic at $z=a$; if $f$ has a pole then determine the order of the pole and the singular part at that pole.

**25.4.5 :** Let $f$ have an isolated singularity at $z=a$. Prove that if either

$$\lim_{z\to a}|z-a|^s\,|f(z)|=0 \text{ -------- } (1)$$

or $$\lim_{z\to a}|z-a|^s\,|f(z)|=\infty \text{ ------- } (2)$$

holds for some real $s$ (i.e. $s\in\mathbb{R}$), then there is an integer m such that (1) holds if $s>m$ and (2) holds if $s<m$

**25.4.6 :** Let $f$, a and m be as in Exercise 25.4.5. Show :

(i) $m=0$ if and only if $z=a$ is a removable singularity and $f(a)\neq 0$;

(ii) $m<0$ if and only if $z=a$ is a removable singularity and $f$ has a zero of order $-m$ at $z=a$.

(iii) $m>0$ if and only if $z=a$ is a pole of $f$ of order m.

**25.4.7 :** A function $f$ has an essential singularity if and only if neither (1) nor (2) (of Exercise 25.4.5) holds for any real s.

## 25.5 ANSWERS TO SAQ'S

**25.2.1 :** Clearly, $f(z)$ is analytic at every $z\neq 0$ in $\mathbb{C}$. Infact, $f(z)$ is not defined at $z=0$. So, $f$ has an isolated singularity at $z=0$.

**25.2.2 :** Yes; Since

$$\lim_{z\to 0}(z-0)f(z)=\lim_{z\to 0}\sin z=0$$

and by the necessary and sufficient condition given in Theorem 25.1.3.

**25.2.3 :** Clearly, $f(z)$ is analytic at every $z \neq 0$ in $\mathbb{C}$ and

$$\lim_{z \to 0} |f(z)| = \infty.$$

So, $f$ has a pole at $z = 0$. Now

$$\lim_{z \to 0} z\, f(z) = \lim_{z \to 0} \cos z = 1 \neq 0 \cdot$$

So, $f$ has a simple pole at $z = 0$.

**25.2.4 :** See Definition 25.1.2.

**25.2.5 :** See Definition 25.1.6.

**25.2.6 :** See Definition 25.1.11

**25.2.7 :** See Definition 25.1.10.

**25.2.8 :** For any $z$ in $\mathbb{C}$, we have

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots.$$

For any $z \neq 0$ in $\mathbb{C}$,

$$f(z) = \frac{e^z - 1}{z^3} = \frac{1}{z^3}\left[\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \frac{1}{1!\, z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!} + \cdots$$

Hence the singular part of $f(z)$ at $z = 0$ is

$$\frac{1}{1!\, z^2} + \frac{1}{2!\, z}$$

Thus, $f$ has a pole of order 2 at $z = 0$.

**25.2.9 :** (a) : 2 and (b) $\dfrac{1}{(z-2)^2}$

**25.2.10 :** Clearly, $z=1$ is an isolated singular point of $f(z)$ and $m=2$ is the smallest positive integer such that $(z-1)^2 f(z)$ has removable singularity. So, $f(z)$ has a pole of order 2 at $z=1$. The singular part of $f(z)$ at $z=1$ is

$$\frac{-1}{z-1}+\frac{2}{(z-1)^2}$$

**25.2.11 :** Clearly, $f(z)$ has an isolated singularity at $z=0$ (as $f(0)$ is not defined and is analytic at every $z \neq 0$). We know that for any $z \in \mathbb{C}$,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

So, for any $z \neq 0$ in $\mathbb{C}$,

$$f(z)=\frac{e^z-1-z}{z^2}=\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\cdots \quad \text{---------} (1)$$

Since the right hand side of (1) is a power series expansion which converges and hence analytic so that $f(z)$ has removable singularity at $z=0$. Obviously, if we define $f(0)=\frac{1}{2}$ then, $f$ becomes an entire function.

**Another Method :**

Clearly $z=0$ is an isolated singular point $f(z)$. Now,

$$\lim_{z\to 0}(z-0)f(z)=\lim_{z\to 0}\frac{e^z-1-z}{z}$$

$$=\lim_{z\to 0}\frac{e^z-1}{1} \quad \text{(by L' Hospital's Rule)}$$

$$=0$$

By Theorem 25.1.3, $f'(z)$ has removable singularity at $z=0$.

So, we have to define $f(0)$ as

$$f(0)=\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{e^z-1-z}{z^2}$$

$$=\lim_{z\to 0}\frac{e^z-1}{2z} \quad \text{(by L' Hospital's Rule)}$$

$$=\lim_{z\to 0}\frac{e^z}{2} \quad \text{(by L' Hospital's Rule)}$$

$$=\frac{1}{2}.$$

**25.2.12 :** Clearly, $z=0$ and $z=1$ are isolated singular points of $f(z)$ and

$f(z)=\dfrac{z^2+1}{z(z-1)}=1-\dfrac{1}{z}+\dfrac{2}{z-1}$. Clearly, $z=0$ and $z=1$ are simple poles of $f$.

**25.2.13 :** 1 (one)

**25.2.14 :** Any rational function $r(z)$ can be written as

$r(z)=$ some polynomial (entire function) + sum of the singular parts of $r(z)$.

**REFERENCE BOOK :**

J.B. Conway : Functions of one complex variable - Second Edition - Springer International Student Edition.

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## Lesson - 26

# LAURENT SERIES DEVELOPMENT

### 26.0 INTRODUCTION

In this lesson, we study the Laurent series expansion of a function $f$ which is analytic in the annulus ann $(a;R_1, R_2)$ where $0 \le R_1 < R_2 \le \infty$ (See Theorem 26.1.4) and we determine the type of singularity of a function $f$ at the isolated singular point $z = a$ on observing the Laurent Series Expansion of $f$ in the annulus $\operatorname{ann}(a;0,R) = B'(a;R) =$ punetwred disk with center a and radius R (for some R>0) (see corollary 26.1.4.1). Further, we study the Casorati-Weierstrass Theorem (See Theorem 26.1.9). For a function defined in a neighborhood of $\infty$ (infinity), we study the notion of isolated singularity at infinity and the concepts - removable singularity, pole, essential singularity at infinity.

### 26.1 LAURENT SERIES DEVELOPMENT :

### 26.1.1 DEFINITION :

Let $\{z_n / n = 0, \pm 1, \pm 2, \cdots\}$ be a doubly infinite sequence of complex numbers. We say that the series $\sum\limits_{n=-\infty}^{\infty} z_n$ is absolutely convergent if both the series $\sum\limits_{n=0}^{\infty} z_n$ and $\sum\limits_{n=0}^{\infty} z_{-n}$ are absolutely convergent and in this case we write

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n} .$$

### 26.1.2 DEFINITION :

Let $u_n$ be a function on a set S for $n = 0, \pm 1, \cdots$ and let $\sum\limits_{n=-\infty}^{\infty} u_n(s)$ converges absolutely, for each s in S. We say that $\sum\limits_{n=-\infty}^{\infty} u_n$ converges uniformly over S if both $\sum\limits_{n=0}^{\infty} u_n$ and $\sum\limits_{n=1}^{\infty} u_{-n}$ converges uniformly on S.

### 26.1.3 DEFINITION :

Let $0 \le R_1 < R_2 \le \infty$ and a be any complex number. Define

$\text{ann}(a;R_1,R_2)=\{z\in\mathbb{C}/R_1<|z-a|<R_2\}$ = The set of all points between the circles having radii $R_1$ and $R_2$, centered at a (called the annulus).

**26.1.3.1 NOTE :**

$\text{ann}(a;0,R)=\{z\in\mathbb{C}/0<|z-a|<R\}=B'(a;R)=$ the punctured disk with center a and radius R.

Now, we study the most important theorem.

**26.1.4 THEOREM (LAURENT SERIES DEVELOPMENT) :**

Let $f$ be analytic in the annulus $\text{ann}(a;R_1;R_2)$. Then

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-a)^n$$

where the convergence is uniform over $\overline{\text{ann}(a;r_1,r_2)}$ if $R_1<r_1<r_2<R_2$. Also, the coefficients $a_n$ are given by the formula.

$$a_n=\frac{1}{2\pi i}\int_\gamma\frac{f(w)dw}{(w-a)^{n+1}}$$

where $\gamma$ is the circle $|z-a|=r$ for any r with $R_1<r<R_2$. Moreover, this series is unique.

**Proof :** If $R_1<r_1<r_2<R_2$ and $\gamma_1:|z-a|=r_1$, $\gamma_2:|z-a|=r_2$ are circles then $\gamma_1\sim\gamma_2$ (i.e. $\gamma_1$ and $\gamma_2$ are homotopic) in $\text{ann}(a;R_1,R_2)$ and hence $\int_{\gamma_1}g=\int_{\gamma_2}g$ for any analytic function g in $\text{ann}(a;R_1,R_2)$ (by Cauchy's Theorem) so that the $a_n$s are independent of r with $R_1<r<R_2$.

Define $f_2:B(a;R_2)\to\mathbb{C}$ by

$$f_2(z)=\frac{1}{2\pi i}\int_{|w-a|=r_2}\frac{f(w)}{w-z}dw$$

where $|z-a|<r_2$, $R_1<r_2<R_2$. Clearly, this is well defined. By Lemma $f_2$ is analytic in $B(a;R_2)$. Let

$$G = \{z \in \mathbb{C} / |z-a| > R_1\}$$

Define $f_1 : G \to \mathbb{C}$ by

$$f_1(z) = -\frac{1}{2\pi i} \int_{|w-a|=r_1} \frac{f(w)}{w-z}\, dw$$

where $|z-a| > r_1$, $R_1 < r_1 < R_2$. Clearly, $f_1$ is well defined and is analytic in G.

Let $z \in \operatorname{ann}(a;R_1,R_2)$ i.e. $R_1 < |z-a| < R_2$. Choose $r_1$, $r_2$ such that $R_1 < r_1 < |z-a| < r_2 < R_2$. Let $\gamma_1$ and $\gamma_2$ be the positively oriented circles $|z-a| = r_1$, $|z-a| = r_2$ respectively i.e. $\gamma_1(t) = a + r_1 e^{it}$, $\gamma_2(t) = a + r_2 e^{it}$ $(0 \le t \le 2\pi)$. Choose a straight line segment $\lambda$ going from a point on $\gamma_1$ onto a point on $\gamma_2$ not passing through $z$. Clearly, $\gamma_1 \sim \gamma_2$ in $\operatorname{ann}(a;R_1,R_2)$. Put $\gamma = \gamma_2 - \lambda - \gamma_1 + \lambda$. Then $\gamma$ is a closed curve and $\gamma \sim 0$ i.e. $\gamma$ is homotopic to zero. Clearly, $n(\gamma_1;z) = 0$ and $n(\gamma_2;z) = 1$. For any w not in $\operatorname{ann}(a;R_1,R_2)$, $n(\gamma;w) = 0$. By Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z}\, dw$$

$$= f_2(z) + f_1(z).$$

Since $f_2$ is analytic in $B(a;R_2)$, $f_2$ has a power series which is valid in $B(a;R_2)$ i.e.

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad (z \in B(a;R_2)) \text{ ----------- } (1)$$

where $a_n = \dfrac{f_2^{(n)}(a)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_2} \frac{f(w)\,dw}{(w-a)^{n+1}}$

for $n = 0, 1, 2, \cdots$

Define $g : B'\left(0;\frac{1}{R_1}\right) \to \mathbb{C}$ by

$$g(z) = f_1\left(a+\frac{1}{z}\right),$$

$z \in B'\left(0;\frac{1}{R_1}\right)$ implies $0<|z|<\frac{1}{R_1}$ i.e. $\left|\frac{1}{z}\right|>R_1$

i.e. $\left|\frac{1}{z}+a-a\right|>R_1$ i.e. $\frac{1}{z}+a \in G$. Since $f_1$ is analytic in G, $g$ is analytic in $B'\left(0;\frac{1}{R_1}\right)$ i.e $z=0$ is an isolated singularity of $g$. Now, we show that $g$ has removable singularity at $z=0$.

Let $r>R_1$. Let $e(z)=d(z,C)$, where C is the circle $|z-a|=r$. Put $M=\max\left\{|f(w)| \middle/ w\in C\right\}$ (Which exists since f is continuous on the compact set C). Then, for any z with $|z-a|>r$,

$$|f_1(z)| = \left|\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}dw\right|$$

$$\leq \frac{1}{2\pi}\times\frac{M}{e(z)}\times 2\pi r \quad \text{(for any w on C, } |w-z|\geq d(z,C)=e(z))$$

$$=\frac{Mr}{e(z)}$$

Clearly, $\lim\limits_{z\to\infty} e(z)=\infty$ and hence $\lim\limits_{z\to\infty} f_1(z)=0$.

So, $\lim\limits_{z\to 0} g(z) = \lim\limits_{z\to 0} f_1\left(a+\frac{1}{z}\right)=0$

Thus, $g$ has removable singularity at $z=0$. Since $\lim\limits_{z\to 0} g(z)=0$, if we define $g(0)=0$

then $g$ is analytic in $B\left(0;\frac{1}{R_1}\right)$. So, $g$ has power series representation about 0 i.e.

$$g(z)=\sum_{n=1}^{\infty} B_n z^n \qquad \left(z\in B\left(0;\frac{1}{R_1}\right)\right)$$

Now, for $z\in G$ i.e. $|z-a|>R_1$,

$$f_1(z)=f_1\left(a+\frac{1}{(z-a)^{-1}}\right)=g\left((z-a)^{-1}\right)$$

$$=\sum_{n=1}^{\infty} B_n (z-a)^{-n}$$

$$=\sum_{n=-\infty}^{-1} B_{-n}(z-a)^n$$

$$=\sum_{n=-\infty}^{-1} B_{-n}(z-a)^n$$

$$=\sum_{n=-\infty}^{-1} a_n (z-a)^n \qquad \text{(where } a_n=B_{-n}\text{)} \text{ -------------- (2)}$$

Let $R_1<r<R_2$. Now

$$B_n=\frac{g^{(n)}(0)}{n!}=\frac{1}{n!}\cdot\frac{n!}{2\pi i}\int_{|w|=\frac{1}{r}\left(<\frac{1}{R_1}\right)}\frac{g(w)\,dw}{(w-0)^{n+1}}$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{g\left(\frac{1}{r}e^{it}\right)\cdot\frac{1}{r}e^{it}\cdot i\,dt}{\frac{1}{r^{n+1}}\cdot e^{it(n+1)}}$$

$$=\frac{r^n}{2\pi}\int_0^{2\pi}\frac{f_1\left(a+re^{-it}\right)dt}{e^{itn}}$$

$$=\frac{r^n}{2\pi}\int_\Gamma\frac{f_1(z)\,dz}{-i(z-a)\left(\frac{r}{z-a}\right)^n}\quad \left(\text{where } \Gamma(t)=a+re^{-it}\ (0\le t\le 2\pi)\right)$$

$$=-\frac{1}{2\pi i}\int_\Gamma\frac{f_1(z)}{(z-a)^{-n+1}}\,dz.$$

Let $n<0$.

$$a_n=B_{-n}=-\frac{1}{2\pi i}\int_\Gamma\frac{f_1(z)}{(z-a)^{n+1}}dz$$

Since $\dfrac{f_2(z)}{(z-a)^{n+1}}$ is analytic in $B(a;R_2)$

$$\int_\Gamma\frac{f_2(z)}{(z-a)^{n+1}}dz=0 \text{ (by Cauchy's Theorem).}$$

So, $$a_n=-\frac{1}{2\pi i}\int_\Gamma\frac{f_1(z)+f_2(z)}{(z-a)^{n+1}}dz$$

$$=-\frac{1}{2\pi i}\int_\Gamma\frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$=\frac{1}{2\pi i}\int_{-\Gamma}\frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$=\frac{1}{2\pi i}\int_\gamma\frac{f(w)\,dw}{(w-a)^{n+1}}$$ (where $\gamma=-\Gamma$ i.e. $\gamma:[0,2\pi]\to\mathbb{C}$ is the circle defined by $\gamma(t)=a+re^{it}$)

Hence, $f(z)=f_1(z)+f_2(z)$

$$= \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{------------ (3)}$$

By the convergence properties of the series (1) and (2) on the proper annuli, (3) converges uniformly and absolutely on proper smaller annuli.

Now, we prove the uniquness of $a_n s$,

Assume that

$$f(z)= \sum_{n=-\infty}^{\infty} b_n (z-a)^n \quad (z \in \operatorname{ann}(a; R_1, R_2))$$

Let m be an integer. Let $\gamma$ be the positively oriented circle with center 'a' and radius 'r' where $R_1 < r < R_2$. Now,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)dw}{(w-a)^{m+1}} = \sum_{n=-\infty}^{\infty} b_n \frac{1}{2\pi i}\int_\gamma (z-a)^{n-(m+1)} dz = b_m$$

{since : the value of the integral is 0 for $n-(m+1) \neq -1$ (as $(z-a)^{n-(m+1)}$ has a primitive) and is $2\pi i$ otherwise}.

Hence, $a_m = b_m$. Thus, the series (3) is unique.

**26.1.4.1 COROLLARY :**

Let $z=a$ be an isolated singularity of $f$ and let

$f(z)= \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ be its Laurent series expansion in $\operatorname{ann}(a;0,\delta)(=B'(a;\delta))$. Then :

(a) $z=a$ is a removable singularity of $f$ iff $a_n=0$ for $n \le -1$;

(b) $z=a$ is a pole of order m iff $a_{-m} \neq 0$ and $a_n=0$ for $n \le -(m+1)$;

(c) $z=a$ is an essential singularity of $f$ iff $a_n \neq 0$ for infinitely many negative integers $n$.

**Proof : (a)** Assume that $a_n = 0$ for $n \le -1$. Define $g$ on $B(a;\delta)$ by

$$g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

By a known theorem, $g$ is analytic in $B(a;\delta)$. Clearly, $g(z) = f(z)$ for all $z$ in $B'(a;\delta)$. So, $f$ has removable singularity at $z = a$.

Conversely assume that $f$ has removable singularity at $z = a$. So, there exists an analytic function $g$ on $B(a;\delta)$ such that $f(z) = g(z)$ for all $z$ in $B'(a;\delta)$. By Theorem 17.3.10, $g$ has power series expansion in $B(a;\delta)$. Let it be

$$g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

Since $f(z) = g(z)$ for all $z$ in $B'(a;\delta)$, we have that $a_n = b_n$ $(n \ge 0)$ and $a_n = 0$ for $n \le -1$.

**(b)** Assume that $f$ has a pole of order m at $z = a$. So, m is the least positive integer such that $(z-a)^m f(z)$ has removable singularity at $z = a$. So, $(z-a)^m f(z)$ has power series expansion about, $z = a$ in $B(a;\delta)$ i.e.

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \left(z \in B(a;\delta)\right).$$

For any $z$ in $B'(a;\delta)$,

$$f(z) = \frac{b_0}{(z-a)^m} + \frac{b_1}{(z-a)^{m-1}} + \cdots + \frac{b_{m-1}}{z-a} + \sum_{n=0}^{\infty} b_{n+m} (z-a)^n$$

But $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad \left(z \in B'(a;\delta)\right)$$

So, $a_n = 0$ for $n \le -(m+1)$. Clearly, $b_0 = a_{-m} \ne 0$ (otherwise $(z-a)^{m-1} f(z)$ has removable singularity, a contradiction to that m is the least positive integer such that $(z-a)^m f(z)$ has removable singularity).

Conversely assume that $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$. So,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-a)^n \qquad (z \in B'(a;\delta))$$

Define $g : B(a;\delta) \to \mathbb{C}$ by

$$g(z) = a_{-m} + a_{-(m-1)}(z-a) + \cdots\cdots\cdots$$

Clearly, $g(z) = (z-a)^m f(z)$ for $z \in B'(a;\delta)$; $g(z)$ has power series expansion (which converges in $B(a;\delta)$) and hence analytic in $B(a;\delta)$. So, $(z-a)^m f(z)$ has removable singularity at $z = a$. If $j$ is a positive integer such that $j < m$, then

$$(z-a)^j f(z) = \sum_{n=-m}^{\infty} a_n (z-a)^{n+j}$$

$$= \frac{a_{-m}}{(z-a)^{m-j}} + \cdots\cdots\cdots$$

and hence $(z-a)^j f(z)$ has no removable singularity (since $a_{-m} \neq 0$). So $f$ has a pole of order m at $z = a$.

**(c)** Assume that $f$ has an essential singularity at $z = a$. Suppose $a_n \neq 0$ holds only for finitely many negative integers n. Then we can choose a positive integer m such that $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$. By (b), $f$ has a pole of order m at $z = a$, a contradiction to our assumption that $f$ has an essential singularity at $z = a$.

Conversely assume that $a_n \neq 0$ for infinitely negative integers n. By (a), $f$ cannot have removable singularity at $z = a$. By (b), $f$ cannot have a pole at $z = a$. So, $f$ has an essential singularity at $z = a$.

In the following examples 26.1.5, 26.1.6 and 26.1.7, we first write the Laurent series expansion about the isolated singular points and then determine the type of singularity in each example.

**26.1.5 EXAMPLE :**

Consider the function

$$f(z)=\frac{e^z-1}{z}$$

Clearly, $f$ is not defined at $z=0$ and $f$ is analytic at every $z\neq 0$. So $f$ has an isolated singularity at $z=0$. We, now write the Laurent Series Expansion for $f(z)$ in $\text{ann}(0;0,\infty)$. We know that for any $z\in\mathbb{C}$

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots\cdots$$

For any $z\neq 0$ in $\mathbb{C}$,

$$f(z)=\frac{e^z-1}{z}=\frac{1}{1!}+\frac{z}{2!}+\frac{z^2}{3!}+\cdots\cdots$$

This is Laurent Series Expansion of $f(z)$ in $\text{ann}(0;0,\infty)$. Since this Laurent Series Expansion contain no negative powers of $z-0$ (i.e. $z$), $f(z)$ has removable singularity at $z=0$.

**26.1.6 EXAMPLE :**

Consider the function

$$f(z)=\frac{\cos z}{z^4}$$

Clearly, $f$ is analytic at every $z\neq 0$ in $\mathbb{C}$ and $f$ is not defined at $z=0$. So, $f$ has an isolated singularity at $z=0$. We know that for any $z$ in $\mathbb{C}$,

$$\cos z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

$$=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\cdots$$

For $z\neq 0$ in $\mathbb{C}$,

$$f(z)=\frac{\cos z}{z^4}=\frac{1}{z^4}-\frac{1}{2!}\frac{1}{z^2}+\frac{1}{4!}-\frac{1}{6!}z^2+\cdots\cdots$$

This is the Laurent Series Expansion of $f(z)$ in $\operatorname{ann}(0;0,\infty)$. Clearly, $f$ has a pole of order 4 at $z=0$.

**26.1.7 EXAMPLE :**

Consider the function

$$f(z)=\exp\left(z^{-1}\right)\left(=e^{\frac{1}{z}}\right).$$

Clearly, $f$ is analytic at $z\neq 0$ in $\mathbb{C}$ and $f$ is not defined at $z=0$. So, $f$ has an isolated singularity at $z=0$. For any $z\neq 0$ in $\mathbb{C}$,

$$f(z)=\exp\left(z^{-1}\right)=\sum_{n=0}^{\infty}\frac{\left(z^{-1}\right)^{n}}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{1}{z^{n}}$$

This is Laurent Series Expansion of $f(z)$ in $\operatorname{ann}(0;0,\infty)$. Since this series contains infinite number of negative powers of $z-0(=z)$, $f$ has an essential singularity at $z=0$.

**26.1.8 EXAMPLE :**

Consider the function

$$f(z)=\frac{1}{z(z-1)(z-2)}$$

We now, write all possible Laurent Series Expansions. Clearly, $z=0,1,2$ are isolated singular points of $f$. At each of these points $f$ has a simple pole (This fact is clear).

Now, we obtain partial fraction expansions for $f(z)$. Let

$$f(z)=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}$$

So, $\quad 1\equiv A(z-1)(z-2)++Bz(z-2)+Cz(z-1)$

$z=0 \Rightarrow 1 = A(-1)(-2)$ i.e $A = \frac{1}{2}$

$z=0 \Rightarrow 1 = B(-1)$ i.e $B=-1$

$z=2 \Rightarrow 1 = C\cdot 2\cdot(1)$ i.e $C = \frac{1}{2}$

Hence $f(z) = \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\cdot\frac{1}{z-2}$.

I: About $z=0$, we have Laurent Series Expansions in

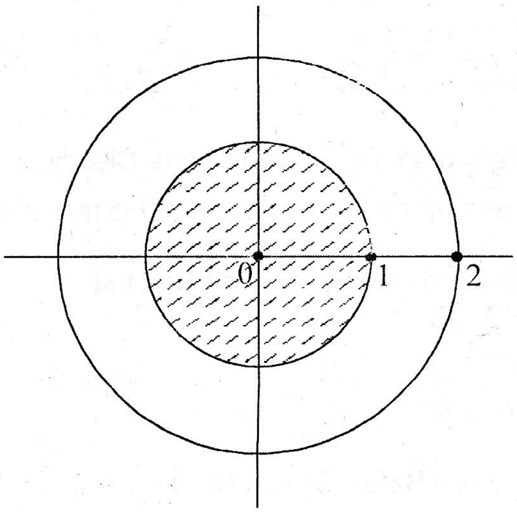
(i) $\operatorname{ann}(0;0,1)$ (ii) $\operatorname{ann}(0;1,2)$ (iii) $\operatorname{ann}(0;2,\infty)$

(i) $\operatorname{ann}(0;0,1)$ : Let $z \in \operatorname{ann}(0;0,1)$ i.e. $0<|z-0|<1$. i.e. $0<|z|<1$.

$$f(z) = \frac{1}{2}\cdot\frac{1}{z} + \frac{1}{1-z} - \frac{1}{4}\cdot\frac{1}{1-\frac{z}{2}}$$

$$= \frac{1}{2}\cdot\frac{1}{z} + \sum_{n=0}^{\infty} z^n - \frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \frac{1}{2z} + \sum_{n=0}^{\infty}\left[1-\frac{1}{2^{n+2}}\right]z^n$$

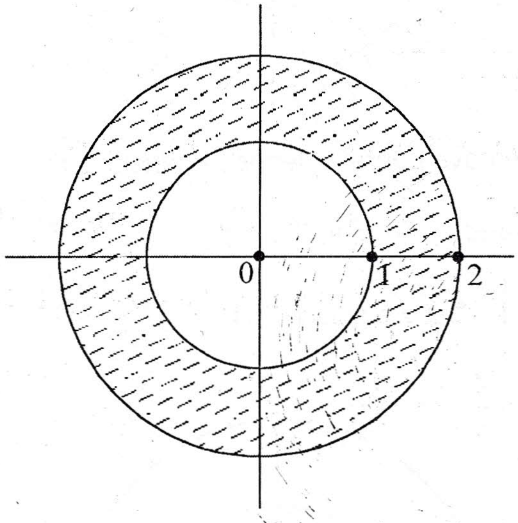



(ii) In $\operatorname{ann}(0;1,2)$ : Let $z \in \operatorname{ann}(0;1,2)$ i.e. $1<|z|<2$

$$f(z) = \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} - \frac{1}{4}\cdot\frac{1}{1-\frac{z}{2}}$$

$$=\frac{1}{2}\cdot\frac{1}{z}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n-\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$=-\sum_{n=1}^{\infty}\frac{1}{z^{n+1}}-\frac{1}{2}\cdot\frac{1}{z}-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}.$$



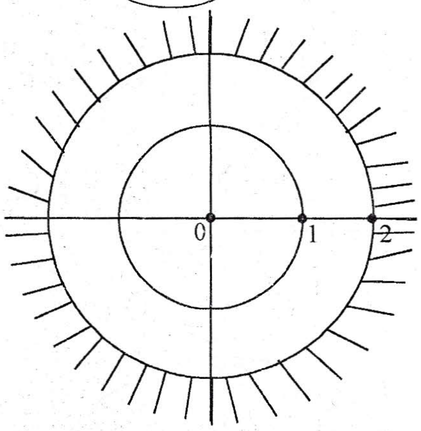

(iii) In $\operatorname{ann}(0;2,\infty)$: Let $z\in\operatorname{ann}(0;2,\infty)$ i.e. $|z|>2$.

$$f(z)=\frac{1}{2}\cdot\frac{1}{z}-\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}+\frac{1}{2z}\cdot\frac{1}{1-\frac{2}{z}}$$

$$=\frac{1}{2}\cdot\frac{1}{z}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n+\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$=\frac{1}{z}\sum_{n=1}^{\infty}\left[-1+2^{n-1}\right]z^{-n}$$

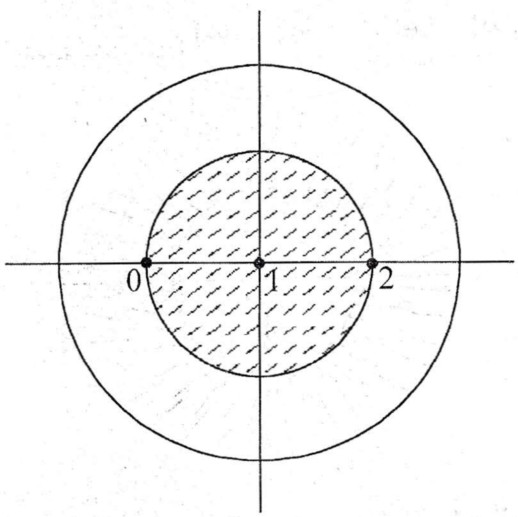
$$=\sum_{n=2}^{\infty}\left[2^{n-1}-1\right]z^{1-n}$$




II. About $z=1$, we have Laurent Series Expansions in

(i) $\operatorname{ann}(1;0,1)$ and (ii) $\operatorname{ann}(1;1,\infty)$

**(i)** In $\operatorname{ann}(1;0,1)$ : Let $z \in \operatorname{ann}(1;0,1)$,




So, $0<|z-1|<1$. Let $u=z-1$.

Then $0<|u|<1$.

Now, $$f(z)=\frac{1}{2}\cdot\frac{1}{z}-\frac{1}{1-z}+\frac{1}{2}\cdot\frac{1}{z-2}$$

$$=\frac{1}{2}\cdot\frac{1}{1+u}-\frac{1}{u}+\frac{1}{2}\cdot\frac{1}{u-1}$$

$$=\frac{1}{2}\cdot\frac{1}{1+u}-\frac{1}{u}-\frac{1}{2}\cdot\frac{1}{1-u}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}(-u)^n-\frac{1}{u}-\frac{1}{2}\sum_{n=0}^{\infty}u^n$$

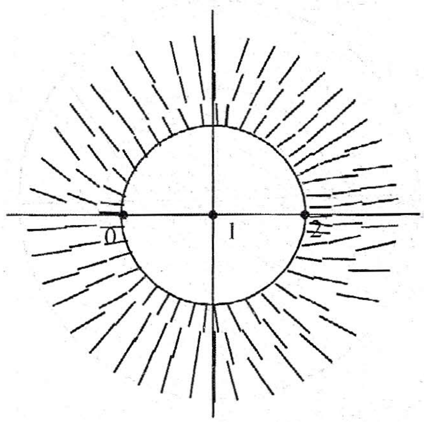
$$=-\frac{1}{u}+\frac{1}{2}\sum_{n=0}^{\infty}\left[(-1)^n-1\right]u^n$$

$$=-\frac{1}{u}+\frac{1}{2}\sum_{\text{n is odd}}-2\,u^n$$

$$=-\frac{1}{u}-\sum_{n=0}^{\infty}u^{2n+1}$$

$$=-\frac{1}{z-1}-\sum_{n=0}^{\infty}(z-1)^{2n+1}$$

**(ii)** In $\operatorname{ann}(1;1,\infty)$ : Let $z\in\operatorname{ann}(1;1,\infty)$




i.e. $|z-1|>1$. Put $u=z-1$. Then $|u|>1$.

$$f(z)=\frac{1}{2}\cdot\frac{1}{1+u}-\frac{1}{u}+\frac{1}{2}\cdot\frac{1}{u-1}$$

$$=\frac{1}{2u}\cdot\frac{1}{1+\frac{1}{u}}-\frac{1}{u}+\frac{1}{2u}\cdot\frac{1}{1-\frac{1}{u}}$$

$$=\frac{1}{2u}\sum_{n=0}^{\infty}\left(-\frac{1}{u}\right)^n-\frac{1}{n}+\frac{1}{2u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n$$

$$= \frac{1}{2u} \sum_{n=1}^{\infty} \left[ (-1)^n + 1 \right] u^{-n} = \frac{1}{2} \sum_{n \text{ is even}} 2u^{-(n+1)}$$

$$= \sum_{n \text{ is even}} u^{-(n+1)} = \sum_{n=1}^{\infty} u^{-(2n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(z-1)^{2n+1}}$$

**III** About $z = 2$, we have Laurent Series Expansions in $\operatorname{ann}(2;0,1)$, $\operatorname{ann}(2;1,2)$, $\operatorname{ann}(2;2,\infty)$.

(i) In $\operatorname{ann}(2;0,1)$ : Let $z \in \operatorname{ann}(2,0,1)$ i.e. $0 < |z-2| < 1$. Put $u = z-2$.

So, $0 < |u| < 1$.

Now, $f(z) = \frac{1}{2} \cdot \frac{1}{u+2} - \frac{1}{u+1} + \frac{1}{2} \cdot \frac{1}{u}$

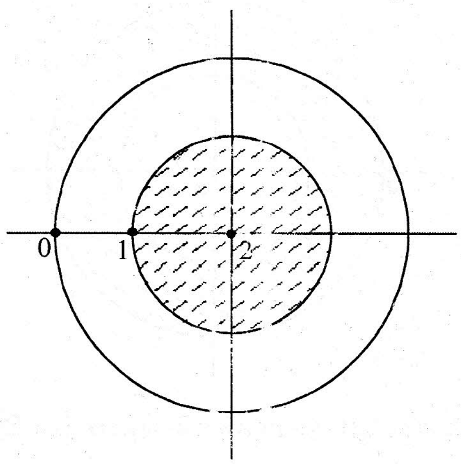
$$= \frac{1}{4} \cdot \frac{1}{1+\frac{u}{2}} - \frac{1}{1+u} + \frac{1}{2} \cdot \frac{1}{u}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} \left( -\frac{u}{2} \right)^n - \sum_{n=0}^{\infty} (-u)^n + \frac{1}{2} \cdot \frac{1}{u}$$

$$= \sum_{n=0}^{\infty} \left[ \frac{1}{2^{n+2}} - 1 \right] (-u)^n + \frac{1}{2u}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{2^{n+2}} - 1 \right] u^n + \frac{1}{2u}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{2^{n+2}} - 1 \right] (z-2)^n + \frac{1}{2(z-2)}$$




(ii) In $\operatorname{ann}(2;1,2)$ : Let $z \in \operatorname{ann}(2;1,2)$ i.e. $1<|z-2|<2$. Put $u=z-2$.

So, $1<|u|<2$.

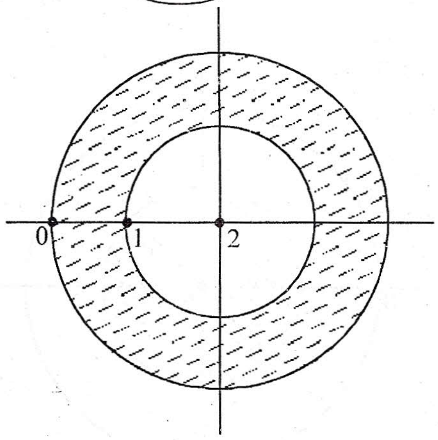
$$f(z)=\frac{1}{2}\cdot\frac{1}{u+2}-\frac{1}{u+1}+\frac{1}{2}\cdot\frac{1}{u}$$

$$=\frac{1}{4}\cdot\frac{1}{1+\frac{u}{2}}-\frac{1}{u}\cdot\frac{1}{1+\frac{1}{u}}+\frac{1}{2}\cdot\frac{1}{u}$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{u}{2}\right)^{n}-\frac{1}{u}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{u}\right)^{n}+\frac{1}{2}\cdot\frac{1}{u}$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{u^{n+1}}-\frac{1}{2}\cdot\frac{1}{u}+\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2^{n+2}}u^{n}$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(z-2)^{n+1}}-\frac{1}{2}\cdot\frac{1}{z-2}+\sum_{n=0}^{\infty}\frac{(-1)^{n}}{2^{n+2}}(z-2)^{n}$$




(iii) In $\operatorname{ann}(2;2,\infty)$ : Let $z \in \operatorname{ann}(2;2,\infty)$ i.e. $|z-2|>2$ i.e. $|u|>2$ where $u=z-2$.

Now, $$f(z)=\frac{1}{2}\cdot\frac{1}{u+2}-\frac{1}{u+1}+\frac{1}{2}\cdot\frac{1}{u}$$

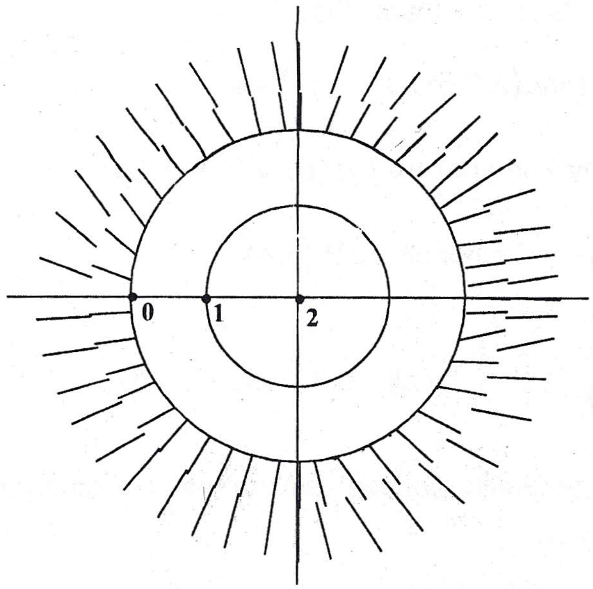
$$=\frac{1}{2u}\cdot\frac{1}{1+\frac{2}{u}}-\frac{1}{u}\cdot\frac{1}{1+\frac{1}{u}}+\frac{1}{2}\cdot\frac{1}{u}$$

$$=\frac{1}{2u}\sum_{n=0}^{\infty}\left(-\frac{2}{u}\right)^n-\frac{1}{u}\sum_{n=0}^{\infty}\left(-\frac{1}{u}\right)^n+\frac{1}{2}\cdot\frac{1}{u}$$

$$=\sum_{n=0}^{\infty}\left[2^{n-1}-1\right](-1)^n\frac{1}{u^{n+1}}+\frac{1}{2}\cdot\frac{1}{u}$$

$$=\sum_{n=1}^{\infty}\left[2^{n-1}-1\right](-1)^n\frac{1}{u^{n+1}}$$

$$=\sum_{n=1}^{\infty}\left[2^{n-1}-1\right](-1)^n\cdot\frac{1}{(z-2)^{n+1}}$$




**26.1.8.1 NOTE :**

In the above example 26.1.8, the singular part of $f$ at

(i) $z=0$ is $\dfrac{1}{2z}$;

(ii) $z=1$ is $-\dfrac{1}{z-1}$; and

(iii) $z=2$ is $\dfrac{1}{2(z-2)}$

**26.1.9 THEOREM (CASORATI-WEIERSTRASS THEOREM) :**

If $f$ has an essential singularity at $z=a$ then for every $\delta>0$,

$$\overline{f(\operatorname{ann}(a;0,\delta))}=\mathbb{C}$$

i.e. $f(\operatorname{ann}(a;0,\delta))$ is dense in $\mathbb{C}$.

**Proof :** Suppose $f$ has an essential singularity at $z=a$. Assume that the theorem is false. So, there exists $\delta>0$ such that

$$\overline{f(\operatorname{ann}(a;0,\delta))}\neq\mathbb{C}.$$

i.e. there exists $w\in\mathbb{C}$ such that w is not in $\overline{f(\operatorname{ann}(a;0,\delta))}$.

Hence, there exists an $\in > 0$ such that

$$f(\text{ann}(a;0,\delta)) \cap B(w;\in) = \phi$$

i.e. for any $z$ in $\text{ann}(a;0,\delta), f(z) \notin B(w;\in)$

i.e. $|f(z)-w| \geq \in$ for all $z \in B'(a;\delta)$

i.e. $\left|\dfrac{1}{f(z)-w}\right| \leq \dfrac{1}{\in}$ for all $z \in B'(a;\delta)$.

Without loss of generality, we can assume that $f$ is analytic in $B'(a;\delta)$ (as $f$ has an isolated singularity at $z=a$).

Define $g$ on $B'(a;\delta)$ by

$$g(z) = \frac{1}{f(z)-w}$$

Clearly, $g$ is analytic and bounded on $B'(a;\delta)$. By Theorem 25.1.4, $g$ has removable singuarity at $z=a$. By defining

$$g(a) = \lim_{z \to a} g(z)$$

$g$ becomes analytic in $B(a;\delta)$.

**Case 1 :** $g(a) \neq 0$: $\dfrac{1}{g(a)} = f(a) - w$ i.e. $f(a) = \dfrac{1}{g(a)} + w$. Since $g(a) \neq 0$ and $g$ is continuous at $z=a$, there exists $\delta_1 > 0$ such that $\delta_1 < \delta$ and $g(z) \neq 0$ for any $z$ in $B(a;\delta_1)$. Define

$$h(z) = \frac{1}{g(z)} + w \qquad (z \in B(a;\delta_1)).$$

Since $g$ is analytic in $B(a;\delta_1)$ and $g(z) \neq 0$ for all $z$ in $B(a;\delta_1)$, we have that $h$ is analytic in $B(a;\delta_1)$. Clearly, $h(z) = f(z)$ for any $z$ in $B'(a;\delta_1)$. So $f$ has removable singularity at $z=a$, a contradiction to that $f$ has essential singularity at $z=a$.

**Case 2 :** $g(a)=0$ : Suppose $z=a$ is a zero of $g$ of multiplicity m. So, there exists an analytic function $h$ on $B(a;\delta)$ such that

$$g(z)=(z-a)^m h(z) \quad (z\in B(a;\delta))$$

and $h(a)\neq 0$. So there exists $\delta_2>0$ such that $\delta_2<\delta$ and $h(z)\neq 0$ for any $z$ in $B(a;\delta_2)$

For any $z$ in $B'(a;\delta_2)$,

$$f(z)-w=\frac{1}{g(w)}=\frac{1}{(z-a)^m h(z)}.$$

Clearly,

$$\lim_{z\to a}|f(z)-w|=\infty.$$

Since $|f(z)-w|\leq|f(z)|+|w|$,

$$\lim_{z\to a}|f(z)|=\infty$$

Thus, $f$ has a pole at $z=0$, a contradiction to that $f$ has an isolated singularity at $z=a$. Hence our assumption is false i.e. the theorem is true.

## 26.2 SINGULARITIES AT INFINITY :

### 26.2.1 DEFINITION :

Let $R>0$ and $G=\{z\in\mathbb{C}/|z|>R\}$. Let $f:G\to\mathbb{C}$ be a function (i.e. $f$ is defined in a neighborhood of $\infty$ except at $\infty$). We say that

(a) $f(z)$ has removable singularity at $\infty$ if $f\left(\frac{1}{z}\right)$ has removable singularity at $z=0$;

(b) $f(z)$ has a pole at infinity if $f\left(\frac{1}{z}\right)$ has a pole at $z=0$; in this case, the pole of order at infinity for $f(z)$ is defined as the pole of order at $z=0$ for $f\left(\frac{1}{z}\right)$.

(c) $f(z)$ has essential singularity at infinity if $f\left(\frac{1}{z}\right)$ has essential singularity at $z=0$.

**26.2.2 THEOREM :**

An entire function has removable singularity at infinity if and only if it is constant.

**Proof :** Let $f$ be an entire function. So,

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \qquad (z\in\mathbb{C}).$$

For any $z\neq 0$ in $\mathbb{C}$,

$$f\left(\frac{1}{z}\right)=\sum_{n=0}^{\infty} a_n \frac{1}{z^n}$$

Now,

$f$ has removable singularity at infinity

$\left(\frac{1}{z}\right)$ has removable singularity at $z=0$

$\Leftrightarrow a_n=0(n=1,2,\cdots\cdots)$ (By Corollary 26.1.4.1(a))

$\Leftrightarrow f(z)=a_0$ for all (for all $z$ in $\mathbb{C}$) i.e. $f$ is constant.

**26.2.3 THEOREM :**

An entire function has a pole of order m at infinity if and only if it is a polynomial of degree m.

**Proof :** Let $f$ be an entire function. So,

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \qquad (z\in\mathbb{C}).$$

For any $z\neq 0$ in $\mathbb{C}$,

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{z^n}$$

Now,

f has a pole of order m at infinity

$\Leftrightarrow f\left(\frac{1}{z}\right)$ has a pole of order m at $z=0$

$\Leftrightarrow a_m \neq 0$ and $a_n = 0$ for $n \geq m+1$

$$\Leftrightarrow f(z) = a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_m z^m$$

i.e. $f$ is a polynomial of degree m.

**26.2.4 THEOREM :**

Consider the rational function

$$r(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials, $q(z) \neq 0$. Then :

(a) $r(z)$ has removable singularity at infinity if and only if $\deg p(z) \leq \deg q(z)$

(b) $r(z)$ has a pole of order m at infinity if and only if $\deg p(z) = m + \deg q(z)$.

**Proof :** Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_n z^n$, $q(z) = b_0 + b_1 z + b_2 z^2 + \cdots\cdots + b_\ell z^\ell$, $a_n \neq 0$, $b_\ell \neq 0$. Now,

$$r\left(\frac{1}{z}\right) = \frac{p\left(\frac{1}{z}\right)}{q\left(\frac{1}{z}\right)} = z^{\ell-n} \times \frac{a_0 z^n + a_1 z^{n-1} + \cdots\cdots + a_n}{b_0 z^\ell + b_1 z^{\ell-1} + \cdots\cdots + b_\ell}$$

$$= z^{\ell-n} f(z)$$

where $f(z)=\dfrac{a_0 z^n + a_1 z^{n-1} + \cdots\cdots + a_n}{b_0 z^\ell + b_1 z^{\ell-1} + \cdots\cdots + b_\ell}$

Clearly, $\lim\limits_{z\to 0} f(z)=\dfrac{a_n}{b_\ell}$.

(a) $r(z)$ has removable singularity at infinites.

$\Leftrightarrow r\left(\dfrac{1}{z}\right)$ has removable singularity at $z=0$

$\Leftrightarrow \lim\limits_{z\to 0} r\left(\dfrac{1}{z}\right)$ exists

$\Leftrightarrow \ell \geq n$ i.e. $\deg p(z) \leq \deg q(z)$.

(b) $r(z)$ has a pole of order m at infinity.

$\Leftrightarrow r\left(\dfrac{1}{z}\right)$ has a pole of order m at $z=0$

$\Leftrightarrow m$ is the least positive integer such that $z^{mr}\left(\dfrac{1}{z}\right)$ has removable singularity and

$\lim\limits_{z\to 0} z^{mr}\left(\dfrac{1}{z}\right) \neq 0$

$\Leftrightarrow m+\ell-n=0$

i.e. $m+\ell=n$

i.e. $m+\deg q(z)=\deg p(z)$

i.e. $\deg p(z)-\deg q(z)=m$.

## 26.3 SHORT ANSWER QUESTIONS :

**26.3.1 :** The Laurent Series Expansion of $f(z)=\dfrac{1}{z}+\dfrac{1}{z^2}$ at the isolated singular point $z=0$ is __________________

**26.3.2 :** The Laurent Series Expansion of $f(z)$ at $z=1$ is

$$f(z)=\sum_{n=0}^{\infty}\frac{(z-1)^{n-3}}{n!}$$

Then the type of singularity of $f(z)$ at $z=1$ is ______________

**26.3.3 :** Write the type of singularity of the function

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{n!z^{n+2}} \text{ at } z=0$$

**26.3.4 :** If $f(z)=e^{\frac{1}{z}}\left(=\exp\left(\frac{1}{z}\right)\right)$ then $\overline{f\left(B'\left(0;\frac{1}{2}\right)\right)}=$ ______________

**26.3.5 :** Write the type of singularity of the function

$$f(z)=\frac{e^z-1}{z^4} \text{ at } z=0.$$

**26.3.6 :** Let $f(z)=\dfrac{1}{(z-1)(z-2)(z-3)}$

Write the annulus in which the Laurent Series expansion is to be considered in the determination of the type of singularity at $z=2$.

**26.3.7 :** State Casorati-Weierstrass Theorem.

**26.3.8 :** Determine the isolated singular points of the function $f(z)=\text{Tan } z$.

**26.3.9 :** Write the type of singularity at infinites of a constant function.

**26.3.10 :** If an entire function $f(z)$ has removable singularities at infinity, what can you say about the function $f(z)$ ?

**26.3.11 :** Write down the type of singularity of a polynomial of degree m at infinity.

**26.3.12 :** Let $f$ be an entire function with pole of order m at infinity. What can you say about $f$ ?

**26.3.13 :** Write down a necessary and sufficient condition for a rational function $r(z)=\dfrac{p(z)}{q(z)}$

to have

(i) removable singularity at infinity

(ii) pole of order m at infinity

**26.3.14 :** State Laurent Series Development Theorem.

**26.3.15 :** State Casorati-Weierstrass Theorem.

**26.3.16 :** Laurent Series Expansion of the function

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^5} \text{ in } \operatorname{ann}(0;0,\infty) \text{ is}$$

**26.3.17 :** Write the annulus in which the Laurent Series Expansion of $f(z) = \exp\left(\frac{1}{z-1}\right)$ is valid.

**26.3.18 :** Define the annulus $\operatorname{ann}(a;R,R_2)$

**26.3.19 :** Is the following statement true ?

"If $f$ has an essential singularity at $z = a$ then for every $\delta > 0$, $f(\operatorname{ann}(a;0,\delta))$ is dense in $\mathbb{C}$"

**26.3.20 :** Write down the type of singularity of the rational function

$$r(z) = \frac{z^3 + z^2 + 1}{z^6 + z^5 + 3} \text{ at infinity.}$$

**26.3.21 :** Let

$$r(z) = \frac{z^6 + z^5 + 1}{z^4 + z^3 + z + 1}$$

What is the order of the pole of $r(z)$ at infinity ?

## 26.4 MODEL EXAMINATION QUESTIONS :

**26.4.1 :** State and prove Laurent Series Development Theorem.

**26.4.2 :** State Laurent Series Development Theorem. Write the Laurent Series Expansion of the function

$$f(z)=\frac{1}{z(z-1)(z-2)}$$

in the annulus $\operatorname{ann}(1;0,1)$ and hence determine the type of singularity of $f$ at $z=1$ and write the singular part of $f$ at $z=1$.

**26.4.3 :** State and prove Casorat-Weierstrass Theorem.

**26.4.4 :** Define the isolated singularity of a function at infinity. Prove that an entire function has removable singularity at infinity if and only if it is constant.

**26.4.5 :** Prove that an entire function has a pole of order m at infinity if and only if it is a polynomial of degree m.

**26.4.6 :** Let $r(z)=\dfrac{p(z)}{q(z)}$ be the rational function (where $p(z), q(z)$ are polynomials, $q(z)\neq 0$). Prove that

(i) $r(z)$ has removable singularity at infinity if and only if $\deg p(z)\leq \deg q(z)$;

(ii) $r(z)$ has a pole of order m at infinity if and only if $\deg p(z)=m+\deg q(z)$.

## 26.5 EXERCISES :

**26.5.1 :** Consider the Exercise 25.4.1. In each of the problems in 25.4.1, obtain Laurent Series Expansion and hence determine the type of singularity at $z=0$.

**26.5.2 :** Suppose $f$ has essential singularity at $z=a$. Prove the following strengthed version of the Casorati-Weierstrass Theorem : If $c\in\mathbb{C}$ and $\in>0$ are given for each $\delta>0$, there is a number $\alpha$ with $|c-\alpha|<\in$ such that $f(z)=\alpha$ has infinitely many solutions in $B(a;\delta)$.

**26.5.3 :** Give Laurent Series Expansion for the function

$$f(z)=\frac{1}{(z-1)(z-2)(z-3)}$$

in each of the following annulii.

(a) $\operatorname{ann}(1;0,1)$ (b) $\operatorname{ann}(2;0,1)$ (c) $\operatorname{ann}(3;0,1)$ (d) $\operatorname{ann}(3;0,1)$

(e) $\operatorname{ann}(3;2,\infty)$ (f) $\operatorname{ann}(2;1,\infty)$

**26.5.4 :** Obtain the singular part of the function $f(z)$ defined in Exercise 26.5.3 at each of its isolated singular points and hence determine the type of singularity.

**26.5.5 :** Consider the Exercise 25.4.1. Determine those problems of Exercise 25.4.1 for which $z=0$ is an essential singular point; compute (or determine) $\overline{f(\{z \in \mathbb{C}/0<|z|<\delta\})}$ for arbitrary small values of $\delta$.

(Hint : See Casorati - Weierstrass Theorem)

**26.5.6 :** Give the Laurent Series Expansion of $f(z)=\exp\left(\frac{1}{z}\right)$. Can you generalize this result?

**26.5.7 :** (a) Let $\lambda \in \mathbb{C}-\{0\}$. Show that $\exp\left(\frac{1}{2}\lambda\left(z+\frac{1}{z}\right)\right)=a_0+\sum_{n=1}^{\infty} a_n\left(z^n+\frac{1}{z^n}\right)$

For $0<|z|<\infty$, where $n \geq 0$

$$a_n=\frac{1}{\pi}\int_0^{\pi} e^{\lambda \cos t} \cos nt\, dt ;$$

(b) Similarly, show that

$$\exp\left(\frac{1}{2}\lambda\left(z-\frac{1}{z}\right)\right)=b_0+\sum_{n=1}^{\infty} b_n\left(z^n+\frac{(-1)^n}{z^n}\right)$$

for $0<|z|<\infty$, where

$$b_n=\frac{1}{\pi}\int_0^{\pi} \cos\,(nt-\lambda \sin t)\,dt$$

**26.5.8 :** Let $G=\{z \in \mathbb{C}/0<|z|<1\}$ and let $f: G \rightarrow \mathbb{C}$ be analytic. Let $\gamma$ be a closed rectifiable curve in G such that $n(\gamma; a)=0$ for all 'a' in $\mathbb{C}-G$. What is $\int_{\gamma} f$ ? why is it so ?

**26.5.9 :** Let f be analytic in $G=\{z \in \mathbb{C}/0<|z-a|<r\}$ except that there is a sequence of poles $\{a_n\}$ in G with $a_n \rightarrow a$. Show that for any w in $\mathbb{C}$, there is a sequence $\{z_n\}$ in G

with $a = \lim z_n$ and $w = \lim f(z_n)$.

**26.5.10 :** Determine the regions in which the functions $f(z) = \left(\sin\frac{1}{z}\right)^{-1}$ and $g(z) = \int_0^1 (t-z)^{-1}\,dt$ are analytic. Do they have any isolated singularities ? Do they have any singularities that are not isolated ?

## 26.6 ANSWERS TO SHORT ANSWER QUESTIONS :

**26.3.1 :** Ifself i.e. $f(z) = \frac{1}{z} + \frac{1}{z^2}$

**26.3.2 :** Pole of order 3 at $z = 1$

**26.3.3 :** Essential singularity

**26.3.4 :** $\mathbb{C}$ (by Casorati - Weierstrass Theorem and $f(z)$ has essential singularity at $z = 0$)

**26.3.5 :** Pole of order 3

**26.3.6 :** $\text{ann}(2;0,1)(= B'(2;1))$

**26.3.7 :** See Statement of the Theorem 26.1.9.

**26.3.8 :** $z = n\pi + \frac{\pi}{2}$ ($n$ is an integer).

**26.3.9 :** Removable singularity (See Theorem 26.2.2)

**26.3.10 :** $f$ is constant (see Theorem 26.2.2).

**26.3.11 :** Pole of order m (see Theorem 26.2.3).

**26.3.12 :** $f$ is a polynomial of degree m (see Theorem 26.2.3)

**26.3.13 :** (i) degree $p(z) \le \deg q/2$ (See Theorem 26.2.4)

(ii) $\deg\, p(z) = \deg q(z) + m$

**26.3.14 :** See Statement of Theorem 26.1.4.

**26.3.15 :** See Statement of Theorem 26.1.9.

**26.3.16 :** $f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^5}$

**26.3.17 :** $\operatorname{ann}(1;0,\infty)\left(=B'(1;\infty)\right.$ = Punctured disk at center 1 with radius $\left.\infty\right)$

**26.3.18 :** See Definition 26.1.3.

**26.3.19 :** Correct (i.e. True). This is nothing but the statement of Casrorati-Weierstrass Theorem.

**26.3.20 :** Removable singularity at infinity since $\deg p(z) \le \deg q(z)$ where $r(z) = \dfrac{p(z)}{q(z)}$ where

$r(z) = \dfrac{p(z)}{q(z)}$ (by Theorem 26.2.4)

**26.3.21 :** Let $r(z) = \dfrac{p(z)}{q(z)}$ where $p(z) = z^6 + z^5 + 1,\ q(z) = z^4 + z^3 + z + 1$. The order of the pole of $r(z)$ at infinity is $\deg p(z) - \deg q(z) = 6 - 4 = 2$.

**REFERENCE BOOK :**

J.B. Conway : Functions of one complex variable - Second Edition - Springer International Student Edition.

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Lesson - 27

# RESIDUE THEOREM

## 27.0 INTRODUCTION

In this lesson, we study the notion of residue and Residue Theorem - which plays as important role in the evaluation of integrals.

## 27.1 RESIDUE THEOREM :

In this section we study Residue Theorem (see 27.1.2) which plays an important role in the evaluation of Integrals. We start with the following.

### 27.1.1 DEFINITION :

Suppose $f$ has an isolated singularity at $z=a$ and let

$$f(z)=\sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

be the Laurent Series Expansion of $f$ about $z=a$ (i.e. in a deleted neighborhood of a i.e. $\operatorname{ann}(a;0,R)$ for some $R>0$). Residue of $f$ at $z=a$ is defined as $a_{-1}$ i.e. the coefficient of $(z-a)^{-1}$. We denote $a_{-1}$ by $\operatorname{Res}(f;a)$.

Now, we prove the cruicial Theorem which will be useful in evaluating the integrals.

### 27.1.2 RESIDUE THEOREM :

Let $f$ be analytic in the region $G$ except for isolated singularities $a_1, a_2, \cdots\cdots, a_n$ in $G$. Let $\gamma$ be a closed rectifiable curve in $G$ not passing through any of the $a_i$s such that $\gamma \approx 0$ in $G$. Then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{j=1}^{n} n(\gamma;a_j)\operatorname{Res}(f;a_j).$$

**Proof :** Let $m_j = n(\gamma;a_j)$ $(1 \le j \le n)$. Choose positive numbers $r_1, r_2, \cdots\cdots, r_n$ such that no two disks $\overline{B(a_j;r_j)}$ intersect and none of them intersect the trace $\{\gamma\}$ of $\gamma$ and each $\overline{B(a_j;r_j)}$ is contained in $G$. For $j=1,2,\cdots\cdots,n$, define $\gamma_j:[0,1]\to\mathbb{C}$ by

$$\gamma_j(t) = a_j + \exp\left(-2\pi i\, m_j t\right).$$

i.e. $\gamma_j$ is the circle centered at $a_j$ with radius $r_j$ taken $m_j$ rounds in the direction of opposite to that of $\gamma$. Clearly, for $1 \le j \le n$,

$$n\left(\gamma_j;\, a_j\right) + \sum_{k=1}^{n} n\left(\gamma_k; a_j\right) = 0$$

Since $\gamma \approx 0$ in $G$ and each $\overline{B}\left(a_j; r_j\right) \subseteq G$, we have that

$$n(\gamma; a) + \sum_{j=1}^{n} n\left(\gamma_j; a\right) = 0$$

for any $a$ not in $G - \{a_1, a_2, \cdots\cdots, a_n\}$. Since $f$ is analytic in $G - \{a_1, \cdots\cdots, a_n\}$, by Cauchy's theorem, we have that

$$\therefore \quad f + \sum_{j=1}^{r} \int_{\gamma_j} f \quad \text{------------} \quad (1)$$

Fix $j$ such that $1 \le j \le n$. Let

$$f(z) = \sum_{n=-\infty}^{\infty} b_n \left(z - a_j\right)^n$$

be the Laurent Series Expansion of $f$ about $z = a_j$. So,

$$\int_{\gamma_j} f = \sum_{n=-\infty}^{\infty} b_n \int_{\gamma_j} \left(z - a_j\right)^n dz = b_{-1} \int_{\gamma_j} \left(z - a_j\right)^{-1} dz$$

(since $\left(z - a_j\right)^n$ has a primitive for $n \ne -1$, we have that $\int_{\gamma_j} \left(z - a_j\right)^n dz = 0$ for $n \ne -1$).

$$= b_{-1} 2\pi i\, n\left(\gamma_j; a_j\right)$$

$$= b_{-1} 2\pi i\left(-m_j\right)$$

$$= -2\pi i\, n(\gamma; a_j)\, b_{-1}$$

$$= -2\pi i\, n(\gamma; a_j)\, b_{-1}$$

$$= -2\pi i\, n(\gamma; a_j) \text{Res}(f; a_j).$$

From (1),

$$\frac{1}{2\pi i}\int_{\gamma} f = -\sum_{j=1}^{n}\frac{1}{2\pi i}\int_{\gamma_j} f$$

$$= \sum_{j=1}^{n} n(\gamma; a_j) \text{Res}(f; a_j)$$

The following theorem gives a principle to evaluate the residue of $f$ at $z=a$, when $f$ has a pole of order m at $z=a$.

**27.1.3 THEOREM :**

Suppose $f$ has a pole of order m at $z=a$. Let $g(z)=(z-a)^m f(z)$. Then

$$\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$$

$$= \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}(g(z))$$

**Proof :** Since $f$ has a pole of order m, m is the least positive integer such that $(z-a)^m f(z)$ has removable singularity at $z=a$. So, $g(z)$ has removable singularity at $z=a$. Without loss of generality, we can assume that $g$ is analytic in a neighborhood of a. So, there exists a neighborhood $B(a;\delta)$ of a in which $g(z)$ has power series expansion. Let it be

$$g(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \qquad (z \in B(a;\delta))$$

For $z$ in $B'(a;\delta)$,

$$f(z)=\frac{g(z)}{(z-a)^m}$$

$$=\frac{b_0}{(z-a)^m}+\frac{b_1}{(z-a)^{m-1}}+\cdots\cdots\cdots\cdots+\frac{b_{m-1}}{z-a}+\sum_{j=0}^{\infty}b_{m+j}(z-a)^j \text{ ----- (1)}$$

Clearly, (1) is the Laurent series Expansion of $f(z)$ about $z=a$ and

$b_0 \neq 0$ (since m is the least positive integer such that $(z-a)^m f(z)$ has removable singularity). By definition,

$$\text{Res}(f;a)=b_{m-1}$$

$$=\frac{g^{(m-1)}(a)}{(m-1)!}=\frac{1}{(m-1)!}\lim_{z\to a} g^{(m-1)}(z)$$

$$=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}z}{dz^{m-1}}.$$

**27.1.3.1 NOTE :**

Suppose $f$ has a simple pole i.e. pole of order 1 at $z=a$. So, 1 is the least positive integer such that $g(z)=(z-a)f(z)$ has removable singularity at $z=a$. So, there exists $\delta>0$ such that

$$g(z)=\sum_{n=0}^{\infty}b_n(z-a)^n \qquad (z\in B(a;\delta))$$

For any $z$ in $B'(a;\delta)$,

$$f(z)=\frac{b_0}{z-a}+b_1+b_2(z-a)+\cdots\cdots\cdots\cdots$$

Clearly,

$$\text{Res}(f;a)=b_0$$

$$=g(a)$$

$$=\lim_{z\to a} g(z)$$

$$= \lim_{z \to a} (z-a) f(z)$$

## 27.2 SHORT ANSWER QUESTIONS :

**27.2.1 :** Suppose a function $f$ has a simple pole at $z=a$ in an open set $G$. Find the residue of $f$ at $z=a$.

**27.2.2 :** If $f$ has a pole of order m at $z=a$ then write the residue of $f$ at $z=a$.

**27.2.3 :** If $f(z) = \frac{1}{z} + e^z$ then what is the residue of $f$ at $z=0$ ?

**27.2.4 :** If $f(z) = \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}$, write the residue of $f$ at $z=1$.

**27.2.5 :** If $f(z) = \frac{1}{z-2} + \frac{1}{(z-2)^4} + \frac{1}{(z-2)^6} + \frac{6}{(z-2)^8}$ then write (a) the order of the pole of $f$ and (b) the residue of $f$ at $z=2$.

**27.2.6 :** Write the residue of $f(z) = e^{\frac{1}{z}}$ at $z=0$.

**27.2.7 :** Evaluate

$$\int_\gamma \exp\left(\frac{1}{z}\right) dz$$

where $\gamma$ is the unit circle $\gamma(t) = e^{it}$ $(0 \le t \le 2\pi)$.

**27.2.8 :** Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

be the Laurent series expansion of $f(z)$ in $B'(a;R)$. Write down the residue of $f$ at $z=a$.

## 27.3 MODEL EXAMINATION QUESTIONS :

**27.3.1 :** Define "Residue". State and prove Residue Theorem.

**27.3.2 :** Find the residues of the following function at its poles.

$$f(z) = \frac{1}{(z-1)(z-2)^3(z-3)^2}$$

**27.3.3 :** Define residue. Obtain the formula for the residue of a function $f(z)$ at $z=a$ when $f$ has a pole of order m at $z=a$.

**27.3.4 :** Evaluate the following integral using (i) Cauchy integral formula and (ii) Residue Theorem.

$$\int_\gamma f(z)\,dz$$

where $\gamma$ is the circle : $\gamma(t) = 2e^{it}$ $(0 \le t \le 2\pi)$ and

$$f(z) = \frac{5z-2}{z(z-1)}.$$

## 27.4 EXERCISES :

**27.4.1 :** In each case, write the pricipal (singular) part of the function at its isolated singular point and determine whether that point is a pole, an essential singular point, or a removable singular point.

(a) $z\exp\left(\frac{1}{z}\right)$ (b) $\frac{z^2}{1+z}$ (c) $\frac{\sin z}{z}$ (d) $\frac{\cos z}{z}$ (e) $\frac{1}{(2-z)^2}$

**27.4.2 :** Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\frac{1-\cos hz}{z^3}$ (b) $\frac{1-\exp(2z)}{z^4}$ (c) $\frac{\exp(2z)}{(z-1)^2}$

(Ans : (a) $m=1,\ B=-\frac{1}{2}$, (b) $m=3, B=-\frac{4}{3}$ (c) $m=2,\ B=2e^2$)

**27.4.3 :** Find the residue at $z=0$ for the following functions.

(a) $\frac{1}{z+z^2}$ (b) $z\cos\left(\frac{1}{z}\right)$ (c) $\frac{z-\sin z}{z}$ (d) $\frac{\cot z}{z^4}$ (e) $\frac{\sin hz}{z^4(1-z^2)}$

(Ans : (a) 1 (b) $-\frac{1}{2}$ (c) 0 (d) $-\frac{1}{45}$ (e) $\frac{7}{6}$)

**27.4.4 :** Use residues to evaluate the following integrals around (over) the circle $\gamma(t) = 3e^{it}\,(0 \le t \le 2\pi)$.

(a) $\dfrac{\exp(-z)}{z^2}$ (b) $z^2 \exp\left(\dfrac{1}{z}\right)$ (c) $\dfrac{z+1}{z^2 - 2z}$

(Ans : (a) $-2\pi i$ (b) $\dfrac{\pi i}{3}$; (c) $2\pi i$)

**27.4.5 :** Let $f$ be a function which is analytic at $z = z_0$. Show that

(a) if $f(z_0) = 0$, then $z_0$ is a removable singular point of the function

$$g(z) = \frac{1}{z - z_0};$$

(b) if $f(z_0) \neq 0$, then $z_0$ is a simple pole (i.e. pole of order one (i.e. 1)) of the function $g(z)$ given in (a).

**27.4.6 :** Evaluate

$$\int_{\gamma} f(z)\,dz$$

where $\gamma$ is the circle : $\gamma(t) = 2e^{it}\,(0 \le t \le 2\pi)$ and when $f(z)$ is

(a) $\dfrac{z^5}{1 - z^2}$ (b) $\dfrac{1}{1 + z^2}$ (c) $\dfrac{1}{z}$

(Ans : (a) $-2\pi i$ (b) 0 (c) $2\pi i$)

## 27.5 ANSWERS TO SHORT ANSWER QUESTIONS :

**27.2.1 :** See Note 27.1.3.1 or

$$\lim_{z \to a} (z - a) f(z)$$

**27.2.2 :** See the statement of theorem 27.1.3.

**27.2.3 :** 1

**27.2.4 :** 1

**27.2.5 :** order = 8, residue = 1

**27.2.6 :** We know that

$$f(z) = \exp\left(\frac{1}{z}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!z^n} \quad (0 < |z| < \infty)$$

So, Residue of $f$ at $z = 0$ is coefficient of $\frac{1}{z}$ i.e. 1.

**22.2.7 :** $$\int_{\gamma} f(z)dz = 2\pi i \text{Res}(f; z = 0)$$

$$= 2\pi i \cdot 1 = 2\pi i$$

**27.2.8 :** Residue of $f$ at $z = \Delta$ i.e. $\text{Res}(f;a) = a_{-1}$.

**REFERENCE BOOK :**

(i) J.B. Conway : Functions of one complex variable - Second Edition - Springer International Student Edition.

(ii) Ruel V. Churchil, James Ward Brown : Complex Variables and Applications - McGraw-Hill International Editions - Fifth Edition.

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Lesson - 28

# EVALUATION OF INTEGRALS

## 28.0 INTRODUCTION

In this lesson, we evaluate integrals using Residue theorem. We prove the relevant theorems ---- (1) Theorem 28.1.1, Jordan Lemma (28.1.3) which play an important role in the evaluation of integrals. We study the procedures of evaluating certain types of integrals mentioned in 28.2.1, 28.2.4 and 28.2.6; further we study the evaluation of some special integrals.

## 28.1 JORDAN'S LEMMA :

In this section, we study theorem 27.2.1, Jordan's inequality (Lemma 28.1.2) and Jordan's Lemma (Lemma 28.1.3), which play an important role in the evaluation of integrals using Residue Theorem (27.1.2).

### 28.1.1 THEOREM :

Let $f$ be analytic except for finite number of singularities which are poles in $\mathbb{C}$. Suppose $z\,f(z) \to \ell$ as $|z| = R \to \infty$. Let $\gamma_R = \left\{ z \in \mathbb{C} / z = Re^{i\theta},\ \theta_1 \le \theta \le \theta_2 \right\}$ i.e $\gamma_R$ is the part of the circle with center 0 and radius R between the angles $\theta_1$ and $\theta_2$ (ofcourse $0 \le \theta_1 \le \theta_2 \le 2\pi$).

Then

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = i\ell(\theta_2 - \theta_1).$$

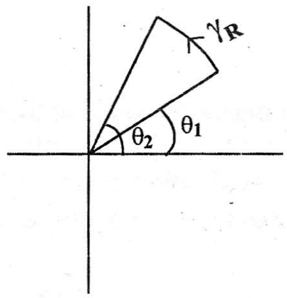
**Proof :** We can choose $R_0$ such that all poles of $f$ lie in the interior of the circle $|z| = R_0$. Let $\in > 0$. Since $zf(z) \to \ell$ as $|z| = R \to \infty$, there exists $R(> R_0)$ such that

$$|z| \ge R \Rightarrow |zf(z) - \ell| < \in.$$

Put $\eta(z) = zf(z) - \ell$. Now, for $|z| \ge R$,

$$\int_{\gamma_R} f(z)\,dz = \int_{\gamma_R} \frac{\eta(z) + \ell}{z}\,dz$$

$$= \int_{\gamma_R} \frac{\eta(z)}{z} dz = \ell \int_{\gamma_R} \frac{dz}{z}$$




$$= \int_{\gamma_R} \frac{\eta(z)}{z} dz + \ell \int_{\theta_1}^{\theta_2} \frac{Re^{i\theta}\, i\, d\theta}{Re^{i\theta}}$$

$$= \int_{\gamma_R} \frac{\eta(z)}{z} dz + i\ell(\theta_2 - \theta_1)$$ and hence

$$\left| \int_{\gamma_R} f(z) dz - i\ell(\theta_2 - \theta_1) \right| = \left| \int_{\gamma_R} \frac{\eta(z)}{z} dz \right|$$

$$\leq \frac{\epsilon}{R} \times \text{length of } \gamma_R$$

(for any $z = Re^{i\theta}$ $(\theta_1 \leq \theta \leq \theta_2)$ on $\gamma_R$, $\left| \frac{\eta(z)}{z} \right| \leq \frac{\epsilon}{R}$)

$$= \frac{\epsilon}{R}(\theta_2 - \theta_1) R = \epsilon(\theta_2 - \theta_1).$$

Hence $\lim_{R \to \infty} \int_{\gamma_R} f(z) dz = i\ell(\theta_2 - \theta_1)$

**28.1.2 LEMMA (JORDAN INEQUALITY) :**

For $0 \leq \theta \leq \frac{\pi}{2}$, $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$.

**Proof :** Define $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$f(\theta) = \begin{cases} \dfrac{\sin\theta}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{otherwise (i.e. } \theta = 0) \end{cases}$$

$$f'(0) = \lim_{\theta \to 0+} \frac{f(\theta) - f(0)}{\theta - 0}$$

$$= \lim_{\theta \to 0+} \frac{\sin\theta - \theta}{\theta^2}$$

$$= \lim_{\theta \to 0} \frac{\cos\theta - 1}{2\theta} \quad \text{(by L'Hospital's Rule)}$$

$$= \lim_{\theta \to 0} \frac{-\sin\theta}{2} \quad \text{(by L'Hospital's Rule)}$$

= 0.

For any $\theta \neq 0$ in $\left[0, \frac{\pi}{2}\right]$,

$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2} = \frac{\cos\theta}{\theta}\left[1 - \frac{\operatorname{Tan}\theta}{\theta}\right]$$

$$\leq 0 \quad \text{(since } \operatorname{Tan}\theta \geq \theta\text{)}$$

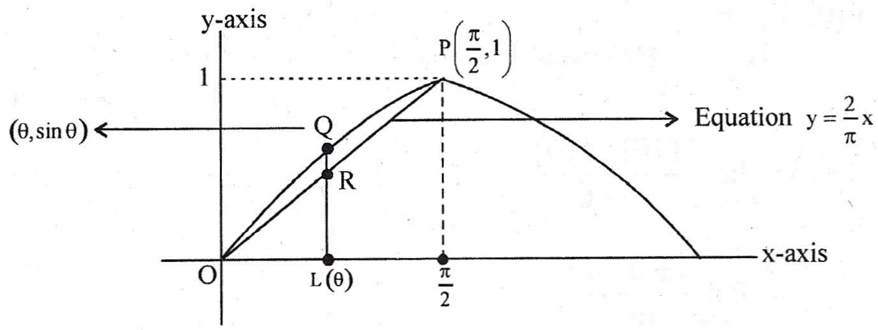
Thus, $f'(\theta) \leq 0$ for all $\theta$ in $\left[0, \frac{\pi}{2}\right]$. So, $f$ is decreasing on $\left[0, \frac{\pi}{2}\right]$ (by Lagrange's Mean value Theorem). So

$$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow f(0) \geq f(\theta) \geq f\left(\frac{\pi}{2}\right)$$

$$\text{i.e. } 1 \geq \frac{\sin\theta}{\theta} \geq \frac{1}{\left(\frac{\pi}{2}\right)} = \frac{2}{\pi}$$

i.e. $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ for $0 < \theta < \frac{\pi}{2}$

**Another Proof :**




The points P, Q, R are $\left(\frac{\pi}{2},1\right)$, $(\theta, \sin\theta)$, $\left(\theta, \frac{2\theta}{\pi}\right)$ respectively (since the equation of OP is $y = \frac{2}{\pi}x$). Clearly,

$$LR \le LQ \le 1$$

i.e. $\frac{2}{\pi}\theta \le \sin\theta \le 1$

and hence

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$$

**28.1.3 LEMMA (JORDAN LEMMA) :**

Let $f(z)$ be analytic except for finite number of poles in the entire complex plane. Suppose $f(z) \to 0$ as $|z| = R \to \infty$. For $m > 0$,

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)e^{imz} = 0$$

Where $\gamma_R$ is the semi circle $|z| = R$ in the upper half plane from +R to -R.

**Proof :** Let $\in > 0$. Since $f(z) \to 0$ as $|z| = R \to \infty$, there exists $R_0 > 0$ such that $R \ge R_0$ implies

$|f(z)| < \in$. Without loss of generality, we can assume that $R_0 > |a|$ for any singular point 'a' of $f(z)$. Now, $R \geq R_0$ implies

$$\left|\int_{\gamma_R} f(z)e^{imz}\,dz\right| = \left|\int_0^{\pi} f\left(Re^{i\theta}\right)e^{imRe^{i\theta}}\,d\left(Re^{i\theta}\right)\right|$$

$$= \left|\int_0^{\pi} f\left(Re^{i\theta}\right)e^{imRe^{i\theta}}\;Re^{i\theta}\;id\theta\right|$$

$$\leq \int_0^{\pi}\left|f\left(Re^{i\theta}\right)\right|\left|e^{imR(\cos\theta+i\sin\theta)}\right|R\,d\theta$$

$$= \int_0^{\pi}\left|f\left(Re^{i\theta}\right)\right|\,e^{-mR\sin\theta}\,Rd\theta$$

$$\leq \int_0^{\pi} \in Re^{-mR\sin\theta}\,d\theta$$

$$= 2R \in \int_0^{\frac{\pi}{2}} e^{-mR\sin\theta}\,d\theta$$

$$\leq 2R \in \int_0^{\frac{\pi}{2}} e^{-mR2\theta/\pi}\,d\theta \quad \text{(By Jordan inequality)}$$

$$= 2R \in \left[\frac{e^{-mR2\theta/\pi}}{\left(\frac{-2mR}{\pi}\right)}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi \in}{m}\left[1-e^{-mR}\right]$$

$$\leq \frac{\pi \in}{m}$$

Hence the conclusion.

## 28.2 EVALUATION OF INTEGRALS :

### 28.2.1 EVALUATION OF INTEGRALS OF THE FORM $\int_{-\infty}^{\infty} f(x)dx$ Where $f(x) = \frac{p(x)}{q(x)}$ :

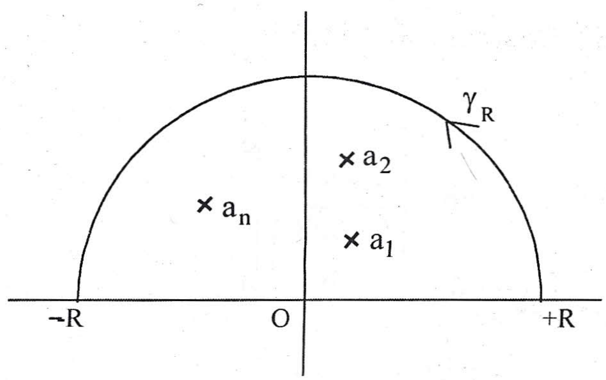
(i) $p(x), q(x)$ are real polynomials;

(ii) $p(x)$ and $q(x)$ have no common zeros;

(iii) $q(x)$ has no real zeros; and

(iv) $\deg q(x) \geq \deg p(x) + 2$.

In this case, we consider $f(z) = \frac{p(z)}{q(z)}$. Clearly, the isolated singularities of $f$ are precisely the zeros of $q(z)$. Since $q(z)$ has no real zeros, each zero of $q(z)$ lies either in the upper half plane or lower half plane. Let $a_1, a_2, \cdots, a_n$ be the zeros of $q(z)$ (i.e. poles of $f$) that lie in the upper half plane (U.H.P.). Choose R such that $R > \text{Max}\{|a_i|/1 \leq i \leq n\}$. Put $\gamma = [-R, R] + \gamma_R$ where $\gamma_R = \{Re^{i\theta}/0 \leq \theta \leq \pi\}$ = semi circle in U.H.P. centered at 0 with radius R from +R to -R.



Clearly, $\gamma$ is a closed curve containing each $a_i \, (1 \le i \le n)$ in the interior of $\gamma$. By Residue Theorem,

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\, dz = \sum_{j=1}^{n} n(\gamma; a_j) \operatorname{Res}(f; a_j)$$

Clearly, $n(\gamma; a_j) = 1$ for $1 \le j \le n$. So,

$$2\pi i \sum_{j=1}^{n} \text{Res } f(f; a_j) = \int_{\gamma} f(z)\, dz$$

$$= \int_{-R}^{R} f(x) dx + \int_{\gamma_R} f(z) dz \text{ --------- (1)}$$

Since deg $q(x) \ge \deg p(x) + 2$, we have that

$$\lim_{|z| \to \infty} z f(z) = \lim_{|z| \to \infty} \frac{z p(z)}{q(z)} = 0$$

By Theorem 28.1.1,

$$\lim_{R \to \infty} \int_{\gamma_R} f(z) dz = \text{i.o.} (\pi - 0) = 0$$

Taking limits on bothsides of (1) as $R \to \infty$, we have

$$2\pi i \sum_{j=1}^{n} \text{Res}(f; a_j) = \int_{-\infty}^{\infty} f(x) dx \cdot$$

**28.2.2 EXAMPLE :**

Now, we evaluate

$$\int_{0}^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx$$

Here the integrand is of the form $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x) = x^2$, $q(x) = x^4 + x^2 + 1$.

(i) Clearly, $p(x)$ and $q(x)$ are polynomials.

(ii) zeros of $p(x)$ are 0, 0. Zeros of $q(x)$ are

$$\alpha_1 = \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \alpha_2 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \alpha_3 = \frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \alpha_4 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

So, $p(x)$ and $q(x)$ have no common zeros.

(iii) Clearly, $q(x)$ has no real zeros.

(iv) $\deg p(x)=2,\ \deg q(x)=4$. So $\deg q(x)=\deg p(x)+2$.

This is of the form given in 28.2.1. Consider the function $f(z)=\frac{p(z)}{q(z)}$. The poles of $f$ (i.e. the zero of $q(z)$) that lie in the upper half plane are $\alpha_1$ and $\alpha_3$. Choose $R>0$ such that $R>\text{Max}\{|\alpha_1|, |\alpha_2|\}=1$. Put $\gamma=[-R, +R] + \gamma_R$, where $\gamma_R$ is the part of the circle in the U.H.P. with center 0 and radius R. Clearly, $\gamma$ is a closed rectifiable curve containing $\alpha_1$ and $\alpha_3$ in its interior. By Residue Theorem,

$$2\pi i\left[\text{Res}(f;\alpha_1)+\text{Res}(f;\alpha_2)\right] = \int_\gamma f$$

$$= \int_{-R}^{R} f(x)dx + \int_{\gamma_R} f(z)dz \text{ ----------- (1)}$$

Clearly, $z\cdot f(z)\to 0$ as $|z|=R\to\infty$ (since $\deg q(x)\geq \deg p(x)+2$).

By Theorem 28.1.1,

$$\lim_{R\to\infty}\int_{\gamma_R} f(z)dz = \text{i.o.}(\pi-0)=0$$

Clearly, $\alpha_1$ and $\alpha_3$ are simple poles of $f$.

$$\text{Res}(f;\alpha_1) = \lim_{z\to\alpha_1}(z-\alpha_1)f(z) \qquad \text{(since } z=\alpha_1 \text{ is a simple pole of } f\text{)}$$

$$= \lim_{z\to\alpha_1} \frac{(z-\alpha_1)z^2}{z^4+z^2+1}$$

$$= \lim_{z\to\alpha_1} \frac{3z^2-2\alpha_1 z}{4z^3+2z} \quad \text{(By L'Hospitals Rule)}$$

$$= \lim_{z\to\alpha_1} \frac{3z-2\alpha_1}{2\left(2z^2+1\right)}$$

$$= \frac{\alpha_1}{2\left(2\alpha_1^{\,2}+1\right)}$$

$$= \frac{1+\sqrt{3}\,i}{4\sqrt{3}\,i}$$

$$\text{Res}\,(f;\alpha_2) = \lim_{z\to\alpha_2} (z-\alpha_2)f(z)$$

$$= \frac{\alpha_2}{2\left(2\alpha_2^{\,2}+1\right)}$$

$$= \frac{1-\sqrt{3}\,i}{4\sqrt{3}\,i}$$

Taking limits as $R \to \infty$ on both sides of (1),

$$2\pi i\left[\frac{1+\sqrt{3}i}{4\sqrt{3}i}+\frac{1-\sqrt{3}i}{4\sqrt{3}i}\right] = \int_{-\infty}^{\infty} f(x)\,dx$$

i.e. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{3}}.$

Since $f(x)$ is an even function,

$$\int_0^{\infty} f(x)\,dx = \frac{\pi}{2\sqrt{3}}$$

### 28.2.3 EXAMPLE :

Prove that

$$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

This integral is of the form $\int_0^{\infty} f(x)dx$ where $f(x) = \frac{p(x)}{q(x)}$, $p(x)=1$, $q(x)=1+x^2$

(i) Clearly, $p(x)$ is a constant polynomial, $q(x)=1+x^2$. The zeros of $q(x)$ are $+i, -i$. The zeros of $q(x)$ are not real.

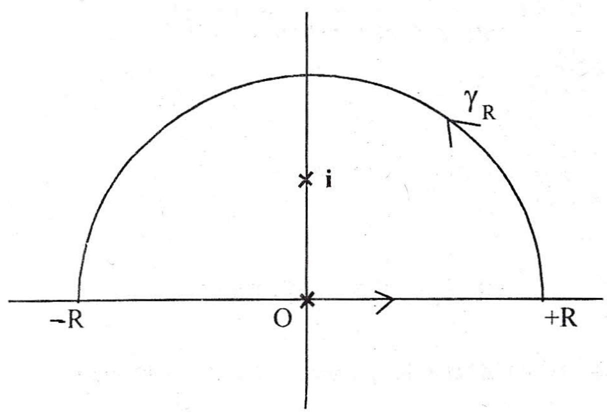
(ii) Clearly $p(x)$ and $q(x)$ have no common zeros.

(iii) Clearly $\deg q(x) = 2 = 0+2 = \deg p(x)+2$

Now, we consider the function $f(z) = \frac{p(z)}{q(z)} = \frac{1}{1+z^2}$.

Clearly, the poles of $f$ are precisely $+i$ and $-i$. Each of these poles is a simple pole of $f$. Choose R such that $R>1$. Put $\gamma = [-R,R]+\gamma_R$ where $\gamma_R$ is the part of the circle $|z|=R$ in the upper half plane from $+R$ to $-R$.

Clearly $\gamma$ is a closed rectifiable curve containing $i$ in its interior. By Residue Theorem




$$\int_{\gamma} f(z)\,dz = 2\pi i \operatorname{Res}(f;i)$$

i.e. $$\int_{-R}^{+R} f(x)dx + \int_{\gamma_R} f(z)dz = 2\pi i \text{ Res}(f;i)$$

i.e. $$2\int_{0}^{R} f(x)dx + \int_{\gamma_R} f(z)dz = 2\pi i \text{Res}(f;i) \text{ ------------ (1)}$$

Since, $\deg q(x) \geq \deg p(x) + 2$, $zf(z) \to 0$ as $|z| = R \to +\infty$,

we have that

$$\lim_{R\to+\infty} \int_{\gamma_R} f(z)dz = i.o\,(\pi - 0) = 0 \text{ ---------- (2).}$$

Now, $\text{Res}(f;i) = \lim_{z\to i} (z-i)f(z)$ $\quad$ ($\because$ $z = i$ is a simple pole of $f$)

$$= \lim_{z\to i} \frac{1}{z+i} = \frac{1}{2i} \text{ ------------- (3)}$$

Taking limit as $|z| = R \to +\infty$ on both sides of (1) and using (2) and (3), we have,

$$\int_{0}^{R} f(x)dx = \frac{\pi}{2}.$$

**28.2.4 EVALUATION OF THE INTEGRALS OF THE FORM :**

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \{\sin mx \text{ or } \cos mx\} dx \text{ or}$$

$$\int_{0}^{\infty} \frac{p(x)}{q(x)} \{\sin mx \text{ or } \cos mx\}$$

where (i) $p(x), q(x)$ are polynomials;

(ii) $p(x)$ and $q(x)$ have no common zeros;

(iii) $q(x)$ has no real zeros;

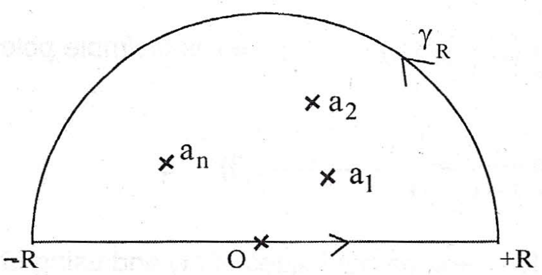
(iv) $\deg q(x) \geq \deg p(x) + 1$ and

(v) $m > 0$.

Consider the function $h(z) = \frac{p(z)}{q(z)} e^{imz}$. The poles of $h$ are precisely the zeros of $q$. Let $a_1, a_2, \cdots, a_n$ be the zeros of $q$ which lie in the U.H.P. Choose R such that

$$R > \text{Max}\{|a_i| / 1 \leq i \leq n\}.$$

Put $\gamma = [-R, R] + \gamma_R$, where $\gamma_R$ is the part of the circle $|z| = R$ in the upper half plane (U.H.P.) from $+R$ to $-R$.




Clearly, $\gamma$ is a closed rectifiable containing $a_1, a_2, \cdots, a_n$ in it's interior. By Residue Theorem,

$$2\pi i \sum_{j=1}^{n} n(\gamma; a_j) \text{ Res}(h; a_j) = \int_{\gamma} h(z)\,dz$$

$$2\pi i \sum_{j=1}^{n} \text{Res}(h; a_j) = \int_{-R}^{+R} h(x)\,dx + \int_{\gamma_R} h(z)\,dz \quad \text{----------- (1)}$$

Now we prove that

$$\lim_{R \to +\infty} \int_{\gamma} h(z)\,dz = 0 \quad \text{---------------- (2)}$$

Write $h(z) = f(z) e^{imz}$ where $f(z) = \frac{p(z)}{q(z)}$. Since $\deg q(x) \geq \deg p(x) + 1$ $f(z) \to 0$ as $|z| = R \to \infty$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_m x^m \; (a_m \neq 0)$,

$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots\cdots + b_n x^n \; (b_n \neq 0)$, $n \geq m+1$. Now,

$$f(z) = \frac{p(z)}{q(z)}$$

$$= \frac{z^m \left[ \frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \cdots\cdots + \frac{a_{m-1}}{z} + a_m \right]}{z^n \left[ \frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \cdots\cdots + \frac{b_{n-1}}{z} + b_n \right]}$$

As $|z| = R \to \infty$, $f(z) \to 0$

By Jordans' Lemma 28.1.3, (2) holds

Taking limits on both sides of (1) as $R \to \infty$,

$$2\pi i \sum_{j=1}^{n} \text{Res}(f; a_j) = \int_{-\infty}^{\infty} h(x)\,dx \qquad \text{(by (2))} \text{ ------------- } (3)$$

Equating the real and imaginary parts on both sides of (3), we have the values of

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos mx \, dx \text{ and } \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin mx \text{ respectively.}$$

**28.2.5 EXAMPLE :**

Now, we evaluate $\int_0^{\infty} \frac{x \sin mx}{x^2 + a^2} dx \; (a > 0)$

Take $p(x) = x$, $q(x) = x^2 + a^2$.

(i) $p(x)$ and $q(x)$ are polynomials

(ii) zero of $p(x)$ is $z = 0$ only. Zeros of $q(x)$ are $\pm ai$. So, $p(x)$ and $q(x)$ have no common zeros.

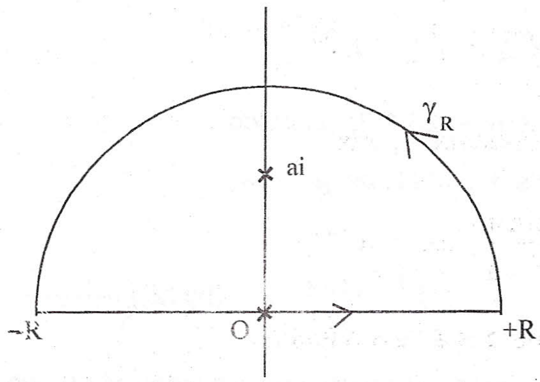
(iii) $q(x)$ has no real zero.

(iv) $\deg q(x) = 2$, $\deg p(x) = 1$. $\deg q(x) = \deg p(x) + 1$

Consider the function $-h(z) = f(z)e^{imz}$, where $f(z) = \dfrac{p(z)}{q(z)}$.

Clearly the poles of $h$ are the zeros of $q(z)$ i.e. $\pm ai$. Clearly, $ai$ is the only pole of $h$ that lies in the U.H.P. Choose a positive real R such that $R > a$. Put $r = [-R, R] U \ \gamma_R$, where $\gamma_R$ is the part of the circle $|z| = R$ in the U.H.P. from $+R$ to $-R$.

Clearly, $\gamma$ is a closed rectifiable curve containing $ai$ in its interior. By Residue Theorem,




$$2\pi i \ \text{Res}(h; ai) = \int_\gamma h$$

$$= \int_{-R}^{R} h(x)dx + \int_{\gamma_R} h(z)dz$$

Res $(h; ai) = \lim\limits_{z \to ai} (z - ai)h(z)$ (since $ai$ is a simple pole of $h$)

$$= \lim_{z \to ai} \frac{z}{z + ai} e^{imz}$$

$$= \frac{1}{2} e^{-am}$$

From (1),

$$2\pi i \cdot \frac{1}{2} e^{-am} = \int_{-R}^{R} h(x)\,dx + \int_{\gamma_R} h(z)\,dz \text{ ----------- (2)}$$

Taking limits as $R \to \infty$ on both sides of (2), we have

$$\pi i e^{-am} = \int_{-\infty}^{\infty} h(x)\,dx$$

$$\deg q(z) = 2 = \deg p(z) + 1 \Rightarrow \frac{p(z)}{q(z)} \to 0 \text{ as } |z| = R \to \infty.$$

By Jordan Lemma, $\lim\limits_{R\to\infty} \int_{\gamma_R} h(z)\,dz = 0$

Equating the imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2}\,dx = \pi e^{-am}$$

Since the integrand here is an even function,

$$\int_{0}^{\infty} \frac{x \sin mx}{x^2 + a^2}\,dx = \frac{1}{2}\pi e^{-am}$$

### 28.2.6 EVALUATION OF THE INTEGRAL OF THE FORM :

$$\int_{0}^{2\pi} F(\cos\theta, \sin\theta)\,d\theta \text{ or } \int_{0}^{\pi} F(\cos\theta, \sin\theta)\,d\theta \text{ or } \int_{-\pi}^{\pi} F(\cos\theta, \sin\theta)\,d\theta.$$

In this case, put $z = e^{i\theta} = \cos\theta + i\sin\theta$. Then,

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta,\ dz = e^{i\theta} \cdot i\,d\theta = iz\,d\theta,$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),\ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Now, $\int\limits_{-\pi}^{\pi}$ or $\int\limits_{0}^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int\limits_{|z|=1} F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{iz}$

We know - how to evaluate this integral.

**28.2.7 EXAMPLE :**

We, now evaluate $\int\limits_{0}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta}\ (a>0)$

$$\int\limits_{0}^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta} = \int\limits_{0}^{\frac{\pi}{2}} \frac{d\theta}{a+\frac{1-\cos 2\theta}{2}} = \int\limits_{0}^{\frac{\pi}{2}} \frac{2d\theta}{1+2a-\cos 2\theta}$$

$$= \int\limits_{0}^{\pi} \frac{dt}{1+2a-\cos t}$$ (on substituting $2\theta = t$)

$$= \frac{1}{2}\int\limits_{-\pi}^{\pi} \frac{dt}{1+2a-\cos t}$$ (since the integrand is an even function)

$$= \frac{1}{2}\int\limits_{|z|=1} \frac{\frac{dz}{iz}}{1+2a-\frac{1}{2}\left(z+\frac{1}{z}\right)}$$ (substitute $z = e^{it}$)

$$= \frac{1}{i}\int\limits_{|z|=1} \frac{dz}{2z(1+2a)-\left(z^2+1\right)}$$

$$= -\frac{1}{i}\int\limits_{|z|=1} \frac{dz}{z^2-(1+2a)2z+1}$$

$$= i\int\limits_{|z|=1} f(z)\,dz$$

where $f(z)=\dfrac{1}{z^2-(1+2a)2z+1}$

Clearly, the poles of $f$ are

$$\frac{2(1+2a)\pm\sqrt{4(1+2a)^2-4}}{2}$$

$$=(1+2a)\pm\sqrt{(1+2a)^2-1}$$

$$=(1+2a)+2\sqrt{a^2+a},\ (1+2a)-2\sqrt{a^2+a}$$

$$=\left(\sqrt{a}+\sqrt{a+1}\right)^2,\ \left(\sqrt{a+1}-\sqrt{a}\right)^2$$

$=\alpha_1,\ \alpha_2$ (respectively say).

Clearly, $\alpha_2$ lies in the interior of $|z|=1$ and $\alpha_1$ lies exterior to $|z|=1$. Further $f$ has a simple pole at $z=\alpha_2$. So,

$$\text{Res}(f;\alpha_2)=\lim_{z\to\alpha_2}(z-\alpha_2)f(z)$$

$$=\lim_{z\to\alpha_2}\frac{1}{z-\alpha_1}=\frac{1}{\alpha_2-\alpha_1}$$

$$=-\frac{1}{4\sqrt{a^2+a}}.$$

By Residue Theorem,

$$\int_{|z|=1} f(z)dz = 2\pi i\,\text{Res}(f;\alpha_2)$$

$$=\frac{-2\pi i}{4\sqrt{a^2+a}}$$

$$\text{Hence, } \int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta} = i\cdot\frac{-2\pi i}{4\sqrt{a^2+a}}$$

$$= \frac{\pi}{2\sqrt{a^2+a}}$$

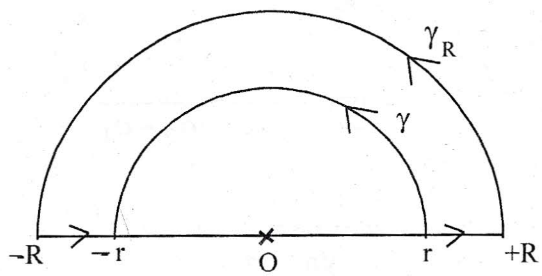
## 28.3 SOME SPECIAL INTEGRALS :

Now, we evaluate some special integrals.

### 28.3.1 EXAMPLE :

We, now evaluate $\int_0^{\infty} \frac{\sin x}{x}\,dx$.

Let $f(z) = \frac{e^{iz}}{z}$. Clearly, $f(z)$ has a pole of order one at $z=0$. Draw two semicircles of radii R and $r(R>r)$ with center 0 in the upper half plane. Put $\gamma = [r,R]+\gamma_R+[-R,-r]-\gamma_r$. Here $\gamma_r$ is the part of the circle $|z|=R$ in the upper half plane from $+R$ to $-R$. Similarly, $\gamma_r$. Clearly,




$\gamma$ is a closed rectifiable curve and $f$ is analytic with in and on $\gamma$. By Cauchy's theorem,

$$0 = \int_\gamma f = \int_r^R \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx - \int_{\gamma_r} \frac{e^{iz}}{z}\,dz$$

$$= \int_r^R \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz - \int_r^R \frac{e^{-ix}}{x}\,dx - \int_{\gamma_r} \frac{e^{iz}}{z}\,dz$$

$$= \int_r^R \frac{e^{ix} - e^{-ix}}{x} dx + \int_{\gamma_R} \frac{e^{iz}}{z} dz - \int_{\gamma_r} \frac{e^{+iz}}{z} dz$$

$$= 2i \int_r^R \frac{\sin x}{x} dx = \int_{\gamma_R} \frac{e^{iz}}{z} dz - \int_{\gamma_r} \frac{e^{iz}}{z} dz.$$

$$\text{So, } 2i \int_r^R \frac{\sin x}{x} dx = \int_{\gamma_r} \frac{e^{iz}}{z} dz - \int_{\gamma_R} \frac{e^{iz}}{z} dz$$

Taking limits as $r \to 0$ and as $R \to \infty$ on both sides,

$$2i \int_0^\infty \frac{\sin x}{x} dx = \lim_{r \to 0} \int_{\gamma_r} \frac{e^{iz}}{z} dz - \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{iz}}{z} dz \text{ ---------- (1)}$$

Now, we prove that

$$\lim_{z \to 0} \int_{\gamma_r} \frac{e^{iz}}{z} dz = \pi i \text{ ------------ (2)}$$

Define $g : \mathbb{C} - \{0\} \to \mathbb{C}$ by

$$g(z) = \frac{e^{iz} - 1}{z}.$$

Clearly, $g$ is analytic in $\mathbb{C} - \{0\}$. Now,

$$\lim_{z \to 0} (z - 0) g(z) = \lim_{z \to 0} \left(e^{iz} - 1\right) = 0.$$

By Theorem 25.1.3, $g(z)$ has removable singularity at $z = 0$. So, $g$ is an entire function. By Cauchy's Theorem,

$$\int_{\gamma_r} g(z) dz = 0$$

i.e. $\int\limits_{\gamma_r} \frac{e^{iz}-1}{z} dz = 0$

i.e. $\int\limits_{\gamma_r} \frac{e^{iz}}{z} dz = \int\limits_{\gamma_r} \frac{1}{z} dz$

$$= \int_0^{\pi} \frac{1}{re^{i\theta}} re^{i\theta} \cdot i\,d\theta$$

$$= i\int_0^{\pi} d\theta$$

$$= i\pi$$

Hence,

$$\lim_{r\to 0} \int\limits_{\gamma_r} g(z)dz = i\pi$$

Now, we prove that

$$\lim_{R\to +\infty} \int\limits_{\gamma_R} \frac{e^{iz}}{z} dz = 0 \text{ --------- (3)}$$

Since $\frac{1}{z} \to 0$ as $|z| = R \to \infty$, by Jordan Lemma (taking m=1), we have (3) holds. Substituting (2) and (3) in (1), we have that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

**28.3.2 EXAMPLE :**

Show that

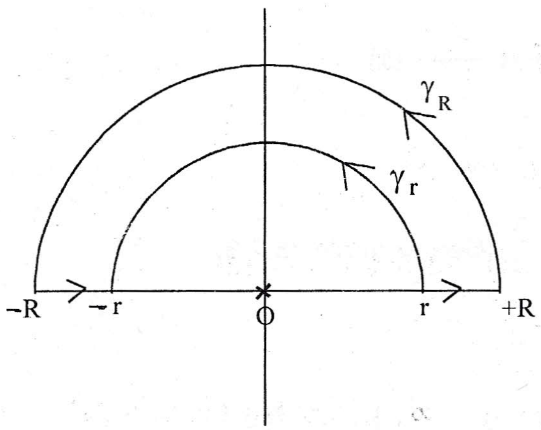
$$\int_0^{\infty} \frac{\log x}{1+x^2} dx = 0$$

To solve this problem, we do not use the principle branch of $\log z$. We consider $\log z$ for z belonging to

$$G=\left\{z\in\mathbb{C}\big/ z\neq 0 \text{ and } -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right\}$$

Let $f(z)=\dfrac{\ell(z)}{1+z^2}$. Clearly, $0,\pm i$ are isolated singular points of $f(z)$ and $f(z)$ has a simple pole at each of the points $i,-i$. Clearly, $i$ is the only isolated singular point (infact pole of $f$) that lies in the upper half plane. Choose $r,R$ such that $0<r<1<R$. Let $\gamma=[-R,-r]-\gamma_r+[r,R]+\gamma_R$ where $\gamma_\rho$ is the part of the circle in the upper half plane from $+\rho$ to $-\rho$ $(\rho=R,r)$. Clearly $\gamma$ is a closed rectifiable curve with in and on which $f$ is analytic except at $z=i$, interior of $\gamma$. By Residue Theorem,

$$\int_\gamma f = 2\pi i \operatorname{Res}(f;i)$$




$\operatorname{Res}(f;i) = \lim\limits_{z\to i}(z-i)f(z)$ (since $z=i$ is a simple pole of $f$).

$$=\lim_{z\to i}\frac{\ell(z)}{z+i}=\frac{\ell(i)}{2i}$$

$$=\frac{i\frac{\pi}{2}}{2i} \qquad \left(\text{since } i=e^{i\frac{\pi}{2}}\right)$$

$$=\frac{\pi}{4}$$

$$\int_{\gamma} f = \int_{-R}^{-r} f(x)\,dx - \int_{\gamma_r} f(z)\,dz + \int_{r}^{R} f(z)\,dz + \int_{\gamma_R} f(z)\,dz$$

$$= \int_{-R}^{-r} \frac{\log|x| + i\pi}{1+x^2}\,dx - \int_{\gamma_r} \frac{\ell(z)\,dz}{1+z^2} + \int_{r}^{R} \frac{\log x\,dx}{1+x^2} + \int_{\gamma_R} \frac{\ell(z)}{1+z^2}\,dz$$

$$= 2\int_{r}^{R} \frac{\log x}{1+x^2}\,dx + i\pi \int_{r}^{R} \frac{dx}{1+x^2} - \int_{0}^{\pi} \frac{\log r + i\theta}{1+r^2 e^{2i\theta}}\, re^{i\theta} \cdot i d\theta + \int_{0}^{\pi} \frac{\log R + i\theta}{1+R^2 e^{ri\theta}} \cdot R\, e^{i\theta} \cdot i\, d\theta$$

$$= 2\int_{r}^{R} \frac{\log x}{1+x^2}\,dx + i\pi \int_{r}^{R} \frac{dx}{1+x^2} - ir\int_{0}^{\pi} \frac{\log r + i\theta}{1+r^2 e^{2i\theta}}\, e^{i\theta}\, d\theta + iR\int_{0}^{\pi} \frac{\log R + i\theta}{1+R^2 e^{2i\theta}}\, e^{i\theta} d\theta$$

$$= 2\pi i \frac{\pi}{4}$$

$$= \frac{\pi^2 i}{2} \text{ ----------------- (1)}$$

We know that

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} \text{ (See example 28.2.3).}$$

Now, we show that

$$\rho \int_{0}^{\pi} \frac{\log \rho + i\theta}{1+\rho^2 e^{2i\theta}} \cdot e^{i\theta}\, d\theta \to 0 \text{ -------- (2)}$$

as either $\rho \to 0$ or $\rho \to \infty$.

Let $\rho > 0$. Now,

$$\left| \rho \int_{0}^{\pi} \frac{\log \rho + i\theta}{1+\rho^2 \rho^{2i\theta}}\, e^{i\theta d\theta} \right| \le \rho|\log \rho| \left| \int_{0}^{\pi} \frac{e^{i\theta}\, d\theta}{1+\rho^2 e^{2i\theta}} \right| + |\rho| \left| \int_{0}^{\pi} \frac{\theta e^{i\theta} d\theta}{1+\rho^2 e^{2i\theta}} \right| \le \frac{\rho|\log \rho|\pi}{|1-\rho^2|} + \frac{\rho\pi^2}{|1-\rho^2|}$$

Whether $e \to 0+$ or $e \to \infty$, we have

$$\frac{\pi\rho\left|\log\rho\right|}{\left|1-\rho^2\right|}+\frac{\rho\pi^2}{2\left|1-\rho^2\right|}\to 0.$$

Hence, (2) holds as $\rho\to 0+$ or $\rho\to\infty$.

Taking limits on both sides of (1) as $r\to 0$ and $R\to\infty$,

$$2\int_0^\infty \frac{\log x}{1+x^2}dx+i\pi\cdot\frac{\pi}{2}-0+0=\frac{\pi^2 i}{2}$$

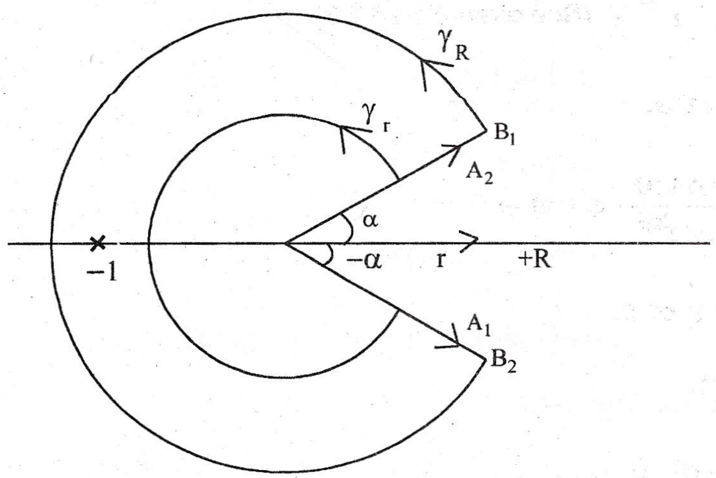
Hence $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}dx=0$.

**28.3.3 EXAMPLE :**

Prove that $\displaystyle\int_0^\infty \frac{x^{-c}}{1+x}dx=\frac{\pi}{\sin\pi c}\ (0<c<1)$.

To evaluate this integral, we must consider the function

$f(z)=z^{-c}=\exp(-c\log z)\ (=\exp(-c\ell(z)))$, when $\ell$ is the branch of the logarithm.




To evaluate the integral, we have to consider the function $g(z)=\dfrac{f(z)}{1+z}$. Clearly $z=-1$ is a simple pole of $g$. If we consider the principal branch of longarithm, then the branch cut contains

$-1$. That is why we consider that branch of logarithm $\ell$ defined on

$$G=\{z\in\mathbb{C}/z\neq 0,\ 0<\arg z<2\pi\}$$

(The branch cut considered here is $\theta=0$ or $\arg z=0$)

So, $\ell$ is defined on G by $\ell\left(r\rho^{i\theta}\right)=\log r+i\theta$ (where $0<\theta<2\pi$). Hence, the $\ell$ which we are considering in the definition of $f$ in this $\ell$ which we are considering in the definition of $f$ in this $\ell$. Choose $r,R$ such that $r<1<R$. Let $\alpha>0$. Consider the closed curve

$$\gamma=\overrightarrow{A_1B_1}+\gamma_R+\left(-\overrightarrow{A_2B_2}\right)+\left(-\gamma_r\right)$$

Where $\gamma_R$ and $\gamma_r$ are the parts of the circles $|z|=R,\ |z|=r$ from $B_1$ to $B_2$ (in anticlockwise direction) and $A_1$ to $A_2$ (in anticlockwise direction) respectively as shown in the figure. Here $[A_1,B_1]=\left[r\,e^{i\alpha},\ Re^{i\alpha}\right],\ [A_2,B_2]=\left[re^{i(2\pi-\alpha)},Re^{i(2\pi-\alpha)}\right]$. Clearly, $g$ is analytic with in and on $\gamma$ except at $z=-1$ interior to $\gamma$ at which $g$ has a simple pole. Clearly, $\gamma\sim 0$ (i.e. $\gamma$ is homotopic to 0) in $G$.

Hence

$$\text{Res}(g;-1)=\lim_{z\to-1}(z+1)g(z)$$

$$=\lim_{z\to-1}f(z)$$

$$=f(-1)$$

$$=\exp(-c\ell(-1))$$

$$=e(-i\pi c)\qquad \text{(since } -1=1e^{i\pi}\text{)}$$

By Residue Theorem

$$\int_\gamma g(z)\,dz=2\pi i\,\text{Res}\left(g_j-1\right)$$

$$=2\pi i\,e^{-i\pi c}$$

i.e. $$\int_{A_1B_1}\frac{f(z)dz}{1+z}+\int_{\gamma_R}\frac{f(z)dz}{1+z}-\int_{A_2B_2}\frac{f(z)dz}{1+z}-\int_{\gamma_R}\frac{f(z)}{1+z}dz=2\pi i e^{-i\pi c} \text{ -------- (1)}$$

$$\int_{A_1B_1}\frac{f(z)}{1+z}dz=\int_{t=r}^{R}\frac{f\left(te^{i\alpha}\right)d\left(te^{i\alpha}\right)}{1+te^{i\alpha}}=e^{i\alpha}\int_{r}^{R}\frac{f\left(te^{i\alpha}\right)}{1+te^{i\alpha}}dt$$

$$=e^{i\alpha(1-c)}\int_{r}^{R}\frac{t^{-c}}{1+te^{i\alpha}}dt \text{ ------------- (2)}$$

$$\int_{A_2B_2}\frac{f(z)}{1+z}dz=\int_{t=r}^{R}\frac{f\left(te^{(2\pi-\alpha)i}\right)}{1+te^{(2\pi-\alpha)i}}d\left(te^{(2\pi-\alpha)i}\right)$$

$$=e^{i(2\pi-\alpha)}\int_{r}^{R}\frac{f\left(te^{i(2\pi-\alpha)}\right)}{1+te^{i(2\pi-\alpha)}}dt$$

$$=e^{i(1-c)(2\pi-\alpha)}\int_{r}^{R}\frac{t^{-c}}{1+te^{i(2\pi-\alpha)}}dt \text{ --------- (3).}$$

Clearly,

$$\lim_{\alpha\to 0}\int_{A_1B_1}\frac{f(z)}{1+z}dz=\int_{r}^{R}\frac{t^{-c}}{1+t}dt \text{ ------------ (4)}$$

$$\lim_{\alpha\to 0}\int_{A_2B_2}\frac{f(z)}{1+z}dz=e^{i(1-c)2\pi}$$

$$=e^{-i2\pi c} \text{ ------------- (5).}$$

Let $\rho>0$ and $\rho\neq 1$. Let $\gamma_\rho$ be the part of the circle $|z|=\rho$ from $\rho e^{i\alpha}$ to $\rho e^{i(2\pi-\alpha)}$. Then

$$\left|\int_{\gamma_\rho}\frac{f(z)}{1+z}dz\right|=\left|\int_{\theta=\alpha}^{2\pi-\alpha}\frac{\left(\rho e^{i\theta}\right)^{-c}}{1+\rho e^{i\theta}}d\left(\rho e^{i\theta}\right)\right|$$

$$= \left| \int_{\theta=\alpha}^{2\pi-\alpha} \frac{\rho^{1-c} e^{i\theta(1-c)}}{1+\rho e^{i\theta}} d\theta \right|$$

$$\leq \frac{\rho^{1-c}}{|1-\rho|} 2\pi$$

Clearly

$$\frac{\rho^{1-c}}{|1-\rho|} \to 0 \text{ as } \rho \to 0 \text{ or as } \rho \to \infty .$$

So, $$\lim_{\substack{\rho\to 0 \\ \text{or} \\ \rho\to\infty}} \int_{\gamma_\rho} \frac{f(z)dz}{1+z} = 0 \text{ ------------ (6)}$$

(i.e. (6) holds and $\rho = r \to 0$ and $\rho = R \to \infty$)

Now taking limit as $\alpha \to 0, r \to 0$ and $R \to \infty$ on both sides of (1), we have

$$\left(1-e^{-i2\pi c}\right) \int_0^\infty \frac{t^{-c}}{1+t} dt = 2\pi i e^{-i\pi c}$$

i.e. $$\int_0^\infty \frac{t^{-c}}{1+t} dt = \frac{2\pi i e^{-i\pi c}}{1-e^{-i2\pi c}}$$

$$= \pi \cdot \frac{2i}{e^{i\pi c} - e^{-i\pi c}}$$

$$= \frac{\pi}{\sin \pi c} .$$

## 28.4 SHORT ANSWER QUESTIONS :

**28.4.1 :** State Jordan inequality.

**28.4.2 :** State Jordan Lemma.

**28.4.3 :** While evaluating $\int_0^{\infty} \frac{dx}{1+x^2}$, we consider the function $f(z)$ where $f(z) =$ ________

**28.4.4 :** Let $f(z)$ be analytic except for finite number of poles in the entire complex plane. Suppose $f(z) \to 0$ as $|z| = R \to \infty$. If $m > 0$ then

$$\lim_{|z|=R\to\infty} \int_{\gamma_R} f(z)\, e^{imz}\, dz = \text{-------------------}$$

where $\gamma_R$ is the semi circle in the upper half plane with center 0 and radius R from $+R$ to $-R$.

## 28.5 MODEL EXAMINATION QUESTIONS :

**28.5.1 :** Evaluate the following integral

(a) $\int_0^{\infty} \frac{x^2}{x^4+x^2+1} dx$ (b) $\int_0^{\infty} \frac{\sin x}{x} dx$

**28.5.2 :** Evaluate the following integrals

(a) $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$ $(a > 0,\ b > 0)$

(b) $\int_0^{\infty} \frac{\log x}{1+x^2} dx$

**28.5.3 :** Prove the following

(a) $\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2} = \frac{\pi a^2}{1-a^2}$

(b) $\int_0^{\infty} \frac{\log x}{\left(1+x^2\right)^2} dx = -\frac{\pi}{4}$

**28.5.4 :** Prove the following

(a) $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

(b) $\int_0^\infty \frac{x^{-c}}{1+x} dx = \frac{\pi}{\sin \pi c} \quad (0 < c < 1)$

## 28.6 EXERCISES :

**28.6.1 :** Evaluate the following integrals

(a) $\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$

(b) $\int_0^\infty \frac{dx}{\left(x^2 + a^2\right)^2} \; (a > 0) \left(= \frac{\pi}{4a^2}\right)$

(c) $\int_0^\infty \frac{dx}{1+x^2} \left(= \frac{\pi}{2}\right)$

(d) $\int_0^\infty \frac{dx}{x^4 + 1} \left(= \frac{\pi}{2\sqrt{2}}\right)$

(e) $\int_0^\infty \frac{x^2 dx}{\left(x^2+1\right)\left(x^2+4\right)} \left(= \frac{\pi}{6}\right)$

(f) $\int_0^\infty \frac{x^2 dx}{x^6 + 1} \left(= \frac{\pi}{6}\right)$

**28.6.2 :** Evaluate the following integrals.

(a) $\int_0^\infty \frac{\cos x - 1}{x^2} dx$

(b) $\int_0^\infty \frac{\cos ax}{\left(1+x^2\right)} dx \left(= \frac{\pi(a+1)e^{-a}}{4}\right) \quad (a > 0)$

(c) $\int_0^\infty \frac{x \sin 2x}{x^2 + 3} dx \left(= \frac{\pi}{2} e^{-2\sqrt{3}}\right)$

(d) $\int_{-\infty}^\infty \frac{x^3 \sin ax}{x^4 + 4} dx \; (a > 0) \left(= \pi \cos a e^{-a}\right)$

(e) $\int_{-\infty}^\infty \frac{\cos x \; dx}{\left(x^2 + a^2\right)\left(x^2 + b^2\right)} \left(= \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)\right)$

**28.6.3 :** Evaluate the following integrals.

(a) $\int_0^\pi \frac{\cos 2\theta \; d\theta}{1 - 2a \cos\theta + a^2} \left(a^2 < 1\right)$

(b) $\int_0^\pi \frac{d\theta}{\left(a + \cos\theta\right)^2} \; (a > 1) \left(= \frac{\pi a^2}{1 - a^2}\right)$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta} \left(=\frac{\pi}{2\sqrt{a(a+1)}}\right)$ (d) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} \left(=\frac{2\pi}{\sqrt{1-a^2}}\right)$

(e) $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta \left(=\frac{(2n)!}{2^{2n}(n!)^2}\,\pi\right)$

**28.6.4 :** Evaluate the following Integrals

(a) $\displaystyle\int_0^{\infty} \frac{(\log x)^3}{1+x^2}\,dx \;(=0)$ (b) $\displaystyle\int_0^{\infty} \frac{\log x}{\left(1+x^2\right)^2}\,dx \left(=-\frac{\pi}{4}\right)$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx \;(0<a<1) \left(=\frac{\pi}{\sin a\pi}\right)$

(d) $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx \left(=\frac{\pi}{2}\right)$

**28.6.5 :** Show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin a\pi} \text{ if } 0<a<1$$

**28.6.6 :** Prove that

$$\int_0^{2\pi} \log \sin^2 2\theta\, d\theta = 4\int_0^{\pi} \log \sin\theta\, d\theta = -4\pi\log 2$$

**28.6.7 :** Find all possible values of

$$\int \exp\left(z^{-1}\right) dz$$

where $\gamma$ is any closed curve not passing through $z=0$.

**28.6.8 :** Suppose that $f$ has a simple pole at $z=a$ and let $g$ be analytic in an openset containing $a$. Show that

$$\text{Res}(fg;a) = g(a)\ \text{Res}(f;a).$$

**28.6.9 :** Let G be a region and let $f$ be analytic in G except for simple poles at $a_1, a_2, \cdots, a_n$ and also let $g$ be analytic in G. Show that

$$\frac{1}{2\pi i}\int_\gamma fg = \sum_{k=1}^{n} n(\gamma;a_k)\, g(a_k)\operatorname{Res}(f;a_k)$$

for any closed rectifiable curve $\gamma$ not passing through $a_1, a_2, \cdots, a_n$ such that $\gamma \approx 0$ in G.

**28.6.10 :** Let $\gamma$ be the rectangular path

$$\left[n+\frac{1}{2}+ni,\ -n-\frac{1}{2}+ni, -n-\frac{1}{2}-ni,\ n+\frac{1}{2}-ni,\ n+\frac{1}{2}+ni\right].$$

(a) Evaluate the integral

$$\int_\gamma \pi(z+a)^{-2} \cot \pi z\, dz \quad \text{for } a \neq \text{an integer.}$$

(b) Show that

$$\lim_{n\to\infty} \int_\gamma \pi(z+a)^{-2} \cot \pi z\, dz = 0$$ and deduce that

$$\frac{\pi^2}{\sin^2 a\pi} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$$

(**Hint** : Use the fact that for $z = x+iy$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$ to show that $|\cot \pi z| \leq 2$ for $z$ or $\gamma$ if $n$ is sufficient by large)

**28.6.11 :** Using exercise 28.6.10, prove that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

**28.6.12 :** Let $\gamma$ be the polygonal path defined in exercise 28.6.10. Evaluate

$$\int_\gamma \pi\left(z^2-a^2\right)^{-1} \cot \pi z\, dz \quad \text{for } a \neq \text{an integer. Show that}$$

$$\lim_{n\to\infty} \int_{\gamma} \pi\left(z^2 - a^2\right)^{-1} \cot \pi z \, dz = 0 \text{ and consequently}$$

$$\pi \cot \pi a = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2} \text{ for } a \neq \text{an integer.}$$

**28.7 ANSWERS TO SHORT ANSWER QUESTIONS :**

**28.4.1 :** See Statement of Lemma 28.1.2.

**28.4.2 :** See Statement of Lemma 28.1.3.

**28.4.3 :** $f(z) = \dfrac{1}{1+z^2}$

**28.4.4 :** 0 (by Jordan Lemma).

**REFERENCE BOOK :**

J.B. Conway : Functions of one complex variable - Second Edition - Springer International Student Edition.

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Lesson - 29

# THE ARGUMENT PRINCIPLE AND ROUCHE'S THEOREM

## 29.0 INTRODUCTION

In this lesson, we study - Argument Principle (see Theorem 29.1.3). Furhter, we study Rouche's Theorem (see 29.1.9) and it's consequence - Fundamentai Theorem of algebra (see Corollary 29.1.9.1)

## 29.1 THE ARGUMENT PRINCIPLE AND ROUCHE'S THEOREM :

Now, we start this section with the following :

### 29.1.1 DISCUSSION :

(i) Suppose $f$ is analytic in a neighborhood of $a$ and $a$ in zero of $f$ of order m. So, $f(z)=(z-a)^m g(z)$ for same analytic function $g(z)$ in a neighborhood of $a$ such that $g(a)\neq 0$. Since $g$ is continuous at $z=a$, there exists $\delta>0$ such that $g(z)\neq 0$ for any $z$ in $B(a;\delta)$. For any $z$ in $B(a;\delta)$,

$$f'(z)=m(z-a)^{m-1}\ g(z)+(z-a)^m g'(z).$$

For any $z$ in $B'(a;\delta)$,

$$\frac{f'(z)}{f(z)}=\frac{m}{z-a}+\frac{g'(z)}{g(z)}$$

Clearly, $\dfrac{g'(z)}{g(z)}$ is analytic in $B(a;\delta)$.

(ii) Suppose $f$ has a pole of order m at $z=a$. So, there exists an analytic function $g(z)$ with $g(a)\neq 0$ and $f(z)=(z-a)^{-m} g(z)$. Since $g$ is continuous at $a$ and $g(a)\neq 0$, there exists $\delta>0$ such that $g(z)\neq 0$ for any $z$ in $B(a;\delta)$. For any $z$ in $B'(a;\delta)$,

$$f'(z)=-m(z-a)^{-m-1}+(z-a)^{-m} g'(z) \text{ and hence}$$

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)}.$$

Clearly, $\frac{g'(z)}{g(z)}$ is analytic in $B(a;\delta)$.

Now, we introduce the notion of a meromorphic function to avoid the sentence - "analytic except for poles".

**29.1.2 DEFINITION :**

Let $G$ be an open set. Any function defined and analytic on $G$ except for poles is called a meromorphic function on $G$.

If $f$ is a meromorphic function defined on an open set $G$ and if we define $f(z_0) = \infty$ whenever $z_0$ is a pole of $f$ in $G$, then $f$ is continuous at each pole in $G$ also. (we leave this as an exercise).

**29.1.3 ARGUMENT PRINCIPLE :**

Let $f$ be meromorphic in $G$ with poles $p_1, p_2, \cdots\cdots, p_m$ and zeros $z_1, z_2, \cdots\cdots, z_n$ counted according to multiplicity. Let $\gamma$ be a closed rectifiable curve in $G$ not passing through any of the points $p_1, p_2, \cdots\cdots, p_m$, $z_1, z_2, \cdots\cdots, z_n$ with $\gamma \approx 0$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{\ell=1}^{m} n(\gamma, z_\ell) - \sum_{j=1}^{m} n(\gamma; p_j)$$

**Proof :** By the discussion,

$$\frac{f'(z)}{f(z)} = \sum_{\ell=1}^{n} \frac{1}{z - z_\ell} - \sum_{j=1}^{m} \frac{1}{z - p_j} + \frac{g'(z)}{g(z)} \quad \text{----------- (1)}$$

where $g(z)$ is an analytic function on $G$ which neither vanishes in $G$ nor has a pole in $G$.

So, $\frac{g'}{g}$ is analytic on $G$. By Cauchy's Theorem, $\int_\gamma \frac{g'(z)}{g(z)} dz = 0$. From (1),

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{\ell=1}^{n} \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_\ell} - \sum_{j=1}^{m} \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - p_j}$$

$$= \sum_{\ell=1}^{n} n(\gamma; z_\ell) - \sum_{j=1}^{m} n(\gamma; p_j)$$

### 29.1.3.1 NOTE :

Suppose $\gamma$ is a simple closed contour in the Argument principle. Then, the conclusion of the Argument principle is

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = z_f - p_f$$

where $z_f$ and $p_f$ are the number of zeros and poles in side $\gamma$ respectively.

### 29.1.4 EXAMPLE :

Consider the function

$$f(z) = \frac{\left(z^2+1\right)^2}{\left(z^2+2z+2\right)^3}$$

we, now evaluate $\int_\gamma \frac{f'(z)}{f(z)} dz$ for various chlosed rectifiable curves $\gamma$ s.

First we compute the zeros of $f$ and the poles of $f$.

$$f(z) = 0 \Rightarrow \left(z^2+1\right)^2 = 0$$

$$\Rightarrow z = i, i, -i, -i$$

$$\left(z^2+2z+2\right)^3 = 0 \Rightarrow \left((z+1)^2+1\right)^3 = 0$$

$$\Rightarrow -1+i, -1+i, -1+i, -1-i, -1-i, -1-i.$$

Hence, the zeros of $f$ are $i, i, -i, -i$ and the poles of $f$ are $-1+i, -1+i, -1-i, -1-i$ (written according to multiplicity).

(i) Suppose $\gamma$ is the positively oriented circle $|z| = 4$ i.e. $\gamma(t) = 4e^{it}$ $(0 \le t \le 2\pi)$. Clearly, all the zeros and all the poles of $f$ lie inside $\gamma$. By the Argument principle,

$$\int_{\gamma}\frac{f'(z)}{f(z)}dz = 2\pi i\,[z_f - p_f]$$

$$= 2\pi i[4-6]$$

$$= -4\pi i.$$

(ii) Suppose $\gamma$ is the circle $\gamma(t) = 4e^{3it}\ (0 \le t \le 2\pi)$. (i.e. $\gamma$ completes three complete rounds in anticlockwise direction along the circle $|z| = 4$). Clearly, all the zeros and poles of $f$ lie interior to $\gamma$. By the Argument principle,

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \sum_{\ell=1}^{n} n(\gamma_j\ z_\ell) - \sum_{j=1}^{m} n(\gamma; p_j)$$

(where $z_1, \cdots\cdots, z_n$ are zeros of $f$ and $p_1, p_2, \cdots\cdots, p_m$ are poles of $f$).

$$= \sum_{\ell=1}^{4} n(\gamma; z_j) - \sum_{j=1}^{6} n(\gamma; p_j)$$

$$= \sum_{\ell=1}^{4} 3 - \sum_{j=1}^{6} 3 = 12 - 18 = -6$$

Hence, $\int_{\gamma}\frac{f'(z)}{f(z)}dz = 12\pi i$

(iii) Suppose $\gamma$ is the positively oriented circle $|z| = \alpha$ (where $1 < \alpha < \sqrt{2}$) i.e. $\gamma(t) = \alpha e^{it}\ (0 \le t \le 2\pi)$. Clearly, each zero of $f$ will be inside $\gamma$ and each pole will be out side $\gamma$. So

$$\int_{\gamma}\frac{f'(z)}{f(z)}dz = 2\pi i[z_f - p_f]$$

(where $z_f$ and $p_f$ are the number of zeros and poles inside $\gamma$ respectively).

$$= 2\pi i[4-0] = 8\pi i$$

**29.1.5 THEOREM :**

Let $f$ be meromorphic in the region $G$ with zeros $z_1, \cdots\cdots, z_n$ and $p_1, p_2, \cdots\cdots, p_m$ counted according to multiplicity. Let $g$ be an analytic function in $G$ and $\gamma$ be a closed rectifiable curve in $G$ with $\gamma \approx 0$ and not passing through any $z_i$ or $p_j$. Then

$$\frac{1}{2\pi i}\int_{\gamma} g\frac{f'}{f} = \sum_{\ell=1}^{n} g(z_\ell)\, n(\gamma; z_\ell) - \sum_{j=1}^{m} g(p_j)\, n(\gamma; p_j)$$

**Proof :** By the discussion 29.1.1, there is an analytic function $h$ on $G$ which neither vanishes on $G$ nor has a pole in $G$ such that for any $z$ in $G-\{z_1, z_2, \cdots\cdots, z_n, p_1, p_2, \cdots\cdots, p_m\}$,

$$\frac{f'(z)}{f(z)} = \sum_{\ell=1}^{n}\frac{1}{z-z_\ell} - \sum_{j=1}^{m}\frac{1}{z-p_j} + \frac{h'(z)}{h(z)}$$

and hence

$$g(z)\frac{f'(z)}{f(z)} = \sum_{\ell=1}^{n}\frac{g(z)}{z-z_\ell} - \sum_{j=1}^{m}\frac{g(z)}{z-p_j} + \frac{g(z)h'(z)}{h(z)} \quad \text{------------ (1)}$$

Since $\frac{g(z)h'(z)}{h(z)}$ is analytic on $G$, by Cauchy's theorem,

$$\int_{\gamma}\frac{g(z)h'(z)}{h(z)}\,dz = 0$$

From (1),

$$\frac{1}{2\pi i}\int_{\gamma} g\frac{f'}{f} = \sum_{\ell=1}^{n}\frac{1}{2\pi i}\int_{\gamma}\frac{g(z)}{z-z_\ell}\,dz - \sum_{j=1}^{m}\frac{1}{2\pi i}\int_{\gamma}\frac{g(z)}{z-p_j}\,dz$$

$$= \sum_{\ell=1}^{n} g(z_\ell)\, n(\gamma; z_\ell) - \sum_{j=1}^{m} g(p_j)\, n(\gamma; p_j)$$

(By Cauchy Integral Formula).

**29.1.5.1 NOTE :**

Argument principle can be obtained from Theorem 29.1.5, by taking $g$ as the constant function 1 (one).

**29.1.6 EXAMPLE :**

Let $f(z)=z^4-2z^3+z^2-12z+20$ and $\gamma$ be the positively oriented simple circle $|z|=5$ i.e. $\gamma(t)=5e^{it}\ (0\le t\le 2\pi)$. We now evaluate

$$\int_\gamma z^2\frac{f'(z)}{f(z)}dz$$

The zeros of $f$ are precisely $2,2,-1+2i,\ -1-2i$. Clearly, all the zeros of $f$ are inside $\gamma$. Since $f(z)$ is a polynomial, it is an entire function. So, $f(z)$ has no poles. According to Theorem 29.1.5, $g(z)=z^2$

Now,

$$\int_\gamma g(z)\frac{f'(z)}{f(z)}dz=2\pi i\left[g(2)h(\gamma;2)+g(2)n(\gamma;2)\right.$$

$$\left.+g(-1+2i)\,n(\gamma;-1+2i)+g(-1-2i)n(\gamma;-1-2i)\right]$$

$$=2\pi i\left[4+4+(-1+2i)^2+(-1-2i)^2\right]$$

$$=2\pi i\left[8+2\left[(-1)^2+(2i)^2\right]\right]$$

$$=4\pi i$$

**29.1.7 THEOREM :**

Let $f$ be analytic on an openset containing $\bar{B}(a;R)$ and suppose that $f$ is one - one on $B(a;R)$. If $\Omega=f(B(a;R))$ and $\gamma$ is the circle $|z-a|=R$ then $f^{-1}(w)$ is defined for each $w$ in $\Omega$ by the formula

$$f^{-1}(w)=\frac{1}{2\pi i}\int_\gamma\frac{zf'(z)}{f(z)-w}dz$$

**Proof :** Let $w\in\Omega$. So, $w=f(\mathscr{E})$ for some $\mathscr{E}\in B(a;R)$. Since $f$ is one - one on $B(a;R), f(z)-w$ is one - one on $B(a;R)$. Also, $f(z)-w$ is analytic on (the) open set containing

$\bar{B}(a;R)$. So $\mathscr{E}$ is the only zero of $f(z)-w$ inside $\gamma$. Taking $g(z)=z,\ f(z)-w$ for $f(z)$ in Theorem 29.1.5,

$$\frac{1}{2\pi i}\int_{\gamma}\frac{z\,f'(z)}{f(z)-w}dz = n(\gamma;\mathscr{E})g(\mathscr{E})$$

$$= g(\mathscr{E}) \text{ (since } n(\gamma;\mathscr{E})=1\text{)}$$

$$= \mathscr{E} = f^{-1}(w)$$

**29.1.8 DISCUSSION :**

Let $G$ be an open set and let $f$ be a meromorphic function on $G$. Let $\gamma$ be a path in $G$ not passing through any pole or zero of $f$. Let $z\in\{\gamma\}$. So, $f(z)\neq 0$ and $f(z)\neq\infty$. Since the zeros and poles can be isolated, there exists a neighborhood of $z$ not containing any pole or zero. Thus, to each $z\in\{\gamma\}$, there is a neighborhood of $z$ not containing any pole or zero of $f$. All such neighborhoods form an open cover for $\{\gamma\}$. (which is compact as $\{\gamma\}$ is the continuous image of the compact set $[0,1]$). So, this open cover has a Lebesgue number $\delta$ say. Choose $\in$ such that $0<2\in<\delta$. Since $\gamma$ is continuous on the compact set $[0,1]$, $\gamma$ is uniformly continuous. So, there exists a partition

$0=t_0<t_1<\cdots<t_n=1$ of $[0,1]$ such that for any $s,t$ in $[t_{j-1},t_j]\ (1\leq j\leq n)$,

$|\gamma(s)-\gamma(t)|<\in$.

Clearly, $\gamma(t_j)\in B(\gamma(t_{j-1});\in)\quad (j=1,2,\cdots\cdots,n)$. Since $\operatorname{diam}B(\gamma(t_{j-1});\in)$ is $2\in<\delta$, $B(\gamma(t_{j-1});\in)$ is contained in some number of the open cover and hence $B(\gamma(t_{j-1});\in)$ contains neither a zero nor a pole of $f$.

This holds for $j=1,2,\cdots\cdots,n$. Hence $\bigcup_{j=1}^{n} B(\gamma(t_{j-1});\in)$ is a neighborhood of $\{\gamma\}$ containing neither a zero nor a pole of $f$. Let $\ell_j$ be the branch of $\log f(z)$ on $B(\gamma(t_{j-1});\in)\ (1\leq j\leq n)$. Since the $j^{th}$ and $(j+1)^{th}$ spheres contain $\gamma(t_j)$, we can choose $\ell_1,\ell_2,\cdots\cdots,\ell_n$ such that

$$\ell_1(\gamma(t_1)) = \ell_2(\gamma(t_1)),\ \ell_2(\gamma(t_2)) = \ell_3(\gamma(t_2)) \cdots\cdots \ell_{n-1}(\gamma(t_{n-1})) = \ell_n(\gamma(t_{n-1})).$$

Suppose, for $j = 1, 2, \cdots\cdots, n$

$$\gamma_j = \gamma/_{[t_{j-1}, t_j]} = \text{ the restriction of } \gamma \text{ to } [t_{j-1}, t_j].$$

Since $\ell'_j = \dfrac{f'}{f}$,

$$\int_{\gamma_j} \frac{f'}{f} = \ell_j(\gamma(t_j)) - \ell_j(\gamma(t_{j-1}))\ (1 \le j \le n).$$

Hence,

$$\int_{\gamma} \frac{f'}{f} = \sum_{j=1}^{n} \int_{\gamma_j} \frac{f'}{f}$$

$$= \sum_{j=1}^{n} \left[\ell_j(\gamma(t_j)) - \ell_j(\gamma(t_{j-1}))\right]$$

$$= \ell_n(\gamma(t_n)) - \ell_1(\gamma(t_0))$$

$$= \ell_n(\gamma(1)) - \ell_1(\gamma(0))$$

when $\gamma$ is closed, $\gamma(0) = \gamma(1)$ and hence

$$\int_{\gamma} \frac{f'}{f} = \ell_n(\gamma(0)) - \ell_1(\gamma(0))$$

$= 2\pi i k$ for some integer $k$.

(since $\ell_1$ and $\ell_n$ are two branches of $\log f(z)$ on $B(\gamma(0);\in) \cap B(\gamma(t_{n-1});\in)$, they differ by $2\pi i k$ for some integer $k$ as we know that "any two branches of logarithm on a region $G$ differ by $2\pi i k$, where $k$ is an integer").

Thus, when $\gamma$ is a closed rectifiable curve, as $z$ traces out $\gamma$, $\arg f(z)$ changes by $2\pi k$.

Now, we prove the crucial Theorem.

**29.1.9 ROUCHE'S THEOREM :**

Suppose $f$ and $g$ are meromorphic functions in a neighborhood of $\bar{B}(a;R)$ with no zeros and poles on the circle $\gamma = \{z/|z-a| = R\}$. If $Z_f, Z_g\ (p_f, p_g)$ are the number of zeros (poles) of $f$ and $g$ inside $\gamma$ counted according to their multiplicities and if

$$|f(z)+g(z)| < |f(z)| + |g(z)|$$

on $\gamma$, then

$$Z_f - p_f = Z_g - p_g$$

**Proof :** Assume the hypothesis of the theorem. So, for any $z$ on $\gamma$,

$$|f(z)+g(z)| < |f(z)| + |g(z)|$$

i.e. $$\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$$

(since $\gamma$ contains neither a zero nor a pole of $g$).

Put $\lambda = \dfrac{f(z)}{g(z)}$. If $\lambda$ is a nonnegative real number then $\lambda+1 < \lambda+1$, a contradiction. So, $\dfrac{f}{g}$ is a meromorphic function and maps $\{\gamma\}$ onto $\Omega = \mathbb{C} - [0,\infty)$. If $\ell$ is a branch of logarithm on $\Omega$,

then $\ell\left(\dfrac{f(z)}{g(z)}\right)$ is a well defined primitive of $\dfrac{\left(\frac{f}{g}\right)'}{\left(\frac{f}{g}\right)}$ in a neighborhood of $\gamma$. Thus,

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{\left(\frac{f}{g}\right)'}{\left(\frac{f}{g}\right)} \quad \text{(by Cauchy's Theorem).}$$

$$= \frac{1}{2\pi i}\int_\gamma \left(\frac{f'}{f} - \frac{g'}{g}\right)$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{f'}{f} - \frac{1}{2\pi i}\int_{\gamma}\frac{g'}{g}$$

$$= \left(Z_f - p_f\right) - \left(Z_g - p_g\right) \text{ (by Note 29.1.3.1)}$$

Hence the theorem.

Now, we obtain the Fundamental Theorem of Algebra using Rouche's Theorem.

**29.1.9 COROLLARY (FUNDAMENTAL THEOREM OF ALGEBRA) :**

Let $p(z) = a_0 + a_1 z + \cdots\cdots\cdots + a_{n-1}z^{n-1} + z^n$ be a polynomial with complex coefficients. Then p has $n$ zeros in $\mathbb{C}$.

**Proof :** Clearly,

$$\lim_{z\to\infty} \frac{p(z)}{z^n} = 1.$$

So, there exists $R > 0$ such that

$$|z| \geq R \Rightarrow \left|\frac{p(z)}{z^n} - 1\right| < 1$$

$$\Rightarrow \left|p(z) - z^n\right| < \left|z^n\right|$$

$$\Rightarrow \left|p(z) + \left(-z^n\right)\right| < \left|-z^n\right|$$

$$< \left|p(z)\right| + \left|-z^n\right|$$

In particular,

$$\left|p(z) + \left(-z^n\right)\right| < \left|p(z)\right| + \left|-z^n\right|$$

for $|z - 0| = R$.

Since $p(z)$ is a polynomial, it is an entire function. So, it has no poles. Clearly $-z^n$ is an entire function and hence has no poles. The number of zeros of $-z^n$ is $n$ (infact all these $n$ zeros are same and equal to 0) and all these zeros lie inside $|z - 0| = R$. By Rouche's Theorem,

The number of zeros of $p(z)$ (inside $|z| = R$)

= The number of zeros of the function $-z^n$ (inside $|z| = R$)

$= n$.

## 29.2 SHORT ANSWER QUESTIONS :

**29.2.1 :** State Argument Principle.

**29.2.2 :** State Rouche's Theorem.

**29.2.3 :** If $f(z)=z^2+1$, then find $\int_\gamma \frac{f'(z)}{f(z)}dz$ where $\gamma$ is the circle

$$\gamma(t)=\frac{1}{2}i+e^{it} \quad (0\le t\le 2\pi).$$

**29.2.4 :** Define a meromorphic function.

## 29.3 MODEL EXAMINATION QUESTIONS :

**29.3.1 :** State and prove argument principle.

**29.3.2 :** State and prove Rouche's theorem. Deduce Fundamental Theorem of Algebra.

**29.3.3 :** State Argument principle and evaluate the following integral

$$\int_\gamma \frac{f'(z)}{f\ )}dz$$

where $f(z)=\dfrac{z^2+1}{\left(z^2+2z+2\right)^2}$ and $\gamma$ is the positively oriented circle with center at 0 and radius $\alpha$ where $1<\alpha<\sqrt{2}$.

## 29.4 EXERCISES :

**29.4.1 :** Let $f$ be a meromorphic function defined on an openset $G$. Define $f(z)=\infty$ whenever $z$ is a pole of $f$. Prove that $f:G\to\mathbb{C}_\infty$ is continuous (i.e. $f$ is continuous at each pole).

**29.4.2 :** Let $f$ be a meromorphic function on an openset $G$. Show that neither the zeros of $f$ nor the poles of $f$ have a limit point.

**29.4.3 :** Suppose $f$ is analytic on $\overline{B}(0;1)$ and satisfies $|f(z)|<1$ for $|z|=1$. Find the number of solutions (counting multiplicaties) of the equation $f(z)=z^n$ where n is an integer greaterthan or equal 0 to 1.

**29.4.4 :** Let $f$ be analytic in $\bar{B}(0;R)$ with $f(0)=0,\ f'(0)\neq 0$ and $f(z)\neq 0$ for $0<|z|\leq R$. Put $\rho=\min\{|f(z)|/|z|=R\}>0$. Define $g:B(0;\rho)\to\mathbb{C}$ by

$$g(w)=\frac{1}{2\pi i}\int_{\gamma}\frac{zf'(z)}{f(z)-w}dz$$

where $\gamma$ is the circle $|z|=R$. Show that $g$ is analytic and discuss the properties of $g$.

**29.4.5 :** State and prove a more general version of Rouche's Theorem for curves other than circles in $G$.

**29.4.6 :** Is a non constant meromorphic function on a region $G$ an open mapping of $G$ into $\mathbb{C}$ ? Is it an open mapping of $G$ into $\mathbb{C}_\infty$ ?

**29.4.7 :** Let $\lambda>1$. Show that the equation $\lambda-z-e^{-z}=0$ has exactly one solution in the half plane $\{z/\operatorname{Re} z>0\}$. Show that this solution must be real. What happens to the solution as $\lambda\to 1$?

**29.4.8 :** Let $f$ be analytic in a neighborhood of $D=\bar{B}(0;1)$. If $|f(z)|<1$ for $|z|=1$, show that there is a unique $z$ with $|z|<1$ and $f(z)=z$. If $|f(z)|\leq 1$ for $|z|=1$, what can you say ?

## 29.5 ANSWERS TO SHORT ANSWER QUESTIONS :

**29.2.1 :** See Statement of Theorem 29.1.3.

**29.2.2 :** See Statement of Theorem 29.1.9.

**29.2.3 :** The only zero of $f$ that lies interior to $\gamma$ is $i$. Since $f(z)$ is a polynomial, it is an entire function. So, $f$ has no poles. By Note 29.1.3.1, the value of the integral is

$$2\pi i[z_f-p_f]=2\pi i[1-0]=2\pi i$$

**29.2.4 :** See Definition 29.1.2

## REFERENCE BOOK :

(i) J.B. Conway : Functions of one complex variable - Second Edition - Springer International Student Edition.

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## Lesson - 30

# MAXIMUM MODULUS THEOREM

### 30.0 INTRODUCTION

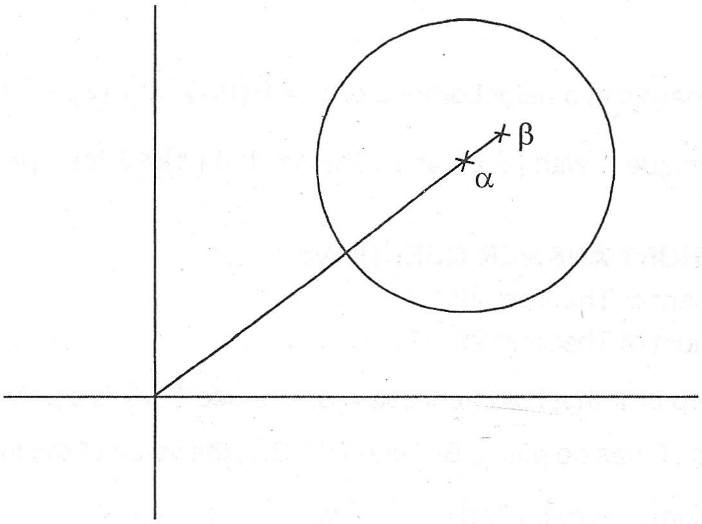
In this lesson, we study various versions of maximum modulus theorems (see theorems 30.1.2, 30.1.3, 30.1.7).

### 30.1 MAXIMUM MODULUS THEOREM :

We start this lesson with the following.

### 30.1.1 DISCUSSION :

Let $\Omega$ be a subset of $\mathbb{C}$. Let $\alpha \in \Omega$ be an interior point of $\Omega$. So, there exists $\delta > 0$ such that $B(\alpha;\delta) \subseteq \Omega$. So, there exists $\beta \in B(\alpha;\delta)$ such that $|\beta| > |\alpha|$. In otherwords, if $\alpha \in \Omega$ is such that $|\alpha| \geq |\beta|$ for all $\beta \in \Omega$ then $\alpha \in \partial\Omega$ = boundary of $\Omega$.




### 30.1.2 THEOREM (MAXIMUM MODULUS THEOREM) - FIRST VERSION :

Let $f$ be an analytic function on a region $G$ and let $a \in G$ be such that $|f(a)| \geq |f(z)|$ for all $z$ in $G$. Then $f$ must be constant.

**Proof :** Suppose $f$ is not constant. Since $G$ is a region, $G$ is open. By open mapping Theorem, $\Omega = f(G)$ is open. Put $\alpha = f(a)$. Then $|\alpha| \geq |\beta|$ for all $\beta \in \Omega$. By the discussion, 30.1.1, $\alpha \in \partial\Omega \cap \Omega = \varphi$ (since $\Omega$ is open), a contradiction. So, $f$ must be constant.

**30.1.3 THEOREM (MAXIMUM MODULUS THEOREM) - SECOND VERSION :**

Let $G$ be a bounded open set in $\mathbb{C}$ and suppose $f$ is a continuous function on $\bar{G}$ which is analytic in $G$. Then

$$\operatorname{Max}\left\{|f(z)| / z \in \bar{G}\right\}$$

$$= \operatorname{Max}\left\{|f(z)| / z \in \partial G\right\}.$$

**Proof :** If $f$ is constant, then the conclusion is clear. Suppose $f$ is not constant. Since $G$ is a bounded open set, $\bar{G}$ is compact. Since $f$ is continuous on $\bar{G}$, $|f|$ is continuous on $\bar{G}$. Since continuous image of compact set is compact, $|f|(\bar{G}) = \left\{|f(z)| / z \in \bar{G}\right\}$ is compact. We know that every compact subset of a metric space is closed and bounded. So, there exists $a \in \bar{G}$ such that $|f(z)| \leq |f(a)|$ for all $z$ in $\bar{G}$. If $a \in G$ then $f$ is constant by Maximum Modulus Theorem first version, a contradiction. So, $a \in \partial G$. Hence the theorem i.e.

$$\operatorname{Max}\left\{|f(z)| / z \in \bar{G}\right\} = \operatorname{Max}\left\{|f(z)| / z \in \partial G\right\}$$

**30.1.4 DEFINITION :**

Let $f : G \to \mathbb{R}$ and let $a \in \bar{G}$ or $a = \infty$. Then the limit superior of $f(z)$ as $z$ approaches a denoted by $\lim\limits_{z \to a} \sup f(z)$ or $\overline{\lim\limits_{z \to a}} f(z)$ is defined by

$$\overline{\lim_{z \to a}} f(z) = \lim_{r \to 0+} \sup \left\{f(z) / z \in G \cap B(a;r)\right\}.$$

Similarly, the limit inferior of $f(z)$ as $z$ approaches a denoted by

$\lim\limits_{z \to a} \inf f(z)$ or $\underline{\lim\limits_{z \to a}} f(z)$ is defined by

$$\underline{\lim_{z \to a}} f(z) = \lim_{r \to 0+} \inf \left\{f(z) / z \in G \cap B(a;r)\right\}$$

**30.1.4.1 NOTE :**

(1) When $a = \infty$, $B(a;r)$ is the open ball considered in $\mathbb{C}_\infty$.

(2) Clearly,

$$\lim_{z\to a} f(z)=\alpha \text{ iff } \overline{\lim_{z\to a}} f(z)=\underline{\lim_{z\to a}} f(z)=\alpha.$$

(3) Let $G\subseteq\mathbb{C}$. The boundary of $G$ in $\mathbb{C}_\infty$ denoted by $\partial_\infty G$ and is called the extended boundary of $G$. We know that $\partial G$ denotes the boundary of $G$ in $\mathbb{C}$. Clearly,

When (i) $G$ is bounded in $\mathbb{C}$ then $\partial_\infty G=\partial G$

(ii) $G$ is unbounded in $\mathbb{C}$ then $d_\infty G = dG U\{\infty\}$.

**30.1.5 LEMMA :**

Let $f: G\to\mathbb{R}$. Suppose

$$\lim_{z\to a} \sup f(z)<M\cdot$$

Then there exists $r_1>0$ such that

$$0<r<r_1 \Rightarrow f(z)<M \text{ for all } z\in G\cap B(a;r)$$

**Proof :** Let $\alpha=\lim\limits_{z\to a}\sup f(z)$

$$=\lim_{r\to 0+}\sup\{f(z)/z\in G\cap B(a;r)\}$$

$$=\lim_{r\to 0+} x_r$$

where $x_r=\sup\{f(z)/z\in G\cap B(a;r)\}$.

So, there exists $r_1>0$ such that

$$0<r<r_1\Rightarrow |x_r-\alpha|<M-\alpha$$

$$\Rightarrow x_r-\alpha<M-\alpha$$

i.e. $x_r<M$

$$\Rightarrow f(z)<M \text{ for all } z\in G\cap B(a;r)$$

**30.1.6 LEMMA :**

Let $f : G \to \mathbb{R}$. Suppose

$$M < \liminf_{z \to a} f(z).$$

Then there exists $r_2 > 0$ such that

$$0 < r < r_2 \Rightarrow M < f(z) \text{ for all } z \in G \cap B(a;r).$$

**Proof :** Exercise

Infact we prove Note 30.1.3.1(2) by using the Lemmas 30.1.5 and 30.1.6.

**30.1.7 THEOREM (MAXIMUM MODULUS THEOREM - THIRD VERSION) :**

Let $G$ be a region in $\mathbb{C}$ and $f$ be an analytic function on $G$. Suppose there is a constant M such that

$$\limsup_{z \to a} |f(z)| \le M$$

for all $a \in \partial_\infty G$. Then $|f(z)| \le M$ for all $z$ in $G$.

**Proof :** Let $\delta > 0$. Write

$$H = \{z \in G / |f(z)| > M + \delta\}.$$

Now, we prove that $H = \phi$.

Suppose $H \neq \phi$. Clearly, $H = |f|^{-1}((M+\delta, \infty))$ is open (since $|f|$ is continuous and $(M+\delta, \infty)$ is open). Let $a \in \partial_\infty G$. Since

$$\limsup_{z \to a} |f(z)| \le M < M + \delta,$$

there exists $r > 0$ such that

$$|f(z)| < M + \delta \text{ --------- (1)}$$

for all $z \in G \cap B(a;r)$ (by Lemma 30.1.5)

Now, we prove that $\bar{H} \subseteq G$. Let $a \in \bar{H}$. Then $\bar{H} \subseteq \bar{G}$. So, $a \in \bar{G}$. If $a \in G$, it is well and good. Suppose $a \notin G$. So, $a \in \partial G \subseteq \partial_\infty G$. By hypothesis, there exists $r > 0$ such that (1) holds for all $z \in G \cap B(a;r)$. Since $a \in \bar{H}$, $H \cap B(a;r) \neq \phi$. Let $z \in H \cap B(a;r) \subseteq G \cap B(a;r)$. So, $|f(z)| > M + \delta$, a contradiction to (1) as $z$ belongs to $G \cap B(a;r)$. Hence $a \in G$. Thus, $\bar{H} \subseteq G$.

Now, we prove that H is bounded. If $G$ is bounded then H is bounded (since $H \subseteq G$). Suppose $G$ is not bounded in $\mathbb{C}$. So, $\infty \in \partial_\infty G$. By hypothesis there exists $r > 0$ such that (1) holds for all $z \in G \cap B(\infty;r)$.

Now,

$$z \in B(\infty;r) \Leftrightarrow d(z,\infty) < r$$

$$\text{i.e. } \frac{2}{\sqrt{|z|^2 + 1}} < r$$

$$\Leftrightarrow |z| > \sqrt{\frac{4}{r^2} - 1}$$ (without loss of generality we can assume that $r < 1$).

$z \in H \Rightarrow z \in G$ and $|f(z)| > M + \delta$

$\Rightarrow z \notin B(\infty, r)$ (By (1))

$$\Rightarrow |z| \leq \sqrt{\frac{4}{r^2} - 1}$$

$$\Rightarrow z \in \bar{B}\left(0; \sqrt{\frac{4}{r^2} - 1}\right).$$

Thus,

$H \subseteq \bar{B}\left(0; \sqrt{\frac{4}{r^2} - 1}\right)$ and hence H is bounded.

Since H is closed and bounded, $\bar{H}$ is compact. Clearly,

$$\bar{H} \subseteq \{z \in G / |f(z)| \geq M + \delta\}.$$

If $z \in \partial H (\subseteq \bar{H})$ then $|f(z)| = M + \delta$ (otherwise $|f(z)| > M + \delta$ and hence $z \in H \cap \partial H = \phi$ (since H is open, a contradiction).

By maximum modulus theorem second version (applying $f$ to H instead of $G$) we have that ____________

for any $z$ in H,

$$|f(z)| \leq \text{Max}\{|f(w)| / w \in \bar{H}\}$$

$$\leq \text{Max}\{|f(w)| / w \in \partial H\}$$

$$= M + \delta,$$

a contradiction to $|f(z)| > M + \delta$ as ($z \in H$). So, $H = \phi$.

Hence, $|f(z)| \leq M + \delta$ --------- (2)

for all $z \in G$. Thus, (2) holds for all $\delta > 0$. Hence

$|f(z)| \leq M$ for all $z$ in $G$.

## 30.2 SHORT ANSWER QUESTIONS :

**30.2.1 :** State Maximum Modulus Theorem - First Version.

**30.2.2 :** State Maximum Modulus Theorem - Second Version.

**30.2.3 :** State Maximum Modulus Theorem - Third Version.

**30.2.4 :** What is Minimum Principle ?

**30.2.5 :** If $G$ is an open set in $\mathbb{C}$, what is the relation between the boundary $\partial G$ of $G$ in $\mathbb{C}$ and the extended boundary $\partial_\infty G$ of $G$ in $\mathbb{C}_\infty$.

**30.2.6 :** Let $f : G \to \mathbb{R}$ be a function, where $G$ is an open set in $\mathbb{C}$ and $a \in \bar{G}$.

(i) Define $\lim\limits_{z \to a} \sup f(z)$

(ii) Define $\lim\limits_{z \to a} \inf f(z)$

(iii) What is the relation between $\lim_{z\to a} f(z)$, $\lim_{z\to a} \sup f(z)$ and $\lim_{z\to a} \inf f(z)$.

## 30.3 MODEL EXAMINATION QUESTIONS :

**30.3.1 :** State and prove Maximum Modulus Theorem - First Version

**30.3.2 :** State and prove Maximum Modulus Theorem - Second Version

**30.3.3 :** State and prove Maximum Modulus Theorem - Third Version

**30.3.4 :** State and prove the Minimum Principle.

## 30.4 EXERCISES :

**30.4.1 :** Prove the following Minimum Principle. If $f$ is a non-constant analytic function on a bounded open set $G$ and is continuous on $\bar{G}$, then either $f$ has a zero in $G$ or $|f|$ assumes minimum value on $\partial G$. (Hint : If $f$ has no zero in $G$, then apply Maximum Modulus Theorem Second Version (30.1.3) to $\frac{1}{f}$).

**30.4.2 :** Let $G$ be a bounded region and suppose $f$ is continuous on $\bar{G}$ and analytic in $G$. Show that if there is a constant $c \geq 0$ such that $|f(z)| = c$ for all $z$ on the boundary of $G$ then either $f$ is a constant function or $f$ has a zero in $G$.

**30.4.3 :** (a) Let $f$ be an entire function and $f$ is non constant. For any positive real number c, prove that the closure of the set $\{z \in \mathbb{C} / |f(z)| < c\}$ is the set $\{z \in \mathbb{C} / |f(z)| \leq c\}$.

(b) Let $p$ be a polynomial and show that each component of $\{z \in \mathbb{C} / |p(z)| < c\}$ contains a zero of $p$ (Hint : Use exercise 30........2.).

(c) if $p$ is a polynomial and $c > 0$ show that the set $\{z \in \mathbb{C} / |p(z)| = c\}$ is the union of a finite number of closed paths. Discuss the behavior of these paths as $c \to \infty$.

**30.4.4 :** Let $0 < r < R$ and put $A = \{z \in \mathbb{C} / r \leq |z| \leq R\}$. Show that there is a positive number $\in > 0$ such that for each polynomial $p$,

$$\sup\left\{\left|p(z)-z^{-1}\right| : z\in A\right\}\geq\in$$

i.e. $z^{-1}$ is the uniform limit of polynomials on A.

**30.5 ANSWERS TO SHORT ANSWER QUESTIONS :**

**30.2.1** : See statement of Theorem 30.1.2.

**30.2.2** : See statement of Theorem 30.1.3.

**30.2.3** : See statement of Theorem 30.1.7.

**30.2.4** : See Exercise 30.4.1.

**30.2.5** : $\partial_\infty G \subseteq \partial_\infty G$. If $G$ is bounded then $\partial G = \partial_\infty G$; otherwise $\partial_\infty G = \partial G \cup \{\infty\}$.

**30.2.6 :** (i) See Definition

(ii) See Definition

(iii) all are equal.

**REFERENCE BOOK :**

(i) J.B. Conway : Functions of one complex variable - Second Edition - Springer International Student Edition.

**Lesson writer :**

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