THEORY OF ORDINARY DIGGERENTIAL EQUATIONS (DM04)

(MSC MATHEMATICS)

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UNIT-I

LESSON-I

INITIAL VALUE PROBLEMS FOR THE HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

1.0 **Introduction**: Lessons 1 to 6 devoted for the linear differential equations with variable coefficients. In this lesson we study the initial value problems for the homogeneous linear differential equations with variable coefficients. We state the existence theorem for solutions of these initial value problems and establish the uniqueness theorem.

1.1 Linear differential equations with variable coefficients:

1.1.1: Definition: A Linear differential equation of order n with variable coefficients is an equation of the form

$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots\ldots + a_n(x)y = b(x)$, where $a_0, a_1, a_2 \ldots a_n$ are complex valued functions on some interval I.

Throughout the lessons 1 to 6 we take $a_0(x) \neq 0$ on I. By dividing by a_0 we can obtain an equation of the same form, but with a_0 replaced by the constant 1. Thus we consider the equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots\ldots + a_n(x)y = b(x) \quad \text{---------- (1)}$$

Let $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \ldots\ldots + a_n(x)y$ ---------- (2). Then (1) can be written as $L(y) = b(x)$. If $b(x) = 0$, for all $x \in I$, then the equation becomes $L(y) = 0$ and is called a homogeneous equation. If $b(x) \neq 0$, for some $x \in I$ then $L(y) = b(x)$ is said to be non-homogeneous equation.

It may be noted that L is an operator which maps a function ϕ, which has n derivatives on I into the function $L(\phi)$ on I. Whose value at x is given by:

$$L(\phi)(x) = \phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \ldots + a_n(x)\phi(x).$$

1.1.2 **Definition:** A function ϕ on I which has n derivatives on I is said to be a solution of (1) if $L(\phi) = b$.

Through the lessons 1 to 6, we assume that $a_1, a_2 \ldots a_n$ and b are complex valued continuous functions on some real interval I and L(y) denotes the expression (2).

1.1.3 Initial value problems for the Homogeneous equations: An initial value problem for $L(y) = 0$ is a problem of finding a solution ϕ of it satisfying $\phi(x_0) = \alpha_1$, $\phi^1(x_0) = \alpha_2$, ... $\phi^{(n-1)}(x_0) = \alpha_n$. Where x_0 is some real number and $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are given constants.

In many cases it is not possible to find a solution of (1) interms of elementary functions, but it can be proved that solutions always exist. We shall now assume the following existence theorem (the proof of which will be discussed later) and proceed further.

1.1.4 Theorem: Existence Theorem:

Let $a_1, a_2, \ldots\ldots a_n$ be continuous functions on an interval I containing the point x_0. If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are any n constants, there exists a solution ϕ of

$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \ldots\ldots\ldots + a_n(x)y = 0$ on I satisfying $\phi(x_0) = \alpha_1$, $\phi^1(x_0) = \alpha_2$, ... $\phi^{(n-1)}(x_0) = \alpha_n$.

1.1.5 Note:

i) The solution exists on the entire interval I where $a_1, a_2, \ldots\ldots a_n$ are continuous.

ii) Every initial value problem has a solution.

Neither of these results may be true if the coefficient of $y^{(n)}$ vanishes some where in I. (See the following example).

1.1.6 Example: Consider the equation $xy^1+y=0$, the coefficients of this equation are continuous for all real x. The initial value problem $xy^1+y=0$, $y(1)=1$ has the solution ϕ_1, where $\phi_1(x) = \frac{1}{x}$.

But this solution exists in (0, 1) but not at x=0. That is the solution does not exist for all real x.

If ϕ is any solution of $xy^1+y=0$, then $x\phi(x)=c$ where c is a constant. Therefore at x=0, only the trivial solution exists (c=0). This implies that the initial value problem $xy^1+y=0$, $y(0)=\alpha_1$, has a solution only for $\alpha_1=0$. Thus neither of the results mentioned in (i) and (ii) of the not are true.

To demonstrate the uniqueness we need the following estimate for $\|\phi(x)\|$,

where $\|\phi(x)\| = [\,|\phi(x)|^2 + |\phi^1(x)|^2 + \ldots + |\phi^{(n-1)}(x)|^2\,]^{1/2}$.

1.1.7 Theorem: Let $b_1, b_2, \ldots, b_n$ be non-negative constants such that $|a_j(x)| \le b_j$ for all $x \in I$, $1 \le j \le n$ and $k = 1 + b_1 + b_2 + \ldots + b_n$. If x_0 is a point in I and ϕ is a solution of L(y)=0 on I then $\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k|x-x_0|}$ for all $x \in I$.

Proof: Since ϕ is a solution of L(y)=0, we have $L(\phi(x)) = 0 \ \forall \ x \in I$.

i.e. $\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \ldots + a_n(x)\phi(x) = 0$.

i.e. $\phi^{(n)}(x) = -a_1(x)\phi^{(n-1)}(x) - \ldots - a_n(x)\phi(x)$.

Therefore $|\phi^n(x)| \le |a_1(x)||\phi^{(n-1)}(x)| + \ldots + |a_n(x)||\phi(x)|$

i.e. $|\phi^n(x)| \le b_1|\phi^{(n-1)}(x)| + \ldots + b_n|\phi(x)|$. Let $u(x) = \|\phi(x)\|^2$.

$\therefore u(x) = |\phi(x)|^2 + |\phi^1(x)|^2 + \ldots + |\phi^{(n-1)}(x)|^2$.

i.e. $u = \phi\bar{\phi} + \phi^1\overline{\phi^1} + \ldots + \phi^{(n-1)}\overline{\phi^{(n-1)}}$.

$u^1 = \phi^1\bar{\phi} + \phi\overline{\phi^1} + \phi^{11}\overline{\phi^1} + \phi^1\overline{\phi^{11}} + \ldots + \phi^n\overline{\phi^{(n-1)}} + \phi^{(n-1)}\overline{\phi^{(n)}}$.

This implies,

$$|u^1(x)| \le |\phi^1(x)||\bar{\phi}(x)| + |\phi(x)||\overline{\phi^1}(x)| + \ldots + |\phi^{(n-1)}(x)||\overline{\phi^{(n)}}(x)|.$$

$$= 2|\phi(x)||\phi^1(x)| + 2|\phi^1(x)||\phi^{11}(x)| + \ldots + 2|\phi^{n-1}(x)||\phi^n(x)|. \ (\because |\bar{\alpha}| = |\alpha|)$$

$$\le 2|\phi(x)||\phi^1(x)| + 2|\phi^1(x)||\phi^{11}(x)| + \ldots\ldots\ldots 2|\phi^{(n-2)}(x)||\phi^{(n-1)}(x)|$$

$$+ 2b_1|\phi^{(n-1)}(x)|^2 + 2b_2|\phi^{(n-2)}(x)||\phi^{(n-1)}(x)| + \ldots + b_n|\phi(x)|^2 + b_n|\phi^{(n-1)}(x)|^2.$$

($\because 2|b||c| \le |b|^2 + |c|^2$ for any numbers b and c).

$$= (1+2b_1+b_2+\ldots+b_n)|\phi^{(n-1)}(x)|^2 + (1+b_n)|\phi(x)|^2 + (2+b_{n-1})|\phi^{(1)}(x)|^2 + \ldots$$

$$+ (2+b_2)|\phi^{(n-2)}(x)|^2.$$

This implies that $|u^1(x)| \le 2k\, u(x)$,

Since $(1+2b_1+b_2+\ldots+b_n) \le 2k$ and $(1+b_n) \le 2k, (2+b_i) \le 2k, 2 \le i \le n-1$.

Therefore,

$-2k\,u(x) \le u^1(x) \le 2k\,u(x)$ -------- (1)

Consider the right inequality. It can be written as $u^1 - 2ku \le 0$

$$\Rightarrow e^{-2kx}(u^1 - 2ku) \le 0$$

$$\Rightarrow (e^{-2kx}\,u(x))^1 \le 0$$

If $x > x_0$, integrating from x_0 to x, we get $e^{-2kx}\,u(x) - e^{-2kx_0}\,u(x_0) \le 0$, or

$u(x) \le e^{2k(x-x_0)}\,u(x_0)$

i.e. $\|\phi(x)\| \le \|\phi(x_0)\| e^{k(x-x_0)}$ $(\because u(x) = \|\phi(x)\|^2)$. The left inequality of (1) similarly yields,

$\|\phi(x_0)\| e^{-k(x-x_0)} \le \|\phi(x)\|$ for $x > x_0$.

Therefore $\|\phi(x_0)\| e^{-k(x-x_0)} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k(x-x_0)}$ for $x > x_0$.

Again considering (1) for $x < x_0$, together with an integration from x to x_0 gives,

$\|\phi(x_0)\| e^{k(x-x_0)} \le \|\phi(x)\| \le e^{-k(x-x_0)} \|\phi(x_0)\| (x < x_0)$. Therefore, for all $x \in I$, we get

$\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k|x-x_0|}$.

Hence the theorem.

1.1.8 **<u>Remark:</u>** If I = [a, b] and a_j are continuous ($1 \le j \le n$) then they are bounded and there always exist finite constants b_j such that $|a_j(x)| \le b_j$ for all $x \in I$, $1 \le j \le n$.

1.1.9 **<u>Theorem:</u> Uniqueness Theorem:** Let $x_0 \in I$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any n constants. Then there is at most one solution of L(y)=0 on I satisfying $\phi(x_0) = \alpha_1, \phi^1(x_0) = \alpha_2, \ldots\ldots$

$\phi^{(n-1)}(x_0) = \alpha_n$ ------- (1)

Proof: Let ϕ and ψ be two solutions of L(y)=0 on I satisfying (1). Putting $\sigma = \phi - \psi$.

We prove that $\sigma(x)=0$ for all $x \in I$. Even though $a_i(1 \le i \le n)$ are continuous on I, they need not be bounded on I. Hence we cannot apply Theorem 1.1.7 directly. Let $x \in I$, $x \ne x_0$ and let J be any closed and bounded interval on I such that $x \in J$ and $x_0 \in J$. Then $a_j(1 \le j \le n)$ are bounded.

i.e. $|a_j(x)| \le b_j$ $(1 \le j \le n)$, $x \in J$.

Applying theorem 1.1.7, we get

$L(\sigma) = L(\phi - \psi) = L(\phi) - L(\psi) = 0$ and $\sigma(x_0) = \phi(x_0) - \psi(x_0) = \alpha_1 - \alpha_1 = 0$. Therefore by theorem 1.1.7, we get $\|\sigma(x)\| = 0 \,\forall x$ in J. Therefore $\sigma(x) = 0 \Rightarrow \phi(x) = \psi(x)$ since x is arbitrary, $\phi(x) = \psi(x)$ for all $x \in I$. Thus the theorem is proved.

1.2 Short Answer questions:

1.2.1 State Existence theorem for n^{th} order linear homogeneous differential equation.

1.3 Model Examination questions.

1.3.1 Let $b_1, b_2, \ldots, b_n$ be non-negative constants such that $|a_j(x)| \le b_j$ for all $x \in I$, $1 \le j \le n$ and $k = 1 + b_1 + b_2 + \ldots + b_n$. If x_0 is a point in I and ϕ is a solution of L(y)=0 on I. Then

$$\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k|x-x_0|} \text{ for all } x \in I.$$

1.3.2 Let I be an interval containing a point x_0 and let $\alpha_1, \alpha_2, \ldots \alpha_n$ be any n-constants. Prove that there is atmost one solution ϕ of the equation. $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \ldots + a_n(x) y = 0$ on I satisfying $\phi(x_0) = \alpha_1, \phi^1(x_0) = \alpha_2, \ldots \phi^{(n-1)}(x_0) = \alpha_n$.

1.3 Answers to short answer questions.

For 1.2.1, see the statement of theorem 1.1.4.

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LESSON-2

SOLUTIONS OF THE HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH VARIAPLE COEFFICIENTS

2.0 Introduction: This lesson deals with the solutions of $L(y)=0$, we prove that the set of all solutions of $L(y)=0$ is an n-dimensional linear space.

2.1 Solutions of the homogeneous linear differential equations

2.1.1 First we note the following:

Any linear combination of solutions of $L(y)=0$ is again a solution of $L(y)=0$.

For, suppose that $\phi_1, \phi_2, \ldots\ldots \phi_n$ are any n-solutions of the n^{th} order linear differential equation $L(y)=0$ and $C_1, C_2, \ldots\ldots C_n$ are any n constants then

$L(C_1\phi_1+C_2\phi_2+\ldots\ldots+C_n\phi_n) = C_1 L(\phi_1)+C_2 L(\phi_2)+\ldots+C_n L(\phi_n)=0 \; (\because L(\phi_i)=0,\ i=1,2,\ldots.n)$.

Therefore any linear combination of solutions of $L(y)=0$ is again a solution of $L(y)=0$.

2.1.2 **Definition:** The functions $\phi_1, \phi_2, \ldots\ldots, \phi_n$ defined on an interval I are said to be linearly independent if the only constants $C_1, C_2, \ldots\ldots, C_n$ such that $C_1 \phi_1(x)+C_2 \phi_2(x)+\ldots..+C_n \phi_n(x)=0$ for all $x \in I$, are the constants $C_1=C_2=\ldots\ldots=C_n=0$.

2.1.3 **Theorem:** There exist n linearly independent solutions of $L(y)=0$ on I.

Proof: Let x_0 be a point in I. By the existence theorem, there exists a solution ϕ_1 of $L(y)=0$ such that $\phi_1(x_0)=1$, $\phi_1^{1}(x_0)=0, \ldots\ldots, \phi_1^{(n-1)}(x_0)=0$. In general for each $i=1, 2, \ldots.., n$ there is a solution ϕ_i satisfying $\phi_i^{(i-1)}(x_0)=1$, $\phi_i^{(j-1)}(x_0)=0$, $j\neq i$.

Now we show that the solutions $\phi_1, \phi_2, \ldots.., \phi_n$ are linearly independent on I.

Suppose there are constants $C_1, C_2, \ldots\ldots, C_n$ such that $C_1 \phi_1(x)+C_2 \phi_2(x)+\ldots..+C_n \phi_n(x)=0$ for all $x \in I$.

Differentiating we get

$C_1 \phi_1^{1}(x) + C_2 \phi_2^{1}(x) + \ldots\ldots + C_n \phi_n^{1}(x)=0$

$C_1 \phi_1^{11}(x) + C_2 \phi_2^{11}(x) + \ldots\ldots + C_n \phi_n^{11}(x)=0$

....

$C_1 \phi_1^{(n-1)}(x) + C_2 \phi_2^{(n-1)}(x) + \ldots\ldots + C_n \phi_n^{(n-1)}(x)=0$ for all $x \in I$.

Putting $x=x_0$, we get $C_1=C_2=\ldots\ldots=C_n=0$. Thus the solutions $\phi_1, \phi_2, \ldots.., \phi_n$ are linearly independent.

Hence the theorem.

2.1.4 **Theorem:** Let $\phi_1, \phi_2, \ldots\ldots, \phi_n$ be the n solutions of $L(y)=0$ on I satisfying $\phi_i^{(i-1)}(x_0)=1$, $\phi_i^{(j-1)}(x_0)=0$, $j\neq i$, $1 \leq i \leq n$, $1 \leq j \leq n$, where x_0 is a point on I. If ϕ is any solution of $L(y)=0$ on I, there exist n constants $C_1, C_2, \ldots\ldots, C_n$ such that $\phi = C_1\phi_1 + [illegible] + C_n\phi_n$.

Proof: Let $\phi(x_0)=\alpha_1, \phi^1(x_0)=\alpha_2, \ldots\ldots\ldots \phi^{(n-1)}(x_0)=\alpha_n$.

Consider the function $\psi=\alpha_1\phi_1+\alpha_2\phi_2+\ldots\ldots\ldots+\alpha_n\phi_n$ then (2.2

$L(\psi) = L(\alpha_1\phi_1+\alpha_2\phi_2+\ldots\ldots\ldots+\alpha_n\phi_n$

$= \alpha_1 L(\phi_1)+\alpha_2 L(\phi_2)+\ldots\ldots.+\alpha_n L(\phi_n)$ ($\because L$ is linear)

$= 0$ ($\because \phi_i$ is a solution of $L(y)=0$) for each $i=1, 2, \ldots. n$)

Now $\psi(x_0) = \alpha_1\phi_1(x_0)+\alpha_2\phi_2(x_0)+\ldots.+\alpha_n\phi_n(x_0)$.

$= \alpha_1$ ($\because \phi_1(x_0)=1, \phi_2(x_0)=\ldots.. =\phi_n(x_0)=0$)

$\psi^1(x_0) = \alpha_1\phi_1{}^1(x_0)+\alpha_2\phi_2{}^1(x_0)+\ldots\ldots+\alpha_n\phi_n{}^1(x_0)$

$= \alpha_2$ ($\phi_2{}^1(x_0)=1, \phi_1{}^1(x_0)=\phi_3{}^1(x_0)\ldots\ldots.\phi_n{}^1(x_0)=0$)

....

$\psi^{(n-1)}(x_0)=\alpha_n$. Thus ψ is a solution of $L(y)=0$ and ($\psi(x_0)=\alpha_1, \psi^1(x_0)=\alpha_2, \ldots$

$\psi^{(n-1)}(x_0)=\alpha_n$) has the same initial conditions at x_0 as ϕ.

By uniqueness theorem $\phi=\psi$. Therefore $\phi=\alpha_1\phi_1+\alpha_2\phi_2+\ldots\ldots+\alpha_n\phi_n$.

The theorem is proved with the constants $C_i=\alpha_i$ $(1\le i\le n)$.

Hence the theorem.

2.1.5 **Solution Space**: Let S be the set of all solutions of $L(y)=0$. If $\phi_1, \phi_2 \in S$, then we have seen that $C_1\phi_1+C_2\phi_2 \in S$, for any constants C_1 and C_2. Therefore S forms a linear space called the solution space of $L(y)=0$. It has n-linearly independent solutions $\phi_1, \phi_2, \ldots.., \phi_n$ and every solution of $L(y)=0$ is a linear combination of $\phi_1, \phi_2, \ldots.., \phi_n$.

Therefore $B=\{\phi_1, \phi_2, \ldots.., \phi_n\}$ is a basis for S and the dimension of S is n. By theorem 2.1.4, it is clear that the functions $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ satisfy the initial conditions. $\phi_i{}^{(i-1)}(x_0)=1$, $\phi_i{}^{(j-1)}(x_0)=0$ $(j\ne i)$, $1\le i\le n$, $1\le j\le n$, and forms a basis of S. Thus S is a linear space.

2.1.6 **Problem**: Consider the equation $y^{11}+\frac{1}{x}y^1+\frac{1}{x^2}y=0$ for $x>0$. Show that a) there is a solution of the form x^r, where r is constant. b) Find two linearly independent solutions for $x>0$, and prove that they are linearly independent.

c) Find the two solutions ϕ_1, ϕ_2 satisfying.

$\phi_1(1)=1, \phi_2(1)=0$

$\phi_1{}^1(1)=1, \phi_2{}^1(1)=1.$

Solution: Given equation is $L(y)= y^{11}+\frac{1}{x}y^1+\frac{1}{x^2}y=0, x>0 \rightarrow (1)$.

Let us assume that the solution of (1) is of the form x^r then we have

$$(x^r)^{11}+\frac{1}{x}(x^r)^1-\frac{1}{x^2}x^r=0$$

$$\Rightarrow r(r-1)x^{r-2}+rx^{r-2}-x^{r-2}=0$$

$\Rightarrow (r(r-1)+r-1)x^{r-2}=0$

$\Rightarrow (r^2-1)x^{r-2}=0$ ($\because$ Since $x>0 \Rightarrow x^{r-2}>0$)

$\Rightarrow r^2-1=0 \quad \therefore r=\pm 1.$

The two solutions are x^{-1}, x^1.

To show x^{-1}, x are linearly independent.

Let C_1, C_2 be two constants s.t. $C_1 x^{-1}+C_2 x=0$ differentiate with respect to 'x', we get

$C_1(-\frac{1}{x^2})+C_2=0$. Again differentiate with respect to 'x' $C_2\left(\frac{2}{x^3}\right)=0$

$\Rightarrow C_2=0$ ($\because x>0$) $\quad \therefore C_1=0$.

Hence $x, \frac{1}{x}$ are linearly independent.

From (a), $x, \frac{1}{x}$ are solution of (1). $\quad \phi_1(x)=C_1x+C_2\frac{1}{x}$

$\phi_1^1(x)=C_1+C_2\left(-\frac{1}{x^2}\right)$

given that $\phi_1(1)=1 \Rightarrow C_1+C_2=1$

$\phi_1^{1}(1)=0 \Rightarrow C_1-C_2=0 \quad \therefore C_1=C_2=\frac{1}{2}$

So the solution is $\phi_1(x)=\frac{1}{2}x+\frac{1}{2x}$. Similarly, we try ϕ_2.

2.1.7 **Problem:** Consider the equation $L(y)=y^{11}+a_1(x)y^1+a_2(x)y=0$, where a_1, a_2 are continuous functions on some interval I, and a_1 has a continuous derivative there.

a) If ϕ is a solution of $L(y)=0$. Let $\phi=u\psi$, and determine a differential equation for u which will make ψ the solution of an equation in which the first derivative term is absent.

b) Solve this differential equation.

c) Show that ψ will satisfy the equation. $y^{11}+\alpha(x)y=0$, where $\alpha=a_2-\frac{a_1}{2}-\frac{a_1^2}{4}$.

Solution: Since ϕ is a solution of $L(y)=0$, we have $\phi^{11}+a_1\phi^1+a_2\phi=0$. Let $\phi=u\psi$. Then $\psi^{11}u+\psi^1(2u^1+a_1u)+\psi(u^{11}+a_1u^1+a_2u)=0$.

The required differential equation for u which makes the solution of an equation in which the first derivative vanishes is $2u^1+a_1u=0$.

b) $\quad 2u^1+a_1u=0$

$\Rightarrow \frac{u^1}{u}=-\frac{a_1}{2}$

Integrating on both sides, we get $u = Ce^{-\int \frac{a_1(t)}{2} dt}$

c) We have $a_1 = -\dfrac{2u^1}{u}$

$$u^1 = -\frac{a_1}{2}u.$$

$$u^{11} = -\frac{a_1^1}{2}u - \frac{a_1}{2}u^1$$

$$\frac{u^{11}}{u} = -\frac{a_1^1}{2} - \frac{a_1}{2}\frac{u^1}{u} = -\frac{a_1^1}{2} + \frac{a_1^2}{4}$$

ψ satisfies the equation $y^{11} + \left(\dfrac{u^{11}}{u} + a_1\dfrac{u^1}{u} + a_2\right)y = 0.$

i.e. $y^{11} + \left(-\dfrac{a_1^1}{2} + \dfrac{a_1^2}{4} - \dfrac{a_1^2}{2} + a_2\right)y = 0$

i.e., $y^{11} + \left(a_2 - \dfrac{a_1^1}{2} - \dfrac{a_1^2}{4}\right)y = 0.$

2.2 Short Answer questions:

2.2.1 Prove that any linear combination of n solutions of L(y)=0 is again a solution of L(y)=0.

2.3 Model Examination questions:

2.3.1 Find two linear independent solutions of the equation $(3x-1)^2 y^{11} + (9x-3)y^1 - 9y = 0$ for $x > \frac{1}{3}$

2.4 Exercises:

2.4.1 The equation $y^1 + a(x)\,y = 0$ has for a solution $\phi(x) = \exp\left[-\int_{x_0}^{x} a(t)dt\right]$ (Here let a be continuous on an interval I containing x_0). This suggests trying to find a solution of $L(y) = y^{11} + a_1(x)y^1 + a_2(x)y = 0$ of the form $\phi(x) = \exp\left[\int_{x_0}^{x} p(t)dt\right]$, where p is a function to be determined. Show that ϕ is a solution of L(y)=0 if and only if, p satisfies the first order non-linear equation $y^1 = -y^2 - a_1(x)y - a_2(x)$.

2.5 Answers to short answer questions:

For 2.2.1, see note 2.1.1.

2)

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Lesson-3

The Wronskian and Linear Independence of the Solutions of the Homogeneous Equation

3.0 **Introduction:** In this lesson we define the wronskian of any n solutions $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ and show that any set of n solutions of L(y)=0 on an interval I is a linearly independent set if and only if their wronskian does not vanish on I. Further some results involving wronskian are established.

3.1 **Linear independence of the solutions of the homogeneous equations:**

3.1.1 **Definition**: The wronskian $w(\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n)$ of n functions $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ having n-1 derivatives on an interval I is defined to be the determinant function

$$w(\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots\cdots & \phi_n \\ \phi_1^1 & \phi_2^1 & \cdots\cdots & \phi_n^1 \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots\cdots & \phi_n^{(n-1)} \end{vmatrix}$$

Its value at any point $x \in I$ is $w((\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n)(x)$.

3.1.2 **Theorem**: If $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ are n solutions of L(y)=0 on an interval I, they are linearly independent if and only if $w(\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n)(x) \neq 0$ for all x in I.

Proof: First suppose that $w(\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n)(x) \neq 0 \ \forall\ x \in I$.

If there are constants $C_1, C_2, \ldots\ldots C_n$ such that

$C_1 \phi_1(x) + C_2 \phi_2(x) + \ldots\ldots + C_n \phi_n(x) = 0$ for all $x \in I$ ---- (1)

$$\left.\begin{array}{l} C_1 \phi_1^1(x) + C_2 \phi_2^1(x) + \ldots\ldots + C_n \phi_n^1(x) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1 \phi_1^{(n-1)}(x) + C_2 \phi_2^{(n-1)}(x) + \ldots\ldots + C_n \phi_n^{(n-1)}(x) = 0 \text{ for all } x \in I. \end{array}\right\} \text{---- (2)}$$

For a fixed $x \in I$ the equations (1), (2) are n linear homogeneous equations satisfied by C_1, $C_2 \ldots C_n$. The determinant of the coefficients is $w(\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n)(x)$ which is not zero. Therefore there is only one solution to the system (1), (2).

i.e. $C_1=C_2= \ldots\ldots = C_n = 0$

Hence $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ are linearly independent on I.

Conversely, suppose that $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ are linearly independent on I. Suppose there is an x_0 in I such that $w(\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n)(x_0)=0$. Then this implies that the system of n linear equations

$$\left.\begin{array}{l} C_1 \phi_1(x_0) + C_2 \phi_2(x_0) + \ldots\ldots + C_n \phi_n(x_0) = 0 \\ C_1 \phi_1^1(x_0) + C_2 \phi_2^1(x_0) + \ldots\ldots + C_n \phi_n^1(x_0) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1 \phi_1^{(n-1)}(x) + C_2 \phi_2^{(n-1)}(x) + \ldots\ldots + C_n \phi_n^{(n-1)}(x_0)=0 \end{array}\right\} \text{---- (3)}$$

has a solution $C_1, C_2, \ldots C_n$ where not all the constants $C_1, C_2, \ldots C_n$ are zero.

Let $C_1, C_2, \ldots C_n$ be such a solution, and consider the function $\psi = C_1\phi_1 + C_2\phi_2 + \ldots + C_n\phi_n$.

Then $L(\psi)=0$, since $\phi_1, \phi_2, \ldots, \phi_n$ are solutions of $L(y)=0$.

From (3), we have $\psi(x_0)=0, \psi^1(x_0)=0, \ldots \psi^{n-1}(x_0)=0$. By the uniqueness theorem $\psi(x)=0$ for all $x \in I$. Since not all C_i $(1 \le i \le n)$ are zero, $\phi_1, \phi_2, \ldots, \phi_n$ are linearly dependent on I, a contradiction. Therefore our assumption that there is a point $x_0 \in I$ such that $w(\phi_1, \phi_2, \ldots, \phi_n)(x_0)=0$ is false. Hence $w(\phi_1, \phi_2, \ldots, \phi_n)(x) \neq 0 \ \forall \ x \in I$.

Hence the theorem.

3.1.3 **Theorem**: Let $\phi_1, \phi_2, \ldots, \phi_n$ be n linearly independent solutions of $L(y)=0$ on an interval I. If ϕ is any solution of $L(y)=0$ on I, it can be represented in the form $\phi = C_1\phi_1 + C_2\phi_2 + \ldots + C_n\phi_n$, where $C_1, C_2, \ldots C_n$ are constants. Thus any set of n linearly independent solutions of $L(y)=0$ on I is a basis for the solutions of $L(y)=0$ on I.

Proof: We have by theorem 3.1.2

$\phi_1, \phi_2, \ldots \phi_n$ are linearly independent on I if and only if $w(\phi_1, \phi_2, \ldots \phi_n)(x) \neq 0$ for all $x \in I$.

Let ϕ be any solution of $L(y)=0$ and x_0 be a point in I.

Suppose that $\phi(x_0) = \alpha_1, \phi^1(x_0) = \alpha_2, \ldots, \phi^{(n-1)}(x_0) = \alpha_n$.

Consider the following system of n linearly equations in $C_1, C_2, \ldots C_n$.

$$\left.\begin{aligned} C_1\phi_1(x_0) + C_2\phi_2(x_0) + \ldots + C_n\phi_n(x_0) &= \alpha_1 \\ C_1\phi_1^1(x_0) + C_2\phi_2^1(x_0) + \ldots + C_n\phi_n^1(x_0) &= \alpha_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_1\phi_1^{(n-1)}(x) + C_2\phi_2^{(n-1)}(x) + \ldots + C_n\phi_n^{(n-1)}(x_0) &= \alpha_n \end{aligned}\right\} \quad \text{---- (1)}$$

There is a unique solution for $C_1, C_2, \ldots C_n$.

Since the determinant of the coefficients of $C_1, C_2, \ldots C_n$ is $w(\phi_1, \phi_2, \ldots, \phi_n)$ which is not zero. With these $C_1, C_2, \ldots, C_n$, we take $\psi = C_1\phi_1 + C_2\phi_2 + \ldots + C_n\phi_n$ we observe that

i) ψ is a solution of $L(y)=0$ (Since $\phi_1, \phi_2, \ldots, \phi_n$ are solutions of $L(y)=0$)

ii) $\psi(x_0) = \alpha_1, \psi^1(x_0) = \alpha_2, \ldots \psi^{(n-1)}(x_0) = \alpha_n$. (From the equations of (1))

By the uniqueness theorem, we have $\psi = \phi$.

Thus for any solution ϕ of $L(y)=0$, there exists a unique set of n constants such that $\phi = C_1\phi_1 + C_2\phi_2 + \ldots + C_n\phi_n$ i.e. any solution of $L(y)=0$ is a linear combination of n linearly independent solutions $\phi_1, \phi_2, \ldots, \phi_n$. Therefore any set of n linearly independent solutions of $L(y)=0$ on I is a basis of the solution space of $L(y)=0$. Hence the theorem.

3.1.4 **Theorem**: Let $\phi_1, \phi_2, \ldots, \phi_n$ be n solutions of $L(y)=0$ on an interval I and x_0 be any point in I. Then $w(\phi_1, \phi_2, \ldots, \phi_n)(x) = w(\phi_1, \phi_2, \ldots, \phi_n)(x_0) \exp\left(-\int_{x_0}^{x} a_1(t)\,dt\right)$.

Proof: We shall give the proof of theorem when n=2 and then give proof of theorem in the general case.

Proof of the theorem when n=2.

In this case $w(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1^1 & \phi_2^1 \end{vmatrix} = \phi_1 \phi_2^1 - \phi_2 \phi_1^1$.

$w^1(\phi_1, \phi_2) = \phi_1 \phi_2^1 + \phi_1 \phi_2^{11} - \phi_2^1 \phi_1^1 - \phi_2 \phi_1^{11} = \phi_1 \phi_2^{11} - \phi_2 \phi_1^{11}$.

Since ϕ_1, ϕ_2 satisfy $y^{11} + a_1(x) y^1 + a_2(x) y = 0$,

We get $\phi_1^{11} = -a_1\phi_1^1 - a_2 \phi_1$

$\phi_2^{11} = -a_1\phi_2^1 - a_2 \phi_2$.

$\therefore w^1(\phi_1, \phi_2) = \phi_1(-a_1\phi_2^1 - a_2 \phi_2) - \phi_2(-a_1\phi_1^1 - a_2 \phi_1)$

$= - a_1(\phi_1 \phi_2^1 - \phi_2 \phi_1^1) = - a_1 w(\phi_1, \phi_2)$

$\Rightarrow w(\phi_1, \phi_2)$ satisfies the first order differential equation $y^1+a_1(x)y=0$

its solution is $w(\phi_1, \phi_2)(x) = C \exp\left(-\int_{x_0}^{x} a_1(t)\,dt\right)$

putting $x = x_0$, we get $C = w(\phi_1, \phi_2)(x_0) \exp\left(\int_{x_0}^{x_0} a_1(t)\,dt\right)$

$= w(\phi_1, \phi_2)(x_0)$.

$\therefore w(\phi_1, \phi_2)(x) = C \exp\left(-\int_{x_0}^{x} a_1(t)\,dt\right)$

$= w(\phi_1, \phi_2)(x_0) \exp\left(-\int_{x_0}^{x} a_1(t)\,dt\right)$.

Hence the theorem when n = 2.

Proof for a general n: Let us write w for $w(\phi_1, \phi_2, \ldots, \phi_n)$ for convenience. We know that the derivative w^1 is a sum of n determinants.

$w^1 = v_1 + v_2 + \ldots + v_n$ where v_k differs from w only in its k^{th} row, and the k^{th} row of v_k is obtained by differentiating the k^{th} row of w.

$$\text{Thus } w^1 = \begin{vmatrix} \phi_1^1 \ldots \phi_n^1 \\ \phi_2^1 \ldots \phi_n^1 \\ \phi_1^{11} \ldots \phi_n^{11} \\ \ldots \\ \phi_1^{(n-1)} \ldots \phi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \phi_1 \ldots \phi_n \\ \phi_1^{11} \ldots \phi_n^{11} \\ \phi_1^{11} \ldots \phi_n^{11} \\ \ldots \\ \phi_1^{(n-1)} \ldots \phi_n^{(n-1)} \end{vmatrix} + \ldots + \begin{vmatrix} \phi_1 \ldots \phi_n \\ \phi_1^1 \ldots \phi_n^1 \\ \phi_1^{11} \ldots \phi_n^{11} \\ \ldots \\ \phi_1^{(n)} \ldots \phi_n^{(n)} \end{vmatrix}$$

We see that the first (n-1) determinants $v_1, v_2, \ldots, v_{n-1}$ are all zero, since they each have two identical rows. Since $\phi_1, \phi_2, \ldots, \phi_n$ are solutions of L(y)=0 we have

$\phi_i^{(n)} = -a_1 \phi_i^{(n-1)} - \ldots - a_n\phi_i$ (i = 1,2, ….., n),

$$\text{Therefore } w^1 = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1^1 & \cdots & \phi_n^1 \\ \cdots & \cdots & \cdots \\ \phi_1^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ -\sum_{j=0}^{n-1} a_{n-j}\phi_1^{(j)} & \cdots & -\sum_{j=0}^{n-1} a_{n-j}\phi_n^{(j)} \end{vmatrix}$$

The value of this determinant is unchanged if we multiply any row by a number and add to the last row. Multiplying the first row by a_n, the second by a_{n-1}, ….., the $(n-1)^{th}$ – row by a_2 and adding these to the last row, we get

$$w^1 = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1^1 & \cdots & \phi_n^1 \\ \cdots & \cdots & \cdots \\ \phi_1^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ -a_1\phi_1^{(n-1)} & \cdots & -a_1\phi_n^{(n-1)} \end{vmatrix} = -a_1 w$$

$\Rightarrow w^1 + a_1 w = 0$

Thus w satisfies the linear first order differential equation $y^1 + a_1(x)\, y = 0$ and therefore

$w(x) = C \exp\left(-\int_{x_0}^{x} a_1(t)\, dt \right)$ putting $x = x_0$ we get $w(x_0) = C$.

Therefore $w(x) = w(x_0) \exp\left(-\int_{x_0}^{x} a_1(t)\, dt \right)$.

Hence the theorem.

3.1.5 **Corollary**: If the coefficients a_k of L are constants,

then $w(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) = e^{-a_1(x-x_0)} w(\phi_1, \ldots \phi_n)(x_0)$

Proof: Since $a_1, a_2, \ldots.., a_n$ are constants, we have $w(x) = w(x_0) \exp\left(-\int_{x_0}^{x} a_1(t)\, dt \right)$

$= w(x_0) \exp(-a_1(x - x_0))$.

3.1.6 **Note**: The n solutions $\phi_1, \phi_2, \ldots\ldots, \phi_n$ of $L(y)=0$ on an interval I are linearly independent if and only if $w(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) \neq 0$ for any particular x_0 in I.

For, by theorem 3.1.2, $\phi_1, \phi_2, \ldots\ldots\ldots, \phi_n$ are linearly independent on I if and only if $w(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) \neq 0$ for all $x \in I$.

$\Leftrightarrow \exp\left(-\int_{x}^{x_0} a_1(t)\, dt \right) w(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) \neq 0$ for any $x_0 \in I$. (By theorem 3.1.4)

$\Leftrightarrow w(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) \neq 0$ for any $x_0 \in I$.

3.1.7 **Problem**: Consider the equation $L(y) = y^{11}+a_1(x)y^1+a_2(x)y = 0$ where a_1, a_2 are continuous on some interval I. Let ϕ_1, ϕ_2 and ψ_1, ψ_2 bases for the solutions of L(y)=0. Show that there is a non-zero constant k such that $w(\psi_1, \psi_2)(x) = k\, w(\phi_1, \phi_2)(x)$.

Solution: Let $\{\phi_1, \phi_2\}$ be a basis for the solution of L(y)=0. Let $\{\psi_1, \psi_2\}$ be another basis for the solution of L(y)=0. Therefore there exists constants C_1, d_1, C_2, d_2 such that $\psi_1=C_1\phi_1 + C_2\phi_2$ and $\psi_2=d_1\phi_1 + d_2\phi_2$.

Consider $w(\psi_1, \psi_2)(x) = \psi_1(x)\, \psi_2^1(x) - \psi_2(x)\, \psi_1^1(x)$.

$$= (C_1\phi_1 + C_2\phi_2)(d_1 \phi_1^1+ d_2\phi_2^1) - (C_1 \phi_1^1+ C_2\phi_2^1)(d_1\phi_1 + d_2\phi_2).$$

$$= C_1\phi_1 d_1 \phi_1^1 + C_1d_2 \phi_1\phi_2^1 + C_2d_1\phi_2\phi_1^1 + C_2d_2 \phi_2\phi_2^1 - C_1d_1\phi_1\phi_1^1 - C_1d_2\phi_2\phi_1^1 - C_2d_1\phi_1\phi_2^1 - d_2C_2\phi_2\phi_2^1.$$

$$= (C_1d_2 - C_2d_1)(\phi_1\phi_2^1 - \phi_2\phi_1^1)$$

$$= k(w(\phi_1\phi_2)) \text{ where } k = C_1d_2 - C_2d_1.$$

Hence the problem.

3.1.8 **Problem**: Consider the equation

$L(y)= y^{11}+a_1(x)\, y^1 + a_2(x)\, y=0$

Where a_1, a_2 are continuous on some interval I. Show that a_1, a_2 are uniquely determined by any basis ϕ_1, ϕ_2 for the solutions of L(y)=0.

Solution: Given that ϕ_1, ϕ_2 are two Linearly independent solutions of L(y)=0 then we have $L(\phi_1)=0$ and $L(\phi_2)=0$

$\therefore\ \phi_1^{11}+ a_1(x)\, \phi_1^1 + a_2(x)\, \phi_1=0 \quad \rightarrow (1)$

$\phi_2^{11}+ a_1(x)\, \phi_2^1 + a_2(x)\, \phi_2=0 \quad \rightarrow (2).$

Multiplying equation (1) by ϕ_1 and equation (2) by ϕ_2 and subtracting

We get $(\phi_1^{11}\phi_2 - \phi_2^{11}\phi_1) + a_1(x)(\phi_2\, \phi_1^1 - \phi_1\, \phi_2^1)=0$

$$\Rightarrow a_1(x) = \frac{\phi_1\phi_2^{11} - \phi_2\phi_1^{11}}{\phi_2\phi_1^1 - \phi_1\phi_2^1} = \frac{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1^{11} & \phi_2^{11} \end{vmatrix}}{-w(\phi_1,\phi_2)}$$

multiplying equation (1) by ϕ_2^1 and equation (2) by ϕ_1^1 and subtracting, we get

$(\phi_1^{11}\phi_2^1 - \phi_1^1\phi_2^{11}) + a_2(x)(\phi_1\, \phi_2^1 - \phi_2\, \phi_1^1)=0.$

$$\Rightarrow a_2(x) = \frac{\phi_1^1\phi_2^{11} - \phi_1^{11}\phi_2^1}{\phi_1\phi_2^1 - \phi_2\phi_1^1} = \frac{\begin{vmatrix} \phi_1^1 & \phi_2^1 \\ \phi_1^{11} & \phi_2^{11} \end{vmatrix}}{w(\phi_1,\phi_2)}$$

Thus a and a_2 are uniquely determined.

3.2 Short Answer questions:

3.2.1 Define wronskian of $\phi_1, \phi_2, \ldots, \phi_n$.

3.2.2 If the coefficients a_k of L(y)=0 are constants, then

$$w(\phi_1, \phi_2, \ldots, \phi_n)(x) = e^{-a_1(x-x_0)} w(\phi_1, \phi_2 \ldots \phi_n)(x_0).$$

3.2.3 Prove that the n solutions $\phi_1, \phi_2, \ldots, \phi_n$ of L(y)=0 on an interval I are linearly independent iff $w(\phi_1, \phi_2, \ldots, \phi_n)(x_0) \neq 0$ for any particular x_0 in I.

3.3 Model Examination Questions:

3.3.1 Let $\phi_1, \phi_2, \ldots, \phi_n$ be n solutions of $y^{(n)} + a_1(x)\, y^{(n-1)} + a_2(x)\, y^{(n-2)} + \ldots + a_n(x)\, y = 0$ on an interval I and let x_0 be any point in I then show that $w(\phi_1, \phi_2, \ldots, \phi_n)(x) = \exp\left(-\int_{x_0}^{x} a_1(t)\, dt\right) w(\phi_1, \phi_2, \ldots, \phi_n)(x_0)$.

3.3.2 If $\phi_1, \phi_2, \ldots, \phi_n$ are n solutions of L(y)=0 on an interval I, then show that they are linearly independent if and only if $w(\phi_1, \phi_2, \ldots, \phi_n)(x) \neq 0$ for all x in I.

3.3.3 Consider the equation $L(y) = y^{11} + a_1(x)\, y^1 + a_2(x)\, y = 0$, where $a_1(x)$ & $a_2(x)$ are continuous functions on I. Let ϕ_1, ϕ_2 and ψ_1, ψ_2 be two basis for the solutions of L(y)=0. Show that there is a non-zero constant k such that $w(\psi_1, \psi_2)(x) = k\, w(\phi_1, \phi_2)(x)$.

3.4 Answers to Short Answer questions:

For 3.2.1, See Definition 3.1.1

For 3.2.2, See Corollary 3.1.5

For 3.2.3, See Note 3.1.6

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LESSON-4
REDUCTION OF THE ORDER OF A HOMOGENEOUS EQUATION

4.0 **Introduction:** If one solution ϕ_1 of the equation $L(y) = y^{(n)}+a_1(x)\, y^{(n-1)}+ \ldots + a_n(x)\, y=0$ is known then it is possible to reduce the order of the equation by one. The idea is the some one as employed in the variation of constants method. We wish to find all solutions ϕ of $L(y)=0$ of the form $\phi = u\phi_1$, where u is some function. When n=2, the second solution ϕ_2 of $L(y)=0$ can be obtained.

4.1 **Reduction of the order of the homogeneous equation:**

Suppose we have a solution ϕ_1 of

$L(y) = y^{(n)}+a_1(x)\, y^{(n-1)}+ \ldots + a_n(x)\, y=0$ with $\phi_1(x) \neq 0$. If $\phi = u\phi_1$, where u is some function, is a solution of $L(y)=0$, then

$$(u\phi_1)^{(n)} + a_1(u\phi_1)^{(n-1)} + \ldots + a_{n-1}\,(u\phi_1)' + a_n(u\phi_1)=0$$

i.e. $[u^{(n)}\phi_1 + {}^nC_1\, u^{(n-1)}\phi_1{}^1 + \ldots + u\phi_1{}^n] + a_1[u^{(n-1)}\phi_1 + {}^{n-1}C_1\, u^{(n-2)}\phi_1{}^1 + \ldots + u\phi_1{}^{(n-1)}] +$

$\ldots + a_{n-1}[u\phi_1{}^1 + u^1\phi_1] + a_n\, u\phi_1 = 0.$

i.e. $\phi_1\, u^{(n)} + \ldots + [n\,\phi_1{}^{(n-1)} + {}^{(n-1)}a_1\,\phi_1{}^{(n-2)} + \ldots + a_{n-1}\,\phi_1]u^1 +$

$[\phi_1{}^{(n)} + a_1\,\phi_1{}^{(n-1)} + \ldots + a_{n-1}\,\phi_1 + a_n\phi_1]u = 0 \quad \rightarrow (1)$

Since ϕ_1 is a solution of $L(y)=0$, the coefficient of u is zero. Putting $v=u^1$, the above equation becomes a linear equation of order (n-1) in v.

$$\phi_1 v^{(n-1)} + \ldots + [n\,\phi_1{}^{(n-1)} + a_1{}^{(n-1)}\,\phi_1{}^{(n-2)} + \ldots + a_{n-1}\,\phi_1]v = 0 \quad \text{-----------} \ (1)$$

4.1.1 **Theorem**: Let ϕ_1 be a solution of $L(y)=0$ on an interval I, and suppose $\phi_1(x) \neq 0$ on I. If $v_2, v_3, \ldots, v_n$ is any basis on I for the solutions of the linear equation (1) of order (n-1), and if $v_k = u_k^1\ (k = 2,3\ldots n)$ Then $\phi_1, u_2\phi_1, \ldots u_n\phi_1$ is a basis for the solutions of $L(y)=0$ on I.

Proof: If ϕ_1 is a solution of $L(y)=0$, and $\phi = u\phi_1$ where u is a function, is a solution of $L(y)=0$ then we get equation (1) where $v = u^1$.

The leading coefficient of (1) is ϕ_1. Since $\phi_1(x) \neq 0$ on I, the equation (1) has (n-1) linearly independent solutions, say $v_2, v_3, \ldots v_n$. If x_0 is some point in I and $u_k(x) = \int_{x_0}^{x} v_k(t)\, dt$,

(k=2, 3, …., n). Then $v_k = u_k^1$ $(2 \leq k \leq n)$ and $\phi_1, u_2\phi_1 \ldots u_n\phi_1$ are solutions of $L(y)=0$.

To show $\phi_1, u_2\phi_1 \ldots u_n\phi_1$ are linearly independent.

Suppose we have constants $C_1, C_2, \ldots C_n$ such that $C_1\phi_1 + C_2 u_2\phi_1 + \ldots + C_n u_n\phi_1 = 0$.

Since $\phi_1(x) \neq 0$ on I, we have $C_1 + C_2 u_2 + \ldots + C_n u_n = 0 \quad \rightarrow (2)$

Differentiating we get $C_2u_2^{1}+.......+C_nu_n^{1}=0$

i.e. $C_2v_2 + ...+ C_nv_n=0$.

Since $v_1, v_2, ..., v_n$ are linearly independent on I, we have $C_2 = C_3 = ...=C_n = 0$.

Substituting in (2) we get $C_1=0$.

Thus $\phi_1, u_2\phi_1 .....u_n\phi_1$ are linearly independent on I. By theorem 3.1.3, $\{\phi_1, u_2\phi_1 .....u_n\phi_1\}$ is a basis for the solutions of L(y)=0 on I.

Hence the theorem.

In the case when n=2, the equation (1) for v is a first order linear differential equation and can be solved explicitly. The expression for the second solution is given in the following theorem.

4.1.2 **Theorem**: If ϕ_1 is a solution of $L(y) = y^{11}+a_1(x)\,y^{1}+a_2(x)\,y=0 \rightarrow (1)$ on an interval I and $\phi_1(x)\neq 0$ on I then a second solution of (1) on I is given by

$$\phi_2(x)=\phi_1(x)\int_{x_0}^{x}\frac{1}{[\phi_1(s)]^2}\exp\left[-\int_{x_0}^{x}a_1(t)dt\right]ds.$$

The functions ϕ_1, ϕ_2 form a basis for the solutions of (1) on I.

Proof: Since ϕ_1 is a solution of (1) on I, we have $\phi_1^{11}+a_1\phi_1^{1}+a_2\phi_1=0$.

Consider $L(u\phi_1)=(u\phi_1)^{11}+a_1(u\phi_1)^{1}+a_2(u\phi_1)$

$$= u^{11}\phi_1 + 2u^{1}\phi_1^{1} + u\phi_1^{11} + a_1u^{1}\phi_1 + a_1u\phi_1^{1} + a_2u\phi_1$$

$$= u^{11}\phi_1 + u^{1}(2\phi_1^{1}+a_1\phi_1) + u(\phi_1^{11}+a_1\phi_1^{1}+a_2\phi_1)$$

$$= u^{11}\phi_1 + u^{1}(2\phi_1^{1}+a_1\phi_1).$$

Taking u such that $L(u\phi_1)=0$ and if $v=u^{1}$ then we get $\phi_1v^{1} + (2\phi_1^{1}+a_1\phi_1)v = 0 \quad \rightarrow (2)$

The above equation is a first order linear differential equation, which can be solved explicitly (since $\phi_1(x)\neq 0$ on I). Multiplying the above equation by ϕ_1 we get

$$\phi_1^2v^1 + (2\phi_1\,\phi_1^{1} + a_1\,\phi_1^2)\,v = 0$$

$$\Rightarrow \phi_1^2v^1 + 2\phi_1\,\phi_1^{1}v + a_1\phi_1^2v = 0$$

$$\Rightarrow (\phi_1^2v)^1 + a_1(\phi_1^2v) = 0 \qquad \rightarrow (3)$$

$\therefore\ [\phi_1^2(x)\,v(x)]\exp\left[\int_{x_0}^{x}a_1(t)\,dt\right]=C$ where x_0 is a point in I and C is a constant,

is a solution of (3).

$$\Rightarrow \phi_1^2(x)v(x)=C\exp\left[-\int_{x_0}^{x}a_1(t)\,dt\right]$$

$$\Rightarrow v(x)=\frac{1}{\phi_1^2(x)}\exp\left[-\int_{x_0}^{x}a_1(t)\,dt\right]$$

$$\Rightarrow v(x) = \frac{1}{\phi_1^2(x)} \exp[-\int_{x_0}^{x} a_1(t)\,dt] \text{ is a solution of (2) or (3)}$$

$$\therefore u(x) = \int_{x_0}^{x} \frac{1}{[\phi_1^2(s)]^2} \exp[-\int_{x_0}^{x} a_1(t)dt]ds.$$ The second solution ϕ_2 of (1) is $u\,\phi_1$

$$\therefore \phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2} \exp[-\int_{x_0}^{x} a_1(t)dt]ds.$$

From theorem (4.1.1) ϕ_1, ϕ_2 from a basis for the solution space of (1) on I_0.

4.1.3 **Problem**: Find a basis for the solution space of the differential equation

$y^{11} - \frac{2}{x^2} y = 0, (0 < x < \infty)$ given that $\phi_1(x) = x^2$ is a solution.

Solution: Note that $\phi_1(x) \neq 0$ for $x \in (0, \infty)$. By theorem 4.1.2, there is another independent solution ϕ_2 of the form $\phi_2 = u\phi_1$. Then $x^2 u^{11} + (4x) u^1 = 0$.

Putting $u^1 = v$ we get $x^2 v^1 + 4xv = 0$

i.e. $xv^1 + 4v = 0 \quad \rightarrow (1)$

solution is $v \exp[\int_{x_0}^{x} \frac{4}{t} dt] = C$

$\Rightarrow v \exp\left(\log t^4\right)_0^x = C$

$\Rightarrow v = Cx^{-4}$

$\Rightarrow v = x^{-4}$ is a solution for (1)

$\therefore u^1 = v = \frac{1}{x^4}$

$u = -\frac{1}{3x^3}$

$\therefore \phi_2(x) = u\phi_1 = -\frac{1}{3x}$

Since any constant times a solution is a solution, we may choose for a second solution,

$\phi_2(x) = \frac{1}{x}$.

$\therefore x^2, x^{-1}$ form a basis for the solutions on $(0, \infty)$.

4.1.4 **Problem**: Solve $x^2y^{11} - 7xy^1 + 15y = 0$, $\phi_1(x) = x^3$ $(x > 0)$.

Solution: The given equation is $x^2y^{11} - 7xy^1 + 15y = 0$

$$\Rightarrow y^{11} - \frac{7}{x} y^1 + \frac{15}{x^2} y = 0$$

which is in the form $y^{11}+a_1(x)\,y^1 + a_2(x)\,y=0$.

Here $a_1(x) = -\frac{7}{x}$, $a_2(x) = -\frac{15}{x^2}$.

The function $\phi_1(x) = x^3$ is a solution (verify).

We note that $\varphi_1(x) \neq 0$ on $(0, \infty)$. By theorem 4.1.2, there is another independent solution ϕ_2 of the form $u\phi_1$. Then $x^3 u^{11} + \left(2x^3 - \frac{7}{x}x^3\right)u^1=0$ putting $u^1 = v$ we get $x^3 v^1+(2x^3-7x^2)\,v = 0$.

$$\therefore\quad v = \frac{1}{\phi_1^2(x)} \exp\left(-\int_0^x a_1(t)dt\right)$$

$$= \frac{1}{x^6} \exp\left(-\int_0^x \frac{7}{t}dt\right) = \frac{1}{x^6}.x^7 = x .$$

$$\therefore\ u = \frac{x^2}{2}$$

$\phi_2 = u\phi_1 = x^5$ (Taking out constants)

Hence the solution of the given equation is $\phi(x)=C_1\,x^3+C_2\,x^5$ for some constants C_1, C_2.

4.2 Short Answer questions:

4.2.1 If ϕ_1 is a solution of $L(y)=y^{11}+a_1(x)y^1+a_2(x)y=0$ on an interval I and $\phi_1(x) \neq 0$ on I, then write the formula to find the second solution ϕ_2 of $L(y)=0$ on I.

4.3 Model Examination Questions:

4.3.1 Let ϕ_1 be a solution of $L(y)=0$ on an interval I and suppose that $\phi_1(x) \neq 0$ on I. If $v_2, v_3, \ldots, v_n$ is any basis on I for the solution of the linear equation.

$\phi_1\, v^{(n-1)} + \ldots + [n\,\phi_1^{(n-1)} + (n-1)\,\phi_1^{(n-1)} + \ldots + a_{n-1}]\, v = 0$ of order (n-1) and if $v_k = u_k^1$ for $k = 2, 3, \ldots n$. Then $\phi_1, u_2\phi_1, \ldots, u_n\,\phi_1$ is a basis for the solutions of $L(y)=0$ on an interval I.

4.3.2 One solution of $x^2y^{111}-3x^2y^1+6xy^1-6y=0$ for $x > 0$ is $\phi_1(x)=x$. Find a basis for the solutions for $x > 0$.

4.3.3 Solve $x^2y^{11}- 7xy^1+ 15y=0$, $\phi_1(x)=x^3$.

4.4 Exercises:

4.4.1 A differential equation and a function ϕ_1 are given in each of the following. Verify that the function ϕ_1 satisfies the equation and find a second independent solution.

a) $x^2\,y^{11} - xy^1 + y = 0$, $\phi_1(x) = x$ $(x > 0)$

Answer: $\phi_2(x) = x \log x$

b) $y^{11} - 4xy^1 + (4x^2 - 2)\,y = 0$, $\phi_1(x) = e^{x^2}$

Answer: $\phi_2(x) = x\,e^{x^2}$

c) $xy^{11} - (x+1)\,y^{1} + y = 0,\ \phi_1(x) = e^x\ (x > 0)$

Answer: $\phi_2(x) = -x - 1$

d) $(1-x^2)\,y^{11} - 2xy^{1} + 2y = 0,\ \phi_1(x) = x\ \ (0 < x < 1)$

Answer: $\phi_2(x) = \frac{x}{2}\log\left(\frac{x+1}{x-1}\right) - 1.$

e) $y^{11} - 2xy^{1} + 2y = 0,\ \phi_1(x) = x\ (x > 0)$

Answer: $\phi_2(x) = x\int_1^x t^{-2} e^{t^2}\, dt.$

4.5 Answer to short answer questions:

For 4.2.1, see theorem 4.1.2.

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LESSON-5

SOLUTIONS OF THE NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

5.0 **Introduction:** Let $a_1, a_2, \ldots\ldots, a_n$, b be continuous functions on an interval I and consider the non-homogeneous linear differential equation

$$L(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots\ldots. + a_n(x)\, y = b(x) \quad \rightarrow (1)$$

If ψ_p is a particular solution of (1) then we prove that any solution of (1) is of the form.

$\psi = \psi_p + C_1\phi_1 + C_2\phi_2 + \ldots\ldots + C_n\phi_n$, where $C_1, C_2, \ldots.., C_n$ are constants and $\{\phi_1, \phi_2, \ldots., \phi_n\}$ is a basis for the solution space of L(y)=0. The form of ψ_p is derived.

5.1 **Solutions of the non-homogeneous Linear Differential Equations:**

5.1.1 **Theorem:** Let $a_1, a_2, \ldots.., a_n$ and b be continuous on an interval I and let $\phi_1, \phi_2, \ldots., \phi_n$ be a basis for the solutions of L(y)=0. Every solution ψ of L(y) = b(x) can be written as

$\psi = \psi_p + \sum_{k=1}^{n} C_k\, \phi_k$, where ψ_p is a particular solution of L(y) = b(x), and $C_1, C_2, \ldots.., C_n$ are constants. Every such ψ is a solution of L(y) = b(x). A particular solution ψ_p is given by

$$\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{w_k(t)\, b(t)}{w(\phi_1,\phi_2,\ldots..\phi_n)(t)}\, dt,$$

where $w(\phi_1, \phi_2, \ldots., \phi_n)$ is the wronskian of the basis $\phi_1, \phi_2, \ldots., \phi_n$ and w_k is the determinant obtained from $w(\phi_1, \phi_2, \ldots., \phi_n)$ by replacing the k^{th} column $((\phi_k, \phi_k^{1}, \ldots\ldots, \phi_k^{(n-1)})$ by column (0, 0, ……, 1).

Proof: If ψ_p is a particular solution of (1) and ψ is any solution of (1) then

$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b(x) - b(x) = 0$

Thus $\psi - \psi_p$ is a solution of the homogeneous equation L(y)=0. Therefore $\psi - \psi_p$ is a linea combination of $\phi_1, \phi_2, \ldots., \phi_n$ (Since $\{\phi_1, \phi_2, \ldots., \phi_n\}$ is a basis for the solution space c L(y)=0).

$\therefore \psi - \psi_p = \sum_{k=1}^{n} C_k\, \phi_k$, where $C_1, C_2, \ldots.\ C_k$ are constants.

This implies that any solution ψ of (1) is of the form $\psi = \psi_p + \sum_{k=1}^{n} C_k\, \phi_k$.

Thus we see that the problem of finding all solutions of (1) reduces to finding a particu solution ψ_p of (1), unless b(x)=0 on I. We shall now allow $C_1, C_2, \ldots.\ C_n$ to be functic $u_1, u_2, \ldots\ldots, u_n$ on I. So that ψ_p is a solution of (1) in the form $\sum_{1}^{n} C_k\, \phi_k$.

i.e. $\psi_p = u_1\phi_1 + u_2\phi_2 + \dots\dots + u_n\phi_n$ is a solution of (1).

If $u_1^1\phi_1 + u_2^1\phi_2 + \dots\dots + u_n^1\phi_n = 0$ then $\psi_p^1 = u_1\phi_1^1 + u_2\phi_2^1 + \dots\dots + u_n\phi_n^1$.

and if $u_1^1\phi_1^1 + u_2^1\phi_2^1 + \dots\dots + u_n^1\phi_n^1 = 0$, we have

$\psi_p^{11} = u_1\phi_1^{11} + u_2\phi_2^{11} + \dots\dots + u_n\phi_n^{11}$.

Thus if $u_1^1, u_2^1, \dots\dots, u_n^1$ satisfy

$$\left.\begin{aligned}
&u_1^1\phi_1 + u_2^1\phi_2 + \dots\dots + u_n^1\phi_n = 0\\
&u_1^1\phi_1^1 + u_2^1\phi_2^1 + \dots\dots + u_n^1\phi_n^1 = 0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&u_1^1\phi_1^{(n-2)} + u_2^1\phi_2^{(n-2)} + \dots\dots + u_n^1\phi_n^{(n-2)} = 0\\
&u_1^1\phi_1^{(n-1)} + u_2^1\phi_2^{(n-1)} \dots\dots + u_n^1\phi_n^{(n-1)} = b
\end{aligned}\right\} \quad \text{--- (2)}$$

We see that $\psi_p = u_1\phi_1 + u_2\phi_2 + \dots\dots + u_n\phi_n$

$\psi_p^1 = u_1\phi_1^1 + u_2\phi_2^1 + \dots\dots + u_n\phi_n^1$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

$\psi_p^{(n-1)} = u_1\phi_1^{(n-1)} + u_2\phi_2^{(n-1)} + \dots\dots + u_n\phi_n^{(n-1)}$

$\psi_p^{(n)} = u_1\phi_1^{(n)} + u_2\phi_2^{(n)} + \dots\dots + u_n\phi_n^{(n)} + b$

$\therefore L(\psi_p) = u_1 L(\phi_1) + u_2 L(\phi_2) + \dots\dots + u_n L(\phi_n) + b = b$

$\therefore$ ψ_p is a solution of (1). The whole problem is now reduced to solving the system (2) for $u_1^1, u_2^1, \dots\dots u_n^1$.

The determinant of the coefficients is $w(\phi_1, \phi_2, \dots, \phi_n)$, which is never zero. ($\because$ $\phi_1, \phi_2, \dots, \phi_n$ are linearly independent solutions of $L(y)=0$). Therefore there exists a unique solution for $u_1^1, u_2^1, \dots\dots u_n^1$ satisfying (2) and $u_k^1(x) = \dfrac{w_k(x)b(x)}{w(\phi_1, \phi_2, \dots \phi_n(x))}$ $(1 \le k \le n)$ where w_k is the determinant obtained from $w(\phi_1, \phi_2, \dots \phi_n)$ by replacing the column (i.e. $\phi_k, \phi_k^1, \dots\dots \phi_k^{(n-1)}$) by column $(0, 0, \dots\dots 1)$. If x_0 is any point in I, we take for u_k the functions given by

$$u_k(x) = \int_{x_0}^{x} \frac{w_k(t)b(t)}{w(\phi_1, \phi_2, \dots \phi_k(t))} dt, \quad (1 \le k \le n).$$

The particular solution ψ_p now takes the form $\psi_p(x) = \sum_{1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{w_k(t)b(t)}{w(\phi_1, \phi_2, ... \phi_k(t))} dt$

Hence the theorem.

5.1.2 **Problem:** Find all solutions of $y^{11} - \frac{2}{x^2} y = x (0 < x < \infty)$.

Solutions: We have already seen in Problem 4.1.3, that a basis for the solution space of L(y)=0 is given by $\phi_1(x) = x^2$, $\phi_2(x) = x^{-1}$.

A solution ψ_p of the non-homogeneous equation has the form $\psi_p = u_1x^2 + u_2x^{-1}$ where u_1 and u_2 satisfy the equations.

$$x^2 u_1^1 + x^{-1} u_2^1 = 0$$

$$2x\ u_1^1 - \frac{1}{x^2} u_2^1 = x$$

Now $w(\phi_1, \phi_2)(x) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -x^{-2} \end{vmatrix} = -3$

and $u_1^1 = \frac{\begin{vmatrix} 0 & x^{-1} \\ x & -x^{-2} \end{vmatrix}}{-3} = \frac{1}{3}$

$$u_2^1 = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & x \end{vmatrix}}{-3} = -\frac{x^3}{3}$$

$\therefore u_1 = \frac{x}{3}$ and $u_2 = -\frac{x^4}{12}$

So the particular solution is $\psi_p = \frac{x}{3} x^2 - \frac{x^4}{12} x^{-1}$

$$= \frac{x^2}{3} - \frac{x^3}{12} = \frac{x^3}{4}.$$

$\therefore$ Every solution has the form $\psi(x) = \psi_p + C_1\phi_1 + C_2\phi_2 = \frac{x^3}{4} + C_1x^2 + C_2x^{-1}$ for some constants C_1, C_2.

5.2 Short Answer Questions:

5.2.1 Write the formula for the particular solution of the n^{th} order non-homogeneous equation.

5.2.2 Find the particular solution of $y^{11} - \frac{2}{x^2} y = x$

5.3 Model Examination Questions:

5.3.1 Let b be continuous on an interval I, and let $\phi_1, \phi_2, \ldots, \phi_n$ be a basis for the solutions of $L(y)=0$ on I. Prove that every solution ψ of $L(y) = b(x)$ can be written as $\psi = \psi_p + C_1\phi_1 + C_2\phi_2 + \ldots + C_n\phi_n$, where ψ_p is a particular solution of $L(y) = b(x)$, and $C_1, C_2, \ldots, C_n$ are constants.

5.3.2 One solution of $x^2 y'' - 2y=0$ on $0 < x < \infty$ is $\phi_1(x) = x^2$. Then find all solutions of $x^2 y'' - 2y = 2x - 1$ on $0 < x < \infty$.

5.4 Exercises:

5.4.1 One solution of $x^2 y'' - 2y=0$ on $0 < x < \infty$ is $\phi_1(x) = x^2$. Find all solutions of $x^2 y'' - 2y = 2x-1$ on $0 < x < \infty$.

Answer: $\phi(x) = C_1x^2 + C_2x^{-1} + \frac{1}{2} - x$, C_1, C_2 are any constants.

5.4.2 One solution of $x^2y'' - xy' + y = 0, (x > 0)$ is $\phi_1(x) = x$. Find the solution ψ of $x^2y'' - xy' + y = x^2$ satisfying $\psi(1)=1, \psi'(1)=0$

Answer: $\psi(x) = x^2 - 2x \log x$.

5.4.3 a) Show that there is a basis ϕ_1, ϕ_2 for the solutions of $x^2y'' + 4xy' + (2+x^2) y = 0, (x > 0)$ of the form $\phi_1(x) = \frac{\psi_1(x)}{x^2}$, $\phi_2(x) = \frac{\psi_2(x)}{x^2}$.

b) Find all solutions of $x^2y'' + 4xy' + (2+x^2) y = x^2$ for $x > 0$

Answer: a) $\phi_1(x) = x^{-2} \cos x$, $\phi_2(x) = x^{-2} \sin x$.

b) $\phi(x) = C_1x^{-2} \cos x + C_2 x^{-2} \sin x + 1 - 2x^2$, C_1, C_2 are any constants.

5.5 Answers to short answer questions:

For 5.2.1, $\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{w_k(t)\, b(t)}{w(\phi_1, \ldots \phi_n)(t)} dt$

Here $w(\phi_1, \phi_2, \ldots, \phi_n)$ is the wronskian of the basis $\phi_1, \phi_2, \ldots, \phi_n$ and w_k is the determinant obtained from $w(\phi_1, \phi_2, \ldots, \phi_n)$ by replacing the k^{th} column $(\phi_k, \phi_k^1, \ldots \phi_k^{(n-1)})$ by $(0, 0, \ldots, 0, 1)$.

For 5.2.2, see problem 5.1.2.

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LESSON-6
HOMOGENEOUS EQUATIONS WITH ANALYTIC COEFFICIENTS

6.0 **Introduction**: In this lesson, the existence theorem for analytic coefficients is proved

6.1 **Homogeneous equations with analytic coefficients:**

6.1.1 **Definition:** If g is a function defined on an interval I containing a point x_0, we say that g is analytic at x_0 if g can be expanded in a power series about x_0, which has its radius of convergence a positive real number. That is, g is analytic at x_0 if it can be expressed in the form

$g(x) = \sum_{k=0}^{\infty} C_k (x-x_0)^k \rightarrow (1)$ where c_k are the constants and the series $\sum_{k=0}^{\infty} C_k (x-x_0)^k$ converges for $|x-x_0| < r_0$, $r_0 > 0$.

Recall that one of the important properties of an analytic function g is that all of its derivatives exist on $|x-x_0| < r_0$ and they may be computed by differentiating the series (1) term by term. Thus, for example $g^1(x) = \sum_{k=1}^{\infty} k C_k (x-x_0)^{k-1}$

and $g^{11}(x) = \sum_{k=2}^{\infty} k(k-1) C_k (x-x_0)^{k-2}$ and the differentiated series converges on $|x-x_0| < r_0$ also.

If the coefficients $a_1, a_2, \ldots, a_n$ of $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y$ at x_0, it follows that the solutions of the differential equation are also analytic at x_0. In fact solutions can be computed by a formal algebraic process. We shall illustrate this by means of an example.

6.1.2 **Example**: Consider the equation

$$L(y) = y^{11} - xy = 0 \quad \rightarrow (1)$$

Here $a_1(x)=0$, $a_2(x)= -x$ and hence a_1, a_2 are analytic for all real x_0.

Now we try to find a solution of (1) in the series

$$\phi(x) = C_0 + C_1 x + C_2 x^2 + \ldots\ldots\ldots\ldots = \sum_{k=0}^{\infty} C_k x^k$$

$$\phi^1(x) = \sum_{k=1}^{\infty} k C_k x^{k-1}$$

$$\phi^{11}(x) = \sum_{k=2}^{\infty} k(k-1) C_k x^{k-2}$$

$$= \sum_{k=0}^{\infty} (k+1)(k+2) C_{k+2} x^k$$

Also $x\,\phi(x) = \sum_{k=0}^{\infty} C_k\, x^{k+1} \quad = \sum_{k=1}^{\infty} C_{k-1}\, x^k$

$\therefore\ \phi''(x) - x\,\phi(x) = 2c_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)C_{k+2} - C_{k-1}\right]x^k.$

In order for ϕ to be a solution of (1) we must have $\phi''(x) - x\,\phi(x) = 0$.

i.e. $2c_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)C_{k+2} - C_{k-1}\right]x^k = 0$

comparing the coefficients of likely powers of x, we get $2c_2 = 0$

$(k+1)(k+2)\ C_{k+2} - C_{k-1} = 0 \rightarrow (2)$ for k =1, 2, ...

Now for k = 1 from (2) we have $3.2.\ C_3 = C_0$ or $C_3 = \frac{1}{3.2}C_0$

putting k = 2 in (2) we get $C_4 = \frac{C_1}{4.3}$

continuing in this way we see that

$$C_5 = \frac{C_2}{5.4} = 0$$

$$C_6 = \frac{C_3}{6.5} = \frac{C_0}{6.5.3.2}$$

$$C_7 = \frac{C_4}{7.6} = \frac{C_1}{7.6.4.3}$$

It can be shown by induction that

$$C_{3m} = \frac{C_0}{2.3.5.6....(3m-1)3m} \quad (m = 1, 2, ...)$$

$$C_{3m+1} = \frac{C_1}{3.4.6.7...3m(3m+1)} \quad (m = 1, 2, ...)$$

$$C_{3m+2} = 0 \quad (m = 0, 1, 2, ...)$$

Thus all the constants are determined in terms of C_0 and C_1. Using all these in the expression for ϕ (x) and collecting together terms with C_0 and C_1 as factor, we have

$$\phi(x) = C_0\left[1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + ...\right] + C_1\left[x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + ...\right]$$

Define $\phi_1(x) = 1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + ...$

$$\phi_2(x) = 1 + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + ... \rightarrow (3)$$

So that $\phi(x) = C_0\phi_1(x) + C_1\phi_2(x)$ → (4)

We have shown in a formal way, that ϕ satisfies for any two constants C_0 and C_1. In particular the choice $C_0=1$, $C_1=0$ shows that ϕ_1 satisfies this equation and the choice $C_0=0$, $C_1=1$ implies ϕ_2 also satisfies the equation.

We shall now describe the convergence of the series defining $\phi_1(x)$ and $\phi_2(x)$. It will be seen by the ratio test that both the series converge for all finite x.

For example, let us consider the series for $\phi_1(x)$.

Writing $\phi_1(x)$ as $\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{x^{3m}}{2.3.5.6\ (3m-1)3m}$

$$= 1 + \sum_{m=1}^{\infty} d_m(x).$$

We see that $\frac{d_{m+1}(x)}{d_m(x)} = \frac{x^{3m+3}}{2.3.5.6.....(3m-1)(3m)(3m+2)(3m+3)} \times \frac{2.3.5......(3m-1)(3m)}{x^{3m}}$

$$\Rightarrow \left|\frac{d_{m+1}(x)}{d_m(x)}\right| = \frac{|x|^3}{(3m+2)(3m+3)}$$

which tends to zero as $m \to \infty$, provided only that $|x| < \infty$. Hence the series for $\phi_1(x)$ converges for all finite x. Similarly we can show that the series for $\phi_2(x)$ also converges for all finite x. Thus the functions ϕ_1, ϕ_2 given by (4) are solutions of the equation $y'' - xy=0$ on $-\infty < x < \infty$. We have

$$\phi_1(0) = 1, \qquad \phi_2(0) = 0$$
$$\phi_1'(0) = 0, \qquad \phi_2'(0) = 1$$

and therefore $w(\phi_1, \phi_2)(0) = 1 \neq 0$.

Hence ϕ_1 and ϕ_2 are linearly independent solutions of (1).

The method illustrated by this example works in general when the coefficients are analytic and always yields a convergent power series solution for any initial value problem. We shall now state and prove a result on this.

6.1.3 **Theorem**: (Existence theorem for analytic coefficients)

Let x_0 be a real number, and suppose that the coefficients $a_1, a_2, \ldots\ldots, a_n$ in $L(y) = y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots\ldots + a_n(x)\,y = 0$ have a convergent power series expansions in powers of $x - x_0$ on an interval $|x - x_0| < r_0$, $r_0 > 0$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any n constants, there exists a solution ϕ of the problem $L(y)=0$, $y(x_0) = \alpha_1$, $y'(x_0) = \alpha_2 \ldots\ldots y^{(n-1)}(x_0) = \alpha_n$ with a power series expansion $\phi(x) = \sum_{k=0}^{\infty} C_k (x - x_0)^k$ that converges for $|x - x_0| < r_0$.

Proof: We shall now give a proof for the case when n=2 and x_0=0. All the essential ideas appear in this case. We shall make use of two results concerning power series. The first is that if we have two power series $\sum_{k=0}^{\infty} C_k x^k$ and $\sum_{k=0}^{\infty} C_k^1 x^k$ with $|C_k| \le C_k^1$, $C_k^1 \ge 0$, (k=0,1,2,3, ……..) and if the series $\sum_{k=0}^{\infty} C_k x^k$ converges for $|x| < r$, for some r > 0, then the series $\sum_{k=0}^{\infty} C_k x^k$ also converges for $|x| < r$. This is usually called the comparison test for convergence. The second result is that if a series $\sum_{k=0}^{\infty} \alpha_k x^k \to (1)$ is convergent for $|x| < r_0$, then for any x, $|x| = r < r_0$, there exists a constant M > 0 such that

$$r^k |\alpha_k| \le M \,(k=0,1,2,\ldots\ldots) \qquad \to (2)$$

Now consider the equation

$$L(y) = y^{11} + a(x)\, y^1 + b(x)\, y = 0 \qquad \to (3)$$

Where a, b are functions with series expansions $a(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, $b(x) = \sum_{k=0}^{\infty} \beta_k x^k \to (4)$

Which converge for $|x| < r_0$ for some $r_0 > 0$. We are required to produce a solution of (3) in the form of a convergent power series for $|x| < r_0$ say $\phi(x) = \sum_{k=0}^{\infty} C_k x^k \to (5)$, which satisfies $\phi(0)=a$, $\phi^1(0)=a_2$ for any given constants a_1 and a_2. If the series in (5) is convergent, we have $C_0 = a_1$, $C_1 = a_2$ and the constants C_k for $k \ge 2$ will satisfy a recursion relation. From (5) we have $\phi^1(x) = \sum_{k=0}^{\infty} (k+1) C_{k+1} x^k$ and $\phi^{11}(x) = \sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2} x^k \to (6)$

From (4) we obtain $a(x)\,\phi^1(x) = \left(\sum_{k=0}^{\infty} \alpha_k x^k\right)\left(\sum_{k=0}^{\infty} (k+1) C_{k+1} x^k\right)$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty} \alpha_{k-j}(j+1) C_{j+1}\right) x^k \qquad \to (7)$$

and $b(x)\,\phi(x) = \left(\sum_{k=0}^{\infty} \beta_k x^k\right)\left(\sum_{k=0}^{\infty} C_k x^k\right)$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \beta_{k-j} C_j\right) x^k \qquad \to (8)$$

Adding (6), (7) and (8) we get

$$L(\phi(x)) = \sum_{k=0}^{\infty}\left[(k+2)(k+1)C_{k+2} + \sum_{j=0}^{k}\alpha_{k-j}(j+1)C_{j+1} + \sum_{j=0}^{k}\beta_{k-j}C_j\right]x^k = 0$$

which implies that

$$(k+2)(k+1)C_{k+2} = -\sum_{j=0}^{k}\left[\alpha_{k-j}(j+1)C_{j+1} + \beta_{k-j}C_j\right] \text{ for } k=0,1,2,\ldots\ldots \rightarrow (9)$$

Now we have to show that the series $\sum_{k=0}^{\infty} C_k x^k \rightarrow (10)$ with C_k for $k \geq 2$ given by (9) is convergent for $|x| < r_0$. To do this we make use of the two results concerning power series we mentioned earlier.

Let r be any number such that $0 < r < r_0$. Since the series in (4) are convergent for $|x| = r$. We can find a constant $M > 0$ such that $|\alpha_j| r^j \leq M$, $|\beta_j| r^j \leq M$, $j = 0,1,2,\ldots\ldots$ using these in (9)

we see that $(k+2)(k+1)|C_{k+2}| \leq \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)|C_{j+1}| + |C_j|\right]r^j$

$$\leq \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)|C_{j+1}| + |C_j|\right]r^j \leq +M|C_{k+1}|r. \rightarrow (11)$$

Let us define $C_0^1 = |C_0|, C_1^1 = |C_1|$ and C_k^1 for $k \geq 2$ by

$$(k+2)(k+1)C_{k+2}^1 = \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)C_{j+1}^1 + C_j^1\right]r^j + M\,C_{k+1}^1\,r, \ (k=0,1,2,\ldots\ldots) \rightarrow (12)$$

comparing (12) with (11) we see by induction that $|C_k| \leq C_k$, $C_k \geq 0$ $(k=0,1,2,\ldots\ldots) \rightarrow (13)$

we shall now investigate for what values of x the series $\sum_{k=0}^{\infty} C_k x^k \rightarrow (14)$ is convergent.

From (12) we find that $(k+1)kC_{k+1} = \frac{M}{r^{k-1}}\sum_{i=0}^{k-1}\left[(j+1)C_{j+1} + C_j\right]r^j + MC_k r$

and $k(k-1)C_k^1 = \frac{M}{r^{k-2}}\sum_{i=0}^{k-2}\left[(j+1)C_{j+1}^1 + C_j^1\right]r^j + MC_{k-1}^1 r$, for large k. From these expressions we obtain

$$r(k+1)kC_{k+1}^1 = \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\left[(j+1)C_{j+1}^1 + C_j^1\right]r^j + M\left[kC_{k-1}^1\right]r + MC_k^1 r^2$$

$$= k(k-1)C_k^1 - MC_{k-1}^1 r + MkC_k^1 r + MC_{k-1}^1 r + MC_k^1 r^2$$

$= [k(k-1)+Mkr+Mr^2]\,C_k^1$.

Hence $\left|\dfrac{C_{k+1}^1\, x^{k+1}}{C_k^1\, x^k}\right| = \dfrac{[k(k-1)+Mkr+Mr^2]}{r(k+1)k}|x|$.

Which tends to $\dfrac{|x|}{r}$ as $k \to \infty$. Thus by ratio test, the series (14) converges for $|x|<r$. This implies that the series (10) converges for $|x|<r$, and since r is any number satisfying $0<r<r_0$, we see that the series (10) converges for $|x|<r_0$. This completes the theorem.

6.1.4 **Problem**:

Find two linearly independent power series solutions of the equation $y^{11} - x^2y=0 \to (1)$

For what values of x do the series converge?

Solution: Here $a_1=0$, $a_2= -x^2$ are analytic on R. Let us assume that $y = \sum_{n=0}^{\infty} C_n x^n$ is a power series solution of (1). Then $y^1 = \sum_{n=0}^{\infty} C_{n+1}(n+1)x^n$ and $y^{11} = \sum_{n=0}^{\infty} C_{n+2}(n+1)(n+2)x^n$

using these in (1) we have $\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n - x^2\sum_{n=0}^{\infty} C_n x^n = 0$

i.e. $\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n - \sum_{n=0}^{\infty} C_n x^{n+2} = 0$

i.e. $2C_2 + 6C_3x + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}x^n - \sum_{n=2}^{\infty} C_{n-2}x^n = 0$

i.e. $2C_2 + 6C_3x + \sum_{n=2}^{\infty}[(n+2)(n+1)C_{n+2} - C_{n-2}]x^n = 0$.

Equating the coefficients of likely powers of x, we get $2C_2=0$, $6C_3=0$ and

$(n+2)(n+1)C_{n+2} - C_{n-2} = 0$, $n = 2, 3, 4, \ldots\ldots \to (2)$

Hence $C_2=0$, $C_3=0$ and for $n = 2, 3, 4, \ldots\ldots$ From (2) we have

$C_4 = \dfrac{1}{3.4} C_0$

$C_5 = \dfrac{1}{4.5} C_1$

$C_6 = 0$

$C_7 = 0$

$C_8 = \dfrac{1}{3.47.8} C_0$

$$C_9 = \frac{1}{4.5.8.9} C_1, \; C_{10} = 0, \; C_{11} = 0, \ldots\ldots$$

Hence the power series solution of (1) is

$$y = C_0\left[1 + \frac{1}{3.4}x^4 + \frac{1}{3.4.7.8}x^8 + \ldots\ldots\right] + C_1\left[x + \frac{1}{4.5}x^5 + \frac{1}{4.5.8.9}x^9 + \ldots\ldots\right]$$

$$= C_0\left[1 + \sum_{k=1}^{\infty} \frac{1}{3.4.6.7\ldots\ldots(4k-1)(4k)} x^{4k}\right] + C_1\left[x + \sum_{k=1}^{\infty} \frac{x^{4k+1}}{4.5.8.9\ldots\ldots4k(4k+1)}\right]$$

Thus, the functions ϕ_1 and ϕ_2 defined by $\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{4k}}{3.4\ldots\ldots(4k-1)4k}$ → (3) and

$\phi_2(x) = x + \sum_{k=1}^{\infty} \frac{x^{4k+1}}{4.5.8.9\ldots\ldots4k(4k+1)}$ → (4) are solutions of (1).

They are linearly independent, since $w(\phi_1, \phi_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$.

Observe that the series for $\phi_1(x)$ and $\phi_2(x)$ given respectively by (3) and (4) converge for all real x.

6.1.5 **Problem**: Compute the solution ϕ of y^{111}- xy=0 which satisfies $\phi(0)=1$, $\phi^1(0)=0$ and $\phi^{11}(0)=0$.

Solution: Let $\phi(x) = \sum_{n=0}^{\infty} C_n x^n$ be a power series solution of y^{111}- xy=0 → (1)

Substituting into this, $\phi(x) = \sum_{n=0}^{\infty} C_n x^n$,

we have $\sum_{n=0}^{\infty} (n+3)(n+2)(n+1)C_{n+3} x^n - x \sum_{n=1}^{\infty} C_n x^n = 0$.

i.e. $6C_3 + \sum_{n=1}^{\infty} (n+3)(n+2)(n+1)C_{n+3} x^n - \sum_{n=1}^{\infty} C_{n-1} x^n = 0$.

i.e. $6C_3 + \sum_{n=1}^{\infty} [(n+3)(n+2)(n+1)C_{n+3} - C_{n-1}]x^n = 0$.

Comparing the coefficients of likely powers of x, we get $C_3=0$, and

$(n+3)(n+2)(n+1)C_{n+3} - C_{n-1} = 0$ (n = 1, 2, 3, …….) → (2)

For n=1, we have $4.3.2\,C_4 = C_0$ (Or) $C_4 = \frac{1}{2.3.4} C_0$

For n=2, we see that $5.4.3\,C_5 = C_1$ (Or) $C_5 = \frac{1}{3.4.5} C_1$ continuing in this way we obtain

$$C_6 = \frac{1.3.2}{6.5..4.3.2} C_2$$

$$C_7 = \frac{1}{7.6.5} C_3 = 0$$

$$C_8 = \frac{5}{8!} C_0, \quad C_9 = \frac{2.6}{9!} C_1$$

$$C_{10} = \frac{1.2.3.7}{10!} C_2$$

$$C_{11} = \frac{1}{11.10.9} C_7 = 0$$

$$C_{12} = \frac{1.5.9}{12!} C_0$$

$$C_{13} = \frac{2.6.10}{13!} C_1, \text{ etc.}$$

∴ The complete solution of (1) is

$$\phi(x) = C_0\left[1 + \sum_{n=1}^{\infty} \frac{5.9.....(4n-3)}{(4n)!} x^{4n}\right] + C_1\left[x + \sum_{n=1}^{\infty} \frac{2.6.10.....(4n-2)}{(4n-1)!} x^{4n+1}\right] +$$

$$C_2\left[x^2 + \frac{6}{6!}x^6 + \frac{6.7}{10!}x^{10} + ...\right]$$

Now we have to determine the particular solution ϕ of (1) satisfying $\phi(0)=1$, $\phi^{1}(0)=0$, $\phi^{11}(0)=0$

$\phi(0)=1$ implies that $C_0=1$, $\phi^{1}(0)=0$ implies that $C_1=0$ and $\phi^{11}(0)=0$ implies that $C_2=0$

Hence the required solution is $\phi(x) = 1 + \sum_{n=1}^{\alpha} \frac{5.9.......(4n-3)}{(4n)!} x^{4n}$

6.2 Short Answer questions:

6.2.1 When do you say that a function is analytic at a point x_0 in an interval I?

6.3 Model Examination questions:

6.3.1 State and prove existance theorem for analytic coefficients.

6.3.2 Solve $(1 - x^2) y^{11} - 2xy^{1} + \alpha(\alpha+1) y = 0$, where α is a constant.

6.3.3 Find two linear independent solutions of $y^{11} - xy^{1} + y = 0$.

6.4 Exercies:

6.4.1 Find two linearly independent power series solutions of the following equations. For what values do the series converge?

a) $y^{11} - xy^{1} + y = 0$

Answer: $\phi_1(x) = x$, $\phi_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!\, 2^n (2n-1)}$

b) $y^{11} + x^3 y^{1} + x^2 y = 0$

Answer: $\phi_1(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k 1.5.9....(4k-3)}{3.4.7.8....(4k-1)4k} x^{4k}$

$$\phi_2(x) = x + \sum_{k=1}^{\infty} \frac{(-1)^k 2.6.10....(4k-2)}{4.5.8....4k(4k+1)} x^{4k+1}$$

c) $y^{11} + 3x^2y^1 - xy = 0$

Answer: $$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 8.17....(9k-10)}{2.3.5.6....(3k-1)3k} x^{3k}$$

$$\phi_2(x) = x + \sum_{k=1}^{\infty} \frac{(-1)^k 2.11.20....(9k-7)}{3.4.6.7....3k(3k+1)} x^{3k+1}$$

d) $y^{11} + y = 0$

Answer: $$\phi_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \text{Cos}\, x$$

$$\phi_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \text{Sin}\, x$$

All series converge for all real x.

6.4.2 Find the solution ϕ of $y^{11}+(x-1)^2 y^1 - (x-1) y = 0$ in the form $\phi(x) = 1 + \sum_{k=0}^{\infty} C_k (x-1)^k$ which satisfies $\phi(1)=1$, $\phi'(1)=0$ (Hint: use the substitution $x - 1=t$)

Answer: $$\phi(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (x-1)^{3k}}{3^k k!(3k-1)}.$$

6.4.3 Find the solution ϕ of $(1+x^2) y^{11} + y = 0$ of the form $\phi(x) = \sum_{k=0}^{\infty} C_k x^k$, which satisfies $\phi(0)=0$, $\phi'(0) = 1$.

Answer: $$\phi(x) = x + \sum_{k=1}^{\infty} \frac{(-1)^k (2.3+1)(4.5+1)......((2k-2)(2k-1)+1)}{(2k+1)!} x^{2k+1}$$

6.4.4 The equation $y^{11}+ e^x y = 0$ has a solution ϕ of the form $\phi(x) = \sum_{n=0}^{\infty} C_k x^k$ which satisfies $\phi(0)=1$, $\phi'(0)=0$. Compute $C_0, C_1, C_2, C_3, C_4, C_5$.

(Hint: $C_k = \frac{\phi^{(k)}(0)}{k!}$ and $\phi^{11}(x) = -e^x \phi(x)$)

Answer: $C_0=1$, $C_1=0$, $C_2= -\frac{1}{2}$, $C_3= -\frac{1}{6}$, $C_4=0$, $C_5= \frac{1}{40}$.

6.5 Answers to short answer questions:

For 6.2.1, see definition 6.1.1.

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LESSON-7
FIRST ORDER DIFFERENTIAL EQUATIONS

7.0 **Introduction**: In this lesson, we consider the general first order equation $y^1 = f(x, y) \to (1)$, where 'f' is some continuous function. Only in rather special cases it is possible to find explicit analytic expressions for the solutions of (1). We have already considered one such special case, namely, the linear equation $y^1 = g(x)\, y + h(x) \to (2)$, where g, h are continuous on some interval I. Any solution ϕ of (2) can be written in the form

$$\phi(x) = e^{Q(x)} \int_{x_0}^{x} e^{-Q(t)} h(t)\,dt + C e^{Q(x)} \to (3) \text{ where } Q(x) = \int_{x_0}^{x} g(t)\,dt,\ x_0 \text{ is in I,}$$

and C is a constant.

In next section and lesson-8, we indicate procedures which can be used to solve other important special cases.

7.1 **First order Differential Equations:**

7.1.1 **Definition:** A differential equation of first order may be written as $F(x, y(x), y^1(x)) = 0$ (or) $y^1(x) = f(x, y(x))$ (or) $y^1 = f(x, y)$ where F (or) f is a given continuous function. This differential equation associated with an initial condition say $y(x_0) = y_0$ is called an initial value problem (or IVP for short).

7.1.2 **Definition:** A solution of the initial value problem $y^1 = f(x, y)$, $y(x_0) = y_0$ is any continuous function on some open interval I which satisfies the given differential equation together with the initial condition on I.

7.1.3 **Definition:** We say a solution ϕ of $y^1 = f(x, y)$; $y(x_0)=y_0$ is unique if every other solution agrees with x as far as both are defined.

Only in some special cases it is possible to find explicit analytic expressions for the solutions of (7.1.3). One such special case is a linear equation. The general form of a linear equation is $y^1 = g(x)\, y + h(x) \to (1)$ where g, h are continuous on some interval I. Any solution ϕ of (1) can be written in the form $\phi(x) = e^{Q(x)} \int_{x_0}^{x} e^{-Q(t)} h(t)\,dt + C e^{Q(x)} \to (2)$

Where $Q(x) = \int_{x_0}^{x} g(t)\,dt$, $x_0 \in I$ and C is a constant.

It should be noted that, in particular examples, the integral occuring in equation (2) may be difficult to evaluate. They may even be impossible to evaluate in terms of elementary functions. Thus even though (2) exactly represents the unique solution of (1) with $y(x_0)=C$, it may not be easy to interpret the result.

7.1.3 **Problem**: Solve the Initial value problem.

$$y^1 + \frac{1}{x} y = 3x, \; y(2) = 3 \qquad \rightarrow (1)$$

which is in the form of $y^1 = g(x)\, y + h(x)$, $y(x_0) = y_0$. Here $g(x) = -\frac{1}{x}$ and $h(x) = 3x$ are continuous in the intervals $(-\infty, 0)$ and $(0, \infty)$. Hence x may take $I = (0, \infty)$. We multiply the differential equation in (1) by the integrating factor $e^{\int \frac{1}{x} dx} = x$, to obtain $xy^1 + 1.y = 3x^2$

(Or) $\frac{d}{dx}(xy) = 3x^2$

integrating both sides from $x_0 = 2$ to x, we get $x\, y(x) - 2y(2) = x^3 - 8$.

Since $y(2) = 3$, we have $x\, y(x) = x^3 - 8 + 6 = x^3 - 2$ for $x > 0$.

7.1.4 **Definition**: A differential equation $y^1 = f(x, y)$ is said to have the variables separated if f can be written in the form $f(x, y) = \frac{g(x)}{h(y)}$, where g and h are functions of singles variables. In this case we may write the equation $f(x, y) = \frac{g(x)}{h(y)}$ as $h(y)\frac{dy}{dx} = g(x) \Rightarrow h(y)dy = g(x)dx$.

We shall now describe the validity of the method of solving separable equations.

7.1.5 **Theorem**: Let g and h be continuous real valued of functions of x and y respectively defined on the respective intervals [a, b] and [c, d], satisfying the equation $h(y)\, y^1 = g(x) \rightarrow (1)$

If G, H are any functions such that $G^1 = g$, $H^1 = h$ and C is any constant such that the relation $H(y) = G(x)+C$ defines a real valued differentiable function ϕ for x in some interval I contained in $a \le x \le b$, then ϕ will be a solution of (1) on I.

Conversely, if ϕ is a solution of (1) on I, it satisfies the relation $H(y) = G(x)+C$ on I, for some constant C.

Proof: Suppose G, H are functions of x and y satisfying the relation

$$\int_{y_0}^{y} H^1(u)du = \int_{x_0}^{x} G^1(t)dt \quad \text{(Or)}$$

$\int_{y_0}^{y} h(u)du = \int_{x_0}^{x} g(t)dt$, for some $x_0 \in [a, b]$ and $y_0 \in [c, d]$. Also assume that the relation

$\int_{y_0}^{y} h(u)du = \int_{x_0}^{x} g(t)dt$ defines implicitly a differentiable function ϕ for all $x \in I \subseteq [a, b]$.

(In fact we say that a relation $F(x, y) = C$ defines a function ϕ implicitly for x in some interval I, if for each x in I, there is a $y = \phi(x)$ such that $F(x, \phi(x)) = C$). Then this function satisfies

$\int_{y_0}^{\phi(x)} h(u)du = \int_{x_0}^{x} g(t)dt$ for all $x \in I \rightarrow (2)$.

Differentiating (2) gives $h(\phi(x))\, \phi^1(x) = g(x)$.

This shows that ϕ is a solution of (1) on I.

Conversely suppose that ϕ is a real valued solution of (1) on I containing the point x_0. Then we have $h(\phi(x))\, \phi^1(x) = g(x)$ for all $x \in I$.

Integrating this form x_0 to x we get $\int_{x_0}^{x} h(\phi(t))\phi^1(t)dt = \int_{x_0}^{x} g(t)dt, x \in I \quad \rightarrow (3)$

Letting $u = \phi(x)$, then (3) reduces to $\phi(x_0)$ $h(u)du = \int_{x_0}^{x} g(t)dt$.

(or) $\int_{\phi(x_0)}^{x} H^1(u)du = \int_{x_0}^{x} G^1(t)dt$

which implies that $H(\phi(x)) = G(x) + C$, here $C = H(\phi(x_0)) - G(x_0)$.

Thus the theorem is complete.

7.1.6 **Problem:** Solve the following differential equations for their real valued solutions.

i) $y^1 = \dfrac{x + x^2}{y - y^2}$

ii) $y^1 = x^2y^2 - 4x^2$

Solution:

i) The given equation is $y^1 = \dfrac{x + x^2}{y - y^2}$ can be written as

$(y - y^2)\dfrac{dy}{dx} = (x + x^2)$

$\Rightarrow (y - y^2)dy = (x+x^2)dx.$

Integrating on both sides we get

$\dfrac{y^2}{2} - \dfrac{y^3}{3} = \dfrac{x^2}{2} + \dfrac{x^3}{3} + C$ (Or)

$3y^2 - 2y^3 = 3x^2+2x^3+C_1$, the required solution.

ii) The given equation is $y^1 = x^2y^2 - 4x^2$ can be written as $\frac{1}{y^2-2^2}\frac{dy}{dx} = x^2$

$$\Rightarrow \frac{1}{y^2-2^2}dy = x^2\,dx$$

Integrating on both sides, we get $\log\left|\frac{y-2}{y+2}\right| = \frac{4x^3}{3} + C$, for some constant C the required solution.

7.1.7 Problem:

i) Show that the solution ϕ of $y^1 = y^2$ which passes through the point (x_0, y_0) is given by

$$\phi(x) = \frac{y_0}{1-y_0(x-x_0)}$$

ii) For which x is ϕ a well-defined function?

iii) For which x is ϕ a solution of the problem $y^1 = y^2$, $y(x_0) = y_0$.

Solution: i) The given equation can be written as $\frac{1}{y^2}\frac{dy}{dx} = 1$

Let ϕ be a solution of the given equation on some interval I. Then we have $\frac{1}{\phi^2(x)}\frac{d\phi}{dx} = 1$ on

integrating we get $-(\phi(x))^{-1} = x + c$ i.e. $\phi(x) = \frac{-1}{x+c}$

Since the solution passes through the point (x_0, y_0), we have $y_0 = \phi(x_0) = -\frac{1}{x_0+C}$

$\Rightarrow C = \frac{x_0y_0+1}{y_0}$ Hence $\phi(x) = -\frac{1}{x - \frac{x_0y_0+1}{y_0}}$

i.e. $\phi(x) = \frac{y_0}{1-y_0(x-x_0)}$, the required solution.

ii) From (i) we notice that $\phi(x) = 0$ for all real x if $y_0 = 0$. Thus ϕ is well defined for all x, if $y_0=0$. If $y_0 \neq 0$, ϕ is well defined as long $x \neq x_0 + \frac{1}{y_0}$.

iii) If $y_0 > 0$, ϕ is well defined for all x if $x \in \left(-\infty, x_0 + \frac{1}{y_0}\right)$.

If $y_0 < 0$, ϕ is well defined for all x if $x \in \left(x_0 + \frac{1}{y_0}, \infty\right)$.

7.1.8 Problem:

Solve the Initial value problem $y^{1} = 1 + y^{2}$, $y(0) = 1$.

Solution: The given equation can be written as $\frac{1}{1+y^2}\frac{dy}{dx} = 1$

Integrating with respect to x from 0 to x, we get $\int_0^x \frac{1}{1+y^2}\frac{dy}{dx}\,dx = x$

Using the substitution y(x) = u, we see that $\int_1^{y(x)} \frac{1}{1+u^2}\,du = x$ i.e. $\left.\text{Tan}^{-1}u\right)_1^{y(x)} = x$

$\text{Tan}^{-1} y(x) - \frac{\pi}{4} = x$.

$y(x) = \text{Tan}(x + \frac{\pi}{4})$

We notice that y(x) is defined as long as $-\frac{3\pi}{4} < x < \frac{\pi}{4}$. Hence y(x) defined by

$y(x) = \text{Tan}(x + \frac{\pi}{4})$ is the unique solution of the Initial Value problem, since every solution

that exists is given by $y(x) = \text{Tan}(x + \frac{\pi}{4})$

7.1.9 Problem: Find all real valued solutions of the following equations

a) $y^{1} = x^{2}y$

b) $yy^{1} = x$

c) $y^{1} = \frac{e^{x-y}}{1+e^{x}}$

Solution:

a) Given equation is $y^{1} = x^{2}y \Rightarrow \frac{1}{y}dy = x^{2}\,dx$.

Integrating on both sides, we get $\log y = \frac{x^3}{3} + C$ i.e. $y = e^{\left(\frac{x^3}{3}+C\right)} = Ce^{\frac{x^3}{3}}$

b) Given equation is $yy^{1} = x \Rightarrow y\,dy = x\,dx$

Integrating on both sides, we get $\frac{y^2}{2} = \frac{x^2}{2} + \frac{C}{2}$ i.e. $y^2 = x^2 + C$, the required solution.

c) Given equation is

$$\Rightarrow \frac{dy}{dx} = \frac{e^x . e^{-y}}{1+e^x}$$

$$\Rightarrow e^y \, dy = \frac{e^x}{1+e^x} dx$$

Integrating on both sides, we get $e^y = \log(1+e^x)+C$, the required solution.

7.1.10 a) Find the solution of $y' = 2y^{1/2}$ passing through the point (x_0, y_0), where $y_0 > 0$.

b) Find all solutions of this equation passing through $(x_0, 0)$.

Solution:

a) The Given equation is $y' = 2y^{1/2} \Rightarrow y^{-1/2} dy = 2\,dx$

Integrating on both sides, we get $2y^{1/2} = 2x + 2C \Rightarrow y^{1/2} = x + C$

$\therefore y = (x+C)^2$

Since the solution passes through the point (x_0, y_0),

We have $y_0 = (x_0 + C)^2$

$$= x_0^2 + 2x_0C + C^2$$

$\therefore C^2 + 2x_0C + x_0^2 - y_0 = 0$

$$C = \frac{-2x_0 \pm \sqrt{4x_0^2 - 4x_0^2 + 4y_0}}{2} = -x_0 \pm \sqrt{y_0}$$

$$= -x_0 - \sqrt{y_0},\ -x_0 + \sqrt{y_0}$$

Therefore $\phi(x) = (x - x_0 + \sqrt{y_0})^2, \ (x \geq x_0 - \sqrt{y_0})$

$$\phi(x) = -(x - x_0 + \sqrt{y_0})^2, \ (x < x_0 - \sqrt{y_0})$$

b) The solution $y = (x+C)^2$ passing through $(x_0, 0)$.

Now $0 = (x_0+C)^2 \Rightarrow x_0 = -C$

$\therefore \quad \phi(x) = (x - x_0)^2$.

7.2. Equations reducible to variable separable form:

A function f defined for real x, y is said to be homogeneous of degree k if $f(tx, ty) = t^k f(x, y)$ for all t, x, y. In case f is homogeneous of degree zero we have $f(tx, ty) = f(x, y)$, and then we say the equation $y' = f(x, y)$ is homogeneous. (Unfortunately this terminology, which is rather standard, conflicts with the use of the word homogeneous in connection with linear equations). Homogeneous differential equations can be reduced to variable separable form by taking the substitution $y = ux$ in $y' = f(x, y)$. Then we obtain

$xu^1 + u = f(x, ux) = f(1, u)$ and hence $u^1 = \frac{f(1,u) - u}{x}$, which is an equation for u with variables separated.

7.2.1 **Problem:** Find all real-valued solutions of the following equations:

a) $y^1 = \frac{x+y}{x-y}$

b) $y^1 = \frac{y^2}{xy + x^2}$

c) $y^1 = \frac{x^2 + xy + y^2}{x^2}$

d) $y^1 = \frac{y + xe^{-2y/x}}{x}$.

Solution: a) Given equation is $y^1 = \frac{x+y}{x-y}$.

Put $y = ux$ then $y^1 = u + xu^1$

Therefore $y^1 = \frac{x+ux}{x-ux} \Rightarrow u + xu^1 = \frac{1+u}{1-u}$

$\Rightarrow xu^1 = \frac{1+u}{1-u} - u = \frac{1+u^2}{1-u}$

$\Rightarrow \frac{1-u}{1+u^2}du = \frac{1}{x}dx$ Integrating on both sides, we get

$\int \frac{1-u}{1+u^2}du = \int \frac{1}{x}dx$

$\Rightarrow \int \frac{1}{1+u^2}du - \int \frac{u}{1+u^2}du = \int \frac{1}{x}dx$

$\Rightarrow \mathrm{Tan}^{-1}u - \frac{1}{2}\log(1+u^2) = \log x + C$

$\Rightarrow \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(1 + \frac{y^2}{x^2}\right) = \log x + C$

$\Rightarrow 2\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = \log\left(1 + \frac{y^2}{x^2}\right) + 2\log x + C$

$\Rightarrow 2\,\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = \log(x^2 + y^2) + C$

b) Given equation is $y^1 = \frac{y^2}{xy + x^2}$

Put $y = ux$ then $y^1 = u + xu^1 . x$

$$\therefore\ u + xu^1 = \frac{u^2x^2}{x(ux)+x^2} = \frac{u^2}{u+1}$$

$$\Rightarrow xu^1 = \frac{u^2}{u+1} - u = \frac{u^2 - u^2 - u}{u+1} = -\frac{u}{u+1}$$

$$\Rightarrow \left(\frac{u+1}{u}\right) du = -\frac{dx}{x}$$

Integrating on both sides, we get u + log u = - log x + C

$$\frac{y}{x} + \log\left(\frac{y}{x}\right) = -\log x + C$$

$$\Rightarrow \frac{y}{x} + \log y - \log x = -\log x + C$$

$$\Rightarrow \frac{y}{x} + \log y = C$$

$\therefore\ y + x\ \log y = xc$, the required solution.

c) Given equation is $y^1 = \dfrac{x^2 + xy + y^2}{x^2}$

Put $y = ux$ then $y^1 = u + xu^1$.

$$\therefore\ u + xu^1 = \frac{x^2 + x(ux) + (ux)^2}{x^2}$$

$$= 1 + u + u^2$$

$$\Rightarrow xu^1 = 1 + u^2$$

$$\Rightarrow \frac{1}{1+u^2}\, du = \frac{1}{x} dx$$

Integrating on both sides, we get $\mathrm{Tan}^{-1}u = \log x + C$

$$\therefore\ \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = \log x + C$$

d) The Given equation is $y^1 = \dfrac{y + x e^{-2y/x}}{x}$

Put $y = ux$ then $y^1 = u + xu^1$

$$\therefore\ u + xu^1 = \frac{ux + x e^{-2ux/x}}{x} = u + e^{-2u}$$

$$\Rightarrow xu^1 = e^{-2u}$$

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$$\Rightarrow e^{2u}\, du = \frac{1}{x} dx$$

Integrating on both sides, we get $\frac{e^{2u}}{2} = \log x + C$

$\Rightarrow e^{2u} = 2\log x + C$

$\Rightarrow e^{\frac{2y}{x}} = 2\log x + C$, the required solution.

7.2.2 **Remark:** The equation $y^1 = \frac{a_1x + b_1y + C_1}{a_2x + b_2y + C_2}$ where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants (C_1, C_2 not both zero) can be reduced to a homogeneous equation.

Letting $x = \varepsilon + h$

$y = \eta + k$, where h, k are constants,

$$y^1 = \frac{a_1x + b_1y + C_1}{a_2x + b_2y + C_2} \text{ reduces to } \frac{d\eta}{d\varepsilon} = \frac{a_1\varepsilon + b_1\eta + (a_1h + b_1k + C_1)}{a_2\varepsilon + b_2\eta + (a_2h + b_2k + C_2)} \rightarrow (*)$$

If h, k satisfy $a_1h + b_1k + C_1 = 0$ and

$a_2h + b_2k + C_2 = 0$ (**)

The equation (*) becomes homogeneous. If the equation (**) have no solution, then $a_1b_2 - a_2b_1 = 0$, and in this case either the substitution $u = a_1x + b_1y + C_1$ (Or) $u = a_2x + b_2y + C_2$, leads to a separation of variables.

7.2.3 **Problem:** Solve the following equations.

a) $y^1 = \frac{x+y+2}{x+y-1}$ b) $y^1 = \frac{2x+3y+1}{x-2y-1}$ c) $y^1 = \frac{x+y+1}{2x+2y-1}$

Solution: a) The given equation is $y^1 = \frac{x-y+2}{x+y-1}$

Let $x = X + h$, $y = Y + k$ then $\frac{dy}{dx} = \frac{dY}{dX}$

$$\therefore \frac{dY}{dX} = \frac{(x+h)-(y+k)+2}{(x+h)+(y+k)-1} = \frac{x-y+h-k+2}{x+y+h+k-1}$$

If h, k satisfy $h - k + 2 = 0$

$h + k - 1 = 0$

Solving these two equations we get $h = -\frac{1}{2}$, $k = \frac{3}{2}$

$\therefore \frac{dY}{dX} = \frac{X-Y}{X+Y}$ which is a Homogeneous equation.

Put $Y = UX$ then $\frac{dY}{dX} = U + X\frac{dU}{dX}$

$$U + X\frac{dU}{dx} = \frac{X-UX}{X+UX} = \frac{1-U}{1+U}$$

$$\Rightarrow X\frac{dU}{dX} = \frac{1-U}{1+U} - U = \frac{1-2U-U^2}{1+U}$$

$$\Rightarrow \frac{1+U}{1-2U-U^2}dU = \frac{dX}{X}$$

Integrating on both sides, we get $\int \frac{1+U}{1-2U-U^2}dU = \int \frac{dx}{x}$

$$\Rightarrow -\frac{1}{2}\log(1-2U-U^2) = \log X + \log C$$

$$\Rightarrow \log(1-2U-U^2) + 2\log X = \log C^{-2}$$

$$\Rightarrow (1-2U-U^2)X^2 = C^{-2}$$

$$\Rightarrow X^2 - 2XY - Y^2 = C$$

$$\Rightarrow \left(x+\frac{1}{2}\right)^2 - 2\left(x+\frac{1}{2}\right)\left(y-\frac{3}{2}\right) - \left(y-\frac{3}{2}\right)^2 = C$$, the required solution.

b) Similar to above

c) The Given equation is $y^1 = \frac{x+y+1}{2x+2y-1}$

Put $x+y=u \quad \Rightarrow 1+\frac{dy}{dx} = \frac{du}{dx}$

$$\therefore \frac{du}{dx} - 1 = \frac{u+1}{2u-1}$$

$$\frac{du}{dx} = \frac{u+1}{2u-1} + 1 = \frac{u+1+2u-1}{2u-1} = \frac{3u}{2u-1}$$

$$\Rightarrow \frac{2u-1}{3u}du = dx$$

Integrating on both sides, we have $\frac{2}{3}u - \frac{1}{3}\log u = x + C$

$$\Rightarrow 2u - \log u = 3x + C$$

$$\Rightarrow 2x + 2y - \log(x+y) = 3x + C$$

$\therefore \log(x+y) + x - 2y = C$, the required solution.

7.3 Short Answer Questions:

7.3.1 Solve $y^1 = x^2y^2 - 4x^2$

7.3.2 Find the solution of $y^1 = 2y^{1/2}$ passing through the point (x_0, y_0), where $y_0 > 0$.

7.3.3 Find the solution of $y^1 = y^2$

7.4 Model Examination Questions:

7.4.1 a) Show that the solution ϕ of $y^1 = y^2$ which passes through the point (x_0, y_0) is given by

$$\phi(x) = \frac{y_0}{1 - y_0(x - x_0)}$$

b) For which x is ϕ a well-defined function?

c) For which x is ϕ a solution of the problem $y^1 = y^2$, $y(x_0) = y_0$?

7.5 Exercises:

7.5.1 Solve the following equations:

1. $y^1 = \frac{2x + 3y + 1}{x - 2y - 1}$

2. $y^1 = \frac{x + y + 1}{2x + 2y - 1}$

7.6 Answers to short answer questions:

For 7.3.1, See problem 7.1.6 (i)

For 7.3.2, See problem 7.1.10 (a)

For 7.3.3, See problem 7.1.7 (i)

7.7 Reference Book:

An Introduction to ordinary differential Equations – Earl A. Coddington, Prinlice Hall Mathematics Series.

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Lesson - 8

EXACT EQUATIONS

8.0 INTRODUCTION

In this lesson, we study the notion of an exact differential equation and a nessasary and sufficient condition for an equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be exact (see theorem 8.1.5).

8.1 EXACT EQUATIONS

8.1.1 Definition : Suppose the first order equation $y' = f(x,y)$ is written in the form

$$y' = -\frac{M(x,y)}{N(x,y)}, \quad \text{---------------- } 8.1.1\ (1)$$

or equivalently,

$$M(x,y) + N(x,y)\,y' = 0$$

or equivalently,

$$M(x,y)\,dx + N(x,y)dy = 0$$

where M, N are real valued functions of x, y defined on some rectangle R. The equation 8.1.1(1) is said to be exact in R if there exists a function F having continuous first partial derivatives in R such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N \text{ in } R.$$

8.1.2 Theorem : Suppose the equation

$$M(x,y) + N(x,y)y' = 0 \quad \text{------------- } 8.1.2\ (1)$$

is exact in a rectangle R and F is a real-valued function such that

$$\frac{\partial F}{\partial x} = M, \ \frac{\partial F}{\partial y} = N \text{ ------------ } 8.1.2\ (2)$$

in R. Every differentiable function ϕ, defined by a relation

$$F(x, y) = c, \ (c = \text{constant}),$$

is a solution of 8.1.2(1) and every solution of 8.1.2(1) whose graph lies in R arises in this way

Proof : By hypothesis,

$$\frac{\partial F}{\partial x}(x, y) + \frac{\partial F}{\partial y}(x, y)\, y' = 0 \text{ ------------- } 8.1.2\ (3)$$

Suppose ϕ $\left(y = \phi(x)\right)$ is a solution on an interval I.

So,

$$\frac{\partial F}{\partial x}\left(x, \phi(x)\right) + \frac{\partial F}{\partial \phi}\left(x, \phi(x)\right) \phi'(x) = 0 \text{ ------------ } 8.1.2\ (4)$$

for all x in I. Put $\Phi(x) = F\left(x, \phi(x)\right)$. Then 8.1.2(4) says

$$\Phi'(x) = 0,$$

and hence

$$\Phi(x) = c$$

i.e., $F\left(x, \phi(x)\right) = c$

where c is constant. Thus, the solution ϕ is a function given implicitly by the relation

$$F(x, y) = c.$$

Suppose ϕ is a differentiable function on an interval I implicitly given by

$$F(x, y) = c.$$

where c is constant, So,

$$F\left(x, \phi(x)\right) = c \text{ ------------ } 8.1.2(5).$$

Differentiating on both sides of 8.1.2(5), we have 8.1.2(4) and hence 8.1.2(3) which is nothing bu 8.1.2(1). So, 8.1.2(5) is a solution of 8.1.2(1).

8.1.2.1 Note : In view of Theorem 8.1.2, the problem of solving an exact equation 8.1.2(1) is reduced to the problem of determining a function F satisfying 8.1.2(2) such that

$$M(x, y)\,dx + N(x, y)dy = \frac{\partial F}{\partial x}(x, y)\,dx + \frac{\partial F}{\partial y}(x, y)\,dy = dF = 0 \quad \text{-------------- 8.1.2.1(1)}$$

That is if 8.1.2(1) is exact, then its left hand side is an exact differential of a function F. Some times, on multiplying both sides of 8.1.2(1) by some function, the equation becomes exact; in this case such function is called integrating factor.

Some times the function F in 8.1.2.1(1) can be determined by inspection. Consider the following.

8.1.3 Example : Consider the equation

$$y' = -\frac{x}{y}$$

This can be written as

$$x\,dx + y\,dy = 0$$

Clearly, the left hand side of this equation is the differntial of $\left(x^2 + y^2\right)/2$. Thus, any differentiable function which is defined by the relation

$$x^2 + y^2 = c \quad (c = \text{constant}),$$ is a solution.

8.1.3.1 Note : The equation in example 8.1.3, does not make sense when $y=0$. In fact it is a special case of an equation with variables seperated. Indeed any such equation is a special case of an exact equation. Consider the following

8.1.4 Example : Consider the equation

$$g(x)dx = h(y)dy.$$

Clealry,

$$F(x, y) = G(x) - H(y)$$

where $G' = g,\ H' = h$ is a solution.

The following theorem gives a necessary and sufficient condition for an equation to become exact.

8.1.5 Theorem : Let M and N be two real valued functions which have continuous first partial derivatives on some rectangle

$$R : |x - x_0| \le a, \; |y - y_0| \le b.$$

Then, the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0 \quad \text{--------------- } 8.1.5\ (1)$$

is exact in R if, and only if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{---------------- } 8.1.5\ (2)$$

in R

Proof : Assume that 8.1.5(1) is exact. So, ther exists a function F which has continuous first partial derivatives such that

$$\frac{\partial F}{\partial x} = M, \; \frac{\partial F}{\partial y} = N \text{ in } R.$$

Since M and N have continuous first partial derivatives, F has continuous second partial derivatives and hence

$$\frac{\partial^2 F}{\partial y\, \partial x} = \frac{\partial^2 F}{\partial x\, \partial y}$$

i.e. $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Conversely assume that 8.1.5(2) holds in R. To prove that 8.1.5(1) is exact, we have to find a function F satisfying

$$\frac{\partial F}{\partial x} = M, \; \frac{\partial F}{\partial y} = N.$$

If we had such a function F, then

$$F(x, y) - F(x_0, y_0) = F(x, y) - F(x_0, y) + F(x_0, y) - F(x_0, y_0)$$

$$= \int_{x_0}^{x} \frac{\partial F}{\partial x}(s, y)\, ds + \int_{y_0}^{y} \frac{\partial F}{\partial y}(x_0, t)\, dt$$

$$= \int_{x_0}^{x} M(s,\ y)\,ds + \int_{y_0}^{y} N(x_0, t)\,dt \quad \text{---------} \ 8.1.5\ (3)$$

and $F(x, y) - F(x_0, y_0) = F(x, y) - F(x, y_0) + F(x, y_0) - F(x_0, y_0)$

$$= \int_{y_0}^{y} \frac{\partial F}{\partial y}(x, t)\,dt + \int_{x_0}^{x} \frac{\partial F}{\partial x}(s, y_0)\,ds$$

$$= \int_{y_0}^{y} N(x, t)\,dt + \int_{x_0}^{x} M(s, y_0)\,ds \quad \text{------} \ 8.1.5\ (4)$$

Define F by

$$F(x, y) = \int_{x_0}^{x} M(s, y)\,ds + \int_{y_0}^{y} N(x_0, t)\,dt \quad \text{---------} \ 8.1.5\ (5)$$

Clearly, $F(x_0, y_0) = 0$ and

$$\frac{\partial F}{\partial x}(x, y) = M(x, y)$$

for all (x, y) in R. From 8.1.5(3) and 8.1.5(4), we have that

$$F(x, y) = \int_{y_0}^{y} N(x, t)\,dt + \int_{x_0}^{x} M(s, y_0)\,ds \quad \text{------------} \ 8.1.5(6)$$

$(\text{since } F(x_0, y_0) = 0)$ and hence

$$\frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. Thus, we have found F.

8.1.6 Example : Consider the equation

$$y' = \frac{3x^2 - 2xy}{x^2 - 2y}$$

i.e. $\left(3x^2-2xy\right)dx+\left(2y-x^2\right)dy=0$.

This is of the form $M\,dx+N\,dy=0$, where

$$M(x,\ y)=3x^2-2xy, \qquad N(x,\ y)=2y-x^2$$

Now,

$$\frac{\partial M}{\partial y}(x,\ y)=-2x, \qquad \frac{\partial N}{\partial x}(x,y)=-2x.$$

Hence,

$$\frac{\partial M}{\partial y}(x,y)=\frac{\partial N}{\partial x}(x,y).$$

By theorem 8.1.5, the given equation is exact. To find an F we could use either of the two formulas 8.1.5(5) and 8.1.5(6), but the following way is often simpler. We know that there is a F such that

$$\frac{\partial F}{\partial x}=M, \qquad \frac{\partial F}{\partial y}=N.$$

So,

$$\frac{\partial F}{\partial x}(x,y)=M(x,\ y)=+3x^2-2xy.$$

For each fixed y,

$$F(x,y)=x^3-x^2y+f(y) \text{ ---------------- 8.1.6 (1)}$$

where f is a function of y alone (and hence f is independent of x). Now,

$$\frac{\partial F}{\partial y}(x,\ y)=N(x,y)$$

$$\text{i.e., } -x^2+f'(y)=2y-x^2$$

$$\text{i.e., } f'(y)=2y.$$

Thus, a choice of f is given by $f(y)=y^2$. Writing this $f(y)$ in 8.1.6 (1),

$$F(x, y) = x^3 - x^2 y + y^2.$$

Any differentiable function ϕ which is defined implicitly by a relation

$$x^3 - x^2 y + y^2 = c$$

will be a solution of the given equation.

8.1.6.1 Note : Consider the differential equation

$M(x, y)\, dx + N(x, y)\, dy = 0$ ------------ 8.1.6.1 (1).

If 8.1.6.1 (1) is not exact, if may be possible to find a function $\mu(x, y)$ such that

$$\mu(x, y)\, M(x, y) dx + \mu(x, y)\, N(x, y) dy = 0$$

is exact ----- in this case $\mu(x, y)$ is called integrating factor.

So, we have that

$$\frac{\partial}{\partial y}\left(\mu(x, y)\, M(x, y)\right) = \frac{\partial}{\partial x}\left(\mu(x, y)\, N(x, y)\right)$$

i.e. $\mu \dfrac{\partial M}{\partial y} + \dfrac{\partial M}{\partial y} M = \mu \dfrac{\partial N}{\partial x} + \dfrac{\partial \mu}{\partial x} N$

i.e. $\mu \left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} \right) = N \dfrac{\partial \mu}{\partial x} - M \dfrac{\partial \mu}{\partial y}$

8.2 SHORT ANSWER QUESTIONS

8.2.1 : Define exact equation.

8.2.2 : Give a necessary and sufficient condition for the differential equation

$M(x, y) dx + N(x, y) dy = 0$ to be exact.

8.2.3 : Define integrating factor.

8.2.4 : Write the integrating factor of the equation

$$\left(xy^2 - e^{\frac{1}{x^2}} \right) dx - x^2 y\, dy = 0$$

8.2.5 : Write (obtain) an integrating factor of the equation

$$\left(xy^3+y\right)dx + 2\left(x^2y^2+x+y^4\right)dy = 0$$

8.2.6 : Prove that the equation $\left(2y^3+2\right)dx + 6xy^2\,dy = 0$ is exact.

8.3 MODEL EXAMINATION QUESTIONS

8.3.1 : Define an exact differential equation.

Let M , N be two real - valued functions which have continuous first partial derivatives on some rectangle

$$R : |x - x_0| \le a, \ |y - y_0| \le b$$

Prove that the equation

$M(x, y)\,dx + N(x, y)\,dy = 0$ is exact if, and only if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

8.3.2 : Prove that the differential equation

$\left(3x^2-2xy\right)dx + \left(2y - x^2\right)dy = 0$ is exact and solve it.

8.4 EXERCISES

8.4.1 : The equuations below are written in the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where M , N exist on the whole plane. Determine which equations are exact there, and solve these.

(a) $2xy\,dx + \left(x^2+3y^2\right)dy = 0$

(b) $\left(x^2+xy\right)dx + xy\,dy = 0$

(c) $e^x\,dx + e^y\left(y+1\right)dy = 0$

(d) $\cos x \cos^2 y\,dx - \sin x \sin 2y\,dy = 0$

(e) $x^2y^3\,dx - x^3y^2\,dy = 0$

(f) $(x+y)dx + (x-y)dy = 0$

(g) $\left(2y\, e^{2x} + 2x \cos y\right)dx + \left(e^{2x} - x^2 \sin y\right)dy = 0$

(h) $\left(3x^2 \log|x| + x^2 + y\right)dx + x\,dy = 0$

8.4.2 : Even though an equation $M(x, y)dx + N(x, y)dy = 0$ may not be exact, some times it is not difficult to find a function u, no where zero, such that

$u(x,y)M(x, y)\, dx + u(x, y)N(x,y)\, dy = 0$ is exact. Such a function is called an integrating factor.

Find an integrating factor for each of the following equations and solve them.

(a) $(2y+2)dx + 3xy^2\, dy = 0$

(b) $\cos x \cos y\, dx - 2 \sin x \sin y\, dy = 0$

(c) $\left(5x^3 y^2 + 2y\right) dx + \left(3x^4 y + 2x\right)dy = 0$

(d) $\left(e^y + x\, e^y\right) dx + x\, e^y\, dy = 0$

8.4.3 : Consider the equation $M(x, y)dx + N(x, y)dy = 0$, where M, N have continuous first partial derivatives on some rectangle R. Prove that a function u on R, having continuous first partial derivatives, is an integrating factor if, and only if,

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} \text{ on } R \quad \text{(Hint : see Note 8.1.6.1)}$$

8.4.4 : (a) Under the same conditions as in exercise 8.4.3, show that if the equation

$$M(x, y)dx + N(x, y)dy = 0$$

has an integrating factor u, which is a function of x alone, then

$$p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \text{ is a continuous function of } x \text{ alone.}$$

(b) If p is continuous and independent of y, show that an integrating factor is given by

$$u(x) = e^{p(x)}$$

where p is any function satisfying $p' = p$.

8.4.5 : (a) Under the same conditions as in exercise 8.4.3, show that if

$M(x, y)dx + N(x, y)\,dy = 0$ has an integrating factor u, which is a function of y alone, then

$$q = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \text{ is a continuous function of } y \text{ alone.}$$

(b) If q is continuous, and is independent of x, show that an integrating factor is given by

$$u(y) = e^{Q(y)}$$

where Q is a function such that $Q' = q$.

8.4.6 : Consider the linear equation of first order

$$y' + a(x)y = b(x),$$

where a, b are continuous on some interval I.

(a) Show that there is an integrating factor which is a function of x alone.

(Hint : Exercise 8.4. 4)

(b) Solve this equation, using an integrating factor.

8.5 ANSWERS TO SHORT ANSWER QUESTIONS

8.2.1 : See Definition 8.1.1

8.2.2 : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

8.2.3 : See Note 8.1.6.1.

8.2.4 : $M = xy^2 - e^{\frac{1}{x^2}}, \quad N = -x^2 y$.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2xy - (-2xy) = 4xy.$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{4xy}{-x^2 y} = -\frac{4}{x} \text{ (= a function of } x \text{ only)}$$

$= p(x)$ say.

Integrating Fact (I.F.) = $e^{\int p(x)dx} = e^{\int -\frac{4}{x}} = x^{-4}$

(Here we have used exercise 8.4.4)

8.2.5 : $M = xy^3 + y,\ N = 2\left(x^2 y^2 + x + y^4\right)$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1, \qquad \frac{\partial N}{\partial x} = 2\left(2xy^2 + 1\right)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = xy^2 + 1.$$

So, $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{xy^2+1}{xy^3+y} = \frac{1}{y} = q(y)$ (= a function of y alone).

$\therefore I.F. = e^{\int q(y)\,dy} = e^{\log y} = y$ (by exercise 8.4.5)

8.2.6 : $M = 2y^3 + 2, \quad N = 6xy^2$

$\frac{\partial M}{\partial y} = 6y^2,\ \frac{\partial N}{\partial x} = 6y^2$. So, the equation is exact.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 9

THE METHOD OF SUCCESSIVE APPROXIMATIONS

9.0 INTRODUCTION

In this lesson, we study the successive approximations to a solution of a given initial value problem.

9.1 THE METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the general problem of finding soutions of the equation

$$y' = f(x, y) \text{ -------- } 9.1(a)$$

where f is any continuous real valued function defined on some rectangle

$R: |x - x_0| \le a, \ |y - y_0| \le b \quad (a, b > 0)$, in the real $(x, y)-$ plane. Our object is to show that on some interval I containing x_0 there is a solution ϕ of 9.1(a) satisfying

$$\phi(x_0) = y_0 \text{ ------ } 9.1(b)$$

By this we mean there is a real valued function ϕ satisfying 9.1(b) such that the points $(x, \phi(x))$ are in R for x in I, and

$\phi'(x) = f(x, \phi(x))$ for all x in I. Such a function ϕ is called a solution of the initial value problem

$$y' = f(x, y), \ y(x_0) = y_0 \text{ --------- } 9.1(c).$$

Our first step will be to show that the initial value problem is equivalent to an integral equation, namely,

$$y = y_0 + \int_{x_0}^{x} f(t, y)\, dt \text{ ----------- } 9.1(d)$$

on I. By a solution of this equation on I, we mean a real continuous function ϕ on

I such that $(x, \phi(n))$ is in R for all x in I, and

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt \quad \text{----------- 9.1 (e)}$$

for all x in I.

9.1.1 Theorem : A function ϕ is a solution of the initial value problem 9.1(c) on an interval I if, and only if it is a solution of the integral equation 9.1(d) on I.

Proof : Assume that ϕ is a solution of the initial value problem on I. Then,

$$\phi'(t) = f(t, \phi(t)) \quad \text{-------------- 9.1.1 (1)}$$

on I. Since ϕ is continuous on I, and f is continuous on R, the function F defined by

$$F(t) = f(t, \phi(t))$$

is continuous on I. Integrating 9.1.1(1) from x_0 to x, we obtain

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))dt,$$

since $\phi(x_0) = y_0$, ϕ is a solution of 9.1(d).

Conversely assume that ϕ satisfies 9.1(e) on I. Differentiating,

$$\phi'(x) = f(x, \phi(x))$$

for all x in I (by the fundamental theorem of integral calculus). Moreover, $\phi(x_0) = y_0$. Hence, ϕ is a solution of the initial value problem 9.1(c).

9.1.2 Definition : Consider the problem of solving 9.1(d)

$$\text{i.e. } y = y_0 + \int_{x_0}^{x} f(t, y)dt$$

Define ϕ_0 by $\phi_0(x) = y_0$. This satisfies the initial condition $\phi_0(x_0) = y_0$ but does not satisfy 9.1(d). Now, we compute

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$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, dt$$

$$= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$$

Now, we define ϕ_2 by

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\, dt$$

Having defined ϕ_i, we define ϕ_{i+1} by

$$\phi_{i+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_i(t))\, dt$$

Thus, we have the functions -- $\phi_0, \phi_1, \phi_2, \ldots\ldots\ldots\ldots$ we call these functions as successive approximations to a solution of the initial value problem 9.1(c).

We now show that there is an interval I containing x_0 where all the functions ϕ_K $(K = 0, 1, 2, \ldots\ldots)$ defined in the definition 9.1.2 exist. Since f is continuous in R, there exists $M > 0$ such that

$$|f(x, y)| \le M$$

for all (x, y) in R. Let $\alpha = \min\{a, b/M\}$. Now we prove that all ϕ_K s are defined on $|x - x_0| \le \alpha$.

9.1.3 Theorem : The successive approximations ϕ_K, given in the Definition 9.1.2, exist as continuous functions on

$$I : |x - x_0| \le \alpha = Min\{a, b/M\},$$

and $(x, \phi_K(x))$ is in R for $x \in I$. Indeed ϕ_K satisfy

$$|\phi_K(x) - y_0| \le M|x-x_0| \text{ ---------- } 9.1.3\ (1)$$

for all x in I.

Proof : Clearly, ϕ_0 is a continuous function on I and

$$|\phi_0(x) - y_0| = |y_0 - y_0| = 0 \le M|x-x_0|.$$

for all x in I. We have

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

Since f is continuous on R, the function F_0 defined by $F_0(t) = f(t, y_0)$ is continuous on I. Thus ϕ_1 is continuous on I and

$$|\phi_1(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_0)\,dt\right|$$

$$\le \left|\int_{x_0}^{x} |f(t, y_0)|\,dt\right|$$

$\le M|x-x_0|$, which shows that ϕ_1 satisfies the inequality 9.1.3(1).

Clearly,

$$\phi_1(t) = y_0 + \int_{x_0}^{x} F_0(t)\,dt.$$

Assume that the theorem has been proved for the functions $\phi_0, \phi_1, ..., \phi_n$. We prove that it is valied for ϕ_{n+1}. We know that $(t, \phi_n(t))$ is in R for t in I. Thus the function F_n given by

$F_n(t) = f(t, \phi_n(t))$ exists for t in I. It is continuous on I since f is continuous on R, and ϕ_n is continuous on I. Therefore ϕ_{n+1}, which is given by

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} F_n(t)\,dt,$$

exists as a continuous function on I. Moreover,

$$\left|\phi_{n+1}(x) - y_0\right| = \left|\int_{x_0}^{x} F_n(t)\,dt\right| \le M\left|x - x_0\right|$$

for all x in I. Hence the theorem by the principle of mathematical induction.

9.1.3.1 Note : Our next step is to show that the successive approximations converge on I to a solution of our initial value problem. In order to do this, we must impose a further restriction on f. We discuss this in the next lesson - 10.

9.1.4 Example : Consider the initial value problem

$$y' = 3y + 1, \qquad y(0) = 2$$

(a) Show that all the successive approximations $\phi_0, \phi_1, \ldots\ldots\ldots$ exist for all x.

(b) Compute the first four approximations $\phi_0, \phi_1, \phi_2, \phi_3$ to the solution.

(a) : Clearly, $\phi_0(x) = y(0) = 2$ exists as a continuous function for all x. We have

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt \qquad \text{(here } f(x, y) = 3y + 1\text{)}$$

$$= 2 + \int_0^x f(t, 2)\,dt$$

$$= 2 + \int_0^x 7\,dt$$

$$= 2 + 7x$$

which exists as continuous function for all real x. Similarly, we can show that $\phi_2, \phi_3, \ldots\ldots$ exist as continuous functions for all real x.

(b) : First approximation ϕ_0 is defined by

$$\phi_0(x) = y(0) = 2.$$

Now, $$\phi_1(x) = y(0) + \int_0^x f(t, \phi_0(t))\,dt$$

$$= 2 + \int_0^x f(t, 2)\,dt$$

$$= 2 + \int_0^x (3 \times 2 + 1)\,dt$$

$$= 2 + \int_0^x 7\,dt = 2 + 7x,$$

$$\phi_2(x) = y(0) + \int_0^x f(t, \phi_1(t))\,dt$$

$$= 2 + \int_0^x (3\phi_1(t) + 1)\,dt$$

$$= 2 + \int_0^x (7 + 21t)\,dt$$

$$= 2 + 7x + \frac{21}{2}x^2,$$

$$\phi_3(x) = y(0) + \int_0^x f(t, \phi_2(t))\,dt$$

$$= 2 + \int_0^x (3\phi_2(t) + 1)\,dt$$

$$= 2 + \int_0^x \left(7 + 21t + \frac{63}{2}t^2\right)dt$$

$$= 2 + 7x + \frac{21}{2}x^2 + \frac{21}{2}x^3$$

9.1.5 Example : We, now find the first four approximations $\phi_0, \phi_1, \phi_2, \phi_3$ for the initial value problem

$$y' = xy, \quad y(0) = 1.$$

$$\phi_0(x) = y(0) = 1;$$

$$\phi_1(x) = y(0) + \int_0^x f(t, \phi_0(t))\,dt$$

$$= 1 + \int_0^x t\phi_0(t)\,dt \qquad \text{(here } f(x, y) = xy\text{)}$$

$$= 1 + \int_0^x t\,dt$$

$$= 1 + \frac{x^2}{2};$$

$$\phi_2(x) = y(0) + \int_0^x f(t, \phi_1(t))\,dt$$

$$= 1 + \int_0^x t\,\phi_1(t)\,dt$$

$$= 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right)dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8};$$

$$\phi_3(x) = y(0) + \int_0^x f(t, \phi_2(t))\,dt$$

$$= 1 + \int_0^x t\left(1 + \frac{t^2}{2} + \frac{t^4}{8}\right) dt$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{6 \cdot 8}.$$

9.2 SHORT ANSWER QUESTIONS :

9.2.1 : State an initial value problem

9.2.2 : Define successive approximations of an initial value problem $y' = f(x, y),\ y(x_0) = y_0$.

9.2.3 : Write down the first approximation for the IVP :

$$y' = x^2 + y^2,\ y(0) = 0$$

9.2.4 : Write the second approximation for the IVP :

$$y' = 3y + 1,\quad y(0) = 2$$

9.3 MODEL EXAMINATION QUESTIONS

9.3.1 : Obtain the first five successive approximations for the initial value problem :

$$y' = x^2 + y^2,\ y(0) = 0$$

9.3.2 : Define successive approximations of an initial value problem and find the first three successive approximations for the initial value problem

$$y' = y^2,\quad y(0) = 1.$$

9.3.3 : Define the successive approximations $\phi_0, \phi_1, \ldots\ldots\ldots$ for the initial value problem :

$$y' = f(x, y),\ y(0) = y_0$$

Further, determine the interval I such that

$$(x, \phi_i(x)) \in R$$

$|\phi_i(x) - y_0| \le M|x - x_0|$ for all $x \in I$ and $i = 0, 1, 2, \ldots$ (where M is some constant)

9.4 EXERCISES :

9.4.1 : For each of the following problems compute the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$:

(a) $y' = x^2 + y^2, \quad y(0) = 0$ (b) $y' = 1 + xy, \quad y(0)=1$

(c) $y' = y^3, \; y(0) = 0$ (d) $y' = y^2, \; y(0) = 1$

9.4.2 : (a) Show that all the successive approximations for the problem

$y' = y^2, \quad y(0) = 1$ exist for all real x.

(b) Find a solution of the initial value problem in (a). On what interval does it exist ?

(c) Assuming that there is just one solution of the problem in (a), indicate why the successive approximations found in (a) cannot converge to a solution for all real x.

9.4.3 : Consider the problem

$y' = x^2 + y^2, \; y(0)=0$ on

$R : |x| \leqq 1, \; |y| \leqq 1$.

(a) Compute an upper bound M for the function $f(x, y) = x^2 + y^2$ on R.

(b) On what interval containing 0, will all successive approximations exist, and be such that their graphs are in R ?

9.5 ANSWERS TO SHORT ANSWER QUESTIONS :

9.2.1 : $y' = f(x, y), \; y(x_0) = y_0$

9.2.2 : See Definition 9.1.

9.2.3 : The first approximation ϕ_0 is given by

$\phi_0(x) = y(0) = 0$ for all x.

9.2.4 : (i) $\phi_0(x) = y(0) = 2$ for all x

(ii) $\phi_1(x) = y(0) + \int_0^x f(t, \phi_0(t))\, dt$

$= 2 + \int_0^x (3\phi_0(t) + 1)\, dt$

$= 2 + \int_0^x 7\, dt$

$= 2 + 7x$

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Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 10

THE LIPSCHITZ CONDITION

10.0 INTRODUCTION

In this lesson, we study the concepts - Lipschitz condition, Lipschitz constant.

10.1 THE LIPSCHITZ CONDITION

10.1.1 Definition : Let f be a function defined for (x, y) in a set S. We say that f satisfies Lipschitz condition on S if there exists a constant $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqq K |y_1 - y_2|$$

for all $(x, y_1), (x, y_2)$ in S. The constant K is called Lipschitz constant.

If f is continuous and satisfies Lipschitz condition on a rectangle R, then the successive approximations converge to a solution of the initial value problem on $|x - x_0| \leqq \alpha$. Before proving this, let us remark that a Lipschitz condition is rather mild restriction of f

10.1.2 Theorem : Suppose S is either a rectangle

$$|x - x_0| \leqq a, \quad |y - y_0| \leqq b. \quad (a,\ b > 0) \text{ or a strip}$$

$$|x - x_0| \leqq a, \quad |y| < \infty \ (a > 0)$$

and that f is a real valued function defiend on S such that $\dfrac{\partial f}{\partial y}$ exists and is continuous on S. Sup' ose there is a constant $K > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leqq K$$

for all (x, y) in S. Then f satisfies Lipschitz condition on S with Lipschitz constant K.

Proof : $|f(x, y_1) - f(x, y_2)| = \left| \int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x, t)\, dt \right|$

$$\leq \left| \int_{y_2}^{y_1} \left| \frac{\partial f}{\partial y}(x, t) \right| dt \right|$$

$$\leq K\, |y_1 - y_2|$$

for all $(x, y_1), (x, y_2)$ in S. Hence the theorem.

10.1.3 Example : Consider the function

$$f(x, y) = x^2 \cos^2 y + y \sin^2 x \text{ on the set}$$

$$S : |x| \leq 1,\ |y| < \infty.$$

Now, we show that f satisfies Lipschitz condition on S and we determine Lipschitz constant.

For any $(x, y) \in S$,

$$\left| \frac{\partial f}{\partial y}(x, y) \right| = \left| x^2 \cdot 2 \cos y (-\sin y) + 1 \cdot \sin^2 x \right|$$

$$\leq \left| 2x^2 \sin y \cos y \right| + \left| \sin^2 x \right|$$

$$\leq 2 + 1 = 3.$$

By Theorem 10.1.2, f satisfies Lipschitz condition and the Lipschitz constant is 3.

10.1.4 Example : Consider the function

$$f(x, y) = x y^2 \text{ on the set}$$

$$R : |x| \leq 1,\ |y| \leq 1.$$

$$\left| \frac{\partial f}{\partial y}(x, y) \right| = |2 x y| \leq 2,$$

for any $(x, y) \in S$. By Theorem 10.1.2, f satisfies Lipschitz condition and Lipschitz constant is 2.

Take $S : |x| \leq 1,\ |y| < \infty$.

Then, $\left|\frac{\partial f}{\partial y}(x, y)\right| = |2xy| = 2|x||y|$.

Clearly,

$\frac{\partial f}{\partial y}(x, y)$ is not bounded (as it tends to ∞ as y tends ∞ for a fixed x with $0 < |x| \leq 1$).

So, f does not satisfy the Lipschitz condition.

10.1.5 Example : Consider the function

$$f(x, y) = y^{\frac{2}{3}} \text{ on the rectangle}$$

$$R : |x| \leq 1, |y| \leq 1.$$

Let $y_1 > 0$.

$$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \frac{y_1^{\frac{2}{3}}}{y_1} = \frac{1}{\frac{1}{y_1^{\frac{1}{3}}}}$$

which is unbounded as $y_1 \to 0$. So, f does not satisfy Lipschitz condition on R.

Note : In the examples 10.1.3 and 10.1.4, $\frac{\partial f}{\partial y}(x, y)$ exists and is continuous at each (x, y) in the domain of the definition. That is why we have considered $\frac{\partial f}{\partial y}(x, y)$.

10.2 SHORT ANSWER QUESTIONS :

10.2.1 : Define Lipschitz condition and Lipschitz constant.

10.2.2 : Assume the hypothesis of Theorem 10.1.2, what is Lipschitz constant ?

10.3 MODEL EXAMINATION QUESTIONS :

10.3.1 : Define Lipschitz condition, Lipschitz constant. Show that the function f defined by

$f(x, y) = x y^2$ satisfies Lipschitz condition on

$R: |x| \leq 1, |y| \leq 1$

but not on

$S: |x| \leq 1, \quad |y| < \infty.$

10.3.2 : Verify whether the following functions f satisfy Lipschitz condition on the set S mentioned.

(a) $f(x, y) = x^2 \cos^2 y + y \sin^2 x$, on $S: |x| \leq 1, \quad |y| < \infty$.

(b) $f(x, y) = 4x^2 + y^2$, on $S: |x| \leq 1, |y| \leq 1$.

10.4 EXERCISES :

10.4.1 : By computing appropriate Lipschitz constants, show that the following functions satisfy Lipschitz conditions on the set S indicated.

(a) $f(x, y) = 4x^2 + y^2$, on $S : |x| \leq 1, \quad |y| \leq 1$

(b) $f(x, y) = x^2 \cos^2 y + y \sin^2 x$, on $S: |x| \leq 1, \quad |y| < \infty$

(c) $f(x, y) = x^3 e^{-xy^2}$, on $S: 0 \leq x \leq a, |y| < \infty, (a>0)$

(d) $f(x, y) = a(x) y^2 + b(x) y + c(x)$ on $S: |x| \leq 1, |y| \leq 2$

(a, b, c are continuous functions of x on $|x| \leq 1$)

(e) $f(x, y) = a(x) y + b(x)$, on $S: |x| \leq 1, |y| < \infty$,

(a, b are continuous functions on $|x| \leq 1$)

10.4.2 : (a) Show that the function f given by

$$f(x, y) = y^{\frac{1}{2}}$$

does not satisfy Lipschitz condition on

$$R: |x| \leqq 1, \quad 0 \leq y \leq 1.$$

(b) Show that the function f defined in (a) satisfies Lipschitz condition on any rectangle of the form

$$R: |x| \leqq a, \quad b \leqq y \leqq c \quad (a, b, c > 0)$$

10.4.3 : (a) Show the function f given by

$f(x, y) = x^2 |y|$ satisfies Lipschitz condition on

$$R: |x| \leqq 1, \quad |y| \leqq 1.$$

(b) Show that $\frac{\delta f}{\delta y}$ does not exist at $(x, 0)$ if $x \neq 0$.

10.4.4 : Show that the assumption that $\frac{\delta f}{\delta y}$ be continuous on S is super fluous in theorem 10.1.2.

(Hint : For each find x, the mean value theorem implies that

$$f(x, y_1) - f(x, y_2) = \frac{\partial f}{\partial y}(x, \eta)(y_1 - y_2)$$

where η (which may depend up on x, y_1, y_2) is between y_1 and y_2.

10.5 ANSWERS TO SHORT ANSWER QUESTIONS :

10.2.1 : See Definition 10.1.1.

10.2.2 : K

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 11

CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS

11.0 INTRODUCTION

In lesson 9, we have defined successive approximations to a solution of an initial value problem. In this solution, we prove that these successive approximations converge on a particular interval I to a solution of the initial value problem.

11.1 CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS

11.1.1 Theorem (Existence Theorem) : Let f be a continuous real valued function on the rectangle

$$R : |x - x_0| \leqq a, \ |y - y_0| \leqq b \ (a, b > 0) \text{ and let}$$

$$|f(x, y)| \leqq M$$

for all (x, y) in R. Further suppose that f satisfies Lipschitz condition with constant K in R. Then the successive approximations.

$$\phi(x) = y_0,$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt \qquad (k = 0, 1, 2, \ldots..)$$

converge on the interval

$$I : |x - x_0| \leqq \alpha = \text{Min}\left\{a, \, {}^{b}\!/\!_{M}\right\}$$

to a solution ϕ of the initial value problem.

$$y' = f(x, y), \ y(x_0) = y_0 \text{ on } I.$$

Proof : We divide the proof into four parts.

(a) convergence of $\{\phi_k(x)\}$: Clearly,

$$\phi_k = \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \cdots\cdots\cdots + (\phi_k - \phi_{k-1}).$$

So, the sequence $\{\phi_k(x)\}$ is the partial sum sequence of the series.

$$\phi_0(x) + \sum_{p=1}^{\infty}\left[\phi_p(x) - \phi_{p-1}(x)\right] \quad \text{-------------- 11.1.1(1)}$$

Hence to show that the sequence $\{\phi_p(x)\}$ converges, it is enough if we prove that the series 11.1.1(1) converges.

By theorem 9.1.3, the functions ϕ_p all exist as continuous functions on I, and for each $x \in I$, $(x, \phi_p(x)) \in R$. Further,

$$|\phi_1(x) - \phi_0(x)| \leq M|x - x_0| \quad \text{---------- 11.1.1(2)}$$

for all x in I. Writing down the relations defining ϕ_1 and ϕ_2 and subtracting, we obtain

$$|\phi_2(x) - \phi_1(x)| = \left|\int_{x_0}^{x}\left[f(t, \phi_1(t)) - f(t, \phi_0(t))\right] dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left|f(t, \phi_1(t)) - f(t, \phi_0(t))\right|\right|$$

$$\leq K\left|\int_{x_0}^{x}|\phi_1(t) - \phi_0(t)|\, dt\right|$$

(since f satisfies Lipschitz condition, we have that for any $(x, y_1), (x, y_2)$ in R,

$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$)

$$\leq KM\left|\int_{x_0}^{x}|t - x_0|\, dt\right| \quad \text{(by 11.1.1(2))}$$

Thus, if $x \geq x_0$, then

$$|\phi_2(x) - \phi_1(x)| \leq KM\int_{x_0}^{x}(t - x_0)\, dt$$

$$= \frac{KM \times (x - x_0)^2}{2}$$

The same in valid even if $x \leqq x_0$.

Now, we shall prove by induction that

$$\left|\phi_p(x) - \phi_{p-1}(x)\right| \leqq \frac{M K^{p-1} \left|x - x_0\right|^p}{p!} \quad \text{------------ 11.1.1(3)}$$

for all x in I.

We have already proved this for $p=1$ and $p=2$. Assume this for $p=m$. Now, we prove this for $p=m+1$. Let $x \geqq x_0$. (The proof in similar when $x \leqq x_0$). Using the definitions of ϕ_{m+1}, ϕ_m, we have

$$\left|\phi_{m+1}(x) - \phi_m(x)\right| = \left|\int_{x_0}^{x} \left[f(t, \phi_m(t)) - f(t, \phi_{m-1}(t))\right] dt\right|$$

$$\leqq \int_{x_0}^{x} \left|f(t, \phi_m(t)) - f(t \phi_{m-1}(t))\right| dt$$

$$\leqq K \int_{x_0}^{x} \left|\phi_m(t) - \phi_{m-1}(t)\right| dt$$

(since f satisfies Lipschitz condition)

$$\leqq \frac{K\ M\ K^{m-1}}{m!} \int_{x_0}^{x} (t - x_0)^m \, dt$$

$$= \frac{M\ K^m (x - x_0)^{m+1}}{(m+1)!}$$

Thus 11.1.1(3) holds when $p=m+1$. By the principle of mathematical induction, 11.1.1(3) holds for all $p = 1, 2, \ldots\ldots$

Now, for any $x \in I$,

$$|\phi_p(x) - \phi_{p-1}(x)| \leqq \frac{M}{K} \cdot \frac{k^p |x-x_0|^p}{p!}$$

$$\leqq \frac{M}{K} \cdot \frac{K^p \alpha^p}{p!} \quad \text{(since } |x-x_0| \leqq \alpha \text{)}$$

for $p = 1, 2, \ldots\ldots\ldots$ Hence

$$\sum_{p=1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)| \leqq \frac{M}{K} \sum_{p=1}^{\infty} \frac{K^p \alpha^p}{p!}$$

$$\leqq \frac{M}{K} e^{K\alpha}$$

So, the series 11.1.1(1) converges absolutely and uniformly on I and hence the sequence $\{\phi_p(x)\}$ of successive approximations converges to a limit $\phi(x)$ (say) for each $x \in I$.

(b) Properties of the limit ϕ :

For any x_1, x_2 in I,

$$|\phi_{p+1}(x_1) - \phi_{p+1}(x_2)| = \left| \int_{x_1}^{x_2} f(t)\, \phi_p(t)\, dt \right|$$

$$\leqq M |x_1 - x_2|$$

(since $|f(x, y)| \leqq M$ for all (x, y) in R).

which implies that - by letting $p \to \infty$,

$$|\phi(x_1) - \phi(x_2)| \leqq M |x_1 - x_2| \quad \text{---------------} \quad 11.1.1(4)$$

This shows that " $x_2 \to x_1 \Rightarrow \phi(x_2) \to \phi(x_1)$" and hence f is continuous on I. Also, letting $x_1 = x,\ x_2 = x_0$ in 11.1.1(4), we have

$$|\phi(x) - y_0| \leqq M |x - x_0| \quad \text{(for any } x \text{ in } I \text{)}$$

which implies that $(x, \phi(x))$ in R for all x in I

(c) Estimate for $|\phi(x) - \phi_p(x)|$:

Now,

$$|\phi(x) - \phi_p(x)| = \left| \sum_{q=p+1}^{\infty} \left[\phi_q(x) - \phi_{q-1}(x)\right] \right|$$

$$\leq \sum_{q=p+1}^{\infty} |\phi_q(x) - \phi_{q-1}(x)|$$

$$\leq \frac{M}{K} \sum_{q=p+1}^{\infty} \frac{(K\alpha)^q}{q!}$$

$$\leq \frac{M}{K} \frac{(K\alpha)^{p+1}}{(p+1)!} \sum_{q=0}^{\infty} \frac{(K\alpha)^q}{q!}$$

$$= \frac{M}{K} \cdot \frac{(K\alpha)^{p+1}}{(p+1)!} e^{K\alpha} \qquad \text{------------ } 11.1.1(5)$$

Let $\in_p = (K\alpha)^{p+1}/(p+1)!$. Then $\in_p \to 0$ as $p \to \infty$, since $\in_p$ is a general term of the series for $e^{K\alpha}$. In terms of $\in_p$, 11.1.1(5) can be written as

$$|\phi(x) - \phi_p(x)| \leq \frac{M}{K} e^{K\alpha} \in_p \quad (\in_p \to 0 \text{ as } p \to \infty) \qquad \text{----------- } 11.1.1(6)$$

(d) The limit ϕ is a solution : To complete the proof, we must show

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt \qquad \text{-------------- } 11.1.1(7)$$

for all x in I. The right side of 11.1.1(7) makes sense as ϕ is continuous on I, f is continuous on R and thus, the function F on I defined by

$$F(t) = f(t, \phi(t))$$

is continuous on I. Now,

$$\phi_{p+1}(x) = y_0 + \int_{x_0}^{x} f\left(t, \phi_p(t)\right) dt$$

and $\phi_{p+1}(x) \to \phi(x)$ as $p \to \infty$. Thus, to prove 11.1.1(7), we must show that for each $x \in I$,

$$\int_{x_0}^{x} f\left(t, \phi_p(t)\right) dt \to \int_{x_0}^{x} f\left(t, \phi(t)\right) dt \ (p \to \infty) \text{ -------- } 11.1.1(8)$$

we have

$$\left| \int_{x_0}^{x} f\left(t, \phi(t)\right) dt - \int_{x_0}^{x} f\left(t, \phi_p(t)\right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \left| f\left(t, \phi(t)\right) - f\left(t, \phi_p(t)\right) \right| dt \right|$$

$$\leq K \left| \int_{x_0}^{x} \left| \phi(t) - \phi_p(t) \right| dt \right|$$

(since f satisfies Lipschitz condition)

$$\leq M \, e^{K\alpha} \in_p |x - x_0| \qquad \text{(by 11.1.1(6))}$$

which tends to 0 as $p \to \infty$ (since $\in_p \to 0$ as $p \to \infty$) for each x in I. Thus, we have proved 11.1.1(8). Hence the theorem.

11.1.2 Theorem : The p^{th} successive approximation ϕ_p is the solution ϕ of the initial value problem of Theorem 11.1.1 satisfies

$$\left| \phi(x) - \phi_p(x) \right| \leq \frac{M}{K} \frac{(K\alpha)^{p+1}}{(p+1)!}$$

Proof : The proof of this theorem is included in the proof of theorem 11.1.1.

11.1.3 Example : Consider the initial value problem

$$y' = 1+y^2,\ y(0) = 0 \text{ ---------- } 11.1.3(1)$$

(a) Using separation of variables, find the solution ϕ of this problem. On what interval does ϕ exist ?

(b) Show that the successive approximations $\phi_0, \phi_1, \phi_2, ...$ exist for all real x.

(c) Show that $\phi_p(x) \to \phi(x)$ for each x satisfying

$$|x| \leq \frac{1}{2}$$

Now we answer these (a), (b) and (c)

(a) : 11.1.3(1) can be written as

$$\frac{y'}{1+y^2} = 1$$

Interesting with respect to x from 0 to x, we have

$$\int_0^x \frac{y'(s)}{1+y^2(s)}\, ds = x.$$

Letting $u = y(s)$,

$$\int_0^{y(x)} \frac{du}{1+u^2} = x$$

i.e. $\left[Tan^{-1} u\right]_0^{y(x)} = x$

i.e. $Tan^{-1} y(x) = x$

i.e. $y(x) = Tan\, x$.

$y(x)$ is defined for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ i.e. $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(b) : Here, $y' = 1+y^2 = f(x, y)$. Clearly,

$$\phi_0(x) = y(0) = 0;$$

$$\phi_1(x) = \int_0^x (1+0)\,dt = x;$$

$$\phi_2(x) = \int_0^x \left[1+\phi_1(t)^2\right] dt$$

$$= \int_0^x \left[1+t^2\right] dt = \frac{x^2}{2} + \frac{x^3}{3};$$

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and hence each $\phi_p(x)$ exists for all real x and in a poly nomial in x.

(c) : For any (x, y) in R (i.e. $|x| \leqq \frac{1}{2}$ and $|y| \leq 1$)

$$|f(x, y)| = \left|1 + y^2\right| \leq 2$$

Take $K = 2$. For any (x, y_1) and (x, y_2) in R,

$$|f(x, y_1) - f(x, y_2)| = \left|y_1^2 - y_2^2\right|$$

$$= |y_1 + y_2| \left|y_1^2 - y_2^2\right|$$

$$\leq (|y_1| + |y_2|)\,|y_1 - y_2|$$

$$= 2|y_1 - y_2|.$$

So, f satisfies Lipschitz condition and Lipschitz constant is $K = 2$. In view of the existence theorem,

$$\phi_p(x) \to \phi(x) \text{ as } p \to \infty$$

for all x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

11.2 SHORT ANSWER QUESTIONS

11.2.1 : State the existence theorme for the convergence of successive approximations of an initial value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

11.2.2 : Write the first three successive as pronimations ϕ_0, ϕ_1, ϕ_2 of the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

11.2.3 : If $\phi_0, \phi_1, ...$ are the successive approximations of the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0,$$

and if $|f(x, y)| \leq M$ for all (x, y) in the rectangle

$$R: |x - x_0| \leq a,\ |y - y_0| \leq b\ (a, b > 0)$$

and if f satisfies Lipschitz condition on R, then for any x in I,

$$|\phi_p(x) - \phi_{p-1}(x)| \leq T,$$

where $T =$ ....

11.3 MODEL EXAMINATION QUESTIONS :

11.3.1 : State and prove the existence theorem of the successive approximations.

11.3.2 : Consider the initial value problem :

$$y' = 1 - 2xy,\ y(0) = 0.$$

(a) Solve this equation as it is a linear equation let us solution be ϕ.

(b) Consider the problem on the rectangle

$$R: |x| \leq \frac{1}{2},\ |y| \leq \frac{1}{2}.$$

If $f(x, y) = 1 - 2xy$, show that

$$|f(x, y)| \leqq 2$$

for all (x, y) in R and that all the successive approximations to the solution exist on $|x| \leqq \frac{1}{2}$, and their graphs remain in R.

(c) Show that f satisfies Lipschitz condition on R with Lipschitz constant $K = 1$.

(d) Show that the approximation ϕ_2 satisfies

$$|\phi(x) - \phi_2(x)| < 0.01 \text{ for } |x| \leqq \frac{1}{2}.$$

11.4 EXERCISES

11.4.1 : Consider the problem :

$$y' = 1 - 2xy, \quad y(0) = 0.$$

(a) Since the differential equation is linear, an expression can be found for the solution. Find it

(b) Consider the above problem on

$$R: |x| \leqq \frac{1}{2}, \; |y| \leqq 1.$$

If $f(x, y) = 1 - 2xy$, show that

$$|f(x, y)| \leqq 2 \quad ((x, y) \text{ in } R),$$

and that all the successive approximations to the solution exist on $|x| \leqq \frac{1}{2}$ and their graphs remain in R.

(c) Show that f satisfies Lipschitz condition on R with Lipschitz constant $K = 1$, and therefore by theorem 11.1.1, the successive approximations converge toa limit (solution) of the initial value problem on $|x| \leqq \frac{1}{2}$.

(d) Show that the approximation ϕ_2 satisfies

$$|\phi(x)-\phi_2(x)| \leqq 0.1 \text{ for } |x| \leqq \frac{1}{2}.$$

(e) Compute ϕ_3.

11.4.2 : Consider the problem

$$y' = 1 + y^2, \; y(0) = 0$$

(a) Using separation of variables, find the solution ϕ of this problem. On what interval does ϕ exist ?

(b) Show that all the successive approximations $\phi_0, \phi_1, \phi_2, ...$ exist for all real x.

(c) Show that $\phi_k(x) \to \phi(x)$ for each x satisfying $|x| \leqq \frac{1}{2}$. (Hint : Consider $f(x, y) = 1 + y^2$ on $R: |x| \leqq \frac{1}{2}, |y| \leqq 1$. Show that $\alpha = \frac{1}{2}$).

11.4.3 : On the square

$R: |x| \leqq 1, \; y \leqq 1$, let f be defined by

$$f(x, y) = 0, \text{ if } x = 0, |y| \leqq 1,$$

$$= 2x, \text{ if } 0 < x \leqq 1, \; -1 \leqq y < 0,$$

$$= 2x - \frac{4y}{x}, \text{ if } 0 < |x| \leqq 1, \; x^2 \leqq y \leqq x^2$$

$$= -2x, \text{ if } 0 < |x| \leqq 1, \; x^2 \leq y \leq 1.$$

(a) Show that f is continuous on R, and $|f(x, y)| \leqq 2$ on R.

(b) Show that this f does not satisfy Lipschitz condition R.

(c) Show taht the successive approximations $\phi_0, \phi_1, ...$ for the problem.

$y' = f(x, y), \; y(0) = 0$ satisfy

$$\phi_0(x) = 0, \; \phi_{2m-1}(x) = x^2, \phi_{2m}(x) = -x^2 \; (m = 1, 2, ...)$$

(d) Prove that neither of the convergent subsequences in (c) converges to a solution of the initial value problem. (Note : This problem has a solution, but the above shows that it cannot be obtained by using successive approximations).

11.5 ANSWERS TO SHORT ANSWER QUESTIONS

11.2.1 : See statement of theorem 11.1.1

11.2.2 : Write

$$\phi_p(x) = y_0 + \int_{x_0}^{x} f\left(t, \phi_{p-1}(t)\right) dt \text{ for } p = 0, 1, 2.$$

11.2.3 : $T = \dfrac{M K^{p-1} |x - x_0|^p}{p!}$, where K is Lipschitz constant.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 12

NON LOCAL EXISTENCE OF SOLUTIONS

12.0 INTRODUCTION

In lesson - 11, Theorem 11.1 is called local existence theorem since it guarantees a solution only for x near to x_0. There are many cases when a solution to the initial value problem exists on the internal $|x-x_0| \leq a$ and in such cases we say that a solution exists non-locally.

Consider the linear equation

$$y' + g(x)y = h(x) \text{ ----------- } (12.0(a))$$

The solution exist on every interval where g and h are continuous. Suppose g and h are continuous on the interval $|x-x_0| \leq a$ and that there is a constant $K(>0)$ such that

$$|g(x)| \leqq K \qquad (|x-x_0| \leqq a)$$

We can write 12.0(a) as

$$y' = f(x,y) = -g(x)y + h(x)$$

For any (x,y) in the strip

$$S : |x-x_0| \leqq a, \ |y| < \infty,$$

$$|f(x, y_1) - f(x, y_2)| = |-g(x)(y_1 - y_2)|$$

$$\leqq K|y_1 - y_2|$$

Thus, f satisfies Lipschitz condition on the strip S instead of a rectangle R. By looking carefully at the Theroem 12.1.1, we can show that there exists a solution on the entire interval $|x-x_0| \leqq a$.

12.1 NON LOCAL EXISTENCE OF SOLUTIONS

12.1.1 Theorem : Let f be a real-valued continuous function on the strip

$$S: |x - x_0| \leqq a,\ |y| < \infty \qquad (a > 0),$$

and suppose that f satisfies on S a Lipschitz condition with constant $K > 0$. The successive approximations $\{\phi_k\}$ for the problem.

$$y' = f(x, y),\ y(x_0) = y_0 \text{ --------------- } 12.1.1(a)$$

exist on the interval $|x - x_0| \leq a$, and converge there to a solution ϕ of 12.1.1 (a).

Proof : The successive approximations are given by

$$\phi_0(x) = y_0,$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\,dt,\ (k = 0, 1, 2, ...).$$

We can establish the existence of each ϕ_k by induction as in theorem for

$$|x - x_0| \leq a.$$

Define F_0 on $[x_0 - a, x_0 + a]$ by

$$F_0(x) = f(x, y_0)$$

Since f is continuous on S, F_0 is continuous for $|x - x_0| \leqq a$ and hence bounded on $[x_0 - a, x_0 + a]$. So, there is a constant M such that

$$|F_0(x)| = |f(x, y_0)| \leq M \qquad (|x - x_0| \leq a).$$

The proof of convergence of $\{\phi_k(x)\}$ follows from part (a) of theorem 11.1.1 as

$$|\phi_1(x) - \phi_0(x)| = \left| \int_{x_0}^{x} f(t, y_0)\,dt \right|$$

$$\leq \left| \int_{x_0}^{x} |f(t, y_0)|\,dt \right| \leq M|x - x_0|.$$

Let ϕ be the limit of the sequence $\{\phi_k\}$ (On $|x-x_0|\leqq a$).

Now,

$$|\phi_k(x)-y_0| = \left|\sum_{i=1}^{k}\left[\phi_i(x)-\phi_{i-1}(x)\right]\right|$$

$$\leq \sum_{i=1}^{K}|\phi_i(x)-\phi_{i-1}(x)|$$

$$\leqq \frac{M}{K}\sum_{i=1}^{k}\frac{K^i|x-x_0|^i}{i!}$$

$$\leqq \frac{M}{K}\sum_{i=1}^{\infty}\frac{K^i|x-x_0|^i}{i!}$$

$$= \frac{M}{K}\left(e^{K\alpha}-1\right) \qquad \text{(where } \alpha=|x-x_0|\text{)}$$

$$\leqq \frac{M}{K}\left(e^{Ka}-1\right)$$

$$= b \text{ (say) ----------- (12.1.1(b))}$$

for any x such that $|x-x_0|\leqq a$. Taking limit as $k\to\infty$ in 12.1.1(b), we have

$$|\phi(x)-y_0|\leqq b$$

for any x with $|x-x_0|\leqq a$. Since f is continuous on

$$R:|x-x_0|\leqq a,\ |y-y_0|\leqq b,$$

there exists $N>0$ such that

$$|f(x,y)|\leqq N$$

for all (x, y) in R. For any x_1, x_2 in the interval $|x-x_0|\leqq a$,

$$\left|\phi_{k+1}(x_1)-\phi_{k+1}(x_2)\right| = \left|\int_{x_1}^{x_2} f(t, \phi_k(t))dt\right|$$

$$\leqq N|x_1 - x_2|$$

and hence, as $k \to \infty$, we have that

$$|\phi(x_1)-\phi(x_2)| \leqq N|x_1 - x_2|$$

so that ϕ is continuous (in fact uniformly continuous) on the interval $|x-x_0| \leqq a$. The remainder of the proof is a repetition of parts (c) and (d) of the proof of Theorem 11.1.1 with α replaced by a and M by N.

12.1.1.1 Corollary : Suppose f is a real valued continuous function on the plane $|x|<\infty, |y|<\infty$, which satisfy a Lipschitz condition on each strip

$$S_a : |x| \leqq a, |y| < \infty$$

for any positive number a. Then every initial value problem

$$y' = f(x, y), y(x_0) = y_0$$

has a solution which exists for all x in $\mathbb{R}$.

Proof : Let x be a real number. Then there exists a number $a>0$. $|x-x_0| \leqq a$ i.e. $x_0 - a \leq x \leq x_0 + a$ i.e. $x \in [x_0 - a, x_0 + a]$. For this a, f is continuous on $[x_0 - a, x_0 + a] X \mathbb{R} (= \widehat{S_a} \text{ say})$. By hypothesis, f satisfies Lipschitz conditon on the strip

$$S_b : |x| \leqq |x_0| + a(=b), |y| < \infty$$

Clearly, $\widehat{S_a} \subseteq S_b$. So, f satisfies Lipschtiz condition on $\widehat{S_a}$. Thus, the hypotehsis of Theorem 12.1.1 in satisfied. Hence, there exists a solution ϕ to the initial value problem.

12.1.2 Example : Consider the initial value problem

$$y' = \frac{y^3 e^x}{1+y^2} + x^2 \cos y, \; y(x_0) = y_0$$

using corollary 12.1.1.1, we show that this problem has a solution for all real x. Let

$$f(x, y) = y' = \frac{y^3 e^x}{1+y^2} + x^2 \cos y$$

Clearly f is continuous on $\mathbb{R} \times \mathbb{R}$. Consider the strip

$$S_a : |x| \leq a, \ |y| < \infty$$

For any (x, y) in S_a,

$$\left|\frac{df}{dy}(x, y)\right| = \left|\frac{y^4 + 3y^2}{\left(1+y^2\right)^2} - x^2 \sin y\right|$$

$$\leq 3\, e^a + a^2.$$

By theorem 12.1.1, f satisfies Lipschitz condition on S_a, with Lipschitz constant.

$$K_a = 3e^a + a^2.$$

By corollary 12.1.1.1, there is a solution ϕ to the given initial value problem which is defined for all real x.

12.1.3 Example : Consider the function f on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = y^2$.

(a) Now, we observe that f does not satisfy Lipschitz condition on any strip

$$S_a : |x| \leq a, \ |y| < \infty$$

but satisfies the same on any rectangle.

Let $a > 0$. If $y_1 > 0$, then

$$\frac{|f(x, y_1) - f(x, \mathrm{v})|}{|y_1 - \mathrm{v}|} = \frac{|y_1{}^2 - \mathrm{v}|}{|y_1|} = y_1$$

which is unbounded on S_a as $y_1 \to \infty$ and is clearly bounded on the rectangle.

$$S_{a,b} : |x| \leq a,\ |y| \leq b;$$

infact bounded by b (i.e. The Lipschitz constant is b).

(b) Consider the initial value problem :

$$y' = f(x,y) = y^2,\ y(1) = -1$$

Now, we solve this problem.

$$\frac{dy}{dx} = y^2 \text{ i.e. } -\frac{dy}{y^2} + dx = 0$$

on Integrating,

$$\frac{1}{y} + x = C \text{ i.e. } y = \frac{1}{C-x}$$

which is a solution of $y' = y^2$ (provided $x \neq C$).

If $y(1) = -1$ then $-1 = \frac{1}{C-1}$ i.e. $C = 1$. So, $y = \frac{1}{1-x}$ is a solution. Thus, solution of this initial value problem exists when $x \neq 1$.

12.1.4 Example : Let

$$f(x,y) = \frac{\cos y}{1-x^2} \quad (|x| < 1)$$

(a) Show that f satisfies Lipschtiz condition on every strip

$$S_a : |x| \leq a(0 < a < 1),\ y < \infty.$$

(b) Show that every initial value problem

$$y' = f(x, y),\ y(0) = y_0 \qquad (|y_0| < \infty)$$

has a solution which exists for $|x| < 1$.

Ans : (a) : Clearly, f is continuous on the strip

$$|x| < 1,\ |y| < \infty$$

Now, for any (x, y) in S_a,

$$\frac{\partial f}{\partial y}(x,y) = \frac{-\sin y}{1-x^2} \text{ and hence}$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \leqq \frac{1}{1-a^2}.$$

In view of theorem 12.1.1, f satisfies Lipschitz condition with Lipschitz constant

$$K_a = \frac{1}{1-a^2}.$$

(b) Since f is continuous on every strip S_a and satisfies Lipschitz condition on S_a, by Theorem 12.1.1, the initial value problem

$$y' = f(x, y),\ y(0) = y_0\ (|y_0| < \infty)$$

has a solution which exists for $|x| < 1$.

12.2 SHORT ANSWER QUESTIONS :

12.2.1 : State non-local existence Theorem.

12.2.2 : Consider the function $f(x, y)$ defined by

$$f(x, y) = \frac{\cos y}{1-x^2}\ (|x| < 1)$$

Let $0 < a < 1$. Prove that f satisfies Lipschitz conditon on the strip

$$S_a : |x| \leqq a.$$

12.3 MODEL EXAMINATION QUESTIONS

12.3.1 Consider the equation

$$y' = \left(3x^2 + 1\right)\cos^2 y + \left(x^3 - 2x\right)\sin 2y\ \left(= f(x, y)\right) \text{ on the strip}$$

$S_a : |x| \leqq a \ (a > 0)$.

Show that f satisfies Lipschitz condition on the strip S_a, and hence every initial value problem

$y' = f(x, y),\ y(x_0) = y_0$ has a solution which exists for all $x \in \mathbb{R}$.

12.3.2 : State and prove Non-local existence Theorem :

12.3.3 : Suppose f is a real value continuous function on the plane

$$|x| < \infty,\ |y| < \infty$$

such that f satisfies Lipschitz condition on each strip

$$S_a : |x| \leqq a,\ |y| < \infty$$

where a is any positive real number. Prove that every initial value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

has a solution which exists for all real x.

12.4 EXERCISES

12.4.1 : Consider the equation

$$y' = f(x)\, p(\cos y) + g(x)\, q(\sin y)$$

where f, g are continuous functions for all real x and p, q are polynomials. Show that every initial value problem for this equation has a solution which exists for all real x.

12.4.2 : Let f be a real valued function on the strip

$$S : |x - x_0| \leqq a,\ |y| < \infty \quad (a > 0),$$

and suppose that f satisfies a Lipschitz condition with constant $K\,(> 0)$. Show that the successive approximations

$$\phi_0(x) = y_0,$$

$$\phi_{k+1}(x) = y_0 + (x - x_0) y_1 + \int_{x_0}^{x} (x - t) f(t, \phi_k(t)) dt$$

$(k = 0, 1, 2, \ldots\ldots\ldots\ldots)$,

exist as continuous functions on the whole interval $I : |x - x_0| \leqq a$, and converge on I to a solution ϕ for the initial value problem

$$y'' = f(x, y), \quad y(x_0) = y_1; \quad y'(x_0) = y_1$$

12.4.3 : Prove that the corollary to theorem 12.1.1 for the initial value problem

$$y'' = f(x, y), \; y(x_0) = y_0, \; y'(x_0) = y_0$$

12.4.4 : Let f be a real valued continuous function on the strip

$$S : |x| \leqq a, \; |y| < \infty \quad (a > 0),$$

and suppose f satisfies a Lipschitz condition on S with constant $K > 0$. Show that the successive approximations.

$$\phi_0(x) = 0,$$

$$\phi_{k+1}(x) = \frac{\sin \lambda x}{\lambda} + \int_0^x \frac{\sin \lambda (x-1)}{\lambda} f(t, \phi_k(t)) dt \quad (\lambda > 0) \; (k = 0, 1, 2, \ldots\ldots\ldots).$$

exist as continuous functions on $I : |x| \leq a$, and converge there to a solution ϕ of the initial value problem.

$$y'' + \lambda^2 y = f(x, y), \; y(0) = 0, \; y'(0) = 1.$$

(**Hint :** See exercise of Lesson 11) (Note : The existence of a solution to the initial value problem can be demonstrated by applying exercise 12.4 to the problem

$$y'' = f(x, y) - \lambda^2 y, \; y(0) = 0, \; y'(0) = 1$$

12.4.5 : Prove the corollary to Theorem 12.4.1 for the initial value problem

$$y'' + \lambda^2 y = f(x, y), \; y(0) = 0, \; y'(0) = 1.$$

12.4.6 : Let g be a real valued continuous function on $I:|x|\leq a$, where $a>0$. Consider the initial value problem

$$y''+\lambda^2 y = g(x)y\ (\lambda \geqq 0),\ y(0)=0,\ y'(0)=1 \qquad (*)$$

(a) Show that there is a solution ϕ of $(*)$ on I and give an integral equation which ϕ also satisfies.

(b) If ϕ is continuous for all real x, show that there is a solution of $(*)$ for all real x (Hint : see exercises 12.4.2, 12.4.3, 12.4.4. and 12.4.5).

12.5 ANSWERS TO SHORT ANSWER QUESTIONS

12.2.1 : See statement of the Theorem 12.1.1.

12.2.2 : See example 12.1.4.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

Lesson Writer :

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Lesson - 13

APPROXIMATIONS TO SOLUTIONS AND UNIQUENESS OF THE SOLUTIONS

13.0 INTRODUCTION

Consider the initial value problem

$$y' = f(x, y),\ y(x_0) = y_0$$

(where f is a continuous real valued function on a rectangle

$$R: |x - x_0| \leqq a,\ |y - y_0| \leqq b$$

satisfying Lipschitz condition) on an interval I containing x_0. In this lesson, we show that the solution of this problem on the interval I is unique.

13.1 APPROXIMATIONS TO SOLUTIONS AND UNIQUENESS OF THE SOLUTIONS

13.1.1 Theorem : Let f, g be continuous real valued functions on the rectangle

$$R: |x - x_0| \leqq a,\ |y - y_0| \leqq b,\ (a,\ b > 0)$$

Suppose f satisfies Lipschitz condition on R with Lipschitz constant K. Suppose ϕ and ψ be solutions of the initial value problems :

$$y' = f(x, y),\ y(x_0) = y_1 \text{ -------------- } 13.1.1\ (a)$$

and $$y' = g(x, y),\ y(x_0) = y_2 \text{ -------------- } 13.1.2\ (b)$$

respectively on an interval I containing x_0 with graphs contained in R. Suppose there exist positive constants $\in, \delta$ such that

$$|f(x, y) - g(x, y)| \leqq \in \text{ (for all } (x, y) \text{ in } R\text{), ------------ } 13.1.1(c)$$

and $|y_1 - y_2| \leq \delta$ ----------- 13.1.1 (d)

Then $|\phi(x) - \psi(x)| \leq \delta e^{K|x-x_0|} + \frac{\in}{K}\left(e^{K|x-x_0|} - 1\right)$ ----------- 13.1.1 (e)

Proof : Since ϕ and ψ are solutions of 13.1.1(a), 13.1.1(b) respectively, we have that

$$\phi(x) = y_1 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

$$\psi(x) = y_2 + \int_{x_0}^{x} g(t, \psi(t))\, dt$$

and hence

$$|\phi(x) - \psi(x)| = \left| y_1 - y_2 + \int_{x_0}^{x} \left[f(t, \phi(t)) - g(t, \psi(t)) \right] dt \right|$$

$$\leq |y_1 - y_2| + \left| \int_{x_0}^{x} \left[f(t, \phi(t)) - g(t, \psi(t)) \right] dt \right|$$

$$+ \left| \int_{x_0}^{x} \left[f(t, \psi(t)) - g(t, \psi(t)) \right] dt \right|$$

$$\leq \delta + K \int_{x_0}^{x} |\phi(t) - \psi(t)|\, dt + \in |x - x_0| \quad \text{--------- 13.1.1 (f)}$$

(Using 13.1.1 (c), 13.1.1 (d) and using the fact that f satisfies Lipschitz condition with Lipschitz constant K and for $x \geq x_0$).

Let $E(x) = \int_{x_0}^{x} |\phi(t) - \psi(t)|\, dt$

For $x \geq x_0$,

$$E'(x)-K\,E(x) \leqq \delta + \in (x-x_0) \text{ ----------- } 13.1.1\text{ (g)}$$

This is a first order linear differential inequality which may be solved in the same way we solve first order linear differential equations. Multiplying 13.1.1(g) by $e^{-K(x-x_0)}$ and replacing by the dummy variable t, we get

$$\frac{d}{dt}\left(e^{-K(t-x_0)}\,E(t)\right) \leqq \delta\, e^{-K(t-x_0)} + \in (t-x_0)\, e^{-K(t-x_0)}$$

Integrating from x_0 to x, we have

$$e^{-K(x-x_0)}\,E(x) \leqq \frac{\delta}{K}\left(1-e^{-K(x-x_0)}\right) + \frac{\in}{K^2}\left[-K(x-x_0)-1\right]e^{-K(x-x_0)} + \frac{\in}{K^2},$$

since $E(x_0)=0$. Multiplying both sides of this inequality by $e^{K(x-x_0)}$, we get

$$E(x) \leqq \frac{\delta}{K}\left(e^{K(x-x_0)}-1\right) - \frac{\in}{K^2}\left[K(x-x_0)+1\right] + \frac{\in}{K^2}e^{K(x-x_0)}$$

So, $|\phi(x)-\psi(x)| = E'(x) \leqq \delta\, e^{K(x-x_0)} + \frac{\in}{K}\left[e^{-K(x-x_0)}-1\right].$

Thus, we have proved the conclusion when $x \geqq x_0$. A similar proof holds when $x \leqq x_0$. Hence the conclusion.

13.1.1.1 : Corollary (Uniqueness thoerem) : Let f be a continuous and satisfy Lipschitz condition on

$$R: |x-x_0| \leq a,\ |y-y_0| \leq b \quad (a, b > 0)$$

If ϕ and ψ are two solutions of the initial value problem

$y' = f(x,y)$, $y(x_0)=y_0$ on an interval I containing x_0 then $\phi(x)=\psi(n)$ for all x in I.

Proof : Taking $g=f$ and $y_0=y_1=y_2$ in Theorem 13.1.1, we see that we may choose $\in=0,\ \delta=0$. By Theorem 13.1.1, $\phi=\psi$ on I.

13.1.1.2 Remark : In addition to continuity, some restriction on f is required inorder to gurantee uniqueness. Consider the following.

13.1.2 Example : Consider the initial value problem

$$y' = 3y^{\frac{2}{3}},\ y(0)=0\cdot$$

Here $f(x,y)=3y^{\frac{2}{3}}$. Clearly, f is continuous on the (x,y) plane. Consider the functions ϕ and ψ defined by

$$\phi(x)=x^3,\ \psi(x)=0\ (-\infty<x<\infty)$$

Clearly, both ϕ and ψ are solutions of the problem. That is the initial value problem has no unique solution. We can easily observe that f does not satisfy Lipschitz condition on any rectangle containing the origin (see).

13.1.1.2 Corollary : Let f be continuous and satisfy Lipschitz on R. Let $g_k\ (k=1,2,......)$ be continuous on R and assume that there exist constants $\in_k$ such that

$$|f(x,\ y)-(1,\ y)|\leqq\in_k$$

for all (x,y) in R, where $\in_k\to 0$ as $k\to\infty$. Let $y_k\to y_0$ as $k\to\infty$. If ψ_k is a solution of

$$y' = g_k(x,\ y),\ y(x_0)=y_k$$

on an interval I containing x_0 and ϕ is a solution of

$$y' = f(x,\ y),\ y(x_0) = y_0$$

or I, then

$$\psi_k(x)\to\phi(x)$$

for all x in I.

Proof : Inview of theorem 13.1.1,

$$|\phi(x)-\psi_k(x)|\leq|y_k-y_0|\,e^{K|x-x_0|}+\frac{\in_k}{k}\left[e^{K|x-x_0|}-1\right]$$

where K is LIpschitz constant. Since $y_k\to y_0$ and $\in_{k\to 0}$ as $k\to\infty$, we have

$$\lim_{k\to\infty}\psi_k(x)=\phi(x),\ \text{for all } x\in I.$$

13.1.2 Example : Consider the initial value problem

$$y' = xy + y^{10},\ y(0) = \frac{1}{10}$$

(a) Show that a solution for this problem exists for $|x| \leq \frac{1}{2}$.

(b) For small $|y|$, the given problem can be approximated by the problem

$$y' = xy,\ y(0) = \frac{1}{10}$$

compute a soluton of this problem and show that its graph is in the rectangle

$$R : |x| \leq \frac{1}{2},\ \left|y - \frac{1}{10}\right| \leq \frac{1}{10}$$

(c) Show that $|\phi(x) - \psi(x)| \leq \frac{2}{5^{10}}\left[e^{|x|/2} - 1\right]$ for $|x| \leq \frac{1}{2}$

(d) Prove that $|\phi(x) - \psi(x)| \leq \frac{1}{5^{10}}\left[e^{|x|} - 1\right]$

Solution : (a) Let $g(x, y) = y' = xy + y^{10}$. For any (x, y) in $R : |x| \leq \frac{1}{2},\ \left|y - \frac{1}{10}\right| \leq \frac{1}{10}$;

$$|g(x, y)| = |xy + y^{10}|$$

$$\leq |x||y| + |y|^{10}$$

$$\leq \frac{1}{2} \times \frac{2}{10} + \left(\frac{2}{10}\right)^{10}$$

$$= \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$$

Hence, as in the main existence theorem,

$$\alpha = \min\left\{a, \frac{b}{M}\right\}$$

$$=\min\left\{\frac{1}{2}, \frac{1}{10} \times \frac{5}{1}\right\} = \frac{1}{2}$$

Since g is continuous on R and $|g(x, y)| \leq \frac{1}{5}$ on R, there exists a solution ψ of the given $I \vee P$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

(b) Clearly, the solution of the $I \vee P$ is given by

$$\phi(x)=\frac{1}{10}e^{x^2/2}.$$

We have

$$\left|\phi(x)-\frac{1}{10}\right| = \left|\frac{1}{10}e^{x^2/2} - \frac{1}{10}\right|$$

$$= \frac{1}{10}\left[e^{1/8} - 1\right] < \frac{1}{10}$$

Hence the graph of ϕ lies in R.

(c) Clearly, the function f is given by

$f(x,y)=xy$ is continuous and satisfies Lipschitz condition with Lipschitz contact $K=\frac{1}{2}$ on

$$k=|x|\leq\frac{1}{2}, \left|y-\frac{1}{10}\right|\leq\frac{1}{10}$$

By Theorem 13.1.1,

$$|\phi(x)-\psi(x)|\leq 0 + \left|\frac{f(x,y)-g(x,y)}{K}\right|\left(e^{K|x|}-1\right)$$

$$\leq 2\left|y^{10}\left[e^{|x|/2}-1\right]\right|$$

$$\leq 2\left(\frac{2}{10}\right)^{10}\left[e^{|x|/2}-1\right]$$

$$= \frac{2}{5^{10}}\left[e^{|x|/2} - 1\right]$$

(d) We have

$$e^{|x|} - 1 = \left(e^{|x|/2}\right)^2 - 1$$

$$= \left(e^{|x|/2} + 1\right)\left(e^{|x|/2} - 1\right)$$

$$\geqq e^{|x|/2} - 1$$

$$|\phi(x) - \psi(x)| \leqq \frac{1}{5^{10}}\left(e^{|x|} - 1\right).$$

13.2 SHORT ANSWER QUESTIONS

13.2.1 : State uniqueness theorem concerning the solution of the $I \vee P$.

$$y' = f(x, y),\ y(0) = y_0.$$

13.3 MODEL EXAMINATION QUESTIONS

13.3.1 : Let f be continuous and satisfy Lipschitz condition on the rectangle

$$R: |x - x_0| \leq a,\ |y - y_0| \leq b \qquad (a, b > 0).$$

If ϕ and ψ are two solutions of

$$y' = f(x, y),\ y(x_0) = y_0$$

on an interval I containing x_0, prove that $\phi(x) = \psi(x)$ for all x in I.

13.3.2 : consider the initial value problem

$$y' = xy + y^{10},\ y(0) = \frac{1}{10}.$$

(a) Show that a solution ψ of this problem exists for $|x| \le \frac{1}{2}$.

(b) For small $|y|$, this problem can be approximated by the problem

$$y' = xy,\ y(0) = \frac{1}{10}$$

Compute a solution ϕ of this problem, and show that its graph is in R for $|x| \le \frac{1}{2}$.

13.4 EXERCISES :

13.4.1 : Consider the problem

$$y' = y + \lambda x^2 \sin y,\quad y(0) = 1,$$

where λ is some real parameter, $|\lambda| \le 1$.

(a) Show that the solution for the problem exists for $|x| \le 1$.

(b) Prove that $\left|\psi(x) - e^x\right| \le \lambda\left(e^{|x|} - 1\right)$ for $|x| \le 1$.

13.4.2 : Let f be a continuous function for (x, y, λ) in $R: |x - x_0| \le a,\ |y - y_0| \le b,\ |\lambda - \lambda_0| \le c$,

where $a, b, c > 0$, and suppose that there exists a constant $K > 0$ such that

$$|f(x, y_1, \lambda) - f(x, y_2, \lambda)| \le K|y_1 - y_2|$$

for all $(x, y_1, \lambda), (x, y_2, \lambda)$ in R. Further suppose that $\frac{\partial f}{\partial \lambda}$ exists and there is a constant $L > 0$ such that

$$\left|\frac{\partial f}{\partial \lambda}(x, y, \lambda)\right| \le L$$

for all (x, y, λ) in R. If ϕ_λ represents the solution of

$$y' = f(x, y, \lambda),\ y(x_0) = y_0,$$ show that

$$|\phi_\lambda(x) - \phi_\mu(x)| \le \frac{L|\lambda-\mu|}{K}\left(e^{|x-x_0|K} - 1\right)$$

for all x for which ϕ_λ, ϕ_μ exist.

13.5 ANSWERS TO SHORT ANSWER QUESTIONS

13.2.1 : See the statement of corolary 13.1.1.1.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 14

SYSTEMS OF DIFFERENTIAL EQUATIONS

14.0 INTRODUCTION

Consider the system

$$\left.\begin{array}{l} y_1' = f_1(x, y_1, y_2, \cdots\cdots\cdots, y_n) \\ y_2' = f_2(x, y_1, y_2, \cdots\cdots\cdots, y_n) \\ \qquad \cdots\cdots \\ \qquad \cdots\cdots \\ \qquad \cdots\cdots \\ y_n' = f_n(x, y_1, y_2, \cdots\cdots\cdots, y_n) \end{array}\right\} \quad \text{------- } 14.0(a)$$

of n ordinary differential equations of first order where the derivatives $y_1', y_2', \cdots\cdots\cdots, y_n'$ appear explicitly. The system 14.0(a) can be represented as

$$y' = f(x, y)$$

where $f = \text{col}(f_1, f_2, \cdots\cdots, f_n)$ in a given complex-valued vector function defined in some set R in the $(x, (y_1, y_2, \cdots\cdots, y_n))$ space, x is real and $y_1, y_2, \cdots\cdots, y_n$ are complex.

14.0.1 : Consider the sytem 14.0(a) of n ordinary differential equations. If there exist n differentiable functions $\phi_1, \phi_2, \cdots\cdots, \phi_n$ on some interval I such that for any x in I,

(a) $\quad (x, \phi_1(x), \phi_2(x), \cdots\cdots \phi_n(x)) \in R$

(b) $\quad \phi_1'(x) = f_1(x, \phi_1(x), \phi_2(x), \cdots\cdots, \phi_n(x)),$

$$\phi_2'(x) = f_2(x, \phi_1(x), \phi_2(x), \cdots\cdots, \phi_n(x))$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\phi_n'(x) = f_n(x, \phi_1(x), \phi_2(x), \cdots\cdots, \phi_n(x)),$$

then we say that $(\phi_1, \phi_2, \cdots\cdots \phi_n)$ is a solution of the system 14.0(a).

One of the most famous system of the type (14.0) results from Newton's second law of motion for a particle of mass m. Using rectangular coordinates (x, y, z), their law is usually written as

$$m\,x'' = X,\ my'' = Y,\ nz'' = Z \text{ ---------- } 14.0(b)$$

Here differentiation is with respect to the time t, and x'', y'', z'' represent the acceleration of the particle in x, y, z directions respectively, where as x, y, z represent the forces activity on the particle in these directions. In general, $X,\ Y,\ Z$ are functions of t, x, y, z, x', y', z'. To see how the system 14.0(b) can be viewed as a system of type 14.0(a), let us make the following substitutions in 14.0 (b).

$$t \to x,\ x \to y_1,\ y_1 \to y_2,\ z \to y_3$$

$$x' \to y_4,\ y' \to y_5,\ z' \to y_6.$$

Then 14.0(b) is equivalent to the following system of sin equations

$$y_1' = y_4$$

$$y_2' = y_5$$

$$y_3' = y_6$$

$$y_4' = \frac{1}{m} X(x, y_1, y_2, \cdots\cdots, y_6)$$

$$y_5' = \frac{1}{m} Y(x, y_1, y_2, \cdots\cdots, y_6)$$

$$y_6' = \frac{1}{m} Z(x, y_1, y_2, \cdots\cdots, y_6)$$, which is of type 14.0.

Note : An equation of the n^{th} order

$$y^{(n)} = f\left(x,\ y,\ y',\ y'', \cdots\cdots y(n-1)\right) \text{ ---------- } 14.0(d)$$

may also treated as a syste of type 14.0(a). To see this, in 14.0(d), we substituts

$$y_1 = y,\ y_2 = y', y_3 = y_2', \cdots\cdots y_n = y^{(n-1)}.$$

Then the system 14.0(d) in equivalent to the system

$$y_1' = y_2$$

$$y_2' = y_3$$

....

....

$$y_{n-}' = y_n$$

$$y_{n-}' = f(x, y_1, y_2, \cdots\cdots, y_n)$$

which of type 14.0 (a)

14.1 AN EXAMPLE - CENTRAL FORCES AND PLANETARY MOTION

We now discuss an intensity example of a system of equations which gives a model for the motion of planets about the sun.

14.1.1 : An example - central forces and planetary motion : Suppose a particle of mass m moves in a plane, and subjected to a force which is directed along the line joining the particle with origin, and which has a magnitude depending only on the distance betweent he particle and the origin. Then, we say that we have a central force. The functions x, y (of the time t) which describe the path the particle taken satisfy, according to Newton's second law,

$$\left.\begin{aligned} mx'' &= \frac{x}{r}F(r), \\ my'' &= \frac{y}{r}F(r) \end{aligned}\right\} \quad 14.1.1(a)$$

where $r = \sqrt{x^2 + y^2}$, and $|F(r)|$ represents the magnitude of the force on the particle when it is at a distance of r units from the origin.

The system 14.1.1(a) is equivalent to a system of four first order equations in x, y, x', y'. However, since F is a function of r alone, it is advantageous to introduce polar - coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta$$

The components of acceleration in the radial and angular directions are given by

$$r'' - r(\theta')^2, \quad 2r'\theta' + r\theta''$$

respectiely. Since the components of the force int hese directions are $F(r)$ and 0, equation 14.1.1(a) are replaced by

$$\left.\begin{array}{l} m\left[r'' - r(\theta')^2\right] = F(r), \\ m\left[2r'\theta' + r\theta''\right] = 0 \end{array}\right\} \quad 14.1.1\text{ (b)}$$

Upon multiplying the second equation in 14.1.1(b) by $\frac{1}{m}$, we have

$$\left(r^2\theta'\right)^1 = 0$$

and hence

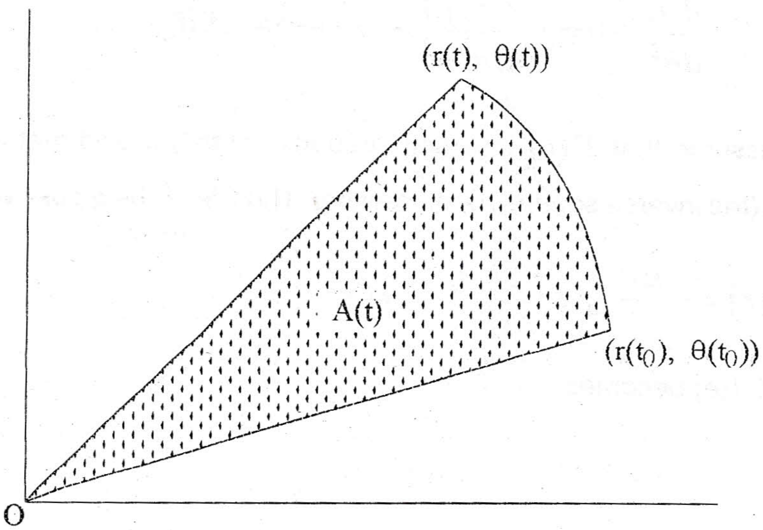
$$r^2\theta' = h \qquad 14.1.1\text{(c)}$$

where h is constant. The equation 14.1.1(c) has an interesting geometrical meaning. The area $A(t)$ transversed by the line segment from the origin to $(r(s), \theta(s))$ as s goes from to tot is given by

$$A(t) = \int_{t_0}^{t} \frac{1}{2} r^2(s)\theta'(s)ds,$$

since the element of area in polar co-ordinates is

$$dA = \frac{1}{2}r^2 d\theta \quad \text{(see figure)}$$



8)

since $r^2\theta' = h,$ we see that

$$A(t)=\frac{1}{2}h(t-t_0) \text{ ------------- } 14.1.1\text{ (d)}$$

Thus, if $h \neq 0$, the line segment from the origin to the particle sweeps out several areas in equal times.

Now supposing that $h>0$, let us analyse the first equation in 14..... (b). Now, we introudce a function υ defined for θ of the form $\theta(t)$ by

$$\upsilon(\theta(t)) = \frac{1}{r(t)} \text{ ------------- } 14.1.1\text{ (e)}$$

Then

$$r'(t) = -\frac{1}{\upsilon^2(\theta(t))}\left[\frac{d\upsilon}{d\theta}(\theta(t))\right]\theta'(t)$$

$$= -h\frac{d\upsilon}{d\theta}(\theta(t)),$$

and $$r''(t) = -h\frac{d^2\upsilon}{d\theta^2}(\theta(t))\,\theta'(t) = -h^2\upsilon^2(\theta(t))\frac{d^2\upsilon}{d\theta^2}(\theta(t))$$

where we have used 14.1.1 (c). Thus, the first equation in 14.1.1(b) becomes the following equation for υ :

$$\frac{d^2\upsilon}{d\theta^2} + \upsilon = -\frac{F\left(\frac{1}{\upsilon}\right)}{mh^2\upsilon^2} \text{ ----------- } 14.1.1\text{ (f)}$$

Now, we assume that $F(r)$ inversely proportional to r^2, and that the force is directed towards the origin (the inverse square law of Newton). Thus, let k be a positive constant such that

$$F(r) = -\frac{km}{r^2} \text{ or } F\left(\frac{1}{\upsilon}\right) = -Km\upsilon^2$$

Thus, 14.1.1(e) becomes

$$\frac{d^2\upsilon}{d\theta^2}+\upsilon=\frac{k}{h^2} \text{ ---------- 14.1.1 (g).}$$

All solutions of this linear equation may be written in the form

$$\upsilon(\theta)=\frac{k}{h^2}+B\cos(\theta-w),$$

where B, w are constants. Using the definition of υ, 14.1.1 (e) can be written as

$$r=\frac{h^2/K}{1+e\cos(\theta-w)} \text{ ---------- 14.1.1 (h)}$$

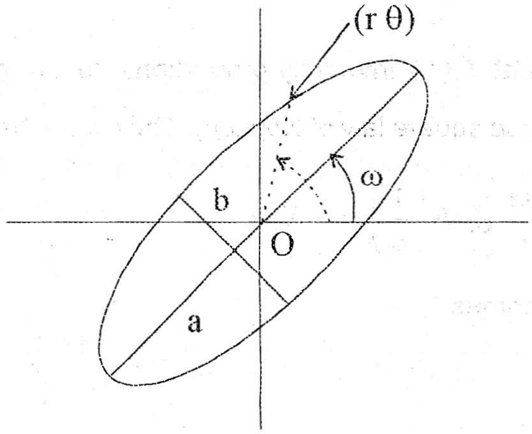
where $e=\frac{Bh^2}{K}$. For $\frac{h^2}{K}>0$ ana $e\geq 0$ the equation 14.1.1(h) is the equation of a conic with one focus at origin, with eccentricity e, this comic is an ellipse, parabole or hyperbola according as $0\leq e\leq 1$, $e>1$ respectively.

Let us analyse the case when the comic is an ellipse having major and minor semi axes a, b. See the following figure. Then $2a$ must be the sum of the largest and smallest values that r can assume, namely

$$2a=\frac{h^2}{k}\left(\frac{1}{1-e}+\frac{1}{1+e}\right)=\frac{2h^2}{k\left(1-e^2\right)}$$

The eccentricity is related to a, b via $b^2=a^2\left(1-e^2\right)$ and hence

$$b^2=\frac{h^2a}{k} \text{ ---------- 14.1.1(i)}$$



Now, the area of the ellipse is πab, and this is related to the time T required for hte particle to transverse the ellipse once by

$$\frac{1}{2}hT = \pi ab.$$

$$T^2 = \frac{4\pi^2}{k} a^3 \text{ ----------- } 14.1.1 \text{ (j)}$$

Kepler, on the basis of observations of Tycho Brahe on the motion of the planets about the sun, deduced his famous three laws of planetary motion :

(1) the line segment from the sun to a planet sweeps out equal areas in equal times,

(2) the planets move along ellipses with the sun as a focus,

(3) the squares of the periods are proportional to the cubes of the major axis of the ellipses.

If we idealize the motion of a planet about the sun as a plane motion, with sun fined at origin and exerting an attractive central force on the planet (thought of as a particle of mass m), then, we see that Newton's second law implies that the motion of the planet is governed by the system of equations 14.1.1(a) Kepler's first law is a consequence of the central force assumption. His second and third laws then result from the assumption that the central force is proportional to $\frac{1}{r^2}$.

Newoton discovered that Kepler's first two laws imply the inverse square law. Indeed, it was this that led Newton to the formulation of his famous law of universal gravitation. The first law

$$r^2 \theta' = h$$

implies that there is no force acting perpendicular to the line segment from the origin to the particle i.e. the second equation of 14.1.1(b) is valid. Hence the particle is acted by a force which acts in the radial direction only. If $F(r, \theta)$ is the radial component of this force at (r, θ), we have equation

$$m\left[r'' - r(\theta')^2\right] = F(r, \theta) \text{ -------------------- } 14.1.1 \text{ (k)}$$

as the analogue of the equation 14.1.1(b). Inroducing υ as in 14.1.1 (e), we see that 14.1.1 (k) implies the following equation for υ :

$$\frac{d^2\upsilon}{d\theta^2} + \upsilon = -\frac{F\left(\frac{1}{\upsilon}, \theta\right)}{m\, h^2 \upsilon^2} \text{ ----------------------- } 14.1.1(\ell)$$

Now Kepler's second law implies that r is related to via an - equatin of theorem 14.1.1 (b) with $0 \le e > 1$, and then υ will satisfy the equation 14.1.1 (g). A comparison of the equations 14.1.1(g) and 14.1.1(ℓ) shows that

$$F\left(1/\upsilon, \theta\right) = -km\,\upsilon^2$$

or that

$$F(r, \theta) = -\frac{km}{r^2}.$$

Thus, F depends only on r according to Newton's square law.

14.1.2 Example : A particle of man m moves in a plane, and is attatched to the origin with a force proportional to its distance r from the origin. Then, if

$$F(r) = -k^2 mr \qquad (k > 0),$$

the equations describing the path of the particle

$$mx'' = \frac{x}{r} F(r),\ K_a = \frac{1}{1-a^2},\ my'' = \frac{y}{r} F(r) \text{ reduced to}$$

$$x'' = -k^2 x,\ y'' = -k^2 y.$$

Show that the path of the aprticle is an ellipose if it satisfies the initial conditions

$$x(0) = a,\ x'(0) = 0,\ y(0) = 0,\ y'(0) = b\ \ (a > 0,\ b > 0)$$

Solution : Clearly, the solution of the equation

$$x'' = -K^2 x \text{ is}$$

$$x(t) = A\cos Kt + B\sin Kt$$

$$x'(t) = -AK\sin Kt + BK\cos t$$

$$x'(0) = \theta \Rightarrow \theta = BK \Rightarrow B = 0 \text{ (since } K \ne 0).$$

$$x(0) = a \Rightarrow a = A.$$

Hence,

$$x(t) = a\cos Kt \text{ ---------------- (1)}$$

Similarly, we can prove that

$$y(t) = \frac{b}{K} \sin Kt \text{ ------------- (2)}$$

On eliminating t from (1) & (2), we have

$$\frac{x^2}{a^2} + \frac{K^2 y^2}{b^2} = 1$$

which is an ellipse with a and $\frac{b}{K}$ as the length of the semi major axis semi major axis respectively if $K > 1$, we have their rolls interchanged if $K < 1$.

14.2 EXERCISES

14....1 : A particle mass m moves ina verticle plane near the surface of the earth, and is acted on by the force of gravities alone. The equations for the motion assume the form

$$m x'' = 0, \; my'' = -mg$$

where g is constant.

(a) Find the solution of these equations satisfying

$$x(0) = 0, \; y(0) = 0, \; x'(0) = \upsilon_0 \cos\alpha,$$

$y'(0) = \upsilon_0 \sin\alpha$, when $\upsilon_0 > 0$ and α are constants, $0 < \alpha < \frac{\pi}{2}$.

(b) Show that the particles path in a parabola.

(c) Compute the vertex of this parabola and the time required to each this vertex.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 15

SOME SPECIAL FUNCTIONS

15.0 INTRODUCTION

There are a number of problems which led to rather special types of second order equations, or systems of such equations. We consider two of these types in this lesson.

15.1 SOME SPECIAL EQUATIONS

15.1.1 Type (1) : The equation is of the form

$$y'' = f(x, y') \text{ ---------------------- } 15.1.1(a)$$

This second order equation has an f which is independent of y, and hence really a first order equation in y'. Indeed, this equation is equivalent to the system of two equations of first order.

$$y' = z,\ z' = f(x, z) \text{ ---------------------- } 15.1.1\ (b)$$

in that ϕ will be a solution of 15.1.1(a) on an interval I if, and only if, the functions ϕ and ϕ' satisfy the system 15.1.1 (b) on I. Now, the system 15.1.1(b) can be solved by first solving the first order equation

$$z' = f(x, z)$$

for ϕ', and then integrating to obtain ϕ.

Now, we consider the following example

15.1.2 Example : Consider the equation

$$xy'' - y' = 0 \text{ -------------------- } 15.1.2\ (a)$$

Let $y' = z$. Then the equation 15.1.2(a) becomes

$$xz' - z = 0$$

i.e. $$\frac{z'}{z} - \frac{z}{x} = 0$$

which is a first order linear equation. Clearly, the solution of this equation is given by

$$\phi'(x) = cx \qquad (x>0)$$

where c is any constant. Thus,

$$\phi(x) = \frac{cx^2}{2} + d \qquad (x>0)$$

where $c,\ d$ are constants.

Now study the equation which is of

15.1.3 Type (2) : The equation is of the form

$$y'' = f(y, y') \text{ -------------- } 15.1.3\ (a).$$

Here f is independent of x, and the strategy is some what different than in Type (1). Suppose we have a solution ϕ of 15.1.3 (a) and there is a differentiable function ψ, defined for all y of the form $y=\phi(x)$, such that

$$\phi'(x) = \psi(\phi(x)).$$

Then ϕ would be a solution of the first order equation

$$\frac{dy}{dx} = \psi(y) \text{ ------------------- } 15.1.3\ (b)$$

Also,

$$\phi''(x) = \phi'(x)\frac{d\psi}{dy}(\phi(x)) = \psi(\phi(x))\frac{d\psi}{dy}(\phi(x)) \text{ and more over}$$

$$\phi''(x) = f(\phi(x), \phi'(x)) = f(\phi(x), \psi(\phi(x))).$$

That ψ must satisfy the equation

$$\psi(y)\frac{d\psi}{dy}(y) = f(y, \psi(y))$$

for all $y=\phi(x)$, and hence must be a solution of

$$z\frac{dz}{dy} = f(y, z) \qquad 15.1.3\ (c)$$

Now, we consider the example

15.1.4 Example : Consider the equation

$$y\, y'' = (y')^2$$

and suppose we seek a solution ϕ satisfying

$$\phi(0) = 1,\ \phi'(0) = 2$$

So, $$y'' = \frac{(y')^2}{y} \quad (y \neq 0)$$

$$= f(y, y').$$

This is of the form Type (2). Let $y = \phi(x)$ be a solution of the given equatin and ψ be a differentiable function such that

$$\phi'(x) = \psi(\phi(x))$$

on some interval I. Then ψ satisfies the equation

$$z \cdot \frac{dz}{dy} = f(y, z) = \frac{z^2}{y}$$

Thus, $\psi \dfrac{d\psi}{dy} = \dfrac{\psi^2}{y}$;

On solving this equation, we have

$\psi(y) = c\,y$ where c is constant. So, $\phi'(x) = \psi(\phi(x)) = c\,\phi(x)$

on solving this equation, we have

$$\phi(x) = K\, e^{cx}$$

where K, c are constants

$$\phi(0) = 1 \Rightarrow K = 1$$

$$\phi'(0) = 2 \Rightarrow K \cdot C \cdot e^{C \cdot 0} = 2 \Rightarrow C = 2$$

Hence $\phi(x) = e^{2x}$ is a solution of the given equation

15.1.5 Example : We now solve the equation

$$y'' + e^x y' = e^x$$

Put $y' = z$. Then the given equation can be written as

$$z' + e^x z = e^x$$

i.e. $$\frac{z'}{1-z} = e^x$$

On integrating, we have

$$\log(z-1) + e^x = c$$

i.e. $$z - 1 = e^{+C-e^x} = C_1 e^{-e^x}$$

where $C_1 = e^c$ (here C is constant and hence C_1 is constant).

i.e. $$z - 1 = e^{+C - e^x} = C_1 e^{-e^x}$$

Hence $y = x + C_1 \int e^{-e^x} dx + C_2$

15.1.6 Example : Find the solution of

$$y'' = 1 + (y')^2$$

which satisfies $\phi(0) = 0,\ \phi'(0) = 0$.

Let $y' = z$. So the given equation can be written as

$$z' = 1 + z^2$$

i.e. $$\frac{z'}{1+z^2} = 1$$

On integrating, we get

$$z = \tan x + C_1$$

where C_1 is constant. That is

$$y'(x) = \tan x + C_1$$

On integrating,

$$y(x) = C_1 x - \log \cos x + C_2$$

where C_2 is constant

$$y(0) = 0 \Rightarrow C_2 = 0,$$

$$y'(0) = 0 \Rightarrow C_1 = 0.$$

Hence the required solution ϕ is given by

$$\phi(x) = \log \sec x$$

for all x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

15.2 SHORT ANSWER QUESTIONS

15.2.1 : If the equation $y'' = f(x, y')$, write down the substitution which we have to do.

15.2.2 : In solving the equation $v'' = f(y, y')$, write down necessary substitution.

15.3 MODEL EXAMINATION QUESTIONS

15.3.1 : Solve (a) $y'' + y' = 1$

(b) $y^2 y'' = y'$

15.3.2 : Solve (a) $y\, y'' + 4(y')^2 = 0$

(b) $y'' + K^2 y = 0 \qquad (K > 0)$

15.3.3 : Solve (a) $y'' + e^x . y' = e^x$

(b) $y'' = y\, y'$

15.4 EXERCISES

15.4.1 : Suppose that f is a continuous function on an interval $[x_0 - a,\ x_0 + a]$. Show that the solution of the initial value problem.

$$y'' = f(x),\ y(x_0) = \alpha,\ y'(x_0) = \beta$$

can be written as

$$\phi(x) = \alpha + \beta(x - x_0) + \int_{x_0}^{x} (x - t) f(t)\, dt$$

15.4.2 : (a) Let f be a continuous function for $|y - y_0| \leq b\ (b > 0)$, and consider the equation

$$y'' = f(y).$$

Show that the equation 15.3.3 (b) has a solution ψ, in this case given by

Show that the equation 15.3.3 (b) has a solution ψ, in this case given by

$$\psi^2(y) = \psi^2(y_0) + 2\int_{y_0}^{y} f(t)\, dt$$

(b) Consider the special case

$$y'' + \sin y = 0,$$

which is an evaluation associated with the oscillation s of a pendulum. If ϕ is a solution satisfying

$$\phi(0) = 0,\ \phi'(0) = \beta(> 0),$$

show that ϕ satisfyies the condition

$$y' = \beta\sqrt{1 - K^2 \sin^2(y/2)}, \qquad (*) \text{ where } K = 2/\beta .$$

(c) Solve the equation $(*)$ in the case $K = 1$.

(d) Can you solve this equation if $K \neq 1$?

15.5 ANSWERS TO SHORT ANSWER QUESTIONS

15.2.1 : $z = y'$.

15.2.2 : To find $\phi(x)$ such that

$$\phi'(x) = \psi(\phi(x))$$

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 16

COMPLEX N- DIMENSIONAL SPACE

16.0 INTRODUCTION

Let $\mathbb{C}$ be the set of all complex numbers. Let n be a positive integer. Let

$$\mathbb{C}^n = \{(y_1, y_2, \cdots\cdots, y_n)/y_i \in \mathbb{C}(1 \le i \le n)\}$$

i.e. $\mathbb{C}^n$ is the set of all n- tuples of complex numbers. Now, we define the following

(i) Two complex numbers $y=(y_1,\cdots, y_n)$ and $z=(z_1,\cdots, z_n)$ are equal and we write $y=z$ if, and only if $y_i=z_i\,(1\le i\le n)$;

(ii) The binary operation + on $\mathbb{C}$ is defined by

$$y+z = (y_1+z_1, y_2+z_2, \ldots\ldots, y_n+z_n)$$

for any $y=(y_1, y_2,\cdots, y_n), z=(z_1, z_2,\cdots, z_n)$ in $\mathbb{C}$

(iii) Define the scalar multiplication $\cdot : \mathbb{C}\times\mathbb{C}^n \to \mathbb{C}$ by

$$C(y_1, y_2,\cdots, y_n)=(Cy_1, Cy_2,\cdots, Cy_n)$$

for any $C\in\mathbb{C}$ and $(y_1, y_2,\ldots, y_n)\in\mathbb{C}^n$

Now, we have state the following theorem with out proof.

16.0.1 Theorem : $(\mathbb{C}^n, +)$ is a vector space over the field of complex numbers $(\mathbb{C}, +, -)$.

Proof : Exercise

16.0.1.1 Note :

(i) : $O=(0, 0,\cdots\cdots, 0)$ ($n-$tuple) is the identity element in $\mathbb{C}^n$ with respect to '+'.

(ii) : If $y=(y_1, y_2,\cdots, y_n)$ is in $\mathbb{C}^n$, then the inverse of y with respect to + is given by

$$-y = (-y_1, -y_2, \cdots\cdots, -y_n)$$

which is, infact, $(-1)\,y$

(iii) : If $y = (y_1, y_2, \cdots\cdots, y_n)$, $z = (z_1, z_2, \cdots, z_n)$ are in $\mathbb{C}^n$ then,

$$y - z\left(\text{means } y + (-z)\right) = (y_1 - z_1,\ y_2 - z_2, \cdots\cdots y_n - z_n)$$

16.0.2 Definition : If $y = (y_1, y_2, \cdots\cdots, y_n)$ is in $\mathbb{C}^n$ then we define the magnitude or the length of the vector or theorem of y denoted by $\| y \|$ is defined by

$$\| y \| = \sum_{i=1}^{n} |y_i|$$

where $|z|$ (for any $z \in \mathbb{C}$) denotes the absolute value of z.

We assume that we know the properties of real numbers.

We can easily prove the following theorem (and so we leave it as an exercise).

16.0.3 Theorem : $\mathbb{C}^n$ is a normed linear space with respect to the norm given in the definition 16.0.2 in the sense that

(1) $\| y \| \geq 0$ for any y in $\mathbb{C}^n$

(2) $\| y \| = 0$ if, and only if $y = 0$

(3) $\| c\, y \| = |c|\, \| y \|$ for any c in $\mathbb{C}$ and y in $\mathbb{C}^n$

(4) $\| y + z \| \leq \| y \| + \| z \|$ for any x, y, z in $\mathbb{C}^n$.

Proof : Exercise

In view of theorem, we have the following

16.0.4 Theorem : Define $d : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ by $d(x, y) = \| x - y \|$.

Then $\left(\mathbb{C}^n, d\right)$ is a metric space.

Proof : Exercise

16.1 COMPLEX N-DIMENSIONAL SPACE

In section 16.0, we have observed that $\mathbb{C}^n$ is a vector space over the field $\mathbb{C}$ of complex numbers, further $\mathbb{C}^n$ is a normed linear space and hence a metric space. So, we can take of the concepts - convergent sequence, Cauchy sequence etc.

16.1.1 Theorem : Let $\{y_m\}$ be a sequence in $\mathbb{C}^n$ where

$$y_m = (y_{1m}, y_{2m}, \cdots\cdots, y_{mn}) \ (m = 1, 2, \cdots\cdots)$$

Let $y = (y_1, y_2, \cdots\cdots, y_n) \in \mathbb{C}^n$. Then

$$\lim_m y_m = y \text{ if, and only if } \lim_m y_{im} = y_i \text{ for } i = 1, 2, \cdots\cdots, n.$$

Proof : Exercise

16.1.2 Definition : Let ϕ be function defined on an interval I with values in $\mathbb{C}^n$. Then, we can define functions $\phi_1, \phi_2, \cdots\cdots, \phi_n$ on I with values in $\mathbb{C}$ such that

$$\phi(x) = (\phi_1(x), \phi_2(x), \cdots\cdots, \phi_n(x)) \ (x \in I).$$

In this can we write

$$\phi = (\phi_1, \phi_2, \cdots\cdots, \phi_n)$$

and we call ϕ_i as the i^{th} component of $\phi(1 \le i \le n)$.

16.1.3 Definition : Let ϕ be a vector valued function with components $\phi_1, \phi_2, \cdots\cdots, \phi_n$ (i.e. $\phi(x) = (\phi_1(x), \phi_2(x), \cdots\cdots, \phi_n(x)(x \in I))$. We say that

(i) ϕ is continuous on I if each ϕ_i is continuous on I.

(ii) ϕ is differentiable if each ϕ_i is differentiable on I and we write the derivative of ϕ as $\phi' = (\phi_1', \phi_2', \cdots\cdots, \phi_n')$.

16.1.4 Example : Define $\phi : [0, 1] \to \mathbb{C}^2$ by

$$\phi(x) = (x^2, x - ix^3).$$

Then

$$\phi'(x) = \left(2x,\ 1-3ix^2\right)\ (0 \le x \le 1)$$

16.1.5 Definition : If $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ is continuous on interval $c \le x \le d$, then we define the integral of ϕ over $[c, d]$ as the n - vector.

$$\int_c^d \phi(x)\,dx = \left(\int_c^d \phi_1(x)\,dx,\ \int_c^d \phi_2(x)\,dx,\ \cdots\cdots,\ \int_c^d \phi_n(x)\,dx\right)$$

16.1.6 Theorem : Let $\phi : [c, d] \to \mathbb{C}_n^n$ with components $\phi_1, \phi_2, \cdots, \phi_n$. Then

$$\left\|\int_c^d \phi(x)\,dx\right\| \le \int_c^d |\phi(x)|\,dx$$

Proof :

$$\left\|\int_c^d \phi(x)\,dx\right\| = \left|\int_c^d \phi_1(x)\,dx\right| + \cdots\cdots + \left|\int_c^d \phi_n(x)\,dx\right|$$

$$\le \int_c^d |\phi_1(x)\,dx| + \cdots\cdots\cdots + \int_c^d |\phi_n(x)|\,dx$$

$$= \int_c^d \left[|\phi_1(x)| + \cdots\cdots + |\phi_n(x)|\right] dx$$

$$= \int_c^d |\phi(x)|\,dx$$

16.1.7 Example : Let $\phi : [0, 1] \to \mathbb{C}^2$ be defined by

$$\phi(x) = \left(x^2,\ x - i\,x^3\right).$$

9)

So,

$$\int_0^1 \phi(x)\,dx = \left(\int_0^1 x^2\,dx,\ \int_0^1 \left(x - i\,x^3\right) dx\right)$$

$$= \left(\frac{1}{3}, \frac{1}{2} - \frac{i}{4}\right)$$

and hence

$$\left\|\int_0^1 \phi(x)\,dx\right\| = \left|\frac{1}{3}\right| + \left|\frac{1}{2} - \frac{i}{4}\right|$$

$$= \frac{1}{3} + \frac{\sqrt{5}}{2}$$

16.2 SHORT ANSWER QUESTIONS

16.2.1 : Suppose y, z, w are in $\mathbb{C}^3$ given by

$$y = (8+i,\ 3i,\ -2),\ z = (i, -i,\ 2).$$

Compute (a) $y+z$ (b) $y-z$ (c) $\|y+z\|$ (d) $\|y-z\|$

16.2.2 : Let $\phi : [0,4] \rightarrow \mathbb{C}^3$ be defined by

$$\phi(x) = \left(x,\ x^2,\ i\,x^4\right)$$

Compute (a) $\phi(1)$, (b) $\phi'(1)$

16.3 MODEL EXAMINATION QUESTIONS

16.3.1 : Let ϕ be the vector valued function for all real x by

$$\phi(x) = \left(x,\ x^2,\ ix^4\right)$$

Compute the following :

(a) $\phi(1)$

(b) $\phi'(x)$, $\phi'(2)$

(c) $\int_{-1}^{1} \phi(x)\,dx$

(d) Verify that $\left\| \int_{-1}^{1} \phi(x)\,dx \right\| \leqq \int_{-1}^{1} |\phi(x)|\,dx$

16.3.2 : For each i, $1 \le i \le n$, let e_i be the vector with lasits i^{th} component and 0 for its other components. Thus,

$$e_1 = (1, 0, \cdots\cdots, 0),\ e_2 = (0, 1, 0, \cdots\cdots 0), \cdots\cdots$$

$$\cdots\cdots\ e_n = (0, 0, \cdots\cdots 1).$$

(a) If $y = (y_1, y_2, \cdots\cdots y_n)$ prove that

$$y = y_1 e_1 + y_2 e_2 + \cdots\cdots + y_n e_n$$

16.4 EXERCISES

16.4.1 : Prove Theorem 16.0.1

16.4.2 : Prove Theorem 16.0.2

16.4.3 : Prove Theorem 16.0.3

16.4.4 : Prove Theorem 16.0.4

16.4.5 : If ϕ is a continuously differentiable vector valued function defined for real x in an interval $a \le x \le b$, the values of ϕ are in $\mathbb{R}^n$, show that :

(a) ϕ' has values in $\mathbb{R}^n$

(b) $\int_{a}^{x} \phi(t)\,dt$ is in $\mathbb{R}^n$ for each x, $a \le x \le b$

16.4.6 : For each $y = (y_1, y_2, \cdots\cdots, y_n)$ in $\mathbb{C}^n$ let

$$\|y\| = \left(y_1 \bar{y}_1 + y_2 \bar{y}_2 + \cdots\cdots + y_n \bar{y}_n\right)^{\frac{1}{2}}$$

(where $\bar{y}_i$ stands for the conjugate of y_i),

the positive square root being understood. This is called the Euclidean length of y.

(a) : Show that

$$\| y \| \leq |y| \leq \sqrt{n} \, \| y \|$$

(Hint : Show that

$$\|y\| \leq |y^2| \leq n \|y^2\|.$$

Use the inequality $2|a||b| \leq |a|^2 + |b|^2$)

(b) : Show that a sequence $\{y_m\}$ $(m=1,2,\cdots\cdots)$ in $\mathbb{C}^n$ is such that

$$|y_m - y| \to 0 \qquad (\text{as } m \to \infty)$$

if and only it $\|y_m - y_n\| \to 0 \quad (mn \to \infty)$

16.4.7 : For any $y=(y_1, y_2, \cdots\cdots, y_n)$, $z=(z_1, z_2, \cdots\cdots, z_n)$ in $\mathbb{C}^n$ define the inner product $y \cdot z$ to be the number given by

$$y \cdot z = y_1 \bar{z}_1 + y_2 \bar{z}_2 + \cdots\cdots + y_n \bar{z}_n$$

(a) Show that $z \cdot y = \overline{(y.z)}$.

(b) Show that $(y_1 + y_2) \cdot z = (y_1 \cdot z) + (y_2 \cdot z)$

(c) Show that if c is a complex number,

$$(c\, y) \cdot z = c(y \cdot z) = y \cdot (\bar{c}\, z)$$

(d) Show that $\|y\|^2 = y \cdot y$

(e) Prove that

$$|y \cdot z| \leq \|y\| \, \|z\|$$

(This is called the Schwarz inequalitiy. Hint : If $z=0$, the result is obvious. If $z \neq 0$,

let $u = z / \|z\|$. Then $\|u\|=1$, use the fact that

$\|y - (y \cdot u)\|^2 \geqq 0$).

16.4.8 : Show that the Euclidean length satisfies the same rules as magnitude, namely :

(i) $\|y\| \geqq 0$, and $\|y\| = 0$ if, and only if $y = 0$,

(ii) $\|c\,y\| = |c|\|y\|$; for any complex number c,

(iii) $\|y+z\| \leqq \|y\| + \|z\|$

(Hint : Interms of the inner product $\|y\|^2 = y \cdot y$. Use Schwarz inequality of Exercise 16.4.7)

16.5 SHORT ANSWER QUESTIONS

16.2.1 : (a) $y+z=(8+3i,\ 2i,\ 0)$

(b) $y-z = (8,\ 4i,\ -4)$

(c) $\|y+z\| = |8+3i| + |2i| + |0|$

$= \sqrt{73} + 4$

(d) $\|y-z\| = |8| + |4i| + |-4|$

$= 8+4+4 = 16$

16.2.2 : (a) $\phi(1) = (1,\ 1,\ i)$

(b) $\phi'(x) = \left(1,\ 2x,\ 4ix^3\right)$

$\phi'(1) = (1, 2, 4i)$.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

Lesson Writer :

Prof. P. Ranga Rao.

Lesson - 17

SYSTEMS AS VECTOR EQUATIONS

17.0 INTRODUCTION

Consider the following system of first order equations :

$$\left.\begin{array}{l} y_1' = f_1(x, y_1, y_2, \cdots, y_n) \\ y_2' = f_2(x, y_1, y_2, \cdots, y_n) \\ ---------------- \\ ---------------- \\ ---------------- \\ y_n' = f_n(x, y_1, y_2, \cdots, y_n) \end{array}\right\} \text{--------- 17.0 (a)}$$

and assume that $f_1, f_2, \cdots, f_n$ are given complex valued functions defined for $(x, y_1, y_2, \cdots, y_n)$ in some set R, where x is real and $y_1, y_2, \cdots, y_n$ are complex. We can assume that f_1 is a function x and the vector

$$y = (y_1, y_2, \cdots, y_n) \text{ in } \mathbb{C}_n.$$

Therefore we write

$$f(x, y) = f_1(x, y_1, y_2, \cdots, y_n).$$

Similarly, we can write each f_i as

$$f_i(x, y) = f_i(x, y_1, y_2, \cdots, y_n).$$

$$f_i(x, y) = f_i(x, y_1, y_2, \cdots, y_n) \quad (1 \le i \le n)$$

Put $f = (f_1, f_2, \cdots, f_n)$ and define

$$f(x, y) = (f_1(x, y), f_2(x, y), \cdots, f_n(x, y))$$

If we let $y' = (y_1', y_2', \cdots, y_n')$ the the system 17.0 (a) can be written as

$$y' = f(x, y) \text{ -------- } 17.0 \text{ (b)}$$

This (17.0(b)) is a vector differential equation.

17.1 SYSTEMS AS VECTOR EQUATION

17.1.1 Definition : Consider the system of equations 17.0(a), equivalently 17.0(b). A solution of his system is defined as vector valued function

$$\phi = (\phi_1, \phi_2, \cdots\cdots, \phi_n)$$

which is differentiable on a real interval I and such that

(i) $(x, \phi(x)) \in R$ for all x in I.

(ii) $\phi'(x) = f(x, \phi(x))$ for all x in I.

.7.1.2 Example : Consider the system of two equations

$$y_1^{1} = x^2 + y_1^{2} + y_2$$

$$y_2^{1} = y_1 + y_2 - y_1 y_2 .$$

Let $y = (y_1, y_2)$. Then

$$f_1(x, y) = f_1(x, y_1, y_2) = y_1^{1} = x^2 + y_1^{2} + y_2,$$

$$f_2(x, y) = f_2(x, y_1, y_2) = y_2^{1} = y_1 + y_2 - y_1 y_2$$

Hence

$$f(x, y) = (f_1(x, y) \; f_2(x, y))$$

$$= \left(x^2 + y_1^{2} + y_2, \; y_1 + y_2 - y_1 y_2\right)$$

If we write $y' = \left(y_1^{1}, y_2^{1}\right)$ then

$$y' = f(x, y).$$

17.1.3 Example : Consider the system of equations

$$y_1' = y_2,\ y_2' = -y_1.$$

This can be written in the form 17.0 (b) as :

$$y_1' = y_2 = f_1(x, y),$$

$$y_2' = -y_1 = f_2(x, y)$$

Where $f_1(x, y) = f_1(x, y_1, y_2) = y_2,$

$$f_2(x, y) = f_2(x, y_1, y_2) = -y_1,$$

and hence

$$y' = f(x, y)$$

i.e. $(y_1', y_2') = (f_1(x, y), f_2(x, y))$

$$= (y_2, -y_1).$$

Take $\phi(x) = (\sin x, \cos x) \quad (x \in (-\infty, \infty)).$

Clearly, ϕ is a solution of the system $y' = f(x, y)$.

17.1.4 Definition : A vector valued function f defined on some set $S \subseteq \mathbb{R} \times \mathbb{C}_n$ is said to be continuous on S if each of its components is continuous on S. We say that f satisfies a Lipschitz condition on S if there exists a constant $K > 0$ such that

$$\|f(x, y) - f(x, z)\| \leq K\|y - z\|$$

for all $(x, y), (x, z)$ in S. The constant K is called Lipschitz constant for f on S.

17.1.5 Example : Consider the function f defined by

$$f(x, y) = (3x + 2y_1, y_1 - y_2)$$

on $S: |x| < \infty,\ |y| < \infty$. For any $(x, y), (x, z)$ in S,

$$\|f(x,y)-f(x,z)\| = \|(3x+2y_1,\ y_1-y_2)-(3x+2z_1,\ z_1-z_2)\|$$

$$= \|(2(y_1-z_1),\ (y_1-z_1)-(y_2-z_2))\|$$

$$\le 2|y_1-z_1| + |y_1-z_1| + |y_2-z_2|$$

$$\le 3\,|y_1-z_1| + 3|y_2-z_2|$$

$$= 3\|y-z\|.$$

Here the Lipschitz constant $K=3$.

17.1.6 Theorem : Suppose f is a vector valued function defined for $(x,\ y)$ in a set

$$S : |x-x_0| \le a,\ \|y-y_0\| \le b \qquad (a>0,\ b>0),$$

or $\quad S : |x-x_0| \le a,\ \|y\| < \infty\ (a>0)$

If, for each $i=1,2,\cdots\cdots,n$, $\dfrac{\partial f}{\partial x_i}$ exists, is continuous on S and there is a constant $K>0$ such that

$$\left\|\frac{\partial f}{\partial x_i}(x,\ y)\right\| \le K \quad \text{----------- 17.1.6 (a)}$$

for $i=1,2,\cdots\cdots,n$ and for all (x,y) in S, then f satisfies a Lipschitz condition with Lipschitz constant K on S.

Proof : The proof of this is a direct consequence of that of Theorem 10.1 : in Lesson 10. Let $(x,\ y)$ be any points in S. For any real number $s,\ 0 \le s \le 1$, define a vector valued function F by

$$F(s) = f(x,\ z+s(y-z)),\quad (0 \le s \le 1) \quad \text{--------- 17.1.6(b)}$$

Since $(x,\ y),\ (x,\ z)$ are in S, we have

$$|x-x_0| \le a,\ \|y-y_0\| \le b,\ \|z-z_0\| \le b$$

and $\quad \|z+s(y-z)\| \le (1-s)\ \|z\| + s\|y\|$

$$\leq \|z\| + \|y\|$$

$$< \infty .$$

Thus, all the points $(x,\, z+s(y-z)),\ 0 \leq s \leq 1$ are in S and that the function F given by 17.1.6(b) is well defined. Using the fact, if $G(s) = f(x,\, g(s))$, that

$$G'(s) = \frac{\partial f}{\partial g}(x,\, g(s))\,\frac{dg}{ds}(s).$$

So, $$F'(s) = (y_1 - z_1)\frac{\partial f}{\partial y_1}(x,\, z+s(y-z)) + \cdots\cdots\cdots$$

$$\cdots\cdots\cdots + (y_n - z_n)\frac{\partial f}{\partial y_n}(x,\, z_n + s(y_n - z_n)),$$

where $y=(y_1,\, y_2, \cdots\cdots y_n),\ z=(z_1,\, z_2, \cdots\cdots, z_n)$. Using 17.1.6(a),

$$\|F'(s)\| \leq K\,\|y-z\|,\quad (0 \leq s \leq 1) \text{ ------------ } 17.1.6(c).$$

Note that $F(1) = f(x,\, y),\ F(0) = f(x, z)$ and that

$$f(x,\, y) - f(x,\, z) = F(1) - F(0) = \int_0^1 F'(s)\,ds.$$

So, $$\|f(x, y) - f(x,\, z)\| \leq K\,\|y-z\|.$$

This completes Theorem.

17.1.7 Example : Consider the system of two equations

$$\left.\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned}\right\} \text{ ---------- } 17.1.7(a)$$

where a, b, c, d are constants.

(i) If this system is written as $y' = f(x,\, y)$, what is f ?

(ii) Show that f of (i) satisfies a Lipschitz condition for all (x, y) where x is real and y is in $\mathbb{C}_2$.

(iii) Show that f of (i) is linear in y.

Solution : (i) System 17.1.7 (a) can be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(ii) For any real x and y, z in $\mathbb{C}_2$, we have

$$\|f(x, y) - f(x, z)\|$$

$$= \|(ay_1 + by_2, cy_1 + dy_2) - (az_1 + bz_2, \; cz_1 + dz_2)\|$$

$$= \|a(y_1 - z_1) + b(y_2 - z_2), c(y_1 - z_1) + d(y_2 - z_2)\|$$

$$\leq |a(y_1 - z_1) + b(y_2 - z_2)| + |c(y_1 - z_1) + d(y_2 - z_2)|$$

$$\leq |a||y_1 - z_1| + |b||y_2 - z_2| + |c| \, |(y_1 - z_1)| + |d| \, |y_2 - z_2|$$

$$= (|a| + |c|) \, |y_1 - z_1| + (|b| + |d|) \, |y_2 - z_2|$$

$$\leq K \, \|y - z\|$$

where $K = \max\{|a| + |c|, \; |b| + |d|\}$.

(iii) For all real x and y, z in $\mathbb{C}_2$ and complex numbers α, β we have

$$f(x, \alpha y + \beta z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (\alpha y + \beta z)$$

$$= \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} y + \beta \begin{bmatrix} a & b \\ c & d \end{bmatrix} z$$

$$= \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \beta \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \alpha\, f(x,y) + \beta\, f(x,z)$$

Hence, f is linear in y.

17.1.8 Example : Find a solution ϕ of the system

$$y_1' = y_1,$$

$$y_2' = y_1 + y_2.$$

satisfying $\phi(0) = (1, 2)$.

Solution : Soving $y_1' = y_1$, we have

$$y_1(x) = c\, e^x$$

where c is constant.

Now, the second equation becomes

$$y_2' = c e^x + y_2$$

i.e. $y_2' - y_2 = c\, e^x$

I.F (Integrating Factor) = e^{-x}. So,

$$\frac{d}{dx}\left(y_2 \cdot e^{-x}\right) = c \cdot e^{-x} \cdot e^x = c\,.$$

Hence, $y_2\, e^{-x} = cx + d$, where c, d are constants. So, $y_2 = (cx+d)e^{-x}$.

Now, $\phi(0) = (1,2)$ implies $c=1,\ d=2$. Thus,

$\phi(x) = \left(e^x, (x+2)e^x\right)$ is the required solution.

17.2 SHORT ANSWER QUESTIONS

17.2.1 : Consider the system of equations :

$$y_1' = x^2 + y_1^2 + y_2$$

$$y_2' = y_1 + y_2 - y_1 y_2 .$$

Put these equations in the form $y' = f(x, y)$.

17.2.2 : Define solution of the system $y' = f(x, y)$

17.3 MODEL EXAMINATION QUESTIONS

17.3.1 : Find a solution ϕ of the system

$$y_1' = y_1,$$

$$y_2' = y_1 + y_2$$

satisfying $\phi(0) = (1, 2)$.

17.3.2 : Find a solution ϕ of the system of equations

$$y_1' = y_2,$$

$$y_2' = 6y_1 + y_2,$$

satisfying $\phi(0) = (1, -1)$

17.3.3 : Find a solution ϕ of the system

$$y_1' = y_1 + y_2,$$

$$y_2' = y_1 + y_2 + e^{3x}$$

satisfying $\phi(0) = (0, 0)$. (Hint : Let $z = y_1 + y_2$)

17. 4 EXERCISES

17.4.1 : Let f be the vector - valued function defined on

$R: |x| \le 1,\ |y| \le 1 \qquad (y \text{ in } \mathbb{C}_2)$

by $f(x, y) = \left(y_2^2 + 1,\ x + y_1^2\right)$.

(a) Find an upper bound M for $|f(x,y)|$ for (x,y) in R

(b) Compute a Lipschitz constant K for f on R.

17.4.2 : Let f be a vector valued function defined for (x, y) in a set S, with x real, y in $\mathbb{C}_n$.

(a) Show that f is continuous at a point (x_0, y_0) in S if, and only if

$$|f(x,y) - f(x_0, y_0)| \to 0$$

as $\quad 0 < |x - x_0| + |y - y_0| \to 0$.

(b) Show that f sa fies a Lipschitz condition on in S if, and only if each component of f satisfies Lipchitz condition in S.

17.5 ANSWERS TO SHORT ANSWER QUESTIONS

17.2.1 : See example 17.1.2. Clearly, $f(x,y) = (f_1(x,y),\ f_2(x, y))$

$$= \left(x^2 + y_1^2 + y_2,\ y_1 + y_2 - y_1 y_2\right).$$

17.2.2 : See Definition 17.1.1

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Prof. P. Ranga Rao.

Lesson - 18

EXISTENCE AND UNIQUENESS OF *SOLUTIONS OF SYSTEMS*

18.0 INTRODUCTION

Let f be a continuous vector valued function defined on

$$R: |x - x_0| \leqq a,\ |y - y_0| \leqq b \qquad (a,\ b > 0)$$

We know that an initial value problem

$$y' = f(x,\ y),\ y(x_0) = y_0 \text{ ----------- } 18.0\ (a)$$

is the problem of finding a solution ϕ of $y' = f(x, y)$ on an interval I containing x_0 such that $\phi(x_0) = y_0$. If

$$y_0 = (\alpha_1,\ \alpha_2, \cdots\cdots, \alpha_n),$$

the problem 18.0(a) written out becomes

$$y_1' = f_1(x, y_1,\ y_2, \cdots\cdots, y_n),$$

$$y_n' = f_n(x, y_1,\ y_2, \cdots\cdots, y_n),$$

$$y_1(x_0) = \alpha_1,\ y_2(x_0) = \alpha_2, \cdots\cdots, y_n(x_0) = \alpha_n.$$

If f is continuous on R, the problem 18.0(a) always has a solution on some interval containing x_0. If, in addition, f satisfies a Lipchitz conition on R, this fact may be demonstrated exactly as in Lesson - 9 by introducing the successive approximations $\phi_0,\ \phi_1, \cdots\cdots$, where

$$\phi_0(x) = y_0$$

$$\phi_{i+1}(x) = y_0 + \int_{x_0}^{x} f(t,\ \phi_i(t))dt \qquad (i = 0, 1, 2, \cdots\cdots)$$

consider the following.

18.0.1 Example : Consider the problem

$$y_1' = y_2,$$

$$y_2' = -y_1,$$

$$y(0) = (0, 1)$$

Here, $f(x, y) = (f_1(x, y), f_2(x, y)) = (y_1', y_2')$

$$= (y_2 - y_1).$$

$$\phi_0(x) = (0, 1)(= y_0)$$

$$\phi_1(x) = y_0 + \int_0^x f(t, \phi_0(t))dt$$

$$= (0, 1) + \int_0^x (1, 0)dt$$

$$= (0, 1) + \int_0^x (1, 0)dt$$

$$= (0, 1) + (x, 0) = (x, 1),$$

$$\phi_2(x) = y_0 + \int_0^x f(t, \phi(t))dt$$

$$= (0, 1) + \int_0^x (1, -t)dt$$

$$= (0, 1) + \left(x, -\frac{x^2}{2}\right)$$

$$= \left(x, 1 - \frac{x^2}{2}\right),$$

$$\phi_3(x) = (0, 1) + \int_0^x f(t, \phi_2(t))\,dt$$

$$= (0, 1) + \int_0^x \left(1 - \frac{t^2}{2}, -t\right) dt$$

$$= \left(x - \frac{x^3}{3!}, 1 - \frac{x^2}{2}\right)$$

Continuing this process, we can observe that $\phi_i(x)$ exist for all real x and

$$\phi_i(x) \to \phi(x) = (\sin x, \cos x)$$

where ϕ is the solution of the problem.

18.1 EXISTENCE AND UNIQUENESS OF SOLUTIONS OF SYSTEMS

18.1.1 Theorem (Local existence) : Let f be a continuous vector valued function defined on

$$R: |x - x_0| \leqq a, \; |y - y_0| \leqq b \qquad (\Delta, b > 0),$$

and suppose f satisfies Lipchitz condition on R. It M is constant such that

$$|f(x, y)| \leqq M$$

for all (x, y) in R. Then the successive approximations $\{\phi_i\}$, $(i = 0, 1, 2, \cdots)$ given by 18.0 (b) converge on the interval

$$I: |x - x_0| \leq \alpha = \text{Minimum} \left\{a, \frac{b}{M}\right\},$$

to a solution ϕ of the initial value problem

$$y' = f(x, y),\; y(x_0) = y_0, \text{ on } I.$$

Proof : The proof is the same on that of theorem 11.1 of lesson 11 with y, f, ϕ replaced by the vector y, f, ϕ.

18.1.2 Theorem : If f satisfies the same conditions as in Theorem 18.1.1, and K is a Lipschitz constant for f in R, then

$$|\phi(x)-\phi_i(x)| \leqq \frac{M}{K}\cdot\frac{(K\alpha)^{i+1}}{(i+1)!}\,e^{K\alpha} \text{ for all } x \text{ in } I.$$

Proof : This is an analogue of Theorem 11.1.1 of Lesson 11 and the proof is the same.

18.1.3 Theorem (Non-local existence) : Let f be a continuous vector valued function defined on

$$S:|x-x_0|\leqq a,\ |y|<\infty \quad (a>0),$$

and satisfy there a Lipschitz condition. Then the successive approximations $\{\phi_i\}$ for the problem

$$y' = f(x,y),\ y(x_0)=y_0 \qquad (|y_0|<\infty)$$

exist on the interval $|x-x_0| \leqq a$, and converge there to a solution ϕ of this problem.

18.1.3.1 Corollary : Suppose f is continuous vector valued function defined on

$$|x|<\infty,\ |y|<\infty,$$

where a is any positive number. Then every initial value problem

$$y' = f(x,y),\ y(x_0)=y_0,$$

has a solution which exists for all real x.

18.1.4 Theorem (Approximation and Uniqueness) : Let f and g be two continuous vector valued functions defined on

$$R:|x-x_0|\leqq a,\ |y-y_0|\leqq b \qquad (a,\ b>0)$$

and suppose f satisfies a Lipschitz condition on R with Lipchitz constant K. Suppose $\phi,\ \psi$ are solutions of the problems.

$$y' = f(x,\ y_j,\ \ y(x_0)=y_1$$

$$y' = g(x,\ y),\ \ y(x_0)=y_2$$

respectively, on some interval I containing x_0. If for $\in,\ \delta \geqq 0$,

$$|f(x,y)-g(x,y)| \leqq \in$$

for all (x, y) in R, and

$$|y_1-y_2| \leqq \delta,$$

then $|\phi(x)-\psi(x)| \leqq \delta\, e^{K|x-x_0|}+\frac{\in}{K}\left[e^{K|x-x_0|}-1\right]$ for all x in I. In particular, the problem

$$y' = f(x,y),\ y(x_0)=y_0$$

has atmost one solution on any interval containing x_0.

Proof : The proof of this theorem is analogue of Theorem of lesson 12.

18.1.5 Example : Consider the system of equations

$$y_1' = 3y_1+xy_3$$

$$y_2' = y_2+x^3y_3$$

$$y_3' = 2x\,y_1-y_2+e^x\,y_3 .$$

Show that every initial value problem for this system has a unique solution which exists for all real x.

Solution : Here

$$y' = f(x,y)$$

i.e. $(y_1', y_2', y_3') = f(x,y) =$

$$\left(3y_1+xy_3,\ y_2+x^3y_3, 2xy_1-y_2+e^x\,y_3\right).$$

Let $b>0$. Fix a real x_0. Clearly, f is continuous on

$$S_b -: |x-x_0| \leqq b,\ |y_i| < \infty \qquad (i=1,2,3,.....).$$

For any $(x,y),(x,z)$ in S_a,

$$\|f(x, y) - f(x, z)\|$$

$$=\left\|\left(3(y_1-z_1)+x(y_3-z_3),\ (y_2-z_2)+x^3(y_3-z_3),\ 2x(y_1-z_1)-(y_2-z_2)+e^x(y_3-z_3)\right)\right\|$$

$$=\left|3(y_1-z_1)+x(y_3-z_3)\right|+\left|(y_2-z_2)+x^3(y_3-z_3)\right|+\left|2x(y_1-z_1)-(y_2-z_2)+e^x(y_3-z_3)\right|$$

$$\leq 3|y_1-z_1|+a|y_3-z_3|+|y_2-z_2|+a^3|y_3-z_3|+2a|y_1-z_1|+|y_2-z_2|+e^a|y_3-z_3|$$

(Here $a=|b|+|x_0|$)

$$\leq\left(3+2a+a^3+e^a\right)\left[|y_1-z_1|+|y_2-z_2|+|y_3-z_3|\right]$$

$$=\left(3+2a+a^3+e^a\right)\|y-z\|.$$

So, f is a Lipschitzian with Lipschitz constant

$$K_a=3+2a+a^3+e^a$$

on S_a for any $a>0$.

Hence by the existence and uniqueness Theorem, every initial value problem

$$y'=f(x,y),\ y(x_0)=y_0$$

has a unique solution which exists for all real x.

18.2 SHORT ANSWER QUESTIONS

18.2.1 : Assume the hypothesis of Theorem (Local existence) 18.1.1. Write down the interval I on which the successive approximations converge to a soluton of I.V.P.

$$y'=f(x,y),\ y(x,0)=y_0$$

18.2.2 : Consider the I.V.P.

$$y_1'=y_2,\ y_2'=-y_1,\ y_1(0)=0,\ y_2(0)=1$$

Express this problem in the form $y'=f(x,y),\ y(x_0)=y_0$

18.2.3 : State local existence thoerem for the initial value problem : $y'=f(x,y),\ y(x_0)=y_0$.

18.3 MODEL EXAMINATION QUESTIONS

18.3.1 : Consider the initial value problem

$$y_1' = y_2^2 + 1,$$

$$y_2' = y_1^2,$$

$$y_1(0)=0,\ y_2(0)=0$$

(a) If this problem is denoted by

$$y' = f(x, y),\ y(0) = y_0,$$

what are f and y_0 ?

(b) Show that there exists

(i) a positive constant M such that

$$|f(x, y)| \leqq M$$

and

(ii) a Lipschitz constant K

on $R: |x| \leqq 1, |y| \leqq 1$.

(c) Compute the first three successive approximations ϕ_0, ϕ_1, ϕ_2.

18.4 EXERCISES

18.4.1 : Prove Theorem 18.1.1.

18.4.2 : Prove Theorem 18.1.2

18.4.3 : Prove Theorem 18.1.3

18.4.4 : Prove Corollary 18.1.3.1

18.4.5 : Prove Theorem 18.1.4

18.4.6 : Consider the initial value problem

$$y_1' = y_2^2 + 1,$$

$y_2' = y_1^2$,

$y_1(0) = 0,\ y_2 = 0$.

(a) If this problem is denoted by

$$y' = f(x, y),\ y(0) = y_0$$

what are f and y_0?

(b) Show that the f of (a) satisfies the conditions of Theorem 18.1.1 on

$$R : |x| \leq 1,\ |y| \leq 1.$$

Compute a bound M, a Lipschitz constant K and an α.

(c) Compute the first three successive approximations ϕ_0, ϕ_1, ϕ_2.

18.4.7 : Consider the system

$y_1' = y_1 + y_2$

$y_2' = \in y_1 + y_2$,

where $\in$ is a positive constant.

(a) Show that every solution exists for all real x.

(b) Let ϕ be the solution satisfying $\phi(0) = (1, -1)$, and let ψ be a solution of

$$y_1' = y_1,\ y_2' = y_2$$

satisfying $\psi(0) = (1, -1)$. Without solving the original system show that

$$|\phi(x) - \psi(x)| \to 0 \qquad (\text{as } \in \to 0) \text{ for each real } x.$$

(c) Find all solutions of the original system.

(Hint : If ϕ is a solution show that $\psi(x) = e^{-x}\phi(x)$ satisfies

$$z_1' = \in z_2,\ z_2' = \in z_1).$$

(d) Find the solutions ϕ and ψ of (b), and verify the conclusions in (b).

18.4.8 : Show that all solutions with values in $\mathbb{R}_2$ of the following system exist for all real x :

$$y_1' = a(x)\cos y_1 + b(x)\sin y_2,$$

$$y_2' = c(x)\sin y_1 + d(x)\cos y_2,$$

where a, b, c, d are polynomials with real coefficients.

18.5 ANSWERS TO SHORT ANSWER QUESTIONS

18.2.1 : $I: |x-x_0| \leq \alpha = \min\left\{a, \frac{b}{M}\right\}$

18.2.2 : Let $y=(y_1, y_2)$

$$f(x, y) = y' = (y_1', y_2') = (y_2, -y_1).$$

$$y(0) = (y_1(0), y_2(0)) = (0, 1).$$

18.2.3 : See statement of Theorem 18.1.1.

REFERENCE BOOK

Earl. A. Coddington : An Introduction to Ordinary Differential Equations; Prentice Hall, India.

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Lesson - 19

THE ADJOINT EQUATION

19.0 INTRODUCTION

In this lesson, we define the adjoints of first and second order linear differential equations; and we study the self-adjoints of these equations. Particularly, it is shown that, every first and second order linear differential equations can be put in their respective self-adjoint floms, under certain conditions on the coefficients of the equations considered.

19.1 FINDING ADJOINT AND SELF-ADJOINT OF FIRST ORDER LINEAR DIFFERENTIAL EQUATION

We consider a first order linear differential equation

$$a_0(x)y' + a_1(x)y = f(x) \qquad 19.1(1)$$

where a_0, a_0', a_1 and f are continuous functions on an interval I and $a_0(x) \neq 0$ for all x in I.

Let us first find integrating factor of 19.1(1).

If z is an integrating factor of 19.1(1) on I, then after multiplying 19.1(1) by z, it becomes exact, so that

$$z(x)\left[a_0(x)y' + a_1(x)y\right] = \left[r(x)y\right]' \text{ for some } r(x).$$

On equating the coefficients of y' and y on bothsides, we get

$$z(x)\, a_0(x) = r(x) \qquad 19.1(2)$$

$$z(x)\, a_1(x) = r'(x) \qquad 19.1(3)$$

These equations lead at once to the differential equation

$$\left(z(x)\, a_0(x)\right)' - z(x)a_1(x) = 0 \qquad 19.1(4)$$

since it is a first order linear homogeneous differential equation, by solving 19.1(4), we obtain the inegrating factor z of 19.1(1).

From 19.1(2),

$$\left(z(x)\,a_0(x)\right)' = r'(x) = z(x)\,a_1(x).$$

Hence

$$z'(x)\,a_0(x) + z(x)a_0'(x) = z(x)a_1(x).$$

This implies

$$z'(x) + \frac{1}{a_0(x)}\left(a_0'(x) - a_1(x)\right) z(x) = 0 \qquad 19.1(5)$$

A solution z of (19.1(5)) is given by

$$z(x) = e^{-\int\left(\frac{a_0'(x) - a_1(x)}{a_0(x)}\right)dx}$$

$$= \frac{1}{a_0(x)}\, e^{-\int \frac{a_1(x)}{a_0(x)}dx}$$

Hence the function r is known from (19.1(2)), and it is given by

$$r(x) = e^{-\int \frac{a_1(x)}{a_0(x)}dx}$$

19.1.1 Definition : We write

$$L_1(y) = a_0(x)\,y' + a_1(x)\,y$$

and $$M_1(z) = -\left(a_0\, z\right)' + (a,\, z).$$

$M_1(z)$ is said to be the adjoint of $L_1(y)$.

The differential equation $L_1(y)=0$ is said to be self-adjoint if $L(y) = -\,M_1(y)$

19.1.2 Theorem : The differential equation $L_1(y)=0$ is self-adjoint if and only if $2a_1(x) = a_0'(x)$.

Proof : $L_1(y)=0$ is self-adjoint $\Leftrightarrow L_1(y) = -M_1(y)$

$$\Leftrightarrow a_0(x)\,y' + a_1(x)\,y = \left(a_0(x)\,y\right)' - a_1(x)\,y$$

$$= a_0'(x)y + a_0(x)y' - a_1(x)y$$

$$\Leftrightarrow 2a_1(x) = a_0'(x).$$

19.1.3 Remark : By theorem 19.1.2, the differential equation $L_1(y)=0$ is self-adjoint if and only if $L_1(y)=0$ can be written in the form

$$a_0(x)y' + \frac{1}{2}a_0'(x)y = 0$$

19.1.4 Theorem : The differential equation $L_1(y)=0$ can always be put in the self-adjoint form by writing it as

$$g(x)\,y' + \frac{a_1(x)g(x)}{a_0(x)}y = 0$$

where $g(x) = e^{2\int \frac{a_1(x)}{a_0(x)}dx}$

Proof : We have the differential equation

$$L_1(y) \equiv a_0(x)y'(x) + a_1(x)y = 0.$$

Dividing throughout by $a_0(x)$ and multiplying by $g(x)$, we get

$$g(x)y' + \frac{a_1(x)\,g(x)}{a_0(x)}y = 0$$

By theorem 19.1.2, this equation is self-adjoint if and only if

$$2\frac{a_1(x)\,g(x)}{a_0(x)} = g'(x).$$

That is $\dfrac{g'(x)}{g(x)} = 2\dfrac{a_1(x)}{a_0(x)}$.

This implies $g(x) = e^{2\int \frac{a_1(x)}{a_0(x)}dx}$ Hence the theorem follows.

19.2 FINDING ADJOINT AND SELF-ADJOINT OF SECOND ORDER LINEAR DIFFERENTIAL EQUATION

We consider a second order linear differential equation

$$L_2(y) \equiv a_0(x)\, y'' + a_1(x) y' + a_2(x) y = 0 \qquad 19.2(1)$$

where $a_0, a_0', a_0'', a_1, a_1', a_2$ are continuous functions on an interval I, and $a_0(x) \neq 0$ for all $x \in I$.

Let us find integrating factor of (19.2(1)).

If z is an integrating factor of (19.2(1)) on I, then

$$z(x)\left[a_0(x) y'' + a_1(x) y' + a_2(x) y\right] = \left[k(x) y' + m(x) y\right]'$$

for some functions k and m.

On equating the coefficients of y'', y' and y, we get

$$a_0(x) z(x) = k(x) \qquad 19.2(2)$$

$$a_1(x) z(x) = k'(x) + m(x) \qquad 19.2(3)$$

and $\quad a_2(x) z(x) = m'(x).$

Using these equations, we have

$$\left(a_1(x)\, z(x)\right)' = k''(x) + m'(x)$$

$$= \left(a_0(x)\, z(x)\right)'' + a_2(x)\, z(x)$$

So that $\quad M_2(z) \equiv \left(a_0(x)\, z(x)\right)'' - \left(a_1(x)\, z(x)\right)' + a_2(x) z(x) = 0 \qquad 19.2(5)$

By solving (19.2(5)), we obtain the integrating factor z of $L(y)=0$. Once z is known, it is easy to evaluate k and m from (19.2(2)) and (19.2(4)) respectively.

19.2.1 Definition : The expression $M_2(z)$ in (19.2(5)) is called the adjoint of $L_2(y)$.

When $L_2(y) \equiv M_2(y)$, the equation

$L_2(y) \equiv a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ is said to be self-adjoint.

19.2.2 Theorem : The differential equation $L_2(y) = 0$ is self-adjoint if and only if $a_1(x) = a_0'(x)$.

Proof : $L_2(y) = 0$ is self-adjoint $\Leftrightarrow L_2(y) = M_2(y)$

$$\Leftrightarrow a_0 y'' + a_1 y' + a_2 y = (a_0 y)'' - (a_1 y)' + a_2 y$$

$$= a_0'' y + 2a_0' y' + a_0 y'' - a_1' y - a_1 y' + a_2 y$$

$\Leftrightarrow a_1 = a_0'$ and $a_0'' = a_1'$ $\Leftrightarrow$ $a_1 = a_0'$ on I.

This completes the proof of the theorem.

19.2.3 Theorem : If the differential equation $L_2(y) = 0$ is self-adjoint then it is of the form

$$(r(x)y')' + p(x)y = 0 \qquad 19.2.3(1)$$

where $r(x) = a_0(x)$ and $p(x) = a_2(x)$.

Proof : $L_2(y) = 0$ is self-adjoint $\Leftrightarrow a_1(x) = a_0'(x)$ for all $x \in I$ (by theorem 19.2.2)

Hence $L_2(y) = 0$ becomes

$a_0(x)y'' + a_0'(x)\,y = 0$. That is $(a_0(x)y')' + a_2(x)y = 0$.

That is

$$(r(x)y')' + p(x)y = 0$$

where $r(x) = a_0(x)$ and $p(x) = a_2(x)$.

19.2.4 Theorem : Let r, r' and p are continuous functions on an interval I and $r(x) > 0$ for all $x \in I$, then the equation

$$(r(x)\,y')' + p(x)y = 0$$

Can be written in the form

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where $a(x)=r(x)$, $b(x)=r'(x)$ and $c(x)=p(x)$ for all $x \in I$.

Conversely, every equation

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad 19.2.4(1)$$

where a, b, c are continuous functions on an interval I and $a(x)>0$ for all $x \in I$, can be written in the form

$$(r(x)y')' + p(x)y = 0,$$ that is in self-adjoint form.

Proof : First part is clear.

For the second part, dividing through out by $a(x)$ and multiplying by $g(x)$, we get

$$g(x)y'' + \frac{b(x)\,g(x)}{a(x)}y' + \frac{c(x)\,g(x)}{a(x)}y = 0. \qquad 19.2.4(2)$$

If this equation is in the self-adjoint form then by theorem 19.2.2, we have

$$\frac{b(x)g(x)}{a(x)} = g'(x).$$

This implies

$$\frac{g'(x)}{g(x)} = \frac{b(x)}{a(x)}.$$

Hence $g(x) = e^{\int \frac{b(x)}{a(x)}dx}$ 19.2.4(3)

Now, (19.2.4(2)) becomes

$$(g(x)y')' + \frac{c(x)\,g(x)}{a(x)}y = 0.$$

It is of the form

$$(r(x)\,y')' + p(x)y = 0$$

where $r(x) = g(x) = e^{\int \frac{b(x)}{a(x)} dx}$

and $p(x) = \dfrac{c(x)\, g(x)}{a(x)} = \dfrac{c(x)}{a(x)} e^{\int \frac{b(x)}{a(x)} dx}$

19.2.4.1 Note : Theorem 19.2.4 is not true for general linear differential equations of order higher than the second.

19.2.4.2 Remark : When $a_1(x) = 0$ for all x in I, then the equation (19.2(1)) may be put in self-adjoint form simply by dividing through by $a_0(x)$.

19.3 EXAMPLES

19.3.1 Example : Put the differential equation

$$x^2 y'' + x y' + y \quad), \; x>0 \qquad \text{19.3.1(1)}$$

in self adjoint form.

Solution : Comparing (19.3.1(1)) with (19.2.4(1)), we have $a(x) = x^2$, $b(x) = x$ and $c(x) = 1$.

Dividing through by x^2, we get

$$y'' + \frac{1}{x} y' + \frac{y}{x^2} = 0.$$

Multiplying throughout by

$$g(x) = e^{\int \frac{b(x)}{a(x)} dx} = e^{\int \frac{dx}{x}} = x \qquad \text{(using (19.2.4(3))}$$

We get

$$x\, y'' + y' + \frac{1}{x} y = 0.$$

That is $(x y')' + \dfrac{1}{x} y = 0$ and this equation is in self-adjoint form.

19.3.2 Example : Find functions $z(x), k(x)$ and $m(x)$ such that

$$z(x)\left[x^2 y'' - 2xy' + 2y\right] \equiv \frac{d}{dx}\left[k(x)y' + m(x)\, y\right], \qquad 19.3.2(1)$$

and hence solve

$$x^2\, y'' - 2x\, y' + 2y = 0, \;\; x > 0 \cdot$$

Solution : Based on the theory discussed in 19.2, by comparing (19.3.2(1)) and 19.2(1), we get

$$a_0(x) = x^2,\; a_1(x) = -2x \text{ and } a_2(x) = 2\,.$$

we find $z(x)$ satisfying $M(z) = 0$.

That is, $(a_0\; z)'' - (a, z)' + a_2 z = 0$

That is, $\left(x^2\; z\right)'' + (2xz)' + 2z = 0$

Hence $x^2\, z'' + 6x\, z' + 6z = 0$ 19.3.2(2)

This is Euler's 'equidimensional equation'.

[The general form of 'Euler's equidimensional equation' is $x^2\, y'' + p\, x\, y' + gy = 0$ where p, g are constants. To solve such equations, we use the change of independent variable given by $x = e^t$ transforms it into an equation with constant coefficients, and we know the method of solving it.]

Let us solve 19.3.2 (2).

Put $x = e^t$. Then $t = \ln x$.

$$\frac{dz}{dx} = \frac{dz}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \cdot \frac{dz}{dt}$$

$$\frac{d^2 z}{dx^2} = \frac{1}{x}\left(\frac{d^2 z}{dt^2} \cdot \frac{dt}{dx}\right) - \frac{1}{x^2} \cdot \frac{dz}{dt}$$

$$= \frac{1}{x^2}\left(\frac{d^2 z}{dt^2} - \frac{dz}{dt}\right)$$

Substituting these expressions in (19.3.2(2)), we get

$$\frac{d^2 z}{dt^2} + 5 \cdot \frac{dz}{dt} + 6z = 0, \qquad 19.3.2(3)$$

a second order linear equation with constant coefficients. On solving it, we get e^{-2t}, e^{-3t} are solutions of (19.3.2(3)). Now, by substituting $t - \ln x$, we have

$$z_1(x) = e^{-2\ln x} \text{ and } z_2(x) = e^{-3\ln x}$$

i.e., $z_1(x) = \frac{1}{x^2}$ and $z_2(x) = \frac{1}{x^3}$; and are linearly independent.

Let $z(x) = \frac{1}{x^2}$. Then using the expressions 19.2(2) to (19.2(4)), we get $k(x) = 1$ and $m(x) = -\frac{2}{x}$. This information would enable us to solve the diffrential equation.

$$x^2 y'' - 2xy' + 2y = 0,$$

An equaivalent form of this equation with these k and m is given by

$$\frac{d}{dx}\left[y' - \frac{2}{x} y\right] = 0$$

(we obtain this equation by substituting the values k and m in the given equation).

This implies

$$y' - \frac{2}{x} y = -c_1 \qquad (c_1 \text{ is a constant})$$

An integrating factor of this first order linear differential equation is

$$e^{-\int \frac{2}{x} dx} = e^{-2\ln x} = \frac{1}{x^2}.$$

Thus $\frac{1}{x^2} y' - \frac{2}{x^3} y = -c_1 \frac{1}{x^2}$

That is $\left(\frac{1}{x^2} y\right)' = -c_1 \cdot \frac{1}{x^2}$.

Hence $\frac{y}{x^2} = \frac{c_1}{x} + c_2$, c_2 is a constant.

This implies $y(x) = c_1 x + c_2 x^2$.

19.4 SHORT ANSWER QUESTIONS

19.4.1 : Show that the differential equation

$$a_0(x) y' + a_1(x) y = 0$$

where a_0 and a_1 are functions on an interval I such that a_0, a_0' and a_1 are functions on an interval I, is self-adjoint if and only if $2a_1(x) = a_0'(x)$ for $x \in I$.

19.4.2 : Which of the following differential equations are self-adjoint ?

(i) $2xy' + y = 0$ (ii) $xy' = y$ (iii) $2(\sin x) y' + (\cos x) y = 0$

19.4.3 : Show that the differential equation

$$a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$$

where $a_0, a_0', a_0'', a_1, a_1'$ and a_2 are continuous functions of I with $a_0(x) \neq 0$ for all x in I, is self-adjoint if and only if $a_1(x) = a_0'(x)$ for all x in I.

19.4.4 : Put the differential equation

$x^2 y'' + x y' + y = 0, \quad x > 0$ in self-adjoint form.

19.5 MODEL EXAMINATION QUESTIONS :

19.5.1 : Show that the differential equation

$a_0(x) y' + a_1(x) y = 0$ is self-adjoint if and only if $2a_1(x) = a_0'(x)$.

19.5.2 : Show that the differential equation

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$$a_0(x) y' + a_1(x) y = 0, \quad a_0(x) \neq 0$$

can always be put in the self-adjoint form by writing it as

$$g(x)y' + \frac{a_1(x)\,g(x)}{a_0(x)}\,y = 0 \quad \text{where } g(x) = e^{2\int \frac{a_1(x)}{a_0(x)}\,dx}$$

19.5.3 : Prove that if r, r' and p are continuous functions on an interval I and $r(x)>0$ for all $x \in I$, then the equation

$$\left(r(x)\,y'\right)' + p(x)y = 0 \text{ can be written in form}$$

$$a(x)y'' + b(x)y' + c(x)y = 0, \quad \text{where} \quad a(x) = r(x),\ b(x) = r'(x) \quad \text{and}$$

$c(x) = p(x)$ for all $x \in I$.

Conversely, prove that every equation $a(x)y'' + b(x)y' + c(x)y = 0$

where a, b, c are continuous functions on an interval I and $a(x)>0$ for all $x \in I$ can be written in the form

$$\left(r(x)\,y'\right)' + p(x)y=0, \text{ i.e. in the self-adjoint form.}$$

19.5.4 : Put the differential equation

$$x^2 y'' + xy' + y = 0,\ x>0 \text{ in self-adjoint form.}$$

19.5.5 : Find functions $z(x)$, $k(x)$ and $m(x)$ such that

$$z(x)\left[x^2 y'' - 2xy' + 2y\right] \equiv \frac{d}{dx}\left[k(x)y' + m(x)y\right], \text{ and hence solve}$$

$$x^2 y'' - 2xy' + 2y = 0,\ x>0$$

19.5.6 : Put the following differential equation in self-adjoint form.

$$\left(1-x^2\right)y'' - 2xy' + \left(n^2+n\right)y = 0,\ |x| < 1.$$

19.5.7 : Find functions $z(x)$, $k(x)$ and $m(x)$ such that

$$z(x)\,[y'' - y] \equiv \frac{d}{dx}\left[k(x)\,y' + m(x)y\right].$$

19.6 EXERCISES

19.6.1 : Put the following differential equations in self-adjoint form.

(a) $xy'' - y' + x^2 y = 0,\ x > 0$ Answer : $\left(\frac{1}{x}y'\right)' + y = 0$.

(b) $k(x)\,y'' + m(x)y = 0$, where $k(x) > 0$. Answer : $(y')' + \frac{m(x)}{k(x)}\,y = 0$

(c) $y'' - 3y' + 2y = 0$ Answer : $\left(e^{-3x}\,y'\right)' + 2e^{-3x}\,y = 0$

(d) $x^2\,y'' + x\,y' + \left(x^2 - n^2\right)y = 0,\ x > 0$ Answer $(xy')' + \left(\frac{x^2 - n^2}{x}\right)y = 0$

(e) $\left(1-x^2\right)y'' - 2xy' + \left(n^2 + n\right)y = 0,\ |x| < 1$ Answer $\left(\left(1-x^2\right)y'\right)' + \left(\frac{n^2+n}{1-x^2}\right)y = 0$

19.6.2 : Find functions $z(x)$, $k(x)$ and $m(x)$ such that

(a) $$z(x)[y'' + y] \equiv \frac{d}{dx}\left[k(x)y' + m(x)y\right]$$

Answer : when $z(x) = \sin x$, we have $k(x) = \sin x$, $m(x) = -\cos x$

when $z(x) = \cos x$, we have $k(x) = \cos x$, $m(x) = \sin x$

(b) $$z(x)\,[y'' - y] \equiv \frac{d}{dx}\left[k(x)y' + m(x)y\right]$$

Answer : when $z(x) = e^x$, we have $k(x) = e^x$, $m(x) = -e^x$

when $z(x) = e^{-x}$, we have $k(x) = e^{-x}$, $m(x) = e^{-x}$

(c) $z(x)[y'' + 3y' + 2y] \equiv \frac{d}{dx}[k(x)y' + m(x)y]$

Answer : when $z(x) = e^{x}$, we have $k(x) = e^{x}$; $m(x) = 2e^{x}$

when $z(x) = e^{2x}$, we have $k(x) = e^{2x}$; $m(x) = e^{2x}$

19.7 ANSWERS TO SHORT ANSWER QUESTIONS

19.4.1 : Proof of Theorem 19.1.2

19.4.2 : (i) Here $a_0(x) = 2x$; $a_1(x) = 1$ and satisfies $a_0'(x) = 2a_1(x)$. Hence $2xy' + y = 0$ is self-adjoint.

(ii) Here $a_0(x) = x$; $a_1(x) = -1$ and

$a_0'(x) = 1 \neq 2 \ a_1(x) = -2$. Hence

$xy' = y$ is not self-adjoint.

(iii) $a_0(x) = 2 \sin x$; $a_1(x) = \cos x$ and satisfies $a_0'(x) = 2a_1(x)$. Hence

$2(\sin x)y' + (\cos x)y = 0$ is self-adjoint,

19.4.3 : Proof of theorem 19.2.2.

19.4.4 : Solution of example 19.3.1

19.8 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Lesson - 20

ABEL'S FORMULA

20.0 INTRODUCTION

In this lesson, we derive Abel's formula and we use it to obtain second linearly independent solution of the second order linear differential equation when one of its solution is known. Also, we review the method of reduction of the order ofa second order linear differential equation, that you might have already studied.

20.1 REDUCTION OFTHE ORDER OF A DIFFERENTIAL EQUATION - A REVIEW

20.1.1 Theorem : If a solution $y_1(x) \neq 0$ for all $x \in I$ of a second order linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0 \text{ -------------- } 20.1.1(1)$$

where $a(x) \neq 0$ for all $x \in I$, a, b, c are continuous functions on an interval I is known, then a second linearly independent solution is given by

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} \exp\left(-\int \frac{b(x)}{a(x)} dx\right) dx$$

Proof : Since $a(x) \neq 0$ for all $x \in I$, we have $y'' + \frac{b(x)}{a(x)} y' + \frac{c(x)}{a(x)} y = 0$ (from 20.1.1(1)). For simplicity, we write $P(x) = \frac{b(x)}{a(x)}$ and $Q(x) = \frac{c(x)}{a(x)}$, so that the given equation reduces to

$$y'' + P(x)y' + Q(x)y = 0 \text{ ----------- } 20.1.1(2)$$

Since y_1 is a solution of 20.1.1(1), which in term y_1 is a solution of 20.1.1(2).

The following is the method of order of reduction of 20.1.1(2).

Assume that $y_2 = \mathrm{v}\, y_1$ is a solution of 20.1.1(2),

so that

$$y_2'' + P(x) y_2' + Q y_2 = 0, \text{ ------------ } 20.1.1(3)$$

and we try to discover the unknown function $v(x)$. We have

$$y_2' = v y_1' + v' y_1$$

$$y_2'' = v y_1'' + 2 v' y_1' + v'' y$$

By substantially thee expressions in 20.1.1(3), and rearranging, we get

$$v(y_1'' + P y_1' + Q y_1) + v'' y_1 + v'(2y_1' + P y_1) = 0 \text{ ------- } 20.1.1(4)$$

Since y_1 is a solution of 20.1.1(2), equation (20.1.1(4)) reduces to

$$v'' y_1 + v'(2y_1' + P y_1) = 0 \text{ ------------ } 20.1.1(5)$$

Put $u = v'$. Then (20.1.1(5)) becomes

$$u' y_1 + u(2y_1' + P y_1) = 0.$$

It is a first order onear differential equation so that the order of the differential equatior 20.1.1(2) is reduced by one and this imples

$$\frac{u'}{u} = -2 \frac{y_1'}{y_1} - P.$$

This implies

$$\log u = -2 \log y_1 - \int P\, dx$$

Thus $v' = u = \frac{1}{y_1^2} \exp\left(-\int P\, dx\right).$

Hence $v' = u = \frac{1}{y_1^2} e^{-\int P\, dx}\, dx.$ --------------- 20.1.1(6)

Thus $y_2 = v y_1$

$$= y_1 \int \frac{1}{y_1^2} e^{-\int P\, dx}\, dx \text{ --------------- } 20.1.1(7)$$

We now show that y_1 and y_2 are linearly independent. For this purpose, we consider

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_1 \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \\ y_1^1 & \frac{1}{y_1} e^{-\int P\,dx} + y_1^1 \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx \end{vmatrix}$$

$$= e^{-\int P\,dx} + y_1\, y_1^1 \int \frac{1}{y_1^2} e^{-\int P\,dx} - y_1\, y_1^1 \int \frac{1}{y_1^2} e^{-\int P\,dx}\,dx$$

$$= e^{-\int P\,dx} \neq 0,$$

so that y_1 and y_2 are linearly independent on I.

Hence, a second linearly independent solution of 20.1.1(2) is given by 20.1.1(7), and hence

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{-\int \frac{b(x)}{a(x)}dx}\,dx$$

is a second linearly independent solution of 20.1.1(1).

20.1.1.1 Note : The case, when $x_0 \in I$ is a zero of y_1 is discussed in section 20.3.

20.1.1.2 Example : Given that one solution of

$$y'' - xy' + y = 0 \text{ -------------- } 20.1.1.2(1)$$

is x, find a second linearly independent solution.

Solution : We have given that $y_1(x) = x$ is one solution of (20.1.1.2(1)).

Let $y_2 = y_1(x)\,\mathrm{v}(x)$ be a second solution of (20.1.1.2(1)), where v has to be determined.

Thus $y_2 = x\,\mathrm{v}$

$$y_2^1 = \mathrm{v} + x\,\mathrm{v}'$$

$$y_2'' = 2v' + xv''$$

Substituting these expressions in equation (20.1.1.2(1)) we get

$$(xv'' + 2v') - x(v + xv') + xv = 0$$

This implies

$$xv'' + (2 - x^2)v' = 0.$$

Put $u = v'$. Then $xu' + (2 - x^2)u = 0$.

It is a linear differential equation of order one in u. It has a solution

$u(x) = \frac{1}{x^2} e^{\frac{x^2}{2}}$, on any interval not containing the origin. Hence

$$v(x) = \int_1^x \frac{1}{x^2} e^{\frac{x^2}{2}} dx.$$

Thus

$$y_2(x) = x\,v(x)$$

$$= x \int_1^x \frac{1}{x^2} e^{\frac{x^2}{2}} dx$$

Observe that y_1 and y_2 are linearly independent solutions of (20.1.1.2(1)).

20.2 ABEL'S FORMULA

20.2.1 Theorem (Abel's Formula) : If u and v are any two solutions of a self-adjoint linear differential equation of the form

$$(r(x)\,y')' + p(x)y = 0 \text{ ---------- } 20.2.1(1)$$

where r and p are continuous functions on an interval I and $r(x) > 0$ for all x in I. Then

$$r(x)\left[u(x)\mathrm{v}'(x)-u'(x)\mathrm{v}(x)\right]\equiv k \quad \text{------------ } 20.2.1(2)$$

whre k is a constant.

The Identity (20.2.1(2)) is known as 'Abel's formula'.

Proof : Since u and v are solutions of (20.2.1(1)), we have

$$\left(r(x)\,u'(x)\right)' + p(x)u(x) = 0 \quad \text{------------ } 20.2.1\ (3)$$

$$\left(r(x)\mathrm{v}'(x)\right)' + p(x)\mathrm{v}(x) = 0 \quad \text{------------ } 20.2.1\ (4)$$

On multiplying (20.2.1(3)) by $-\mathrm{v}(x)$; and (20.2.1(4)) by $\mathrm{v}(x)$; and adding, we get

$$u(x)\left[r(x)\,\mathrm{v}'(x)\right]' - \mathrm{v}(x)\left[r(x)\,u'(x)\right]' \equiv 0.$$

On integrating this identity from a to x, and using integration by parts, we get

$$\left[u(x)\,r(x)\mathrm{v}'(x)\right]_a^x - \int_a^x u'(x)\,r(x)\mathrm{v}'(x)\,dx$$

$$-\mathrm{v}(x)r(x)u'(x) + \int_a^x \mathrm{v}'(x)\,r(x)\,u'(x)\,dx = c,$$

where c is an integrating constant. This implies

$$r(x)\left[u(x)\mathrm{v}'(x)-u'(x)\mathrm{v}(x)\right] = c + r(a)\left[u(a)\mathrm{v}'(a)-u'(a)\mathrm{v}(a)\right]$$

so that

$$r(x)\left[u(x)\mathrm{v}'(x)-u'(x)\mathrm{v}(x)\right] = k,$$

where

$$k = c + r(a)\left[u(a)\mathrm{v}'(a) - u'(a)\,\mathrm{v}(a)\right], \text{ a constant.}$$

This proves Abel's formula.

20.2.1.1 Remark : From Abel's formula, we have $r(x)W(u,\ \mathrm{v})(x) = k$,

where $W(u,\mathrm{v})$ is the Wronskian of the two solutions u and v of (20.2.1(1)). Hence

$k=0 \Leftrightarrow W(u,v)(x)=0$ for all x in I

$\Leftrightarrow u,v$ are linearly independent on I.

20.3 METHOD OF FINDING SECOND LINEARLY INDEPENDENT SOLUTION OF A SECOND ORDER LINEAR DIFFERENTIAL EQUATION WHEN ONE OF ITS SOLUTION IS KNOWN, BY USING ABEL'S FORMULA.

Let us consider a second order linear differential equation (20.1.1(1)), i.e.,

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where a, b, c are continuous functions on an interval I and $a(x)>0$ for all $x \in I$.

We write it in self-adjoint form

$$(r(x)y')' + p(x)y = 0 \text{ ------------ } 20.3(1)$$

where $r(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right)$

and $p(x) = \frac{c(x)}{a(x)} \exp\left(\int \frac{b(x)}{a(x)} dx\right)$ by theorem 19.2.4.

Here we observe that $r(x)>0$ and r and p are continuous on I.

If $y_1(x) \neq 0$ is a known solution of (20.3(1)), then we find a second linearly independent solution y_2 of (20.3(1)), that satisfies the Abel's formula. That is

$$r(x)\left[y_1(x)y_2^1(x) - y_1^1(x)\, y_2(x)\right] = 1 \text{ ------------ } 20.3(2).$$

That is, y_2 is a solution of

$$y_2^1 - \frac{y_1^1}{y_1} y_2 = \frac{1}{r\, y_1}.$$

This implies $\left(\frac{y_2}{y_1}\right)' = \frac{1}{r\, y_1^2}$

On integrating both sides, we have

$$\left(\frac{y_2}{y_1}\right) = \int_a^x \frac{1}{r\, y_1^2}\, dx\,.$$

This implies

$$y_2(x) = y_1(x) \int_a^x \frac{dx}{r(x)\, y_1^2(x)} \quad \text{------- } 20.3(3)$$

except possibly at the zeros of $y_1(x)$.

Let x_0 be a zero of y_1. Then

$$\lim_{x \to x_0} y_2(x) = \lim_{x \to x_0} \frac{\displaystyle\int_a^x \frac{dx}{r(x)\, y_1^2(x)}}{\dfrac{1}{y_1(x)}}$$

$$= \lim_{x \to x_0} \frac{\dfrac{1}{r(x)\, y_1^2(x)}}{-\dfrac{y_1^1(x)}{y_1^2(x)}} \quad \text{(By using L' Hospital's rule)}$$

$$= \lim_{x \to x_0} \left(- \frac{1}{r(x)\, y_1^1(x)} \right)$$

$$= -\frac{1}{r(x_0)\, y_1^1(x_0)} \quad \text{----------- } 20.3(4)$$

Here we observe that $y_1^1(x_0) \neq 0$, for if $y_1^1(x_0) = 0$ then by (20.3(4)),

$$0 = r(x_0)\left[y(x_0)\, y_1^1(x_0) - y_1^1(x_0)\, y_2(x_0)\right] = 1, \text{ a contradiction.}$$

Hence (20.3(3)) with $y_2(x_0)$ defined by (20.3(4)) provides a second solution of (20.3(1)) on the entire interval I

Clearly y_1 and y_2 are linearly independent, since the quotient.

$$\frac{y_2}{y_1} = \int_a^x \frac{dx}{r(x)\, y_1^2(x)} \text{ is not identically constant.}$$

20.3.1 Example : Given that one solution of $y'' - xy' + y = 0$ is x, find a second linearly, independent solution by using Abel's formula.

Solution : Let us first transform the given differential equation into its self-adjoint form.

By corresponding the given differential equation with (20.1.(1)), we have $a(x) = 1$, $b(x) = -x$ and $c(x) = 1$. From theorem 19.2.4, the self-adjoint form of the given differential equation is

$$\left(r(x)y'\right)' + p(x)y = 0$$

where

$$r(x) = \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp\left(-\int x\, dx\right) = \exp\left(-\frac{x^2}{2}\right)$$

and $$p(x) = \frac{c(x)}{a(x)} \exp\left(\int \frac{b(x)}{a(x)} dx\right) = \exp\left(-\frac{x^2}{2}\right).$$

If $y_1(x) = x$ is a solution of the given equation, then its second linearly independent solution $y_2(x)$ satisfies Abel's formula so that

$$r(x)\left[y_1 y_2^1 - y_1^1 y_2\right] = 1.$$

i.e., $$e^{-\frac{x^2}{2}}\left[x y_2^1 - y_2\right] = 1.$$

This implies $xy_2^1 - y_2 = e^{\frac{x^2}{2}}$

i.e. $$\left(\frac{y_2}{x}\right)' = \frac{e^{\frac{x^2}{2}}}{x^2}$$

This implies $\frac{y_2}{x} = \int \frac{e^{\frac{x^2}{2}}}{x^2} dx$.

Thus $y_2(x) = x\int \frac{e^{\frac{x^2}{2}}}{x^2} dx$

At $x = 0$,

$$y_2(0) = \lim_{x \to 0} y_2(x) = -\frac{1}{r(0)\, y_1^1(0)} = -1$$

Thus the two linearly independent solutions of the given equation are

$$y_1(x) = x$$

and $$y_2(x) = \begin{cases} x\int \frac{1}{x^2} e^{\frac{x^2}{2}} dx & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

20.3.1.1 Remark : The domain of the solution is not mentioned in examples (20.1.1.2) and (20.3.1). In example (20.1.1.2), the solution y_2 is given on an interval not containing the origin, whereas in example 20.3.1, the solution y_2 is given on the entire real time.

20.4 SHORT ANSWER QUESTIONS

20.4.1 : Given that one solution of $y'' - xy' + y = 0$ is x, find a second linearly independent solution on an interval not containing the origin, by using the method of reduction of order.

20.4.2 : State Abel's formula.

20.4.3 : In Abel's formula, $r(x)\left[u(x)v'(x) - u'(x)\, v(x)\right] = k,\ k$ constant, where u, v are solutions of $\left(r(x)y'\right)' + p(x)y = 0$.

Then show that u and v are linearly independent if and only if $k = 0$.

20.4.4 : Given that one solution of $y'' - xy' + y = 0$ is x, find a second linearly independent solution on the real line by using Abel's formula.

20.5 MODEL EXAMINATION QUESTIONS

20.5.1 : If u and v are any two solutions of a self-adjoint linear differential equation of the form

$\left(r(x)y'\right)' + p(x)y = 0,$ where r and p are continuous functions on an interval $[a, b]$ and $r(x) > 0$ for all x in I, then prove that

$$r(x)\left[u(x)v'(x) - u'(x)v(x)\right] \equiv k$$, where k is a constant (or)

state and prove Abel's formula related to self-adjoint linear differential equation of the form

$$\left(r(x)y'\right)' \quad p(x)y = 0.$$

20.5.5 : Given that one solution of $y'' - xy' + y = 0$ is x; find a second linearly independent solution by using Abel's formula.

20.5.3 : Given that $x+1$ is a solution of

$y'' - 2(x+1)y' + 2y = 0$; find the general solution.

20.5.4 : Find the general solution of the differential equation $x^4 y'' - x^2 y' + xy = 1 \quad (x > 0)$.

20.6 EXERCISES

20.6.1 : Given that one soluton of $y'' + a^2 y = 0$ is $\sin ax$, find a second linearly independent solution by using Abel's formula.

Answer : $\cos ax$.

20.6.2 : Given that one solution of $y'' - 4y' + 4y = 0$ is e^{2x}, use Abel's formula to find a second linearly independent solution.

Answer : $x\, e^x$

20.6.3 : Given that one solution of $y'' - 2ay' + a^2 y = 0$ is $e^{ax}\,(a \neq 0)$, use Abel's formula to find a second linearly independent solution.

Answer : $x\, e^{ax}$

20.6.4 : Given that $x+1$ is a solution of $y''-2(x+1)y'+2y=0$, find the general solution.

Answer : $c_1(x+1)+c_2(x+1)\int_0^x \frac{1}{(x+1)^2} e^{(x+1)^2}\,dx$

20.6.5 : Find the general solution of the differential equation $x^3 y'' - xy' + y = 0 \quad (x>0)$.

(Hint : Guess one solution).

Answer : $c_1 x + c_2 x e^{-\frac{1}{x}}$.

20.6.6 : Use exercise 20.6.5 to solve the differential equation $x^4 y'' - x^2 y' + xy = 1 \quad (x>0)$.

Answer : A particular solution is $-1+\frac{1}{2}x^{-1}$. The general solution is $c_1 x + c_2 x e^{-\frac{1}{x}} + \frac{1}{2x} - 1$.

20.6.7 : One solution of $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0,\ x>0$ is $x^{-\frac{1}{2}} \sin x$. Find the general solutior

Answer : $c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}$

20.7 ANSWERS TO SHORT ANSWER QUESTIONS

20.4.1 : Solution of example 20.1.1.2

20.4.2 : Statement of theorem 20.2.1

20.4.3 : Remark 20.2.1.1

20.4.4 : Solution of example 20.3.1

20.8 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Lesson - 21

THE RICCATI EQUATION

21.0 INTRODUCTION

In this lesson, we introduce Riccati differential equation and a method to find its solution.

21.1 RICCATI EQUATION

Consider the selfadjoint differential equation

$$(r(x)y')' + p(x)y = 0 \text{ --------------- } 21.1(1)$$

where r and p are continuous functions on an interval $I = [a, b]$ with $r(x) > 0$ on $[a, b]$.

Let us substitute $z = \frac{r(x)y'}{y}$, $y \neq 0$ in (21.1(1)). Then (21.1(1)) becomes

$$(yz)' + p(x)y = 0$$

That is $\quad y'z + yz' + p(x)y = 0$

or $\quad z' + \frac{y'}{y} z + p(x) = 0$

That is $\quad z' + \frac{1}{r(x)} z^2 + p(x) = 0 \text{ ------------- } (21.1(2))$

Equation (21.1(2)) is called a Riccati Equation.

The equation

$$z' + a(x)z + b(x)z^2 + c(x) = 0 \text{ -------------- } 21.1(3)$$

where a, b and c are continuous functions on an interval $I = [a, b]$, is called the general Riccati equation.

Equation (21.1(3)) is only apparently more general than equation (21.1(2)). Since the substitution

$$w = z \cdot e^{\int_a^x a(x)\,dx} \text{ --------------- } 21.1(4)$$

in (21.1(3)), reduces to

$$w' + q(x)\, w^2 + p(x) = 0 \quad \text{-------} \; 21.1(5)$$

where $$q(x) = b(x)\, e^{-\int_a^x a(x)\,dx}$$ and

$$p(x) = c(x)\, e^{\int_a^x a(x)\,dx}$$

For, by differentiating w of (21.1(4)), we get

$$w' = z'\, e^{\int_a^x a(x)\,dx} + z\, a(x)\, e^{\int_a^x a(x)\,dx}$$

$$= z'\, e^{\int_a^x a(x)\,dx} + a(x)\, w(x)$$

i.e., $$z' = w'\, e^{-\int_a^x a(x)\,dx} - a(x) z$$

Substituting it in (21.1(3)), we get

$$w'\, e^{-\int_a^x a(x)\,dx} - a(x) z + a(x) z + b(x) z^2 + c(x) = 0$$

$$\Rightarrow w'\, e^{-\int_a^x a(x)\,dx} + b(x) w^2\, e^{-2\int_a^x a(x)\,dx} + c(x) = 0$$

$$\Rightarrow w' + b(x) w^2\, e^{-\int_a^x a(x)\,dx} + c(x)\, e^{\int_a^x a(x)\,dx} = 0$$

i.e. $w' + q(x)\, w^2 + p(x) = 0$

If $b(x) \equiv 0$ in (21.1(3)), then the equation is linear and it is immediately integrable.

If $b(x) \not\equiv 0$ on any subinterval of $[a, b]$ in (21.1(3)), then we employ the substitution (21.1(4)) to reduce (21.1(3)) to the form (21.1(5)).

The substitution $\frac{y'}{y}$ then reduces (21.1(5)) to the form (21.1(1)), where $r(x) = \frac{1}{q(x)}$; for,

substituting $qw = \frac{y'}{y}$ in (21.1(5)), we get

$$\left(\frac{1}{q}\cdot\frac{y'}{y}\right)' + \frac{1}{q}\cdot\left(\frac{y'}{y}\right)^2 + p(x) = 0$$

i.e., $\left(\frac{1}{q}y'\right)'\frac{1}{y} - \frac{1}{y^2}\cdot\frac{1}{q}(y')^2 + \frac{1}{q}\left(\frac{y'}{y}\right)^2 + p(x) = 0$

$$\Rightarrow \left(\frac{1}{q}y'\right)'\frac{1}{y} + p(x) = 0.$$

i.e. $(r(x)\, y')' + p(x)y = 0$ where $r(x) = \frac{1}{q(x)}$, which is equation (21.1(1))

Here we observe that we have made the successive substitutions

$$w = z\, e^{\int_a^x a(x)\, dx}$$

and $\quad qw = \frac{y'}{y}$.

These may be replaced by

$$\frac{y'}{y} = bz, \text{ since}$$

$$\frac{y'}{y} = qw = qz\, e^{\int_a^x a(x)\,dx} = b(x)\, z\,.$$

21.2 EXAMPLES

21.2.1 Example : Study the solutions of the Riccati equation $w' - w^2 - 1 = 0$ ------- 21.2.1(1)

Solution : The given equation is in the form (21.1(5)) where $q(x) = -1$ and $p(x) = -1$.

Put $qw = \frac{y'}{y}$, $y \neq 0$. i.e., $-w = \frac{y'}{y}$; then

$$-w' = \left(\frac{y'}{y}\right)'$$ so that (21.2.1(1)) becomes

$$-\left(\frac{y'}{y}\right)' - \left(-\frac{y'}{y}\right)^2 - 1 = 0.$$

This implies

$$\frac{-y''\,y + (y')^2}{y^2} - \frac{(y')^2}{y^2} - 1 = 0.$$

i.e., $-\frac{y''}{y} - 1 = 0$. Hence $y'' + y = 0$

Its general solution is

$$y(x) = c_1 \sin x + c_2 \cos x$$

The null solution $(c_1 = c_2 = 0)$ leads to no solution w of the given Riccatie equation (21.2.1(1)). All other solutions y provide solutions.

$$w = -\frac{y'}{y} = -\frac{c_1 \cos x - c_2 \sin x}{c_1 \sin x + c_2 \cos x}, \quad c_1 \neq 0,\ c_2 \neq 0$$

The choice $c_1 = 0$ leads to the particular solution $w = \tan x$.

21.2.2 Example : Study the solutions of the Riccati equation

$$z' + z - e^x z^2 - e^{-x} = 0 \text{ ------------- } (21.2.2(1))$$

Solution : Comparing (21.2.2(1)) to the general Riccati equation (21.1(3)), we have

$$a(x) = 1,\ b(x) = -e^{-x} \text{ and } c(x) = -e^{-x}$$

substituting $w = ze^{\int a(x)dx} = ze^x$,

i.e., $z = w\,e^{-x}$ so that $z' = w'\,e^{-x} - we^{-x}$ in (21.2.2(1)) we get

$$w' e^{-x} - w e^{-x} + w e^{-x} - e^{x} \cdot w^2 e^{-2x} - e^{-x} = 0 .$$

This implies $w' - w^2 - 1 = 0$.

Now, from example 21.2.1, its solution w is given by

$$w(x) = -\frac{c_1 \cos x - c_2 \sin x}{c_1 \sin x + c_2 \cos x}, \quad c_1 \neq 0,\ c_2 \neq 0$$

Hence $z = w e^{-x}$

$$= -e^{-x} \frac{c_1 \cos x - c_2 \sin x}{c_1 \sin x + c_2 \cos x}, \quad c_1 \neq 0,\ c_2 \neq 0$$

21.2.2.1 Observations : The method explained above enables us to find an infinity of solutions of a Riccati equation (by varying the constants c_1, c_2 (parameters)). Now the first question is :

(i) may there not be solutions of the Riccati equation other than those obtained in this way?

The second question is :

(ii) The solutions obtained in the above examples contain two arbitrary constants c_1 and c_2. As the Riccati equation is a first order differential equation, we expect only one arbitrary constant in the solution. Why is it happening ?

Let us first deal with the second question. By the method of this lesson, we shall be led to solutions z of (21.1(3)) fo the form

$$z(x) = -f(x) \frac{c_1 u'(x) + c_2 v'(x)}{c_1 u(x) + c_2 v(x)} \quad \text{-----------} \quad (21.2.2.1(1))$$

where $f(x) \neq 0$, c_1, c_2 are not both zero, and $u(x)$ and $v(x)$ are linearly independent solutions of the related linear differential equation (21.1(1)). By taking $c_1 = 0$ (hence $c_2 \neq 0$) in (21.2.2.1(1)), we get the solution

$$z = -\frac{f(x) v'(x)}{v(x)}$$

For all other solutions given by (21.2.2.1(1)), $c_1 \neq 0$, and (21.2.2.1(1)) may accordingly written in the form

$$z = -f(x) \frac{u'(x) + k v'(x)}{u(x) + k v(x)} \quad \text{-----------} \quad (21.2.2.1(2))$$

where $k = \frac{c_2}{c_1}$ is an arbitrary constant.

This shows that the solutions depend essentially on one arbitrary constant. The exceptional solution $-\frac{f(x)\mathrm{v}'(x)}{\mathrm{v}(x)}$ may be regarded (and commonly is) as being obtained from (21.2.2.1(2)) when k is equal to infinity.

21.2.2.2 Remark : The form (21.2.2.1(2)) yields all solutions.

21.3 A THEOREM ON THE SOLUTION OF $w' + q(x)w^2 + p(x) = 0$

If z is a solution of (21.1(3)) then $w(x)$ defined by (21.1(4)) is a solutin of (21.1(5)) and conversely. We then have the following result.

21.3.1 Theorem : If $q(x) \neq 0$ and $p(x)$ and $q(x)$ are continuous on the interval $a \leq x \leq b$, every solution $w(x)$ of (21.1(5)) may be written in the form

$$\frac{1}{q(x)} \frac{c_1 u'(x) + c_2 \mathrm{v}'(x)}{c_1 u(x) + c_2 \mathrm{v}(x)} \text{ ---------------- } 21.3.1(1)$$

where c_1 and c_2 are constants, not both zero, and where $u(x)$ and $\mathrm{v}(x)$ are linearly independent solutions of the differential equation

$$\left(\frac{1}{q(x)} y'\right)' + p(x)\, y = 0 \text{ -------------- } 21.3.1(2)$$

Conversely, if $u(x)$ and $\mathrm{v}(x)$ are linearly independent solutions of (21.3.1(2)) and if c_1 and c_2 are any constants, not both zero, the function (21.3.1(1)) is a solution of (21.1(5)) on any interval in which $c_1\, u(x) + c_2\, \mathrm{v}(x) \neq 0$.

Proof : Let w be an arbitrary solution of $w' + q(x)\, w^2 + p(x) = 0$.

Put $w = \frac{1}{q} \cdot \frac{y'}{y}$; i.e., $y = e^{\int q(x)\, w(x)\, dx}$

Then

$$w' = \left(\frac{1}{q}y'\right)'\frac{1}{y} - \frac{1}{q}\left(\frac{y'}{y}\right)^2$$

$$= \left(\frac{1}{q}y'\right)'\frac{1}{y} - qw^2.$$

Hence $w' + q(x)\,w^2 + p(x) = \left(\frac{1}{q}y'\right)'\frac{1}{y} - qw^2 + qw^2 + p(x) = 0$

That is $\left(\frac{1}{q}y'\right)'\frac{1}{y} + p(x) = 0.$

This implies $\left(\frac{1}{q}y'\right)' + p(x)y = 0.$

This shows that $y = e^{\int q(x)w(x)\,dx}$ is a solution of (21.3.1(2)). Hence y is a linear contribution of the linearly independent solutions $u(x)$ and $v(x)$ of (21.3.1(2)). Hence

$$e^{\int q(x)\,w(x)\,dx} = c_1\,u(x) + c_2\,v(x),\ (c_1 \neq 0,\ c_2 \neq 0).$$

Differentiating, we get

$$q(x)\,w(x)\,e^{\int q(x)\,w(x)\,dx} = c_1\,u'(x) + c_2\,v'(x).$$

Therefore $w(x) = \dfrac{1}{q(x)} \cdot \dfrac{c_1\,u'(x) + c_2\,v'(x)}{c_1\,u(x) + c_2\,v(x)}.$

Conversely, we assume $u(x)$ and $v(x)$ are two linearly independent solutions of (21.3.1(2)).

Let $L(x) \equiv c_1 u(x) + c_2 v(x) \neq 0$ on $\alpha \le x \le \beta$. Then $m(x) \equiv \dfrac{L'(x)}{q(x)\,L(x)}$ is a solution of (21:1(5)), for consider

$$m'(x) + q(x)\,m^2(x) + p(x) =$$

$$\left(\frac{L'(x)}{q(x)\,L(x)}\right)' + q(x)\left(\frac{L'(x)}{q(x)L(x)}\right) + p(x)$$

$$= \left(\frac{L'(x)}{q(x)}\right)' \frac{1}{L(x)} - \frac{1}{q(x)}\left(\frac{L'(x)}{L(x)}\right)^2 + \frac{1}{q(x)}\left(\frac{L'(x)}{L(x)}\right)^2 + p(x)$$

$$= \left(\frac{L'(x)}{q(x)}\right)' \frac{1}{L(x)} + p(x)$$

= 0 (since $L(x)$ is a solution of (21.3.1(2))

This shows that

$$m(x) = \frac{L'(x)}{q(x)\,L(x)}$$

$$= \frac{1}{q(x)} \frac{c_1 u'(x) + c_2 v'(x)}{c_1 u(x) + c_2 v(x)}$$

is a solution of $w' + q(x)\,w^2 + p(x) = 0$.

21.4 A SOLVED PROBLEM

Show that if z, z_1, z_2 and z_3 are any four different solutions of the Riccati's equation

$z' + a(x)z + b(x)z^2 + c(x) = 0$ then show that

$$\frac{z - z_2}{z - z_1} \cdot \frac{z_3 - z_1}{z_3 - z_2} = \text{constant.}$$

Solution : From hypotheses, we have

$$z' + a(x)z + b(x)\,z^2 + c(x) = 0$$

$$z_1' + a(x)z_1 + b(x)\,z_1^2 + c(x) = 0.$$

On subtracting second equation from the first one, we get

$$(z - z_1)' + a(x)(z - z_1) + b(x)(z - z_1)(z + z_1) = 0$$

i.e. $$\frac{(z - z_1)'}{(z - z_1)^2} + a(x)\frac{1}{z - z_1} + \frac{b(x)(z + z_1)}{z - z_1} = 0$$

That is

$$-\frac{(z-z_1)'}{(z-z_1)^2} - a(x)\frac{1}{z-z_1} - b(x)\left[\frac{z_1 - z_1 + z + z_1}{z-z_1}\right] = 0$$

Hence $-\dfrac{(z-z_1)'}{(z-z_1)^2} - a(x)\dfrac{1}{z-z_1} - \dfrac{2b(x)z_1}{z-z_1} = b(x).$

i.e. $u = \dfrac{1}{z-z_1}$ is a solution of

$$u' - (a + 2b\,z_1)\,u = b$$

Similarly,

$u_2 = \dfrac{1}{z_2 - z_1}$ and $u_3 = \dfrac{1}{z_3 - z_1}$ are solutions of $u' - (a + 2b\,z_1)u = b$

Now $\dfrac{u-u_2}{u_3-u_2} = \dfrac{z-z_2}{z-z_1} \cdot \dfrac{z_3-z_1}{z_3-z_2}$

Observe that

$$(u-u_2)' - (a+2b\,z_1)(u-u_2) = 0$$

and $(u_3-u_2)' - (a+2b\,z_1)(u_3-u_2) = 0$

Hence $\dfrac{(u-u_2)'}{u-u_2} = \dfrac{(u_3-u_2)'}{u_3-u_2}$

On integrating both sides, we get

$$\log(u-u_2) - \log(u_3-u_2) = \log c$$

$$\Rightarrow \frac{u-u_2}{u_3-u_2} = c \text{ (a constant)}$$

That is $\dfrac{z_2-z}{z-z_1} \cdot \dfrac{z_3-z_1}{z_2-z_3}$ = constant

This solves the problem.

21.5 SHORT ANSWER QUESTIONS

21.5.1 : Transform the differential equation

$$(r(x)\, y')' + p(x)y = 0$$

where r and p are continuous functions on an interval $I=[a,\, b]$, $r(x)>0$ on $[a,\, b]$, into Riccati equation.

21.5.2 : Solve $w' - w^2 - 1 = 0$

21.5.3 : Solve that if $u(x)$ and $v(x)$ are linearly independent solutions of

$$\left(\frac{1}{q(x)}\, y'\right)' + p(x)y = 0$$, and if c_1 and c_2 are any constants, not both zero, then the function

$$m(x) = \frac{1}{q(x)} \frac{c_1 u'(x) + c_2 v'(x)}{c_1 u(x) + c_2 v(x)}$$

is a solution of $w' + q(x)w^2 + p(x) = 0$ on any interval I in which $c_1 u(x) + c_2 v(x) \neq 0$.

21.6 MODEL EXAMINATION QUESTIONS

21.6.1 : Transform the differential equation $(r(x)\, y')' + p(x)y = 0$, where r and p are continuous functions on an interval $I=[a,\, b]$ into Riccati equation.

21.6.2 : Write the general Riccati equation and study the solutions of the Riccati equation

$$z' + z - e^{x} z^2 - e^{-x} = 0$$

21.6.3 : If $q(x) \neq 0$ and p and q are continuous functions on an interval $[a,\, b]$ then prove that every solution w of

$w' + q(x)w^2 + p(x) = 0$ may be written in the form

$$\frac{1}{q(x)} \cdot \frac{c_1 u'(x) + c_2 v'(x)}{c_1 u(x) + c_2 v(x)} \quad \text{------------------ (A)}$$

where c_1 and c_2 are constants, not both zero, and where u and v are linearly independent solutions of

$$\left(\frac{1}{q(x)}y'\right)' + p(x)y = 0 \quad \text{-------------------- (B)}$$

Conversely, if u and v are linearly independent solutions of (B) and if c_1 and c_2 are any constants, not both zero, then prove that the function (A) is a solution of $w' + q(x)w^2 + p(x) = 0$ on any interval in which $c_1 u(x) + c_2 v(x) = 0$.

21.6.4 : Find all solutions of $z' + z^2 - z - 2 = 0$

21.6.5 : Find all solutions of $x^2 z' - 2xz + x^2 z^2 + 2 = 0$.

21.7 EXERCISES

21.7.1 : Find all solutions of the following Riccati equations

(a) $z' + z^2 - 1 = 0$

Answer : $\dfrac{c_1 e^x - c_2 e^{-x}}{c_1 e^x + c_2 e^{-x}}$

(b) $z' + z^2 - z - 2 = 0$

Answer : $\dfrac{2c_1 e^{2x} - c_2 e^{-x}}{c_1 e^{2x} + c_2 e^{-x}}$

(c) $z' + z^2 - 2z + 2 = 0$

Answer : $\dfrac{(c_1 - c_2)\sin x + (c_1 + c_2)\cos x}{c_1 \sin x + c_2 \cos x}$

(d) $x^2 z' - 2xz + x^2 z^2 + 2 = 0$

Answer : $\dfrac{c_1 + 2c_2 x}{c_1 x + c_2 x^2}$

(e) $x^2 z' - 3xz + z^2 + 2x^2 = 0$

Answer : $$\frac{(c_1-c_2)x \sin \log|x| + (c_1+c_2)x \cos \log|x|}{c_1 \sin\log|x| + c_2 \cos \log|x|}$$

(f) $xz' - z + xz^2 + x^3 = 0$

Answer : $$\frac{x\left[c_1 \cos\left(\frac{x^2}{2}\right) - c_2 \sin\left(\frac{x^2}{2}\right)\right]}{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}.$$

21.8 ANSWERS TO SHORT ANSWER QUESTIONS

21.5.1 : See section 21.1. The equation (21.1(1)) is transformed to (21.1(2)).

21.5.2 : Example 21.2.1

21.5.3 : Proof of the converse part of theorem 21.3.1

21.9 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Lesson - 22

GREEN'S FUNCTION

22.0 INTRODUCTION

In this lesson, we define Green's function and show that it provides a particular solution of the nonhomogeneous linear differential equation of second order. Examples are provided to illustrate the procedure. We derive Lagrange's theorem and Green's theorem.

22.1 GREEN'S FUNCTION AND GENERAL SOLUTION OF 2ND ORDER LINEAR DIFFERENTIAL EQUATION IN TERMS OF GREEN'S FUNCTION

22.1.1 Definition : We consider the differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x) \quad \text{--------------------------------- (22.1.1(1))}$$

where a_0, a_1, a_2 and f are continuous functions on $[a, b]$ and $a_0(x) \neq 0$ for all x in $[a, b]$.

The corresponding homogeneous equation of (22.1.1(1)) is

$$a_0(x)y'' + a_1(x)\, y' + a_2(x)y = 0 \quad \text{--------------------- (22.1.1(2))}$$

Let y_1, y_2 be linearly independent solutions of (22.1.1(2)). Let $x_0 \in [a, b]$. Define the function $G(x, t)$ by

$$G(x,t) = -\frac{1}{a_0(t)} \frac{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1^1(t) & y_2^1(t) \end{vmatrix}}$$

$$= -\frac{1}{a_0(t)W(y_1, y_2)(t)} \begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}, \quad x_0 \leq t \leq x \quad \text{-------------------- (22.1.1(3))}$$

Hence $W(y_1, y_2)$ denotes the Wronskian of y_1 and y_2; and we observe that $W(y_1, y_2)(t) \neq 0$.

The function $G(x, t)$ defined by (22.1.1(3)) is known as 'Green's function' for the homogeneous equation (22.1.1(2)).

22.1.2 Theorem : A particular solution of (22.1.1(1)) is

$\int_{x_0}^{x} G(x, t) f(t) dt$, and the general solution y of (22.1.1(1)) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_{x_0}^{x} G(x,t) f(t) dt \quad \text{------------ (22.1.2(1)).}$$

Where y_1 and y_2 are linearly independent solutions of the homogeneous equation (22.1.1(2).

Proof : Let y be an arbitrary solution of (22.1.1(1)). Let y_0 be a particular solution of (22.1.1(1)), and consider the function

$$z(x) = y(x) - y_0(x).$$

Then z is a solution of the corresponding homogeneous equation (22.1.1(2)). Hence z can be written as

$z(x) = c_1 y_1(x) + c_2 y_2(x)$, where c_1, c_2 are constants.

Hence $y(x) = c_1 y_1(x) + c_2 y_2(x) + y_0(x)$.

We now find $y_0(x)$ by using the method of variation of parameters, in terms of Green's function. For this purpose, we determine functions u_1 and u_2 such that

$$y_0(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

where u_1 and u_2 are satisfying

$$u_1^1(x) y_1(x) + u_2^1(x) y_2(x) = 0 \quad \text{--------------------- (22.1.2(2))}$$

substituting y_0 in (22.1.1(1)) and using (22.1.2(2)), we get

$$u_1^1(x) y_1^1(x) + u_2^1(x) y_2^1(x) = \frac{f(x)}{a_0(x)} \quad \text{---------------- (22.1.2(3),}$$

On solving (22.1.2(2)) and (22.1.2(3)) for $u_1^1(x)$ and $u_2^1(x)$, we get

$$u_1^1(x) = -\frac{y_2(x)\, f(x)}{a_0(x)\, W(y_1\, y_2)(x)} \quad \text{and}$$

$$u_2^1(x) = -\frac{y_1(x)\, f(x)}{a_0(x)\, W(y_1\, y_2)(x)},$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2; and since y_1, y_2 are linearly independent solutions of (22.1.1(2)), it never zero. Hence

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)\, f(t)}{a_0(t)\, W(t)}\, dt \quad \text{and}$$

$$u_2(x) = -\int_{x_0}^{x} \frac{y_1(t)\, f(t)}{a_0(t)\, W(t)}\, dt$$

where we are denoting $W(y_1, y_2)\ (t)$ by $W(t)$; and the point $x = x_0$ may be any convenient point of I. Hence a pa ticular solution of (22.1.1(1)) is

$$y_0(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$= -\, y_1(x) \int_{x_0}^{x} \frac{y_2(t)\, f(t)}{a_0(t)\, W(t)}\, dt + y_2(x) \int_{x_0}^{x} \frac{y_1(t)\, f(t)}{a_0(t)\, W(t)}\, dt$$

$$= \int_{x_0}^{x} \left[y_1(t)\, y_2(x) - y_1(x)\, y_2(t) \right] \frac{f(t)}{a_0(t)\, W(t)}\, dt$$

$$= \int_{x_0}^{x} \begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix} \frac{f(t)}{a_0(t)\, W(t)}\, dt$$

$$= \int_{x_0}^{x} G(x, t)\, f(t)\, dt$$

Therefore, the general solution of (22.1.1(1)) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_{x_0}^{x} G(x,t) f(t) dt$$

22.2 OBSERVATIONS :

For each t, the function $G(x,t)$ is a solution of the equation (22.1.1(2)).

Proof : We have

$$G(x,t) = -\frac{1}{a_0(t)W(t)} \begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}$$

$$= -\frac{1}{a_0(t)W(t)} \left(y_1(x) y_2(t) - y_2(x) y_1(t) \right)$$

Let $t \in I$ be fixed.

$$G(x,t) = -\frac{1}{a_0(t)W(t)} \left(y_1^1(x) y_2(t) - y_2^1(x) y_1(t) \right)$$

$$G_{xx}(x,t) = -\frac{1}{a_0(t)W(t)} \left(y_1''(x) y_2(t) - y_2''(x) y_1(t) \right)$$

Consider

$$a_0(x) G_{xx}(x,t) + a_1(x) G_x(x,t) + a_2(x) G(x,t)$$

$$= -\frac{1}{a_0(t)W(t)} \{ a_0(x)(y_1''(x) y_2(t) - y_2''(x) y_1(t))$$

$$+ a_1(x)\left(y_1'(x) y_2(t) - y_2^1(x) y_1(t) \right)$$

$$+ a_2(x)(y_1(x) y_2(t) - y_2(x) y_1(t)) \}$$

$$= -\frac{1}{a_0(t)W(t)} \{ y_2(t) \left(a_0(x) y_1''(x) + a_1(x) y_1^1(x) + a_2(x) y_1(x) \right)$$

$$- y_2(t)(a_0(x) y_2''(x) + a_1(x) y_2'(x) + a_2(x) y_2(x)) \}$$

= 0 (since y_1, y_2 are solutions of 22.1.1(2)).

22.2.2 Observation :

$$G_x(t,\, t) \equiv \frac{\partial}{\partial x} G(x,t)\Big|_{x=t} = \frac{1}{a_0(t)}$$

Proof : $$G_x(t,\, t) \equiv \frac{\partial}{\partial x} G(x,t)\Big|_{x=t}$$

$$= -\frac{1}{a_0(t)\, W(t)}\left(y_1^1(x)\, y_2(t) - y_2^1(x)\, y_1(t)\right)_{x=t}$$

$$= -\frac{1}{a_0(t)\, W(t)}\left(y_1^1(t)\, y_2(t) - y_2^1(t)\, y_1(t)\right)$$

$$= -\frac{1}{a_0(t)\, W(t)}\left(-W(t)\right)$$

$$= -\frac{1}{a_0(t)}.$$

22.2.3 Observation : When $x=t$, G vanishes; i.e. $G(t,\, t) = 0$.

Proof : Clear from the definition of G.

22.2.4 Observation : The particular solution

$y_0(x) = \int_{x_0}^{x} G(x,t) f(t) dt$ of equation (22.1.1(1)) is the unique particular solution

$y_0(x)$ of (22.1.1(1)) that has the property that $y_0(x_0) = y_0^1(x_0) = 0$.

Proof : We have $y_0(x_0) = \int_{x_0}^{x} G(x,\, t)\, f(t) dt = 0$ and

$y_0^1(x) = G(x,\, x)\, f(x) + \int_{x_0}^{x} G_x(x,t) f(t)\, dt$ and hence

$y_0^1(x_0) = G(x_0,\, x)\, f(x_0) = 0$, and hence by the existence and uniqueness of solutions, this observation follows.

22.3 EXAMPLES :

If two linearly independent solutions of the homogeneous equation (22.1.1(2)) are known, $G(x, t)$ can be constructed from (22.1.1(3)), and then (22.1.2(1)) provides the general solution of the non homogeneous equation (22.1.1(1)) for any function f which is continuous on $[a, b]$. We follow the procedure to construct particular solution of (22.1.1(1)). The following examples illustrate the above procedure of finding general solution of (22.1.1(1)).

22.3.1 Example : Find the general solution of

$$y'' - 3y' + 2y = f(x) \qquad (-\infty < x < \infty) \text{ ------------ } \quad (22.3.1(1))$$

where f is a continuous function; and then evaluate the general solution of (22.3.1(1)) when $f(x)=x$.

Solution : Consider the homogeneous corresponding to (22.3.1(1)). That is

$$y'' - 3y' + 2y = 0 \qquad (-\infty < x < \infty) \text{ -------------------- } (22.3.1(2))$$

It is easy to see that

$$y_1(x) = e^x \text{ and } y_2(x) = e^{2x}$$

are linearly independent solutions of (22.3.1(2)). Comparing (22.3.1(1)) to (22.3.1(2)), we have

$$a_0(t) = 1;\ a_1(t) = -3;\ a_2(t) = 2;$$

Now $W(t) = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$

By (21.1.1(3)),

$$G(x, t) = -e^{-3t} \begin{vmatrix} e^x & e^{2x} \\ e^t & e^{2t} \end{vmatrix}$$

$$= -e^{-3t}\left(e^{x+2t} - e^{2x+t}\right)$$

$$= e^{2(x-t)} - e^{(x-t)},\ x_0 \le t \le x.$$

Now, from theorem 22.1.2, the particular solution y_0 of (22.3.1(1)) is given by

$$y_0(x) = \int_0^x G(x,t) f(t)\, dt$$

$$= \int_0^x \left(e^{2(x-t)} - e^{(x-t)}\right) f(t)\,dt$$

where we have taken $x_0 = 0$.

Thus the general solution of (22.3.1(1)) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_0(x)$$

$$= c_1 e^x + c_2 e^{2x} + \int_0^x \left(e^{2(x-t)} - e^{(x-t)}\right) f(t)\,dt,$$

where c_1, c_2 are constants.

when $f(x) = x$, y_0 becomes

$$y_0(x) = \int_0^x \left(e^{2(x-t)} - e^{(x-t)}\right) t\,dt = -e^x + \frac{1}{4}e^{2x} + \frac{x}{2} + \frac{3}{2}.$$

Thus, when $f(x) = x$, the general solution of (22.3.1(1)) is given by

$$y(x) = c_1 e^x + c_2 e^{2x} - e^x + \frac{1}{4}e^{2x} + \frac{x}{2} + \frac{3}{2} \text{ (or)}$$

$$y(x) = k_1 e^x + k_2 e^{2x} + \frac{x}{2} + \frac{3}{4}, \text{ where}$$

$$k_1 = c_1 - 1 \text{ and } k_2 = c_2 + \frac{1}{4}$$

22.3.2 Example : Given the differential equation

$$y'' - 4y' + 3y = f(x),\ (-\infty < x < \infty) \text{ ----------- } (22.3.2(1))$$

Find two linearly independent solutions of

$$y'' - 4y' + 3y = 0 \text{ ------------------------ } (22.3.2(2))$$

Complete Green's function $G(x, t)$ and then compute the particular solution

$$\int_0^x G(x,t) f(t)\,dt$$

of (22.3.2(1)) when (a) $f(x)=3$; (b) $f(x)=x$. Check your answers. What is the general solution of (22.3.2(1)) in each case ?

Solution : It is easy to see that the two linearly independent of (22.3.2(2)) are $y_1(x)=e^x$; $y_2(x)=e^{3x}$.

Here $a_0(x)=1, \ -\infty < x < \infty$.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{vmatrix} = 3e^{4t} - e^{4t} = 2e^{4t}$$

$$G(x,t) = -\frac{1}{a_0(t)} \frac{\begin{vmatrix} e^t & e^{3t} \\ e^t & 3e^{3t} \end{vmatrix}}{W(y_1, y_2)(t)}$$

$$= -\frac{e^{-4t}}{2}\left(e^{x+3t} - e^{3x+t}\right)$$

$$= \frac{1}{2}\left(e^{3(x-t)} - e^{(x-t)}\right)$$

Hence the particular solution.

$$y_0(x) = \int_0^x G(x,t) f(t)\, dt$$

$$= \frac{1}{2}\int_0^x \left(e^{3(x-t)} - e^{(x-t)}\right) f(t)\, dt$$

(a) $f(x)=3$. Then

$$y_0(x) = \frac{1}{2}\int_0^x \left(e^{3(x-t)} - e^{(x-t)}\right) 3\, dt$$

$$= \frac{3}{2}\left[e^{3x}\cdot\left(\frac{e^{-3t}}{-3}\right)_0^x - e^x\left(\frac{e^{-t}}{-1}\right)_0^x\right]$$

$$=\frac{3}{2}\left(\frac{e^{3x}-1}{3}+1-e^{x}\right)$$

$$=\frac{1}{2}\left(e^{3x}-3e^{x}+2\right)$$

Hence the general solution is

$$y(x)=c_1 e^{x}+c_2 e^{3x}+y_0(x)$$

$$=c_1 e^{x}+c_2 e^{3x}+\frac{1}{2}\left(e^{3x}-3e^{x}+2\right)$$

$$=k_1 e^{x}+k_2 e^{3x}+1$$

where $k_1=c_1-\frac{3}{2}$ and $k_2=c_2+\frac{1}{2}$.

(b) $f(x)=x$.

Now $y_0(x)=\frac{1}{2}\int_0^x\left(e^{3(x-t)}-e^{(x-t)}\right)t\,dt$

$$=\frac{1}{2}e^{3x}\int_0^x t\,e^{-3t}\,dt-\frac{1}{2}e^{x}\int_0^x t\,e^{-t}\,dt$$

we have

$$\int_0^x t\,e^{-3t}\,dt=-\frac{x}{3}e^{-3x}-\frac{1}{9}e^{-3x}+\frac{1}{9}$$

and

$$\int_0^x t\,e^{-t}\,dt=-x\,e^{-x}-e^{-x}+1$$

Hence $y_0(x)=\frac{1}{2}e^{3x}\left[-\frac{x}{3}e^{-3x}-\frac{1}{9}e^{-3x}+\frac{1}{9}\right]-\frac{1}{2}e^{x}\left[-x\,e^{-x}-e^{-x}+1\right]$

$$=-\frac{x}{6}-\frac{1}{18}+\frac{1}{18}e^{3x}+\frac{x}{2}+\frac{1}{2}-\frac{1}{2}e^{x}.$$

$$=\frac{1}{3}x+\frac{2}{9}+\frac{1}{18}e^{3x}-\frac{1}{2}e^{x}.$$

Therefore the general is

$$y(x)=c_1\,e^{x}+c_2\,e^{3x}+y_0(x)$$

$$=c_1\,e^{x}+c_2\,e^{3x}+\frac{1}{3}x+\frac{2}{9}+\frac{1}{18}e^{3x}-\frac{1}{2}e^{x}$$

$$=k_1\,e^{x}+k_2\,e^{3x}+\frac{1}{3}x+\frac{2}{9}$$

where $k_1=\left(c_1-\frac{1}{2}\right)$ and $k_2=\left(c_2+\frac{1}{18}\right)$.

Note : Checking of particular solution is a solution of (22.3.2(1)) in each case is left as an exercise.

22.4 LANGUAGE THEOREM :

22.4.1 Theorem (Language Theorem) : Let $L(y)\equiv a_0(x)y''+a_1(x)y'+a_2(x)y$

where $a_0,\ a_0',\ a_0''\ a_1,\ a_1'$ and a_2 are continuous functions and $a_0(x)\neq 0$ on $[a,\ b]$. By (19.2(5)), let $M(z)=\left(a_0(x)z\right)''-\left(a_1(x)\ z\right)'+a_2(x)z$ be the adjoint of $L(y)$. Then

$$\text{v}\ L(u)-u\ M(\text{v})\equiv\frac{d}{dx}P(u,\text{v})$$

where $P(u,\ \text{v})=u\left[a_1\,\text{v}-(a_0\ \text{v})'\right]+u'(a_0\ \text{v})$.

Proof : Let us consider

$$\text{v}L(u)-uM(\text{v})=\text{v}\ a_0\,u''+\text{v}\ a_1\,u'+\text{v}\ a_2\ u$$

$$-u\left[(a_0\ \text{v})''-(a_1\,\text{v})'+a_2\,\text{v}\right]$$

$$=u'\ a_1\,\text{v}+u(a,\text{v})'-\left(u(a_0\ \text{v})'\right)'+u'(a_0\,\text{v})'+\text{v}\ a_0\ u''$$

$$=\left(u(a_1\ \text{v})\right)'-\left(u(a_0\ \text{v})'\right)'+\left(u'(a_0\ \text{v})\right)'$$

$$= \frac{d}{dx} P(u, \mathrm{v})$$

where $P(u, \mathrm{v}) = u(a_1 \mathrm{v} - a_0 \mathrm{v}')' + u'(a_0 \mathrm{v})$.

22.5 GREEN'S FORMULA :

22.5.1 Theorem (Green's Formula) :

Let $L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y$, where

$a_0, a_0', \cdots, a_0^{(n-1)}, a_1, a_1', \cdots, a_1^{(n-2)}, \cdots, a_{n-1}, a_{n-1}'$ and a_n are continuous on $[a, b]$.

Let $M(z) = (-1)^n (a_0 z)^{(n)} + (-1)^{n-1} (a_0 z)^{n-1} + \cdots - (a_{n-1} z)' + a_n z$ be the adjoint of $L(y)$.

Then $\int_a^b [\mathrm{v}\, L(u) - u M(\mathrm{v})] dx = [p(u, \mathrm{v})]_a^b$, where

$$p(u, \mathrm{v}) = u\left[a_{n-1}\mathrm{v} - (a_{n-2}\mathrm{v})' + \cdots + (-1)^{n-1} (a_0 \mathrm{v})^{(n-1)}\right]$$

$$+u'\left[a_{n-2}\mathrm{v} - (a_{n-3}\mathrm{v})' + \cdots + (-1)^{n-2} (a_0 \mathrm{v})^{(n-2)}\right]$$

$$+ \cdots + u^{(n-1)} a_0 \mathrm{v}.$$

Proof : As in the case of Lagrange theorem (Theorem 22.4.1), it can be shown that

$$\mathrm{v} L(u) - u M(\mathrm{v}) = \frac{d}{dx} P(u, \mathrm{v}).$$

Now, by integrating from $x = a$ to $x = b$ we get

$$\int_a^b [\mathrm{v}\,L(u) - u\,M(\mathrm{v})]\,dx = \int_a^b \frac{d}{dx} P(u, \mathrm{v})\,dx$$

$$= [P(u, \mathrm{v})]_a^b$$

This formula is known as Green's formula.

22.6 SHORT ANSWER QUESTIONS

22.6.1 : Show that the function y given in (22.1.2(1)) is a solution of (22.1.1(2)).

22.6.2 : Define Green's function.

22.6.3 : For each t, the function $G(x,t)$ defined by (22.1.1(3)) is a solution of (22.1.1(2)).

22.6.4 : Show that $G_x(t, t) = \dfrac{1}{a_0(t)}$.

22.6.5 : State Lagrange's theorem.

22.7 MODEL EXAMINATION QUESTIONS

22.7.1 : Define Green's function. Show that a particular solution of

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x) \quad \text{--------------------} \quad (22.7.1(1))$$

is $\int_{x_0}^{x} G(x, t)\, f(t)\, dt$ and the general solution y of (22.7.1(1)).

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \int_{x_0}^{x} G(x, t) f(t) dt$$

where y_1, y_2 are linearly independent solutions of the homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \quad \text{------------------} \quad (22.7.1(2))$$

and G is the Green's function associated with (22.7.1(2)).

22.7.2 Find the general solution of

$$y'' - 3y' + 2y = f(x), \quad (-\infty < x < \infty) \text{ ---------------- } (22.7.2(1))$$

where f is a continuous function; and then evaluate the general solutions of (22.7.2(1)) when $f(x) = x$.

22.7.3 : Given the differential equation

$$y'' + y = f(x), \quad -\infty < x < \infty;$$

compute Greens' function and then compute the particular solution. Find general solutions when (a) $f(x) = 3$; (b) $f(x) = x$.

22.7.4 : Given the differential equation

$$y'' + 4y' + 4y = f(x), \ -\infty < x < \infty,$$

Compute Greens' function and then compute the particular solution. Find general solution when

(a) $f(x) = 3;$ (b) $f(x) = x$

22.7.5 : Given the differential equation

$$x^2 y'' - 2xy' + 2y = f(x), \quad 1 \le x < \infty$$

compute Green's function and then compute the particular solution. Find general solution when

(a) $f(x) = 3$ (b) $f(x) = x$

22.7.6 : Given the differential equation

$$4x^2 y'' + y = f(x), \quad 1 \le x < \infty$$

compute Green's function and then compute the particular solution. Find general solution when

(a) $f(x) = 3$ (b) $f(x) = x$

22.7.7 : $L(y) \equiv a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y$

where $a_0, a_0', a_0'', a_1, a_1'$ and a_2 are continuous functions and $a_0(x) \ne 0$ on $[a, b]$. Then prove that

$$v < (u) - M(v) \equiv \frac{d}{dx} P(u, v)$$

where $P(u, v) = u\left[a_1 v - (a_0 v)'\right] + u'(a_0 v)$

and M is the adjoint of L.

22.7.8 : State Green's formula.

22.8 ANSWERS TO SHORT ANSWER QUESTIONS

22.6.1 Hint : Differentiate twice the given function y in (22.1.2(1)) and then substitute in (22.1.1(2)).

22.6.2 : Write the definition 22.1.1.

22.6.3 : Write the proof of observation 22.2.1.

22.6.4 : Write the proof of observation 22.2.2.

22.6.5 : Write statement of Lagange's theorem 22.4.1.

22.9 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

Lesson Writer :

Dr. G.V.R. Babu.

Lesson - 23

OSCILLATION THEORY FOR LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

23.0 INTRODUCTION

Oscillation theory for linear differential equations of second order is to be discussed in lessons 23 to 30. The self adjoint form of linear differential equations of second order arises naturally in mechanics; it has a central role in the Calculus of Nariations. The student will observe its use throughout the lessons from 23 to 30 as we study the behaviour of solutions of the linear differential equation of second order.

In this lesson, we introduce the concept of oscilatory solutions of second order linear differential equation. Also we study the zeros of the equation.

$$(r(x)y')' + p(x)y = 0 \qquad \text{(A)}$$

where r and p are continuous, and $r(x)>0$ on an interval I, which is a named theorem known as 'The sturm separation theorem'. In fact, this theorem tells us that if u and v are linearlyindependent solutions of (A) then between two consecutive zeros of u, there will be precisely one zero of v. We also derive some consequences of the sturm separation theorem.

23.1 SELF-ADJOINT LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER

A self-adjoint linear differential equation of second order is a differential equation of the form

$$(r(x)y')' + p(x)y = 0 \text{ --------------- (23.1(1))}$$

where $r(x)$ and $p(x)$ are continuous and $r(x)>0$ on an interval I

The results will apply to the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0 \text{ ------------ (23.1(2))}$$

where $a(x)>0$, and $a(x)$, $b(x)$ and $c(x)$ are continuous on the interval I because, as we have seen in lesson 19, (theorem 19.2.4) that equation (23.1(2)) can be put in self-adjoint form by multiplying both members of the equation (23.1(2)) by the function $\dfrac{1}{a(x)} \exp\left\{\int_{x_0}^{x} \dfrac{b(x)}{a(x)}\, dx\right\}$.

Then

$$r(x) = e^{\int_{x_0}^{x}\left(\frac{b(x)}{a(x)}\right)dx} \text{ and } p(x) = \frac{c(x)}{a(x)}r(x)$$

Let us now recall the Abel's formula whose proof was given in lesson 20.

23.1.1 Abel's Formula : If u and v are any two solutions of a self-adjoint linear differential equation of the form (23.1(1)), then

$$r(x)\left[u(x)\,\mathrm{v}'(x) - u'(x)\,\mathrm{v}(x)\right] \equiv k, \qquad \text{------------------ (23.1.1(1))}$$

where k is a constant.

23.2 THE NUMBER OF ZEROS ON A FINITE INTERVAL

23.2.1 Theorem : If $r(x)>0$ and $r(x)$ and $p(x)$ are continuous on the interval $a \le x \le b$, then the only solution of equation (23.1(1)) which vanishes infinitely often on this interval is the null solution.

Proof : Suppose $y(x)$ is a solution of (23.1(1)) which has an infinity of zeros on the interval $[a, b]$. Let Z be the set of all zeros of the solution $y(x)$ on the interval $[a, b]$. Since $[a, b]$ is a bounded, infinite subset of real numbers, by Bolzano - Weierstrass theorem, Z has a limit point, say x^* on the interval $[a, b]$. Hence the exists a sequence $\{x_n\}_{n=1}^{\infty}$ in Z with $x_n \ne x^*$ $(n=0,1,2,..........)$ and $x_n \to x^*$ as $n \to \infty$. Since y is continuous and $y(x_n) = 0$ for all $n = 0, 1, 2,$ .... it follows that $y(x^*) = 0$.

Also,

$$\lim_{x \to x^*} \frac{y(x) - y(x^*)}{x - x^*} = y'(x^*)$$

Since $y'(x^*)$ is known to exist, we evaluate the above limit by letting x tend to x^* through the members of the sequence $x_0, x_1, x_2,$........... Then $y'(x^*) = 0$

Hence the solution $y(x)$ of (23.1(1)) has the following properties at x^* of $[a, b]$.

$$y(x^*)=0 \text{ and } y'(x^*)=0.$$

Thus by the existence and uniqueness thoerem, it follows that $y(x)=0$ s for all x in $[a, b]$. Thus $y(x)$ is the null solution.

Hence the theorem follows.

23.2.2 Definition : If the differential equation (23.1(1)) has a non null solution that vanishes infinitely often on I then the solutions of (23.1(1)) are said to be oscillatory on I.

23.2.3 Definition : Let $u : I \to \mathbb{R}$, we say that $x_0 \in I$ is a simple zero of u if $u(x_0) = 0$ and $u'(x_0) = 0$.

23.3 THE STURM SEPARATION THEOREM

We now prove a fundamental result due to sturm, which is now nown as the sturm separation theorem.

23.3.1 Theorem : (The Sturm separation Theorem)

If $u(x)$ and $v(x)$ are linearly independent solutions of (23.1(1)) then between any two consecutive zeros of $u(x)$ there will be precisely one zero of $v(x)$.

The following lemma is helpful in proving theorem 23.3.1.

23.3.2 Lemma : If two solutions $u(x)$ and $v(x)$ of (23.1(1)) have a common zero they are linearly independent. Conversely, if $u(x)$ and $v(x)$ are linearly independent solutions, neither of them identically zero, then if one of them vanishes at $x = x_0$, so does the other.

Proof of the Lemma : Suppose $u(x)$ and $v(x)$ are solutions of (23.1(1)). Then by Abel's formula (23.1.1(1)) we have

$$r(x)\left[u(x)\, v'(x) - u'(x)\, v(x)\right] \equiv k$$

where k is a constant.

Let x_0 be the common zero of $u(x)$ and $v(x)$. If $x = x_0$ in Abel's formula, we get $k=0$. That is, the Wronskian of $u(x)$ and $v(x)$ at $x=x_0$, vanish and hence they are linearly dependent. This proves the first part of the Lemma.

Conversely, suppose $u(x)$ and $v(x)$ are linearly dependent solutions of (23.1(1)). Hence there exist constants c_1 and c_2, not both zero such that

$$c_1 u(x) + c_2 v(x) = 0.$$

Since we have neither $u(x)$ nor $v(x)$ is identically zero, we must have both c_1 and c_2 are different from zero. Thus, if $u(x_0) = 0$ then $v(x_0) = 0$ also.

This prove sthe lemma.

Proof of the Sturm Separation Theorem : It show be noted that not all self-adjoint linear differential equations of the form (23.1(1)) admit a nonnull solution that vanishes twice on I.

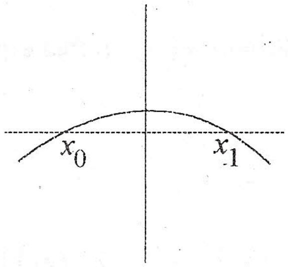


Fig. 23(i)

Suppose that $x = x_0$ and $x = x_1$ be two consecutive zeros of $u(x)$ (see Fig. 23(i)) and let $x_0 < x_1$. Since $-u(x)$ is also a solution of (23.1(i) with the same zeros of $u(x)$, we may assume without loss of generality that $u(x) > 0$ for $x \in (x_0, x_1)$.

Since $u(x)$ is continuous, and x_0, x_1 are zeros of $u(x)$, we have $u(x_0 + h) > 0$, and

$u(x_1 + h) < 0$ when $h > 0$, and

$u(x_0 + h) < 0$, and $u(x_1 + h) > 0$ when $h < 0$.

Therefore $\dfrac{u(x_0+h) - u(x_0)}{h} = \dfrac{u(x_0+h)}{h} > 0$

and $\dfrac{u(x_1+h) - u(x_1)}{h} = \dfrac{u(x_1+h)}{h} < 0$ for all h

This implies $u'(x_0) \geq 0$ and $u'(x_0) \leq 0$.

If $u'(x_0) = 0$ then, since $u(x)$ is a solution of (23.1(1)) with $u(x_0) = 0$, by the existance and uniqueness theorem $u(x_0) \equiv 0$ on I, which is not the case. Thus $u'(x_0) > 0$. With the same reason, $u'(x_1) < 0$.

If $\mathrm{v}(x_0) = 0$ then by lemma 23.3.2, $u(x)$ and $\mathrm{v}(x)$ are linearly dependent, a contradiction (since $u(x)$ and $\mathrm{v}(x)$ are linearly independent). Therefore $\mathrm{v}(x_0) \neq 0$. We assume without loss of generality that $\mathrm{v}(x_0) > 0$.

By Abel's formula, we have

$$r(x)\left[u(x)\,\mathrm{v}'(x) - u'(x)\,\mathrm{v}(x)\right] \equiv k,$$

where k is a constant. Substituting $x = x_0$ in this equation we get $k < 0$, since $r(x_0) > 0$, $u'(x_0) > 0$ and $\mathrm{v}(x_0) > 0$.

Now, by substituting $x = x_1$,

$$r(x_1)\left[u(x_1)\,\mathrm{v}'(x_1) - u'(x_1)\,\mathrm{v}(x_1)\right] = k < 0$$

But $u(x_1) < 0$. Thus

$$r(x_1)\,u'(x_1)\,\mathrm{v}(x_1) \quad 0$$

Since $u'(x_1) < 0$ and $r(x_1) > 0$ it follows that $\mathrm{v}(x_1) < 0$.

Hence $\mathrm{v}(x_0) > 0$ and $\mathrm{v}(x_1) < 0$. Since $\mathrm{v}(x)$ is a continuous function, it must have atleast one zero between x_0, where $\mathrm{v}(x)$ is positive, and x_1, where $\mathrm{v}(x)$ is negative. This means that $\mathrm{v}(x)$ must have at least one zero x^* (say) between x_0 and x_1.

We now show that $\mathrm{v}(x)$ vanishes at most once between x_0 and x_1.

Assume that $\mathrm{v}(x)$ vanishes at two points x^*, x^{**} in (x_0, x_1). By the argument above, with the roles $u(x)$ and $\mathrm{v}(x)$ reversed, we get that between x^* and x^{**} there must be at least one zero of $u(x)$, a contradiction to the fact that x_0, x_1 are consecutive zeros of $u(x)$.

Therefore, there cannot be more than one zero of $\mathrm{v}(x)$ in (x_0, x_1). Thus there exists precisely one zero of $\mathrm{v}(x)$ between two consecutive zeros of $u(x)$.

The proof of the Sturum separation theorem is complete.

More generally, we have the following result.

23.3.3 Theorem : Let $u(x)$ and $v(x)$ be functions of class $\mathscr{C}'$ on $[a, b]$ and suppose that $u(a) = v(b) = 0,\ v(x) \neq 0$ on (a, b). If

$w(x) = u(x)\, v'(x) - v(x)\, u'(x) \neq 0$ on $[a, b]$ then $u(x)$ vanishes precisely once on (a, b).

Proof : Without loss of generality, we may assume $w(x) > 0$ on $[a, b]$ and $v(x) > 0$ on (a, b). Then, since $v(a) = v(b) = 0$, it follows that a and b are consecuritve zeros of $v(x)$. Since $v(x) > 0$ on (a, b), we have $v'(a) > 0$ and $v'(b) < 0$ (as in the proof of theorem 23.3.1).

We have

$w(a) = u(a)\, v'(a) > 0$ and $w(b) = u(b)\, v'(b) > 0$. Since $v'(a) > 0$, it follows that $u(a) > 0$; and since $v'(b) < 0$, it follows that $u(b) < 0$.

Since u is continuous, u takes every value between $u(a)$ and $u(b)$ at least once. Therefore $u(x)$ must vanish at least once in (a, b), say at α. If α is not simple then $u(\alpha) = u'(\alpha) = 0$ and hence $w(\alpha) = 0$, which is not possible. Thus the zero $\alpha \in (a, b)$ of u is simple.

Assume that $u(x)$ has a second zero β on (a, b). By the same argument as above, with the roles of $u(x)$ and $v(x)$ interchanged, we get a zero of $v(x)$ on (α, β), and $(\alpha, \beta) \subset (a, b)$, a contradiction to the fact that $v(x) \neq 0$ on (a, b). This shows that $u(x)$ has precisely one zero on (a, b).

This completes the proof of the theorem.

The following corollary is an immediate consequence of the theorem 23.3.3.

23.3.4 Corollary : If $v(x)$ has an infinity of zeros and if $w(x) \neq 0$ on $[a, b)$ then

(i) $u(x)$ has an infinity of zeros on $[a, b)$ also;

(ii) the zeros of both $u(x)$ and $v(x)$ are simple;

(iii) the zeros of $u(x)$ and $v(x)$ separate each other. Here b may be finite or infinite.

Note that if $b = \infty$, and if $w(x)$ is eventually (for x sufficiently large) of one sign, the conclusions of the corollary are valied x sufficiently large.

23.4 A FUNDAMENTAL LEMMA

The following result is of fundamental importance in many oscillation studies. It is the complementary to theorem 23.3.3.

23.4.1 Lemma : Let us $u(x)$ and $v(x)$ be functions of class $\mathscr{C}'$ on $[a, b]$ and suppose $u(a) = u(b) = 0$, and let $v(x) \neq 0$ on $[a, b]$. Then there exist constants c_1 and c_2, not both zero, such that the function $c_1 u(x) + c_2 v(x)$ has a double zero on (a, b)

Proof : Consider the function

$$w(x) = u(x)v'(x) - u'(x)\, v(x)$$

Suppose without loss of generality that $v(x)>0$ on $[a, b]$, and that $x=a$ and $x=b$ are consecutive zeros of $u(x)$, and hence $u(x)>0$ on (a, b). Therefore, we have $u'(a)>0$ and $u'(b)<0$.

Case (i) : Suppose that the zeros of $u(x)$ are simple.

Then $w(a) = -u'(a)\, v(a)$ (since $u(a) = 0$ and $u'(a) \neq 0$)

and $w(b) = -u'(b) v(b)$ (since $u(b)=0$ and $u'(b)\neq 0$)

Therefore $w(a)<0$ and $w(b)>0$.

Since $u, v \in \mathscr{C}'$, we have u, u' and v, v' are continuous on $[a, b]$ and hence $w(x)$ is continuous on $[a, b]$. Therefore $w(x)$ must vanish at some point x_0 of (a, b). That is,

$$w(x_0) = \begin{vmatrix} u(x_0) & v(x_0) \\ u'(x_0) & v'(x_0) \end{vmatrix} = 0$$

It follows that there exist constants c_1 and c_2 not both zero such that

$$c_1 u(x_0) + c_2 v(x_0) = 0$$

$$c_1 u'(x_0) + c_2 v'(x_0) = 0$$

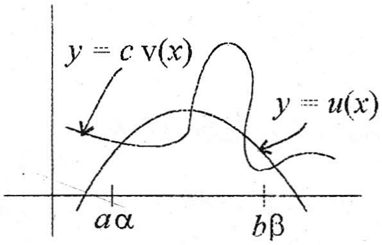
That is, the function $c_1 u(x) + c_2 v(x)$ has a double zero at $x = x_0$.

Case (ii) : Suppose the zeros of $u(x)$ are not simple.

Then we can choose a constant $c>0$, small enough that the curve $y=c\,v(x)$ cuts the curve $y=u(x)$ in at least two distinct points (Fig. 23(ii)). Let $x=\alpha$ be the smallest and $x=\beta$ be the largest of such points. Then

$$c\,v(\alpha)=u(\alpha) \text{ and } c\,v(\beta)=u(\beta)$$

Further, $c\,v'(\alpha)<u'(\alpha)$ and $c\,v'(\beta)>u'(\beta)$. This is clear geometrically and is readily argued analytically) since $u,\ v \in \mathscr{C}'$.



Now

$$w(\alpha)=u(\alpha)\,v'(\alpha)-u'(\alpha)\,v(\alpha)$$

$$=c\,v(\alpha)\,v'(\alpha)-u'(\alpha)\,v(\alpha)$$

$$=v(\alpha)\left[c\,v'(\alpha)-u'(\alpha)\right]<0$$

(since $u(\alpha)>0$ and $c\,v'(\alpha)<u'(\alpha)$).

Now $w(\beta)=v(\beta)\left[c\,v'(\beta)-u'(\beta)\right]>0$

Since w is continuous on $[a,\ b]$, there is a point $x_0\in(\alpha,\beta)$ such that $w(x_0)=0$.

Now the conclusion of theorem follows by case (i).

Note that from (23.4.1(1)) that if c_1 where zero, then $c_2\neq 0$ and $v(x_0)=0$, contrary to the hypothesis. Accordingly, we may conclude that $c_1\neq 0$ and the conclusion of the theorem may be given as there exists a constant k such that the function $u(x)-k\,v(x)$ has a double zero on $(a,\ b)$, for k may be taken as $-\dfrac{c_2}{c_1}$. Geometrically, the lemma may be interpreted to mean that there exists a constant k with the property that the curves $y=k\,v(x)$ and $y=u(x)$ are tangent at a point $x=x_0$ of $(a,\ b)$.

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23.5 SHORT ANSWER QUESTIONS

23.5.1 : Prove that between every pair of consecutive zeros of $\sin x$, there is one zero of $\sin x + \cos x$.

23.5.2 : Construct an eample of a differential equation in the form (23.1(1)) such that no nonnull solution has more than one zero [thus, there are equations (23.1(1)) to which Sturm's theorem does not apply].

23.5.3 : Show that between every pair of consecutive zeros of $\sin \log x$ there is a zero of $\cos \log x$.

23.6 MODEL EXAMINATION QUESTIONS

23.6.1 : If $r(x) > 0$ and $r(x)$ and $p(x)$ are continuous functions on the interval $a \leq x \leq b$, then prove that the only solution of the equation

$$(r(x)\, y')' + p(x) y = 0$$

which vanishes infinitely often on this interval is the null solution.

23.6.2 : Define oscillatory solution of $(r(x) y')' + p(x) y = 0$. State and prove Sturm separation theorem.

23.6.3 : Let $u(x)$ and $v(x)$ be functions of class $\mathscr{C}'$ on $[a, b]$ and suppose that $v(a) = v(b) = 0$ and $v(x) \neq 0$ on (a, b). If $w(x) = u(x)\, v'(x) - v(x)\, u'(x) \neq 0$ on $[a, b]$ then prove that $u(x)$ vanishes precisely once on (a, b).

23.6.4 : If $u(x)$ is a solution of $(r(x) y')' + p(x) y = 0$ such that $u(x_0) = u(x_1) = 0$ and $u(x) > 0\ (x_0 < x < x_1)$, prove that $u'(x_0) > 0$ and that $u'(x_1) < 0$.

23.6.5 : Prove that between every pair of consecutive zeros of $\sin x$ there is one zero of $\sin x + \cos x$.

26.6.6 : Show that between every pair of consecutive zeros of $\sin \log x$ there is a zero of $\cos \log x$.

23.7 EXERCISE

23.7.1 : Find a self-adjoint differential equation which has the solution $r(x)\, y'(x)$ where $y(x)$ is any solution of

$$\left(r(x)y'\right)' + p(x)y = 0$$

Answer : $\left(\frac{1}{p} z'\right)' + \left(\frac{1}{r}\right) z = 0$

23.8 ANSWERS TO SHORT ANSWER QUESTIONS

23.5.1 : Let $u(x) = \sin x + \cos x$

and $v(x) = \sin x$

Clearly, $u,\, v \in \mathscr{C}'([0, \pi])$;

Also $v(0) = 0$ and $v(\pi) = 0$ and $v(x) \neq 0$ on $(0, \pi)$.

Now

$$w(x) = u(x)\, v'(x) - v(x)\, u'(x)$$

$$= (\sin x + \cos x)\cos x - \sin x(\cos x - \sin x)$$

$$= \sin x \cos x + \cos^2 x - \sin x \cos x + \sin^2 x$$

$$= 1 \neq 0 \text{ on } [0, \pi].$$

Thus, by theorem 23.3.3, the conclusion follows.

23.5.2 : The equation (23.1(1)) is of the form $\left(r(x)y'\right)' + p(x)y = 0$.

Put $r(x) = 1$ for every x and $p(x) = -1$ for every x. Then the equation reduces to

$y'' - y = 0;\; y(0) = 0,\; y'(0) = 1;$ and its two solutions are $u(x) = e^x,\, v(x) = e^{-x}$

Then the equation reduces to

$y'' - y = 0;\; y(0) = 0,\; y'(0) = 1;$ and its two solutions are $u(x) = e^x,\, v(x) = e^{-x}$

Therefore, the general solution is

$$y(x) = c_1 e^x + c_2 e^{-x}$$

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = c_1 - c_2 = 1$$

By adding these two equations, we get

$2c_1 = 1$. This implies $c_1 = \frac{1}{2}$;

Hence $c_2 = -\frac{1}{2}$

Thus $y(x) = \frac{1}{2}\left[e^x - e^{-x}\right]$.

This y is anontrivial solution of $y'' - y = 0$, which has exactly one zero $x=0$.

23.5.3 : Write $u(x) = \cos(\log x)$

$$\mathrm{v}(x) = \sin(\log x)$$

$\mathrm{v}(1) = \sin\left(\log e^0\right) = \sin 0 = 0$ and $\mathrm{v}\left(e^\pi\right) = \sin\left(\log e^\pi\right) = \sin \pi = 0$

Also, $\mathrm{v}(x) \neq 0$ on $\left(1, e^\pi\right)$.

$$w(x) = u(x)\,\mathrm{v}'(x) - \mathrm{v}(x)\,u'(x)$$

$$= \frac{1}{x}\cos^2(\log x) + \frac{1}{x}\sin^2(\log x) = \frac{1}{x} \neq 0 \text{ on } \left[1, e^\pi\right]$$

Hence, by theorem 23.3.3, $u(x)$ vanishes precisely once in $\left(1, e^\pi\right)$.

Now by using corollary 23.3.4, since v has infinity of zeros and $w(x) \neq 0$ on $\mathbb{R}$ it follows that u has infinity of zeros on $\mathbb{R}$ and the zeros of $u(x)$ and $\mathrm{v}(x)$ separate each other.

23.9 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Dr. G.V.R. Babu.

Lesson - 24

THE STURM COMPARISON THEOREM

24.0 INTRODUCTION

In this section we continue the study of solutions of differential equations of the form

$$(r(x)\,y')' + p(x)\,y = 0$$

where $r(x) > 0$, and $r(x)$ and $p(x)$ are continuous on the closed interval $a \le x \le b$.

The student will recall that the Sturm separation theorem asserts that between two consecutive zeros of a solution of 24.0(1) there appears one zero of every linearly independent solution. Thus, speaking roughly, the number of zeros on an interval of any solution of 24.0(1) is about the same as the number of any other possesses. Thus, the solutions of 24.0(1) oscillate with the same rate of oscillation.

On the other hand, it is clear that solutions (for example, $\sin 2x$) of

$$y'' + 4y = 0 \text{ -------------------- } 24.0(2)$$

Oscillate more frequently (that is, have more zeros) on the interval $0 \le x \le 2\pi$ than do the solutions of

$$y'' + y = 0 \text{ -------------------- } 24.0(3)$$

on that interval. A typical solution of (24.0(3)) is, of couse, $\sin x$.

The Sturm comparison theorem compares the rates of oscillation of solutions of two equations.

$$(r(x)\,y')' + p(x)\,y = 0 \text{ ---------------- } 24.0(4)$$

$$(r(x)\,z')' + p(x)\,z = 0 \text{ --------------- } 24.0(5)$$

where $r(x) > 0$, $r(x), p(x)$ and $p_1(x)$ are continuous on $a \le x \le b$.

24.1 THE STURM COMPARISON THEOREM

24.1.1 Theorem (Sturm Comparison Theorem) : If a solution $y(x)$ of 24.0(4) has consecutive zeros at $x = x_0$ and $x = x_1$ $(x_0 < x_1)$, and if $p_1(x) \ge p(x)$ with strict inequality holding for at least one point of the closed interval $[x_0, x_1]$, a solution $z(x)$ of 24.1(5) which vanishes at $x=x_0$ will vanish again on the interval $x_0<x<x_1$.

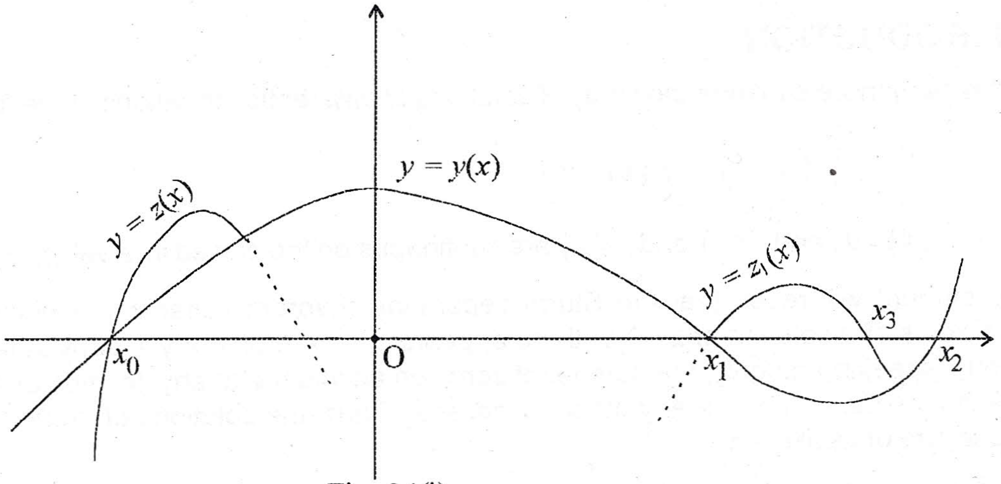

Fig. 24(i)

Proof : Suppose without loss of generality that $y(x)>0$ for all $x \in (x_0, x_1)$, and that $y'(x_0)>0, y'(x_1)<0$, and $z'(x_0) > 0$. Since $y(x)$ and $z(x)$ are solutions of equations 24.0(4) and 24.0(5) respectively, we have the identities

$$\left(r(x)\, y'(x)\right)' + p(x) y(x) \equiv 0$$

$$\left(r(x)\, z'(x)\right)' + p_1(x) z(x) \equiv 0$$

Multiplying 24.1.1(1) by $-z(x)$; and 24.1.1(2) by $y(x)$; and adding we get

$$y(x)\left(r(x)\, z'(x)\right)' - z(x)\left(r(x) y'(x)\right)' + y(x)\, z(x)\left(p_1(x) - p(x)\right) = 0.$$

Integrating both sides over $[x_0, x_1]$, we get

$$\left[y(x)\, r(x)\, z'(x)\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} y'(x)\, r(x)\, z'(x)\, dx - \left[z(x)\, r(x)\, y'(x)\right]_{x_0}^{x_1}$$

$$+ \int_{x_0}^{x_1} z'(x)\, r(x)\, y'(x)\, dx + \int_{x_0}^{x_1} \left(p_1(x) - p(x)\right) y(x)\, z(x)\, dx = 0$$

(Here integration by parts formula is used) i.e. $y(x_1)\, r(x_1)\, z'(x_1) - y(x_0)\, r(x_0)\, z'(x_0) - z(x_1)\, r(x_1)\, y'(x_1)$

$$+z(x_0)\, r(x_0)\, y'(x_0) = -\int_{x_0}^{x_1} (p_1(x) - p(x))\, y(x)\, z(x)\, dx$$

since x_0, x_1 are zeros of $y(x)$ and z vanishes at x_0, we have

$$r(x_1)\, y'(x_1) z(x_1) = \int_{x_0}^{x_1} (p_1(x) - p(x))\, y(x)\, z(x)\, dx \quad \text{---------------} \quad 24.1.1(3)$$

Now suppose $z(x)>0$ on (x_0, x_1). Since $y(x)>0$ on (x_0, x_1) and $p_1(x) \geq p(x)$ on (x_0, x_1) and $p_1(x) \geq p(x)$ for at least at one point of (x_0, x_1), the integral in 24.1.1(3) is positive.

Since $r(x_1)>0$, the left hand members of 24.1.1(3) is non-positive. This is a contradiction. (Here $z(x_1) \geq 0$). Therefore $z(x) < 0$ at some point of (x_0, x_1).

This completes the proof of the Sturm comparison theorem.

24.1.1.1 Note : Under the hypothesis of theorem 24.1.1, further if $y(x)$ vanishes again at $x = x_2 > x_1$ with $p_1(x) > p(x)$ on (x_1, x_2), it follows that $z(x)$ will vanish at some point on the interval (x_1, x_2). For, consider a second solution $z_1(x)$ of 24.0(5), defined by the conditions $z_1(x) = 0,\ z_1'(x_1) = 1$. As in the proof of Sturm comparison theorem, $z_1(x)$ has a zero x_3 on the interval (x_1, x_2). Now by applying Sturm separation theorem, we conclude that $z(x)$ has a zero on the interval (x_1, x_3), (see Fig. 24(i)).

24.1.1.2 Note : Sturm comparison theorem says that, the larger is $p(x)$ of 24.0(1), the more rapidly the solutions of 24.0(1) oscillate (see Fig. 24(i)).

24.1.1.3 Example : If nonnull solutions $y(x)$ and $z(x)$, respectively of the equations

$$x^2 y'' + xy' + (x^2 - 1) y = 0$$

$$xz'' + z' + xz = 0$$

vanish at $x=1$, which solution will vanish first after $x=1$?

Solution : First we put both the given differential equations in the form of self-adjoint linear differential equation.

We have

$$x^2 y'' + xy' + \left(x^2 - 1\right) y = 0 \text{ ------------------ } 24.1.1.3(1)$$

Comparing 24.1.1.3(1) with the differential equation

$$a(x) y'' + b(x) y' + c(x) y = 0,$$

we get $a(x) = x^2$, $b(x) = x$; $c(x) = x^2 - 1$

$a(x) > 0$ on $(0, \infty)$, $b(x)$ and $c(x)$ are continuous on $(0, \infty)$.

Multiplying 24.1.1.3(1) by $\frac{1}{a(x)} e^{\int_0^x \frac{b(x)}{a(x)} dx}$, i.e. $\frac{1}{x^2} e^{\int_0^x \left(\frac{x}{x^2}\right) dx} = \frac{x}{x^2} = \frac{1}{x}$, we get

$$xy'' + y' + \frac{x^2 - 1}{x} y = 0$$

i.e., $(x\, y')' + \left(\frac{x^2 - 1}{x}\right) = 0$

i.e. $\left(r(x) y'\right)' + p(x) y = 0$ where $r(x) = x > 0$ on $(0, \infty)$ and

$$p(x) = \frac{x^2 - 1}{x} \text{ on } (0, \infty)$$

The second equation is

$$xz'' + z' + xz = 0.$$

This can be written as

$$(xz')' + xz = 0.$$

It is of the form $\left(r(x) z'\right)' + p_1(x) z = 0$

where $r(x) = x$ and $p_1(x) = x$ on $(0, \infty)$.

Here we observe that $p_1(x) > p(x)$ for all $x \in (0, \infty)$.

Now by applying the Sturm comparison theorem, the solution of the second differential equation vanishes first after $x=1$.

24.2 A GENERALIZATION

24.2.1 Theorem : If $p(x)$ and $q(x)$ are continuous functions on the interval $[a, b]$ with $p(x) \not\equiv q(x)$, if $z(x)$ is a nonnull solution of the system

$$\left.\begin{aligned} z'' + q(x)z &= 0, \\ z(a) = z(b) &= 0 \end{aligned}\right\} \qquad 24.2.1(1)$$

and if $\int_a^b [p(x) - q(x)] z^2(x)\, dx \ge 0$, a nonnull solution $y(x)$ of the system

$$\left.\begin{aligned} y'' + p(x)y &= 0 \\ y(a) &= 0 \end{aligned}\right\} \quad 24.2.1(2)$$

has a zero on the interval $a < x < b$.

Before proving the theorem 24.2.1, we introduce the following two lemmas 24.2.2 and 24.2.3.

24.2.2 Lemma : If two differential equations

$$(r(x)y')' + a(x)y = 0, \quad \text{-----------------} \; 24.2.2(1)$$

$$(r(x)z')' + b(x)z = 0, \quad \text{-----------------} \; 24.2.2\,(2)$$

where $r(x) > 0$ and, $r(x)$, $a(x)$, and $b(x)$ are continuous on I, have a common solution $u(x) \not\equiv 0$, then $a(x) \equiv b(x)$ on I.

Proof : Since $u(x)$ is a common solution of 24.2.2(1) and 24.2.2(2), we have

$$[r(x)\, u'(x)]' + a(x)\, u(x) \equiv 0$$

$$\left[r(x)\,u'(x)\right]' + b(x)\,u(x) \equiv 0$$

This implies

$$\left[a(x) - b(x)\right] u(x) \equiv 0 \text{ for all } x \in I.$$

Since $u(x) \not\equiv 0$ on I, it follows that

$$a(x) - b(x) \equiv 0 \text{ for all } x \in I.$$

Thus

$$a(x) \equiv b(x) \text{ for all } x \in I.$$

Hence the lemma.

24.2.3 Lemma : If $y(x)$ and $z(x)$ are functions of class $\mathscr{C}'$ on I, wtih $y(x) \neq 0$ there, and if

$$y(x)\,z'(x) - y'(x)\,z(x) \equiv 0 \text{ on } I$$

then $z(x) \equiv c\,y(x)$, where c is a constant.

Proof : By hypothesis, we have

$$\left(\frac{z}{y}\right)' = \frac{z'y - y'z}{y^2} = 0$$

This implies $\frac{z}{y} \equiv c$ on I, and the lemma is proved.

We now prove theorem 24.3.1.

Proof of theorem 24.2.1 :

Consider the identities

$$\left.\begin{array}{l} z\left[z'' + qz\right] \equiv 0, \\ -\dfrac{z^2}{y}\left[y'' + py\right] \equiv 0 \end{array}\right\} \text{ ----------------- } 24.2.1(3)$$

on $[a, b]$. Note that the quotient $\frac{y''}{y}$ is continuous on $[a, b]$, when defined suitably at a zero of y. Adding the identities of 24.2.1(3), we get

$$\frac{z}{y}\left(yz'' - zy''\right) = (p - q)z^2$$

That is

$$\frac{z}{y}\left(yz' - zy'\right)' = (p-q)z^2$$

On integrating from a to b we get

$$\int_a^b \frac{z}{y}\left(yz' - zy'\right)' dx = \int_a^b (p - q)z^2\, dx \quad \text{-------------------- 24.2.1(4)}$$

Assume that $y(x)\neq 0$ on $(a,\ b]$, and integrating left hand member of (24.2.1(4)) by using integration by parts, we get

$$\left[\frac{z}{y}\left(yz' - zy'\right)\right]_a^b - \int_a^b \left(z' - \frac{z}{y}y'\right)^2 dx = \int_a^b (p-q)z^2\, dx$$

This implies

$$\frac{z(b)}{y(b)}\left(y(b)\,z'(b) - z(b)\,y'(b)\right) - \frac{z(a)}{y(a)}\left[y(a)\,z'(a) - z(a)\,y'(a)\right]$$

$$-\int_a^b \left(z' - \frac{z}{y}y'\right)^2 dx = \int_a^b (p-q)z^2\, dx \quad \text{-------- 24.2.1(5)}$$

The ratio $\frac{z}{y}$ remains finite at $x=a$ by L'Hospital's rule.

Thus 24.3.1(5) becomes

$$-\int_a^b \left(z' - \frac{z}{y}y'\right)^2 dx = \int_a^b (p-q)z^2\, dx \geq 0 \text{ (by hypothesis)} \quad \text{------------- 24.2.1(6)}$$

Hence the integral

$$-\int_a^b \left(z' - \frac{z}{y}y'\right)^2 dx$$

is positive; other wise $z(x) = c\,y(x)$ by lemma 24.2.3.

Thus 24.2.1(1) and 24.2.1(2) have a common solution. Now, by lemma 24.2.2, it follows that $p(x) \equiv q(x)$, a contradiction.

Therefore $y(x)$ must vanish at some point of (a, b).

Suppose now that $y(x) \neq 0$ on (a, b) and $y(b)=0$. The ratio $\frac{z}{y}$ in (24.2.1(5)) is finite at both $x=a$ and $x=b$, by using L'Hospital's rule. Thus 24.2.1(5) becomes (24.2.1(6)) and proceeding further, we get a contradiction.

Hence $y(x)=0$ for some $x \in (a, b)$

The proof of the theorem is complete.

24.2.1.1 Note : Theorem 24.3.1 is a generalization of theorem 24.2.1 (Sturm Comparison Theorem) when $r(x) \equiv 1$.

24.2.1.2 Remark : An important special case of theorem 24.2.1 occurs when $q(x) = \lambda^2$, where λ is a positive constant.

In this case, from 24.2.1(1), we have $z(x) = \sin \lambda(x-a)$ and

$$\int_a^b [p(x) - q(x)]\, z^2(x)\, dx = \int_a^{a+\frac{\pi}{\lambda}} \left(p(x) - \lambda^2\right) \sin^2 \lambda(x-a)\, dx \geq 0.$$

This implies

$$\int_a^{a+\frac{\pi}{\lambda}} p(x) \sin^2 \lambda(x-a)\, dx \geq \int_a^{a+\frac{\pi}{\lambda}} \lambda^2 \sin \lambda(x-a)\, dx$$

$$= \int_a^{a+\frac{\pi}{\lambda}} \lambda^2 \left[\frac{1-\cos 2\lambda(x-a)}{2}\right] dx$$

$$= \frac{\lambda^2}{2}\left[x - \frac{\sin 2\lambda(x-a)}{2\lambda}\right]_a^{a+\frac{\pi}{\lambda}}$$

$$= \frac{\lambda^2}{2} \cdot \frac{\pi}{\lambda} = \frac{\pi\lambda}{2} \quad (24.2.1.2(1))$$

Put $t = \lambda(x - a)$ then (24.2.1.2(1)) becomes

$$\int_0^{\frac{\pi}{\lambda}} p\left(\frac{t}{\lambda} + a\right) \sin^2 t \cdot \frac{1}{\lambda}\, dt \geq \frac{\pi\lambda}{2}$$

$$\text{i.e. } \int_0^{\frac{\pi}{2}} p\left(\frac{t}{\lambda} + a\right) \sin^2 t \geq \frac{\pi\lambda^2}{2} \quad \text{------------------ } 24.2.1.2(2)$$

Hence, if $q(x) = \lambda^2$ then the condition

$\int_a^b [p(x) - q(x)]\, z^2(x)\, dx \geq 0$ of theorem 24.2.1, becomes

$$\int_0^{\frac{\pi}{\lambda}} p\left(\frac{t}{\lambda} + a\right) \sin^2 t \cdot \frac{1}{\lambda}\, dt \geq \frac{\pi\lambda}{2}$$

$$\text{i.e. } \int_0^{\frac{\pi}{\lambda}} p\left(\frac{t}{\lambda} + a\right) \sin^2 t \geq \frac{\pi\lambda^2}{2} \quad \text{--------- } 24.2.1.2(2)$$

Hence, if $q(x) = \lambda^2$ then the condition

$\int_a^b [p(x) - q(x)] z^2(x) dx \geq 0$ of theorem 24.2.1, becomes

$$\int_0^{\frac{\pi}{\lambda}} p\left(\frac{t}{\lambda} + a\right) \sin^2 t \geq \frac{\pi\lambda^2}{2}.$$

24.2.1.3 Example : Consider the differential equation

$$y'' + x^2 y = 0,\ y(0) = 0.$$

Here $p(x)=x^2$, $a = 0$. By comparing with theorem 24.2.1,

we have

$$\int_a^b [p(x)-q(x)]z^2(x)dx = \int_0^{\frac{\pi}{2}} \left(x^2-\lambda^2\right)\sin^2 \lambda x\, dx \geq 0.$$

Carrying out the indicated integration in as in note 24.2.1.2, we have

$$\lambda^4 \leq \frac{2\pi^2-3}{6}.$$

That is, a nonnull solution $y(x)$ of the given differential equation has a zero on the interval

$0<x<\frac{\pi}{\lambda_0}$ where $\lambda_0 = \left(\frac{2\pi^2-3}{6}\right)^{\frac{1}{4}}$ After simplication we see that $y(x)$ has a zero on the interval (0, 2.44).

24.3 SHORT ANSWER QUESTIONS

24.3.1 : State Sturm comparison theorem.

24.3.2 : Write the significance of Sturm comparison theorem.

24.3.3 : Show that if $p(x) \leq 0$ on $[a, b]$, no nonnull solution of

$$\left(r(x)\, y'\right)' + p(x)y = 0$$

can have more than one zero on $[a, b]$.

24.3.4 : If two differential equations

$$\left(r(x)y'\right)' + a(x)y = 0$$

$$\left(r(x)z'\right)' + b(x)z = 0$$

where $r(x)>0$ and $r(x)$, $a(x)$ and $b(x)$ are continuous on I, have a common solution $u(x) \not\equiv 0$. Then show that $a(x) \equiv b(x)$ on I.

24.3.5 : If $y(x)$ and $z(x)$ are functions of class $\mathscr{C}'$ on I, with $y(x) \neq 0$ there, and if

$$y(x)z'(x) - y'(x)\,z(x) \equiv 0 \text{ on } I$$

then prove that $z(x) \equiv c\,y(x)$, where c is a constant.

24.3.6 : Write the statement of theorem 24.3.1 using note 24.3.1.2.

24.3.7 : Write the differences between the Sturm separation theorem and the Staurm comparison theorem.

24.4 MODEL EXAMINATION QUESTIONS

24.4.1 : State and prove Sturm comparison theorem.

24.4.2 : If nonnull solutins $y(x)$ and $z(x)$, respectively of the equations.

$$x^2 y'' + xy' + \left(x^2 - 1\right)y = 0$$

$$xz'' + z' + xz = 0$$

vanish at $x=1$, which solution will vanish first after $x=1$?

24.4.3 : Are the solutions of the differential equation

$$(2x+1)y'' + (x+1)y = 0 \text{ oscillatory on the interval } (0, \infty) ?$$

24.4.4 : If $p(x)$ and $q(x)$ are continuous on the interval $[a, b]$ with $p(x) \not\equiv q(x)$, if $z(x)$ is a nonnull solution of the system

$$z'' + q(x)z = 0$$

$$z(a) = z(b) = 0, \text{ and if}$$

$\int_a^b [p(x) - q(x)]z^2(x)dx \geq 0$, then prove that a nonnull solution $y(x)$ of the system

$$y'' + p(x)y = 0$$

$$y(a) = 0 \text{ has a zero on the interval } a < x < b.$$

24.4.5 : Find an upper bound for the first positive zero of a solution $y(x)$ of the differential system

$$y'' + \left(4+x^2\right)y = 0$$

$$y(0)=0.$$

24.5 EXERCISES

24.5.1 : Which of the following differential equations posses the more rapidly oscillating solutions on the interval $(1, \infty)$:

$$x^2 y'' + xy' + y = 0,$$

$$y'' + y = 0 \ ?$$

Answer : The second equation.

24.5.2 : Are the solutions of the differential equation $(2x+1)y'' + (x+1)y = 0$ oscillatory on the interval $(0, \infty)$?

Answer : Yes

24.5.3 : The solutions of the differential equation

$\left(\frac{x^2}{1+x^2} y'\right)' + x\sin\frac{1}{x} y = 0$ are oscillatory on the interval $(1, \infty)$. Estimate roughly the number of zeros a nonnull solution has on the interval $(10\pi, 20\pi)$

Answer : 10 or 11

24.5.4 : Using theorem 24.3.1, find an upper bound for the first positive zero of a solution $y(x)$ of the differential system

(a) $y'' + \left(4+x^2\right)y = 0$

$$y(0) = 0;$$

(b) $y'' + \left(7-x^2\right)y = 0$

$$y(0) = 0;$$

(c) $y'' + xy = 0$

$$y(0) = 0.$$

Answer : (a) 1.48

(b) zero of solution is $\frac{\sqrt{6}}{2}$ exactly.

(c) 2.7

24.6 ANSWERS TO SHORT ANSWER QUESTIONS

24.3.1 : Statement of Sturm comparison theorem, 24.2.1

24.3.2 : The larger of $p(x)$ of

$$(r(x)y')' + p(x)y = 0, \quad \text{-------------------- } 24.4.2(1)$$

The more rapidly the solutions of (24.4.2(1)) oscillate.

24.3.3 : Compare the nonnull solutions of

$$(r(x)\,y')' + p(x)y = 0$$

with the nonnull solutions of

$$(r(x)\,y')' = 0$$

by using Sturm comparison theorem, conclusion follows.

24.3.4 : Proof of the lemma 24.3.2.

24.3.5 : Proof of the lemma 24.3.3.

24.3.6 : Sturm separation asserts that between two consecutive zeros of solution of

$$(r(x)y')' + p(x)y = 0, \quad \text{----------------- } (24.4.6(1))$$

$r(x) > 0$, $r(x)$ and $p(x)$ are continuous on the interval $a \le x \le b$, there appears one zero of every linearly independent solution.

On the other hand, Sturm comparison theorem says that the larger is $p(x)$ in (24.4.6(1)), the more rapidly the solutions of (24.4.6(1)) oscillate.

24.7 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Lesson - 25

THE STURM - PICONE THEOREM
AND
THE BÔCHER - OSGOOD THEOREM

25.0 INTRODUCTION

The Sturm - Comparison theorem may be generalized to more general pair of self adjoint linear differential equations. This is done in Sturm Picone theorem. The Bocher - Osgood theoem is established, regarding the amplitudes of the oscillation of solutions. The Bocher-Osgood theorem for a self-adjoint differential equation is also established.

25.1 THE STURM - PICONE THEOREM

25.1.1 Lemma (Picone Formula) : Consider the self-adjoint differential equations

$$\left[r(x)y'\right]' + p(x)y = 0 \quad \text{---------- } 25.1.1\ (1)$$

$$\left[r_1(x)y'\right]' + p_1(x)z = 0 \quad \text{---------- } 25.1.1\ (2)$$

where $r(x)$ and $r_1(x)$ are positive, $r(x)$, $p(x)$, $r_1(x)$, $p_1(x)$ are continuous on an interval $[a, b]$.

If y and z are nonnull solutions of 25.1.1(1) and 25.1.1(2) respectively, then for $x \in (a, b)$,

$$\int_a^x \left[(r_1 - r)z'^2 + (p - p_1)z^2\, dx\right] + \int_a^x r\left[\frac{yz' - y'z}{y}\right]^2 dx$$

$$= \left[\frac{z}{y}(r_1 y z' - r y' z)\right]_a^x \quad \text{------ } 25.1.1(3)$$

except possibly at the zeros of $y(x)$. Formula (25.1.1(3)) is known as 'Picone formula'.

Proof : Assume $y \neq 0$ on $[a, b]$. From 25.1.1(1) and 25.1.1(2), we have

$$z\left[\left(r_1(x)\, z'(x)\right)' + p_1(x)\, z(x)\right] = 0$$

$$-\frac{z^2}{y}\left[\left(r(x)\, y'(x)\right)' + p(x)\, y(x)\right] = 0$$

By adding these two equations, we get

$$\frac{z}{y}\left[\left(r_1(x) z'\right)' y - \left(r(x) y'\right)' z\right] = (p - p_1) z^2$$

This implies

$$\frac{z}{y}\left[r_1(x) z' y - r\, y' z\right]' - r_1 y' z' \frac{z}{y} + r\, y' z' \frac{z}{y} = (p - p_1) z^2$$

Thus

$$\frac{z}{y}\left[r_1 z' y - r\, y' z\right]' = (r_1 - r) y' z' . \frac{z}{y} + (p - p_1) z^2$$

By using integration by parts, we get

$$\left[\frac{z}{y}(r_1 z' y - r\, y' z)\right]_a^x - \int_a^x \left(\frac{z}{y}\right)' [r_1 z' y - r y' z]\, dx$$

$$= \int_a^x \left[(r_1 - r) y' z' \cdot \frac{z}{y} + (p - p_1) z^2\right] dx$$

Hence

$$\int_a^x \left[\frac{yz' - y'z}{y^2}\right][r_1 z' y - r\, y'\, z]\, dx + \int_a^x \left[(r_1 - r) y'\, z' \frac{z}{y} + (p - p_1) z^2\right] dx$$

$$= \left[\frac{z}{y}(r_1 z' y - r y' z)\right]_a^x \quad \text{------------- (25.1.1(4))}$$

Let us now simplify LHS of (25.1.1(4)).

$$\text{LHS of (25.2.1(4))} = \int_a^x \left[r_1 z'^2 - 2r\, y'\, z' \frac{z}{y} + r\, y'^2 \frac{z^2}{y^2} + (p - p_1) z^2 \right] dx$$

$$= \int_a^x \left[(r_1 - r) z'^2 + (p - p_1) z^2 \right] dx + \int_a^x \frac{r}{y^2} \left[y^2 z'^2 - 2r\, y\, y' z\, z' + y'^2 z^2 \right] dx$$

$$= \int_a^x \left[(r_1 - r) z'^2 + (p - p_1) z^2 \right] dx + \int_a^x r \left[\frac{y z' - y' z}{y} \right]^2 dx \text{ --------- 25.2.1.(5)}$$

Now, from 25.2.1(4) and 25.2.1(5), clearly Picone formula follows.

25.1.2 Theorem : Let y and z be two nonnull solutions of 25.1.1(1) and 25.1.1(2) respectively, satisfying the conditions

$$\left. \begin{aligned} y(a) &= 0 \\ z(a) = z(b) &= 0 \end{aligned} \right\} \text{ -------------- (25.1.2(1))}$$

If $\int_a^b \left[(r_1 - r) z'^2 + (p - p_1) z^2 \right] dx > 0$ ------------- (25.1.2(2))

Then $y(x)$ must have a zero on the interval (a, b).

Proof : Under the hypotheses of the theorem, by using b in place of x in Picone formula, and assuming $y(x) \neq 0$ on $(a, b]$, we get

$$\int_a^b \left[(r_1 - r) z'^2 + (p - p_1) z^2 \right] dx + \int_a^b r \left[\frac{yz' - y'z}{y} \right]^2 dx$$

$$= \left[\frac{z}{y} (r_1\, y\, z' - r\, y' z) \right]_a^b$$

$$= 0 \text{ (since } z(a) = z(b) = 0) \text{ ----------- (25.1.2(3))}$$

The ratio $\frac{z}{y}$ is well defined on $[a, b]$ and by using (25.1.2(2)) in (25.1.2(3)), we immediately get a contradiction.

Similarly, if $g(b) = 0$, the ratio $\frac{z}{y}$ would be continuous on $[a, b]$ and (25.1.2(3)) yields a contradiction.

Hence for some $x_0 \in (a, b)$, we have $y(x_0) = 0$.

This completes the proof of theorem 25.1.2.

25.1.3 Corollary (Sturm - Picone Theorem) : If $r_1(x) \geq r(x)$ and $p(x) \geq p_1(x)$ with strict inequality holding in each of these conditions at at least one point of the interval then the solution $y(x)$ must vanish on (a, b).

Proof : Follows from theorem 25.2.2.

25.2 THE BOCHER - OSGOOD THEOREM

Consider the differential equation

$$y'' + p(x)y = 0 \text{ ------------- } (25.2(1))$$

where $p(x)$ is positive and of class $\mathscr{C}'$ on the interval $I = [0, \infty)$. If $p'(x) \geq 0$ on I then $p(x) \geq p(0) > 0$, and every solution of (25.2(1)) vanishes infinitely often on I, by Sturm comparison theorem. It follows from a result due to Bocher and Osgood that under these conditions, the amplitudes of the oscillation of solutions never increase as x increases on I

25.2.1 Theorem (The Bocher - Osgood Theorem) : Suppose that $p(x) \geq 0$ on $I = [0, \infty)$, and let $y(x)$ be an arbitrary solution of (25.2(1)). If $x = a$ and $x = b$ are two consecutive zeros of $y'(x)$, then $|y(b)| \leq |y(a)|$.

Proof : First we observe that if $p'(x) \equiv 0$ for $a \leq x \leq b$, then $p(x)$ is constant on this interval and $|y(b)| = |y(a)|$. The theorem is accordingly true in this case. We may then proceed with the proof when $p'(x) \neq 0$ on $[a, b]$.

Since $y(x)$ is a solution of (25.2(1)), we have

$$y''(x)+p(x)y(x)=0 \text{ ------------ } (25.2.1(1))$$

On multiplying both sides of (25.3.1(1)) by $2y'(x)$ and integrate the result over the interval $[a,b]$, we get

$$y'^2(x)\Big|_a^b + \int_a^b p(x)\left[y^2(x)\right]' dx = 0 \text{ ----------- } (25.2.1(2))$$

i.e. $y'^2(b) - y'^2(a) + p(x)y^2(x)\Big|_a^b - \int_a^b p'(x)\, y^2\, dx = 0$

Since $y'(a) = y'(b) = 0$, it follows that

$$p(b)y'(b) - p(a)y^2(a) = \int_a^b p'(x)\, y^2(x) dx \text{ ----------- } (25.2.1(3))$$

Suppose now that $y^2(b) > y^2(a)$. Then from (25.2.1(3)), we have

$$p(b)y^2(b) - p(a)y^2(a) = \int_a^b p'(x)\, y^2(x) dx$$

$$< y^2(b)\int_a^b p'(x) dx$$

$$= y^2(b)\left[p(b)-p(a)\right],$$

so that

$$-p(a)y^2(a) < -y^2(b)\, p(a)$$

i.e. $p(a)\, y^2(b) < p(a)\, y^2(a)$

i.e. $y^2(b) < y^2(a)$, a contradiction, to our supposition.

Therefore $y^2(b) \le y^2(a)$.

i.e. $|y(b)| \le |y(a)|$.

This completes the proof of the theorem.

25.2.1.1 Remark : Geometrically, the view of the Biocher - Osgood theorem will look like as in Fig. 25(i)

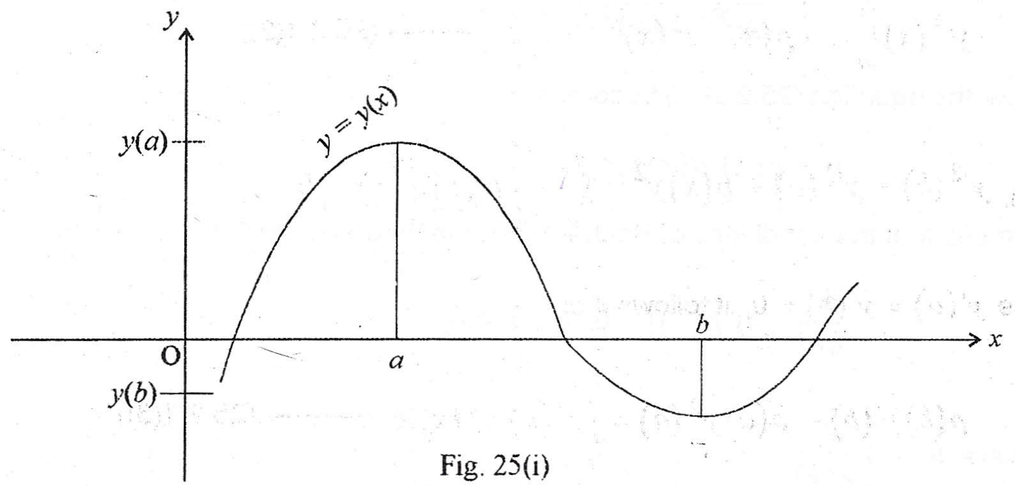

Fig. 25(i)

25.5.2 Theorem (The Bocher - Osgood Theorem for Self-adjoint Differential Equations) :

Suppose that $r(x)$, $p(x)$ are functions of class $\mathscr{C}'$ on $I=[0,\infty)$ and that $p(x)$, and $r(x)$ are positive. If $(r(x)\,p(x))' \ge 0$ on I then for any solution $y(x)$ of

$$(r(x)\,y'(x))' + p(x)\,y(x) = 0 \text{ ------------- } (25.2.2(1))$$

We have $|y(b)| \le |y(a)|$ where $x=a$ and $x=b$ are the consecutive zeros of $y'(x)$ on I.

Proof : We use the transformation

$t = \int_0^x \frac{dx}{r(x)}$. Then

$$\frac{dt}{dx} = \frac{1}{r(x)}; \quad y'(x) = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{1}{r(x)}\frac{dy}{dt}; \text{ i.e. } r(x)\, y'(x) = \frac{dy}{dt}$$

and $$\left(r(x)\, y'(x)\right)' = \frac{d^2 y}{dt^2}\cdot\frac{dt}{dx}$$

$$= \frac{d^2 y}{dt^2}\cdot\frac{1}{r(x)}.$$

Now, the equation (25.2.2(1)) becomes

$$\ddot{y} + r(x)\, p(x) y = 0.$$

We see that the conditions of Bocher - Osgood theorems 25.3.1, will be met if

$$\frac{d}{dt}\left(r(x)\, p(x)\right) \geq 0,\ 0 \leq t \leq T_0$$

where $T_0 = \int_0^\infty \frac{dx}{r(x)} \leq \infty$

We have

$$\frac{d}{dt}\left(r(x)\, p(x)\right) = \left(r(x)\, p(x)\right)' \frac{dx}{dt} = \left(r(x) p(x)\right)' \cdot r(x) \geq 0,$$

since $\left(r(x)\, p(x)\right)' \geq 0$ and $r(x) > 0$.

Hence the conditions of Bocher - Osgood theorem 25.2.1 are satisfied. Therefore the conclusion of the theorem follows.

25.2.1 Note : From theorem 25.2.2, it follows that if $\left[r(x)\, p(x)\right]' \geq 0$ then the solutions of (25.2.2(1)) are bounded. When this inequality is reversed, the solutions are not necessarily bounded. The following result is, however, note worthy.

25.2.3 Theorem : If $r(x)$ and $p(x)$ are positive and of class $\mathscr{C}'$ on I and if $\left(r(x)\, p(x)\right)' \leq 0$, the products $r(x)\, p(x)\, y^2(x)$ and $r^2(x)\, y'^2(x)$, where $y(x)$ is a solution of (25.2.2(1)), are bounded.

Proof : Let y be any solution of (25.2.2(1)). Then $\left(r(x) y'\right)' + p(x) y = 0$.

Multiplying both sides of this equation by $2ry'$ and on integrating for $x \in (a, b)$, we get

$$(r\,y')^2 \Big|_a^x + \int_a^x r\;p\left(y^2\right)'\,dx = 0$$

i.e. $$(r(x)\;y'(x))^2 - (r(a)\;y'(a))^2 + \left(r\;p\,y^2\right)\Big|_a^x - \int_a^x (r(x)\;p(x))'\;y^2(x)\,dx = 0$$

(Here we used integration by parts).

Therefore

$$(r(x)\;y'(x))^2 + r(x)\;p(x)\;y^2(x) = c^2 + \int_a^x (r(x)p(x))'\;y^2(x)\,dx$$

where $c^2 = (r(a)\;y'(a))^2 + r(a)\;p(a)\;y^2(a) > 0$

since $(r(x)\;p(x))' \le 0,$ it follows that each of $(r(x)\;y'(x))^2$ and $r(x)\;p(x)\;y^2(x)$ is less than c^2.

Thus the theorem is proved.

25.3 SHORT ANSWER QUESTIONS

25.3.1 : Show that the Picone formula is true by differentiating the same, by assuming the hypotheses of lemma 25.1.1.

25.4 MODEL EXAMINATION QUESTIONS

25.4.1 : State and prove Picone formula.

25.4.2 : Let y and z be two nonnull solutions of

$$(r(x)\;y')' + p(x)y = 0$$

$(r_1(x)\;y')' + p_1(x)z = 0$ respectively, satisfying the conditions

$$y(a) = 0$$

$$z(a) = z(b) = 0.$$

Here $r(x)$ and $r_1(x)$ are positive, $r(x),\ p(x),\ r_1(x),\ p_1(x)$ are continuous on an interval $[a, b]$

If $\int_a^b \left[(r_1 - r)z'^2 + (p - p_1)\, z^2\right] dx > 0$

then prove that $y(x)$ must have a zero on the interval (a, b).

25.4.3 : State and prove Bocher - Osgood theorem.

25.4.4 : State and prove Bocher - Osgood theorem for selfadjoint differential equations.

25.4.5 : If $r(x)$ and $p(x)$ are positive and of class $\mathscr{C}'$ on $I = [a, b]$ and if $\left(r(x)\ p(x)\right)' \le 0$, then prove that the products $r(x)\ p(x)\ y^2(x)$ and $r^2(x)\ y'^2(x)$ are bounded, where $y(x)$ is a solution of the selfadjoint differential equation

$$\left(r(x)\ y'(x)\right)' + p(x)\ y(x) = 0.$$

25.5 EXERCISES

25.5.1 : Find a differential equation in the form

$$\left(r(x)\ y'\right)' + p(x)y = 0$$

which has the solutions $\frac{1}{x^\alpha}\sin x,\ \frac{1}{x^\alpha}\cos x,\ \alpha$ constant, and show that the Bocher - Osgood theorem for self-adjoint equations (Theorem 25.2.2) applies when $\alpha \ge 1$. Also, show that the conclusion of theorem 25.2.2 is valid for the function

$\frac{1}{x^\alpha}\sin(x - \alpha)$, for $\alpha \ge 0$.

Answer : $\left(x^{2\alpha}\ y'\right)' + \left[x^{2\alpha} + \alpha(\alpha - 1)x^{2\alpha - 2}\right]y = 0$

25.5.2 : Find a lnear differential equation of third crder that is satisfied by the square of an arbitrary solution of equation

$$y'' + p(x)y = 0$$

Assume that $p(x)$ is of class $\mathscr{C}'$

Answer : $z''' + 4pz' + 2p'z = 0$

25.6 ANSWERS TO SHORT ANSWER QUESTIONS

25.3.1 : Since y and z are solutions of (25.1.1(1)) and (25.1.1(2)) respectively, we have

$$r(x)\, y'' + r'(x)\, y' + p(x)\, y = 0 \quad \text{------------- } 25.6(1)$$

$$r_1(x)\, z'' + r_1'(x)\, z' + p_1(x)\, z = 0 \quad \text{------------- } 25.6(2)$$

Let us now consider 25.1.1(3).

$$\int_a^x \left[(r_1 - r) z'^2 + (p - p_1) z^2 \right] dx + \int_a^x r \left[\frac{y z' - y' z}{y} \right]^2 dx$$

$$= \left[\frac{z}{y} (r_1 y z' - r\, y'\, z) \right]_a^x$$

On differentiating the above equation, we get

$$(r_1 - r) z'^2 + (p - p_1) z^2 + r \left(\frac{yz' - y'z}{y} \right)^2 = \left[\frac{z}{y} (r_1 y z' - r\, y'\, z) \right]$$

i.e. $$r_1\, z'^2 - r\, z'^2 + p\, z^2 - p_1 z^2 + r z'^2 + \frac{r\, y'^2\, z^2}{y^2} - 2\, \frac{r y' z z'}{y}$$

$$- r_1\, z\, z' + \frac{r\, y'\, z^2}{y} = 0$$

and on simplifying, we see that

$$z\left(r_1\, z'' + r_1' z' + p_1 z\right) - \frac{z^2}{y}\left(r\, y'' + r' y' + p y\right) = 0\,;$$

which is true, by using (25.6(1)) and (25.6(2)).

25.7 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Lesson - 26

OSCILLATION ON A HALF - AXIS

26.0 INTRODUCTION

In this lesson, we develop a fundamental theorem that will provide a sufficient condition for the solutions of an equation.

$$(r(x)y')' + p(x)y = 0$$

that vanish infinitely often on the interval $0<x<\infty$.

26.1 A SPECIAL PAIR OF SOLUTIONS

Consider the differential equation

$$(r(x)y')' + p(x)y = 0, \text{ ------------ } (26.1.(1))$$

where $r(x)>0$, and $r(x)$ and $p(x)$ are continuous on some interval $I=[a, \infty)$.

We recall that if $y_1(x)$ is a solution of (26.1(1)), a second linearly independent solution $y_2(x)$ of (26.1(1)) is given by (Lesson 20, section 3)

$$y_2(x) = y_1(x)\int_0^x \frac{dx}{r(x)\, y_1^2(x)}.$$

For each zero $x=x_0$ of $y_1(x)$, we define

$$y_2(x_0)= \lim_{x\to x_0}\left[y_1(x)\int_a^x \frac{dx}{s(x)y_1^{\,2}(x)}\right].$$

The solution $y_2(x)$ will then be defined for all $x\geq a$.

These results are needed in our subsequent discussion.

26.2 OSCILLATION ON A HALF-AXIS

26.2.1 Theorem : Let $r(x)$ be positive, and suppose that $r(x)$ and $p(x)$ are continuous on the interval $0<x<\infty$. If the two improper integrals

$$\int_1^\infty \frac{dx}{r(x)} = +\infty, \quad \int_1^\infty p(x)dx = +\infty, \quad \text{------------ (26.2.1(1))}$$

then every solution $y(x)$ of (26.1(1)) vanishes infinitely often on the interval $1<x<\infty$. Similarly, if the integrals

$$\int_0^1 \frac{dx}{r(x)} = +\infty, \quad \int_0^1 p(x)\,dx = +\infty, \quad \text{---------------- (26.2.1(2))}$$

every solution of (26.1.1(1)) vanishes infinitely often on the interval.

Proof : Assume that some solution $y(x)$ of (26.1(1)) is nonoscillatory, then by the Sturm separation theorem, all (nontrivial) solutions are nonoscillatory, and there exists a real number $a>1$ such that $y(x) \neq 0$ on $[a, \infty)$.

Put $z(x) = \dfrac{r(x)\,y'(x)}{y(x)}, \quad a \le x < \infty.$

Then $$z'(x) = \frac{\left(r(x)\,y'(x)\right)'}{y(x)} - \frac{r(x)\,y'^2(x)}{y^2(x)}$$

$$= -p(x) - \frac{z^2(x)}{r(x)} \quad \text{(since } y(x) \text{ is a solution of 26.1(1))}$$

Therefore z satisfies

$$z'(x) + \frac{z^2(x)}{r(x)} + p(x) = 0. \quad \text{--------------- (26.2.1(1))}$$

Integrating from a to x $(x>a)$, we get

$$z(x) + \int_a^x \frac{z^2(x)}{r(x)} dx = z(a) - \int_a^x p(x)dx \text{ ------------ } (26.2.1(2))$$

For x sufficiently large, say $x \geq b > a,$ the right hand member of (26.2.1(2)) is negative Accordingly, $z(x)$ is negative, for $x \geq b$, and

$$z^2(x) > \left[\int_a^x \frac{z^2(x)}{r(x)} dx\right]^2 \text{ --------------- } (26.2.1(3))$$

Write $I(x) = \int_a^x \frac{z^2(x)}{r(x)} dx$

Thus from (26.2.1(3)), we have

$$r(x)\, I'(x) > I^2(x), \quad x \geq b.$$

Thus $\int_b^x \frac{I'(x)}{I^2(x)} dx \geq \int_b^x \frac{dx}{r(x)}.$

i.e. $\frac{1}{I(b)} > \frac{1}{I(x)} + \int_b^x \frac{dx}{r(x)}, \quad x > b$

This is a contradiction to the fact that $\int_1^\infty \frac{dx}{r(x)} = +\infty$

Therefore, our assumption is false.

This shows that every solution (26.1(1)) vanishes infinitely often on the interval $(1, \infty)$.

The second part of the theorem can be proved similarly.

26.2.1.1 Example : Consider the differential equation

$$y'' + a^2 y = 0 \quad (a \neq 0).$$

It is of the form $(r(x)y')' + p(x)y = 0$ where $r(x) = 1,\ p(x) = a^2$. Here $r(x) > 0$, $r(x)$

and $p(x)$ are continuous on $(0, \infty)$. Further, $\int_1^{\infty} \frac{dx}{r(x)} = +\infty$ and $\int_1^{\infty} p(x)\, dx = +\infty$. Thus the conditions of the theorem 26.2.1 are satisfied. Therefore all the solutions of the given equation vanish infinitely often on $(1, \infty)$. This was already known to us, since a solution of the differential equation is $\sin ax$.

26.2.1.2 Example : Consider the differential equation

$$(x y')' + \frac{1}{x} y = 0.$$

On comparing this equation with (26.1(1)), we have $r(x) = x$ and $p(x) = \frac{1}{x}$. Here we observe that, both the conditions (26.2.1(1)) and (26.2.1(2)) are satisfied. Therefore, by theorem 26.2.1, all solutions of the given differential equation, vanish infinitely often on $(1, \infty)$ and also on $(0, 1)$. It is easy to verify that $\sin \log x$ is a solution of this differential equation on the interval $(0, \infty)$.

26.2.1.3 Remark : If we were to consider the differenital equation

$$y'' + \frac{a^2}{x^2} y = 0 \quad \text{---------------} \quad (26.2.1.3(1))$$

We would note that

$$\int_1^{\infty} \frac{dx}{r(x)} = +\infty, \quad \int_1^{\infty} p(x)\, dx < +\infty$$

The test would fail. Since the differential equation is of Euler type, we may solve it. It will be seen that solutions are <u>oscillatory</u> (that is, have an infinity of zeros) on $(1, \infty)$ when $a^2 > \frac{1}{4}$, and are <u>nonoscillatory</u> when $a^2 \leq \frac{1}{4}$.

The reason the test fails is that for this equation $[r(x) = 1]$, the integral $\int_1^{x} \frac{dx}{r(x)}$ becomes infinite too rapidly. To overcome this difficulty, we transform equation (26.2.1.3(1)) by means of the substitution

$$y = x^{\frac{1}{2}} z$$

and obtaining $(xz')' + \dfrac{a^2 - \frac{1}{4}}{x} z = 0$ --------------- (26.2.1.3(2))

solutions z of (26.2.1.3(2)) will be oscillatory if and only if solutions y of (26.2.1.3(1)) are oscillatory. Theorem 26.2.1 applied to (26.2.1.3(2)) yields the result that the solutions of (26.2.1.3(2)) and hence those of (26.2.1.3(1)) are oscillatory if $a^2 > \frac{1}{4}$. If $a^2 = \frac{1}{4}$, all solutions of (26.2.1.3(2)) and of (26.2.1.3(1)) are nonoscillatory. It follows then from the Sturm comparison theorem that all solutions of both equations also are nonscillatory when $a^2 < \frac{1}{4}$.

It is frequently helpful, if the test of theorem 26.2.1 fails to appply to an equation in the form (26.1.(1)) we try the substitution.

$$y = \left(\frac{x}{r}\right)^{\frac{1}{2}} z$$

and to attempt to apply the test given by the theorem to the resulting differenital equation in z.

For completeness of the theory , we add the following result.

26.2.2 Theorem : Every nonnull solution of equation (26.1(1)) has at most a finite number of zeros on the interval $a \le x < \infty$, if

$$\int_a^\infty \frac{dx}{r(x)} < +\infty \text{ and } \left|\int_a^x p(x)\, dx\right| < M,\ a \le x \le \infty$$

where M is any positive constant.

The proof is beyong the scope and is omitted.

26.3 OSCILLATIONS OF THE BESSEL EQUATION

Consider the Besel's equation

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0 \text{ ------------------ } (26.3.(11))$$

where n is a constant. In this section, we show that every solution $y(x)$ of (26.3.(1)) is *oscillatory on* $(1, \infty)$.

Let $y(x)$ be a solution of (26.3.(1))

Put $y(x) = u(x)\, v(x)$ in (26.3(1)), where u and v are twice differentiable functions. Then

$$x^2\left(u v'' + 2u' v' + u'' v\right) + x\left(u v' + u' v\right) + \left(x^2 - n^2\right)u v = 0$$

This implies

$$x^2 u v'' + \left(2x^2 v' + x v\right)u' + \left(x^2 v'' + x v' + \left(x^2 - n^2\right)v\right)u = 0 \quad \text{----------- (26.3(2))}$$

If $2x^2 v' + x v = 0$ then $2x v' + v = 0$

This implies $\dfrac{dv}{v} = -\dfrac{1}{2x}dx$

Thus $v(x) = \dfrac{1}{\sqrt{x}}$ --------------- (26.3(3))

Hence $v'(x) = -\dfrac{1}{2x}\, v(x)$

$$v''(x) = -\frac{1}{2}\left[\frac{x v'(x - v(x))}{x^2}\right]$$

$$= -\frac{1}{2}\left[\frac{-\frac{1}{2} v(x) - v(x)}{x^2}\right]$$

$$-\frac{1}{2x^2}\left[-\frac{3}{2} v(x)\right] = \frac{3}{4x^2} v(x)$$

substituting these values in (26.3(2)), we get

$$x^2 vu'' + \left(\frac{3}{4}v(x) - \frac{1}{2}v(x) + \left(x^2 - n^2\right)v\right)u = 0.$$

This implis

$$x^2 u'' + \left(\frac{1}{4} + \left(x^2 - n^2\right)\right)u = 0$$

$$\text{i.e., } u'' + \left(\frac{1}{4x^2} + 1 - \frac{n^2}{x^2}\right)u = 0$$

$$\text{i.e. } u'' + \left(1 + \frac{1-4n^2}{4x^2}\right)u = 0 \text{ ------------------ } (26.3(4))$$

Now our aim is to show that every nonnull solution of (26.3(4)) has infinitely many positive zeros.

The equation (26.3(4)) is of the form (26.1.(11)) with $r(x) = 1$, and

$$p(x) = \left(1 + \frac{1-4x^2}{4x^2}\right).$$

Clearly, $\int_1^{\infty} r(x)\, dx = +\infty$

Now, $\int_1^{\infty} \phi(x) dx = \lim_{x\to\infty} \int_1^{x} \left(1 + \frac{1-4x^2}{4t^2}\right) dt$

$$= \lim_{x\to\infty} \left\{(x-1) + \frac{\left(1-4x^2\right)}{4}\left(-\frac{1}{t}\right)\Bigg|_1^x\right\}$$

$$= \lim_{x\to\infty} \left\{(x-1) + \frac{\left(1-4n^2\right)}{4}\left(1 - \frac{1}{x}\right)\right\}$$

$= +\infty$.

Thus all the hypothes of Theorem 26.2.1 is satisfied and hence every solution $u(x)$ of (26.3(4)) is oscillatory.

Now, since $y(x) = v(x)\, u(x)$

$$= \frac{1}{\sqrt{x}}\, u(x)$$

and $\frac{1}{\sqrt{x}}$ is nonoscillatory and $u(x)$ is oscillatory, it follows that $y(x)$ is oscillatory.

Thus every solution of Bessel's equation is oscillatory.

26.4 SHORT ANSWER QUESTIONS

26.4.1 : Prove the final statement of theorem 26.2.1.

26.4.2 : Showt hat $\sin(\log x)$ is a solution of

$$(x y')' + \frac{1}{x} y = 0 \text{ on } (0, \infty).$$

26.5 MODEL EXAMINATION QUESTIONS

26.5.1 : Let $r(x)$ be positive, and suppose that $r(x)$ and $p(x)$ are continuous on the inteval $0 < x < \infty$. If the two improper integrals

$$\int_1^\infty \frac{dx}{r(x)} = +\infty, \quad \int_1^\infty p(x)\, dx = +\infty$$

then show that every solution $y(x)$ of

$$(r(x)\, y')' + p(x)\, y = 0$$

vanishes infinitely often on the interval $0 < x < \infty$.

26.5.2 : prove that all solutions of $y'' + a^2 y = 0$ $(a \neq 0)$ are oscillatory on $(1, \infty)$.

26.5.3 : Showt hat the substitution $y=u(x)z$ in the differential equation

$$(r(x)y')' + u\left[(ru')' + pu\right]z = 0$$

26.5.4 : Test the equation

$$xy'' + \left(1+\frac{1}{\log x}\right)y' + y = 0$$

for oscillation of solutions on the interval $(2, \infty\}$.

26.5.5 : Show that all solutions of the equation

$\left(x^q\ y'\right)' + x^q\ y = 0$ (q constant) are oscillatory on the interval $(1, \infty)$.

25.5.6 : Prove that every solution of the Bessel's equationj $x^2\ y'' + xy' + \left(x^2 - n^2\right)y = 0,$ n constant is oscillatory.

25.6 EXERCISES

25.6.1 : Show that solutions of the differential equation $x^p\ y'' + k^2\ y = 0$ (p constatn; $k^2 > 0$)

are oscillatory on the interval $(1, \infty)$ if and only if either $p<2$ or $p=2$, $k^2 > \frac{1}{4}$.

Hint : Consider separately the following cases : $p \le 1,\ 1<p<2,\ p=2,\ p>2$.

25.6.2 : Show that the substitution $y=u(x)z$ in the differential equation

$\left(r(x)\ y'\right)' + p(x)y = 0$ yeilds the equation

$$\left(ru^2\ z'\right)' + u\left[(ru')' + pu\right]z = 0\cdot$$

25.6.3 : Show that all solutions of the equation

$\left(x^q\ y'\right)' + x^q\ y = 0$ (q constant) are oscillatory on the interval $(1, \infty)$.

25.6.4 : Test the equation $xy'' + \left(1+\frac{1}{\log x}\right)y' + y = 0$

for oscilaltion of solutions on the interval $(2, \infty)$.

25.6.5 : Prove that if $\int_1^{\infty}\left[x\,p(x) - \frac{1}{4x}\right]dx = +\infty$ the solutions of $y'' + p(x)y = 0$ are oscillatory on $(1, \infty)$. Show that the solutions are monoscillatory on $(1, \infty)$ if $4x^2\,p(x) - 1 \leq 0$ for x large. Assume that $p(x)$ is continuous on $[1, \infty]$.

26.6.6 : Prove that if

$$\int_0^{1}\left[x\,p(x) - \frac{1}{4x}\right]dx = +\infty,$$

the solutions of $y'' + p(x)y = 0$ are oscillatory on $(0, 1]$. Show that if $4x^2\,p(x) - 1 \leq 0$, for x positive and sufficiently small, the solutions are non oscillatory on $(0, 1)$. Assume $p(x)$ is continuous on $(0, 1]$.

26.7 ANSWERS TO SHORT ANSWER QUESTIONS

26.4.1 : The solution is similar to the proof explained in the proof of theorem 26.2.1

26.4.2 : Write $y = \sin(\log x)$. It is trivial to see that this $y(x)$ satisfies.

$$(xy')' + \frac{1}{x}y = 0 \text{ on } (0, \infty)$$

26.8 REFERENCE BOOK

Walter Leighton - An Introduction to the Theory of Ordinary Differential Equations - Wadsworth Publishing company, Inc. 1970.

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Lesson - 27

TWO TRANSFORMATIONS ON A SELF ADJOINT DIFFERENTIAL EQUATION

27.0 INTRODUCTION

It is frequently useful to transform an equation of the type

$$(r(x)y')' + p(x)y = 0 \quad \text{----------} \; 27.0(1)$$

into an equation of the form

$$y'' + p(x)y = 0 \quad \text{------------} \; 27.0(2)$$

Here we assume that $r(x), r'(x), r''(x)$ and $p(x)$ are continuous, and $r(x) > 0$ in $[a, b]$.

In this lesson, we illustrate the two commonly used transformations.

27.1 THE FIRST TRANSFORMATION

Consider the differential equation 27.0(1); i.e.

$$(r(x)y')' + p(x)y = 0.$$

Put $y = u(x)z$, $(u(x) > 0)$ in (27.0(1)).

Then by using $y' = u'z + uz'$

and $y'' = u''z + 2u'z' + uz''$,

from (27.0(1)), we get

$$r'(u'z + uz') + r(u''z + 2u'z' + uz'') + puz = 0.$$

That is

$$ru''z + 2ru'z' + ruz'' + r'u'z + r'uz' + puz = 0.$$

On multiplying by u and simplifying, we get

$$\left(ru^2 z'\right)' + u\left[\left(ru'\right)' + pu\right]z = 0 \text{ --------- } (27.1(1))$$

(see exercise 26.6.2 of lesson 26).

Let $ru^2 = 1$ or $u = \left(r(x)\right)^{-\frac{1}{2}}$.

Then from (27.1(1)), we have

$$z'' + p(x)z = 0 \text{ where } p(x) = u\left[\left(ru'\right)' + pu\right].$$

27.1.1 Remark : In section 27.1, we used the transformation $y = u(x)z$, and it was a transformation of the dependent variable.

27.2 THE SECOND TRANSFORMATION

An equation of the type (27.0(2)) can also be obtained by transforming the independent variable. In this case, we put

$$t = \int_a^x \frac{dt}{r(t)} \text{ ------------ } (27.2(1))$$

Therefore

$$\frac{dt}{dx} = \frac{1}{r(x)}.$$

Since $r(x) > 0$, t is a strictly increasing function of x, and equation (27.2(1)) also defines x as an increasing function of t. We call this function $g(t)$. Then,

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{r(x)} \frac{dy}{dt} = \frac{1}{r(x)} \dot{y}$$

and $$\left(ry'\right)' = \frac{d}{dx}\dot{y} = \frac{d}{dt}(\dot{y}) \frac{dt}{dx} = \ddot{y}\frac{1}{r(x)}.$$

Now the equation (27.0 (1)) becomes

$$\frac{1}{r}\ddot{y}+py=0;$$

i.e. $\ddot{y}+r\,py=0;$

where the product rp in (27.2(2)) means

$$rp=r(g(t))\,p(g(t)).$$

27.3 EXAMPLES

27.3.1 Example : Solve the differential equation

$$x^2y''-2xy'+(2+x^2)y=0,\ (x>0) \text{ ------------ } (27.3.1(1))$$

Solution : The given differential equation is of the form

$$a(x)y''+b(x)y'+c(x)y=0$$

where $a(x)=x^2,\ b(x)=-2x;\ c(x)=2+x^2$.

To put (27.3.1(1)) into self-adjoint form, multiply (27.3.1(1)) by $\frac{1}{a(x)}\exp\left(\int\frac{b(x)}{a(x)}dx\right)$,

i.e. $\frac{1}{x^2}\exp\left(\int-\frac{2x}{x^2}dx\right)=\frac{1}{x^2}\cdot\frac{1}{x^2}=\frac{1}{x^4}$. Then we get

$$\frac{1}{x^2}y''-\frac{2}{x^3}y'+\left(\frac{2+x^2}{x^4}\right)y=0.$$

It can be written as

$$\left(\frac{1}{x^2}y'\right)'+\left(\frac{2+x^2}{x^4}\right)y=0 \text{ ,-----------} (27.3.1(2))$$

which is of the selfadjoint form

$$(r(x)y')'+p(x)y=0,$$

where $r(x)=\frac{1}{x^2}$ and $p(x)=\frac{2+x^2}{x^4}$.

By using a transformation of the dependent variable, that is $y=u(x)z=(r(x))^{-\frac{1}{2}}z=xz$ in (27.3.1(2))

we get $z''+p(x)z=0$, where

$$p(x)=u\left[(ru')'+pu\right]$$

$$=x\left[\left(\frac{1}{x^2}\right)'+\left(\frac{2+x^2}{x^4}\right)\cdot x\right]$$

$$=x\left[-\frac{2}{x^3}+\frac{2}{x^3}+\frac{1}{x}\right]$$

$$=1$$

Thus the transformed equation is

$$z''+z=0.$$

We know that $\sin x$ and $\cos x$ are two linearly independent solutions of this equation. Suppose that $z_1(x)=\sin x$ and $z_2(x)=\cos x$. Thus the two linearly independent solutions of the given equations are

$$y_1(x)=x\sin x;\text{ and } y_2(x)=x\cos x.$$

27.3.1.1 Remark : By applying the second transformation 27.2(1) and proceeding as in section 27.2, we get the transformed equation of (27.3.1(1)) as

$$\ddot{y}+\left(\frac{2+(3t)^{\frac{2}{3}}}{9t^2}\right)y=0 \text{ --------- } (27.3.1.1\ (1))$$

In this example, the transformation $y=u(x)z$ of the dependent variable led to an easy solution of (27.3.1(1)), while (27.3.1.1(1)) appears to be less interesting. In practice, however, both transformations are useful, and either may have advantages over the other in a given situation.

27.3.2 Example : Employ the transformation $t=\int_a^x \frac{dt}{r(t)}$ to solve the differential equation

$$\left(r(x)y'\right)' + \frac{1}{r(x)}y = 0$$

where $r(x)>0$ and continuous on the interval $[a, b]$.

Solution : By substituting $t=\int_a^x \frac{dt}{r(t)}$, proceeding as in the section 27.2, the given differential equation becomes

$$\frac{1}{r}\ddot{y} + \frac{1}{r}y = 0.$$

Since $r>0$, it follows that $\ddot{y} + y = 0$.

We know that its solutions are $y_1(t) = \sin t$, and $y_2(t) = \cos t$, and are linearly independent. Thus, the two linearly independent solutions of the given equation are $y_1(x) = \sin x$ and $y_2(x) = \cos x$ where $x=g(t)$, an increasing function of t

27.4 SHORT ANSWER QUESTIONS

27.4.1 : Transform the equation

$$\left(\frac{1}{x^2}y'\right) + \left(\frac{2+x^2}{x^4}\right)y=0$$

into the form

$$\ddot{y} + \left(\frac{2+(3t)^{\frac{2}{3}}}{9t^2}\right)y = 0$$

27.5 MODEL EXAMINATION QUESTIONS

27.5.1 : By using suitable transformation, transform the equation

$$(r(x)y')' + p(x)y = 0$$

into an equation of the form

$$y'' + p(x)y = 0$$

where r, r', r'' and p are continuous and

$r(x)>0$ on $[a,b]$.

27.5.2 : Solve the differential equation

$$x^2 y'' - 2xy' + \left(2+x^2\right)y = 0 \ (x>0).$$

27.5.3 : Transform the equation

$$x^2 y'' - 2mxy' + \left[m(m+1) + x^2\right]y = 0$$

by means of $y=u(x)z,\ u(x)>0$,

and thus find two linearly independent solutions of the equation.

27.5.4 : Solve the differential equation

$$u'(x)\,z'' - u''(x)\,z' + u'^3(x)\,z = 0$$

where $u'(x)>0$ and $u''(x)$ is continuous on an interval $a<x<b$.

27.6 EXERCISES

27.6.1 : Transform the equation

$x^2 y'' - 2mxy' + \left[m(m+1)+x^2\right]y = 0$ by means of $y=u(x)z,\ u(x)>0$, and thus find two linearly independent solutions of the equation.

Ans : $x^m \sin x,\ x^m \cos x$.

27.6.2 : Solve the differential equation

$$u'(x)z'' - u''(x)z' + u'^3(x)z = 0,$$

where $u'(x)>0$ and $u''(x)>0$ is continuous on an interval $a<x<b$.

Ans : $c_1 \sin u(x) + c_2 \cos u(x)$

27.6.3 : Use (27.1(1)) to solve the differential equation

$$\left(w^2(x)y'\right)' + w(x)\left[w''(x)+w(x)\right]y = 0,$$

where $w(x) \neq 0$ and of class $\mathscr{C}''$ on $[a, b]$.

Ans : $c_1 \dfrac{\sin x}{w(x)} + c_2 \dfrac{\cos x}{w(x)}$.

27.6.4 : If $u'(x)>0$, $w(x)\neq 0$ and $u(x)$ and $w(x)$ are of class $\mathscr{C}''$ on (a, b), show that linearly independent solutions of the differential equation

$$\left(\frac{w^2(x)}{u'(x)}y'\right)' + \left[\left(\frac{w'(x)}{u'(x)}\right)' + u'(x)\,w(x)\right]w(x)y = 0 \text{ are}$$

$$\frac{\sin u(x)}{w(x)} \text{ and } \frac{\cos u(x)}{w(x)}$$

27.6.5 : Show that the linearly independent solutions of the differential equation

$$x^2 y'' + (2m-n-1)x y' + \left[n^2 x^{2n} - n(m-n)\right]y = 0 \quad (x>0)$$

are $\dfrac{\sin x^n}{x^m}$ and $\dfrac{\cos x^n}{x^m}$, provided $n \neq 0$

27.6.6 : Use the result in Exercise 27.6.5 to solve the differential equation

$$x^2 y'' - 4xy' + \left(x^2+6\right)y = 0 \quad (x>0).$$

27.6.7 : Use the result in Exercise 27.6.5 to solve the differential equation

$$x^2 y'' - xy' + \left(4x^4 - 1\right)y = 0 \quad (x>0).$$

27.6.8 : By a suitable choice of the function $u(x)$, transform the differential equation

$$y'' - b(x)y' + c(x)y = 0$$

by means of the substitution

$$y = u(x)z$$

into a differential equation of the form

$$z'' + p(x)z = 0.$$

Here assume $b'(x)$ and $c(x)$ are continuous on an interval $[a, b]$.

27.6.9 : Given the differential equation

$$y'' + b(x)y' + c(x)y = 0 \text{ ---------- (27.6.9(1))}$$

where $b(x)$ and $c(x)$ are continuous on $I = [a,\ b]$, show that every solution that vanishes at a point $x = x_0$ of I is a constant multiple of the solution

$$y_2(x_0)\, y_1(x) - y_1(x_0)\, y_2(x) \text{ ---------- (27.6.9(2))}$$

where $y_1(x)$ and $y_2(x)$ are any linearly independent solutions of (27.6.9(1)). Then considering x_0 as a variable parameter, show that

$$\frac{\partial}{\partial x_0}\left[y_2(x_0)\, y_1(x) - y_1(x_0) - y_2(x)\right] = y_2'(x_0)\, y_1(x) - y_1'(x_0)\, y_2(x)$$

is a solution of (27.6.9(1)) linearly independent of (27.6.9(2)).

27.7 ANSWERS TO SHORT ANSWER QUESTIONS

27.4.1 : The given equation is of the form

$$\left(r(x)y'\right)' + p(x)y = 0$$

where $r(x) = \dfrac{1}{x^2}$ and $p(x) = \dfrac{2 + x^2}{x^4}$.

Put $t = \int_0^x \frac{dt}{r(t)} = \int_0^x t^2 \, dt = \frac{x^3}{3}$

and using the procedure mentioned in section 27.2, we get

$$\frac{1}{r(x)} \ddot{y} + p\, y = 0$$

i.e. $x^2 \ddot{y} + \left(\frac{2+x^2}{x^4} \right) y = 0$

i.e. $\ddot{y} + \left(\frac{2+x^2}{x^6} \right) y = 0$

Hence,

$$\ddot{y} + \left(\frac{2+(3t)^{\frac{2}{3}}}{9t^2} \right) y = 0.$$

REFERENCE BOOK

Walter Leigliton - An introduction to the Theory of Ordinary differential equations - Wardsworth Publishing Company, Inc. 1970.

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Lesson - 28

MORE ON OSCILLATION

28.0 INTRODUCTION

In this lesson we present a necessary and sufficient condition for the solutions of a self-adjoint differential equation

$$\left(r(x)y'\right)' + p(x)y = 0$$

where $r(x)>0$, and $r(x),\ p(x)$ are continuous on the interval $I=[a, \infty)$, to be oscillatory.

28.1 A THEOREM ON OSCILLATION

28.1.1 Theorem : A necessary and sufficient condition that the solutions of

$$\left(r(x)\, y'\right)' + p(x)y = 0, \quad \text{------------- (28.1.1(1))}$$

where $r(x)>0$, and $r(x),\ p(x)$ are continuous on $I=[a, \infty)$, be oscillatory is that there exist a function $u(x)\neq 0$ of class C' on I for which the following conditions hold :

$$\int_a^{\infty} \frac{dx}{ru^2} = +\infty, \quad \int_a^{\infty} u\left[(ru')' + pu\right]dx = +\infty \quad \text{------- (28.1.1(2))}$$

Proof : If $\mathrm{v}(x)$ and $w(x)$ are solutions of (28.1.1(1)) then

$$r\mathrm{v}'' + r'\mathrm{v}' + p\mathrm{v} = 0$$

and $$rw'' + r'w' + pw = 0.$$

Write $u(x) = \sqrt{\mathrm{v}^2(x) + w^2(x)}$. Then

$$ru^3\left[(ru')' + pu\right] = r^2u^3u'' + rr'u^3u' + rpu^4$$

$$= r^2\left(\mathrm{v}^2 + w^2\right)\left(\mathrm{v}'^2 + w'^2 + \mathrm{v}\mathrm{v}'' + ww''\right)$$

$$-r^2(\mathrm{v}\mathrm{v}'+ww')^2+rr'(\mathrm{v}^2+w^2)(\mathrm{v}\mathrm{v}'+ww')+r(\mathrm{v}^2+w^2)^2p$$

$$=r(\mathrm{v}^2+w^2)\mathrm{v}[r\mathrm{v}''+r'\mathrm{v}'+p\mathrm{v}]$$

$$+r(\mathrm{v}^2+w^2)w[rw''+r'w'+pw]$$

$$+r^2\mathrm{v}^2w'^2+r^2\mathrm{v}'^2w^2-2r^2\mathrm{v}\mathrm{v}'ww'$$

$$=r^2(\mathrm{v}^2w'^2+\mathrm{v}'^2w^2-2\mathrm{v}\mathrm{v}'ww')$$ (since v and w are solutions of (28.1.1(1))

$$=[r(\mathrm{v}w'-\mathrm{v}'w)]^2$$

$$=k^2$$

where $k=r(\mathrm{v}w'-\mathrm{v}'w)$.

Thus $u(x)$ is a solution of the differential equation

$$ru^3\left[(ru')'+pu\right]=k^2,\ a\le x<\infty \qquad (28.1.1(3))$$

with $k=r(\mathrm{v}w'-\mathrm{v}'w)$.

Thus, when $\mathrm{v}(x)$ and $w(x)$ are linearly independent, we have $k=rw(\mathrm{v},w)\neq 0$ on I, and thus $u(x)\neq 0$ on I. Hence $\mathrm{v}(x)$ and $w(x)$ cannot have a common zero.

It may be noted conversely that the differential equation (28.1.1(3)) with $k=1$, namely

$$ru^3\left[(ru')'+pu\right]=1,$$

always has a solution $u(x)\neq 0$ on I [replace $\mathrm{v}(x)$ by $\frac{\mathrm{v}(x)}{k}$, for example].

If $v(x)\neq 0$ is any function of class $\mathbb{C}'$ on I, and if we make the substitution $y=u(x)z$ in (28.1.1(1)), then we get the diffrential equation

$$\left(ru^2z'\right)' + u\left[\left(ru'\right)' + pu\right]z = 0 \text{ ----------------- (28.1.1(5)) (see 27.1(1))}$$

If $u(x)$ is such that

$$\left(ru^2\right)u\left[\left(ru'\right)' + pu\right] = 1 \text{ ---------------- (28.1.1(6))}$$

Then two linearly independent solutions of (28.1.1(5)) are

$$\sin\int_a^x \frac{dx}{r(x)u^2(x)}, \quad \cos\int_a^x \frac{dx}{r(x)u^2(x)}.$$

Clearly (28.1.1(6)) is same as (28.1.1(4)). Therefore, the solutions of (28.1.1(1)) are oscillatory on I if and only if the solutions of (28.1.1(5)) are oscillatory on I.

It follows that, if the solutions of (28.1.1(1)) are oscillatory, then there exists a function $u(x)\neq 0$ of class $\mathbb{C}'$ on I such that

$$\int_a^\infty \frac{dx}{r(x)u^2(x)} = +\infty, \text{ and } \int_a^\infty u\left[\left(ru'\right)' + pu\right]dx = +\infty.$$

Conversely, if there exists a function $u(x)\neq 0$ of class $\mathbb{C}'$ on I such that

$$\int_a^\infty \frac{dx}{r(x)u^2(x)} = +\infty, \text{ and } \int_a^\infty u\left[\left(ru'\right)' + pu\right]dx = +\infty$$

then by theorem 26.2.1, the solutions of (28.1.1(5)) are oscilaltory on I. Hence the solutions of (28.1.1(1)) are oscillatory on I.

This completes the proof of the theorem.

28.2 EXAMPLES

28.2.1 Example : Find the criteria for the existence of oscillatory solutions of $y''+p(x)y = 0$, where p is continuous on $[a, \infty)$.

Solution : By comparing the given equation

$$y'' + p(x)y = 0 \text{ ----------------------------- (28.2.1(1))}$$

(7)

with the equation (28.1.1(1)), we have

$r(x)=1$. We take $a=1$.

Then the conditions (28.1.1(1)) become

$$\int_1^{\infty}\frac{dx}{u^2}=+\infty, \text{ and } \int_1^{\infty}u(u''+pu)\,dx=+\infty.$$

Set $u=x^{\alpha}$. Then these conditions become

$$\int_1^{\infty}\frac{dx}{x^{2\alpha}}=+\infty, \text{ and } \int_1^{\infty}\left[\alpha(\alpha-1)x^{2\alpha-2}+x^{2\alpha}p(x)\right]dx=+\infty \text{ ---------- } (28.2.1(2))$$

From the first condition in (28.2.1(2)), we have $2\alpha\le 1$. Therefore, if there exists a constant $\alpha\le\frac{1}{2}$ such that

$$\int_1^{\infty}\left[\alpha(\alpha-1)x^{2\alpha-2}+x^{2\alpha}p(x)\right]dx=+\infty,$$

then by theorem 28.1.1, it follows that the solutions of (28.2.1(1)) are oscillatory.

28.2.1.1 Note : In example 28.2.1, in particular $\alpha=\frac{1}{2}$, then the solutions of $y''+p(x)y=0$ are oscillatory on $[1,\infty)$ if

$$\int_1^{\infty}\left[x\,p(x)-\frac{1}{4x}\right]dx=+\infty. \quad \text{(see exercise 26.6.6)}$$

28.2.2 Example : Show that the solutions of $y''+(\sin x)y=0$ are oscillatory.

Solution : On comparing

$$y''+(\sin x)y=0$$

with (28.1.1(1)), we have $p(x)=\sin x$, and $r(x)=1$. Let us take $a=1$. The conditions (28.1.1(2)) become

$$\int_1^{\infty} \frac{dx}{u^2} = +\infty, \text{ and } \int_1^{\infty} u(u'' + pu)\,dx = +\infty.$$

Set $u(x) = 2 + \sin x$. Then clearly the first condition is satisfied.

The second condition becomes

$$\int_1^{\infty} (2+\sin x)\left[-\sin x + \sin x\,(2+\sin x)\right]dx = +\infty$$

This condition will be satisfied if

$$\lim_{x\to\infty} \int_1^{x} \sin x(1+\sin x)\,dx = +\infty,$$

which is readily verified.

Thus by theorem 28.1.1, the solutions of

$y'' + (\sin x)y = 0$ are oscillatory.

28.2.3 Example : Show that the solutions of the differential equation

$$y'' + \frac{k}{x^3}y = 0 \quad (1 \le x < \infty) \text{ -------(28.2.3(1))}$$

are non oscillatory on $[1, \infty)$

Solution : On comparing (28.2.3) with the equation (28.1.1(1)), we have $r(x) = 1$ and $p(x) = \frac{k}{x^3}$.

By choosing $u(x) = \sqrt{x}$, we have for sufficiently large x,

$u\left(u'' + \frac{k}{x^3}u\right) < 0$, so that

$$\int_1^{\infty} u\left[(ru')' + pu\right]dx = \int_1^{\infty} u\left(u'' + \frac{k}{x^3}u\right)dx < 0.$$

Thus by theorem 28.1.1, solutions of (28.2.3(1)) are nonoscillatory on $[1, \infty)$.

The following theorem is an immediate consequence of (28.1.1(5)).

28.2.4 Theorem : If there exists a positive function $u(x)$ of class $\mathbb{C}''$ on $[a, \infty)$ such that $(ru')' + pu < 0$ for x large, then the solutions of (28.1.1(1)) are nonoscillatory on that interval.

28.3 SHORT ANSWER QUESTIONS

28.3.1 : Show that the solutions of $y'' + (\sin x)y = 0$ are oscillatory.

28.3.2 : Discuss the oscillation of solutions of $y'' + (\cos 2x)y = 0$.

28.4 MODEL EXAMINATION QUESTIONS

28.4.1 : Prove that a necessary and sufficient condition that the solutions of

$$(r(x)y')' + p(x)y = 0$$

where $r(x) > 0$, and $r(x), p(x)$ are continuous on $I = [a, \infty)$, be oscillatory is that there exist a function $u(x) \neq 0$ of class $\mathbb{C}'$ on I for which the following conditions hold :

$$\int_a^\infty \frac{dx}{ru^2} = +\infty, \qquad \int_a^\infty u\left[(ru')' + pu\right]dx = +\infty$$

28.4.2 : Find the criteria for the existence of oscillatory solutions of $y'' + p(x)y = 0$, where p is continuous on $[a, \infty)$.

28.4.3 : Show that the solutions of $y'' + (\sin x)y = 0$ are oscillatory.

28.4.4 : Show that the solutions of the differential equation

$$y'' + \frac{k}{x^3}y = 0, \quad (1 \le x < \infty),$$

k is a positive constant, are nonoscillatory on $[1, \infty)$.

28.5 EXERCISES

28.5.1 : Show that the solutions of the differential equation

$$y'' + \left[\frac{1}{4x^2} + \frac{k}{x^2 \log^2 x}\right] y = 0 \ (c \le x \le \infty)$$

are oscillatory for $k > \frac{1}{4}$ and nonoscillatory for $k \le \frac{1}{4}$.

28.5.2 : Show that the solutions of $\left(x^{\beta} y'\right)' + k^2 x^{\beta} y = 0$ are oscilaltory on $[1, \infty)$ for every constant β, and each $k \ne 0$.

28.6 ANSWERS TO SHORT ANSWER QUESTIONS

28.3.1 : Solutions of example 28.2.2.

28.3.2 : The solution will be like that of examples 28.2.2. Verify the condition (28.1.1(2)) of theorem 28.1.1 with suitable u.

28.7 REFERENCE BOOK

Walter Leigliton - An introduction to the Theory of Ordinary differential equations - Wardsworth Publishing Company, Inc. 1970.

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Lesson - 29

LIAPUNOV'S INEQUALITY AND GRONWALL'S INEQUALITY

29.0 INTRODUCTION

In this lesson we prove famous inequalities namely Liapunov's inequality and Gronwalls' inequality.

29.1 LIAPUNOV'S INEQUALITY

29.1.1 Theorem (Liapunov's Inequality) : If $p(x)$ is continuous on the interval $I=[a, b]$ and if there exists a solution $y(x)\not\equiv 0$ of the differential equation

$$y'' + p(x)y=0 \quad \text{-------------} \quad (29.1.1(1))$$

that has two zeros on I, then

$$\int_a^b p_+(x)dx \geq \frac{4}{b-a} \quad \text{---------} \quad (29.1.1(2))$$

where $p_+(x) = \frac{1}{2}\left[p(x)+|p(x)|\right]$.

Proof : Consider the differential equation

$$y''+p_+(x)y = 0 \quad \text{----------} \quad (29.1.1(3))$$

We have $p_+(x)\geq p(x)$ on $[a, b]$.

Suppose that there exists a solution $y(x)$ of (29.1.1(3)) such that

$$y(a)=0=y(c)$$

with $y(x)>0$ on $[a, c]\subset I$,

Since $y'' = -p_+(x)y \le 0$ on (a, c), let $x=t$ be a point between a and c at which $y(x)$ attains its absolute maximum. (We note that since $y''(x) \le 0$ on (a, c), $y(x)$ possesses no relative minima on that (open) interval; it may posses a line of relative maxima, however). Draw the two chords that connect the point $p(t, y(t))$ wtih the points $(a,0)$ and $(c,0)$, as shown in the Fig. 29(i).

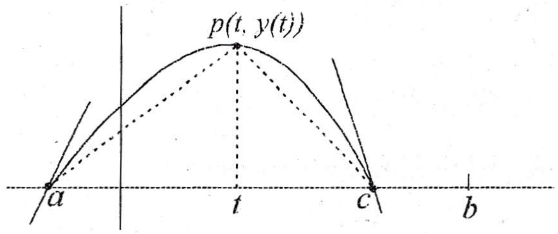

Fig. 29(i)

Then

$$y'(a) > \frac{y(t)}{t-a} \text{ and } -y'(c) > \frac{y(t)}{c-t} \ge \frac{y(t)}{b-t}$$

It follows that

$$y'(a) - y'(c) > \frac{y(t)}{t-a} + \frac{y(t)}{b-t}.$$

That is

$$-\frac{y'(c) - y'(a)}{y(t)} > \frac{b-a}{(t-a)(b-t)}.$$

Thus

$$-\frac{1}{y(t)}\int_a^c y''(x)\,dx = \frac{1}{y(t)}\int_a^c p_+(x)\,y(x)\,dx > \frac{b-a}{(t-a)(b-t)}.$$

Since $y(x) < y(t)$ for all $x \in (a, c)$, we have

$$\frac{1}{y(t)}\int_a^c p_+(x)\,y(x)\,dx < \int_a^c p_+(x)\,dx$$

Thus we have, from the above two inequalities

$$\int_a^b p_+(x)\,dx \geq \int_a^c p_+(x)\,dx > \frac{b-a}{(t-a)(b-t)}.$$

This implies

$$\int_a^b p_+(x)\,dx > \frac{4}{b-a}, \text{ since } \frac{b-a}{(t-a)(b-t)} \geq \frac{4}{b-a};$$

with the equality only if t is the mid-point of (∞, b).

This completes the proof of Liapunov's inequality.

29.2 EXTENSION OF LIAPUNOV'S INEQUALITY TO THE GENERAL SELF-ADJOINT EQUATION

In this section, we extend Lipaunov's inequatlity to the general self-adjoint differential equation

$$\left(r(x)y'\right)' + p(x)y = 0 \text{ --------- } (29.2(1))$$

where $r(x)>0$, and $r(x)$, $p(x)$ are continuous on I.

Put $t = \int_a^x \frac{dx}{r(x)}$ in (29.2(1)), then we get

$$\ddot{y} + r(x)p(x)y = 0. \text{ -------- } (29.2(2))$$

If a solution $y(x)$ of (29.2(1)) has consecutive zeros $x=a$ and $x=c$, then there is a solution $y_1(t)$ of (29.2(2)) with consecutive zeros at $t=0$ and at $t=t_0$, where $t_0 = \int_a^c \frac{dx}{r(x)}$.

According to theorem 29.1.1 applied to (29.2(2)) we get

$$\int_0^{t_0} r(x)p_+(x)\,dt > \frac{4}{t_0}.$$

Evaluation of this integral by means of the substitution $t = \int_a^x \frac{dx}{dx}$, yields

$$\int_a^c \frac{dx}{r(x)} \int_a^c p_+(x)dx > 4 \quad \text{---------- (29.2(3))}$$

This is the Liapunov's inequality for the self-adjoint differential equation (29.2(1)).

29.2.1 Note : The number 4 in (29.1.1(2)) and (29.2(3)) is a "best" constant, in the sense that it cannot be replaced by a larger number.

29.3 GRONWALL'S INEQUALITY

In this section we develop a generalization and an extension of what is known as Gronwall's inequality. This inequality is frequently used to establish the uniqueness of a solution of a system of differential equations.

29.3.1 Theorem (Gronwall's inequality) : Suppose that $u(t)$ and $v(t)$ are continuous and $c(t)$ is of class $\mathbb{C}'$ on an interval $I=[a, b]$.

If $v(t) \geq 0$ and $c'(t) \geq 0$ on I, and if

$$u(t) \leq c(t) + \int_a^t v(t)\, u(t) dt \qquad (a \leq t \leq b), \quad \text{------ (29.3.1(1))}$$

then $u(t) \leq c(t) \exp \int_a^t v(t)\, dt$ ---------- (29.3.1(2))

when $c(t)$ is a constant, a (29.3.1(2)) becomes Gronwalls' inequality.

Proof : Set $q(t) = \int_a^t v(t)\, u(t)\, dt$.

then $q'(t) = v(t)\, u(t) \leq v(t) c(t) + v(t) q(t)$ ------- (29.3.1(3)) (by (29.3.1(1))

Thus $q'(t) - v(t) q(t) \leq c(t) v(t)$.

Multiplying both sides of this inequality by $\exp\left[-\int_a^t v(t) dt\right]$ (and changing t to s) we have

$$\frac{d}{ds}\left\{q(s)\exp\left[-\int_a^s v(t)\,dt\right]\right\} \le c(s)v(s)\exp\left[-\int_a^s v(t)\,dt\right]$$

Integrating both sides of this inequality from a to t, we get

$$q(s)\exp\left[-\int_a^s v(t)\,dt\right]_{s=a}^{s=t} \le \int_a^t c(s)v(s)\exp\left[-\int_a^s v(t)\,dt\right]ds$$

That is

$$q(t) \le \int_a^t c(s)v(s)\exp\left[\int_s^t v(\tau)\,d(\tau)\right]ds, \text{ since } q(a)=0$$

Using integration by parts and applying (29.3.1(1)), we have

$$u(t) \le c(a)\exp\left[\int_a^t v(t)\,dt\right] + \int_a^t c'(s)\exp\left[\int_s^t v(t)\,dt\right]ds \text{ -------- (29.3.1(4))}$$

Since $c'(s)\ge 0$ on I, it follows that

$$u(t)\le c(a)\exp\left[\int_a^t v(t)\,dt\right] + \int_a^t c'(s)\exp\left[\int_a^t v(t)\,dt\right]ds \le c(t)\exp\int_a^t v(t)\,dt.$$

This computes the proof of the theorem.

29.3.2 Theorem : If $v(t)\ge 0,\ c'(t)\le 0$ on I, and if

$$u(t)\ge c(t)+\int_a^t v(t)\,u(t)\,dt, \text{ then}$$

$$u(t)\ge c(t)\exp\int_a^t v(t)\,dt \text{ ------- (29.3.2(1))}$$

If $c(t)$ is a constant c, then

$$u(t) \geq c \exp \int_a^t v(t)\, dt .$$

Proof : The proof is strictly analogous to the proof of theorem 29.3.1. In place of (29.3.1(3)) we have

$$q'(t) \geq u(t)c(t) + v(t)\, q(t)$$

The inequality (29.3.1(4)) becomes

$$u(t) \geq c(a) \exp\left[\int_a^t v(t)\, dt\right] + \int_a^t c'(s) \exp\left[\int_s^t v(t)\, dt\right] ds,$$

and because $v(t) \geq 0$ and $c'(t) \leq 0$ on I, we have

$$u(t) \geq c(a) \exp\left[\int_a^t v(t)\, dt\right] + \int_a^t c'(s) \exp\left[\int_a^t v(t)\, dt\right] ds,$$

and (29.3.2(1)) is then immediate.

29.4 AN EXAMPLE

29.4.1 Example : Suppose that a function $u(t)$ satisfies the inequality

$$u(t) \leq t + \int_0^t u(t)\, dt, \qquad t \geq 0 \text{ --------- } (29.4.1(1))$$

Then prove that

$$e^t - 1 \leq t\, e^t, \qquad t \geq 0 .$$

Solution : On comparing (29.4.1(1)) with (29.3.1(1)), we have $c(t) = t,\ t \geq 0;\ v(t) = 1,\ t \geq 0$. Thus by using Gronwall's inequality (29.3.1(2)), we have

$$u(t) \leq c(t) \exp\left(\int_0^t v(t)\, dt\right),$$

(i.e.) $u(t) \le t \exp\left(\int_0^t dt\right) = t\, e^t$ ------------- (29.4.1(2))

Observe that the solution $e^t - 1$ of the differential equation $u' - u = 1$. Clearly satisfies (29.4.1(1)). Thus substituting $u(t) = e^t - 1$ in (29.4.1(2)), we get

$$e^t - 1 \le t\, e^t,\ t \ge 0.$$

29.5 MODEL EXAMINATION QUESTIONS

29.5.1 : State and prove Liapunov's inequality.

29.5.2 : State and prove Gronwall's inequality.

29.5.3 : Suppose that a function $u(t)$ satisfies the inequality.

$$u(t) \le t + \int_0^t u(t)\,dt, \quad t \ge 0$$

Then prove that

$$e^t - 1 \le t\, e^t, \quad t \ge 0$$

29.6 REFERENCE BOOK

Walter Leigliton - An introduction to the Theory of Ordinary differential equations - Wardsworth Publishing Company, Inc. 1970.

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