TOPOLOGY AND FUNCTIONAL ANALYSIS (DM21)

(MSC MATHEMATICS)

ACHARYA NAGARJUNA UNIVERSITY

CENTRE FOR DISTANCE EDUCATION

NAGARJUNA NAGAR,

GUNTUR

ANDHRA PRADESH

LESSON - 1

PRELIMINARIES

Prerequisites: Notation and set theoretic preliminaries:

We do not attempt to define a set and assume familiarity with the idea of a set. We understand that a set consists of elements. These are objects which make up the set. The relationship between a set A and an element x of A is described variously.

x_0 belongs to A – x is in A – x is an element of A – A contains x.

This situation is described by the symbol $x \in A$.

A set is also described in the form $\{ x : \ldots\ldots \}$ or $\{x / \ldots.\}$

The relationship x does not belong to A is denoted by $x \notin A$.

We assume the existence of one and only one set without elements in it. This set is called the empty set and is denoted by ϕ.

A set with one element only, namely a is called a singleton set and is denoted by $\{a\}$. The set of natural numbers is denoted by N :

$N = \{1, 2, 3, 4, \ldots.\}$

If $n \in N$, I_n stands for the set $\{1, 2, \ldots..n\}$

If A, B are sets we say that A and B are equal, in symbols $A = B$ if they have the same elements : $A = B$ if and only if "$x \in A \Leftrightarrow x \in B$"

A is called a subset of B equivalently, A is contained in B iff $x \in A \Rightarrow x \in B$.

In this case we write $A \subseteq B$. If $A \subseteq B$ and $A \neq B$ we say that A is a proper subset of B and write $A \underset{\neq}{\subset} B$ or simply $A \underset{\neq}{\subset} B$.

If $A \subseteq B$ we also write $B \supseteq A$. $A \underset{\neq}{\subset} B$ is also represented by $B \underset{\neq}{\supset} A$.

If A, B are sets "A union B" in symbols $A \cup B$ is defined by

$A \cup B = \{ x : x \in A \text{ or } x \in B\}$

A intersection B, in symbols $A \cap B$ is defined by

$A \cap B \setminus \{x : x \in A \text{ and } x \in B\}$

The complement of a set B in a set A, in symbols, A - B (or A \ B) is defined by

$A - B = \{x : x \in A \text{ and } x \notin B\}$

Note that there is no need to assume that $B \subseteq A$ to define A - B. Most of the times we consider complements of subsets of a fixed set X. In this situation, i.e. when $A \subseteq X$ we write A^I for X - A.

Indexed Set:

Suppose that I is a set, A is a set and corresponding to each $i \in I$ we are "given" a definite element $x_i \in A$. What we mean by "given" and "how" it is "given", we do not ask.

In this situation we say that $\{x_i\}_{i \in I}$ is an indexed collection of elements from A or $\{x_i\}_{i \in I}$ is an indexed family of elements from A or $\{x_i\}_{i \in I}$ is an indexed set of elements of A. It might happen for some $i \in I$ and $j \in I$, $i \neq j$, $x_i = x_j$. If A is a set and n is a positive integer, the indexed set from A, $\{x_i\}_{i \in I_n}$ is also denoted by $\{x_1, x_2, \ldots x_n\}$ or $\{x_i / 1 \leq i \leq n\}$. Where $I_n = \{1, 2, \ldots, n\}$. The indexed set $\{x_i\}_{i \in \mathbb{N}}$ from A is also represented by $\{x_n / n \in \mathbb{N}\}$ or $\{x_n / n \geq 1\}$ or $\{x_1, x_2, x_3 \ldots\}$ or $\{x_1, x_2, \ldots x_n \ldots\}$. A note of caution to be remembered is that in general. $\{x_1, \ldots x_n\} \neq \{x_1, x_2, \ldots x_n \ldots\}$

We must also keep in mind that the elements in either of the above sets need not be distinct. For example when n > 1,

$\{(-1)^1, (-1)^2, \ldots (-1)^n\} = \{-1, 1\}$ and

$\{(-1)^n\}_{n \geq 1} = \{-1, 1\}$

Arbitrary unions, Arbitrary intersections:

We will be dealing with sets whose elements are subsets of a given set, say X. We use the words "class", "family" for such sets. The class of all subsets of X is called the power set of X and is denoted by P (X).

Suppose X is a set, **C** is a class of subsets of X indexed by a set I : **C** = $\{A_i\}_{i \in I}$. By the union of the sets in **C** or by the union of the indexed family $\{A_i\}_{i \in I}$ we mean the set $\{x : x \in A_i$ for some $i \in I\}$. This set is denoted by $\bigcup_{i \in I} A_i$ or $\bigcup_{A \in C} A$.

When n is a positive integer and **C** = $\{A_1, \ldots A_n\}$ $\bigcup_{A \in C} A$, $\bigcup_{i \in I_n} A_i$, $\bigcup_{i=1} A_i$ and $A_1 \cup A_2 \ldots \cup A_n$ – all these symbols are used for the same set namely the union of the sets in **C**. When the index set I = ℕ, the symbols $\bigcup_{A \in C} A$, $\bigcup_{i \in N} A_i$, $\bigcup_{n=1}^{\infty} A_i$ represent the union of the sets in **C**.

By the intersection of the sets in **C**or the intersection of the indexed family $\{A_i\}_{i \in I}$ we mean the set $\{x : x \in A_i$ for every $I \in I\}$.

This set is denoted by $\bigcap_{i \in I} A_i$ or $\bigcap_{A \in C} A$.

When n is a positive integer and **C** = $\{A_1, \ldots A_n\}$

$\bigcap_{A \in C} A$, $\bigcap_{i \in I_n} A_i$, $\bigcap_{i=1}^{n} A_i$ and $A_1 \cap A_2 \cap \ldots \cap A_n$ – all these symbols are used for the same set namely the intersection of the sets in **C**. When the index set I = ℕ, the symbols $\bigcap_{A \in C} A$, $\bigcap_{i \in N} A_i$, $\bigcap_{n=1}^{\infty} A_i$ represent the intersection of the sets in **C**.

In these definitions we have not said whether we assume the index set I to be non empty or not. There is no need to assume that I is non-empty. Let us consider the situation, when I = ϕ. An element $x \in X$ is an element of $\bigcup_{i \in \phi} A_i$ if and only if $x \in A_i$ for some $i \in \phi$. However there is no i in the empty set. Therefore $\bigcup_{i \in \phi} A_i = \phi$

In a similar way an element $x \in X$ is in $\bigcap_{i \in \phi} A_i$ if and only if $x \in A_i$ for every $i \in \phi$. Equivalently $x \notin \bigcap_{i \in \phi} A_i$ if and only if $x \notin A_i$ for some $i \in \phi$. However there is no i in the empty set. Therefore $\bigcap_{i \in \phi} A_i = X$

These properties are also in accordance with the natural laws:
The bigger the index set the bigger the union and smaller the intersection and the smaller the index set, the smaller the union and bigger the intersection.

A set A is said to be finite if either $A = \phi$ or for some positive integer n, A is in one to one correspondence with I_n, that is there is a bijection from A onto I_n. In this case we say that A is a n-element set and write $A = \{a_1, a_2, \ldots.. a_n\}$ where the a_i are precisely the elements of A. A is said to be infinite if it is not finite. A is called countably infinite if A is in one to one correspondence with the set of natural numbers $\mathbb{N}$ i.e. there is a bijection from A onto $\mathbb{N}$. A set which is either finite or countably infinite is called countable. A set which is not countable is called uncountable.

We list out a few properties of countable and uncountable sets without proofs.

Facts:

1. A subset of a finite set is finite set.
2. If A, B are finite sets so is $A \cup B$.
3. If n is a positive integer and A_i is a finite set for each i, $1 \le i \le n$ then so is $\bigcup_{i=1}^{n} A_i$.
4. A set A is finite iff there is a positive integer m and an injection from A into I_m.
5. A set A is finite iff there is a positive integer m and a surjection from I_n onto A.
6. A is countable iff there is a surjection from $\mathbb{N}$ to A.
7. A is countable iff there is an injection from A into $\mathbb{N}$.
8. A subset of a countable set is a countable set.
9. If $\{A_i\}_{i \in \mathbb{N}}$ is a collection of sets where A_i is a countable $\forall\ i \in N$ then $\bigcup_{i \in N} A_i$ is a countable set.

1.5 Functions: A function consists of three objects: Two nonempty sets X, Y and a "rule" f which assigns to each x in X a unique y in Y. this y determined by x and f is denoted by f(x) and is called the image of x under f or the value of f at x. f is called a mapping or transformation or operator. X is called the domain of the function and $f(X) = \{f(x)/x \in X\}$ is called the range of the function. A function whose range consists of a single element is called a constant function.

A function determined by sets X and Y and a rule f which assigns to each x in X a unique y in Y is usually denoted by $f : X \to Y$ or $X \xrightarrow{f} Y$. Some times it is also represented by $x \to f(x)$ without specifically mentioning x, y. It is also customary to represent a function by its rule f alone without mentioning X and Y or without mentioning Y.

A function f is called an extension of a function g and g is called a restriction of f if domain of f contains domain of g and f(x) = g(x) for all x in dom g.

If $f: X \to Y$ is a function for each subset $B \subseteq Y$ we define the preimage of B under f or inverse image of B under f as the set $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. If $y \in Y$, we write $f^{-1}(y)$ for $f^{-1}(\{y\})$. Clearly $f^{-1}(B) = \bigcup_{y \in B} f^{-1}(y)$. It is possible that $f^{-1}(B) = \phi$

If $f : X \to Y$ is a bijection then $\forall\, y \in Y$, $f^{-1}(y)$ is a singleton set. Thus we have a rule which assigns to each y in Y a unique dement in X namely $f^{-1}(y)$. This rule $f^{-1} : Y \to X$ is called the inverse of f.

If $f: X \to Y$ and $g : Y \to Z$ are mappings we define the product, also called composition $g \circ f : X \to Z$ by $(g \circ f)(x) = g(f(x))$ for $x \in X$.

For $f : X \to X$ and $n \in N$ we define f^n by $f \circ f^{n-1}$ where f^0 is the identity map $i_X = I$ defined by $i_X(x) = x\ \forall\, x \in X$.

Products:

Let A be a non empty set and n a positive integer, $n \geq 2$. An ordered n –tuple is a function from I_n into A and this function is denoted by the corresponding images in a "specific" way the image of i + 1 succeeding that of i and all the images being written in a row within round brackets : $(a_1, \dots a_n)$ where a_i is the image of i and is called the ith component of the n-tuple.

An ordered 2-tuple is called an ordered pair, an ordered 3-tuple is called an ordered triad and so on. If n is a positive integer and $\{A_i\}_{i \in I_n}$ is a family sets, the product of this family in symbols $\prod_{i=1}^{n} A_i$ is defined to be the set of all ordered n – tuples $(a_1, \dots a_n)$ where $a_i \in A_i$ for $i \in I_n$. $\prod_{i=1}^{n} A_i$ is also denoted by $A_1 \times A_2 \times A_n$. If $A_i = A \ \forall \ i$, we write A^n for $\prod_{i \in I_n} A_i$. The mapping $p_j : \prod_{i=1}^{n} A_i \rightarrow A_j$ defined by

$p_j (a_1, \dots a_n) = a_j$ is called the jth projection.

If $\{A_i\}_{i \in I}$ is any family of sets, the collection of all functions $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $\forall \ i \in I, \ f(i) \in A_i$ is called the product of the family of sets $\{A_i\}_{i \in I}$ and is denoted by $\prod_{i \in I} A_i$. In particular we write $\prod_{i \in N} A_i = \prod_{i=1}^{\infty} A_i$. An element $f \in \prod_{i \in I} A_i$ is also represented by $\{ f(i) \}_{i \in I}$ or $\{f_i\}_{i \in I}$. For $j \in I$ the mapping $p_j : \prod_{i \in I} A_i \rightarrow A_j$ defined by $p_j (\{f_i\}_{i \in I}) = f_j$ is called the j-th projection or j-th coordinate function. At times we delete "the j-th" and simply say projection or coordinate function.

Posets:

A relation on a nonempty set A is a subset of A x A. If R is a relation on A we write a R b whenever $(a, b) \in R$ for convenience. A relation R on a nonempty set P is called a partial order relation or partial ordering on P if it satisfies.

(1) Reflexivity : $a R a \ \forall \ a \in P$

(2) Antisymmetry : $a R b \ \& \ b R a \Rightarrow a = b$

(3) Transitivity : $a R b \ \& \ b R c \Rightarrow a R c$.

The pair (P, R) is called a partially ordered set or simply a poset. The most commonly used symbol for a partial ordering on a set P is $\leq$, called less than or equal to.

If (P, $\leq$) is a poset, $a \in P$ $b \in P$ & $a \neq b$ and $a \leq b$ we write $a < b$. In this situation we say that a is less than b. We also use $\geq$ and $>$, respectively called greater than or equal to and greater than as follows.

If $a \leq b$ we write $b \geq a$ and $b > a$ when $a < b$.

If (P, $\leq$) is a poset, $x \in P$ and $y \in P$, there is no guarantee that x, y satisfy one of the following conditions:

$x = y, \ x < y, \ y < x$

For example, the set P (N) is a poset w.r.t. the partial ordering $\leq$ defined by $A \leq B \Leftrightarrow A \leq B$. This ordering is called set inclusion. In this poset the set E of positive even integers and the set 0of positive odd integers do not satisfy any of : $E = 0, E \subset 0, 0 \subset E$.

In a poset (P, $\leq$) elements a, b are said to be comparable (compatible) if $a \leq b$ or $b \leq a$.

A poset in which every pair of elements is comparable is called a totally (linearly) ordered set or a chain.

A subset A of a poset (P $\leq$) is said to be bounded above if there is an element $x \in P$ such that $a \leq x$ for every $a \in A$. Any such x is called an upper bound of A.

An element $x \in P$ is called a least upper bound or supremum of a subset A of P if

(i) x is an upper bound of A i.e. $a \leq x \ \forall \ a \in A$ and

(ii) $x \leq y$ if y is an upper bound of A.

A least upper bound is abbriviated by l.u.b or simply by lub. There is no guarantee that a set $A \subseteq P$ which is bounded above has necessarily a lub. However A cannot have more than one least upper bound.

The least upper bound of A is denoted by any of : sup A, l.u.b. A or lub A.

$A \subseteq P$ is said to be bounded below if $\exists$ a $y \in P$ such that $y \leq x \ \forall \ x \in A$. Any such y is called a lower bound of A in P. y is called a greatest lower bound or infimum of A in $(P \leq)$ if

(i) y is a lower bound of A in P and

(ii) $z \leq y \ \forall$ lower bound z of A in P.

A set which is bounded below need not necessarily have a greatest lower bound in P. However A cannot have more than one infimum. The infimum of A is denoted by inf A or g.l.b.A or glbA.

A poset $(P, \leq)$ in which every pair a, b has both lub and gib is called a lattice.

A poset $(P, \leq)$ in which every nonempty subset has glb and glb, is called a complete lattice.

Zorn's Lemma:

We are frequently encoutered with situations we can satisfy ourselves by showing the existence of an element satisfying preassigned conditions without bothering about other details. The main technique which is handy in this context is the famous Zorn's lemma. This lemma has several equivalent forms such as Axiom of choice, Hausdorff's maximal condition and so on. As none of these is established so far using the existing set theoretic properties and as the acceptances of any of these equivalent statements does not contradict the theories developed so far these are accepted as axioms. For further details see Appendix.

Examples: 1) $(\mathbb{N}, \leq)$

2) $A = \mathbb{N} \setminus \{1\}$. Define $m \leq n \Leftrightarrow m / n$

3) $A = \mathbb{N} \setminus \{1\}$. Define $m \leq n \Leftrightarrow n / m$.

ZORN'S LEMMA: Let $(P, \leq)$ be a poset in which every chain has an upper bound. Then P has a maximal element m in the sense that $x \in P$, $m \leq x$ implies $m = x$.

We now fix the following notation:

Z : The set of integers.

Q : The set of rational numbers

$\mathbb{R}$: The set of real numbers

$\mathbb{R}_+$: The set of real numbers $\alpha \ni \alpha \geq 0$, i.e. non-negative real number

$\mathbb{R}$ - : The set of real numbers $\alpha \ni \alpha \leq 0$. Clearly $0 \in \mathbb{R}_+ \cap \mathbb{R}-$

Exercises: Let $f : X \to Y$ be a mapping. Prove the following.

1) If $A_1 \subseteq A_2 \subseteq x$ then $f(A_1) \subseteq f(A_2) \subseteq Y$.

2) If $A_i \subseteq X$ for $i \in I$ then $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$ and $f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f(A_i)$.

3) Show by means of an example that $f(\bigcap_{i\in I} A_i)$ can be a proper subset of $\bigcap_{i\in I} f(A_i)$

4) If $B_1 \subseteq B_2 \subseteq Y$ then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$

5) If $B_j \subseteq Y \ \forall j \in J$, then $f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j)$, $f^{-1}\left(\bigcap_{j\in J} B_j\right) = \bigcap_{j\in J} f^{-1}(B_j)$

6) Show that $\forall B \subseteq Y$, $f^{-1}(B^1) = \{f^{-1}(B)\}^1$

7) Function $f : X \to Y$ and $g : X \to Y$ are said to be equal : $f = g$, if $\forall x \in X\ f(x) = g(x)$.

   Show that $f : X \to Y$ is one-to-one iff there exists mapping $g : Y \to X$ such that $g . f = i_X$ where i_X is the identity function defined on X by $i_X(t) = t \ \forall t \in X$.

8) Show that $f : X \to Y$ is onto iff there exists a $h : Y \to X$ such that $f h = i_Y$

9) If $f : X \to Y$ is a mapping show that $\forall B \subset Y$.

Lesson Writer: V.J. LAL

LESSON - 2

TOPOLOGICAL SPACES – DEFINITION AND SOME EXAMPLES

2.1 Introduction: The word topology, a branch of Mathematics, which is derived from Greek words has literal meaning, "the science of position". A topological property is a property of a "topological space" which is possessed by all topological spaces that are "homeomorphic" to the space. Topology can be defined as the study of all topological properties of topological spaces.

If we think of a topological space as a diagram drawn on a rubber sheet a homeomorphism may be thought of as any deformation of this diagram. A topological property, then would be any property of the diagram which is invariant under any deformation. That is why topology is also called a rubber sheet geometry.

In this lesson we start with the definition of a topology on a set, a topological space, a subspace of a topological space, the topology generated by a class of subsets of a set and provide a good number of examples. A special type of topological spaces, called metric spaces deserve separate attention because of their resemblance with the real line. We make a preliminary study of these spaces as well.

2.2 Definition: Let X be a nonempty set. A class T of subsets of X is called a topology on X if it satisfies the following conditions.

(i) The union of every class of sets in T is in T. i.e. if $\{A_i / i \in I\}$ is any class of sets in T indexed by a set I. Then $\bigcup_{i \in I} A_i$ O is in T and

(ii) The intersection of every finite class of sets in T is in it i.e. if **C** $\subseteq$ T is any finite class of sets and G is the intersection of the sets in **C** then G $\in$ T.

If T is a topology on X we call the ordered pair (X, T) a topological space.

2.3 Remark: In definition 2.2, condition (i) is described by saying that T is closed under arbitrary unions while condition (ii) is described by saying that T is closed under finite intersections.

2.4 Remark: The empty set is finite and the intersection of a family of subsets of X indexed by the empty set is the universal set X. Like wise the union of a family of sets indexed by the empty set is the empty set. Thus if (X, T) is a topological space, then $\phi \in T$ and $X \in T$.

2.5 SAQ: Show that condition (ii) of definition 2.2 holds if and only if $X \in T$, and $A \in T$, $B \in T \Rightarrow A \cap B \in T$.

2.6 Example: Discrete Topology:

Let X be a nonempty set. For T we take the power set P(X) of X. T is a topology. This is clear since P(X) contains all subsets of X and hence is closed under arbitrary unions and finite intersections. This topology is called the discrete topology on X.

2.7 Example: Let X be a nonempty set and $T = \{\phi, X\}$. Clearly T is closed under arbitrary unions and finite intersections hence T is a topology on X. This topology is called the indiscrete topology on X.

2.8 Example: Suppose X is a nonempty set. We take T to be the class consisting of all $A \subseteq X$ where

(i) either $A = \phi$ or

(ii) X/A is a finite set

Then T is a topology on X. This topology on X is called the confinite topology or the topology of finite complements.

Solution: Let $\{A_i / i \in I\}$ be any family of sets in T.

If $\bigcup_{i \in I} A_i = \phi$ then $\bigcup_{i \in I} A_i \in T$ by (i)

If $\bigcup_{i\in I} A_i \neq \phi$ then $A_{i_0} \neq \phi$ for some i_o . Now

$$X - \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X \setminus A_i) \subseteq X\setminus A_{i_0}$$

Since $X\setminus A_{i_0}$ is finite, $\bigcap_{i\in I}(X \setminus A_i)$ is finite.

$\therefore \bigcup_{i\in I} A_i$ satisfies (ii) so lies in T

Hence T is closed under arbitrary unions.

$X \setminus X = \phi$. So $X \in T$.

If A_1, A_2 are in T and $A_1 \cap A_2 \neq \phi$ then $A_1 \neq \phi \neq A_2$, so $X - A_1$, $X - A_2$ are finite. Hence $X - (A_1 \cap A_2) = (X - A_1) \cup (X - A_2)$ is finite. Hence $A_1 \cap A_2 \in T$. Hence T is closed under finite intersection. Thus T is a topology on X.

2.9 SAQ: We fix a symbol ∞ which is different from every natural number and write $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The set T consists of all sets A where (i) $A \subseteq \mathbb{N}$ or (ii) $A \subseteq \mathbb{N}$, $\infty \in A$ and $\overline{\mathbb{N}} \setminus A$ is a finite subset of $\mathbb{N}$. Then T defines a topology on $\mathbb{N}$.

2.10. SAQ: If T_1, T_2 are topologies on **X it** is not necessarily true that $T_1 \cup T_2$ is a topology on X. Give an example.

2.11 SAQ: If T is a topology on X it is not necessarily true that T is closed under arbitrary intersections. Give an example.

2.12 Proposition: If $\{T_i \,/\, i \in I\}$ is any class of topologies on a nonempty set X and $T = \bigcap_{i\in I} T_i$ then T is a topology on X. Further If T is any topology on X such that

$T^1 \subseteq T_i \; \forall \, i \in I$, then $T^1 \subseteq T$.

Proof: T is closed under arbitrary unions : $\{A_\alpha \,/\, \alpha \in \Delta\} \subseteq T$

$\Rightarrow \{A_\alpha \,/\, \alpha \in \Delta\} \subseteq T_i \;\; \forall \, i \in I$

$\Rightarrow A = \bigcup_{\alpha \in \Delta} A_\alpha \subseteq T_i \ \forall i \in I$ ($\because T_i$ is a topology)

$\Rightarrow A \in T_i \ \forall i \in I$

$\Rightarrow A \in T$

T is closed under finite intersections : If $\mathbf{C} \subseteq T$ is any finite family of subsets of X, $\mathbf{C} \subseteq T_i$ $\forall i \in I$.

$\Rightarrow G = \bigcap_{A \in C} A \in T_i \ \forall i \in I$ (since T_i is topology)

$\Rightarrow G \in \bigcap_{i \in I} T_i$

$\Rightarrow G \in T$

Since T is closed under arbitrary unions and finite intersections T is a topology on X.

If T^1 is a topology on X such that $T^1 \subseteq T_i \ \forall i \in I$, it is clear that $T^1 \subseteq T$.

Comparison of Topologies:

2.13 Definition: Let X be a nonempty set and T_1, T_2 be topologies on X. We say that T_1 is weaker (= coarser) than T_2 and write in symbols $T_1 \leq T_2$ if $T_1 \subseteq T_2$. In this case we also say that T_2 is stronger (= finer) than T_1 and write $T_2 \geq T_1$.

Remark: The indiscrete topology $T = \{\phi, X\}$ alone is contained in every topology on X so that T is weaker than every topology on X. Thus we may say that the indiscrete topology is the "weakest" topology on X.

Like wise the discrete topology P (X) consisting of all subsets of X is the "strongest" topology on X as it is stronger than every topology on X.

2.14 Proposition: Let $A = \{A_\alpha / \alpha \in \Delta\}$ be a collection of subsets of X. There is a unique topology T on X such that

(i) $A \subseteq T$ and

(ii) $T \subseteq T^1$ for every topology T^1 containing A.

Proof: Let **C** be the class of all topologies T^1 on X containing A. since P (X) $\in$ **C**, **C** is nonempty. If $T = \cap T^1$, $T^1 \in$ **C** by proposition 2.12 T is a topology on X.

Since $A \subseteq T^1 \ \forall \ T^1 \in$ **C**, $A \subseteq T$.

If T^1 is any topology on X such that $A \subseteq T^1$ then $T^1 \in$ **C** so $T \subseteq T^1$.

Thus $T = \bigcap_{T^1 \in C} T^1$ satisfies the required conditions.

If T_1 is a topology on X satisfying (i) and (ii), $T_1 \in$ **C** hence $T \subseteq T_1$.

Since T satisfies (i) and (ii) $T \in$ **C** hence $T_1 \subseteq T$.

Thus $T_1 = T$

This proves uniqueness.

2.15 Definition: Given $A = \{A_\alpha / \alpha \in \Delta\} \subseteq P(X)$, by the topology generated by A we mean the topology T which is the smallest topology containing A.

$T = \cap \{T_1 / T_1$ topology on X, $A \subseteq T_1\}$

T is also called the topology generated by A.

2.16 Proposition: Given any collection $\{T_\alpha / \alpha \in \Delta\}$ of topologies on X there is a unique topology T on X such that

(i) $T_\alpha \subseteq T \ \forall \ \alpha \in \Delta$

(ii) If T_1 is any topology such that $T_\alpha \subseteq T_1 \ \forall \ \alpha \in \Delta$ then $T \subseteq T_1$.

Proof: Let **C** be the collection of all topologies on X that contain T_α for every $\alpha \in \Delta$ and T_o be the intersection of all topologies in **C**.

$$T_o = \bigcap_{T^1 \in C} T^1$$

By proposition 2.14 T_o is the smallest topology on X containing $T_\alpha \ \forall \ \alpha \in \Delta$. Thus T_o is he required topology.

2.17 Theorem: Let X be a non-empty set and let T(X) be the class of all topologies on X. Let $\leq$ on T(X) be defined by $T_1 \leq T_2$ iff $T_1 \subseteq T_2$ for $T_1, T_2 \in T(X)$. Then $(T(X), \leq)$ in a complete lattice.

Proof: Clearly the indiscrete topology is the least element and the discrete topology is the greatest element in $(T(X), \leq)$.

Let $\{T_\alpha\}_{\alpha \in \Delta}$ be a non-empty family of topologies on X.

Let $T_1 = \bigcap_{\alpha \in \Delta} T_\alpha$.

Let $A = \{T \in T(X) / T_\alpha \subseteq T \ \forall \ \alpha \in y\}$.

Let $T_2 = \bigcap_{T \in A} T$.

Thus T_1 and T_2 are topologies on X and it is easy to verify that

T_1 = g.l.b. $\{T_\alpha / \alpha \in \Delta\}$ and

T_2 = l.u.b.g $\{T_\alpha / \alpha \in \Delta\}$

Hence $(T(X), \leq)$ is a complete lattice.

2.18 Proposition: Let T be a topology on a nonempty set X.

$Y \subseteq X$ be a nonempty set and

$T_Y = \{V \cap Y / V \in T\}$

Then T_y is a topology on Y.

Proof: (1) T_Y is closed under arbitrary unions: Let $\{A_i / i \in I\}$ be an arbitrary class of sets in T_Y. For each $i \in I$, $\exists \ B_i \in T \ni B_i \cap Y = A_i$

Hence $\bigcup_{i \in I} A_i = \bigcup_{i \in I} (B_i \cap Y) = \left(\bigcup_{i \in I} B_i\right) \cap Y$. Since τ is closed under arbitrary unions

$\bigcup B_i \in \tau$ hence $\bigcup A_i \in \tau_Y$.

(2) τ_Y is closed under fintie intersections : Clearly $Y \in T_Y$. So it is enough to prove that $A \in T_Y$ and $B \in T_Y \Rightarrow A \cap B \in T_Y$. Since $A \in T_Y$, $\exists \ A_1 \in T$ such that

$A = A_1 \cap Y$.

Similarly $\exists B_1 \in T \ni B = B_1 \cap Y$.

Since $A_1 \in T$ and $B_1 \in T$, $A_1 \cap B_1 \in T$. Hence $A \cap B = (A_1 \cap B_1) \cap Y \in T_Y$

Hence T_Y is a topology on Y.

2.19 Definition: The topology $T_Y = \{A \cap Y / A \in T\}$ is called the relative topology on Y and (Y, T_Y) is called a subspace of (X, T). If (Y, T_y) is a subspace of (X, T) it is customary to say that Y is a subspace of X.

In definition 2.15 the topology generated by a given family A of subsets of a set X ($\neq \phi$) is described as the smallest topology on X containing the given family A. In the following problem we provide a characterization for this topology.

2.20 Problem: Let X be a nonempty set and $A \subseteq P(X)$ write T_1 (A) for the family of subsets of X each of which is the intersection of a finite class of sets in A and T_2 (A) for the family of subsets of X each of which is an arbitrary union of sets in A. Prove that $T_2(T_1(A))$ is the topology generated by A on X by providing the following.

1. $T_1(A)$ is closed under finite intersections and $A \subseteq T_1(A)$
2. $T_2(A)$ is closed under arbitrary unions and $A \subseteq T_2(A)$
3. $T = T_2(T_1(A))$ is a topology on X and $A \subseteq T_2(T_1(A))$
4. If T^1 is any topology on X containing A then T is contained in T^1.

Solution: (1) Note that $T_2(T_1(A))$ is the family of all unions of finite intersections of sets in A. If A is empty, then $T_1(A) = \{X\}$ and $T_2(T_1(A)) = \{\phi, X\}$. Clearly $\{\phi, X\}$ is the topology generated by A. So, we may assume that A is nonempty. Let $A = \bigcap_{C \in F_1} C \in T_1(A)$ and $B = \bigcap_{C \in F_2} C \in T_1(A)$ where F_1, F_2 are finite subsets of A.

Then $A \cap B = \bigcap_{C \in F_1 \cup F_2} C \in T_1(A)$ because $F_1 \cup F_2$ is a finite subset of A. Clearly $A \subseteq T_1(A)$.

(2) let $\{A_\alpha / \alpha \in D\}$ be any arbitrary family of sets in $T_2(A)$. For each $\alpha \exists$ a set $F_\alpha \subseteq A \ni A_\alpha = \bigcap_{C \in F_\alpha} C$.

Then $F = \bigcup_{\alpha\in\Delta} F_\alpha \subseteq A$ and $\bigcup_{\alpha\in\Delta} A_\alpha = \bigcup_{\alpha\in\Delta}\left(\bigcup_{C\in F_\alpha} C\right) = \bigcup_{C\in F} C \in T_2(A)$

Hence $T_2(A)$ is closed under arbitrary unions.

Clearly $A \subseteq T_2(A)$

(3) We claim that $T = T_2(T_1(A))$ is a topology on X containing A. Clearly T contains ϕ and X. From(2), T is closed under arbitrary unions. We show that T is closed under finite intersections. For this it is enough to show that $A \in T$, $B \in T$.

$\Rightarrow A \cap B \in T$

Let $A = \bigcup_{i\in I} A_i$ and $B = \bigcup_{j\in J} B_j$ where $A_i \in T_1(A)$ and $B_j \in T_1(A)$ for $i \in I$ and $j \in J$.

Then $A \cap B = \bigcup_{i\in I, j\in J} C_{ij}$ where $C_{i,j} = A_i \cap B_j$. Since $T_1(A)$ is closed under finite intersections (by 1)

$A_i \cap B_j = C_{i,j} \in T_1(A)\ \forall\ i \in I$ and $j \in J$.

Then $A \cap B = \bigcup_{(i,j)\in I\times J} C_{ij} \in T$

Since T is closed under finite intersections and arbitrary unions T is topology on X. Since $A \subseteq T_1(A) \subseteq T_2(T_1(A))$ $A \subseteq T$.

4. Let T^1 be any topology on X containing A. Since T is closed under arbitrary unions and finite intersections, $T_1(A) \subseteq T_1$ and hence $T = T_2(T_1(A)) \subseteq T^1$.

This completes the proof.

We now consider special type of topological spaces called metric spaces. A metric on a set X resembles the distance between real numbers and so several properties of the usual distance on the real line $\mathbb{R}$ may be extended to a metric space.

2.21 Definition: Let X be a nonempty set. A mapping $d: X \times X \to \mathbb{R}$ is said to be a metric on X if d satisfies.

(i) $d(x, y) \geq 0 \ \forall \ x, y$ in X and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x) \ \forall \ x, y$ in X (d is symmetric)

(iii) $d(x, y) \leq d(x, z) + d(z, y) \ \forall \ x,y,z$ in X (triangle inequality)

If d is a metric on X, the pair (X,d) is called a metric space. We also say that X is a metric space with metric d.

2.22 Definitions: Let X be a metric space with metric d. If $x_0 \in X$ and $r > 0$ is a real number, the open sphere $S_r(x_o)$ with center x_0 and radius r is defined by

$$S_r(x_0) = \{x / x \in X, d(x_1 \ x_0) < r\}$$

$S_r(x_0)$ is also called the open sphere centred on x_0 with radius r.

The set $S_r[x_0] = \{x / x \in X, d(x, x_0) \leq r\}$ is called the closed sphere centred on x_0 with radius r.

$G \subset X$ is said to be an open set, simply G is open in X if for any $x \in G$, there exists a real number $r > 0$ such that $S_r(x) \subseteq G$.

2.23 Proposition: Let (X,d) be a metric space and T_d be the class of all open sets in X. Then T_d is a topology on X.

Proof: (1) Clearly ϕ and X are in T_d.

T_d is closed under arbitrary unions: Let $\{G_i / I \in I\}$ be any class of sets in T_d and $G = \bigcup_{i \in I} G_i$. $x \in G \Rightarrow x \in G_i$ for some $i \in I$. Since G_i is open there exists

$r > 0 \ni S_r(x) \subseteq G_i \subseteq G$. Hence $S_r(x) \subseteq G$. Since this holds $\forall \ x \in G$, $G \in T_d$.

(2) T_d is closed under finite intersections : Let $G_1 \in T_d$ and $G_2 \in T_d$. $x \in G_1 \cap G_2$ $\Rightarrow x \in G_1$ and also $\in G_2$.

$\Rightarrow \exists\, r_1 > 0$ and $r_2 > 0 \ni S_{r_1}(x) \subseteq G_1$ and $S_{r_2}(x) \subseteq G_2$

$\Rightarrow S_{r_1}(x) \cap S_{r_2}(x) \subseteq G_1 \cap G_2$.

If $r = \min\{r_1, r_2\}$ and $y \in S_r(x)$ then

$d(x, y) < r \quad \Rightarrow d(x, y) < r_i \ (i = 1, 2)$

$\Rightarrow y \in S_{r_i}(x) \ (i = 1, 2)$

$\Rightarrow y \in S_{r_1}(x) \cap S_{r_2}(x) \subseteq G_1 \cap G_2$

$\Rightarrow S_r(x) \subseteq G_1 \cap G_2$

Since Corresponding to every $x \in G_1 \cap G_2 \ \exists\, r > 0 \ni S_r(x) \subseteq G_1 \cap G_2$, $G_1 \cap G_2 \in T_d$.

Hence τ_d is closed under finite intersections. This shows that τ_d is a topology on X.

2.24 Definition: The topology T_d on X defined in proposition 2.23 is called the topology on X induced by the metric d or simply the metric topology corresponding to d or the usual topology on the metric space X. The sets in T_d are called the open sets generated by the metric d on the space X.

2.25 Example: the usual topology on the Real line $\mathbb{R}$.

By an open interval in $\mathbb{R}$ we mean a set of the form $(a, b) = \{x / x \in \mathbb{R}, a < x < b\}$ where $a \in \mathbb{R}$, $b \in \mathbb{R}$. A closed interval is of the form $[a, b] = \{x/x \in \mathbb{R}, a \leq x \leq b\}$ and an open closed interval is defined to be $(a, b] = \{x/x \in \mathbb{R}, a < x \leq b\}$, and a closed. open interval is defined to be $[a, b) = \{x \text{ i } a \leq x < b\}$. The absolute value or modulus of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Let $T_U = \{G/G \subseteq \mathbb{R}$ and $\forall\, x \in G, \exists\, a\, \delta > 0 \ni (x - \delta, x + \delta) \subseteq G\}$.

T_U is a topology on $\mathbb{R}$. This topology is called the usual topology on $\mathbb{R}$.

Verification of the conditions for a topology.

(i) T_U is closed under arbitrary unions : Let $\{G_i / i \in I\}$ be an arbitrary class of sets in T_U and $G = \bigcup_{i \in I} G_i$. $x \in G \Rightarrow x \in G_i$ for some $i \in I$. Since for any such i, $G_i \in T_0$, $\exists$ a $\delta > 0 \ni$ $(x - \delta\ x + \delta) \subseteq G_i \Rightarrow (x - \delta, x + \delta) \subseteq G$. This is true $\forall\ x \in G$. Hence $G \in T_U$.

(ii) T_N is closed under finite intersections: let $G_1 \in T_U$ and $G_2 \in T_U$

$$x \in G_1 \cap G_2 \Rightarrow x \in G_1 \text{ and } x \in G_2$$
$$\Rightarrow \exists\ \delta_1 > 0 \ni (x - \delta_1, x + \delta_1) \subseteq G_1, \text{ and}$$
$$\delta_2 > 0 \ni (x - \delta_2, x + \delta_2) \subseteq G_2.$$

If $\delta = \min\{\delta_1, \delta_2\}$, $(x - \delta\ x + \delta) \subseteq (x - \delta_1, x + \delta_1) \cap (x - \delta_2, x + \delta_2) \subseteq G_1 \cap G_2$

Thus $x \in G_1 \cap G_2 \Rightarrow \exists\ \delta > 0 \ni (x - \delta, x + \delta) \subseteq G_1 \cap G_2$.

Hence $G_1 \cap G_2 \in T_U$.

Thus T_U is a topology on $\mathbb{R}$.

Consider the real line $\mathbb{R}$. We define a metric on $\mathbb{R}$ by $d(x, y) = |x - y|$. This is called usual metric on $\mathbb{R}$. Note that the usual topology on $\mathbb{R}$ mentioned above is the same as the topology induced by d.

2.26 The Euclidean space $\mathbb{R}^n$:

If n is a positive integer, $\mathbb{R}^n$ stands for the set of all n – tuples $(x_1, x_2, \ldots. x_n)$ where $x_1 \in \mathbb{R}$ for $1 \le i \le n$. If $n = 1$ we write $\mathbb{R}^1 = \mathbb{R}$ and identify (x_1) with x_1. n tuples $x = (x_1, \ldots x_n)$ and $y = (y_1, \ldots., y_n)$ are said to be equal if $x_i = y_i$ for $1 \le i \le n$. We define $x + y = (x_1 + y_1, \ldots., x_n + y_n)$. If $\alpha \in \mathbb{R}$ we define $\alpha\, x = (\alpha\, x_1, \ldots.., \alpha\, x_n)$

We define $||x|| = \sqrt{x_1^{\ 2} + x_2^{\ 2} + \ldots\ldots + x_n^{\ 2}}$ and call this the norm of the vector x.

This norm is called the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{R}^n$ with this norm is called the **Euclidean space.**

Properties of the norm:

(1) $||x|| \ge 0\ \forall\ x \in \mathbb{R}^n$ and $||x|| = 0 \Leftrightarrow x = 0$

Proof: $||x|| = \sqrt{x_1^{\ 2} + \ldots.. + x_n^{\ 2}} \ge 0$ and $||x|| = 0 \Leftrightarrow x_1^2 + \ldots.. + x_n^2 = 0$

$\Leftrightarrow x_i^2 = 0\ \forall\ i \Leftrightarrow x_i = 0\ \forall\ i \Leftrightarrow x = (0,0,\ldots.0) = 0$ vector.

(2) $||\alpha x|| = |\alpha|\ ||x||$

$$||\alpha x||^2 = \alpha^2\ x_1^2 + \ldots\ldots + \alpha^2 x_n^2 = |\alpha|^2 \left(\sqrt{x_1^2 + \ldots\ldots + x_n^2}\right)^2$$

$$\Rightarrow ||\alpha x||^2 = |\alpha|^2\ ||x||^2 \Rightarrow ||\alpha x|| = |\alpha|\ ||x||$$

(3) $||x + y|| \leq ||x|| + ||y||\ \forall\ x, y$ in $\mathbb{R}^n$

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Then $x + y = (x_1, y_1, \ldots\ldots, x_n + y_n)$

So that $||x + y||^2 = (x_1 + y_1)^2 + \ldots\ldots + (x_n + y_n)^2$ and

$$(\ ||x|| + ||y||)^2 = x_1^2 + \ldots\ldots x_n^2 + y_1^2 + \ldots\ldots + y_n^2 + 2\sqrt{x_1^2 + \ldots\ldots + x_n^2}\ \sqrt{y_1^2 + \ldots\ldots + y_n^2}$$

Hence $||x+y||^2 - (||x|| + ||y||)^2 = 2\left\{\sum_1^n x_i\ y_i - \sqrt{\sum_1^n x_i^2}\ \sqrt{\sum_1^n y_i^2}\right\}$

Then it is enough to show that $2\sum_{i=1}^n x_i\ y_i \leq \sqrt{\sum_{i=1}^n x_i^2}\ \sqrt{\sum_{i=1}^n y_i^2}$

This inequality, known as Cauchy Schwarz inequality can be proved directly by considering the square of the difference. For details see SAQ.

2.27 SAQ (Cauchys' Inequality):

If $x_1, \ldots\ldots x_n$ and $y_1, \ldots\ldots y_n$ are complex numbers

$$\sum_{i=1}^n |x_i\ y_i| \leq \sqrt{\sum_{i=1}^n |x_i|^2}\ \sqrt{\sum_{i=1}^n |y_i|^2}$$

The Euclidean distance on $\mathbb{R}^n$ is defined by

$d(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ where $x = (x_1, \ldots\ldots x_n)$ and $y = (y_1, \ldots\ldots y_n)$

d is a metric on $\mathbb{R}$

$d(x, y) = ||x - y|| \geq 0$ and $d(x, y) = 0 \Leftrightarrow ||x - y|| = 0 \Leftrightarrow x = y$

$d(x,y) = ||)x - y)|| = ||(y - x)|| = d(y,x)$

$d(x,y) - d(x, z) - d(z,y)$

$= ||\ x - y\ || - (||x - z|| + ||z - y||) \leq 0$

since $||x - y|| \leq ||x - z|| + ||z - y||$

Hence $d(x,y) \leq d(x,z) + d(z,y)$ Hence this function d satisfies the metric properties. The metric d defined by $d(x,y) = \|x - y\|$ is called the Euclidean metric. The topology induced by the Euclidean metric is called the Euclidean topology on $\mathbb{R}^n$. We now concentrate on the Euclidean plane $\mathbb{R}^2$.

2.28 The Euclidean Plane $\mathbb{R}^2$:

$\mathbb{R}^2 = \{ (x_1, x_2) / x_i \in \mathbb{R}\ 1 \leq i \leq 2\}$ is called the Euclidean plane

If $x = (x_1, x_2)$, $y = (y_1, y_2)$; $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$$S_r(x) = \{y / d(x, y) < r\}$$
$$= \{(y_1, y_2) / (x_1 - y_1)^2 + (x_2 - y_2)^2 < r^2\}$$

$S_r(x)$ is the open sphere with centre at $x = (x_1, x_2)$ and radius r.

The open rectangle $(a, b) \times (c,d) = \{(x_1, x_2) / a < x_1 < b, c < x_2 < d\}$

An open strip is of the form $(a, b) \times \mathbb{R}$ or $\mathbb{R} \times (a, b)$.

2.29 The Unitary space $\mathbb{C}^n$:

If n is a positive integer, $\mathbb{C}^n$ stands for the set of all n – tuples $(z_1, \ldots, z_n)$ where $z_i \in \mathbb{C}\ \forall$ i, $1 \leq i \leq n$. If $n = 1$ we write $\mathbb{C}^1 = \mathbb{C}$ and identify (z) with z. If $z = (z_1, \ldots, z_n)$ and $z^1 = (z_1^1, \ldots z_n^1) \in \mathbb{C}^n$, we say that $z = z^1$ when $z_i = z_i^1$ for $1 \leq i \leq n$. We define $z + z^1 = (z_1 + z_1^1, \ldots z_n + z_n^1)$ and for $\alpha \in \mathbb{C}$, $\alpha z = (\alpha z_1, \ldots, \alpha z_n)$

For $z \in \mathbb{C}^n$ we define $\|z\| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$ where $z = (z_1, \ldots, z_n)$.

We call $\|z\|$, the norm of the vector z.

As in the case of the Euclidean space $\mathbb{R}^n$ we can show that the norm defined above satisfies the properties of the norm:

(i) $\|z\| \geq 0\ \forall\ z \in \mathbb{C}^n$ and $\|z\| = 0$ if and only if $z = 0$

(ii) $\|\alpha z\| = |\alpha|\ \|z\|\ \forall\ \alpha \in \mathbb{C}$ and $z \in \mathbb{C}^n$ and

(iii) $\forall\ z, z^1$ in $\mathbb{C}^n$, $\|z + z^1\| \leq \|z\| + \|z^1\|$

Consequently $d(z, z^1) = \|z - z^1\|$ defines a metric on $\mathbb{C}^n$. $\mathbb{C}^n$ with the metric d is called the Unitary space.

2.30 Problem: If $Q_0 = \{x + iy \,/\, x \in Q, y \in Q\}$, then Q_0 is countable and $z \in \mathbb{C}$, $\in > 0$ $\Rightarrow S_\in(z) \cap (Q_0 \setminus \{z\}) \neq \phi$.

Solution: Let $z = x_1 + i\,x_2 \in \mathbb{C}$, and $\varepsilon > 0$. Choose $\alpha_1 \in Q$, $\alpha_2 \in Q \ni |x_1 - \alpha_1| < \frac{\varepsilon}{\sqrt{2}}$ and

$|x_2 - \alpha_2| < \frac{\varepsilon}{\sqrt{2}}$ and $x_i \neq \alpha_i$ (i=1,2)

$\alpha = \alpha_1 + i\,\alpha_2 \in Q_0$ and $|z - \alpha| = |(x_1 - \alpha_1) + i\,(x_2 - \alpha_2)|$

$$= \sqrt{(x_1 - \alpha_1)^2 + (x_2 - \alpha_2)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon$$

Thus $\alpha \in S_\varepsilon(z) \cap Q_0$ and $\alpha \neq z$. Hence $\alpha \in S_\varepsilon(z) \cap Q_0 \setminus \{z\}$.

2.31 The spaces $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$:

We write K for either $\mathbb{R}$ or $\mathbb{C}$ and $K^{\mathbb{N}}$ for the collection of all sequences $\{x_n\}$ where $x_n \in K \ \forall n \in \mathbb{N}$

We write K^∞ for all sequences $\{x_n\}$ in $K^{\mathbb{N}}$ for which $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. The space $\mathbb{R}^\infty$ is called the infinite dimensional Euclidean space while $\mathbb{C}^\infty$ is called the infinite dimensional Unitary space. In Lesson 2 of functional Analysis some properties of this space, which is denoted by l^2, are studied.

There it is proved that

(1) K^∞ $(= l^2)$ is a vector space

(2) $\|x\| = \left\{\sum_{n=1}^{\infty} |x_n|^2\right\}^{1/2}$ for $x = \{x_n\} \in K^\infty$ $(= l^2)$ has the properties of a norm i.e. for every $x \in K^\infty$,

$\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

$\|\alpha\, x\| = |\alpha|\, \|x\| \ \forall x \in K^\infty$ and $\alpha \in K$ and

$\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in K^\infty$

This shows that d (x,y) = ||x – y|| defines a metric on K^∞. It is also shown that K^∞ is separable. For details see Lesson 2 of Functional Analysis.

2.32 Examples:

1. Let be any nonempty set. The discrete metric d on X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

If $x \in X$ and $0 < r \leq 1$, $S_r(x) = \{y \in X / d(x, y) < r < 1\} = \{x\}$

If $r > 1$, $S_r(x) = \{y \in x / d(x, y) < r\} = \{y / y \in X\} = X$

Consequently, if $G \subseteq X$ and $x \in G$, $S_1(x) \subseteq G$. this implies that the induced topology T consists of all subsets of X. Since T = P (X), the topology induced by the discrete metric is the discrete topology on X.

For any x, y in $\mathbb{R}$, $|x + y| \leq |x|+|y|$. This inequality becomes equality when x,y have the same sign. If $x < 0 \leq y < -x$

$y + x < 0$ so $|x + y| = -x - y \leq -x + y = |x| + |y|$

If $x < 0 < -x < y$ then $0 < x + y$ so $|x + y| = x + y < -x + y = |x| + |y|$

The other cases being similar; it follows that $|x + y| \leq |x| + |y|$

Now define $d(x,y) = |x - y|$. d is a metric on $\mathbb{R}$ (verify)

If $x \in \mathbb{R}$ and $r > 0$, $S_r(x) = \{y \in \mathbb{R} / |y - x| < r\} = (x - r, x + r)$

A set $G \subseteq \mathbb{R}$ is open if and only if $\forall x \in G \; \exists r > 0$ with $S_r(x) \subseteq G$ i.e. $(x - r, x + r) \subseteq G$.

This topology on $\mathbb{R}$ induced by the metric is precisely the usual topology.

2.33 Problem: Suppose X is a nonempty set and d: X x X → $\mathbb{R}$ satisfies

(1) $d(x,y) \geq 0$ for all x, y in X

(2) $d(x, y) = d(y, x)$ for all x, y in X and

(3) $d(x, y) = 0$ iff $x = y$

Show that d is a metric on X iff

$d(x, z) \geq d(x, y) - d(y, z)$ for all x, y, z in X iff

$d(x, z) \geq |d(x, y) - d(y, z)|$ for all x, y, z in X

Solution: Suppose d is a metric on X. Then for all x,y,z in X

d (x, y) ≤ d (x, z) + d(z, y)

⇒ d (x, z) ≥ d(x,y) – d(z,y)

= d (x, y) – d (y, z) by (2)

Assume that d (x, z) ≥ d (x, y) – d (y, z) for all x, y, z. Interchanging z and x we get

d (z, x) ≥ d (z, y) – d (y, x) using (2) we get d (x, z) = d (z, x) ≥ d (y, z) – d (x, y)

Hence d (x, z) ≥ | d (x, y) – d (y, z)|

Assume that d (x, z) ≥ | d (x, y) – d (y, z) | ∀ x, y, z in X

Then d (x, z) ≥ d (x, y) – d (y, z) = d (x,y) – d (z, y) (by 2)

⇒ d (x, z) + d (z, y) ≥ d (x, y) for all x, y, z in x

Since the triangle inequality holds in addition to (1), (2) and (3) d defines a metric on X.

2.34 SAQ: Suppose d : X x X → ℝ satisfies

(1) d (x,x) = 0 ∀ x ∈ ℝ and

(2) d (x, y) ≤ d (x, z) + d (y, z) for all x, y, z in ℝ. Then d is a metric.

2.35 Problem: Let d be a metric on X. Define for x, y in X

$$d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$$

Show that d_1 is a metric on X.

Show also that if $S_r^d(x) = S(x)$ w.r.to d and

$S_r^{d_1}(x) = S_r(x)$ w.r.to d_1 then $S_r^d(x) = S^{d_1}_{\frac{r}{1+r}}(x)$ for $r > 0$

Solution: Since d (x, y) ≥ 0, $d_1(x, y) \geq 0$ ∀ x, y in X

Also $d_1(x, y) = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$

$$d(x, y) = d(y, x) \Rightarrow d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)} = d_1(y, x)$$

To prove the triangle inequality for d_1, let $a > 0$, $b > 0$, $c > 0$ and $b + c \geq a$.

3

Then $\frac{1}{a} \geq \frac{1}{b+c}$

$$\Rightarrow 1 + \frac{1}{a} \geq 1 + \frac{1}{b+c} = \frac{1+b+c}{b+c}$$

$$\Rightarrow \frac{a}{1+a} \leq \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c}$$

Now put a = d (x, y), b = d (x, z), c = d (z, y) where x,y,z are in X and are all distinct.

By the triangle inequality for d we have

d (x, y) ≤ d (x, z) + d (z, y)

Hence d_1 (x, y) ≤ d_1 (x, z) + d_1 (z, y)

If d (x, y) = 0, the above inequality holds trivially.

If one of d_1 (x, z) and d_1 (z, y) is zero, say d (x, z) = 0 then x = z and in this case lhs = rhs = d_1 (z, x)

Hence d_1 is a metric on X.

$$y \neq x \text{ and } y \in S_r^d(x) \Leftrightarrow 0 < d(x, y) < r \Leftrightarrow \frac{1}{d(x,y)} > \frac{1}{r}$$

$$\Leftrightarrow 1 + \frac{1}{d(x,y)} > 1 + \frac{1}{r} = \frac{1+r}{r}$$

$$\Leftrightarrow y \in S_{r/1+r}^{d_1}(x) \text{ and } y \neq x$$

Hence $S_r^d(x) = S_{\frac{r}{1+r}}^{d_1}(x)$

2.36 Problem: Consider the set $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$

Define $d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ if m, n ∈ $\mathbb{N}$

$d(m, \infty) = d(\infty, m) = \frac{1}{m}$ if m ∈ $\mathbb{N}$

and $d(\infty, \infty) = 0$

Show that d is a metric on $\overline{\mathbb{N}}$

Solution: We verify the triangle inequality only. Let m, n, p ∈ ℕ

If m, n, p ∈ ℕ, $d(m, n) + d(n, p) = \left|\frac{1}{m} - \frac{1}{n}\right| + \left|\frac{1}{n} - \frac{1}{p}\right|$

$$\geq \left|\frac{1}{m} - \frac{1}{p}\right| = d(m, p)$$

If $p = \infty$ and m, n ∈ ℕ $d(m, n) + d(m, \infty) = \left|\frac{1}{m} - \frac{1}{n}\right| + \left|\frac{1}{n} - 0\right| \geq \left|\frac{1}{m} - 0\right| = d(m, \infty)$

If m, p ∈ ℕ and $n = \infty$, $d(m, \infty) + d(\infty, p) = \frac{1}{m} + \frac{1}{p}$

$$= \left|\frac{1}{m} - 0\right| + \left|0 - \frac{1}{p}\right| \geq \left|\frac{1}{m} - \frac{1}{p}\right| = d(m, p)$$

If $m = \infty$, p, n ∈ ℕ

$$d(\infty, n) + d(n, p) = \frac{1}{n} + \left|\frac{1}{n} - \frac{1}{p}\right| = \left|0 - \frac{1}{n}\right| + \left|\frac{1}{n} - \frac{1}{p}\right|$$

$$\geq \left|0 - \frac{1}{p}\right| = d(\infty, p)$$

The other cases are clear.

2.37 Answers to SAQ's:

SAQ 2.5: If T is closed under finite intersections, $A \in T$, $B \in T$

$\Rightarrow A \cap B \in T$. As said in Remark 2.4 $X \in T$.

Conversely suppose $A \in T, B \in T \Rightarrow A \cap B \in T$ and also suppose $X \in T$.

Then T contains the intersection of a family sets in T indexed by the empty set.

If = $\{A_1, \ldots, A_n\}$ is a non-empty finite family of sets in T then $\bigcap_{i=1}^{n} A_i = \left(\bigcap_{i=1}^{(n-1)} A_i\right) \cap A_n$

Thus we can apply induction on n. If n = 2 $A_1 \cap A_2 \in T$ by hypothesis. Assume that $\bigcap_{i=1}^{n-1} A_i \in T$ whenever $A_i \in T$ for $1 \leq i \leq n - 1$. Since $A_n \in T$ and $A = \bigcap_{i=1}^{n-1} A_i$, then

$\bigcap_{i=1}^{n} A_i = A \cap A_n \in T$ by induction hypothesis. Hence the intersections of n elements in T is in whenever it happens for n–1. So by induction this holds for all $n \in \mathbb{N}$.

SAQ 2.9: Suppose $A_1 \in T$ and $A_2 \in T$. If one of A_1, A_2 is a subset of $\mathbb{N}$ then so is $A_1 \cap A_2$ so $A_1 \cap A_2 \in T$. If $\infty \in A_1 \cap A_2$ then $\mathbb{N} \setminus A_1$, $\mathbb{N} \setminus A_2$ are finite subsets of $\mathbb{N}$. Hence $\mathbb{N} \setminus (A_1 \cap A_2) = (\mathbb{N} \setminus A_1) \cup (\mathbb{N} \setminus A_2)$ is a finite subset of $\mathbb{N}$. So $A_1 \cap A_2 \in T$. Thus T is closed under finite intersections as $\mathbb{N}$, the intersections of an empty family lies in T by ondition (2) of this question and by SAQ 2.5. That T is closed under arbitrary unions can be proved as in example 2.8.

SAQ 2.10: $X = \{1, 2, 3\}$

$T_1 = \{ \phi, \{1\}, X \}$ $T_2 = \{\phi, \{2\}, X \}$

$T_1 \cup T_2 = \{\phi, \{1\}, \{2\}, X \}$

$\{1, 2\} = \{1\} \cup \{2\} \notin T_1 \cup T_2$

SAQ 2.11: Let $U_n = \{k / k \in \mathbb{N}, k \geq n \} \cup \{\infty\}$ for $n \geq 1$.

U_n satisfies (2) of SAQ 2.9 so $U_n \in T \ \forall\ n$.

$$\{\infty\} = \bigcap_{n=1}^{\infty} U_n \notin T$$

SAQ 2.27: If $x_1, \ldots x_n$ and $y_1, \ldots y_n$ are complex numbers

$$\sum_{i=1}^{n} |x_i\, y_i| \leq \sqrt{\sum_{i=1}^{n} |x_i|^2}\ \sqrt{\sum_{i=1}^{n} |y_i|^2}$$

Solution: We may assume that x_i and y_i are non negative real numbers so that

$|x_i\, y_i| = x_i\, y_i$, $|x_i|^2 = x_i^2$ and $|y_i|^2 = y_i^2$

$$(RHS)^2 - (LHS)^2 = \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

$$= \sum_{i=1}^{n} x_i^2\, y_i^2 - \sum_{i=1}^{n} x_i^2\, y_i^2 + \sum_{i<j} (x_i^2 y_j^2 - 2\, x_i x_j\, y_i y_j + x_j^2\, y_i^2$$

$$= \sum_{i=1}^{n} (x_i y_j - x_j y_i)^2 \geq 0$$

$\Rightarrow (RHS)^2 \geq (LHS)^2$

$\Rightarrow$ RHS $\geq$ LHS

SAQ 2.38:

Put $z = x$ in $d(x, y) \leq d(x, z) + d(y, z)$ ------- (1)

$\Rightarrow d(x, y) \leq d(x, x) + d(y, x) = d(y, x)$ since $d(x, x) = 0$

$\Rightarrow d(x, y) \leq d(y, x)$

Interchange x and y so that $d(y, x) \leq d(x, y)$

Hence $d(x, y) = d(y, x)$

Put $x = y$ in (1) above. Then $0 = d(x, x) \leq d(x, z) + d(x, z)$

$= 2d(x, z)$

Hence $d(x, z) \geq 0\ \forall\ x, z$ in X

Hence d is a metric.

2.39 Model Examination Questions:

1. Define a topology on a nonempty set X and a metric on X. Show that every metric induces a topology on X.
2. Let X be a nonempty set, $T = \{A / A \subseteq X, A = \phi$ or X / A countable$\}$. Show that T is a topology on X.
3. Show that the class of topologies is a complete lattice with set inclusion.
4. If T_1, T_2 are topologies on a set X show that $T_1 \cup T_2$ is not necessarily a topology on X. Show also that there is a topology T on X containing both T_1 and T_2 and which is ⁺ained in every topology containing T_1, T_2.

2.40 Exercises :

1. Let X be a nonempty set and T be the class of all subsets of X whose complements are countable. Also let $\phi \in T$. Then show that T is a topology on X.
2. Let $Z \subseteq Y \subseteq X$. If T is a topology on X and T_Y, T_Z are the relative topologies on Y and Z respectively show that (Z, T_Z) is a subspace of (Y, T_Y) i.e. T_Z is the relative topology on Z with respect to the topology T_Y on Y.
3. Let X = {a, b c} T = {ϕ, {a}, {a, b}, {a, c}, X}

   Show that {X, T) is a topological space.
4. Find all possible topologies on X when

   (a) X = {a} (b) X = {a, b} (c) X = {a, b, c}
5. Compare the topologies obtained in

   (i) 4(a) (ii) 4(b) (iii) 4 (c)
6. Call a topological space metrizable if its topology is "Induced" by a metric. Show that if (X, T) is metrizable then it can be metrizable in infinitely many ways.

   Hint: If $k > 0$ show that d(x,y) and K d(x,y) give rise to the same topology.
7. Let (X, d) be a metric space. Define

   $$d_1, (x, y) = \begin{cases} d(x,y) & \text{if } d(x,y) \leq 1 \\ 1 & \text{if } d(x,y) > 1 \end{cases}$$

   Does d_1 define a metric on X.
8. Show that if (X, d) is a metric space, $x \in X$, $y \in X$ and $x \neq y$ there exist disjoint open spheres S_r (x) and S_r (y) containing x and y respectively. Hint let $0 < r < \dfrac{d(x,y)}{2}$.
9. Using 8 show that the topological space (X, T) in exercise 3 is not metrizable.
10. Let $d(x,y) = |x - y|^p$ where $p \geq 0$. For what values of p does d define a metric on the real line $\mathbb{R}$.
11. Show that the relative topology on the set Z of integers as a subspace of the real line with the usual topology is the discrete topology on Z.
12. Show that the indiscrete topology on a set consisting of at least two elements is not metrizable.
13. Let B be the collection of all open intervals (a, b) in $\mathbb{R}$, show that T_2 (B) (Problem 20) is a topology on $\mathbb{R}$.

14. Let B_0 be the collection of all intervals of the form (a, b) where $a \in Q$ and $b \in Q$. Show that $T_2(B_0)$ is a topology on $\mathbb{R}$.

15. Are the topologies $T_2(B)$, $T_2(B_0)$ in (13) and (14) equal? Is any one of them equal to the usual topology?

16. Let d_1, d_2 be metrics on a set $X \neq \phi$.

    Which of the following are metrics on X?

    (a) $d(x, y) = d_1(x, y) + d_2(x, y)$

    (b) $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$

    (c) $d(x, y) = \sqrt{d_1^2(x, y) + d_2^2(x, y)}$

Lesson Writer : V. J. LAL

LESSON NO - 3

BASIC CONCEPTS IN TOPOLOGICAL SPACES

3.1 Introduction : In this lesson we study some of the basic concepts of a topological space. The terms open set, closed set, closure of a set, dense set, separable set in a topological space are defined and their properties are studied. The first countability axiom, the axiom of second countability are also introduced and the famous Lindelof's theorem along with simple but important consequences are proved. Separability in relation to second countability is also discussed.

A neighbourhood at a point, open base at a point, isolated point, limit point, interior point, boundary point and other concepts are also defined and some of their salient features are proved.

In the process some subtilities in metric spaces in this context are also discussed.

3.2 Definitions :

Let (X,τ) be a topological space. $V \subseteq X$ is said to be an open set or simply V is open in X or V is open if $V \in \tau$.

$F \subseteq X$ is said to be a closed set or simply F is closed in X or F is closed if its complement $F = X \backslash F$ is open in X

If $A \subseteq X$, the closure of A, denoted by $\overline{A}$, is the intersection of all closed supersets of A

Ie., $\overline{A} = \cap \{F \backslash A \subseteq F, \text{F is closed in X}\}$

A is dense in X if $\overline{A} = X$, In this case we simply say that A is everywhere dense or A is dense.

(X,τ) is said to be a separable space or X is said to be separable if X has a countable dense subset.

3.3 Remark : Since the intersection of the empty family of sets (in τ) is the space X, $X\in\tau$. Since the union of the empty family of sets (in τ) is the empty set ϕ, $\phi\in\tau$.

Then ϕ and X are open sets in X. consequently, ϕ and X are closed sets.

3.4 Examples : If X is a nonempty set, every subset of X is open in the discrete topology and hence every subset of X is closed, whereas in the case of the indiscrete topology the only open sets are ϕ and X, hence the only closed sets are ϕ and X.

3.5 SAQ: If (x,d) is a metric space show that every open sphere is an open set.

3.6 Proposition : The class Σ of all closed sets in a topological space (x,τ) has the following properties.

(i) $\phi\in\Sigma$, $X\in\Sigma$

(ii) $A\in\Sigma, B\in\Sigma \Rightarrow A\cup B\in\Sigma$

(iii) $\{A_i \backslash i\in I\} \subseteq\Sigma \Rightarrow \cap_{i\in I} A_i \in \Sigma$

Proof : (I) follows by remark 3.3

We use De Morgan's laws:

$(A\cup B)^1 = A^1 \cap B^1$ and

$(\cap_{i\in I}A_I \in \Sigma$

If $A \in\Sigma$ and $B\in\Sigma \Rightarrow A^1 \in\tau, B^1\in\tau$

$\Rightarrow A^1\cap B^1 \in \tau$

$\Rightarrow (A\cup B)^1 \in\tau$

$\Rightarrow A\cup B \in \Sigma$

$A_i\in\Sigma\ \forall\ i\in I \Rightarrow A_i^1\in\tau\ \forall\ i\in I$

$\Rightarrow \cup_{i\in I} A_i^1\in\tau.$

$\Rightarrow \left(\bigcap_{i\in I} A_i\right)^1 \in\tau$

$\Rightarrow \cap_{i\in I} A_i \in \Sigma$

3.7 Corollary : The class Σ of all closed sets in a topological space in closed under finite unions and arbitrary intersections.

Proof follows from proposition 3.6.

3.8 SAQ : Suppose Σ is a class of subsets of a nonempty set X which is closed under finite unions and arbitrary intersections. Show that

$Y = \{A^1/A \in \Sigma\}$ is a topology on X.

3.9 SAQ : If (X,d) is a metric space $x \in X$ and $r>0$, is $\overline{S_r(x)} = S_r(x)$? Justify your answer.

3.10 SAQ : In a metric space (X,d) show that $\{x\}$ is a closed set $\forall\ x \in X$.

3.11 The Closure Operation:

We have defined the closure of a set A in a topological space (X,τ) to be the intersection of all closed sets containing A. The set X is closed as $\phi \in Y$ so that the collection of subsets of X that are closed in (X,τ) and containing A is nonempty. Moreover, the intersection of any class of closed sets is closed so that $\overline{A}$ is a closed set containing A. Moreover, $\overline{A}$ is the "smallest" closed set containing A since every closed set F that contains A, also contains $\overline{A}$.

We will prove soon that this closure operation assigning $\overline{A}$ to an arbitrary set A in X satisfies "Kuratowski closure axioms". We will also prove that any operation on P(X) satisfying these axioms induces a unique topology on X so that the closed sets in this topology are precisely those subsets of X that are invariant under this operation.

3.12 Proposition : Let (X,τ) be a topological space the operation $A \rightarrow \overline{A}$ from P(X) into P(X), where $\overline{A}$ is the closure of A satisfies the following:

$K_1 : \overline{\phi} = \phi$

$K_2 : A \subseteq \overline{A}$

$K_3 : \overline{\overline{A}} = \overline{A}$ and

$K_4 : \overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof : By definition, $\overline{A} = \cap\{F/F$ is closed and $F \supseteq A\}$. So $\overline{A} \subseteq F$ $\forall$ closed set $F \supseteq A$. In particular when $A = \phi$, $\overline{\phi} \subseteq \phi$ since ϕ is closed. Then $\overline{\phi} = \phi$. This proves K_1.

If $A \subseteq X$, then $A \subseteq \cap$ {F/A⊆F, F is closed}

Hence $A \subseteq \overline{A}$

If $A \subseteq X$, then $\overline{A}$ is a closed set and clearly $\overline{A} \subseteq \overline{A}$. Hence $\overline{\overline{A}} \subseteq \overline{A}$.

Since $\overline{A} \subseteq \overline{\overline{A}}$ by K_2, it now follows that $\overline{\overline{A}} = \overline{A}$

Let $A \subseteq X$ and $B \subseteq X$. clearly by K_2, $A \subseteq \overline{A}$,

$B \subseteq \overline{B}$ so that $A \cup B \subseteq \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is closed,

$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Further $\overline{A \cup B} \supseteq A \cup B \supseteq A$.

Since $\overline{A \cup B}$ is closed, $\overline{A} \subseteq \overline{A \cup B}$. Similarly $\overline{B} \subseteq \overline{A \cup B}$

Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$

Hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$. This process K_4.

3.13 Theorem : Let X be a nonempty set. Suppose that with every subset A of X a set $\overline{A}$ is associated and that this association satisfies the following "Kuratowski closure axioms"

1) $\overline{\phi} = \phi$

2) $A \subseteq \overline{A}$ $\forall$ $A \subseteq X$

3) $\overline{\overline{A}} = \overline{A}$ $\forall$ $A \subseteq X$ and

4) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ $\forall$ $A \subseteq X$ and $B \subseteq X$

Then there is a unique topology τ and X such that a set $A \subseteq X$ is closed in this topology if and only if $A = \overline{A}$.

Proof: Let $\Sigma = \{A / A \subseteq X \text{ and } A = \overline{A}\}$

Clearly $\phi = \overline{\phi}$ and $X \subseteq \overline{X} \subseteq X$ so that $\overline{X} = X$ hence $\phi \in \Sigma$ and $X \in \Sigma$.

$A \in \Sigma, B \in \Sigma \Rightarrow \overline{A} = A$ and $\overline{B} = B$

$$\Rightarrow A \cup B = \overline{A} \cup \overline{B} = \overline{A \cup B}$$

$$\Rightarrow A \cup B \in \Sigma$$

We prove that Σ is closed under finite unions by using the principle of mathematical induction on the number of sets. Clearly this holds when n=1 and that this also holds for n=2 is proved above. Now assume that the union of any n sets in Σ is in Σ.

Let $A_1, \ldots\ldots\ldots A_{n+1}$ sets in Σ. Then

$$\overline{\bigcup_{i=1}^{n+1} A_i} = \overline{(\bigcup_{i=1}^{n} A_i) U A_{n+1}} = \overline{(\bigcup_{i=1}^{n} A_i)} \, U \, \overline{A_{n+1}}$$

$$= \bigcup_{i=1}^{n} A_i U A_{n+1} = \bigcup_{i=1}^{n+1} A_i$$

since $\overline{A_{n+1}} = A_{n-1}$, $\overline{(\bigcup_{i=1}^{n} A_i)} = \bigcup_{i=1}^{n} A_i$ by induction hypothesis. Hence $\bigcup_{i=1}^{n+1} A_i \varepsilon \Sigma$. This shows that whenever Σ is closed under union for n sets, Σ is closed under union for (n+1) sets. Hence by induction Σ is closed under finite unions.

We now show that Σ is closed under arbitrary intersections. Let $\{A_i / i \in I\}$ be any non-empty class of sets in Σ. Then $\overline{A_i} = A_i \forall\, i \in I$.

Clearly $\bigcap_{i \in I} A_i \subseteq \overline{\bigcap_{i \in I} A_i}$ (by 2). To prove the reverse inclusion we first note that $A \subseteq B$ $\Rightarrow A \cup B = B$

$\Rightarrow \overline{A \cup B} = \overline{B} \Rightarrow \overline{A} \cup \overline{B} = \overline{B}, \overline{A} \subseteq \overline{B}$ (by 2).

Since $\bigcap_{i \in I} A_i \subseteq A_i \ \forall\, i \in I$, $\overline{\bigcap_{i \in n} A_i} \subseteq \overline{A_i} = A_i \forall\, i \in I$

Since this is true $\forall\, i \in I$, $\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} A_i$

This shows that $\overline{\bigcap_{i\in I} A_i} = \bigcap_{i\in I} A_i$, hence $\bigcap_{i\in I} A_i \in \Sigma$

Thus Σ is closed under arbitrary intersections. By SAQ 3.8, it follows that $Y=\{A/A^1\in\Sigma\}$ is a topology on X.

$A\subseteq X$ is closed in this topology

$\Leftrightarrow A^1 \in Y$

$\Leftrightarrow (A^1)^1 = A \in \Sigma \Leftrightarrow A = \overline{A}$

For uniqueness, suppose τ_0 is any topology on X such that A is closed in (X, τ_0) if $A = \overline{A}$, for every subset A of X. Then A is closed in $(X, \tau_0) \Leftrightarrow A = \overline{A} \Leftrightarrow$ A is closed in (X, τ) $\therefore \tau = \tau_0$.

This completes the proof of the theorem.

3.14 Corollary : Let Y be the unique topology on X obtained from the given operation $A \to \overline{A}$ from P (X) into itself as in the above theorem. Then for any $A\subseteq X$, the closure of A in (X,τ) is precisely $\overline{A}$.

Proof : For clarity, the closure of A in (X,τ) is denoted by $\tilde{A}$, for any subset A of X. Note that

$$\tilde{A} = \bigcap\{F/F \text{ is closed and } F\supseteq A\}$$

$$= \bigcap\{F/F=\overline{F} \text{ and } F\supseteq A\}$$

Since $\overline{A}=\overline{\overline{A}}$ and $\overline{A}\supseteq A$, $\tilde{A}\subseteq \overline{A}$

But $\tilde{A}$ is closed and $\tilde{A}\supseteq A$,

$$\tilde{A}\supseteq A \Rightarrow \overline{(\tilde{A})}\supseteq \overline{A} \Rightarrow \tilde{A}\supseteq \overline{A}$$

Thus $\tilde{A}=\overline{A}$ as required

3.15 Definition : A neighbourhood of a point x is a topological space (X,τ) is an open set containing x

A class Σ of neighbourhoods of a point x in a topological space (x,Y) is called an open base at the point x (or for the point x) if every neighbourhood of x contains a member of

3.16 Example : If (x, d) is a metric space and $x \in X$, the class of all open spheres $\{S_r(x)/r>0\}$ is an open base at x because by definition every open set containing x contains $S_r(x)$ for some r>0

3.17 Proposition : Let (X,τ) be a topological space and $A \subseteq X$. Then $\overline{A} = \{x/x \in X$ and every neighboured of x intersects A$\}$

Proof : Let B be the set specified on the right hand side. Let $x \in \overline{A}$ and V, any neighbourhood of x. If $V \cap A = \phi$, then $A \subseteq V^1$. Since V is an open set V^1 is a closed set containing A. Hence $\overline{A} \subseteq V^1$.

Since $x \in \overline{A}$, $x \in V^1$ so that $x \notin V$. This is a contradiction so that every neighbourhood of x intersects A, hence $\overline{A} \subseteq B$------------(1)

Now suppose that $x \in B$. We show that $x \in \overline{A}$, if $x \notin \overline{A}$ then $(\overline{A})^1$ is a neighbourhood of x since $A \subset \overline{A}$, $A \cap (\overline{A})^1 = \phi$. This contradicts the assumption that every neighbourhood of x intersects A. Hence $x \in \overline{A}$ as required. Therefore

$B \subseteq \overline{A}$. This, together with (1) yields $\overline{A} = B$.

3.18 Definitions : Let X be a topological space, $A \subseteq X$. A point $x \in A$ is called an isolated point of A if it has a neighbourhood V such that $V \cap A = \{x\}$

A point $x \in X$ is called a limit point of A if each of its neighbourhoods contains a point of A other than x

The set of limit points of A is called the derived set of A and is denoted by D(A)

3.19 Remark : It is customary to call $V\setminus\{x\}$ a deleted neighbourhood of x if V is a neighbourhood of x. Thus, x is a limit point of A if and only if every deleted neighbourhood of x intersects A.

A limit point of a set A is not necessarily a point of A where as an isolated point of A must necessarily belong to A.

3.20 Proposition : Let X be a topological space and A be a subset of X, then

(1) $\overline{A} = A \cup D(A)$

(2) $\overline{A} \setminus D(A)$ is the set of isolated points of A.

(3) $D(A) \subseteq A$ if and only if A is closed.

Proof : (1) If $x \in \overline{A}$, then by proposition 3.17 every neighbourhood of x intersects A so that if $x \notin A$ every neighbourhood of x unlersects A in a point other than x so that x is a limit point of a, hence $x \in D(A)$. Thus $\overline{A} \subseteq A \cup D(A)$. On the other hand $x \in D(A) \Rightarrow$ every neighbourhood of x intersects A in a point other than x so that $x \in \overline{A}$.

Hence $D(A) \subseteq \overline{A}$. Since $A \subseteq \overline{A}$, $A \cup D(A) \subseteq \overline{A}$.

It is now clear that $\overline{A} = A \cup D(A)$.

(1) If $x \in \overline{A}$ and $x \notin D(A)$ there is a nbd V of $x \ni V\setminus\{x\} \cap A = \phi$ so that by (1) $V \cap A = \{x\}$. Hence x is an isolated point of A.

Conversely if x is an isolated point of A, then $x \in A$ and there exists a neighbourhood V of $x \ni V \cap A = \{x\}$ so that $x \notin D(A)$. This implies that $x \in A \setminus D(A) \subseteq \overline{A} \setminus D(A)$. Thus $\overline{A} \setminus D(A)$ is the set of isolated points of A.

(2) Since $\overline{A} = A \cup D(A)$ and A is closed if and only if $A = \overline{A}$, it follows that a is closed if and only if $A = A \cup D(A)$ if and only if $D(A) \subseteq A$.

As a consequence we have the following theorem.

3.21 Theorem : Let X be a topological space. Then any closed subset of X is the disjoint union of the set of its isolated points and the set of its limit points in the sense that it contains these sets, they are disjoint and it is their union

Proof : Let A be a closed subset of X and i(A) the set of isolated points of A. Then $i(A) \subseteq A$, $D(A) \subseteq A$ and by proposition 3.20, $i(A) = \overline{A} \setminus D(A) = A \setminus D(A)$ so that $i(A) \cap D(A) = \phi$ and $i(A) \cup D(A) = A$.

3.22 SAQ : Let X be a nonempty set and T={ ϕ, x} be the indiscrete topology on X. Determine $\overline{\{x_0\}}$ for $x_0 \in X$.

3.23 SAQ : Let $\overline{IN} = N \cup \{\infty\}$ and T be the topology on $\overline{IN}$ described in problem 25 of lesson 2. Determine D(A) and $\overline{A}$ for $A \subseteq \overline{IN}$

3.24 SAQ : Show by an example that D(A) is not necessarily closed for a subset A of a topological space.

3.25 SAQ : Show that if (X, d) is a metric space and $A \subseteq X$ then D(A)is a closed subset of X.

3.26 Definitions : Let (X, T) be a topological space and a be a subset of X. The interior of A, denoted by int(A) is the union of all open sets contained in A. A point $x \in A$ is called an interior point of A if $x \in X$ int (A) ; ie, $x \in V$ for some open set $V \subseteq A$; equivalently some neighbourhood of x is contained in A.

The boundary of A is the set $\overline{A} \cap \overline{A^1}$, where A^1 is the complement of A. $x \in X$ is called a boundary point of A if x is in the boundary of A. Equivalently x is in the closure of A as well as the closure of its complement A^1. It is denoted by $\partial(A)$.

3.27 Remark : It is clear that x is a boundary point of A if and only if every neighbourhood of x intersects A as well as its complement A^1.

3.28 Theorem : Let X be a topological space. Then any closed subset of X is the disjoint union of its interior and its boundary; in the sense that it is their union.

Proof : Let $A \subseteq X$ be a closed set and $\partial(A) = \overline{A} \cap \overline{A^1}$, the boundary of A. When A is closed $D(A) = \overline{A} \cap \overline{A^1} \subseteq A$, clearly int $(A) \subseteq A$. If $x \in int(A)$, then some neighbourhood V of x is contained in A so that $V \cap A^1 = \phi$ and so $x \notin \partial(A)$. On the other hand if $x \in \partial(A)$, $x \in A$ and every neighbourhood V of x intersects A^1 so that $V \not\subset A$. This implies that $x \notin int(A)$. Hence $int(A) \cap \partial(A) = \phi$.

Clearly if $x \in A$ and x is not an interior point of A, every neighbourhood V of x intersects A^1 and A so that $x \in \partial(A)$. Hence $A = int(A) \cup \partial(A)$.
This completes the proof.

3.29 Definition : An open base for a topological space is X is a class $\mathcal{B}$ of open subsets of X such that every open set in X is the union of a class of sets in $\mathcal{B}$. If $\mathcal{B}$ is an open base for X, sets of $\mathcal{B}$ are called Basic open sets.

3.30 Proposition : Let (X, T) be a topological space and $\mathcal{B} \subseteq T$. $\mathcal{B}$ is an open base for (X, T) if and only if $x \in G \in Y \Rightarrow$ there exists a $B \in \mathcal{B}$ such that $x \in B \subseteq G$.

Proof : Let $\mathcal{B}$ be an open base, $x \in G \in T$. By definition, there exists a class $\{B_i / i \in I\} \subseteq \mathcal{B}$. Clearly $B_I \subseteq G$. Thus there exists $i \in I$ such that $x \in B_i \subseteq G$ and $B_i \in \mathcal{B}$.

Conversely suppose this condition is satisfied. Let $G \in T$ for each $x \in G$ there exists a $B_x \in \mathcal{B} \ni x \in B_x \subseteq G$. $\{B_x / x \in G\} \subseteq \mathcal{B}$ and clearly $G = \bigcup_{x \in G} B_x$. Hence $\mathcal{B}$ is an open base for (X, T)

3.31 Remark : Let us recall that for any class of sets B, $T_2(B)$ is the class of sets that are unions of members of B. Thus we may rephrase the definition of an open base as follows:

A class of open sets $\mathcal{B}$ in a topological space (X, T) is an open base if and only if $T_2(\mathcal{B}) = Y$.

4P

3.32 Examples : Let X be a nonempty set and T_d be the discrete topology on X. For each $x \in X$, let $B_x=\{x\}$. Then $\mathcal{B} = \{B_x/x \in x\}$ is an open base for (X, T_d)

Reason : Let us recall that every subset of X is open in the discrete topology. Thus if $G \subseteq x$, $G= \bigcup_{x \in G} B_x$

Since $B_x \in \mathcal{B}\ \forall\ x \in G$, G is the union of a class of sets in $\mathcal{B}$.

Hence $\mathcal{B}$ is an open base for (X, T_d).

3.33 Example : For the real line R with the usual topology T_u, the class B of all open intervals (a, b) where $a \in R$ and $b \in R$ is an open base

Reason : By definition, G is an open set in the usual topology if and only if $\forall\ x \in G\ \exists$ a $\delta_x > 0 \ni$

$I_x= (x-\delta_x, x+\delta_x) \subseteq G$.

Clearly $I_x \in B$ and $G= \bigcup_{x \in G} I_x$

3.34 Definition : Let (X, T) be a topological space. A class Y of open subsets of X is said to be an open sub base for (x, y) if the class B = $T_1(y)$ is an open base for (X,T) where $T_1(y)$ confute of finite intersections of members of Y ie., $A \in T_1(y)$ such that $A= \bigcap_{F \in Y} F$. Elements of Y are called sub basic open sets in (x, y).

3.35 Example : If $a \in R$ write $(-\infty,a) = \{x/x \in R$ and $x<a\}$ and

$(a, \infty) = \{x/x \in R$ and $a<x\}$ $a<b$;

The class $Y=\{(a, b)/a \in R, b=\infty$ or $a=-\infty, b \in R\}$ is an open Subbase for the real line with the usual topology.

Reason : We know that the class $B = \{(a, b)/a \in R, b \in R\}$ is an open base for the space R with the usual topology.

If $a \in R$ and $b \in R$, $(a, b) = (-\infty, b) \cap (a, \infty) \in T_1(y)$.

Further $(-\infty, a) = \bigcup_{x<a} (x, a) \in Y_4$ and

$(b, \infty) = \bigcup_{b<x} (b, x) \in Y_4$

Thus $(-\infty, a)$ and (b, ∞) are sub basic open sets.

Hence $(a, b) \in T_1(y)$.

Thus $B \subseteq T_1(y)$, since B is an open base, $T_1(y)$ is also an open base. Therefore y is an open sub base for R.

3.36 Theorem : Let X be any nonempty set and τ an arbitrary class of subsets of X, then τ can serve an open subbase for a topology on X, in the sense that $T_2(T_1(\tau))$ is a topology on X.

Proof : That $T_2(T_1(\tau))$ is a topology on X containing τ is proved in problem 20 of lesson2. By definition $T_2(T_1(\tau))$ is the collection of all sets which are arbitrary unions of members of $T_1(\tau)$ hence $T_1(\tau)$ is an open base for this topology

Thus τ is an open sub base for this topology.

3.37 Example : The collection $\mathcal{B}_1$ of all open spheres is an open base for the Euclidean topology on R^2.

Reason : By definition $G \subseteq R^2$ is open in the Euclidean topology if $\forall\, x \in G$ $\exists$ an $r>0 \ni x \in S_r(x) \subseteq G$. Hence $\mathcal{B}_1$ is an open base for the Euclidean topology on R^2.

3.38 SAQ : The collection B_2 of all open rectangles is an open base and the collection of all open strips is an open sub base for the Euclidean topology on R^2.

Countability axiom :

3.39 Definition : A topological space (x, T) is said to satisfy the first countability axiom or simply be first countable if every point in x has a countable open base (X, T) is said to satisfy the second countability axiom or simply be second countable if there is a countable open base for (X, T).

3.40 SAQ : Every metric space is first countable.

Second countability axiom implies first countability axiom where as the reverse implication does not hold. Moreover a sub space of a first (or second) countable space is first (or second) countable. These two types of conditions play an important role in reducing the "number" of open sets in test cases.

We now prove the central fact about second countable spaces namely Lindelof's theorem and its conuquence which is mostly used.

3.41 Lindelofs Theorem : Let X be a second countable space. If a nonempty set G of X is represented as the union of a class $\{G_x/i\in I\}$ of open sets then G can be represented as a countable union of G_I's.

Proof : Let $\beta = \{B_n/n\in N\}$ be a countable open base for X. Let $J_0 = \{n\in N/B_n\subset G_I$, for some $i\in I$ such that $B_n\subseteq G_i$. Among all such i's we fix one and denote this by i_n, ie., $B_n\subseteq G_{In}$ since J_0 is countable. Clearly $\{G_{in}/n\in J_0\}$ is a sub class of $\{G_I / i\in I\}$. We claim that $G = \bigcup_{n\in J_0} G_{in}$. Clearly $\bigcup_{n\in J_0} G_{in} \subseteq G_I$.

Let $x\in G$. then $x\in G_I$ for some i. Since P is an open base, there exists an integer $n\geq 1$ such that $x\in B_n \subseteq G_i$. Then $n\in J_0$ and $x\in B_n \subseteq G_{i_n}$, by the choice of I_n. Thus $x\in \bigcup_{n\in J0} G_{i_n}$

Hence $G\subseteq \bigcup_{n\in J0} G_{i_n}$. Therefore $G= \bigcup_{n\in J0} G_{i_n}$

3.42 Theorem : Let X be a second countable space. Then any open base for X has a countable sub class which is also an open base.

Proof : *Suppose* (x,τ) *is a topological space* which is second countable and we are given a basis $\{V_i / i \in I\}$ for τ, indexed by a set I. We show that there is a countable subset I_0 of I such that $\{V_i / i \in I_0\}$ is an open base for τ.

Since (x, τ) is second countable, there is a countable open base $\mathfrak{B} = \{B_n / n \in N\}$ for τ. For each $n \in N$ there is a countable subset I_n of I (by Lindelof's theorem) such that $B_n = \bigcup_{i \in I_n} V_i$. Let $I_0 = \bigcup_{n \in N} I_n$. Then I_0 is a countable subset of I.

We show that $V = \{V_i / i \in I_0\}$ is an open base for τ. Let us recall that for any class $Y \subseteq P(X)$, $T_2(Y)$ stands for the class of all sets which are unions of members of Y. Since $\mathfrak{B}$ is an open base for τ, $T_2(\mathfrak{B}) = \tau$.

Since $B_n = \bigcup_{i \in I_n} V_I$, $B_n = T_2(V)$.

Hence $Y = T_2(B) \subseteq T_2(v)$. Since $v \subseteq y$ and Y is closed under arbitrary unions, $T_2(v) \subseteq \tau$. hence $Y = T_2(v)$./ Thus v is a countable sub class which is an open base for τ.

Separability and Second Countability :

3.43 Proposition : A second countable topological space is separable.

Proof : Let (X, τ) be a topological space with a countable open base $\{B_n / n \in N\}$.

Let $J = \{n / B_n \neq \phi\}$.

For each $n \in J$, choose x_n in B_n. The set $H = \{x_n / n \in J\}$ is clearly countable. If $x \in X$ and V is any open set in X containing x, then there exists a $n \in N \ni x \in B_n \subseteq V$. So that $x_n \in V$. thus every neighbourhood of x intersects H. Hence $\overline{H} = X$ ie., H is dense in X. Thus X is separable.

3.44 Remark : In general separability does not imply second countability (see exercise 7)

For metric spaces these two notions are equivalent as is evident from the following theorem.

3.45 Theorem : Every separable metric space is second countable.

Proof : Let X be a separable metric space with metric d and A be a countable dense set. We may enumerate the elements of A as $(a_1, a_2, \ldots\ldots a_n..\}$.

For a fixed n, let $\mathfrak{B}_n = \{S_r(a_n) \,/\, r\in Q, r>0\}$ clearly $\mathfrak{B}_n$ is countable. Hence $\mathfrak{B} = \bigcup_{n\geq 1} \mathfrak{B}_n$ is countable. We show that $\mathfrak{B}$ is an open base for (X, d). Clearly elements of $\mathfrak{B}$ are open spheres and hence are open sets. If V is any open set and $x\in v$, $\exists$ a $\delta>0 \ni S_\delta(x) \subseteq V$. Since A is dense, $\exists$ an $a_n \in S_{\delta/3}(x)$.

Choose $r\in Q$, r.0 $\ni \delta/3 < r < 2\delta/3$.

Since $d(x, a_n)<\delta/3 <r$, $x\in S_r(a_n)$.

$y\in S_r(a_n) \Rightarrow d(y, a_n) < r$.

$\Rightarrow d(y, x) \subseteq d(y,a_n)+ d(a_n, x) < r + \delta/3 + \delta/3 = \delta$

$\Rightarrow y\in S_\delta(x)$. Hence $S_r(a_n) \subseteq S_\delta(x)$.

Thus $x\in S_r(a_n) \subseteq S\delta(x) \subseteq G$.

Since $S_r(a_n) \in \mathfrak{B}_n \subseteq \mathfrak{B}$, it follows that $\forall$ G $\in$J and

$x\in G$, $\exists$ a $\mathfrak{B} \in \mathfrak{B} \ni x \in B\subseteq G$.

Hence B is a basis for (x, d)

Since B is countable, (X, d) is second countable.

3.46 Example : The Euclidean space R with the usual metric is separable, hence second countable.

Proof : We use Archimedean principle which says that if $\alpha\in\Re$ and $\alpha>0$ there exists a natural number n such that $n>\alpha$.

As a consequence given $a\in R$, $b\in R$, $a<b$ there exists $x\in Q \ni a<x<b$.

From this it follows that if $x\in R$ and $\epsilon>0$, $\exists$ $y\in Q \ni x-\epsilon<y<x$ so that $(x-\epsilon, x+\epsilon)$ contains a point of Q other than x. If V is a neighbourhood of x, $\exists$ an $\epsilon>0 \ni (x-\epsilon, x+\epsilon) \subseteq V$. Since $(x-\epsilon, x+\epsilon)$ contains a $y\in Q - \{x\}$, $y\in V\cap Q-\{x\}$. Hence x is a limit point of Q. Since this is true for every $x\in R$, $R \subseteq \overline{Q} \subseteq R$. Hence $R = \overline{Q}$. Thus R is separable.

3.47 Example : R^n with the Euclidean metric is second countable, hence separable.

Proof : We use the fact that if $A_1, \ldots\ldots A_n$ are

Countable then so is $A_1 x \ldots x\ A_n$. Since Q is separable, $Q^n = Q^x Q^x \ldots x\ Q$ (n times) is countable. Let $x = (x_1, .. x_n) \in R^n$; and v be a neighbourhood of x. Then $\exists \in > 0 \ni S_\in (x) \subseteq$ V. Choose $y_i \in Q \ni x_i - \frac{\in}{\sqrt{n}} < y_i < x_i + \frac{\in}{\sqrt{n}}$ and $x_i \neq y_i$.

$$\text{Then } y = y_1, \ldots.. y_n) \in Q^n \text{ and } d(x, y) = \{\sum_{i=1}^{n} |x_i - y_i|^2\}^{1/2}$$

$$< \{\sum_{i=1}^{n} \in^2/n\}^{1/2} = \in.$$

Thus $y \neq x$ and $y \in S_\in (x) \subseteq V$. Since every neighbourhood of x contains a point of Q^n other than x, $x \in \overline{Q^n}$. This is true for every $x \in R^n$ so that $R^n \subseteq \overline{Q^n} \subseteq R^n$, hence $R^n = \overline{Q^n}$. Hence R^n is separable.

3.48 Solutions to short answer Questions:

SAQ 3.5 : Let $x \in X$, r>o and $v = S_r(x)$, $y \in v$

Then $d(x, y) < r$. let $s = r - d(x, y)$.

We show that $S_r(y) \subseteq S_r(x)$

$$z \in S_s(y) \Rightarrow d(z, y) < s \Rightarrow d(x, z) \subseteq d(x, y) + d(y, z).$$

$$< d(x, y) + s$$

$$= r.$$

Hence $S_s(y) \subseteq S_r(x)$. This is true $\forall\ y \in S_r(x)$.

Hence $S_r(x)$ is an open set.

SAQ 3.8 : Use De Morgan's laws : $(\bigcap_{i \in I} V_i) = \bigcup_{i \in I} V_i^1$ and

$$(\bigcup_{i \in I} V_i)^1 = \bigcap_{i \in I} V_i^1$$

$$V_i \in \tau\ \forall\ i \in I \Rightarrow V_i^1 \in \Sigma \Rightarrow \bigcap_{i \in I} V_i^1 \in \Sigma \Rightarrow (\bigcup_{i \in I} V_i)^1 \in \Sigma$$

$\Rightarrow \bigcup_{i \in I} V_i \in \tau.$

If I is finite and $V_i \in \tau \; \forall \; i \in I$; $V_i^1 \in \Sigma \; \forall \; i \in I$

$\Rightarrow \bigcap_{i \in I} v_I^1 \in \Sigma$

$\Rightarrow (\bigcup_{i \in I} V_i)^1 \in \Sigma \Rightarrow \bigcup_{i \in I} V_i \in \tau.$

SAQ 3.9 : Let x be a set with at least two points and d be the discrete metric on X. if $x \in X$, then $S_1(x) = \{x\}$

Since the topology induced by this metric is the discrete topology every subset of X is open hence closed. Thus $\overline{S_1}(x) = \{x\}$

However $S_1[x] = \{y \in x / d(x, y) \leq 1\} = X$.

Thus it is not necessarily true that in a metric space $\overline{S_r(x)} = S_r[x]$.

SAQ 3.10 : Let (X, d) be a metric space and $x \in x$. If $x = \{x\}$, $\{x\}$ is closed. Suppose $\{x\} \neq X$

Then $X \setminus \{x\} \neq \phi \; \forall \; y \in X \setminus \{x\}$, $r = d(x, y) > 0$.

We show that $S_r/2\,(y) \subseteq x \setminus \{x\}$. This holds since $z \in S_r/2\,(y)$

$\Rightarrow d(y, z) < r/2$. Since $d(y, x) = r$, z; $\neq$ x so $z \in X \setminus \{x\}$

Then $S_r/2\,(y) \subseteq X \setminus \{x\}$. This shows that $X \setminus \{x\}$is an open set. Hence $\{x\}$ is a closed set.

SAQ 3.22 : We consider (x, J), J= $\{\phi, x\}$.

Suppose $x_0 \in X$. If $x \in X$ and $x \neq x_0$, then x is the only neighbourhood of x in X. We have

$\{x_0\} \cap (X \setminus \{x\}) = \{x_0\}$.

Therefore x is a limit point of the set $\{x_0\}$.

We have $\{x_0\} \cap (X \setminus \{x_0\}) = \phi$,

And so x_0 is not a limit point of the set $\{x_0\}$. $\overline{\{x_0\}} = X$.

SAQ 3.23 : We consider $\overline{\mathbb{N}}$

Suppose $A \subseteq \mathbb{N}$ is a finite subset. Then $D(A) = \phi$ and hence $\overline{A} = A$

Suppose A is infinite.

Consider a neighbourhood V of ∞. By definition V^1 = N\V is finite ; therefore there are at least two points x, y in A which are not in v^1. Thus x, y $\in$v. At least one of them is different from ∞. Thus $A \cap (V\backslash\{\infty\}) \neq \phi$.

So ∞ is a limit point of A.

Let n $\in$N, Then {n} is a neighbourhood of n.

We have $A \cap (\{n\} \setminus \{n\}) = \phi$

So n is not a limit point of A.

Hence D(A) = {∞} and $\overline{A} = \cup\{\infty\}$.

SAQ 3.24 : We consider a set x with at least two elements with the topology T = {ϕ,X}. In SAQ 3.22 we have seen that the set X \ {x} is the set of limit points of {x}. Since X contains at least two elements X \ {x} is not empty and is not equal to x. Therefore it is not a closed subset of X:

D ({x}) is not a closed set.

SAQ 3.25 : Let z be a limit point of D(A). To show that z$\in$D(A) we have to show that z is a limit point of A. Let V be a neighbourhood of z. Since z is a limit point of D(A), $\exists$ a y $\in$ D(A) $\cap$ V such that y$\neq$z. Since y $\in$ V $\exists$ an r>o $\ni$ $S_r(y) \subseteq$ V. Since d(y, z)>0 we may choose r $\ni$ 0<r< d(x, z). Since y $\in$D(A), $\exists$ an x $\in$ $S_r(y) \cap A$ $\ni$ y $\neq$x. Since x $\in$ $S_r(y)$, 0<d(x, y) <r<d(y, z) so that x $\neq$ z. Also x $\in$ $S_r(y) \subseteq$V. Thus x$\in$V$\cap$A and x$\neq$z. Since every neighbourhood V of z contains a x$\neq$z $\ni$ x$\in$A, z is a limit point A. Therefore D(A) is a closed subset of X.

SAQ 3.38 : To prove that B_2 is an open base we have to show that every open rectangle is an open set and for every open set G in R^2 and x$\in$G $\exists$ an open rectangle R such that x$\in$R $\subseteq$ G. Towards this end it is enough to show that if

R=(a, b) x (c, d) and x=(x_1, x_2) $\in$ R, $\exists$ a δ>0 $\ni$ $S_\delta(x) \subseteq$R and if r>0 and y$\in S_r(x)$ $\exists$ a rectangle S= (α,β) x (γ,δ) $\ni$ y$\in$S $\subseteq S_r(x)$.

Let x=(x_1,x_2) $\in$(a, b) x (c, d) $\Rightarrow$ a<x_1<b and c<x_2<d.

δ = ½ min { x_1-a, b-x_1, x_2-c, d-x_2}. Y= (y_1, y_2) $\in S_\delta$ (x)

$\Rightarrow d(x, y) < \delta \Rightarrow (x_1-y_1)^2 + (x_2-y_2) < \delta^2$

$\Rightarrow |x_1-y_1| < \delta$ & $|x_2-y_2| < \delta$

$\Rightarrow y_1 \in (x_1-\delta_1, x_1+\delta) \Rightarrow a<x_1-\delta< y_1 < x_1+\delta<b$ and similarly $c<y_2<d$ so that $y\in(a, b)$. Thus $S_\delta(x) \subseteq (a, b) \times (c, d)$.

Again if $r>0$ and $y = (y_1, y_2) \in S_r(x)$, $d(x, y)<r$ if $\delta = r-d(x, y)$, $S_\delta(y) \subseteq S_r(x)$. The above argument shows that S_δ (y) contains a rectangle $S=(a, b) \times (c, d)$ containing y. Hence $y\in S \subseteq S_r(x)$.

SAQ 3.40 : Let (x, d) be a metric space and $x\in X$. Then $\{S_{1/n}(x)/n\in \mathbb{N}\}$ is a countable collection of open sets which form an open base at x. For this let V be a neighbourhood of x is the induced topology $\exists$ an $\varepsilon>0$, ε $x\in S\varepsilon(x) \subseteq v$. If $n\in\mathbb{N}$ and $n>1/\varepsilon$, then $1/n<\in$ $S_{1/n}(x) \subseteq S\varepsilon(x) \subseteq V$. Thus every neighbourhood of x contains $S_{1/n}(x)$ for some $n\in\mathbb{N}$. This completes the proof.

SAQ 3.49 : Model Examination Questions

1. Define an open base for a topology τ. Show that given any nonempty family τ of subsets of a nonempty set x there is a unique topology τ on **** for which y is an open sub base.
2. State and prove Lindelof's theorem.
3. Show that every open base of a second countable topological space contains a countable sub family which is a base.
4. Define first countable topological space and second countable topological space. Show that a second countable topological space is first countable but the converse is not true. Show that in a second countable topological space every open set is a union of a countable family of open sets.
5. State Kuratowski's closure axioms and prove that any closure operation "-" satisfying these axioms induces a topology τ on x such that for any subset A of X, $\overline{A}=A$ iff $X \setminus A\in\tau$.

6. Show that in a topological space (X,τ), for any $A \subseteq X$ $\overline{A} = A \cup D(A) = \{ x \in X /$ every neighbourhood of x intersects $A\}$.

3.50 Examples :

1. Let X be a topological space and B be an open base for X such that $\forall\ x \in X$, $\exists\ B \in \mathfrak{B} \ni B \neq X$ and $x \in B$. Show that $B_1 = B - \{\phi, X\}$ is an open base for X.
2. Give an example of a metric space which is not (a) separable (b) second countable.
3. Is a metrizable space first countable? Justify.
4. Is $S_r[x]$ closed in a metric space $\forall\ x \in X$ and $r>0$.
5. Show that in any topological space (X,τ) int(A) is the "largest" open set contained in A- more precisely,

   (1) int (A) is an open set, int(A) $\subseteq$A.

   (2) If B is any open set $\ni B \subseteq A$, then $B \subseteq$ int(A).
6. Let X be a nonempty set and consider the class τ of subsets of X consisting of the empty set ϕ and all sets whose complements are countable. For definiteness let X =R, the real line

   (1) Is X first countable?

   (2) Is X second countable?

   (3) Find $\overline{A}$ when A is the set of even integers.
7. Answer (1), (2), (3) of exercise 6 where X=z, the set of integers.
8. Let (X,τ) be a topological space, $A \subset X$. Show that A is dense in $\overline{A}$ when $\overline{A}$ is treated as a sub space of (X,τ).
9. Let $Z \subset Y \subset X$ and (X, Y) be a topological space. If Z is closed in Y and Y is closed in X, does it follow that Z is closed in X? Same question with "open" in place of "closed".
10. Show that a subset of a topological space is dense if and only if it intersects every nonempty open set.

11. Let (X,τ) be a topological space and $A \subseteq X$. Show that the following conditions are equivalent.

    (1) A is closed and has no isolated points

    (2) $\overline{A} = D(A)$.

Definition : $A \subset X$ is said to be perfect if A satisfies one of the above two conditions.

12. Let C be the Cantor set in [0, 1] obtained by removing the middle on third at every stage. Show that C is perfect.

13. For any $A \subset (X,\tau)$ show that $int(A^1) = (\overline{A})^1$.

14. Show that $\overline{A} = A$ iff A contains its boundary.

15. Let (X,τ) be a topological space and $A \subseteq X$. Show that boundary of $A = \phi$ if and only if $A \in \tau$ and $A^1 \in \tau$.

16. Let (X,τ) be a topological space. For $A \subset X$ show that int $(\overline{A}) = \phi$ iff every nonempty open set has a nonempty open subset disjoint from A. Such sets A (with int $\overline{A} = \phi$) are called nowhere dense sets.

17. (a) Show that a closed subset a of (X,τ) is nowhere dense iff A^1 is dense

    (b) Consider the real line with the usual topology

    (i) Is Q dense?

    (ii) Is Q nowhere dense/

    (iii) Is Q closed?

    (iv) Is Q open?

18. Show that the boundary of a closed set is nowhere dense. What is the boundary of Q in R with the usual topology?

19. Show that the set of isolated points of a second countable space is either empty or countable.

20. Show that (X,τ) is second countable and $Y \subseteq X$ is uncountable then $D(Y) \neq \phi$.

Lesson Writer: V.J. LAL

LESSON 4

CONTINUITY

4.1. Introduction: In this lesson we make a beginning of learning the concept of continuity which is the most fundamental notion in Topology. We first define continuity of a function from a topological space X into a topological space Y at point x and on X and study some elementary properties of continuous functions. We then turn our attention to continuity in metric spaces and establish equivalence of the two definitions in the context of metric spaces. Finally we establish some basic properties of continuous real or complex valued functions.

4.2 Definition: Let X, Y be topological spaces. A mapping f: $X \to Y$ is said to be continuous at $x_0 \in X$ if for every neighborhood V of $f(x_0)$ there is a neighborhood U of x_0 such that $f(U) \subseteq V$.

4.3 Example: Let X be any nonempty set. Equip X with the discrete topology. If Y is any topological space and $f : X \to Y$ is any map and x_0 is any point of X, f is continuous at x_0 because every subset of X is open with respect to the discrete topology and in particular for every open set V containing $f(x_0)$, $U = f^{-1}(V) \subseteq X$ is an open set.

4.4 Theorem: Let X, Y be topological spaces and $f : X \to Y$ be any map. The following are equivalent.

(a) f is continuous at every $x_0 \in X$

(b) $f^{-1}(V)$ is open in X for every open set V in Y.

Proof: Assume (a) and let V be any open set in Y. If $x \in f^{-1}(V)$, then $f(x) \in V$ so that V is a neighborhood of f(x). By (a) $\exists$ a neighborhood U_x of x such that $f(U_x) \subseteq V$. the set $U = \bigcup_{x \in f^{-1}(V)} U_x$ is open X. since $f(U_x) \subseteq V$ for $x \in f^{-1}(V)$, $U_x \subseteq f^{-1}(V)$ for $x \in f^{-1}(V)$.

Hence $U \subseteq f^{-1}(V)$. On the other hand $x \in f^{-1}(V) \Rightarrow x \in U_x \subseteq U$

$\Rightarrow f^{-1}(V) \subseteq U$. thus $U = f^{-1}(V)$ is open in X. thus (a) $\Rightarrow$ (b).

Assume (b) and let $x \in X$. If V is any neighborhood of f(x), V is an open set in Y so by (b) $f^{-1}(V)$ is an open set in X containing x. Hence $f^{-1}(V)$ is a neighborhood of x. Clearly $f(f^{-1}(V)) = V$. Thus f is continuous at x $\forall x \in X$. Thus (b) $\Rightarrow$ (a).

Hence (a) $\Leftrightarrow$ (b)

4.5 Definitions: Let X and Y be topological spaces and $f : X \to Y$ be a mapping.

1) f is said to be continuous if $f^{-1}(G)$ is open in X $\forall$ open sets G in Y.
2) f is said to be open if f (G) is open in Y $\forall$ open sets G in X.
3) f is said to be homeomorphism if f is a bijection and is both continuous and open.
4) f(X) is said to be a continuous image of X if f is continuous.
5) f(X) is said to be a homeomorphic image of X if f: $X \to Y$ is continuous, one-one and open.

4.6 Remark: Some authors prefer to define continuity of f in terms of continuity at every point of X. However theorem 4.3 confirms equivalence of these two definitions.

4.7 Example: If X is a nonempty set and τ_i , τ_d are respectively the indiscrete topology and discrete topology on X respectively then the identity map I. : $X \to X$ is clearly a bijection.

When the domain space X is equipped with the discrete topology every set in X is open in τ_d where as the only open sets in τ_i are ϕ and X. Thus

(a) If X has more than one point, I : $(x, \tau_i) \to (x, \tau_d)$ is not continuous.

(b) As mentioned in 4, I: $(x, \tau_d) \to (X, \tau_i)$ is continuous.

(c) If X has more than one element and $x \in X$, $I(\{x\}) = \{x\} \neq X$ and is nonempty so that I : $(X, \tau_d) \to (x, \tau_i)$ is not open, hence is not a homeomorphism.

4.8 SAQ: Let (X, τ) and (Y,σ) be topological spaces. For a mapping f: $X \to Y$ prove that the following are equivalent.

(a) f is continuous

(b) $f^{-1}(F)$ is closed in (X, τ) for every closed set F in (Y,σ)

(c) $f(\overline{A}) \subseteq \overline{f(A)}$ $\forall A \subseteq X$.

4.9 Theorem: Let X, Y be topological spaces, $\mathcal{B}$ an open base for X and σ an open subbase for Y. Then the following are equivalent.

(a) $f : X \to Y$ is continuous

(b) $f^{-1}(B)$ is open in X for every basic open set B in Y.

(c) $f^{-1}(B)$ is open in X for every $B \in \sigma$.

Proof: (a) $\Rightarrow$ (b) is clear

(b) $\Rightarrow$ (c) since $\sigma \subseteq T_1(\sigma)$ and $T_1(\sigma)$ is an open base for Y.

(c) $\Rightarrow$ (a) : Let V be open in Y, $x \in X$ and $= f(x) \in V$.

Then since $T_1(\sigma)$ is an open base for the topology on Y, $\exists\, B \in T_1(\sigma)$ such that $y \in B \subseteq V$. since $B \in T_1(\sigma)$, $\exists$ a finite number of subbasic open sets $B_1, \ldots B_n$ such that

$$B = \bigcap_{i=1}^{n} B_i .$$

Since $y \in B$, $y \in B_i$ for $1 \le i \le n$. by (c) $f^{-1}(B_i)$ is open in X $\forall\, i$.

Hence $f^{-1}\left(\bigcap_{i=1}^{n} B_i\right) = \bigcap_{i=1}^{n} f^{-1}(B_i)$ is open in X.

Since $y = f(x) \in B_i$, $x \in f^{-1}(B_i)\ \forall\, i$, so that $x \in \bigcap_{i=1}^{n} f^{-1}(B_i) = f^{-1}\left(\bigcap_{i=1}^{n} B_i\right)$

Thus $x \in G = \bigcap_{i=1}^{n} f^{-1}(B_i)$ and G is open in X.

Further $f(G) = f f^{-1}\left(\bigcap_{i=1}^{n} B_i\right) = f(f^{-1}(B)) \subseteq B \subseteq Y$.

Hence f is continuous at x. Since this holds $\forall\, x \in X$, f is continuous on X.

Continuity in metric spaces:

4.10 Definition: Let (X, d_1) and $(Y. d_2)$ be metric spaces. A mapping $f: X \to Y$ is said to be continuous at $x \in X$ if for every $\in > 0$

there is a $\delta > 0$ such that

$y \in X,\ d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \in$

f is said to be continuous (on X) if f is continuous at every point of X. (compare with 4.5 (1)).

4.11 Proposition: Let (X, d_1), (Y, d_2) be metric spaces, τ_i be the topology induced by d_i . If $x \in X$, then the following are equivalent.

(a) $f: (X, d_1) \to (Y, d_2)$ is continuous at x.

(b) $f: (X, \tau_1) \to (Y, \tau_2)$ is continuous at x.

Proof: If $\varepsilon > 0$ and $x \in X$ we write $S^1_\varepsilon(x) = \{y \in Y / d_1(x, y) < \varepsilon\}$ and if $y \in Y$ we write $S^2_\varepsilon(y) = \{z \in Y / d_2(y, z) < \varepsilon\}$

Assume (a) and let $x \in X$, $V \in \tau_2$ and $y = f(x)$.

Then $\exists\ \varepsilon > 0 \ni S^2_\varepsilon(y) \subseteq V$. By (a) $\exists\ \delta > 0 \ni$

$y \in S^1_\delta(x) \Rightarrow d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$

$\Rightarrow f(y) \in S^2_\varepsilon(f(x))$

Hence $f(S^1_\delta(x)) \subseteq S^2_\varepsilon(f(x)) \subseteq V$.

Thus f is continuous at x i.e. (b) holds. Therefore (a) $\Rightarrow$ (b)

Conversely assume that (b) holds. Let $x \in X$ and $\varepsilon > 0$.

Then $S^2_\varepsilon(f(x)) \in \tau_2$. Since $S^2_\varepsilon(f(x))$ is a neighborhood of f(x), there exists a $G \in \tau_1 \ni$ $x \in G$ and $f(G) \subseteq S^2_\varepsilon(f(x))$.

Since G is open and $x \in G$, $\exists\ \delta > 0 \ni S^1_\delta(x) \subseteq G$

Clearly $f(S^1_\delta(x)) \subseteq S^2_\varepsilon(f(x))$

Thus $d_1(x\ y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$

Hence $f: (X, d_1) \to (Y, d_2)$ is continuous at x.

4.12 SAQ: $f: (X, d_1) \to (Y, d_2)$ is continuous at $x \in X$ if and only if for every $\in > 0$, there a $\delta > 0 \ni f(S^1_\delta(x)) \subseteq S^2_\varepsilon(f(x))$

Remark: Continuity in metric spaces has a special property – namely transforming convergent sequences in the domain into convergent sequences in the codomain along with limits. To establish this we need the definition of convergence of a sequence and of a cauchy sequence in a metric space.

4.13 Definitions: Let (X,d) be a metric space, $\{x_n\}$ a sequence in X and $x \in X$. We say that $\{x_n\}$ is a cauchy sequence in x if for every $\varepsilon > 0$ there corresponds $N(\varepsilon) \in \mathbb{N}$ such that

$d(x_n, x_m) < \varepsilon$ whenever $n > m \geq N(\varepsilon)$

(equivalently whenever $n \geq N(\varepsilon)$ and $m \geq N(\varepsilon)$).

We say that $\{x_n\}$ is a convergent sequence and $x = \lim_n x_n$ or $\{x_n\}$ converges to x if for every $\varepsilon > 0$ there corresponds $N(\varepsilon) \in \mathbb{N}$ such that

$d(x_n, x) < \varepsilon$ whenever $n \geq N(\varepsilon)$

4.14 Proposition: Let (X, d_1) and (Y, d_2) be metric spaces.

A function $f: X \to Y$ is continuous at $x \in X$ if and only if for every sequence $\{x_n\}$ in X with $\lim_n x_n = x$, $\lim_n f(x_n) = f(x)$ in Y.

Proof: Assume that f is continuous at $x \in X$ and $\lim_n x_n = x$ in X.

If $\varepsilon > 0$, $\exists$ a $\delta > 0 \ni d_2(f(y), f(x)) < \varepsilon$ whenever $y \in X$ and $d_1(y, x) < \delta$. Since $\lim_n x_n = x$, for this $\delta > 0$ there corresponds a positive inter N such that

$d_1(x_n, x) < \delta$ whenever $n \geq N$.

Hence $d_2(f(x_n), f(x)) < \varepsilon$ whenever $n \geq N$.

Hence $\lim_n f(x_n) = f(x)$ in Y.

Conversely assume that $\forall$ sequence $\{x_n\}$ in X that converges to x, $\lim_n f(x_n) = f(x)$.

Suppose f is not continuous at x. Then there is a positive number ε such that whatever $\delta > 0$ we choose, there is at least one x_δ, depending on δ, in X such that

$d_1(x_\delta, x) < \delta$ but $d_2(f(x_\delta), f(x)) \geq \varepsilon$

In particular for every positive integer n, there corresponds an element x_n in X such that

$d_1(x_n, x) < \frac{1}{n}$ but $d_2(f(x_n), f(x)) \geq \varepsilon$

This implies that $\lim_n x_n = x$ where as $\lim_n f(x_n) \neq f(x)$ as $\lim_n f(x_n) = f(x)$ would imply that for some $N \in \mathbb{N}$ and all $n \geq N$ $d_2(f(x_n), f(x)) < \varepsilon$ which does not happen. This contradiction yields that f must be continuous at x.

4.15 Corollary: Under the above hypothesis f is continuous on X if and only if $\lim_n x_n = x$ in $(X, d_1) \Rightarrow \lim_n f(x_n) = f(x)$ in (Y, d_2).

Proof: f is continuous on X if and only if f is continuous at every $x \in X$. By 4.14 if $x \in X$, f is continuous at x iff $\lim_n x_n = x$ in $(X, d_1) \Rightarrow \lim_n f(x_n) = f(x)$ in (X, d_2). Hence the result.

4.16 Example: Let d_1, d_2 be metrics on a nonempty set X and assume that $d_1(x, y) \leq d_2(x, y)$ for all x, y in X. Then the identity map I on X is continuous from (X, d_2) to (X, d_1).

Solution: We use 4.15. Let $\{x_n\}$ be any sequence in X, $x \in X$ and $\lim_n x_n = x$ in (X, d_2) in $\lim_n d_2(x_n, x) = 0$. Then $\forall \varepsilon > 0$

$\exists$ a $N(\varepsilon) \in \mathbb{N} \ni d_2(x_n, x) < \varepsilon$ whenever $n \geq N(\varepsilon)$

Hence $d_1(x_n, x) \leq d_2(x_n, x) < \varepsilon$ whenever $n \geq N(\varepsilon)$

So that $\lim_n x_n = x$ in (X, d_1) i.e. $\lim_n I(x_n) = I(x)$ in (X, d_1).

This implies that I is continuous on X.

Continuity of real or complex valued functions:

4.17 Definitions: In what follows X stands for a nonempty set. The collection of all functions from X into K is denoted by F(X). We define the following pointwise operations on F(X).

Let f, g belong to F(x), $\alpha \in K$ and $x \in X$

Pointwise addition:- $f + g : (f + g)(x) = f(x) + g(x)$

Pointwise scalar multiplication:- $\alpha f : (\alpha f)(x) = \alpha . f(x)$

Pointwise multiplication: $fg : (f g)(x) = f(x) g(x)$

When K = R

Maximum f & g : - $(f \vee g)(x) = \text{maximum } \{f(x), g(x)\}$

($f \vee g$ is called f join g)

minimum f & g : - $f \wedge g = (f \wedge g)(x) = \text{minimum } \{f(x), g(x)\}$

($f \wedge g$) is called f meet g)

When $K = \mathbb{C}$ and $f(x) = f_1(x) + i f_2(x)$ where $f_1(x) \in \mathbb{R}$ & $f_2(x) \in \mathbb{R}$

(real f) : (real f) $(x) = f_1(x)$

(Im f) : (Imf) $(x) = f_2(x)$

Complex conjugate $\bar{f}$: $\bar{f}(x) = f_1(x) - i f_2(x)$

Absolute f, $|f|$: $|f|(x) = |f(x)| = \sqrt{f_1^2(x) + f_2^2(x)}$

4.18 Proposition: If X is a topological space and f, g are continuous real or complex valued functions on X then so are

(i) $f + g$ (ii) αf for any scalar α

Proof: It is enough to establish continuity at each $x \in X$. Fix x.

Let $\varepsilon > 0$. By the continuity of f and g at x, there exist neighborhoods V_1, V_2 of x such that $|f(y) - f(x)| < \frac{\varepsilon}{2}$ if $y \in V_1$ and

$$|g(y) - g(x)| < \frac{\varepsilon}{2} \text{ if } y \in V_2$$

$V_1 \cap V_2$ is a neighborhood of x and if $y \in V_1 \cap V_2$

$$|(f+g)(y)-(f+g)(x)| \le |f(y)-f(x)| + |g(y)-g(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence f + g is continuous at x.

To prove continuity of α f at x, given $\varepsilon > 0$ choose a neighborhood V of x such that

$$|f(y)-f(x)| < \frac{\varepsilon}{1+|\alpha|} \text{ if } y \in V$$

For $y \in V$, $|(\alpha f)(y) - (\alpha f)(x)|$

$$= |\alpha|\ |f(y)-f(x)|$$

$$< \frac{|\alpha|}{1+|\alpha|}\varepsilon < \varepsilon$$

Hence α f is continuous at x.

4.19 Proposition: Let X be a topological space. If f and g are continuous real or complex valued functions on X then f g is continuous on X.

Proof: It is enough if we establish continuity of f g at each $x \in X$.

Let $x \in X$. There exist neighborhoods V_1, V_2 of x such that $|f(y) - f(x)| < 1$ if $y \in V_1$ and $|g(y) - g(x)| < 1$ if $y \in V_2$.

$V_1 \cap V_2$ is a neighborhood of x and for $y \in V_1 \cap V_2$

$|f(y)| - |f(x)| \le |f(y) - f(x)| < 1$ and also

$|g(y)| - |g(x)| \le |g(y) - g(x)| < 1$ so that $|f(y)| < 1 + 1\ f(x)|$ and $|g(y)| < 1 + |g(x)|$ for $y \in V_1 \cap V_2$.

Now if $\varepsilon > 0$ there exist neighborhoods U_1 and U_2 of x such that

$$|f(y) - f(x)| < \frac{\varepsilon}{1+|f(x)|+|g(x)|} \text{ for } y \in U_1 \text{ and}$$

$$|g(y) - g(x)| < \frac{\varepsilon}{1+|f(x)|+|g(x)|} \text{ for } y \in U_2$$

$U_1 \cap U_2$ is a neighborhood of x and for $y \in U_1 \cap U_2 \cap V_1 \cap V_2$

$|f(y)\,g(y) - f(x)\,g(x)|$

$\leq |f(y)||g(y) - g(x)| + |g(x)||f(y) - g(y)|$

$< (1 + |f(x)|) \dfrac{\varepsilon}{1 + |f(x)| + |g(x)|} + |g(x)| \dfrac{\varepsilon}{1 + |f(x)| + |g(x)|} = \varepsilon$

Hence f g is continuous at x.

4.20 Proposition: If f and g are real valued continuous functions on a topological space X then so are $f \wedge g$ and $f \vee g$.

Proof: Write $f \wedge g = h$ and $f \vee g = H$.

We prove that for every $x \in X$, H is continuous at x.

Given $\varepsilon > 0$ there exist neighborhood V_1, V_2 of x,

such that $|f(y) - f(x)| < \in$ for $y \in V_1$, and $|g(y) - g(x)| < \varepsilon$ for $y \in V_2$

For $y \in V_1 \cap V_2$,

$$f(x) - \varepsilon < f(y) < f(x) + \varepsilon \text{ and}$$

$$g(x) - \varepsilon < g(y) < g(x) + \varepsilon$$

$\Rightarrow H(x) - \varepsilon \leq H(y) < H(x) + \varepsilon$

$\Rightarrow |H(y) - H(x)| < \varepsilon$ if $y \in V_1 \cap V_2$.

Hence H is continous at x.

Hence $H \in \mathbf{C}(X, \mathbb{R})$

Continuity of h follows from the fact that $\forall\, y \in X$

$h(y) = \min\{f(y), g(y)\}$

$= -\max\{-f(y), -g(y)\}$

4.21 Proposition: Let X be a topological space and $f : X \to \mathbb{C}$ be any function on X. Then the following are equivalent

(i) f is continuous

(ii) Real f and Im f are continuous

(iii) $\bar{f}$ is continuous.

Proof: It is enough to establish the equivalence at each $x \in X$.

Fix $x \in X$. Write f_1 =Real f and f_2 = Im f.

Then $f(y) = f_1(y) + i f_2(y) \ \forall\, y \in X$.

Hence $| f(y) - f(x) | = | (f_1(y) - f_1(x) + i (f_2(y) - f_2(x)) |$

$$= \{| f_1(y) - f_1(x)|^2 + |f_2(y) - f_2(x)|^2\}^{1/2}$$

Since for any complex number $z = \alpha + i\beta$, $| z | = \sqrt{\alpha^2 + \beta^2}$,

$$|\alpha| \le |z| \le |\alpha| + |\beta| \text{ and } |\beta| \le |z| \le |\alpha| + |\beta|$$

we have for $j = 1,2$

$$|f_j(y) - f_j(x)| \le |f(y) - f(x)| \le |f_1(y) - f_1(x)| + |f_2(y) - f_2(x)| \text{ ------ } \quad (1)$$

If f is continuous at x, given $\varepsilon > 0$ there exists a neighborhood V of x such that for $y \in V$.

$|f(y) - f(x)| < \varepsilon$ and hence $|f_j(y) - f_j(x)| < \varepsilon$ for $j = 1, 2$ & $y \in V$.

This implies that f_1, f_2 are continuous at x.

Conversely if f_1 and f_2 are continous at x, and $\varepsilon > 0$

There exist neighborhoods V_1, V_2 for x such that for $j = 1$ & 2.

$$| f_j(y) - f_j(x) | < \frac{\varepsilon}{2} \text{ if } y \in V_j$$

If $y \in V = V_1 \cap V_2$ then by (1)

$$|f(y) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ so f is continuous at x.}$$

Thus (i) $\Leftrightarrow$ (ii),

Hence (ii) $\Leftrightarrow$ (iii) by 4.18

4.22 SAQ: Show that if $f: X \to \mathbb{C}$ is continuous so is $| f |$.

Also give an example of a function for which the converse fails.

4.23 SAQ: If τ is a finite set of continuous real valued functions on a topological space X and

$$g(x) = \sum_{f_i \in \tau_i} f_i(x),\ h(x) = \inf \{f_i(x) / f_i \in \tau\} \text{ and } H(x) = \sup \{f_i(x) / f_i \in \tau\}$$

Shows that g, h, H are continuos.

4.24. Model Examination Questions:

1. If X and Y are topological spaces show that f: $X \rightarrow Y$ is continuous if and only if $f^{-1}(G)$ is open in X for every open set G in Y.
2. If (X, d), (Y,σ) are metric spaces, show that f: $X \rightarrow Y$ is continuous if and only if for every $x \in X$ and every sequence $\{x_n\}$ in X such that $\lim_n x_n = x$, $\lim_n f_n(x) = f(x)$.
3. If f and g are real valued continuous functions on a topological space X show that $f \vee g$ is continuous.

4.25 Answers to Self Assessment Questions:

SAQ 4.8 $a \Rightarrow b$ Assume that f: $X \rightarrow Y$ is continuous. If $F \subseteq Y$ is closed then $G = F^{1}$ is open in Y. Hence $f^{-1}(G)$ is open in X. since $f^{-1}(G) = (f^{-1}(F))^{1}$ $f^{-1}(F)$ is closed in X.

$b \Rightarrow c$ Assume that $f^{1}(F)$ is closed in X whenever F is closed in Y.

If $A \subseteq X$, $\overline{f(A)}$ is closed in Y. Hence $f^{1}(\overline{f(A)})$ is closed in X.

Since $A \subseteq f^{-1}(\overline{f(A)})$, $\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$ since $f^{-1}(\overline{f(A)})$ is closed. Hence

$f(\overline{A}) \subseteq \overline{f(A)}$

$c \Rightarrow b$. Assume that $f(\overline{A}) \subseteq \overline{f(A)}$ $\forall A \subseteq X$. Let F be any closed set in Y. Then

$f(\overline{f^{-1}(F)}) \subseteq (\overline{f^{-1}(F)}) \subseteq \overline{F} = F$.

Hence $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$. Since $f^{-1}(F) \subseteq \overline{f^{-1}(F)}$ it follows that $\overline{f^{-1}(F)} = f^{-1}(F)$ so that $f^{-1}(F)$ is closed in X.

$b \Rightarrow a$. Assume that $f^{-1}(F)$ is closed in X whenever F is closed in Y.

If $G \subseteq Y$ is open then $F = G^{1}$ is closed in Y. Hence $f^{-1}(F)$ is closed in X.

Hence $f^{-1}(G) = f^{-1}(F^{1}) = (f^{-1}(F))^{1}$ is open in X.

SAQ 4.12 If $f : (X, d_1) \rightarrow (Y, d_2)$ is continuous at $x_0 \in X$ and $\varepsilon > 0$

there exists, by definition, $\delta > 0 \ni d_1 (x, x_0) < \delta \Rightarrow d_2 (f(x), f(x_0) < \varepsilon$

$\Rightarrow f (S_\delta (x_o)) \subseteq S_\varepsilon (f (x_0))$

Conversely if $\forall\ \varepsilon > 0$ there exists $\delta > 0$ such that $f (S_\delta (x_0)) \subseteq S_\varepsilon (f (x_0))$

Then $d_1 (x, x_0) < \delta \Rightarrow x \in S_\delta (x_0)) \Rightarrow d_2 (f (x), f(x_0)) < \varepsilon$

Thus f is continuous at x_0

SAQ 4.22 If $f: X \rightarrow \mathbb{C}$ is continuous at x, given $\varepsilon > 0$ there is a neighborhood V of x such that

$| f (y) - f (x) | < \varepsilon$ whenever $y \in V$.

Since $| |f (y) | - | f (x) | | \leq | f (y) - f (x)| < \varepsilon$ whenever $y \in V$ it follows that $|f|$ is continuous at x.

To show that the converse is false

Define f (x) $= 1$ if $x \geq 0$

$= - 1$ if $x < 0$

$| f (x) | = 1\ \forall\ x \in \mathbb{R}$ so $|f|$ is continuous at o. (SAQ 4.3 0)

If $0 \leq x < \delta$, $f (x) - f (0) = 0$ while

If $- \delta < x < 0$ $f (x) - f(0) = -2$

Hence f is not continuous at 0

4.23 SAQ If $\tau = \phi$ or has just one element in it g, h, H are clearly continuous. Assume that τ has at least two elements in it. Let $\tau = \{f_1, ... f_n\}$ where $n > 2$.

Then $g (x) = \sum_{i=1}^{n} f_i (x)$, $h (x) = \inf \{f_1(x), f_2(x),...f_n(x) \{$and $H(x) = \sup \{f_1 (x),.... F_n (x) \}$

Write $g_1(x) = \sum_{i=2}^{n} f_i (x)$, $h_1 (x) = \inf \{f_i (x) | 2 \leq i \leq n\}$

and $H_1 (x) = \sup \{f_i (x) / 2 \leq i \leq n\}$

Then $g = f_1 + g_1$, $h = f_1 \wedge g_1$, and $H = f_1 \vee g_1$

Now apply induction.

4.26 Exercises:

1. Let X, Y be topological spaces, $Z \subseteq X$. If f: $X \to Y$ is continuous show that
   a) The restriction g of f defined by $g(x) = f(x)$ for $x \in Z$ is continuous on Z
   b) $f : X \to f(X)$ is continuous.
2. Let X, Y, Z be topological spaces; $f : X \to Y$ and $g : Y \to Z$ be continuous show that g o f is continuous.
3. Give an example of a continuous map which is not open.
4. Give an example of an open map which is not continuous.
5. If f: $X \to Y$ is a bijection show that f is open if and only if f^{1} is continuous.
6. Let $n \geq 2$; $f_i : \mathbb{R}^n \to \mathbb{R}$ be mappings for $1 \leq i \leq n$.
   Define $f(x) = (f_1(x), f_2(x), \ldots.f_n(x))$.
   Show that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if and only if for each I, $1 \leq i \leq n$, $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.
7. Prove proposition 4.19.
8. If X and Y topological spaces write $X \sim Y$ is there if a homeomorphism from X onto Y. Prove the following.
   a) $X \sim X$
   b) $X \sim Y \Rightarrow Y \sim X$
   Because of this symmetry X and Y are said to be homeomorphic if $X \sim Y$.

(c) $X \sim Y$ and $Y \sim Z \Rightarrow X \sim Z$.

Lesson Writer: V.J. LAL

LESSON - 5

SPACES OF CONTINUOUS FUNCTIONS

5.1 Introduction: In this lesson we continue our study of continuity. The sequence spaces K^n and K^∞ and the function spaces B(X), C(X, K) are defined. These spaces are complete normed linear spaces. It is also shown that in C(X, $\mathbb{R}$) equipped with the supremum norm convergence is equivalent to uniform convergence. Finally some algebraic properties of the space C(X, K) are studied.

5.2 Let X be a nonempty set and K be the field of real or complex numbers. We say that X is a linear space over K if there are operations + and respectively called addition and scalar multiplication on X such that writing α x instead of α x we have

(i) (X, +) is an Abelian group, the scalar multiplication assigns to each pair $(\alpha, x) \in K \times X$ an element $\alpha x \in X$ such that for x, y in X and α, β in K.

(ii) $\alpha (x + y) = \alpha x + \alpha y$

(iii) $(\alpha + \beta) x = \alpha x + \beta x$

(iv) $(\alpha \beta) x = \alpha (\beta x)$ and

(v) $1 x = x$

Elements of X are called vectors and of K are called scalars. If K = $\mathbb{R}$, X is called a real linear space while when K = $\mathbb{C}$, X is called a complex linear space. Some authors call X a vector space as well. We use both these terms in this lesson and subsequent lessons.

5.3 A nonempty subset Y of a linear space X is called a linear subspace of X if x + y is in Y whenever x is in Y and y is in Y and α x is in Y for any scalar α and x in Y.

5.4 A normed linear space is a linear space X on which there is defined a real valued function called norm, denoted usually by the symbol || || which assigns to each x in X, an element ||x|| in $\mathbb{R}$ satisfying.

$\| x \| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$

$\| x + y \| \leq \| x\| + \| y \|$ and

$\| \alpha x \| = |\alpha| \|x \|$

for x, y in X and α in K.

5.5 Proposition: For a nonempty subset Y of a linear space X, the following are equivalent.

a) Y is a linear subspace of X.

b) $\alpha x + \beta y \in Y$ whenever $x \in Y$, $y \in Y$ and α, β are any scalars.

c) $\alpha x + y$ whenever $x \in Y$, $y \notin Y$ and α is any scalar

Proof: Left as an exercise.

5.6 Proposition: If X is a normed linear space and $d(x, y) = \| x - y\|$ then d is a metric on X. This metric is called the metric induced by the norm.

Proof: Left as an exercise.

Let us recall that a sequence $\{x_n\}$in a metric space (X, d) is convergent in X if there exists $x \in X$ such that $\lim d(x_n, x) = 0$, i.e. $\forall\, \varepsilon > 0$ there corresponds a positive integer $N(\varepsilon)$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N(\varepsilon)$

In this case x is called the limit of $\{x_n\}$ and is denoted by $\lim x_n = x$. $\{x_n\}$ is called a cauchy sequence in (X, d) if for every $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for $n > m \geq N(\varepsilon)$, eqivalently for $n \geq N(\varepsilon)$ and $m \geq N(\varepsilon)$.

It is not necessarily true that every cauchy sequence in a metric space (X, d) is convergent. A metric space (X, d) is said to be complete if every cauchy sequence is convergent in (X, d).

5.7 A normed linear space X is said to be a Banach space if it is complete with respect to the induced metric.

5.8 SAQ: Show that a nonempty subset A of a normed linear space X is bounded if and only if there is a constant K such that $\| x \| \leq K \ \forall \ x \in A$.

Euclidean and Unitary Spaces:

Let n be a positive integer and let $\mathbb{R}^n$ be the collection of all ordered n – tuples $(x_1, x_2, \ldots\ldots x_n)$ where each $x_i \in \mathbb{R}$. An ordered n – tuple is the range of a map x from the set $J_n = \{1, \ldots, n\}$ into $\mathbb{R}$ which is arranged in a row in increasing order of the elements of the domain J_n. For the sake of convenience we call an ordered n-tuple simply an n-tuple, omitting the adjective ordered. If $x = (x_1, \ldots\ldots, x_n)$ and $y = (y_1, \ldots\ldots, y_n)$ we say x = y when $x_i = y_i \ \forall \ i, 1 \leq i \leq n$ and write 0 for the ordered n tuple all of whose entries are zero. If $x = (x_1, \ldots\ldots, x_n)$ and $y = (y_1, \ldots\ldots, y_n)$ we write $x + y = (x_1 + y_1, \ldots\ldots, x_n + y_n)$ and for $\alpha \in K$ we write $\alpha x = (\alpha x_1, \ldots\ldots, \alpha x_n)$. These operations are called coordinatewise addition and scalar multiplication respectively. With these operations $\mathbb{R}^n$ is a linear space.

The euclidean norm on $\mathbb{R}^n$ is defined by

$$\| x \| = \sqrt{x_1^2 + x_2^2 + \ldots\ldots + x_n^2} \quad \text{where } x = (x_1, \ldots\ldots, x_n)$$

5.9 Proposition: $\mathbb{R}^n$ is a normed linear space with the Euclidean norm.

(i) If $x = (x_1, \ldots\ldots, x_n) \in \mathbb{R}^n$, $\| x \| = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \geq 0$ and

$$\| x \| = 0 \quad \Leftrightarrow \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} = 0$$

$$\Leftrightarrow x_i = 0 \ \forall \ i, \ 1 \leq i \leq n$$

$$\Leftrightarrow x = 0$$

(ii) if $\alpha \in \mathbb{R}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$ so $\|\alpha x\|^2 = \sum_{i=1}^{n} (\alpha x_i)^2 =$

$$\alpha^2 \sum_{i=1}^{n} x_i^2 = \alpha^2 \| x \|^2 \text{ so that } \| \alpha x \| = |\alpha| \, \| x \|$$

We now prove the triangle inequality

Let $x = (x_1, \ldots, x_n)$ and $y = y_1, \ldots, y_n)$ be elements of $\mathbb{R}^n$. Then

$x + y = (x_1 + y_1, \ldots, x_n + y_n)$

$$\text{So } \| x + y \|^2 = \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2 \sum_{i=1}^{n} x_i y_i$$

$$(\|x\| + \|y\|)^2 = \|x\|^2 + \|y\|^2 + 2 \|x\| \, \|y\|$$

Thus to show that $\|x + y\| \leq \|x\| + \|y\|$ it is enough to verify that $\sum_{i=1}^{n} x_i y_i \leq$

$$\sqrt{x_1^2 + \ldots + x_n^2} \sqrt{y_1^2 + \ldots + y_n^2}$$

$$\| x \|^2 \, \|y\|^2 - \left(\sum_{i=1}^{n} x_i y_i \right)^2$$

$$= \sum_{i=1}^{n} x_i^2 \sum_{j=1}^{n} y_j^2 - \left(\sum_{i=1}^{n} x_i^2 y_i^2 + 2 \sum_{i<j} x_i y_i x_j y_j \right)$$

$$= \sum_{i<j} (x_i y_j - x_j y_i)^2 \geq 0$$

$$\text{Hence } \left(\sum_{i=1}^{n} x_i y_i \right)^2 \leq (\|x\| \, \|y\|)^2$$

$$\Rightarrow | \Sigma x_i y_i | \leq \| x \| \, \| y \|$$

This completes the proof of the triangle inequality. Thus $\mathbb{R}^n$ is normed linear space with the Euclidean norm.

5.10 Definition: The space $\mathbb{R}^n$ equipped with the Euclidean norm is called the n-dimensional Euclidean space.

We define coordinate wise operations on the space $\mathbb{C}^n$ consisting of all n tuples $(z_1, \ldots, z_n)$ where each $z_i \in \mathbb{C}$ and define the unitary norm by

$$\| z \| = \sqrt{| z_1 |^2 + \ldots\ldots + | z_n |^2} \quad \text{where } z = (z_1, \ldots\ldots, z_n)$$

As in the case of $\mathbb{R}^n$, we can show that $\mathbb{C}^n$ is a normed linear space with this unitary norm. The space $\mathbb{C}^n$ equipped with the unitary norm is called the n-dimensional unitary space.

5.11 SAQ: show that if $x = (x_1, \ldots, x_n) \in K^n$ ($K = \mathbb{C}$ or $\mathbb{R}$) then $| x_i | \leq \| x \| \leq \sum_{j=1}^{n} | x_j |$, for $1 \leq i \leq n$.

5.12 SAQ: Show that a sequence $\{x^{(k)}\}$ in K^n where $x^k = (x_1^k, \ldots, x_n^k)$ converges to $x = (x_1, \ldots, x_n)$ if and only if for each $1 \leq i \leq n$, the sequence $\{x_1^{(k)}\}$ converges to x_i.

5.13 Proposition: The space K^n is complete.

Proof: Let $x^{(k)} = (x_1^{(k)}, \ldots\ldots, x_n^{(k)})$ $(k \geq 1)$ and $\{x^{(k)}\}$ be a cauchy sequence in K^n. So given $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that

$$\| x^{(k)} - x^{(r)} \| < \varepsilon \text{ whenever } K \geq r \geq N(\varepsilon)$$

By SAQ 5.11 $| x_i^{(k)} - x_i^{(r)} | < \varepsilon$ for $k \geq r \geq N(\varepsilon)$

So $\{x_i^{(k)}\}$ is a cauchy sequence in K for $1 \leq i \leq n$.

Since K is complete, $\exists$ a $x_i \in K$ $\lim_k x_i^{(k)} = x_i$. Put $x = (x_1, \ldots, x_n)$

Since $0 \leq \|x^{(k)} - x \| \leq \sum_{i=1}^{n} | x_i^{(k)} - x_i |$ and $\lim_k | x_i^{(k)} - x_i | = 0 \ \forall i$

$\lim_k \sum_{i=1}^{n} | n_i^{(k)} - n_i | = 0$ so that;

$\lim_k \|x^{(k)} - x \| = 0$, i.e. $\lim_k x^{(k)} = x$ in K^n.

Thus every cauchy sequence in K^n converges.

5.14 the infinite dimensional Euclidean (Unitary) space K^∞:

The space K^∞ of all sequences $\{x_n\}$ where $x_n \in K \ \forall$ n, such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, is a vector space over K.

Solution: Since K^∞ is a nonempty subset of the linear space S of all sequences with entries in K, it is enough to verify that K^∞ satisfies the conditions of 4.19.

Let $x = \{x_n\} \in K^\infty$, $y = \{y_n\} \in K^\infty$ and α, β be any scalars.

Then $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and $\sum_{n=1}^{\infty} |y_n|^2 < \infty$

Since $(|x_n| - |y_n|)^2 \geq 0$, $|x_n|^2 + |y_n|^2 \geq 2\,|x_n|\,|y_n|$

So by comparison test $\sum_{n=1}^{\infty} 2\,|x_n y_n|$ is convergent.

Now $(\,|x_n + y_n|\,)^2 \leq (\,|x_n| + |y_n|)^2 = |x_n|^2 + |y_n|^2 + 2\,|x_n y_n|$

Since $\sum_{n=1}^{\infty} |x_n|^2$, $\sum_{n=1}^{\infty} |y_n|^2$ and $\sum_{n=1}^{\infty} 2\,|x_n y_n|$ are convergent, by comparison test

$\sum_{n=1}^{\infty} (|x_n + y_n|)^2 < \infty$

Hence $\{x_n + y_n\} \in K^\infty$; i.e., $x + y \in K^\infty$

$\sum_{n=1}^{\infty} |\alpha x_n|^2 = |\alpha|^2 \sum_{n=1}^{\infty} |x_n|^2 < \infty$ since $\sum_{n=1}^{\infty} |x_n|^2 < \infty$

Hence $\{\alpha x_n\} \in K^\infty$ i.e. $\alpha x \in K^\infty$

Hence K^∞ is a subspace of S, hence is a linear space.

Note: This space K^∞ is usually denoted by l^2. For $x \in K^\infty$, $x = \{x_n\}$. $\|x\| = \left\{\sum_{n=1}^{\infty} |x_n|^2\right\}^{1/2}$ defines a norm with respect to which K^∞ is complete. For details the reader may refer to the study material on Functional analysis in Paper I Topology and Functional analysis.

When $K = \mathbb{R}$, K^∞ is called the infinite dimensional Euclidean space.

When $K = \mathbb{C}$, K^∞ is called the infinite dimensional unitary space.

5.15 The space B (X):

Proportion: Let X be a nonempty set. The set B(X) of all bounded K – valued functions is a Banach space with respect to the supremum norm defined by

$||f|| = \text{Sup } \{|f(x)| / x \in X\}$

Proof: Define addition and scalar multiplication on B (X) by point wise operations:

$(f + g)(x) = f(x) + g(x)$ and $(\alpha f)(x) = \alpha f(x)$

for f, g in B (X), $\alpha \in K$ and $x \in X$.

$\forall x \in X, |(f+g)(x)| = |f(x) + g(x)|$

$\leq |f(x)| + |g(s)|$

$\leq ||f|| + ||g||$ .... (1)

So f + g is bounded, hence belongs to B (X).

Similarly $\forall \alpha \in K, |(\alpha f)(x)| = |\alpha|\, |f(x)|$

$\Rightarrow \text{Sup } \{ |(\alpha f)(x)| / x \in X\} = \text{Sup } \{ |\alpha|\, |f(x)| / x \in X\}$

$= |\alpha| \text{ Sup } \{|f(x)| / x \in X\}$

$= |\alpha|\, ||f||$ .... (2)

Therefore $\alpha f \in B(X)$ and B(X) is a linear subspace of the linear space F(X) of all functions on X. thus B(X) is a linear space with respect to the point wise operations.

Verification of norm properties :

That $|| f + g || \leq || f || + || g ||$ and $||\alpha f|| = |\alpha|\, ||f||$ is clear from (1) and (2). It is also clear that $|f(x)| \geq 0 \ \forall \ x \in X$ so that $||f|| \geq 0$. Finally $||f|| = 0 \Leftrightarrow |f(x)| = 0 \ \forall x \in X$.

$\Leftrightarrow f(x) = 0 \ \forall x \in X$

$\Leftrightarrow f = 0$

Hence $|| \ \ ||$ defines a norm and thus B(X) is a normed linear space.

Completion: Let $\{f_n\}$ be a Cauchy sequence in *B(X). If* $\in > 0$ *there is a positive integer* N (ε) such that $\|f_n - f_m\| < \varepsilon$ for $n \geq m \geq N(\varepsilon)$

$\Rightarrow \forall\, x \in X\ |f_n(x) - f_m(x)| < \varepsilon$ for $n \geq m \geq N(\varepsilon)$ .... (3)

$\Rightarrow \forall\, x \in X\ \{f_n(x)\}$ is a Cauchy sequence in K.

$\Rightarrow \{f_n(x)\}$ converges in K. Write $f(x) = \lim\limits_{n} f_n(x)\ (x \in X)$

In (3) fix $m \geq N(\varepsilon)$ and let $n \to \infty$

We get $|f(x) - f_m(x)| \leq \varepsilon$ for $m \geq N(\varepsilon)\ \forall\, x \in X$ .... (4)

In particular $|f(x)| - |f_N(x)| \leq \varepsilon$ where $m \geq N(\varepsilon)$

$\Rightarrow |f(x)| \leq \varepsilon + |f_N(x)|$ where $N = N(\varepsilon)$

This being true $\forall\, x \in X$ it follows that f is bounded so that $f \in B(X)$.

From(4) it follows that $\|f - f_m\| \leq \varepsilon$ for $m \geq N(\varepsilon)$

Hence $\{f_n\}$ converges to f in B(X).

Thus B(X) is complete.

5.16. Uniform convergence:

Definition: A sequence $\{f_n\}$ of functions defined on a set E with values in $\mathbb{R}$ or $\mathbb{C}$ is said to converge uniformly to a function f defined on E with values in $\mathbb{R}$ or $\mathbb{C}$ if for every ε>0 there corresponds a positive integer N(ε) such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x\ \varepsilon\ E \text{ and } n \geq N(\varepsilon)$$

5.17 Proposition: Let $\{f_n\}$ be a sequence of continuous complex (or real) valued functions defined on a topological space X. If $\{f_n\}$ converges uniformly to a complex (or real) valued function f on X then f is continuous on X.

Proof: It is required to prove that f is continuous at every $x \in X$. Fix $x \in X$ and let $\varepsilon > 0$.

Then there exists a positive integer $N(\varepsilon) = N$ such that $|f_n(t) - f(t)| < \frac{\varepsilon}{3}$ for $n \geq N$ and

$t \in X$.

By the continuity of f_N at x, ∃ an open set V of x such that $|f_N(t) - f_N(x)| < \frac{\varepsilon}{3}$ for $t \in V$.

For $t \in V$.

$$| f(t) - f(x)| \leq |f(t) - f_N(t)| + |f_N(t) - f_N(x)| + |f_N(x) - f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Hence f is continuous at x.

5.18 SAQ: If X is a topological spsace and **C** (X, K) is equipped with the supremum norm defined by $||f|| = \text{Sup}\{ |f(x)| / x \in X\}$, then a sequence $\{f_n\}$ in **C**(X, K) converges to f in **C**(X, K) with respect to the induced metric if and only if $\{f_n\}$ converges uniformly to f on X.

Remark: It is because of the equivalence of the two types of convergence in SAQ 5.18, the supremum norm is also called the uniform norm.

5.19 Theorem: Let **C**(X, ℝ) be the set of all bounded continuous real functions defined on a topological space X. Then **C**(X, ℝ) is a real Banach space with respect to point wise addition and scalar multiplication and the norm defined by $||f|| = \text{Sup}\{|f(x)| / x \in X\}$

Proof: Let $\{f_n\}$ be a Cauchy sequence in **C**(X, ℝ). Then for every $\varepsilon > 0$ there corresponds a positive integer N(ε) such that $||f_n - f_m|| < \varepsilon$ for $n > m \geq N(\varepsilon)$ ------- (1)

If $x \in X$, $|f_n(x) - f_m(x)| \leq ||f_n - f_m|| < \varepsilon$ for $n > m \geq N(\varepsilon)$ ...... (2)

Hence $\{f_n(x)\}$ is a Cauchy sequence in ℝ. Therefore there is a number which we denote by f(x) such that $\lim f_n(x) = f(x)$ since this is fixed uniquely $\forall x \in X$, $x \to f(x)$ defines a function from X into ℝ.

We show that $f \in$ **C**(X, ℝ) and $\lim f_n = f$ in **C**(X, ℝ).

Continuity of f: If $x \in X$, and $\varepsilon > 0$, from (2)

$|f_n(x) - f_m(x)| < \varepsilon$ for $n > m \geq N(\varepsilon)$

Keeping m fixed and letting n tend to ∞ we get

$$|f(x) - f_m(x)| \leq \varepsilon \text{ for } m \geq N(\varepsilon)$$

This is true $\forall x \in X$ hence $\{f_m\}$ converges uniformly to f on X.

Boundedness: Corresponding to $\varepsilon = 1$ $\exists$ a N_1 as per (1) above so that

$| f_n (x) - f_{N_1} (x) | \leq \|f_n - f_{N_1}\| < 1$ for $n \geq N_1$.

Letting $n \to \infty$ we get $|f(x) - f_{N_1} (x) | \leq 1$ and this is true for all $x \in X$.

$\Rightarrow |f(x)| - | f_{N_1} (x) | \leq | f (x) - f_{N_1} (x) | \leq 1$ for $x \in X$

$\Rightarrow |f (x) | \leq 1 + | f_{N_1} (x) | \leq 1 + \| f_{N_1} \| \;\forall x \in X$

Hence f is bounded.

Convergence: If $\varepsilon > 0$ by (1) $\| f_n - f_m\| < \varepsilon$ for $n > m \geq N (\varepsilon)$ and $x \in X$.

$\Rightarrow |f_n (x) - f_m (x) | < \varepsilon$ for $n > m \geq N (\varepsilon)$ and $x \in X$

Letting n tend to ∞ we get

$|f (x) - f_m (x) | \leq \varepsilon$ for $m \geq N (\varepsilon)$ and all $x \in X$

$\Rightarrow \|f - f_m\| = \text{Sup } \{|f(x) - f_m (x) | / x \in X\} \leq \varepsilon$ for $m \geq N (\varepsilon)$

Hence $\{f_m\}$ converges to f in **C**(X, **R**)

Since every Cauchy sequence in **C(X, R)** converges to an element in **C(X, R)**, this space is a complete normed linear space, hence a Banach space.

This completes the proof.

5.20 Definition: By an algebra over K we mean a vector space X over K together with a binary operation called multiplication with respect to which, the additive group (X, +) becomes a ring and satisfies

$\alpha (x, y) = (\alpha x) y = x (\alpha y)$ for x, y in X and α in K.

If the ring (X, +, .) is commutative, the algebra is said to be commutative.

If the ring (X, +, .) has multiplicative identity, the algebra is said to have identity.

If K = **R**, A is said to be a real algebra and if K = **C**

A is said to be a complex algebra.

5.21. Definition: If A is an algebra, a nonempty subset B of A is said to be a subalgebra of A if B is linear subspace of A and $x \in B$, $y \in B \Rightarrow xy \in B$.

5.22 Theorem: Let X be a topological space and C(X, ℂ) be the set of continuous bounded real functions defined on X and $||f|| = \text{Sup} \{ |f(x)| / x \in X\}$

(1) If multiplication is defined point wise C(X, ℝ) is a real commutative algebra with identity in which

$|| f g || \leq || f || \, || g ||$ and $|| 1 || = 1$

(2) if $f \leq g$ is defined to mean that $f(x) \leq g(x)$ for all $x \in X$, C(X, ℝ) is a lattice in which the greatest lower bound and the least upper bound of a pair of functions f and g are given by $(f \cap g)(x) = \min \{f(x), g(x)\}$ and

$(f \cup g)(x) = \max \{f(x), g(x)\}$

Proof: Clearly C(X, ℝ) is vector space over ℝ.

That C(X, ℝ) is a commutative ring follows from the properties of ℝ since multiplication and addition are point wise operations.

The function 1 defined on X by $1(x) = 1$ for $x \in X$ is the multiplication identity in C(X, ℝ).

Further for f, g in C(X, ℝ) and $x \in X$

$|(fg)(x)| = |f(x)| \, |g(x)| \leq || f || \, || g ||$

so that $|| fg|| = \text{Sup} \{ |(fg)(x)| / x \in X\} \leq || f || \, || g ||$

this completes the proof of (1).

Proof of (2) follows from the lattice properties of ℝ and 4.29.

5.23 Theorem: Let C(X, ℂ) be the set of all bounded continuous complex functions defined on a topological space X. then

(1) C(X, ℂ) is a complex Banach space with respect to pointwise addition and scalar multiplication and the norm defined by $|| f || = \text{Sup} \{| f(x) | / x \in X\}$

(2) If multiplication is defined pointwise C(X, ℂ) is a commutative complex algebra with identity 1 in which

$|| f g || \leq || f || \, || g ||$ and $|| 1 || = 1$ and

(3) If $\bar{f}$ is defined by $\bar{f}(x) = \overline{f(x)}$ then $f \to \bar{f}$ is a mapping of the algebra $C(X, \mathbb{C})$ into itself which has the following properties:

(a) $\overline{f+g} = \bar{f} + \bar{g}$

(b) $\overline{\alpha f} = \bar{\alpha}\,\bar{f}$

(c) $\overline{fg} = \bar{f}\,\bar{g}$

(d) $\bar{\bar{f}} = f$ and

(e) $\|\bar{f}\| = \|f\|$

Proof: The proof for (1) and (2) is same as that of $C(X, \mathbb{R})$. That for $f \in C(X, \mathbb{C})$, $\bar{f} \in C(X, \mathbb{C})$ is proved in lesson 4. (a), (b), (c), (d) need to be verified for every $x \in X$ and when we take the value at any x we get complex numbers for which (a), (b), (c), (d) hold good. Since $|f(x)| = |\bar{f}(x)|\ \forall x \in X$ and $\bar{f} \in C(X, \mathbb{C})$, (e) follows.

This completes the proof.

5.24. Model Examination Questions:

1. Define an n – dimensional Euclidean space $\mathbb{R}^n$ and show that $\mathbb{R}^n$ is complete.
2. Show that the space $C(X, \mathbb{R})$ of all real valued bounded continuous functions on a topological space X is a Banach space.
3. Show that in $C(X, \mathbb{R})$ where X is a topological space $\lim_n \|f_n - f\| = 0$ if and only if $\{f_n\}$ converges to f uniformly on X.
4. Show that if A is an algebra which is a Banach space and B is a subalgebra of A then so is its closure $\bar{B}$.
5. Show that the space K^n with $\|z\| = \{|z_1|^2 + \ldots\ldots + |z_n|^2\}^{1/2}$ is complete.

5.25 Exercises:

1. Prove that a nonempty subset Y of a linear space X is a linear subspace of X if and only if

   $x \in Y, y \in Y, \alpha \in K, \beta \in K \Rightarrow \alpha x + \beta y \in Y$

   if and only if $x \in Y, y \in Y, \alpha \in K \Rightarrow \alpha x \in Y$ and $x + y \in Y$.

2. Prove that the space F(X) of all functions $X \to K$ is a commutative algebra with identity under point wise operations.
3. Prove that the space S of all sequences of numbers is a commutative algebra with identity under pointwise operations.
4. Define homomorphism and isomorphism of algebras X, Y over the same scalar field and prove that if X is a finite set with n elements ($n > 1$) then $\mathbb{C}(X, \mathbb{R})$ is isomorphic to B(X) over $\mathbb{R}$.
5. Define $\| x \|_\infty = \max \{ |x_i| / 1 \le i \le n\}$ when $x = \{x_1, x_2, \dots, x_n\} \in K^n$

   Show that the algebra K^n is complete with this norm.
6. Prove that K^∞ is complete.
7. Prove that K^∞ is separable.
8. a) Show that $\mathbb{C}(X, K) = B(X)$ when X is equipped with discrete metric.

   b) Prove that $\mathbb{C}(X, \mathbb{R})$ is separable if and only if X is finite when X is equipped with the discrete metric.
9. Let A be an algebra of real or complex functions defined on a nonempty set X. Assume that for each $x \in X$ there exists a $f \in A$ such that $f(x) \neq 0$. Show that if A has identity element e then $e(x) = 1$ for all $x \in X$.
10. Let $f \in \mathbb{C}(X, K)$, $f \neq 0$. Show that the set $Y = \{x \in X / f(x) \neq 0\}$ is open and $\frac{1}{f}$ is continuous on Y.
11. Prove that the closure $\bar{A}$ of a subalgebra A of $\mathbb{C}(X, K)$ is a subalgebra.
12. If A is a subalgebra of $\mathbb{C}(X, \mathbb{C})$ such that $f \in A \Rightarrow \bar{f} \in A$ show that $f \in \bar{A} \Rightarrow \bar{f} \in \bar{A}$.

5.26. Answers to self assessment Questions:

SAQ 5.8: $\phi \neq A \subseteq X$ is bounded if and only if there exists a real number $K_1 > 0$ such that $d(x, y) \le K_1$ for all x, y in A.

Fix y_0 in A. This implies that $d(x, 0) \le d(x, y_0) + d(y_0, 0)$

$$\le K_1 + d(y_0, 0)$$

Hence $\| x \| = d(x, 0) \leq K_1 + \| y_0 \|$ for every $x \in A$.

Conversely if there is a $K > 0$ such that $\| x \| \leq K \ \forall \ x \in A$, then for every $x \in A$ and $y \in A$,

$d(x, y) \leq d(x, 0) + d(0, y) \leq 2K$

Hence A is bounded.

SAQ 5.11: $|x_i|^2 \leq |x_i|^2 + \ldots\ldots + |x_n|^2 \leq (|x_1| + \ldots\ldots + |x_n|)^2 \ \forall \ i, 1 \leq i \leq n$

$\Rightarrow |x_i| \leq \|x\| \leq |x_1| + \ldots\ldots + |x_n|$

SAQ 5.12: $|x_i^{(k)} - x_i| \leq \|x^{(k)} - x\| \leq \sum_{j=1}^{n} | x_j^{(k)} - x_j |$ for $1 \leq i \leq n$.

Thus $\lim x^{(k)} = x$ in $K^n \Rightarrow \lim \|x^{(k)} - x\| = 0$

$\Rightarrow \lim |x_i^{(k)} - x_i| = 0 \ \forall \ i, 1 \leq i \leq n$

$\Rightarrow \lim x_i^{(k)} = x_i \ \forall \ i, \ 1 \leq i \leq n$

Conversely if $\lim x_i^{(k)} = x_i \ \forall \ i, 1 \leq i \leq n$

$\text{Lim} \left| x_i^{(k)} - x_i \right| = 0 \ \forall \ i, 1 \leq i \leq n$

$\Rightarrow \lim \sum_{i=1}^{n} | x_i^{(k)} - x_i | = 0$

$\Rightarrow \lim \| x^{(k)} - x \| = 0$

$\Rightarrow \lim x^{(k)} = x$ in K^n.

SAQ 5.18: If X is a topological space and C(X, K) is equipped with the supremum norm defined by $\|f\| = \text{Sup} \{ |f(x)| / x \in X\}$ then a sequence $\{f_n\}$ is C(X, K) converges to f in C(X, K) with respect to the induced metric d if and only if $\{f_n\}$ converges to f uniformly on X.

Proof: Suppose $d(f_n, f) \to 0$ as $n \to \infty$ i.e. $\lim_n \|f_n - f\| = 0$

Then given $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that

$\|f_n - f\| < \varepsilon$ for $n \geq N(\varepsilon)$

Since $0 \le |f_n(x) - f(x)| \le \|f_n - f\| \; \forall \; x \in X$,

$$|f_n(x) - f(x)| < \varepsilon \text{ for } n \ge N(\varepsilon) \text{ and } x \in X$$

Hence $\{f_n\}$ converges uniformly on X.

Conversely assume that $\{f_n\}$ converges to f uniformly on X. Then for every $\varepsilon > 0$ there corresponds a positive integer $N(\varepsilon)$, such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for $n \ge N(\varepsilon)$ and all $x \in X$. Hence $\|f_n - f\| = \text{Sup}\{|f_n(x) - f(x)| / x \in X\} \le \frac{\varepsilon}{2} < \varepsilon$ for $n \ge N(\varepsilon)$

Hence $d(f_n, f) \to 0$ as $n \to \infty$

Lesson Writer : V.J. LAL

LESSON NO. 6

COMPACT SPACES

6.1 It is well known that closed and bounded sets of real numbers have important properties in analysis. For example continuous real-valued functions defined on closed and bounded sets of real numbers are bounded and uniformly continuous. In contrast to this, the function defined on the open unit interval (0, 1) by $f(x) = \frac{1}{x}$ is neither bounded nor uniformly continuous. An abstractization of this important property possessed by closed and bounded sets of real numbers gives rise to the concept of compactness for tological spaces.

6.2 Definitions: Let X be a topological space. A class $\{G_i\}_{i \in I}$ of open subsets of X is said to be an open cover of X if $X = \bigcup_{i \in I} G_i$.

A sub class of an open cover which is itself an open cover is called a subcover.

A topological space X is called a compact space if every open cover of X has a finite subcover.

A subspace Y of a topological space X is said to be compact if Y is compact as a topological space in its own right.

6.3 Examples:

(1) Every indiscrete space is compact (Ex. 2.6).

Solution: If X is an indiscrete space, since X has only two open sets, every open cover has a finite subcover. Thus X is compact.

(2) Let X be any infinite set and let

$T = \{V \subseteq X / X - V \text{ is finite}\} \cup \{\phi\}$

Then T is a topology on X, called the cofinite topology. The topological space (X, T) is called a cofinite topological space. This cofinite topological space is compact (Ex. 2.8).

Solution: Let $\{V_i\}_{i \in I}$ be an open cover of X. Since $X = \bigcup_{i \in I} V_i$, some V_i is nonempty, say V_{i_0}. Then $X - V_{i_0}$ is finite. Let $X - V_{i_0} = \{x_1, \ldots, x_n\}$. Suppose $x_r \in V_{i_r}$ for r = 1, …, n. Thus $X = V_{i_1} \cup \ldots \cup V_{i_n} \cup V_{i_0}$ So $\{V_{i_1}, \ldots V_{i_n}, V_{i_0}\}$ is a finite subcover. Hence X is compact.

(3) Every finite topological space X (i.e. $|X| < \infty$) is compact.

Solution: Since X has only a finite number of open sets, every open cover has a finite sub cover. ∴ X is compact.

(4) The open un.. interval (0, 1) with usual topology is not compact.

Solution: For each positive integer n, let $V_n = (\frac{1}{n}, 1)$ Then $\{V_n\}_{n \in \mathbb{N}}$ is an open cover of (0, 1), but it has no finite subcover. ∴ (0, 1) is not compact.

(5) The set ℝ of all real numbers with usual topology is not compact.

Solution: Clearly $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$. For each positive integer n, let $U_n = (-n, n)$. Then $\{U_n\}_{i \in \mathbb{N}}$ is an open cover of ℝ, but has no finite subcover. Therefore ℝ is not compact.

6.4 SAQ: Let Y be a subspace of a topological space X. Then Y is compact if and only if for every class $\{H_i\}_{i \in I}$ of open sets in X such that $Y \subseteq \bigcup_{i \in I} H_i$ there is a finite subclass $\{H_i\}_{i \in J}$ ($J \subseteq I$ and J is finite) such that $Y \subseteq \bigcup_{i \in J} H_i$

We now prove two simple, but useful, theorems.

6.5. Theorem: Any closed subspace of a compact space is compact.

Proof: Let Y be a closed subspace of a compact space X. Let $\{H_i\}_{i\in I}$ be a class of open sets in X such that $Y \subseteq \bigcup_{i\in I} H_i$. Then $X = Y \cup Y^1 \subseteq (\bigcup_{i\in I} H_i) \cup Y^1 \subseteq X$, where Y^1 is the complement of Y in X. $\therefore$ $X = (\bigcup_{i\in I} H_i) \cup Y^1$. Since Y is closed, $Y^1$is open. Hence the class $\{H_i\}_{i \in I} \cup \{Y^1\}$ is an open cover of X. Since X is compact, there exists a finite subclass $\{H_{i_1}, \ldots, H_{i_n}\}$ of $\{H_i\}_{i \in I}$ such that $X = H_{i_1} \cup \ldots \cup H_{i_n} \cup Y^1$. Hence $Y = (H_{i_1} \cap Y) \cup \ldots \cup (H_{i_n} \cap Y) \cup (Y^1 \cap Y) \subseteq H_{i_1} \cup \ldots \cup H_{i_n}$. By SAQ 6.4, Y is compact.

6.6 Theorem: Any continuous image of a compact space is compact.

Proof: Let $f : X \to Y$ be a continuous mapping of a compact space X into an arbitrary topological space Y. We claim that f(X) is a compact subspace of Y. Let $\{H_i\}_{i \in I}$ be a class of open sets in Y such that $f(X) \subseteq \bigcup_{i\in I} H_i$. Since f is continuous and H_i is open in Y. $f^{-1}(H_i)$ is open in X, for every $i \in I$. Therefore $\{f^{-1}(H_i)\}_{i \in I}$ is a class of open sets in X. Also $f(X) \subseteq \bigcup_{i\in I} H_i \Rightarrow X \subseteq f^{-1}(\bigcup_{i\in I} H_i) = \bigcup_{i\in I} f^{-1}(H_i) \Rightarrow X = \bigcup_{i\in I} f^{-1}(H_i)$. Since X is compact, there exists a finite subcover $\{f^{-1}(H_{i_1}), \ldots, f^{-1}(H_{i_n})\}$ of $\{f^{-1}(H_i)\}$. Hence $X = f^{-1}(H_{i_1}) \cup \ldots \cup f^{-1}(H_{i_n})$ and this implies that $f(X) \subseteq H_{i_1} \cup \ldots \cup H_{i_n}$. Thus by SAQ 5.4, f(X) is compact.

6.7 Remark: Let us recall that if X is a set and $\{A_i\}_{i \in I}$ is a class of subsets of X, then we have that

$$\left(\bigcup_{i\in I} A_i\right)^1 = \bigcap_{i\in I} A_i^1$$

and, $$\left(\bigcap_{i\in I} A_i\right)^1 = \bigcup_{i\in I} A_i^1$$

We note the following:

$\{A_i\}_{i \in I}$ is a covering of X $\Leftrightarrow X = \bigcup_{i \in I} A_i$

$$\Leftrightarrow X - \bigcup_{i \in I} A_i = \phi$$

$$\Leftrightarrow \left(\bigcup_{i \in I} A\right)^1 = \phi$$

$$\Leftrightarrow \bigcap_{i \in I} A_i^1 = \phi$$

We also note that a subset A of a topological space X is open iff its complement A^1 is closed.

The following theorem is an easy consequence of the definition of compactness of a topological space.

6.8 Theorem: A topological space is compact $\Leftrightarrow$ every class of closed sets with empty intersection has a finite subclass with empty intersection.

6.9 Remark: In remark 6.7, it was observed that if $\{A_i\}$ is a class of subsets of a set X then $\{A_i\}_{i \in I}$ is a covering of X if and only if $\bigcap_{i \in I} A_i^1 = \phi$. As a consequence of this we have that the class $\{A_i\}$ is not a covering of X if and only if $\bigcap_{i \in I} A_i^1 \neq \phi$.

6.10. Definition: A class $\{A_i\}_{i \in I}$ of subsets of a non-empty set X is said to have the finite intersection property (simply f . i . p.) if every finite subclass of $\{A_i\}_{i \in I}$ has non-empty intersection.

In view of this definition, theorem 6.8 can be restated as follows.

6.11. Theorem: A topological space is compact $\Leftrightarrow$ every class of closed sets with the finite intersection property has non-empty intersection.

Let us recall that an open base for a topological space X is a class of open sets with the property that every open set is a union of sets in this class.

6.12 Definition: Let X be a topological space. An open cover of X whose sets are all in some given open base is called a basic open cover.

6.13. Remark: Suppose X is a compact space. Since every basic open set is an open set, every basic open cover is an open cover and hence it has a finite subcover. We prove the converse part in the following.

6.14 Theorem: Suppose $\{B_i\}_{i \in I}$ is an open base for a topological space X. If every basic open cover by sets from $\{B_i\}_{i\in I}$ has finite subcover, then X is compact.

Proof: Let $\{G_i\}_{i\in J}$ be an open cover of X. Since $\{B_i\}_{i \in I}$ is an open base, each G_i is a union of sets from $\{B_i\}_{i \in I}$. So there is a subset $I_r \subseteq I$ such that $G_r = \bigcup_{i\in I_r} B_i$. Put $I_o = \bigcup_{i\in J} I_r$. Therefore $X = \bigcup_{i\in J} G_r = \bigcup_{i\in J}(\bigcup_{i\in I_r} B_i) = \bigcup_{i\in I_0} B_i$. Thus $\{B_i\}_{i\in I_0}$ is a basic open cover of X. By hypothesis, there exists a finite subclass $\{B_{i_1} \ldots\ldots B_{i_n}\}$ of $\{B_i\}_{i\in I_0}$ such that $X = \bigcup_{k=1}^{n} B_{i_K}$. Now for each B_{i_K}, there exists G_{i_K} ($r_k \in J$) such that $B_{i_K} \subseteq G_{r_k}$. So $X = B_{i_1} \cup B_{i_n} \subseteq G_{r_1} \cup \ldots\ldots \cup G_{r_n}$ and hence $X = G_{r_1} \cup \ldots\ldots \cup G_{r_n}$. Thus X is compact.

6.15 Definition: Let X be a topological space. A class $\{F_i\}$ of closed subsets of X is called a closed base if the class $\{F_i^{!}\}$ of all complements of its sets is an open base of X. Sets F_i are called basic closed sets.

Theorem 6.14 can be restated as follows.

6.16 Theorem: A topological space is compact if every class of basic closed sets with the finite intersection property has non-empty intersection.

6.17 Definition: Let X be a topological space. A class $\{S_i\}$ of closed subsets of X is called a closed subbase if the class $\{S_i^{'}\}$ of all complements of its sets is an open subbase.

6.18 Remark: Let us recall that an open subbase is a classes of open subsets of a topological space X whose finite intersections form an open base. This open base is called the open base generated by the open subbases. From the definitions 6.15 and 6.17 it is clear that the class of all finite unions of sets in a closed subbase **C** is a closed base. This is called the closed base generated by the closed subbase **C**.

We now prove a criterion for a topological space to be compact in terms of subbasic closed sets.

6.19 Theorem: A topological space is comapct if and only if every class of subbasic closed sets with finite intersection property has non-empty intersection.

Proof: Let X be a topological space. Since very subbasic closed set is a closed set, it follows from theorem 6.11 that if X is compact then every class of subbasic closed sets with f.i.p. has non-empty intersection. Conversely suppose that every class of subbasic closed sets with f.i.p has nonequity intersection. Let $\{S_\alpha\}_{\alpha\in\Delta}$ be a closed subbase and let $\{B_i\}_{i\in I}$ be the closed base generated by this subbase. So, each B_i is a finite union of $S_\alpha^{'}$s.

By theorem 6.16, to prove the theorem it suffices to show that every class of basic closed sets from $\{B_i\}_{i\in I}$ with f.i.p. has non-empty intersection. So, let $\{B_i\}_{i\in J}$ be a class of B_i's with f.i.p. We have to show that $\bigcap_{i\in J} B_r \neq \phi$. Let $\Im_1$ be the family of all classes of B_i's which contain $\{B_i\}_{i\in J}$ and have the f.i.p. Since the class $\{B_i\}_{i\in J}$ is in $\Im_1$, the family $\Im_1 \neq \phi$. Then $\Im_1$ is a partially ordered set with respect to class inclusion. Let $\{\mathcal{B}_i\}$ be a chain in $\Im_1$. Put $\mathcal{B} = \bigcup_n \mathcal{B}_2$. Since each $\mathcal{B}_N$ is a class of $B_i^{'}$s, $\mathcal{B}$ is also a class of $B_i^{'}$s. Let $\{B_{i_1},$ ...., $B_{i_n}\}$be a finite $\mathcal{B}_N$ class of sets in $\mathcal{B}$. contained in some $\mathcal{B}_N$. Since $\mathcal{B}_N$ has the f.i.p., $B_{i_1} \cap \ldots \cap B_{i_n} \neq \phi$. Since $\{\mathcal{B}_N\}$ is a chain,. The finite class $\{\mathcal{B}_{i_1} \ldots, \mathcal{B}_{i_n}\}$

So β has the f.i.p. Therefore $\mathcal{B} \in \mathfrak{I}_1$ and is an upper bound of $\mathcal{B}_N'$ s. By Zorn's lamma, $\mathfrak{I}_1$ has a maximal element. Let $\{B_k\}_{K\in K}$ be a maximal element in $\mathfrak{I}_1$. Since $\{B_k\}_{K\in K}$ contains $\{B_r\}_{r\in J}$, we have that $\bigcap_{k\in K} B_k \subseteq \bigcap_{i\in J} B_i$. So, it suffices if we show that $\bigcap_{k\in K} B_k \neq \phi$.

Now consider the class $\{B_k\}_{k\in K}$. Each B_k is a finite union of sets in $\{S_\alpha\}_{\alpha\in\Delta}$, for instance let $B_k = S_1 \cup .....\cup S_n$. It now suffices to show that at least one of the sets $S_1, ..., S_n$ belongs to - (*) the class $\{B_k\}_{k\in K}$. For, if we obtain such a set S_{α_k} for each B_k, then the resulting class $\{S_{\alpha_k}\}$ is a class of subbasic closed sets. Since S_{α_k} is a subclass of $\{B_k\}$, $\{S_{\alpha_k}\}$ has f.i.p. By hypothesis, $\bigcap_{k\in K} S_{\alpha_k} \neq \phi$ and hence $\bigcap_{k\in K} B_x \neq \phi$, since each $S_{\alpha_k} \subseteq B_k$. We prove (*) by contradiction. We assume that each of the sets = $S_1, ..., S_n$ is not in the class $\{B_k\}$. Consider S_1. Since each subbasic closed set is a basic closed set, S_1 is a basic closed set.

Since S_1 is not in the class $\{B_k\}$, the class $\{B_k\}_{k\in K} \cup \{S_1\}$ contains the class $\{B_k\}_{k\in K}$ properly. By the maximality property of $\{B_k\}_{k\in K}$, the class $\{B_k\}_{k\in K} \cup \{S_1\}$ fails to have the f.i.p. So there exists a finite subclass Γ_1 of $\{B_k\}_{k\in K}$ such that $S_1 \cap \left(\bigcap_{B\in\Gamma} B\right) = \phi$. If we do this process for each of the sets $S_1, ..., S_n$, we get finite subclasses $\Gamma_1 ....., \Gamma_n$ of $\{B_k\}_{k\in K}$ such that $S_i \cap \left(\bigcap_{B\in\Gamma_i} B\right) = \phi$, for $1 \le i \le n$. Put $\Gamma = \Gamma_1 \cup ..... \cup \Gamma_n$. Now Γ is a finits subclass of $\{B_k\}_{k\in K}$ such that $B_k \cap \left(\bigcap_{B\in\Gamma} B\right) =$ $\left(S_1 \cap \left(\bigcap_{B\in\Gamma} B\right)\right) \cup ......\cup \left(S_n \cap \left(\bigcap_{B\in\Gamma} B\right)\right) = \phi$. Therefore $\Gamma \cup \{B_k\}$ is a finite subclass of $\{B_k\}_{k\in K}$ **with empty intersection. This contradicts the finite intersection property of the** class $\{B_k\}_{k\in K}$. Therefore one of the sets $S_1, ..., S_n$ belongs to the class $\{B_k\}_{k\in K}$ as **defined.**

By remark 6.9, the above theorem can be restated as follows.

6.20. Theorem: A topological space is compact if every subbasic open cover has a finite subcover.

6.21. SAQ Let X be a topological space and Y a subspace of X If $\{V_i\}_{i\in I}$ is an open subbase for X then the class $\{U_i\}_{i\in I}$, where $U_i = V_I \cap Y$, $\forall\, i \in I$, is an open subbase of Y.

We now prove the famous Heine – Borel theorem.

6.22 Theorem: (The Heine – Borel theorem) Every closed and bounded subspace of the real line is compact.

Proof: Let E be a closed and bounded subspace of the real line ℝ. E is bounded $\Rightarrow E \subseteq [-n, n]$ for some positive integer n. Since E is closed in ℝ, it is also closed in [-n, n]. By theorem 6.5, to show that E is compact, it suffices to show that every interval of the form [a, b] is compact. If a = b, then [a, b] = {a} and hence it is compact, because every finite space is compact. So, we may assume that a < b. Clearly the class of all intervals of the form $(c, +\infty)$ and $(-\infty, d)$, where c and d are real numbers is an open base for ℝ. By SAQ 6.21 by droping the empty set, the class of all intervals of the form [a, d) and [c, b] where c and d are real numbers such that a < c < b and a < d < b is an open subbase for [a,b]. Therefore the class of all interals of the form [a, c] and [d, b], where a < c, d < b is a closed subbase for [a, b]. Let $y = \{[a, c_i]\}_{i\in I} \cup \{[d_i, h]\}_{i\in J}$ be a class of subbasic closed sets with f.i.p. It suffices to show that the intersection of all sets in y is nonempty.

If y contains only intervals of the form $[a, c_i]$ then the intersection contains a. Similarly y contains only intervals of the form $[d_i\ b]$, then the intersection contains b. So, we may assume that y contains intervals of both the types.Define $d = \sup \{d_i / [d_i, b] \in y\}$. Cleanly $d \in [d_i, b]$, $\forall$ i. We complete the proof by showing that $d \leq c_i\ \forall\, i$. Suppose that $d > c_{i_0}$ for some i_0. Then c_{i_0} is not an upper bound of the defining set of d. $\therefore$ There exists a d_{r_0} such that $c_{i_0} < d_{r_0}$. Thus $[a, c_{i_0}] \cap [d_{r_0}, b] = \phi$. This contradicts the f.i.p of y. This completes the proof.

6.23. SAQ: Prove the convrse of the Heine – Borel theorem: Every compact subspace of the real line is closed and bounded.

6.24 Definition: A topological space is said to the countably compact of every countable open cover has a finite subcover.

6.25 SAQ: Prove that a second countable space is countably compact ⇔ it is compact.

6.26 Model Examination Questions:

1. Prove that any closed subspace of a compact space is compact.
2. Prove that any continuous image of a continuous space is compact.
3. Prove that a topological space is compact if and only it every class of basic closed sets with the f.i.p has non-empty intersection.
4. Prove that a topological space is compact if and only if every subbasic open cover has a finite subcover.
5. State and prove the Heine – Borel theorem.
6. Prove that every compact subspace of the real line is closed and bounded.

6.27. Exercises:

1. Prove that a compact subspace of a metric space is closed and bounded.
2. Let X be a topological space. If Y_1, and Y_2 are compact subspaces of X, prove that $Y_1 \cup Y_2$ is also a compact subspace of x.
3. If $\{X_i\}$ is a non-empty class of compact subspaces of X each of which is closed, and if $\bigcap_i X_i$ is non-empty,show that $\bigcap_{i \in I} X_i$ is also a compact subspace of x.
4. Show that a continuous real or complex function defined on a compact space is bounded.
5. Show that a continuous real function defined on a compact space X attains its infimum and its supremum.

6. If X is a compact space, and if $\{f_n\}$ is a monotone sequence of continuous real functions defined on X which converges pointwise to a continuous real funtion f defined on X, show that $\{f_n\}$ converges uniformly to f.
7. Prove 6.8
8. Prove that the class of intervals of the form (c, ∞) or (-∞, d) where c, d are real numbers is an open base for ℝ.

6.28 Answers to self assessment questions:

6.4. Suppose that Y is a compact subspace of X. Let $\{H_i\}_{i\in I}$ be a class of open sets in X such that $Y \subseteq \bigcup_{i\in I} H_i$.

Then $\{Y \cap H_i\}_{i\in I}$ is an open cover of Y. Hence, there exists a finite subcover, say $\{Y \cap H_{i_1}, \ldots\ldots, y \cap H_{i_n}\}$. Therefore $Y = (Y \cap H_{i_1}) \cup \ldots\ldots \cup (Y \cap H_{i_n}) = Y \cap (H_{i_1} \cup \ldots\ldots \cup H_{i_n}) \subseteq H_{i_1} \cup \ldots\ldots \cup H_{i_n}$. Since every open set G in Y can be written as G = $Y \cap H$, where H is open in X, the converse part can be proved in a similar way.

6.21 Let H be any non-empty open set in Y and $y \in H$. Then $H = G \cap Y$, where G is open in X.Since $y \in G$, there exists $V_{i_1}, \ldots\ldots, V_{i_n}$ in $\{V_i\}_{i\in I}$ such that $y \in V_{i_1} \cap \ldots\ldots \cap V_{i_n} \subseteq G$. Then, clearly $y \in V_{i_1} \cap \ldots\ldots \cap V_{i_n} \subseteq H$. $\therefore$ $\{U_i\}_{i\in I}$ is an open subbase for Y.

6.23 Let Y be a compact subspace of the real line ℝ. For each positive integer n, let $I_n = (-n, n)$. Then $\{I_n\}_{n\in N}$ is a class of open sets in ℝ such that $Y \subseteq \bigcup_{n=1}^{\infty} I_n$. Since Y is compact, there exists positive integers $n_1, \ldots, n_k$ such that $Y \subseteq I_{n_1} \cup \ldots\ldots \cup I_{n_k}$. Let n be the maximum of $n_1, \ldots, n_k$. Then $Y \leq I_n \Rightarrow Y \subseteq (-n, n) \Rightarrow$ Y is bounded. To show Y is closed it suffices to show that its complement Y^1 is open. Let $x_o \in Y^1$. For each $x \in Y$, since $x \neq x_0$, there exists neighborhoods V_x of x and U_{x_0} of x_o such that $V_x \cap V_{x_0}$. Clearly $Y \leq \bigcup_{x\in y} V_x$. Since Y is compact, there exists $x_1, \ldots\ldots, x_m \in y$ such that

$Y \subseteq V_{x_1} \cup \ldots\ldots V_{x_m}$. Let $V_{x_1}, \ldots\ldots, V_{x_m}$ be the corresponding neighborhoods of x_0. Put $G\ V_{x_1} \cup \ldots\ldots \cup V_{x_m}$ and $H = U_{x_1}, \cap \ldots\ldots \cap, U_{x_m}$. *Then* $x \in G \Rightarrow x \in V_{x_i}$ *for some i* $\Rightarrow x \notin U_{x_1} \Rightarrow x \notin H$. Therefore $G \cap H = \phi$ and hence $x_0 \in H \leq G^1 \leq Y^1$. Thus Y^1 is open .

6.25. Let X be a second countable space. Since every countable open cover is an open cover, it follows that if X is compact, then it is countably compact. Conversely suppose that X is countably compact. Let $\{G_i\}_{i \in I}$ be an open cover of X. Then $X = \bigcup_{i \in I} G_i$. By Lindelof's theorem, there exists a countable subclass $\{G_{i_1}, G_{i_2} \ldots\ldots\}$ such that $x = \bigcup_{i=1}^{\infty} G_{i_i}$. Thus $\{G_{i_r}\}_{r \in N}$ is a countable open cover of X. by hypothesis, there exists a finite subcover, say $\{G_{i_{i_1}} \ldots\ldots\ldots, G_{i_{i_k}}\}$. Since this is a finite subcover of $\{G_i\}_{i \in I}$, we have that X is compact.

Lesson writer: Y.V. REDDY

LESSON - 7

PRODUCT SPACES

7.1 Introduction : In this lesson we introduce the notions of product topology and product spaces. we define these notions initially for two topological spaces for a better understanding and prove that the usual topology on the Euclidean Plant R^2 is precisely the product topology. We then extend these notions to arbitrary class of topological spaces. We prove the main theorem of this lesson, namely the Tychonoff's theorem. As an application of this theorem we obtain the Generalized Heine - Borel Theorem. We also define the notion of locally compact space and some examples of these spaces are given. We obtain an equivalent condition for a topological space to be locally compact.

We begin with the notions of product topology and product space for two topological spaces.

Let X_1 and X_2 be topological spaces. Let us recall that the Cartesian product of the sets X_1 and X_2 is the set of all ordered pairs (x_1, x_2) with $x_1 \in X_1$ and $x_2 \in X_2$ We denote it by $X_1 \times X_2$. Suppose $X = X_1 \times X_2$. Let S be the class of all subsets of X of the form $G_1 \times X_2$ and $X_1 \times G_2$ where G_1 and G_2 are open subsets of X_1 and X_2 respectively. The topology on X generated by the class S is called the product topology. The open sets in the product topology are the unions of finite intersections of sets in S. The set X equipped with the product topology is called the product space or the product of the spaces X_1 and X_2. The product topology has S as an open subbase. It is clear that $(G_1 \times X_2) \cap (X_1 \times G_2) = (G_1 \cap X_1) \times (X_2 \cap G_2) = G_1 \times G_2$ Therefore the open base generated by S is the class of all subsets of the form $G_1 \times G_2$., where G_1 and G_2 are open in X_1 and X_2 respectively.

Define mappings $p_i : X \to X_i$, for $i = 1,2$, by $p(x_1, x_2) = x_i$ for all $(x_1, x_2) \in X$. p_1 and p_2 are called projection mappings (or simply projections)

Let us recall that if X is a non-empty set and if T_1 and T_2 are topologies on X such that $T_1 \subseteq T_2$, we say that T_1 is weaker than T_2. Further, the family of all topologies on X is a complete lattice with respect to the relation 'is weaker than'.

7.2 Theorem. Let X_1 and X_2 be topological spaces and let X be their **product** space. Then the projections p_i, for I = 1,2 are continuous. Moreover the product topology is the weakest topology for which the projections are continuous.

Proof. If G_1 is an open set in X_1, then $p^{-1}(G_1) = G_1 \times X_2$, which is a sub-basic open set in X, so p_1 is continuous. Similarly p_2 is continuous. Suppose T is a topology on X for which the projections p_1 and p_2 are continuous. Then for each pair of open sets G_1 and G_2 in X_1 and X_2 respectively, the set $G_1 \times G_2 = (G_1 \times X_2) \cap (X_1 \times G_2) = p_1^{-1}(G_1) \cap p_2^{-1}(G_2)$ must be open in T since the projections are continuous with respect to T. Thus every set which is open in the product topology must be open in T.

7.3 **Definition :** A mapping ϕ from a topological space X into a topological space Y is called an open mapping if ϕ (G) is open in Y whenever G is open in X.

7.4 **SAQ. Prove that the projections** p_1 and p_2 are open mappings. Let us recall that the Euclidean plane R^2 is a normed real linear space, where R^2 is the set of all ordered pairs (x_1, x_2) of real numbers, under coordinate wise operations and norm given by $\|(x_1, x_2)\| = \sqrt{|x_1|^2 + |x_2|^2}$.

7.5 **Theorem.** The usual topology on the Euclidean plane R^2 is precisely the product topology of the usual topologies on R taken twice.

Proof : we know that the funection d defined by

$$d((r_1, s_1), (r_2, s_2)) = \sqrt{|r_1 - r_2|^2 + |s_1 - s_2|^2}$$

is a metric on R^2. We have to show that the topology induced by the metric d is precisely the product topology. Suppose G is a subset of R^2 which is open with respect to the metric d, and let (r,s) $\in$ G. Then there exists an $\in > 0$ such that the open sphere $S_\in(r,s) \subseteq G$. Let $V = S_{\in/\sqrt{2}}(r)$ and $W = S_{\in/\sqrt{2}}(s)$, which are open sets in R containing r and s respectively. We assert that V x W $\subseteq$G and this will show that G is open in the product topology. If (x,y) $\in$ V x W, then x $\in$V and y $\in$ W; that is $|r-x| < \in/\sqrt{2}$ and $|s-y| < \in/\sqrt{2}$. Thus d (r,s), (x,y)) = $\sqrt{|r-x|^2 + |s-y|^2}$ $< \sqrt{(\in/\sqrt{2})^2 + (\in/\sqrt{2})^2} = \in$ and so (x,y) $\in S_\in(r,s) \subseteq G$, as desired.

Now suppose G is a subset of R^2 which is open with respect to the product topology, and let (x,y) $\in$ G. Then there exists open sets V and W such that (x,y) $\in$ V x W $\subseteq$ G. Thus x $\in$ V and y $\in$ W, so there exist $\in_x, \in_y > 0$ such that $S_{\in_x}(x) \subseteq V$ and $S_{\in_y}(y) \subseteq W$. Let $\in = \min\{\in_x, \in_y\}$, We claim that $S_\in(x,y) \subseteq S_{\in_x}(x) \times S_{\in_y}(y)$ which will show that G is open with respect to the metric d, since $S_{\in_x}(x) \times S_{\in_y}(y) \subseteq V \times W \subseteq G$. Now if $(r,s) \in S_\in(x,y)$ then

$$|x-r| \le \sqrt{|x-r|^2 + |y-s|^2} < \in \le \in_x \quad \text{and} \quad |y-s| \le \sqrt{|x-r|^2 + |y-s|^2} < \in \le \in_y, \quad \text{so}$$

(r,s)$\in S_{\in_x}(x) \times S_{\in_y}(y)$, as desired.

We now prove that the product of two compact spaces is compact.

7.6 Theorem : If X and Y are compact spaces, then their product space X x Y is also compact.

Proof : Let $\{W_\lambda\}_{\lambda\in\wedge}$ be an open covering of X x Y. We choose an x_0 in X and Consider $\{x_0\}$ x Y. Corresponding to each y in Y, there is a $\lambda(y) \in \wedge$ such that $(x_0, y) \in W_{\lambda(y)}$. Then there exists a basic open set $U_y \times V(y)$ such that

$(x_0, y) \in U_y \times V(y) \subseteq W_{\lambda(y)}$

The class $\{V(y)\}_{y\in Y}$ is an open covering of Y. Since Y is compact there exist $y_1, \ldots,$ $y_m \in Y$ such that Y = $V(y_1) \cup \ldots \cup V(y_n)$. Let $U_{y_1}, \ldots, U_{y_m}$ the corresponding neighborhoods of x_0. Put $U(x_0) = U_{y_1} \cap \ldots \cap U_{y_m}$. Then we have

$U(x_0) \times V(y_r) \subseteq W_{\lambda(y_r)}$, for r = 1, …, m. and so $U(x_0) \times y \subseteq W_{\lambda(y_1)} \cup \ldots\ldots \cup W_{\lambda(y_m)}$. It follows that corresponding to each x in X there is a neighborhood U(x) of x and there are finitely many elements $\lambda(x, 1), \ldots, \lambda(x, m(x))$ in $\wedge$ such that $U(x) \times Y \subseteq W_{\lambda(x,1)} \cup \ldots\ldots \cup W_{\lambda(x,\, m(x))}$. Now the class $\{U(x)\}_{x \in X}$ is an open covering of X. Since X is compact, it follows that there are elements $x_1, \ldots, x_n$ in X such that $X = U(x_1) \cup \ldots\ldots \cup U(x_n)$. So we have

$$X \times Y \subseteq (U(x_1) \times Y) \cup \ldots\ldots \cup (U(x_n) \times Y)$$

$$\subseteq \bigcup_{r=1}^{n} \bigcup_{i=1}^{m(x_i)} W_{\lambda(x_i,i)}$$

Thus $\left\{ \left\{ W_{\lambda(x_r,i)} \right\}_{i=1}^{m(x_i)} \right\}_{r=1}^{n}$ is a finite sub covering of X x Y. Therefore X x Y is compact.

7.7 SAQ. Prove that if X and Y are topological spaces such that their product space X x Y is compact, then X and Y are compact.

We now extend the notion of product topology to arbitrary class of topological spaces. Let us recall that the cartesian product $P_{i \in I} X_I$ of a non-empty class of sets $\{X_i\}_{i \in I}$ is the set of all mappings f of I into $\bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for every $i \in I$. If $f \in P_{i \in I} X_i$ then f is denoted by $f = \{x_i\}_{i \in I}$, where $f(i) = x_i$ for each $i \in I$. For each $i \in I$, the projection mapping p_i is the mapping from $P_{i \in I} X_i$ into X_i defined by $p_r(\{x_i\}_{i \in I}) = x_r$ for every $\{x_i\}_{i \in I} \in P_{i \in I} X_i$.

7.8 Definitions: (i) Let $\{X_i\}_{i \in I}$ be a non-empty class of topological spaces and let $X = P_{i \in I} X_i$ be the cartesian product of the sets $\{X_i\}_{i \in I}$. For each $i \in I$, let p_i be the projection of X onto X_i. Let S be the class of all subsets of X of the form $S = p_i^{-1}(G_i)$, where $i \in I$ and G_i is an open subset of X_i. The topology on X generated by the class S is called the product topology. The set X together with the product topology on it is called a product space of the product of the spaces $\{X_i\}_{i \in I}$.

(ii) A subset of X is open with respect to the product topology if and only if it is a union of finite intersections of sets in S. It is clear that S is an open subbase for the product topology and is called the defining open subbase.

(iii) A subset B of X is in the defining open subbase $\Leftrightarrow B = p_i^{-1}(G_i)$, for some $i \in I$ and some open subset G_i of $X_i \Leftrightarrow B = P_{i\in I}G_i$, where $G_i = X_i$ for $i \neq i$ and G_i is an open set in $X_i \Leftrightarrow B = \underset{i\in I}{P} G_i$, where G_i is an open subset of X_i which equals X_i for all i's but one. The class of all complements of open sets in the defining open subbase – namely, the class of all products of the form $P_{i\in I}F_i$, where F_i is a closed subset of X_i which equals X_i for all i's but one – is called the defining closed subbase.

(iv) The open base generated by the defining open subbase, that is, the class of all finite intersections of subbasic open sets, is called the defining open base for the product topology. A subset G of X is in the defining open base if and only if it is of the form $G = P_{i\in I}G_i$, where G_i is an open subset of X_i which equals X_i for all but a finite number of i's.

As in theorem 7.2 one can prove that all the projection mappings p_i are continuous and the product topology is the weakest topology for which the projections are continuous. Also, it is clear that all the projection mappings are open.

7.9 SAQ Let f be a mapping of a topological space X into a product space $P_{i\in I}X_i$. Prove that f is continuous $\Leftrightarrow$ p_i of is continuous for each projection p_i.

7.10 Definition: Let X be a non-empty set, let $\{X_i\}$ be a non-empty class of topological spaces, and for each i let f_i be a mapping of X into X_i. Note that if X is given its discrete topology, then all the $f_i^!$s are continuous. The intersection of all topologies on X with respect to each of which all the $f_i^!$s are continuous is called the weak topology generated by the $f_i^!$s.

It is clear that this is the topology on X which makes all the $f_i^!$s continuous and it is the weakest topology for which all the $f_i^!$s are continuous.

7.11 Remark: In view of the above definition, it is obvious that if $\{X_i\}$ is a non-empty class of topological spaces and if $X = P_i X_i$ is their product space, then the product topology on X is the weak topology generated by the set of all projections.

We now prove the main theorem of this lesson.

7.12 Theorem (Tychonoff's theorem) Let $\{X_i\}_{i\in I}$ be a non-empty class of topological spaces and let $X = P_{i\in I}X_i$ be their product space. Then X is compact if and only if each space X_i is compact.

Proof: If X is compact, then each space X_i is compact since the projections are continuous and onto. Hence, suppose that each space X_i is compact. Let $\{F_j\}_{j\in J}$ be a non-empty class of closed sets from the defining closed subbase for the product topology on X. Therefore each F_j is a product of the form $F_j = P_{i\in I} F_{ij}$, where F_{ij} is a closed subset of X_i which equals X_i for all i's but one. We assume that the class $\{F_j\}_{j\in J}$ has the finite intersection property. To show X is compact, it suffices to show that $\bigcap_{j\in J} F_{ij} \neq \phi$. For a fixed $i \in I$, we show that the class $\{F_{ij}\}_{j\in J}$, all are closed subsets of X_i, has the finite intersection property. If $\{F_{ij_1}, \ldots\ldots\ldots, F_{ij_n}\}$ is a finite subclass of $\{F_{ij}\}_{j\in J}$ then the corresponding subbasic closed sets $F_{j_1}, \ldots\ldots, F_{j_n}$ form a finite subclass of $\{F_j\}_{j\in J}$. Since $\{F_j\}_{j\in J}$ has the finite intersection property, we have that $F_{j_1} \cap \ldots\ldots \cap F_{j_n} \neq \phi$. Choose a point x in $F_{j_1} \cap \ldots\ldots \cap F_{j_n}$. Suppose $x = \{x_i\}_{i\in I}$ where $x_i \in X_i$ for all i. For $1 \leq k \leq n$, $x \in F_{ij_k} = P_{i\in I} F_{ij_k} \Rightarrow x_i \in F_{ij_k}$. Therefore $x_i \in F_{ir_1} \cap \ldots\ldots \cap F_{ir_n}$ and so $F_{ir_1} \cap \ldots\ldots \cap F_{ir_n} \neq \phi$. Thus the class $\{F_{ij}\}_{j\in J}$ has the finite intersection property. Since X_i is compact, $\bigcap_{j\in J} F_{ij} \neq \phi$. Choose a point a_i in $\bigcap_{j\in J} F_{ij}$. Since $i \in I$ was arbitrary, we have that $a_i \in \bigcap_{j\in J} F_{ij}$ for all i. Put $a = \{a_i\}_{i\in I}$. Thus $a_i \in F_{ij}$ for all i and for all $j \Rightarrow a \in \underset{i\in I}{P} F_{ij}$ for all $j \Rightarrow a \in F_j$ for all $j \Rightarrow a \in \bigcap_{j\in J} F_j \Rightarrow \bigcap_{j\in J} F_j \neq \phi$, as desired.

7.13. SAQ Show that the relative topology on a subspace of a product space is the weak topology generated by the restrictions of the projections to that subspace.

Let us recall that the n-dimensional Euclidean space R^n is the normed real linear space, where R^n is the real linear space of all ordered n-tuples $x = (x_1, ..., x_n)$ of real numbers under coordinatewise operations and the norm is given $||x|| = \sqrt{|x_1|^2 + ... + |x_n|^2}$. The topology on R^n obtained from the norm is called the usual topology. As in theorem 7.5, one can prove that the product topology on R^n is precisely the usual topology.

We now prove an important consequence of Tychonoff's theorem, namely, the 'Generalized Heine-Borel thoerem'.

7.14 Definition: Let R^n be the n- dimensional Euclidean space. If (a_i , b_i) is a bounded open interval on the real line for each i = 1, …, n, then the subset of R^n defined by

$$\mathop{P}_{i=1}^{n} (a_i , b_i) = \left\{ {(x_1, \ldots, x_n)} / {a_i < x_i < b_i} \text{ for each } i \right\}$$

is called an open rectangle in R^n. Similarly if $[a_i, b_i]$ is a closed interval on the real line for i = 1, …., n then

$$\mathop{P}_{i=1}^{n} [a_i , b_i] = \left\{ {(x_1, \ldots, x_n)} / {a_i \le x_i \le b_i} \text{ for each } i \right\}$$

is called a closed rectangle in R^n.

7.15 Theorem: (The Generalized Heine-Borel theorem) Every closed and bounded subspace of R^n is compact.

Proof: Let E be a closed and bounded subspace of R^n. Since E is bounded, there exists a real number K > 0 such that $||x|| \le K$ for all $x \in E$. If $x = (x_1, ..., x_n) \in E$, then $|x_i| \le ||x|| \le K$ and hence $x_i \in [-K, K]$ for all i. Thus $E \subseteq \mathop{P}_{i=1}^{n} [-r_i , r_i]$, where $r_i = k$ for all i. Since E is closed in R^n, it is also closed in the subspace $\mathop{P}_{i=1}^{n} [-r_i , r_i]$. Thus E is a closed subspace of the closed rectangle $\prod_{i=1}^{n}[-r_i, r_i]$. To show E is compact, it suffices to show that each closed rectangle is compact as a subspace of R^n.

Let $X = \prod_{i=1}^{n}[a_i, b_i]$ be a closed rectangle in R^n. Each coordinate space $[a_i, b_i]$ is compact by the Heine-Borel theorem. Therefore, by Tychonoff's theorem, $X=\prod_{i=1}^{n}[-a_i, b_i]$ is compact with the product topology. So, to show that X is compact as a subspace of R^n, if suffices to show that the product topology on X is the same as its relative topology as a subspace of R^n. By the above remarks, the product topology on R^n is the same as its usual topology. By SAQ 7.13, the relative topology on X is precisely the weak topology generated by the restrictions of the projections to X. It is clear that the restrictions of the projections on R^n to X are precisely the projections on X. Therefore the relative topology on X as a subspace of R^n is precisely the product topology on X. This is the desired result and the proof of the theorem is complete.

We now discuss about the localization of compactness in topological spaces.

7.16 Definition: A topological space is said to be locally compact if each of its points has a neighborhood whose closure is compact.

7.17 Examples: (i) Every compact space is locally compact. For, if X is a compact space and if $x \in X$, then X itself is a neighborhood of x such that $\overline{X} = X$ is compact. Thus X is locally compact.

The following example shows that every locally compact space need not be compact.

(ii) Let R^n be the n – dimensional Euclidean space. If $x \in R^n$ and if $S_r(x)$ is any open sphere centered on x then $S_r(x)$ is a neighborhood of x. Since the closure $\overline{S_r(x)}$ is closed and bounded, by the Generalized Heine Borel theorem, $\overline{S_r(x)}$ is compat. Hence R^n is locally compact. But R^n is not compact.

(iii) Every discrete space is locally compact.

Let us recall that a class of neighborhoods of a point is called an open base at the point if each neighborhood of the point contains a neighborhood in this class.

We now prove a necessary and sufficient condition for a topological space to be locally compact.

7.18 Theorem: A topological space is locally compact if and only if there is an open base at each point whose sets all have compact closures.

Proof: Let X be a topological space. Suppose that X is locally compact. Let x be a point in X. Let $\mathcal{B}_x$ be the class of all neighborhoods of x whose closures are compact. Since X is locally compact the class $\mathcal{B}_x$ is non-empty. We prove that $\mathcal{B}_x$ is an open base at x. Let G be any neighborhood of x. Since X is locally compact, there is a neighborhood H of x such that its closure $\overline{H}$ is compact. Clearly $G \cap H$ is a neighborhood of x and its closure $\overline{G \cap H}$ is compact, since $\overline{G \cap H}$ is a closed subspace the compact space $\overline{H}$ (theorem 6.5). Thus $G \cap H \in \mathcal{B}_x$ such that $x \in G \cap H \subseteq G$. Therefore $\mathcal{B}_x$ is an open base at x.

Conversely suppose that there is an open base at each point whose sets all have compact closures. Let $x \in X$. then there exists on open base $\mathcal{B}_x$ at x whose sets all have compact closures. Since X is a neighborhood of x, there is a neighborhood B of x in $\mathcal{B}_x$ such that $x \in B \subseteq X$. Now $B \in \mathcal{B}_x$ implies $\overline{B}$ is compact. Thus X is locally compact.

7.19 Answers to self Assessment Questions:

SAQ 7.4 Let G be an open subset of X. If $p_1(G) = \phi$ then clearly it is open. Suppose $P_1(G) \neq \phi$. Let $a \in P_1(G)$. then $a = p_1(x, y)$, for some $(x,y) \in G$. Then there exists a basic open set $G_1 \times G_2$, where G_1 and G_2 are open in X_1 and X_2 respectively, such that $(x, y) \in G_1 \times G_2 \subseteq G$. Thus $p_1(x,y) \in p_1(G_1 \times G_2) \subseteq p_1(G)$ and so $a \in G_1 \subseteq p_1(G)$, since $p_1(G_1 \times G_2) = G_1$. Hence p_1 is an open mapping. Similarly, p_2 is also an open mapping.

SAQ 7.7: Let $p_X : X \times Y \to X$ be the projection mapping. Then P_X is continuous and onto. Since continuous image of a compact space is compact, it follows that $p_X (X \times Y) = X$ is compact. Similarly Y is compact.

SAQ 7.9: Suppose f is continuous. For each i, the projection mapping p_i: $\underset{\lambda \in I}{P} X_\lambda \to X_i$ is continuous. Therefore $p_i of$ is continuous. Conversely suppose that $p_i of$ is continuous for each projection p_i. Let S be any open set from the defining open subbase of the product topology on $\underset{i \in I}{P} X_i$. Then $S = p_i^{-1}(G_i)$, for some i and some open set G_i in X_i. Therefore $f^{-1}(S) = f^{-1} (p_i^{-1}(G_i)) = (p_i of)^{-1} (G_i)$ is open, since p_i of is continuous. Thus f is continuous.

SAQ 7.13 Let $\{X_i\}$ be a non-empty class of topological spaces and let $X = P_i X_i$ be their product space. Suppose Y is a subspace of X. For each i, let p_i: $X \to X_i$ be the projection mapping and let p_i / Y: $Y \to X_i$ be the restriction of p_i to Y. The product topology on X is the topology generated by the class of all subsets of X of the form p_i^{-1} (G_i), where i is an index element and G_i is an open set in X_i. Therefore, the relative topology on Y is the topology generated by the class of all subsets of Y of the form $p_i^{-1}(G_i) \cap Y$, where i is any index element and G_i is any open subset of X_i. It is clear that

$$\left(p_i/_Y\right)^{-1} (G_i) = p_i^{-1} (G_i) \cap Y$$

Hence the relative topology on Y is the weak topology generated by the restrictions $p_i/_Y$ of the projections p_i to y.

7.20 Model Examination Questions:

1. Prove that the usual topology on the n dimensional Euclidean space R^n is the same as the product topology on it.
2. State and prove Tychonoff's theorem.
3. State and prove Generalized Heiene – Borel theorem.

4. (a) Define a locally compact space. Prove that every compact space is locally compact. Is the converse true. Justify your answer.

   (b) Prove that a topological space is locally compact if and only if there is an open base at each point whose sets all have compact closures.

7.21 Exercises

(1) Let X and Y be topological spaces. If Y is compact, prove that the projection mapping of X x Y onto X is a closed mapping (Let A and B be topological spaces. A mapping f: A → B is called a closed mapping if f(F) is closed in B wherever F is closed in A)

(2) Prove that if X and Y are metric spaces with metrics d_1 and d_2 respectively then the mapping d defined by

$$d((a, b), (c, d)) = \sqrt{d_1^2(a,c) + d_2^2(b,d)}$$

is a metric on X x Y which induces the product topology.

(3) Let X be a metric space with metric d. Prove that d is a continuous mapping of X x X into R.

(4) Prove that a closed subspace of a locally compact space is locally compact.

(5) (a) Let X, Y and Z be metric spaces and let f be a mapping of the product space X x Y into the space Z. Prove that f is continuous if and only if $x_n \to$ X and $y_n \to y$ implies $f(x_n, y_n) \to f(x, y)$

(b) Show that if f is continuous, then for any y in Y, the mapping f_y: X → Z defined by $f_y(x) = f(x, y)$ is continuous and for any x in X the mapping $x^f : Y \to Z$ defined by $x^f(y) = f(x, y)$ is also continuous. (If we regard f as a function f(x, y) of two variables x and y, it is customary to say that f is jointly continuous in both the variables x and y whenever f is continuous from the product space X x Y into Z.

(c) What about the converse of the result stated in (b)? Justify your answer.

Lesson writer: Prof.Y. VENKATESWARA REDDY

LESSON NO. 8

COMPACTNESS FOR METRIC SPACES

8.1 Introduction: The famous 'Bolzano – weierstrass theorem' (with its converse) states that a nonempty subset E of the real line is compact if and only if every infinite subset of E has a limit point in E. This motivates the concept of Bolzano – Weierstrass property for metric spaces. In this lesson, we define this concept for metric spaces and prove that a metric space is compact if and only if it has the Bolzano – weierstrass property. We also introduce the notion of sequentially compactness for metric spaces and prove that a metric space is compact if and only if it is sequentially compact. In this lesson, we further define the notion of totally boundedness for metric spaces and prove that a metric space is compact if and only if it is sequentially compact. In this lesson, we further define the notion of totally boundedness for metric spaces and prove that a metric space is compact if and only if it is totally bounded. In the sequel, we define the notion of Lebesgue number of an open cover in a metric space and prove that every open cover of a sequentially compact metric space has a Lebesgue number. By using this as a tool, we prove that any continuous image of a compact metric space is uniformly continuous.

First, let us define the following very important concept.

8.2 Definition: A metric space X is said to have the Bolzano – Weierstrass property if every infinite subset of X has a limit point in X.

8.3 Theorem: Every compact metric space has the Bolzano Weierstrass property.

Proof: Assume that the metric space X is compact. We show that every infinite subset of X has a limit point in X. Suppose that A is a subset of X with no limit points. Since each point $x \in X$ is not a limit point of A, there exists an open sphere $S_{r_x}(x)$ centered on x such that $S_{r_x}(x) \cap A \subseteq \{x\}$. Since the class $\{S_{r_x}(x)\}$ forms an open covering of X, there must be some finite subcovering $X = \bigcup_{i=1}^{n} S_{r_{x_i}}(x_i)$. Therefore $A = A \cap X = \bigcup_{i=1}^{n} (A \cap S_{r_{x_i}}(x_i) \subseteq \{x_1, \ldots, x_n\}$ and so A is finite. Thus every infinite subset of X must have a limit point in X.

8.4 SAQ. Prove that a compact subspace of a metric space is closed.

Let us recall the following definitions. If X is a metric space with metric d and if x is a point and $\{x_n\}$ is a sequence in X, we say that the sequence $\{x_n\}$ has a limit or converges to x, written $\lim x_n = x$ or $x_n \to x$, if for every $\in > 0$ there exists an integer $N(\in) > 0$ such that $d(x_n, x) < \in$ whenever $n > N(\in)$.

If $\{x_n\}$ is a sequence in X and if $\{n_k\}$ is a sequence of positive integers such that $n_1 < n_2 < \ldots\ldots$, then the sequence $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$.

8.5 Definition : A metric space is said to be sequentially compact if every sequence in X has a convergent subsequence.

8.6 Theorem: A metric space is sequentially compact if and only if it has the Bolzano – weierstrass property.

Proof: Let X be a metric space. Assume that X is sequentially compact. We show that every infinite subset A of X has a limit point in X. Since A is infinite, we can choose a sequence $\{x_n\}$ of distinct points from A. Since X is sequentially compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to a point x in X. Since $\{x_{n_k}\}$ is a sequence of distinct points, x is a limit point of the set $\{x_{n_k} / k \geq 1\}$. Since the set $\{x_{n_k} / k \geq 1\} \subseteq A$, it follows that x is a limit point of A.

Conversely suppose that every infinite subset of X has a limit point in X. We prove that X is sequentially compact. Let $\{x_n\}$ be an arbitrary sequence in X. If the sequence $\{x_n\}$ has a point x which is infinitely repeated, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = x$ for all $k \geq 1$. This subsequence $\{x_{n_k}\}$, clearly, converges to x. If no point of $\{x_n\}$ is infinitely repeated, the set A of points of the sequence $\{x_n\}$ is infinite. Since A is infinite. It has a limit point x. Then each open sphere centered on x contains infinitely many points of A. We choose a subsequence $\{x_{n_k}\}$ as follows. Choose n_1 such that $d(x, x_{n_1}) < 1$. Having chosen $n_1, \ldots, n_{k-1}$ such that $n_1 < n_2 < \ldots < n_{k-1}$ and $d(x, x_{n_i}) < \frac{1}{i}$ for $i = 1, \ldots, k-1$, choose an integer n_k such that $n_k > n_{k-1}$ and $d(x, x_{n_k}) < \frac{1}{k}$. By induction, we get a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(x, x_{n_k}) < \frac{1}{k}$ for all $k \geq 1$. Clearly $\{x_{n_k}\}$ converges to x. Thus X is sequentially compact.

Let us recall the following definition.

Let X be a metric space with metric d and let $A \subseteq X$. The diameter $d(A)$ of A is defined by $d(A) = \sup\{d(x,y) / x, y \in A\}$. A is said to have finite diameter if d(A) is a real number. In this case we say that A is bounded. Observe that $A = \phi$ if and only if $d(A) = -\infty$. So, if $A \neq \phi$, then $0 \leq d(A) \leq \infty$.

8.7 Definition: Let $\{G_i\}$ be an open cover of a metric space X. A real number $a > 0$ is called a Lebesgue number for the open cover $\{G_i\}$ if each subset of X whose diameter is less than a is contained in at least one G_i.

8.8 Theorem: (Lebesgue's covering lemma). In a sequentially compact metric space, every open cover has a Lebesbue number.

Proof: Let X be a sequentially compact metric space and let $\{G_i\}$ be an open cover of X. We say that a subset of X is 'big' if it is not contained in any G_i. If there are no big sets, then any positive real number will serve as a Lebesgue number. We may thus assume that big sets do exist. Note that every big set contains at least two points. We define a^1 = glb {d(B) / B is a big set}. Clearly $0 \le a^1 \le \infty$ and $a^1 \le d(B)$, for any big set B. It will suffice to show that $a^1 > 0$; for if $a^1 > 0$ then any real number a such that $0 < a < a^1$ will be a Lebesgue number. We therefore assume that $a^1 = 0$ and we deduce a contradiction from this assumption. For each positive integer n there exists a big set B_n such that $0 < d(B_n) < \frac{1}{n}$. Choose a point x_n in each B_n. Since X is sequentially compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, which converges to some point x in X. Then $x \in G_{i_0}$ for some set G_{i_0} in our open cover $\{G_i\}$. Since G_{i_0} is open, there exists an open sphere $S_r(x)$ such that $S_r(x) \subseteq G_{i_0}$. Since $x_{n_k} \to x$, it follows that $x_{n_k} \in S_{r/2}(x)$ for infinitely many k, that is $x_n \in S_{r/2}(x)$ for infinitely many n. Choose n_0 such that $x_{n_0} \in S_{r/2}(x)$ and $n_0 > \frac{2}{r}$. Thus $0 < d(B_{n_0}) < \frac{1}{n_0} < \frac{r}{2}$. If $y \in B_{n_0}$, then $d(x,y) \le d(x, x_{n_0}) + d(x_{n_0}, y) < \frac{r}{2} + d(B_{n_0}) < \frac{r}{2} + \frac{r}{2} = r$. Hence $B_{n_0} \subseteq S_r(x) \subseteq G_{i_0}$. This contradicts the fact that B_{n_0} is a big set.

8.9 Definitions: (i) Let X be a metric space and $\in > 0$. A subset A of X is called on $\in$ -net for X if A is finite and $X = \bigcup_{a \in A} S_{\in}(a)$.

(ii) A metric space X is said to be totally bounded if it has an $\in$-net for each $\in > 0$.

Let us recall that a subset A of X is said to be bounded if $0 \le d(A) < \infty$, where d(A) is the diameter of A.

8.10.SAQ. Prove that a totally bounded metric space is bounded.

8.11 Theorem: Every sequentially compact metric space is totally bounded.

Proof: Let X be a sequentially compact metric space with metric d. Suppose that X is not totally bounded. Then, for some $\in > 0$, X must have no $\in$-net. Let $x_1 \in X$. Then the finite set $\{x_1\}$ is not an $\in$-net for X, and so there exists a point $x_2 \notin S_\in(x_1)$. Therefore $d(x_1, x_2) \geq \in$. Now the finite set $\{x_1, x_2\}$ is also not an $\in$-net, and so there exists a point $x_3 \notin \bigcup_{i=1}^{2} S_\in(x_i)$. Thus $d(x_1, x_3) \geq \in$ and $d(x_1, x_2) \geq \in$. Proceed by induction. If there exist a set of points $\{x_1, \ldots, x_n\}$ such that $d(x_i, x_r) \geq \in$ whenever $i \neq r$, then this finite set is not an $\in$-net and so there exists a point $x_{n+1} \notin \bigcup_{i=1}^{n} S_\in(x_i)$; that is $d(x_i, x_r) \geq \in$ whenever $i \neq r$. Now, by induction, we have a sequence $\{x_n\}$ of distinct points in X such that $d(x_i, x_r) \geq \in$ whenever $i \neq r$. Since X is sequentially compact, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$, which converges to a point $x \in X$. But then the open sphere $S_{\in/2}(x)$ must contain x_{n_k} for $k > N$, where N is some positive integer; that is $S_{\in/2}(x)$ contains x_n for infinitely many n. This contradicts the fact tht $d(x_i, x_r) \geq \in$ whenever $i \neq r$. Hence X is totally bounded.

8.12. Theorem: Every sequentially compact metric space is compact.

Proof: Let X be a sequentially compact metric space. Let $\{G_i\}$be an open cover of X. By Lebesgue's covering lemma, the open cover $\{G_i\}$ has a Lebesgue number a. Put $\in = a/3$. By theorem 8.11, X has an $\in$-net, say $\{a_1, \ldots, a_n\}$. For each $k = 1,2, \ldots, n$, we have that the diameter $d(S_\in(a_k)) \leq 2\in < a$. Since a is a Lebesgue number of the open cover $\{G_i\}$, for each k, there exists an open set G_{i_k} in $\{G_i\}$ such that $S_\in(a_k) \subseteq G_{i_k}$. Thus $X = S_\in(a_1) \cup \ldots \cup S_\in(a_n) \subseteq G_{i_1} \cup \ldots \cup G_{i_n}$ and hence $X = G_{i_1} \cup \ldots \cup G_{i_n}$.

Therefore X is compact.

8.13 Theorem: If X is a metric space then the following conditions are equivalent.

(i) X is compact

(ii) X is sequentially compact

(iii) X has the Bolzano – Weierstrass property

Proof: (i) ⇒ (iii) follows from the theorem 8.3, (iii) ⇒ (ii) follows from the theorem 8.6 and (ii) ⇒ (i) follows from the theorem 8.12.

8.14 SAQ: Show that a compact metric space is separable.

We now prove an important theorem regarding continuous functions of compact metric spaces into arbitrary metric spaces.

8.15 Theorem: Any continuous mapping of a compact metric space into a metric space is uniformly continuous.

Proof: Let f be a continuous mapping of a compact metric space X into a metric space Y. Let d_X and d_y be the metrics on X and Y respectively. Let $\in > 0$. For each $x \in X$, consider the open sphere $S_{\in/2}(f(x))$ centered on f(x) and radius $\in/2$ in Y. Since f is continuous, $f^{-1}(S_{\in/2}(f(x))$ is an open set in X containing x. Now, the class $\{ f^{-1}(S_{\in/2}(f(x)) \}_{x\in X}$ is an open cover of X. since X is compact, this open cover has a Lebesgue number $\delta > 0$. If $x, x^1 \in X$ are such that $d_X(x, x^1) < \delta$, then the set $\{x, x^1\}$ is a set with diameter $< \delta$. Therefore $\{x, x^1\} \subseteq f^{-1}(S_{\in/2}(f(x_0)))$ for some $x_0 \in X$. Hence $f(x), f(x^1) \in S_{\in/2}(f(x_0))$. This implies that $d_y(f(x), f(x^1)) \leq d_y(f(x), f(x_0)) + d_y(f(x_0), f(y)) < \in/2 + \in/2 = \in$. Thus f is uniformly continuous.

8.16 Answers to Self Assessment Questions:

SAQ8.4: Let Y be a compact subspace of a metric X and let d be the metric on X. To prove that Y is closed, it suffices to show that its complement Y^1 in X is open. Let $z \in Y^1$. For each positive integer n, let $A_n = \{x \in X / d(x, z) > \frac{1}{n}\}$. Then $\{A_n\}$ is an ascending sequence of open sets in X such that $Y \subseteq \bigcup_{n=1}^{\infty} A_n$. Since Y is compact and $\{A_n\}$ is an ascending sequence, there exists a positive integer n such that $Y \subseteq A_n$. Clearly $z \in$

$S_{1/n}(z) \subseteq Y^1$, where $S_{1/n}(z)$ the open sphere with centre z and radius $\frac{1}{n}$. Hence Y^1 is open.

SAQ 8.10:d Let X be a totally bounded metric space with metric d. Let $\in > 0$. Since X is totally bounded, X has an $\in$-net, say $\{a_1, \ldots\ldots, a_n\}$. Then $X = \bigcup_{i=1}^{n} S_{\in}(a_i)$. If x, y $\in$ X then $x \in S_{\in}(a_i)$ any $y \in S_{\in}(a_r)$ for some i, r. Therefore $d(x, y) \leq d(x, a_i) + d(a_i, a_r) + d(a_r, y) \leq d(\{a_1, \ldots, a_n\}) + 2 \in$, where $d(\{a_1, \ldots, a_n\})$ is the diameter of $\{a_1, \ldots, a_n\}$. This implies that $d(X) \leq d(\{a_1, \ldots, a_n\}) + 2 \in < \infty$. Hence X is bounded.

SAQ. 8.14: Let X be a compact metric space. By theorems 8.13 and 8.11, X is totally bounded. For each positive integer n, let C_n be an $\frac{1}{n}$- net of X. Put $D = \bigcup_{n=1}^{\infty} C_n$. Since each C_n is finite, it follows that D is countable.

To prove the result, it suffices to prove that D is dense in X. Let $S_r(x)$ be any open sphere in X. Choose n such that $\frac{1}{n} < r$. Since C_n is an $\frac{1}{n}$ - net for X, we get that $X = \bigcup_{a \in C_n} S_{1/n}(a)$. Therefore $x \in S_{1/n}(a)$ for some $a \in C_n \subseteq D$. Since $\frac{1}{n} < r$, it follows that $a \in S_{1/n}(x) \subseteq S_r(x)$. Thus $D \cap S_r(x) \neq \phi$. Hence D is dense in X.

8.17: Model Examination Questions:

(i) Prove that a metric space is sequentially compact iff it has the Bolzano - Weierstrass property.

(ii) Prove that every open cover of a sequentially compact metric space has a Lebesgue number.

(iii) Prove that eve / compact topological space has the Bolzano – Weierstrass property.

(iv) Prove that every sequentially compact metric space is totally bounded.

(v) Prove that every sequentially compact metric space is compact.

(vi) Prove that any continuous mapping of a compact of a compact metric space is uniformly continuous.

8.18 Exercises:

(i) Prove that every compact topological space has the Bolzano – Weierstrass property.

(ii) Prove that if $\{x_n\}$ is a sequence in a metric space X which converges to a point x in X and if the set $\{x_n / n \geq 1\}$ of points of the sequence $\{x_n\}$ is infinite, then x is a limit point of the set $\{x_n / n \geq 1\}$.

(iii) Let X be a metric space with metric d and let $\in > 0$. Prove that if $x \in X$, then the set $M_{\in}(x) = \left\{ y \in X \middle/ d(x, y) > \in \right\}$ is open in X.

(iv) Let X be the set of all positive integers. Let T be the topology on X generated by the class of all sets of the form $\{2n - 1, 2n\}$, where $n \in X$. Show that with this topology T, X has the Bolzano – Weierstrass property but it is not compact.

(v) Prove that if E is a compact subset of a metric space, then its derived set $\bar{d}(E)$ is also compact ($\bar{d}$ (E) is the set of all limit points of E in X)

(vi) Prove that a subspace A of a metric space X is totally bounded iff $\overline{A}$ is totally bounded.

Lesson Writer

Prof. Y. Venkateswara Reddy

LESSON - 9

ASCOLI'S THEOREM

9.1 Introduction:

In Lesson 8 we established that compactness of a metric space is equivalent to sequential compactness as well as Bolzano - Weierstrass property. The full power of these criteria becomes evident when these are found to be instrumental to characterize compact subsets of the space $\mathbb{C}(X, \mathbb{C})$ of complex valued continuous functions on a compact metric space X. This characterization is known as Ascoli's theorem, also called Arzela – Ascoli theorem and Ascoli – Arzela theorem.

This theorem is based on "Cantor's diagonalization process" which enables us to select a sequence from an array, of sequences in such a way that except for a few terms in the beginning depending on the array all the remaining terms lie in every array.

Technical terms: Compact set – Total boundedness – equicontinuity.

The proof of Ascoli's theorem requires consideration of a countable collection of sequences which, when arranged in a sequence, each one is a subsequence of its predecessor. We recall that a sequence $\{b_n\}$ is a subsequence of a sequence $\{a_n\}$ if there is a strictly increasing map $\phi: \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\phi(n)}$ for every $n \geq 1$. This definition is equivalent to the existence of a strictly increasing sequence of positive integers $\{n_k\}$ such that for every $k \geq 1$, $b_k = a_{n_k}$.

Notation: Suppose X is set and $\{x_1, x_2, \ldots\ldots x_n\ldots\ldots\ldots\}$ is a sequence in X. We write $S_0 = \{x_1, x_2, \ldots\ldots, x_n, \ldots\ldots\}$.

Suppose we are given a countable collection of sequences $\{S_0, S_1, S_2, \ldots\ldots, S_n\ldots\ldots\}$ such that each S_k is a subsequence of its predecessor S_{k-1}. We write

$S_k = \{x_{k,1}, x_{k,2}, \ldots\ldots\ldots x_{k,n}\ldots\ldots\ldots\}$

9.2 SAQ: With notation as above for every $m \geq 1$ the sequence

$\{y_m, y_{m+1}, \ldots\ldots, y_{m+p}, \ldots\ldots\}$ where $y_k = x_{k,k}$ $\forall\, k \geq 1$, is a subsequence of S_m.

In particular $\{y_1, y_2, \ldots, y_k, \ldots\}$ is a subsequence of $\{x_1, x_2, \ldots, x_n, \ldots\}$

In the sequel (X, d) stands for a compact metric space and $\mathbb{C}(X, \mathbb{C})$ for the Banach space of all complex valued continuous functions on X.

Theorem A: A metric space X is compact if and only if X is complete and totally bounded.

Proof: Assume that X is compact. Let $\{x_n\}$ be any Cauchy sequence in X. Since X is sequentially compact, $\{x_n\}$ contains a convergent subsequence say $\{x_{n_k}\}$.

Let $x = \lim \{x_{n_k}\}$. We show that $\{x_n\}$ converges to x.

If $\varepsilon > 0$ there exist positive integers N_0 and N_{k_0} such that

$d(x_n, x_m) < \frac{\varepsilon}{2}$ for $n > m \geq N_0$ and

$d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for $n_k \geq N_{k_0}$

We may choose $N_{k_0} \geq N_0$. We then have for $n \geq N_{k_0}$

$$d(x_n, x) \leq d(x_n, x_{N_{k_0}}) + d(x_{N_{k_0}}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\lim x_n = x$.

Since every Cauchy sequence in X converges in X, X is complete.

To prove that X is totally bounded let $\varepsilon > 0$ be any number. The collection $\{S_\varepsilon(x) / x \in X\}$ is an open cover for X. Since X is compact, there exist finitely many elements of X, say $x_1, \ldots\ldots, x_n$ such that $X = \bigcup_{i=1}^{n} S_\varepsilon(x_i)$. This is true for every $\varepsilon > 0$, so X is totally bounded.

Conversely suppose X is complete and totally bounded. Let $\{x_n\}$ be any sequence in X. We show that $\{x_n\}$ has a convergent subsequence. Since X is complete it is enough to show that $\{x_n\}$ has a subsequence which satisfies Cauchy criterion.

Write $x_n = x_{1,n}$ and S_1 for $\{x_{1,n}\}$.

Since X is totally bounded the collection $\left\{S_{\frac{1}{4}}(x)/x \in X\right\}$ has a finite subcollection which covers X. Denote this finite subcollection by $V_1, V_2, \ldots\ldots\ldots, V_n$. Since the elements $x_{1,n}$ belong to the union of V_i, $1 \le i \le n$, one of these neighborhoods contains $x_{1,n}$ for infinitely many n.

Let $S_2 : \{x_{2,1}, x_{2,2}, \ldots\ldots\ldots, x_{2,n}, \ldots\ldots\ldots\}$ be such a sequence which is included in a single V_i so that $d(x_{2,i}\ x_{2,j}) < \frac{1}{2}$ for all i and j

Apply the above argument to the sequence S_2 and the collection $\left\{S_{\frac{1}{6}}(x)/x \in X\right\}$ has a finite subcollection which covers X. As above we get a subsequence of S_2, say $S_3 = \{x_{3,1}, x_{3,2}, \ldots\ldots\ldots, x_{3,n} \ldots\ldots\ldots\}$ whose elements lie in one of the spheres so that

$d(x_{3,i}, x_{3,j}) < \frac{1}{3}$ $\forall$ i, j.

We repeat this process and get a sequence of sequences $\{S_k\}$ where

$S_k = \{x_{k,1}, x_{k,2}, \ldots\ldots\ldots, x_{k,n} \ldots\ldots\ldots\}$ is a subsequence of its predecessor S_{k-1} and

$$d(x_{k,i}, x_{k,j}) < \frac{1}{k}\ \forall\, i, j$$

The diagonal sequence

$$S = \{y_1, y_2, \ldots\ldots, y_k, \ldots\ldots\ldots\}$$

Where $y_k = x_{k,k}$ $\forall$ k satisfies the conditions of SAQ 9.2.

Thus if $r > s$, $d(x_{r,r}, x_{s,s}) < \frac{1}{s}$

If $\varepsilon > 0$ and $s > \frac{1}{\varepsilon}$ then for $r > s$

$d(y_r, y_s) < \frac{1}{s} < \varepsilon$

Hence $\{y_r\}$ is a Cauchy sequence and as each $y_r = x_{r,r}$ is an element of $\{x_n\}$, $\{y_r\}$ is a subsequence of $\{x_n\}$. This completes the proof.

Since a closed subspace of a complete metric space is complete we have the following theorem as an immediate consequence of theorem A.

Theorem B: A closed subspace of a complete metric space is compact if and only if it is totally bounded.

9.3 Definition: A subset F of $\mathbb{C}(X, \mathbb{C})$ is said to be equicontinuous if for every positive number ε there corresponds $\delta(\varepsilon) > 0$ depending on ε such that for every x, y in X with $d(x, y) < \delta(\varepsilon)$ and $f \in F$:-

$|f(x) - f(y)| < \varepsilon$

Remark: Since every $f \in \mathbb{C}(X, \mathbb{C})$ is uniformly continuous given $\in > 0$ and $f \in \mathbb{C}(X, \mathbb{C})$ there exists $\delta > 0$ depending on ε as well as f such that $x \in X$, $y \in X$ and $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

The property that makes a family of functions F in $\mathbb{C}(X, \mathbb{C})$ equicontinous, is the existence of a common $\delta(\varepsilon) > 0$ depending on ε alone, such that

$d(x, y) < \delta(\in) \Rightarrow |f(x) - f(y)| < \varepsilon$ for all f in F.

9.4 SAQ: Every finite set $F \subseteq \mathbb{C}(X, \mathbb{C})$ is equicontinuous.

9.5 SAQ: If (X, d) is any metric space, not necessarily compact, $A \subseteq X$ and if for every $\delta > 0$ there exist finitely many points $x_1, \ldots. x_m$ in X such that

$$A \subseteq \bigcup_{i=1}^{m} S_\delta(x_i)$$

then there exist finitely many points $a_1, \ldots\ldots, a_n$ in A such that $A \subseteq \bigcup_{i=1}^{n} S_\delta(a_i)$

9.6 Proposition: A totally bounded subset $F_1 \subseteq C(X, \mathbb{C})$ is equicontinuous.

Proof: Since F_1 is totally founded, given $\in > 0$ there exist finitely many elements $f_1, \ldots, f_n$ depending upon ε such that

$$F_1 \subseteq \bigcup_{i=1}^{m} S_{\frac{\in}{3}}(f_i)$$

Since X is compact and each f_i is continuous on X, corresponding to ε and f_i there exists $\delta_i > 0$ such that $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$ for x, y in X satisfying $d(x, y) < \delta_i$.

Let $\delta = \min\{\delta_1, \ldots\ldots, \delta_n\}$

If $f \in F_1$, for some i, $f \in S_{\frac{\varepsilon}{3}}(f_i)$ so that $\forall\, x \in X\ |f(z) - f_i(z)| < \frac{\varepsilon}{3}$.

If $d(x, y) < \delta$ and $x \in X$, $y \in X$ then $d(x, y) < \delta_i$ for some i so that $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$.

Hence
$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_i(x) + f_i(x) - f_i(y) + f_i(y) - f(y)| \\ &\le |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

The proof complete.

9.7 SAQ: If $F \subset C(X,K)$ is totally bounded then F is bounded.

9.8 SAQ: Let (X, d) be a compact metric space. If $f_n \in C(X, \mathbb{C})\ \forall\, n$ and $\{f_n\}$ converges uniformly on X then $\{f_n\}$ is equicontinuous on X.

9.9 Theorem (Ascoli): Suppose F is a closed subset of $C(X, \mathbb{C})$. Then F is compact if and only if F is bounded and equicontinuous.

Proof: Suppose F is compact. Then F is totally bounded hence by 9.7 equicontinuous. Moreover a totally bounded set is bounded. Thus compactness of F implies that F is *bounded and equicontinuous.*

Conversely suppose that F is bounded and equicontinuous. To prove that F is a compact subset of the metric space **C**(X, **C**), it is enough to show that F is sequentially compact in **C**(X, **C**). As we have assumed that F is a closed subset of **C**(X, **C**), F is complete as a metric space so that every Cauchy sequence in F is convergent in F. Thus it is enough to show that every sequence in F contains a subsequence which satisfies Cauchy's criterion for convergence in **C**(X, **C**).

Since X is a compact metric space, X is separable. Hence there is a countable set which is dense in X. Let $D = \{x_1, \ldots\ldots, x_n \ldots\ldots\ldots\}$ be any such countable dense set in X.

Since F is bounded, there exists a real number $K > 0$ such that $|f(x)| \leq K$ for all f in F and $x \in K$ .... (1). Since F is equicontinuous, given $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for all x, y in X and f in F.

$$d(x, y) < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3} \qquad \ldots\ldots\ldots\ldots (2)$$

We claim that the open spheres

$\{S_\delta(x_n) / n \geq 1\}$ where $\delta = \delta(\varepsilon)$

cover X. Since D is dense in X for any $x \in X$ the open sphere $S_\delta(x)$ contains x_m for some m so that $X = \bigcup_{m=1}^{\infty} S_\delta(x_m)$.

Since X is compact this open cover has a finite subcover. So there are integers m_1, m_2, ..., m_r such that $X = \bigcup_{i=1}^{r} S_\delta(x_{m_i})$

If s is the largest integer among $m_1, \ldots\ldots, m_r$ then $X = \bigcup_{i=1}^{s} S_\delta(x_i)$ .... (3)

Now let $S_0 = \{f_1, f_2, \ldots\ldots\ldots f_n \ldots\ldots\}$ be any sequence in F. By (1) the sequence $|f_i(x_1)| \leq K \;\forall\; i$ so $S_0(x_1) = \{f_1(x_1), f_2(x_1) \ldots\ldots\ldots f_n(x_1) \ldots\ldots\}$ is bounded.

By Heine – Borel theorem this sequence of numbers contains a convergent subsequence. Choose any such convergent subsequence say,

$$\{f_{1,1}(x_1), f_{1,2}(x_2), \ldots\ldots\ldots f_{1,n}(x_1) \ldots\ldots\ldots\}$$

Write $S_1 = \{f_{1,1}, f_{1,2}, f_{1,3}, \ldots\ldots f_{1,n} \ldots\ldots\}$

S_1 is a subsequence of S_0 such that $S_1(x_1) = \{f_{1,1}(x_1), f_{1,2}(x_1), \ldots\ldots\}$ converges. We define inductively a sequence of sequences.

$$S_n = \{f_{n,1}, f_{n,2}, \ldots\ldots\ldots f_{n,k} \ldots\ldots\ldots\}$$

such that for each n, S_n is a subsequence of S_{n-1} and

$S_n(x_n) = \{f_{n,1}(x_n), f_{n,2}(x_n), \ldots\ldots\ldots f_{n,k}(x_n) \ldots\ldots\ldots\}$ converges. We have already defined such a sequence when $n = 1$. Assuming that S_{n-1} is already defined.

Then $S_{n-1}(x_n) = \{f_{n-1,1}(x_n), f_{n-1,2}(x_n) \ldots\ldots\ldots f_{n-1,k}(x_n) \ldots\ldots\ldots\}$ is bounded, hence contains a convergent subsequence. We choose any such convergent subsequence and denote this by $\{f_{n,1}(x_n), f_{n,2}(x_n), \ldots\ldots\ldots f_{n,k}(x_n) \ldots\ldots\ldots\}$ we now write $S_n = \{f_{n,1}, f_{n,2}, \ldots\ldots\ldots f_{n,k} \ldots\ldots\ldots\}$

S_n is a subsequence of S_{n-1} and $S_n(x_n)$ is a convergent sequence. The inductive process is complete. We now apply SAQ 9.2 to the countable collection $\{S_0, S_1, S_2, \ldots\ldots, S_n, \ldots\ldots\}$

The sequence $S = \{f_{1,1}\ \ f_{2,2} \ldots\ldots\ f_{n,n} \ldots\ldots\ldots\}$ is a subsequence of $\{f_1, f_2, \ldots\ldots f_n, \ldots\ldots\}$

Also for all $k \geq 1$, $\{f_{k,k}, f_{k+1,k+1}, \ldots\ldots\ldots, f_{k+p_{k+p}}\}$ is a subsequence of

$$\{f_{k,k}, f_{k,k+1}, f_{k,k+2}, \ldots\ldots\ldots, f_{k_{k+p}}, \ldots\ldots\ldots\}$$

Since the sequence $S_k(x_k)$ converges and

$$\{f_{k,k}(x_k), f_{k,k+1}(x_k), f_{k,k+2}(x_k), \ldots\ldots\ldots, f_{k_{k+p}}(x_k), \ldots\ldots\ldots\}$$

is a subsequence of $S_k(x_k)$ this subsequence converges.

Hence $S(x_k) = \{f_{1_1}(x_k), f_{2_2}(x_k), \ldots\ldots, f_{k_k}(x_k) \ldots\ldots\}$ converges for every k.

Write $g_n = f_{n_n}$. Then $\{g_1, g_2, \ldots\ldots, g_n \ldots\ldots\}$ is a subsequence of $\{f_1, f_2, f_3 \ldots\ldots f_n, \ldots\ldots\}$ and the sequence $\{g_n(x_k)\}$ converges for every k.

We now show that the sequence $\{g_n\}$ is a Cauchy sequence in F using (2) and (3).

Since $\{g_n(x)\}$ converges for $1 \leq i \leq s$ (s as in (3)), for each i, $1 \leq i \leq s$ $\exists$ a positive integer N_i such that $| g_n(x_i) - g_m(x_i) | < \frac{\varepsilon}{6}$ for $n \geq m \geq N_i$ .... (4)

Let $N(\varepsilon) = \max \{N_1, ... N_s\}$ and $x \in X$.

By (3) $\exists$ a i $\ni 1 \leq i \leq s$ and $x \in S_\delta(x_i)$

For $n \geq m \geq N(\epsilon)$, $n \geq m \geq N_i$ so

$$|g_n(x) - g_m(x)| = |g_n(x) - g_n(x_i) + g_n(x_i) - g_m(x_i) + g_m(x_i) - g_m(x)|$$

$$\leq |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| + |g_m(x_i) - g_m(x)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} \text{ by (4) and (2).}$$

Since this is true for every $x \in X$ we get for $n > m \geq N(\varepsilon)$

$$d(g_n, g_m) = \sup_{x \in X} |g_n(x) - g_m(x)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} < \varepsilon$$

Hence $\{g_n\}$ is a Cauchy sequence.

The proof is complete.

9.9.1 Corollary:

Let **K** be either $\mathbb{R}$ or $\mathbb{C}$, X a compact metric space and F, a closed subset of **C**(X, **K**). Then F is compact if it is equicontinuous and $F_x = \{f(x) / f \in F\}$ is bounded for every $x \in X$.

Proof: In view of Ascoli's theorem it is enough to show that F is bounded in **C(X, K)** ; that is there exists a $K > 0$ such that $|f(x)| \leq K$ for $x \in X$ and $f \in F$. Since F is equicontinuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < 1$ for all f in F and x, y in X with $d(x, y) < \delta$. The collection of open spheres $\{S_\delta(x) / x \in X\}$ is an open cover for X. So there is a finite number of elements, say, $x_1, ..., x_m$ in X such that $X = \bigcup_{i=1}^{m} S_\delta(x_i)$.

Since F_{x_i} is bounded for every i, $1 \leq i \leq m$, there is a $M > 0$ such that $|f(x_i)| < M$ for every $f \in F$ and $1 \leq i \leq m$. If $x \in X$ there is a i such that $d(x, x_i) < \delta$. This implies that

$|f(x) - f(x_i)| < 1$ for every $f \in F$

Hence $|f(x)| \le |f(x_i)| + |f(x) - f(x_i)| < M + 1 \; \forall \; f \in F$. Since this is true for every $f \in F$ and $x \in X$ it follows that F is bounded.

9.10 Model Examination Questions:

1. Define equicontinuity of a family of functions F in $C(X, \mathbb{C})$ when X is a compact metric space.

   Show that if $F \subseteq C(X, \mathbb{C})$ is totally bounded in $C(X, \mathbb{C})$ then F is equicontinuous.

2. Let (X, d) be a compact metric space and $F \subseteq C(X, \mathbb{C})$. If F is compact then F is equicontinuous.
3. Let D be a countable set and $\{f_n\}$ be a sequence of complex valued functions such that $\{f_n(x)\}$ is bounded for every $x \in D$. Show that there is a subsequence $\{g_k\}$ of $\{f_n\}$ such that $\{g_k(x)\}$ converges for every $x \in D$.

9.11 Solutions to SAQ's

SAQ 9.2: Let $\phi_{k+1} : \mathbb{N} \to \mathbb{N}$ be the strictly increasing map that makes

$S_{k+1} : \{x_{k+1,1}, x_{k+1,2}, \ldots\ldots, x_{k+1,n}, \ldots\ldots\}$ a subsequence of

$S_k : \{x_{k,1}, x_{k,2}, \ldots\ldots, x_{k,n}, \ldots\ldots\}$

Then $x_{k,n} = x_{k-1,\phi(n)}$ for every $k \ge 1$ and $n \ge 1$ where $x_{0,n} = x_n$ We define

$\phi_{k,p} : \phi_{k+1}\, \phi_{k+2} \ldots\ldots \phi_{k+p}$

Then $\phi_{k,p} : \mathbb{N} \to \mathbb{N}$ is strictly increasing and

$x_{k+p,n} = x_{k,\phi_{kp}(n)} \quad \forall \; n \ge 1$

Thus $\{x_{k+p,n}\}$ is a subsequence of $\{x_{k,n}\}$ for all $p \in \mathbb{N}$.

Also if $k + p = r$,

$\{y_r, y_{r+1}, \ldots\ldots y_{r+n}, \ldots\ldots\}$ is a subsequence of

$\{x_{k,k}, x_{k,k+1}, \ldots\ldots x_{k,k+n}, \ldots\ldots\}$, which is a subsequence of S_k.

In particular $\{x_{1,1}, x_{2,2}, \ldots\ldots x_{n,n} \ldots\ldots\}$ is a subsequence of $S_0 = \{x_1, x_2, \ldots\ldots x_n, \ldots\ldots$

SAQ 9.4 Suppose $\varepsilon > 0$. There are $\delta_j(\varepsilon) > 0$ such that $|f_j(x) - f_j(y)| < \varepsilon$ if $d(x,y) < \delta_j(\varepsilon)$.

Set $\delta(\varepsilon) = \text{Min}\{\delta_1(\varepsilon), \quad, \delta_n(\varepsilon)\}$

SAQ 9.5 Suppose $\delta > 0$. Then there are $x_1, \ldots\ldots, x_m$ in X such that

$$A \subseteq S_{\delta/2}(x_1) \cup \ldots\ldots\ldots\ldots\ldots \cup S_{\delta/2}(x_m).$$

Suppose $a \in A \cap S_{\delta/2}(x_j)$ then

$$A \cap S_{\delta/2}(x_j) \subseteq S_\delta(a)$$

Choose one element from each non-empty set $A \cap S_{\delta/2}(x_j)$

Let $a_1, a_2, \ldots\ldots, a_n$ be the points so selected that

$A \subseteq S_{\delta/2}(a_1) \cup \ldots\ldots\ldots\ldots \cup S_\delta(a_n)$.

SAQ 9.7: $1 > 0$. So there are $f_1, \ldots\ldots, f_m$ in F such that $F \subseteq S_1(f_1) \cup \ldots\ldots \cup S_1(f_m)$

Let $K = \| f_1 \| + \ldots\ldots | \| f_m \|$. If $f \in F$ then there is j such that

$$f \in S_1(f_j)$$

So $\|f\| = \| f - f_j + f_j \| \leq \|f - f_j\| + \| f_j \| \leq 1 + K$

SAQ 9.8 Let $\varepsilon > 0$. Since $\{f_n\}$ converges uniformly on X, there is a positive integer **N**

such that $\|f_n - f_m\| = \sup\limits_{x \in X} |f_n(x) - f_m(x)| < \dfrac{\epsilon}{3}$ for $n > m \geq N$.

In particular $\|f_n - f_N\| < \dfrac{\varepsilon}{3}$ for $n \geq N$.

Each of the functions $f_1, \ldots\ldots f_N$ is continuous, hence uniformly continuous on X. Hence

there is $\delta > 0$ such that $|f_i(x) - f_i(y)| < \dfrac{\varepsilon}{3}$ if $d(x, y) < \delta$ and $1 \leq i \leq N$.

If $n \geq N$ and $d(x, y) < \delta$

$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

9.12 Exercises:

1. Show that (0, 1) is bounded but not totally bounded.
2. Let $f: \mathbb{R} \to \mathbb{R}$ be uniformly continuous.

   Define $f_n(x) = f(nx)$ for $n \geq 1$.

   Is $\{f_n\}$ an equicontinuous family?
3. Suppose $\{f_n\}$ is an equicontinuous on a compact metric space (X, d) and $\{f_n(x)\}$ converges for every $x \in X$. Show that $\{f_n\}$ converges in $\mathbb{C}(X, \mathbb{C})$.
4. Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$ $(0 \leq x \leq 1 \text{ and } n \geq 1)$

   a) Show that $\lim_n f_n(x) = 0$ $(0 \leq x \leq 1)$

   b) Show that $|f_n(x)| \leq 1$ $0 \leq x \leq 1$ and $n \geq 1$

   c) Show that $\{f_n\}$ is not equicontinuous.
5. Does equicontinuous imply boundedness?

Reference: Introduction to Topology and Modern Analysis – G.F. Simmons - International student edition - Mc Graw - Hill International Book Company (1963).

Lesson Writer: V.J. LAL

9A

LESSON NO - 10

SEPARATION

10.1 Introduction : In this lesson we introduce three separation axioms and explore some of their properties. These axioms are called separation axioms for the reason that they involve "separating" certain kinds of sets from one another by disjoint open sets.

Consider the fact that in $\mathbb{R}$ and $\mathbb{R}^2$ each one point set is closed. But this is not true in arbitrary topological spaces. For example, consider the topology $\mathfrak{I}=\{\phi,x,\{a,b\},\{b,c\},\{b\}\}$ on the three point set $x=\{a,b,c\}$. In this space the one point set $\{b\}$ is not closed, for its complement is not open therefore, one often imposes an additional condition that will rule out examples like this one, bringing the class of spaces under consideration closer to those to which one's geometric intuition applies. The condition was suggested by the mathematician Felix Hausdorff, so mathematicians have come to call it by his name.

The Hausdorff condition is stronger than the following property, which is usually called the T_1-axiom:

10.2 Definition : A T_1-space is a topological space in which given any pair of distinct points, each has a neighborhood which does not contain the other.

10.3 Examples

(i) Every discrete space with more than one point is a T_1-space.

(ii) Every indiscrete space with more than one point is not a T_1-space.

(iii) Consider the space $X=\{1,2,3\}$, $Y\{\phi,x,\{1\},\{1,2\},\{1,3\}\}$ every open set that contains 2 also contains 1. Hence X is not a T_1-space.

(iv) Let X be any infinite set, and let the topology consist of the empty set ϕ together with all subsets of X whose complements are finite (that is, co-finite topology). This is a T_1-space.

10.4 Self Assessment Question

Show that any subspace of a T_1-space is also a T_1-space.

In the following theorem we will give a simple characterization of T_1-spaces.

10.5 Theorem : A topological space X is a T_1-space if, and only if every subset consisting of exactly one point is closed.

Proof : If x and y are district points of X in which every subset consisting of exactly one point is closed, then $\{x\}^1$ is an open set containing y but not x, while $\{y\}^1$ is an open set containing x but not y. Thus X is a T_1-space.

Conversely, let us suppose that X is a T_1-space and that x is a point of X. Then by definition 8.2 if $y \neq x$, there exists an open set G_y containing y but not x, that is $y \in G_y \leq \{x\}^1$. But then $\{x\}^1 = \cup\{\{y\}: y \neq x\} \leq \{x\}^1$, and so $\{x\}^1$ is a union of open sets, and hence is itself open. Thus $\{x\}$ is a closed set for every $x \in X$.

10.6 Self Assessment Question:

Show that in a T_1-space X, a point x is a limit point of a set E if and only if every open set containing x contains an infinite number of distinct points of E.

10.7 Self Assessment Question.

Show that any finite T_1-space is discrete.

10.8 Self Assessment Question.

Show that a topological space is a T_1-space iff each point of X is the intersection of all open sets containing it.

We now define a separation property which is slightly stronger than the T_1-axiom.

10.9 Definition : A T_2-space or Hausdorff space is a topological space X in which each pair of distinct points can be separated by open sets, in the sense that they have disjoint neighborhoods. That is $x \in X$, $y \in X$ and $x \neq y$, there exists neighborhoods U_x, U_y of X respectively such that $U_x \cap U_y = \phi$.

10.10 Examples :

(i) Every discrete space X is a T_2-space for, if x, $y \in X$, are such that $x \neq y$ $\{x\}$ and $\{y\}$ are open sets, $\{x\} \cap \{y\} = \phi$ and $x \in \{x\}$, $y \in \{y\}$.

(ii) Every metric space is a Hausdorff space.

(iii) Every subspace of a Hausdorff space is a Hausdorff space.

(iv) Every Hausdorff space is a T_1-space but the converse is not true. For example, if T is the co-infinite topology on an infinite set X then (X,T) is a T_1-space but not a Hausdorff space (T_2-space)

By the definition of T, since any finite subset of X is closed, singletons are closed. Hence, (X,T) is a T_1-space.

We will show that in this space we cannot find two disjoint open sets neither of which is empty. For otherwise, suppose G and H are disjoint non-empty open sets then, $X = \phi^1 = (G \cap H)^1 = G^1 \cup H^1$, a contradiction, since G^1 and H^1 are finite, so is their union $G^1 \cup H^1 = X$.

Therefore (X,T) is not a Hausdorff space.

10.11 Theorem : The product of any non-empty class of Hausdorff spaces, is a Hausdorff space.

Proof : Let $X = \prod_i X_i$ be the product of a non-empty class of Hausdorff spaces. Let x and y be two distrinct points in X. Then we must have $x_{i_0} \neq y_{i_0}$ for at least one index i_0. Since X_{i_0} is Hausdorff, there exists disjoint open sets U_{i_0} and V_{i_0} containing x_{i_0} and y_{i_0} respectively. Now, $\Pi_{i_0}^{-1} U_{i_0})$ and $\Pi_{i_0}^{-}$ (V_{i_0}) disjoint open sets in the product space containing x and y respectively.

10.12 Theorem : In a Hausdorff space any point and disjoint compact subspace can be separated by open sets in the sense that they have disjoint neighborhoods.

Proof : Let X be a Hausdorff space, x a point in X and C a compact subspace of X which does not contain x. We exhibit a disjoint pair of open sets G and H such that $x \in G$ and $C \subseteq H$. Let y be a point in C. Since $y \neq x$ and X is a Hausdorff space, there exists disjoint neighborhoods G_y and H_y of x and y respectively. If we allow y to vary over C, we obtain a class $\{H_y\}_{y \in C}$ of open sets such that $C \subseteq \bigcup_{y \in C} H_y$. Since C is compact, there is a finite subclass $\{H_{y_1}, H_{y_2}, \ldots, H_{y_n}\}$ such that.$C \subseteq H_{y_1} \cup H_{y_2} \cup \ldots \cup H_{y_n}$. If $G_{y_1}, G_{y_2}, \ldots\ldots G_{y_n}$ are the neighborhoods of X which correspond to H_{y_i}'s, put $G = \bigcap_{i=1}^{n} G_{y_i}$; and $H = \bigcup_{i=1}^{n} H_{y_i}$. Clearly G and H are open sets containing x and C respectively. For each I = 1,2,……, n $G \cap H_{y_i} \subseteq G_{y_i} \cap H_{y_i} = \phi$. Therefore, $G \cap H = \bigcup_{i=1}^{n} (G \cap H_{y_i}) = \phi$. Hence, G and H are disjoint.

We have proved in theorem 6.4 that every closed subspace of a compact space is compact. By considering the indiscrete space X, we have proved that a compact subspace of a compact space X need not be closed. We now use the preceding theorem to show that compact subspaces of Hausdorff spaces are always closed.

10.13 Corollary : Every compact subspace of a Hausdorff space is closed.

Proof : Let C be a compact subspace of a Hausdorff space X. We prove that C is closed by showing that its complement C^1 is open. C^1 is open if it is empty. So we may assume that C^1 is non-empty. Let x be any point in C^1. By theorem 10.12, x has a neighborhood G_x such that $x \in G_x \subseteq C^1$. Clearly, $C^1 = \bigcup_{x \in C^1} G_x$; therefore C^1 is open.

One of the most useful consequences of this result is the following:

10.14 Theorem : A one-to-one continuous mapping of a compact space on to a Hausdorff space is a homeomorphism.

Proof : Let f: $X \rightarrow Y$ be a one-to-one continuous mapping of a compact space X onto a Hausdorff space Y. We must show that f(G) is open in Y whenever G is open in X. If G is open in X then G^1 is closed in X. Since X is compact, G^1 is compact. Therefore $f(G^1)$ is compact, since f is continuous. Since f is onto $f(G)^1=f(G^1)$ is a compact subspace of a Hausdorff space Y. Hence, by Corllary 10.13, $f(G)^1$ is closed. Therefore, f(G) is open.

10.15 Self Assessment Question.

(a) Give an example of a topological space in which any sequence converges to every point of the space.

(b) If X is a Hausdorff space, show that every convergent sequence in X has a unique limit.

10.16 Definition : Let X be a topological space and consider the set $C(X,\mathbb{R})$ of all bounded continuous real functions defined on X. If for each pair of distinct points x and y in X there exists a function f in $C(X, \mathbb{R})$ such that $f(x)\neq f(y)$, we say that $C(X, \mathbb{R})$ separates points.

10.17 Lemma : If $C(X, \mathbb{R})$ separates points then X is Hausdorff.

Proof : Let $x,y \in X$ such that $x\neq y$. Since $C(X, \mathbb{R})$ separates points there exists a function f in $C(X, \mathbb{R})$ such that $f(x) \neq f(y)$. Suppose $f(x)<f(y)$. Let r be a real number such that $f(x)<r<f(y)$. Now, put $G_x= f^{-1}(-\infty,r)$, $G=f^{-1}(r, \infty)$

Since f is continuous, G_x and G_y are open in X and $x\in G_x$ and $y\in G_y$, $G_x \cap G_y=\phi$. Hence, X is a hausdorff space.

10.18 Definition : A topological space X is said to be a completely regular space if (i) X is a T_1-space (ii) $x\in X$, F is a closed subspace of X such that $x\notin F$ then there exists a function f in $C(X, \mathbb{R})$ such that $0 \leq f(x) \leq 1 \forall x \in X$ and $f(x)=0$ and $f(F)=1$.

Thus completely regular spaces are T_1-spaces in which continuous functions separate points from disjoint closed subspaces.

10.19 Lemma : Every completely regular space is a Hausdorff space.

Proof : Let X be a completely regular space. Then X is a T_1-space, by definition. We will show that C(X, ℝ) separates points. Let $x, y \in X$ such that $x \neq y$. Since X is a T_1-space singletons are closed. Thus $\{y\}$ is closed and $x \notin \{y\}$. Then there exists an f in C(X, ℝ) with values in [0,1] such that $f(x)=0$ and $f(y)=1$. Hence for any x,y in X such that $x \neq y$ there exists f in C(X, ℝ) such $f(x) \neq f(y)$. By Lemma 10.17, X is a Haussdorff space.

10.20 Remark: Any subspace of a completely regular space is completely regular. Our next separation property is similar to that of a Hausdorff space, except that it applies to disjoint closed sets instead of merely distinct points.

10.21 Definition : A T_1-space X is said to be a normal space if for any two disjoint closed sets F_1 and F_2 in X there exist disjoint open sets G_{F_1} and G_{F_2} such that $F_1 \subseteq G_{F_1}$ and $F_2 \subseteq G_{F_2}$.

Note : Any metric space is a normal space (see 10.28.5)

10.22 Theorem : Every compact Hausdorff space is normal.

Proof : Let X be a compact Hausdorff space and F_1 and F_2 be disjoint closed subsets of X. We must produce a disjoint pair of open sets G_{F_1} and G_{F_2} such that $F_1 \subseteq G_{F_1}$ and $F_2 \subseteq G_{F_2}$. If either of the closed sets is empty, we can take the empty set as a neighborhood of F_i and the full space as a neighborhood of the other. We may therefore assume that both F_1 and F_2 are disjoint compact subspaces of X. Let x be a point of F_1 then $x \notin F_2$ hence by theorem 10.12, there exist disjoint open sets G_x and $G^x_{F_2}$ such that $x \in G_x$ and $F_2 \subseteq G^x_{F_2}$. The collection $\{G_x / x \in F_1\}$ covers F_1 and since F_1 is compact there exist $x_1, x_2, \ldots, x_n$ in F_1 such that $F_1 \subseteq \bigcup_{i=1}^{n} G_{x_i}$

Now, put $G_{F_1} = \bigcup_{i=1}^{n} G_{x_i}$, clearly G_{F_1} is an open set containing F_1. Put $G_{F_2} = \bigcap_{i=1}^{n} G_{F_2}{}^{x_i}$

$G_{F_2} \cap G_{x_i} \subseteq G_{F_2} x_i = \phi$. Hence, $G_{F_2} \cap G_{x_i} = \phi$ for i=1,2........,n. Therefore, $G_{F_1} \cap G_{F_2}$ =

$(\bigcup_{i=1}^{n} G_{x_i}) \cap G_{F_2} = \bigcup_{i=1}^{n} (G_{x_i} \cap G_{F_2}) = \phi$.

Hence, G_{F_1} and G_{F_2} are disjoint open sets such that $F_1 \subseteq G_{F_1}$ and $F_2 \subseteq G_{F_2}$, since

$F_2 \subseteq G_{F_2}^{x_i}$ for i = 1,2.........,n.

Therefore X is a normal space.

A characterization of normality is given in the following theorem. Let us recall that by a neighborhood of a set F we mean an open set G containing F.

10.23 Theorem : A topological space X is normal if and only if each neighborhood of a closed set F contains the closure of some neighborhood of F.

Proof : Suppose X is normal and the closed set F is contained in an open set G. Put, $K = x - G$. Now K is a closed set which is disjoint from F. Since X is normal there exist disjoint open sets G_F and G_K such that $F \subseteq G_F$ and $K \subseteq G_k$. Since $G_F \subseteq X \setminus G_k$ and x - G_k is closed, we have $\overline{G_F} \subseteq x - G_k$. Now, $\overline{G_F} \subseteq X - G_k \subseteq x - K = G$. Thus G_F is a desired set. Here G_F is a neighborhood of F and its closure $\overline{G_F} \subseteq G$. Conversely suppose the condition holds and let F_1 be contained in the open set x - F_2, and by hypothesis there exists an open set G^* such that $F_1 \subseteq G^*$ and $\overline{G^*} \subseteq X - F_2$. Clearly G^* and $X - \overline{G^*}$ form a pair of desroint open sets containing F_1 and F_2 respectively.

We now prove the main theorem of the lesson that is commonly called the 'Urysohn's Lemma'. It asserts the existence of certain real-valued continuous functions on a normal space X.

10.24 Theorem (Urysohn's Lemma) : Let X be a normal space and let A and B be disjoint closed subspaces of X. Then there exists a continuous real function f defined on X, all of whose values lie in the closed unit interval [0,1] such that f(A)=0 and f(B)=1

Proof : We shall define, for each rational number r, an open set U_r of X in such a way that whenever r<s we have $\overline{U_r} \subseteq U_s$. For each rational number r such that r<0, define $U_r = \phi$ and for each r>1, define U_r=X. Let $\{r_n\}$ be a listing of all rational numbers in the interval [0,1] such that r_1=0 and r_2=1. Define $U_{r_2} = B^i$. Since A is a closed set contained in the open set U_{r_2}, by theorem 10.23, there is an open set U_{r_1} such that $A \subseteq U_{r_1}$ and $\overline{U_{r_1}} \subseteq U_{r_2}$. Suppose that $U_{r_1}, U_{r_2}, \ldots\ldots, U_{r_n}$ are defined. We define $U_{r_{n+1}}$ as follows: The number r_1 is the smallest element, and r_2 is the largest element of the set $\{r_1, r_2, \ldots\ldots r_{n+1}\}$ and r_{n+1} is neithter r_1 nor r_2. So, r_{n+1} has an immediate predecessor p and an immediate successor q in $\{r_1, r_2, \ldots\ldots r_{n+1}\}$. Since $p<r_{n+1}<q$, the sets U_p and U_q are already defined, and $\overline{U}_p \subseteq U_q$. Since X is normal, there is an open set $U_{r\ n+1}$ of X such that $\overline{U}_p \subseteq U_{r_{n+1}}$ and $\overline{U}_{r_{n+1}} \subseteq U_q$. By induction, we have U_{r_n} defined for all n. We now define f: X→ℝ as follows: Given a point x of X, let us define $Q(x) = \{r / x \in U_r\}$. $r<0 \Rightarrow x \notin \phi = U_r$ so $r \in Q(x)$ ⇒xc. Also Q(x) it contains every number greater than 1, since every x is in U_r for r>1. Therefore Q(x) is bounded below, and its greater lower bound is a point in the internal [0,1]. Define f(x)=gtb Q(x) = g/b $\{r/x \in U_r\}$.

We show that f is the desired function. If $x \in A$, then $x \in U_r$ for every $r \geq 0$, so that Q(x) equals the set of all non-negative rationals and f(x)=g.l.b Q(x)=0. Similarly, if $x \in B$, then $x \in U_r$ for no $r \leq 1$, so that Q(x) consists of all rational numbers greater than 1, and f(x)=1. We finally we show that f is continuous.

For this purpose, we first prove the following elementary facts.

(i) $x \in \overline{U}_r \Rightarrow f(x) \leq r$

(ii) $x \notin U_r \Rightarrow f(x) \geq r$

To prove (i), note that if $x \in \overline{U}_r$, then $n \notin U_s$ for every s>r Q(x) contains all rational numbers greater than r, so that by definition we have f(x)=glb Q(x) ≤r.

To prove (ii), note that if $x \notin U_r$ then $x \notin U_s$ for any $s \leq r$. Therefore Q(x) contains no rational number less than or equal to r. $f(x) = \text{glb } Q(x) \geq r$.

Now we prove continuity of f. Let $x_0 \in X$. Let (c,d) be an open interval containing the point $f(x_0)$. Choose rational numbers p and q such that $c<p<f(x_0)<q<d$. Put $U=U_q \cap (\overline{U}_p)'$. Clearly $x_0 \in U$ (for it $x_0 \notin U_q$ then by (ii) $f(x_0) \geq q$. Also $x_0 \notin \overline{U}_p$, because $x_0 \in \overline{U}_p \Rightarrow f(x_0) \leq p$, by(i)).

U is a nbd of x_0. We show that $f(U) \subseteq (c,d)$.

Let $x \in U$. then $x \in U_q \subseteq \overline{U}_q$ so that $f(x) \leq q$ by (i). And $x \notin \overline{U}_p$, so that $x \notin U_p$ and $f(x) \geq p$ by (ii). Thus $f(x) \in [p,q] \subseteq (c,d)$, as desired.

The following slightly more flexible form of Urysohn's lemma will be useful in applications.

10.25 Theorem : Let X be a normal space, and let A and B be disjoint closed subspaces of X. If [a,b] is any closed interval on the real line, then there exists a continuous real function f defined on X, all of whose values lie in [a,b], such that f(A)=a and f(B)=b.

Proof : If a=b, we have only to define f by f(x)=a for every x, so we may assume that a<b. If g is a function with the properties stated in Urysohn's lemma, then the function f defined by f(x)=(b-a) g(x)+a has the required properties.

10.26 Answers to SAQS

10.4 Let Y be a subspace of a T_1-space X. Let $y_1 \neq y_2$ be distinct elements in Y. Since X is a T_1-space there exists a neighborhood G of y_1 and a neighborhood H of y_2 such that $y_2 \notin G$ and $y_1 \notin H$. Then, $G \cap Y$ and $H \cap Y$ are neighborhoods of y_1 and y_2 in Y such that $y_1 \notin G \cap Y$ and $y_1 \notin H \cap Y$. Hence, Y is a T_1-space.

10.6 Sufficiency of the condition is obvious. To prove the necessity, suppose there were an open set G containing α for which G∩E was finite. If we set $G\cap(E\backslash\{x\})=\bigcup_{i=1}^{\infty}\{x_i\}$, then each set $\{x_i\}$ would also be a closed set. But then $(\bigcup_{i=1}^{n}\{x_i\})^1\cap G$ would be an open set containing x with $[(\bigcup_{i=1}^{n}\{x_i\})^1\cap G]\cap E\backslash\{x\}=(\bigcup_{i=1}^{n}\{x_i\})^1\cap(\bigcup_{i=1}^{n}\{x_i\})=\phi$. Thus x would not be a limit point of E.

10.7 Since X is a T_1-space, singletons are closed. Let A be a subset of X then $A=\bigcup_{a\in A}\{a\}$ is a finite union of closed sets and hence closed. Thus any subset of X is closed and thus any subset of X is open. That is $(X,\Im)$ is a discrete space.

10.8 Let N be the intersection of all open sets containing on arbitrary point x and let y be any point of X different from x. Since the space is T_1, there exists a neighborhood of x not containing y and consequently y cannot belong to N, that is $y\notin N$. Since y is arbitrary, no point of X other then x can belong to N. It follows that N={x}. Now we prove the converse part let x,y be any two distinct points of X. By hypothesis, the intersection of all neighborhoods of x is {x}. Hence there must be a neighborhood of x which does not contain y. It follows that X is a T_1-space.

10.16 We first recall the definition of the convergence of a sequence in a topological space.

Let X be an arbitrary topological space and $\{x_n\}$ a sequence of points in X. This sequence is said to be convergent if there exists a point x in X such that for each neighborhood G of x a positive integer n_0 can be found with the property that x_n is in G for all $n\geq n_0$. The point x is called a limit of the sequence, and we say that $\{x_n\}$ converges to x (and symbolize this by $x_n\to x$).

a) **Example :** Consider the indiscrete topological space X consisting of at least two points. This space is not a Hausdorff space but in this space any sequence converges to every point of the space.

Note : This is the reason why the above point x is called a limit instead of the limit. It is the failure of limits of sequences to be unique that makes this concept unsatisfactory in general topological spaces. The following result shows that this anomalous behavior cannot occur in a Hausdorff space.

b) In a Hausdorff space, a convergent sequence has a unique limit :- Suppose a sequence $\{x_n\}$ converges to two distinct points x and x^* in a Hausdorff space X. Then there exist two disjoint open sets G and G^* such that $x \in G$ and $x \in G^*$. Since $x_n \to x$, there exists a positive integer N such that $x_n \in G$ whenever $n > N$. Since $x_n \to x^*$, there exists an integer N^* such that $x_n \in G^*$ whenever $n > N^*$. If m is any integer greater than both N and N^*, then x_m must be in both G and G^*, which contradicts the fact that G and G^* are disjoint.

10.27 Model Examination Questions:

10.27.1 Show that a topological space is a T_1-space if and only if each point is a closed set,

10.27.2. Show that a one-to-one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.

10.27.3. Define a Hausdorff space. Show that every compact subspace of a Hausdorff space is closed.

10.27.4. Define a completely regular space and a normal space. Prove that every compact Hausdorff space is normal.

10.27.5. State and prove Urysohn's lemma.

10.28 Exercises :

10.28.1 Show that in a T_1-space, no finite set has a limit point.

10.28.2 Show that the co-finite topology defined on an infinite set is a T_1-space but not a Hausdorff space.

10.28.3 If f is a continuous mapping of a topological space X into a Hausdorff space Y, prove that the graph of $f=\{(x,f(x)/x\in X\}$ is a closed subset of the product space X xY.

10.28.4 Show that any metric space is a Hausdorff space.

10.28.5 Show that any metric space is a normal space.

10.28.6 Show that a closed subspace of a normal space is normal.

10.28.7 Let X be a T_1-space, and show that X is normal iff each neighborhood of a closed set F contains the closure of some neighborhood of F.

10.28.8 Is every normal space a Hausdorff space?

10.28.9 Is a normal space completely regular?

10.28.10 Is a completely regular space normal?

Reference Book:

1. Introduction to Topology and Modern Analysis G.F. Simmons
2. Topology by MUNKERS

Lesson Writer : V. J. LAL.

Lesson : 11

Tietze Extension Theorem and Uryshon's Embedding theorem

11.1. Introduction

An important problem in topology is regarding extension of continuous real function on a subspace of a topological space to the whole space. If we consider the function f defined by

$$f(x) = \frac{1}{x} \text{ for } x \neq 0,\ x \in \Re$$

which is continuous on $\Re$ - {o} there is no $\alpha \in \Re$ so that if we write f (o) = α then extended function becomes continues at o.

If we assume that the topological space is normal and the subspace is closed, by applying uryshon's lemma one can get an extension of any continuous function which is bonded on the subspace. This famous theorem, which is knownas as. 'Tietze Extension theorem' is proved is this lesson.

We know that every metric space X is a topology space X. We now look at the converse. Can every topology be induced by a metric. The immediate answer is no because the topology induced by a metric has several nice properties which do not hold in an arbitrary topological space For example Hausdorff's separation property, equivalence of continuity and sequential continuity, equivalence of compactness and sentential compactness and so on. The next natural question is which extra conditions on the topological space allow it to be metrizable or to be imbedded in a metric space ? A partial answer to this question is provided by Uryshon's imbedding theorem which we learn in this lesson.

Let us recall that if X is topological space Y, a subspace of X and $f : Y \to Z$ is a map, we say that a map $g : X \to Z$ is an extension of equivalently f is a restnction of g if f(x) = g (x) for every $x \in Y$.

Key Words : Extension – restriction – imbedding

TIETZE EXTENSION THEOREM : Let X be a normal space, Y a closed subspace of X and f a continuous real function defined on Y whose values lie in a closed interval [a,b]. Then f has a continuous extension f^1 defined on all of X whose values also line in [a, b]

Proof : If a = b then f (x) = $a \forall x \in Y$ and we define in this case $f^1(x) = a \forall x \in X$.

Assume that a < b. Since f is bounded, the set $\{f(x)/x \in Y\}$ has l. u. b. M and g. 1.b.m. since $a \leq f(x) \leq b$ for every $x \in y$ we have $a \leq m \leq M \leq b$. We may therefore assume that [a,b] itself is the smallest closed interval such that $a \leq f(x) \leq b$ for all $x \in Y$.

Since [a,b] is homeomorphic to [-1, 1] we may further assume that a = -1 and b = 1

Thus f is a continuous function from Y into [-1, 1] and g. l. b $\{f(x)/x \in y\}$ = -1 and l.u.b. $\{f(x)/x \in y\} = 1$.

Now let $A_1 = f^{-1}\left[-1, -\frac{1}{3}\right]$ and $B_1 = f^{-1}\left[\frac{1}{3}, 1\right]$. Then A_1, B_1 are closed subsets of Y and hence of X. Since $-1 = \text{g.l.b.}\{f(x)/x \in Y\}$, there exists a sequence $\{a_n\}$ in Y such that $\lim\limits_n f(a_n) = -1$

Similarly there is a sequence $\{b_n\}$ in Y such that $\lim\limits_n f(b_n) = 1$

This implies that A_1, B_1 are non – empty. Since X is normal, by urysohn's lemma there is a continuous function.

$g_1 : X \to [-1, +1]$

Such that

(a) $g_1(A_1) = -1$

(b) $g_1(B_1) = +1$

Define

$$f_1 = f - \frac{1}{3} g_1$$

Then function f_1 is a continuous function

$$f_1 : Y \to \left[-\frac{2}{3}, \frac{2}{3}\right]$$

To see this first we select $x \in A_1$. Then we have

$$f_1(x) = f(x) - \frac{1}{3} g_1(x) = f(x) - \left(-\frac{1}{3}\right)$$
$$= f(x) + \frac{1}{3}$$

and $-1 \le f(x) \le -\frac{1}{3}$

therefore

$$-\frac{2}{3} \le f_1(x) \le O$$

similarly we find for x in B_1

$$O \le f_1(x) \le \frac{2}{3}$$

Now suppose

$x \in Y-(A_1 \cup B_1)$

then

$$-\frac{1}{3}<f(x)<\frac{1}{3}$$

and $-\frac{1}{3} \leq \frac{1}{3} g_1(x) \leq \frac{1}{3}$

Therefore $f(x)-\frac{1}{3} g(x)$

lies between

$$-\frac{1}{3}+\left(-\frac{1}{3}\right) \text{ and } \frac{1}{3}+\frac{1}{3}$$

i.e. $-\frac{2}{3} \leq f_1(x) \leq \frac{2}{3}$

Then we note that

$$\text{g.u.b. } f_1=-\frac{2}{3}, \quad \text{l.u.b. } f_1=\frac{2}{3}.$$

To see this we note that there is an $n_1 \in N$ such that for all $n \geq n_1$

$$f(a_n)<-\frac{1}{3}$$

This implies that for $n \geq n_1$

$a_n \in A_1$

It follows that

$$\underset{A_1}{\text{g.l.b}}\ f=-1$$

and hence $\lim f_1(a_n)=-\frac{2}{3}$ and $\underset{A_1}{\text{g.l.b}}\ f_1=-\frac{2}{3}$

It follows similarly that

$$\lim_{n \to \infty} f_1(b_n)=-\frac{2}{3} \text{ and } \underset{B_1}{\text{l.u.b.}}\ f_1=\frac{2}{3}$$

We now set

$$A_2 = f_1^{-1}\left[-\frac{2}{3}, -\frac{2}{3^2}\right] = f_1^{-1}\left[-\frac{2}{3}, \frac{1}{3}\left(-\frac{2}{3}\right)\right]$$

$$B_2 = f_1^{-1}\left[\frac{2}{3^2}, \frac{2}{3}\right] = f^{-1}\left[\frac{1}{3}.\frac{2}{3}, \frac{2}{3}\right]$$

It is clear that A_2, B_2 are closed subsets of Y and so closed subsets of X.

We claim that A_2 is non-empty.

We know that

$$\lim_{n \to \infty} f_1(a_n) = -\frac{2}{3}$$

Therefore thus is an $n_2 \in N$ such that for all $n \geq n_2$

$$-\frac{2}{3} \leq f_1(a_n) \leq -\frac{2}{3^2}$$

It follows that for all $n \geq n_2$

$$a_n \in A_2 \text{ and } \underset{A_2}{\text{g.l.b}}\, f_1 = -\frac{2}{3}$$

similarly

$$\underset{B_2}{\text{l.u.b.}}\, f_1 = \frac{2}{3}.$$

Since X is normal there is a continuous function

$$g_2 : X \to \left[-\frac{2}{3}, \frac{2}{3}\right]$$

Such that

$$g_2(A_2) = -\frac{2}{3}, \quad g_2(B_2) = \frac{2}{3}.$$

We set

$$f_2 = f_1 - \frac{1}{3} g_2$$

It is checked that

$$\lim_{n \to \infty} f_2(a_n) = -\left(\frac{2}{3}\right)^2, \quad \lim_{n \to \infty} f_2(b_n) = \left(\frac{2}{3}\right)^2$$

Inductively we construct a sequence of continuous function on X

$g_1, \ldots\ldots\ldots\ldots, g_{n_1} \ldots\ldots\ldots\ldots$

Such that

$$g_m : X \to \left[-\left(\frac{2}{3}\right)^{m-1} \left(\frac{2}{3}\right)^{m-1}\right] \text{-------}(1)$$

$$\left\| f - \frac{1}{3}(g_1 + \ldots + g_m) \right\| \leq \left(\frac{2}{3}\right)^m \text{--------}(2)$$

If we set $f_m = f - \frac{1}{3}(g_1 + \ldots + g_m)$ ---(3)

then

$$\underset{Y}{\text{g.l.b.}}\, f_m = -\left(\frac{2}{3}\right)^m \text{ and } \underset{Y}{\text{l.u.b.}}\, f_m = \left(\frac{2}{3}\right)^m$$

Suppose we have defined $g_1, \ldots\ldots, g_m$ having properties 1, 2, 3,

Define

$$A_{m+1} = f_m^{-1}\left(\left[-\left(\frac{2}{3}\right)^m - \frac{1}{3}\left(\frac{2}{3}\right)^m\right]\right)$$

$$B_{m+1} = f_m^{-1}\left(\left[\frac{1}{3}\left(\frac{2}{3}\right)^m \left(\frac{2}{3}\right)^m\right]\right)$$

Then it is easily checked that

4) A_{m+1}, B_{m+1} are closed subsets of X

5) A_{m+1}, B_{m+1} are non – empty

6) $\underset{A_{m+1}}{\text{g.l.b.}}\ f_m = -\left(\frac{2}{3}\right)^m$

$\underset{B_{m+1}}{\text{l.u.b.}}\ f_m = \left(\frac{2}{3}\right)^m$

By Urysohn's lemma there is a continuous function

$$g_{m+1} : X \to \left[-\left(\frac{2}{3}\right)^m \left(\frac{2}{3}\right)^m\right]$$

such that

$$g_{m+1}\left(A_{m+1}\right) = -\left(\frac{2}{3}\right)^m \quad g_{m+1}\left(B_{m+1}\right) = \left(\frac{2}{3}\right)^m$$

We find that

$g_1, \ldots, g_{m+1}$

satisfy conditions (1), (2), (3)

Since

$$\| g_m \| \le \left(\frac{2}{3}\right)^{m+1}$$

by weierstrars M-test we obtain that the series $\sum_{m=1}^{\infty} g_m$ Converges uniformly to a function $g : X \to R$

As a limit of a uniformly convergent series for functions

$$\sum_{m=1}^{\infty} g_m$$

g is continuous. And we have

$$\left\| f - \frac{1}{3} g \right\| = \left\| f - \frac{1}{3} \lim_{m} \left(g_1 + \ldots + g_m\right) \right\|$$

$$= \lim_{m} \left\| f - \frac{1}{3}(g_1 + \dots + g_m) \right\|$$

$$\leq \lim_{m} \left(\frac{2}{3}\right)^m = 0$$

thus we have extended f to X ;

$$\frac{1}{3} g$$

is the extension

11.3. SAQ : Prove the converse of Tietze extension theorem :

Let X be a topological space. Prove that if every real – valued continuous mapping f of a closed subspace F of X into a closed interval [a,b] can be extended to a continuous real-valued mapping f* of X into [a,b] then X is normal.

Proof : suppose F, and F_2 are two disroint nonempty closed subsets of X. Let [a,b] be any closed interval such that a<b. The mapping f, defined by f (x) = a if $x \in F_1$ and f(x) = b if $x \in F_2$ is then a continuous mapping of the closed subs are $F_1 \cup F_2$ into [a,b]. Then there exists a continuous function f* of x into [a,b] such that $f^*/F_1 \cup F_2 = f$. If c is any real number such that a<c<b then f^{*-1} ([a,c]) and f^{*-1} ([c,b]) are disroint open sets containing F_1 and F_2 respectively. Thus X is normal.

The following example shows that the closed ness of Fis essential in the above the theorem.

11.4. Example :

Let X = [0, 1] and F – (0, 1]. Since X is a metric space, it is normal. F is not closed in X. Define $f : F \to [-1, 1]$ by $f(x) = \text{Sin}\left(\frac{1}{x}\right)$. Then f is continuous. Since $\lim_{x \to 0} f(x)$ does not exist, f can not be extended to a continuous mapping f* of X into [-1, 1].

We now turn our attention on the metrization problem. We begin with the following example.

Example : The infinite dimensional unitary space c^∞ consisting of all sequences of complex number $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ is also denoted by l^2 and is a complete metric space with respect to the metric defined by

$$d((x_n), (y_n)) = \left\{ \sum_{n=1}^{\infty} x_n - y_n{}^2 \right\}^{\frac{1}{2}}$$

For details of the proof one may refer to page 2.7, 2.8. of the study material on functional analysis. We denote the topology induced by this metric by T_d.

Clearly $l^2 \subseteq \mathbb{C}^N$ where $\mathbb{C}^N$ is the space of all sequences of complex numbers. Since $\mathbb{C}^N = \prod_{n \in N} \mathbb{C}_N$ where $\mathbb{C}_n = \mathbb{C} \ \forall n$, $\mathbb{C}^N$ has the product topology on it where $\mathbb{C}_n$ is equipped with the usual topology, induced by the metric $d_n(z_1, z_2) = |z_1 - z_2| \ \forall n \geq 1$ and $z_1 \in \mathbb{C}\ z_2 \in \mathbb{C}$ we denote the restriction of the product topology on $\mathbb{C}^N$ to l^2 by T and prove that $T \subsetneq T_d$

For $k \in N$ and $\delta > o$ write

$S(k, \delta) = \{Z = (Z_n) \in \mathbb{C}^N : |Z_K| < \delta\}$ These sets $S(k, \delta)$ are the typical subbasic open sets of $\mathbb{C}^N$ that contain $O \in \mathbb{C}^N$

Let $K_1 < \dots\dots\dots < K_r$ be a sequence of natural numbers and let $\delta_1, \delta_2, \quad , \delta_r$ be a sequence of positive numbers. The sets

$$S(\underline{K}, \underline{\delta}) = S(K_1, K_2, \dots\dots K_r, \delta_1, \delta_2 \dots\dots \delta_r)$$

$$= S(K_1, \delta_1) \cap \dots\dots \cap S(K_r, \delta_r)$$

$$= \{(z_n) \in \mathbb{C}^N : |z_{K_1}| < \delta_1, \dots\dots, |z_{K_r}| < \delta_r\}$$

are the typical basic open sets of $\mathbb{C}^N$ that contain $O \in \mathbb{C}^N$

If $z \in l^2$ and $\|z\| < \delta$ then

$$|z_K| \leq \|z\| < \delta$$

and so

$$S_\delta(O) \subset S(k, \delta)$$

This implies $T \subseteq T_d$.

We claim that there are no sequences

$K_1, K_2, \dots\dots\dots\dots K_r$ of natural numbers and

$\delta_1, \delta_2, \dots\dots\dots, \delta_r$ positive numbers such that

$S(K, \delta) \subseteq S_\delta(O)$

To see this we set $\delta_0 = \text{Min}\{\delta_1, \dots., \delta_r\}$

Chose any positive integer k. Take any sequence

$Z(k, \delta_0) = \{Z_1, \ldots, Z_K, 0, 0 \ldots\ldots\ldots 0\}$

$$|Z_n| < \delta_O$$

This sequence is in l^2. This sequence is also in $S(\underline{K}, \delta)$. However if we take

$$Z_n = \frac{1}{2}\delta_o \quad n = 1, 2, \ldots, K$$

We have

$$\|Z(K, \delta_o)\| = \frac{K}{2}\delta_o$$

If $\frac{K}{2}\delta_o > \delta$ i.e. $K > \frac{2\delta}{\delta_o}$

$$\|Z(K, \delta_o)\| > \delta$$

and so $Z(K, \delta_o) \notin S(\underline{K}, \underline{\delta})$

The following basic fact about product topology will be used in Uryshon's embedding theorem.

11.5. Proposition :

Suppose that I is a set, for each i in I, (A_i, T_i) is a topological space and (A, T) is the product of the (A_i, T_i)

let $p_r : A \to A_j$

be the projection of A onto the $_r$ th factor

$$p_j((x_j)) = x_j$$

Then

1) P_j is continuous

2) P_j is a open map

and

3) If Y is a topological space and

$$f : Y \to A$$

is any map, then f is continuous if and only if p_r of is continuous for all j in I.

Proof : Let us recall that

for $j \in I, I(j) = I - \{\dot{\gamma}\}; j_1, \ldots, j_n \in I, I(j_1, \ldots, j_n) = I - \{\dot{\gamma}, \ldots, \dot{\gamma}\}$

And for $V_{j_1}, \ldots\ldots, V_{j_n}, \ldots\ldots\ldots, V_{j_n}$ in $A_{j_2}, \ldots\ldots A_{j_n}$

the sets

$$V_{j1} \times \ldots\ldots \times V_{jn} \times \prod_{i \in I(j_1,..j_n)} A_i$$

are basic open sets for T.

1) Suppose V_j is an open set in A_j

Then $P_j^{-1}(V_j) = V_j \times \prod_{i \in I(j)} A$

is a sub basic open set of T. This implies that P_j is continuous.

2) It is enough to prove P_j (V) is an open subset for every basic open subset V of T. Let

$$V = V_{j_1} \times \ldots \times V_{j_n} \times \prod_{i \in I\,(j_1..j_n)} A_i$$

Then

$$P_j(V) = \begin{cases} V_{j_r} & \text{if } j = j_r \\ A_j & \text{if } j \notin \{j_1,..,j_n\} \end{cases}$$

and so 2 is proved

3) Consider $W_j \subseteq A_j$ and $(P_j.\, f)^{-1}(W_j)$

We have

$$(P_j,.f)^{-1}(W_j) = f^{-1}\left(W_j \times \prod_{i \in I(j)} A_j\right)$$

Therefore P_γ of is continuous for all j implies

f^{-1} (V)

is an open subset of Y for all sub basic open sets V of A. This implies that f is continuous. The rest is clear.

We recall a definition

11.6. Definition :

Suppose (X,T) is a topological space We say that

(X, T) is metrizable

if there is a metric d on X such that the topology T_d induced by the metric d on X is the same as T :

$T = T_d$

11.7. Proposition

Let $J_n = \left[0, \frac{1}{2^n}\right]$ be given the usual topology and let $J = \prod_{n \in N} J_n$

be the product space, with the product topology T Define

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|$$

where $x = (x_n)$, $y = (y_n)$ are elements of J. Then d is a metric on J and the topology T_d induced by d is the same as T.

Proof : We leave the proof that d is a metric to the reader. In the case of l^2 we have just proved that

$$T \subseteq T_d$$

The same method will give us in this case also

$$T \subseteq T_d$$

We shall now prove

$$T_d \subseteq T$$

To get a clear idea we shall first prove that the open sphere

$$S_\delta(0)$$

contains a basic open neighborhood V of o with respect to the product topology.

We set

$$\delta_o = \text{Min}\{1, \delta\}$$

We can find a positive integer n_o such that

$$\frac{1}{n_o} < \frac{\delta_o}{2}$$

We Set

$$V = \prod_{K=1}^{n_o} \left[o, \frac{\delta_o}{2n_o} \cdot \frac{1}{2^K}\right] \times \prod_{K=n_o+1}^{\infty} J_K$$

For any $x = (x_n)$ in V we have

$$d(x,o) = \Sigma x_n$$

$$= \sum_{K=1}^{n_o} x_K + \sum_{K=n_o 1}^{\infty} x_K$$

$$< \sum_{K=1}^{n_o} \frac{\delta}{2n_o} \cdot \frac{1}{2^K} + \sum_{K=n_o+1}^{\infty} \frac{1}{2^K}$$

$$< \frac{\delta_o}{2} + \frac{1}{2^{n_o}}$$

$$< \frac{\delta_o}{2} + \frac{\delta_o}{2}$$

$$= \delta_o .$$

Thus

$$V \subseteq S_\delta (O)$$

Suppose now

$$\alpha = (\alpha_n) \in J$$

and we are given

$$S_\delta (\alpha)$$

As above we set

$$\delta_o = \text{Min} \{1, \delta\}$$

and choose an n_o such that

$$\frac{1}{n_o} < \frac{\delta_o}{2}$$

If $m \le n_o$ then we choose intervals as follows

$$I_m = \left(\alpha_m - \frac{\delta_o}{2n_o} \frac{1}{2^m}, \alpha_m + \frac{\delta_o}{2n_o} \frac{1}{2^m} \right) \cap J_m$$

This is an open interval of J_m and

$$V = \prod_{m=1}^{n_o} I_m \times \prod_{K=n_o+1}^{\infty} J_K$$

is a basic open set containing α. Suppose $x \in V$. Then

$$\Sigma \left| x_K - \alpha_K \right| = \sum_{R=1}^{n_o} \left| x_K - \alpha_K \right| + \sum_{K=n_o+1}^{\infty} \left| x_K - \alpha_K \right|$$

$$< \frac{\delta_o}{2n_o} \sum_{K=1}^{n_o} \frac{1}{2^K} + \sum_{K=n_o+1}^{\infty} \frac{1}{2^K}$$

$$< \frac{\delta_o}{2} + \frac{\delta_o}{2} = \delta_o$$

Thus

$$V \subset S_\delta(\alpha)$$

This implies that every point α of an open set U with respect to the metric topology T_d has a neighborhood $V(\alpha)$ with respect to the product topology T. So we have

$$U \in T$$

i.e.

$$T_d \subseteq T$$

Thus the proposition is proved

11.8. Theorem

(Uryshon imbedding theorem). Suppose X is a topological space. which is normal and second countable. Then X is metrizable

Proof : Let

$$\{U_n\}_{n \in N}$$

be a basis for the open sets of the topology of X. We consider the ordered pairs of

$$(m,n) \in N \times N$$

of natural numbers such that

$$U_m \subseteq \bar{U}_m \underset{\neq}{\subset} U_n$$

we have assumed that X is normal. By urysohn's lemma there is a continuous function.

$$f_{m,n} : X \to [0,1] \subseteq R$$

Such that

1) $f_{mn} = 1$ on U_m

and

2) $f_{mn} = 0$ outside U_n

The set of ordered pairs (m, n) we have considered is a subset of N x N which is a countable se So the set functions f_{mn} is a countable set. We write them in a sequence

$$\{f_1 f_2, \ldots\ldots, f_n, \ldots\ldots\ldots\}$$

Corresponding to each $p \in N$ there is a unique ordered pair (m, n) such that

$$f_{mn} = f_p$$

and conversely. We define a map

$$F : X \to J$$

by

$$F(x) = \left(\frac{f_1(x)}{2}, \frac{f_2(K)}{2^2}, \ldots, \frac{f_p(x)}{2^p}, \ldots\ldots \right)$$

We recall that

$$p_K : J \to J_K$$

is the projection map and that

$$F : X \to J$$

is continuous if and only if

$$p_j\, F = \frac{f_j}{2^j}$$

is a continuous function on X. If follows that F is a continuous map into the product J of topology spaces

$$J_K$$

We denote F (X) by Y. We consider Y with the topology T_0 induced by the product topology T on J. We have just proved that

$$F : X \to Y$$

is a continuous onto map.

We will now prove that F is one-to-one and open.

F is one-to-one : Let $x, y \in x$ and $x \neq y$. Then there is a basic neighborhood U_n of x. such that

$$x \in U_n \subseteqq \bar{U}_n \text{ and } y \notin U_n$$

Since X is normal we may choose a basic neighborhood U_m of x such that

$$x \in U_m \subseteqq \bar{U}_m \subset U_n$$

Then we have a p = p (m, n) such that

$$f_p(x) = l \text{ and } f_p(y) = 0$$

This implies that

$$F(x) \neq F(y)$$

$F: X \to Y$ **is an open map :** It is enough to show that the image of each U_n is an open subset of Y. Let $v \in F(U_n)$

We shall prove that there is a neighborhood V of v with respect to the topology T on Y induced by T on J such that

$$V \subseteqq F(U_n)$$

since $v \in F(U_n)$ there is an u in U_n such that

$$v = F(u)$$

The space X is normal and $u \in U_n$. Therefore there is a basic open set

U_m

such that

$$x \in U_m \subseteqq \bar{U}_m \subseteqq U_n$$

Corresponding to this pair (m, n) of natural numbers there is a p = p (m, n) such that

$$f_p = f_{mn}$$

We have

$$f_p(x) = 0 \qquad \text{if } x \notin U_n$$

The set

$$V = \left\{ Z \in J : Z_p > \frac{1}{2^{p+1}} \right\}$$

is a sub basic open set of J Since

$$f_p(u) = 1$$

the point

$$\nu = F(u) \in V$$

If x in X satisfies

$$F(x) \in V$$

then we must have

$$\frac{1}{2^p} f_p(x) > \frac{1}{2^{p+1}}$$

that is

$$\frac{1}{2} < f_p(x)$$

This implies that

$$x \in U_n$$

Therefore we have proved that

$$V \cap Y \subseteq F(U_n)$$

That is $F(U_n)$ is an open subset of Y. Since F is continuous and also one-to-one it follows tha

$$F : X \to (Y, T^1)$$

is a homeom orphism

In the previous result we have proved that

$$T = T_d$$

on J. Therefore T on Y is the same as the topology induced by the metric on Y. Thus we have provide that X is homeomorphic to a metrizable space. So X is metrizable.

11.9. SAQ Show that a compact hausdorff space is metrizable it is second countable

Proof : Let X be compact Hausdorff space. Then it is normal Suppose X is second countable. Then by urysohn imbedding theorem, X is metrizable. Conversely suppose that X is metrizable. By SAQ 8.14 since X is a compact metric space, it is separable. Since every separable metric space is second countable it follows that x is second countable.

Remark : Theorem 11.10 provides an example of a metrizable space. In this Theorem we show that the Tychnoff product of metric spaces is metrizable. In example 11.11 we show that the metric topology of the infinite dimensional Euclidean space $\mathbb{C}^{\infty}$ (also denoted by l^2) is stronger than the relative topology inherited from the Tychnoff product $\mathbb{C}^N$

11.10. Theorem :

Suppose for each natural number n, (A_n, d_n) is a metric space and T_n is the topology induced on A_n by d_n. Then the product space (A, T) where $A = \prod_{n \in N} A_n$ and T is the product topology, is metrizable.

Proof : Since the metric $\frac{d_n}{1+d_n}$ and d_n generate the same topology on A_n we may assume that $o \le d_n(x,y) \le 1$ for all x, y in A_n.

For $x = (x_n)$ and $y = (y_n)$ in A define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n)$$

Since $o \le d_n(x,y) \le 1$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent,

The series on the right hand side converges, hence the above definition is meaningful.

For $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ in A we have

i) $d(x,y) = o \Leftrightarrow d_n(x_n, y_n) = o \forall n \Leftrightarrow x_n = y_n \forall n \Leftrightarrow x = y$

ii) $d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} (x_n, y_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(y_n, x_n) = d(y,x)$

and iii) $d(x,z) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, Z_n)$

$$\le \sum_{n=1}^{\infty} \frac{1}{2^n} \{d_n(x_n, y_n) + d_n(y_n, z_n)\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n) + \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, z_n)$$

Thus d is a metric on A. We denote the topology on A induced by d by T^1. What we prove is that $T = T^1$

For this it is enough to prove that

1) given an open set V of T and $\alpha \in V$ there is a neighborhood U' (α) of α with respect to T' such that

$$U'(\alpha) \subseteq V$$

and

2) given an open set U' in T' and $\beta \in U'$ there is a neighborhood $V(\alpha)$ of α with respect to T such that

$$V(\alpha) \subseteqq U'.$$

Further it is clear that the statement (1) if proved for a class of sub basic open sets implies the statement for all open sets of T. So it is enough to prove (1) and (2) for sub basic open sets V and U.

Let V be a subbasic open set with respect to the product topology T. Then there is a natural number m and an open set V (m) of A_m such that

$$V = V(m) \times \prod_{\substack{n \in N \\ n \neq m}} A_n$$

Let $\alpha = (\alpha_n) \in V$. Then $\alpha_m \in V(m)$ since T_m is the topology induced by the metric d_m there is a $\delta > o$ such that

$$S_\delta(\alpha_m) \subseteqq V(m)$$

Then we claim that the sphere of radius

$$\frac{\delta}{2^m} \text{ centered at } \alpha$$

with respect to d is contained in V :

$$S_{\frac{\delta}{2^m}}(\alpha) \subseteqq V$$

Suppose $x \in S_{\frac{\delta}{2^m}}(\alpha)$. Then we have

$$\frac{1}{2^m} d(x_m, \alpha_m) \leq d(x, \alpha)$$

$$< \frac{\delta}{2^m}$$

This implies that

$$x_m \in S_\delta(\alpha_m)$$

and so $x \in v$. We have provide (1). Let U' be an open set in the topology generated by the metric d on A and let $\beta \in U'$. By the definition of the topology T' there is a $\delta > o$ such that

$$S_\delta(\beta) \subseteqq U'$$

We choose a natural number k such that

$$\frac{1}{2^K} < \frac{\delta}{2} \text{ i.e. } 2^{K-1} > \frac{1}{\delta}$$

This is possible because of Archimedian property of R. we claim that

$$V(\beta) = \prod_{k=1}^{K} S_{\frac{1}{2^K}}(\beta_r) \times \prod_{n>K} A_n$$

is contained in $S_\delta(\beta)$ suppose $y \in V(\beta)$ then

$$d(y, \beta) = \sum_{r=1}^{K} \frac{1}{2^r} d_r(y_r, \beta_r) + \sum_{n=K+1}^{\infty} \frac{1}{2^n} d_n(y_n, \beta_n)$$

$$\leq \sum_{r=1}^{K} \frac{1}{2^r} d_r(y_r, \beta_r) + \frac{1}{2^K}$$

$$\leq \sum_{r=1}^{K} \frac{1}{2^r} \cdot \frac{1}{2^K} + \frac{1}{2^K}$$

$$< \frac{1}{2^K} + \frac{1}{2^K} = \frac{1}{2^{K-1}} < \delta$$

Therefore

$$V(\beta) \subseteq S_\delta(\beta) \subseteq U'$$

and we have proved (2)

The result is proved

Model Examination Questions :

1. State and prove Tietze extension theorem
2. State and prove Uryshon's imbedding theorem
3. Show that the product topology of a countable collection of metric space is metrizable.

Exercises :

1. Prove that a separable metric space is second countable
2. Show that every metric space is normal
3. Show that a second countable normal space is metrizable
4. Let $I_x = [0,1] \ (\forall x \in [0,1])$ equipped with the usual topology. Show that the product topology on $\prod_{x \in [0,1]} I_x$ is normal but not metrizable.

5. Given an example of a metric space which is not second countable.

6. Does the continuous function $\frac{1}{x}$ defined on $R-\{O\}$ have a continuous extension to the whole of R ?

7. Let X = [-1, 1], Y = $\left[-\frac{1}{2}, \frac{1}{2}\right]$ Define f, f_1, f_2 by

$$f(x) = |x| \text{ for } x \in Y$$

$$f_1(x) = |x| \text{ for } x \in X \text{ and}$$

$$f_2(x) = \begin{Bmatrix} |x| \text{ if } x \in Y \text{ and} \\ \frac{1}{2} \text{ if } x \in X - Y \end{Bmatrix}$$

Show that f_1 and f_2 are continuous extension of f

8. Define f(x) = x sim $\frac{1}{x}$ for $x \in (0,1]$ Show that f has a unique continuous extension on [0,1],

lesson writer : **V.J. Lal.**

YIA

LESSON NO. 12

CONNECTED SPACES

12.1. Introduction

In this lesson we study connected topological spaces, which is one of the most important topics in topology. Intuitively, a connected space may be thought of as a space consisting of a single piece. We give a formal definition of a connected topological space. We prove that a subspace of the real line R is connected if, an only if, it is an interval. We also prove that the property of connectedness is preserved by continuous functions. We further prove that the product of a non-empty class of connected spaces is connected and hence R^n and C^n are connected. We also introduce the concept of components of a topological space. We study some elementary properties of components.

CONNECTEDNES

12. 2. Definition. A toplogical space (X, T) is said to be connected if X can not be represented as the union of two non-empty disjoint open sets. In other words; if, $X = A \cup B$, A, $B \in T$, $A \neq \phi$, $A \cap B = \phi$ implies $B = \phi$, then X is said to be connected.

12. 3. Definition. Let (X, T) be a topological space. If there exists A, B in T such that $X = A \cup B$, $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$ then this representation of X is called a disconnection of X. If X is not connected we say that X is disconnected, or equivalently X is disconnected if, and only if, X has a disconnection.

12. 4. Definition. A subspace Y of a topological space X is said to be connected if Y is connected with respect to the relative (induced) topology in Y.

12. 5. Lemma. A subspace Y of a topological space X is connected if, and only if, Y is not contained in the union of two open subsets of X whose intersections with Y are non-empty and disjoint.

Proof. Suppose that Y is connected. Let $Y \subseteq A \cup B$ where A and B are open in X, Let C $= A \cap Y$, $D = B \cap Y$. Then $Y = C \cup D$, C and D are open in Y. If $C \cap D = \phi$ and $C \neq \phi$ then $D = \phi$ since Y is connected.

Conversely suppose that the stated condition holds. Let $Y = C \cup D$ where C and D are disjoint open sets in Y. Let $C = Y \cap A$, $D = Y \cap B$ where A and B are open in X. Then $Y \subseteq A \cup B$. If $C \neq \phi$ then $A \cap Y \neq \phi$.

$B \cap Y = \phi$. i.e. $D = \phi$. Hence Y is connected.

12. 6. SAQ. Let X be any non-empty set. Let T be the indiscrete a topology on X. Show that (X, T) is a connected topological space.

12. 7. SAQ. Let X be a set with at least two elements. Let T be the discrete topology on X. Show that (X, T) is disconnected.

12.8. Lemma. Let X = { a, b }, Y = { c, d }. Let T_X be the discrete topology on X and let T_Y be the discrete topology on Y. Then (X, T_X) and (Y, T_Y) are homeomorphic. Thus there exists a unique discrete space with two points upto isometry.

Proof. $T_X = \{ \phi, \{a\} \{b\}, X \}$, $T_Y = \{\phi, \{ c \}, \{ d \}, Y \}$.

Define the mapping $f: X \to Y$ by

f (a) = c and f (b) = d. Then f is a bijection which is continuous and open. Thus f is a homeomorphism.

12. 9. Notation. Let 0 and 1 be two symbols. The discrete two point space is denoted by {0, 1}.

12. 10. Theorem. A topological space X is disconnected if, and only if, there exists a continuous function from X onto the discrete two point space {0, 1}.

Proof: Suppose that X is disconnected and Let $X = A \cup B$ be a disconnection of X. Then A and B are non-empty disjoint open subsets of X.

Define $f : X \to \{ 0, 1\}$ by $f(x) = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \in B \end{cases}$

Them $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = B$ and $f^{-1}(\{0,1\}) = X$ and all these sets are open in X. Hence f is continuous.

Also f is onto, since $A \neq \phi$ and $B \neq \phi$.

Conversely suppose that there exists a continuos surjective function $f : X \to \{0, 1\}$. Let $A = \{x \in X / f(x) = 0\}$ and $B = \{x \in X / f(x) = 1\}$. A and B are non-empty, since f is surjective. Also $A \cap B = \phi$. {0} and {1} are open and

$A = f^{-1}(\{0\})$, $= B = f^{-1}(\{1\})$. Since f is continuous, we have that A and B are open in X. Thus $X = A \cup B$ is a disconnection of X. Thus X is disconnected.

12.11. Theorem: Let $f : X \to Y$ be a continuous mapping of a connected topological space X into a topological space Y. Let $Z = f(X)$ be the (continuous) image of X. Then Z is connected.

Proof: If Z is not connected, then by 12. 10 there exists a continuous function g from Z onto the discrete two point space {o, 1}. Then the mapping $h : X \to \{0,1\}$ defined by $h(x) = g(f(x))$, being the composite of two continuous functions, is continuous and it is also onto. This implies that X is not connected, which is a contradiction to the hypothesis. Hence Z is connected.

12. 12. SAQ. Give a direct proof of theorem 12.11 without using theorem 12.10.

12. 13. Theorem. The product of any non-empty class of connected spaces in connected.

Proof: Let $\{X_i\}$ be a non-empty class [illegible] ted spaces. Let $X = \prod_i X_i$ be the product space of the topological spaces $\{X_i\}$. [illegible] e that X is connected, it is enough to prove that any continuous function from X into the discrete two point space {0,1} is not onto.

Let $f : X \to \{0,1\}$ be a continuous function.

Part A. We first prove that if two elements of X differ in atmost one component then they have the same image under the mapping f.

Let $a = \{a_i\}$ and $x = \{x_i\} \in X$ and let i1 be an index such that $x_i = a_i$ for $i \neq i1$.

Define $f_{i1} : X_{i1} \rightarrow X$ by $f_{i1}(t) = \{y_i\}$ where $y_{i1} = t$ and $y_i = a_i$ for $i \neq i_1$. f_{i1} is a continuous mapping from $X_{i1} \rightarrow X$.

Now $fof_{i1} : X_{i1} \rightarrow \{0,1\}$ is continuous. Since X_{i1} is connected fof_{i1} is a constant.

Now $fof_{i1}(a_{i1}) = f(a)$ and $f(x) = fof_{i1}(x_{i1})$, fof_{i1} is a constant imply that $fof_{i1}(a_{i1}) = fof_{i1}(x_{i1})$, Thus $f(x) = f(a)$.

B. Let $a = \{a_i\}$ be a fixed element of X. Now we prove that if $x \in X$ and x differs from a in atmost n components then $f(x) = f(a)$.

If $n = 1$ then the result is true by part A. Suppose that the result is true for $n = k$ Let $x \in X$ such that x differs from a in atmost k+1 components, say i1, i2, . . . , ik, i (k+1).

Define $y = \{y_i\} \in X$ by

$y_{ij} = x_{ij}$ for $j = 1,2, \ldots, k$.

and $y_i = a_i$ for $i \neq i1, \ldots, ik$.

Then x and y differ in at most in their i(k+1) th component.

∴ By Part **A** we have $f(x) = f(y)$. Also y and a differ at most in their

∴ i1, i2, . . ., ik components. By induction hypothesis we have $f(y) = f(a)$.

∴ $f(x) = f(a)$. Hence the result is true for all n.

C. Fix some $a \in X$.

Let $A = \{ x \in X$ / x differs from a in atmost a finite number of components.$\}$

Then it can be shown that A is a dense subset of X. Also by part B, f is a constant on A. Since $\{0, 1\}$ is a T_1 – space we get that f is a constant mapping on X. Hence f is not onto. Thus there is no continuous mapping of X onto the discrete two point space {0, 1}. Hence X is connected.

12.14. SAQ. Prove that the mapping f_{i1} in Theorem 12.13 part A is continuous.

12.15. SAQ. Prove that the set A in Theorem 12.13 part C is dense in X.

12.16.SAQ. Let X be a topological space and let Y be a T_1 – space. Let $f : X \rightarrow Y$ be a continuous map such that f is a constant on a dense subset A of X. Prove that f is constant on X.

12.17. Theorem. A subspace of the real line $\mathbb{R}$ is connected if, and only if it is an interval. In particular $\mathbb{R}$ is connected.

Proof. Let X be a subspace of $\mathbb{R}$

(i) Suppose that X is not an interval. Then there exist real numbers r, s, t $\ni$ $r < s < t$, $r, t \in X$ and $s \notin X$. The sets $A = X \cap (-\infty, s)$ and $B = X \cap (s, +\infty)$ are non-empty disjoint open sets in X such that $X = A \cup B$ Hence X is not connected.

(ii) Assume that X is not connected. Let $X = A \cup B$ be a disconnection of X. Then A and B are non-empty and disjoint closed, as well as open, subsets of X. We can choose $x \in A$ and $z \in B$ such that $x \neq z$. We may assume that $x < z$. Now $x, z \in X$. $[x, z] \cap A$ is bounded above by z. Hence $y = \sup([x, z] \cap A)$ exists in $\mathbf{R}$. It is clear that $x \le y \le z$. Since X is an interval, $x, z \in X$, we have $y \in X$.

Since A is also closed in X, the definition of y shows that $y \in A$.

$\therefore$ $y < z$. Also if $\varepsilon > 0$ then $y < y + \varepsilon < z$ implies $y + \varepsilon \in B$. Since B is closed in X we get $y \in B$. Thus $y \in A \cap B$, which is a contradiction since A and B are disjoint.

Hence X is connected

The proof is complete from (i) and (ii)

12.18. Theorem: The range of a continuous real valued function on a connected space is an interval.

Proof. Let $f : X \to \mathbf{R}$ be a continuous real valued function. Let $Z = f(X)$.

By theorem 12.11, Z is connected. By theorem 12.17, we get that Z is an interval.

Theorem 12.18 may also be stated as follows: Let f be a real valued continuous mapping on a connected space X. Let $x, y \in X$. Let c be a real number $\ni f(x) \le c \le f(z)$. Then $\exists\, z \in X \ni f(z) = c$. Thus theorem 12.18 is also called "Intermediate value theorem".

12. 19. Theorem. The spaces $\mathbf{R}^n$ and C^n are connected.

Proof. We know that $\mathbf{R}$, being an interval, is connected with the usual topology. We also know that $\mathbf{R}^n$ as a topological space can be regarded as the product of n copies of the connected space $\mathbf{R}$. Hence by theorem 12.13, we get that $\mathbf{R}^n$ is connected. We show that C^n and $\mathbb{R}^{2n}$ are homeomorphic as toplogical spaces. Let $z = (z_1, z_2, \ldots, z_n) \in C^n$. Let $z_k = a_k + ib_k$ for $k = 1,2,\ldots, n$.

Define $f : C^n \to \mathbb{R}^{2n}$ by $f(z) = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$

Clearly f is one-one and onto and $|f(z)| = |z|$.

Thus f is an isometry of C^n onto $\mathbb{R}^{2n}$ and hence f is a homeomorphism. Since $\mathbb{R}^{2n}$ is connected we have C^n is connected.

Answers to SAQs

12.6. SAQ. $T = \{\phi, X\}$. Thus X is the only non-empty open set and hence X can not be represented as the union of two non-empty disjoint open sets.

Hence (X, T) is a connected space.

12. 7. SAQ. Let $a \in X$. Then $B = X \setminus \{a\}$ is non-empty. Since T is the discrete topology, every subset of X is open in (X, T). Thus $X = \{a\} \cup B$ is a disconnection of X.

12.12. SAQ. Let $f : X \to Y$ be a continuous function and suppose X is connected. Let $Z = f(X)$.

Let $Z = A \cup B$ be a disconnection of Z.

Then $\exists$ open sets G and H in Y such that

$A = Z \cap G$ and $B = Z \cap H$.

Let $G_1 = f^{-1}(H)$ and $H_1 = f^{-1}(H)$. Then G_1 and H_1 are open in X.

$A \neq \phi \Rightarrow \exists\, x \in X \ni f(x) \in A = Z \cap G$

$\Rightarrow f(x) \in G \Rightarrow x \in f^{-1}(G) = G_1$

Similarly $\exists\, y \in X \ni y \in f^{-1}(H) = H_1$. Thus $G_1 \neq \phi$ and $H_1 \neq \phi$.

$$t \in G_1 \cap H_1 \Rightarrow f(t) \in G \text{ and, } f(t) \in H$$

$$\Rightarrow f(t) \in Z \cap G = A \text{ and } f(t) \in Z \cap H = B$$

$$\Rightarrow f(t) \in A \cap B \text{ which is a contradiction.}$$

$$\therefore \quad G_1 \cap H_1 = \phi.$$

$$t \in X \Rightarrow f(t) \in Z$$

$$\Rightarrow f(t) \in A \text{ or } f(t) \in B$$

$$\Rightarrow t \in f^{-1}(G) \text{ or } t \in f^{-1}(H)$$

$$\Rightarrow t \in G_1 \text{ or } t \in H_1$$

$$\Rightarrow t \in G_1 \cup H_1$$

Thus $X = G_1 \cup H_1$ is a disconnection of X, which contradicts the hypothesis that X is connected. Hence Z is connected.

12. 14. SAQ. We know that the canonical projection $\prod_i : X \to X_i$ is continuous for each i.

$$\text{Also } \prod_i \circ f_{i1}(t) = \begin{cases} t \text{ for } i = i1 \\ a_i \text{ for } i \neq i1 \end{cases}$$

Thus $\prod_i \circ f_{i1}$ is the identity map on X_{i1} for i = i1 and is the constant map to a_i for $i \neq i1$. Thus $\prod_i \circ f_{i1}$ is continuous for all i. Thus by SAQ 7.9 of lesson 7 f_{i1} is continuous.

12.15. SAQ. Let B be a non-empty basic open set in X. Then $\exists$ finite number of indices i1, . . . in and non-empty open sets $G_{i1}, \ldots, G_{in}$ in X_{i1}, $X_{i2}, \ldots X_{in}$ respectively $\ni$

$B = \bigcap_{j=1}^{n} \prod_{ij}^{-1}(G_{ij})$. Choose $x_{ij} \in G_{ij}$ for j = 1,2, . . ., n.

Define $y = \{y_i\}$ by

$y_{ij} = x_{ij}$ for j = 1, 2, . . ., n and $y_i = a_i$ for $i \notin \{i1, i2, \ldots, in\}$. Then

$$y \in A \cap \bigcap_{j=1}^{n} \prod_{ij}^{-1}(G_{ij}) = A \cap B$$

Thus $A \cap B \neq \phi$ for every non-empty basic open set B in X. Hence A is dense in X.

12.16. SAQ. Let f (x) = a $\forall x \in A$. Since Y is a T_1 – space and $a \in Y$, {a} is closed in Y. Since f is continuous $f^{-1}(\{a\})$ is closed in X. We have

$$A \subseteq f^{-1}(\{a\}) \Rightarrow \bar{A} \subseteq \overline{f^{-1}(\{a\})} = f^{-1}(\{a\})$$

$$\Rightarrow X = \bar{A} \subseteq f^{-1}(\{a\})$$

$$\Rightarrow f(X) = \{a\}.$$

Thus f is a constant map on X.

Exercises:

12. 20. Exercise. Prove that $X = A \cup B$ is a disconnection of a topological space X iff A and B are non-empty disjoint closed sets.

12. 21. Exercise. Show that a topological space X is connected if, and only if, every non-empty proper subset of X has non-empty boundary.

12. 22. Exercise. Show that a topological space X is connected iff for every two points in X there is some connected subspace of X which contains both.

12. 23. Exercise. Prove that a subspace of a topological space X is disconnected iff it can be represented as the union of two non-empty sets each of which is disjoint from the closure in X of the other.

12. 24. Exercise. Show that the graph of a continuous real function defined on an interval is a connected subspace of the Euclidean plane.

12.25. Exercise. If X is a countable, connected topological space, show that constant functions are the only real valued continuous functions on X. (Hint. Use Theorem 12.18 and the fact that every interval with more than one point in $\mathbb{R}$ is uncountable).

12. 26. Exercise. Determine whether the following are connected subspaces of $\mathbb{R}^2$

(i) $\{ (x, y) \in \mathbb{R}^2 / x \neq 0 \}$

(ii) $\{ (x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1 \}$

(iii) $\{ (x, \text{Sin}(1/x)) / 0 \neq x \in \mathbb{R} \}$

(iv) $\{ (x, y) \in \mathbb{R}^2 / x \neq y \}$

12. 27. Exercise. For any completely regular space X, prove that X is connected iff the Stone-Cech compactification β (X) of X is connected.

12. 28. Exercise. If T_1 and T_2 are topologies on X such $T_1 \subseteq T_2$ and (X, T_1) is connected prove that (X, T_2) is also connected.

12. 29. Exercise. Prove that a topological space X is connected iff every continuous function from X into the discrete two point space {0, 1} is constant.

Components

12. 30. Introduction. In this lesson, we will prove that a topological space X can always be decomposed into a disjoint union of maximal connected subspaces of X, which we call the components of X.

12. 31. Definition. Let (X, T) be a topological space. A connected subset A of X is said to be a component of X if A is not properly contained in any other connected subset of X. That is, a subset A of X is a component if it is connected and $A \subseteq B$, B is connected implies A = B.

12. 32. Example. If X is a connected space, then X is the only component of X.

12. 33. Example. In a discrete topological space X, any set with more than one lement is disconnected. Hence singleton sets are the only components of X.

12. 34. Example. In the space Q of rational numbers with usual topology, any subspace A with more than one element is not connected; for if $r, s \in A$ and $r < s$ we can find an irrational $t \ni r < t < s$ and $A = \{A \cap (-\infty, t)] \cup [A \cap (t, +\infty]$ is a disconnection of A. Thus singleton sets are the only components of Q. But the usual topology in Q is not discrete.

We prove the following two theorems before we attempt to decompose a space into its components.

12. 35. Theorem. Let X be a topological space, let $\{C_i\}$ be a non-empty class of connected subspaces of X such that $\bigcap C_i$ is non-empty. Then the subspace $C = \bigcup C_i$ is connected.

Proof: Suppose $C \subseteq A \cup B$ where A and B are open sets in X such that $A_1 = C \cap A$ and $B_1 = C \cap B$ are disjoint. For each i, the connected set $C_i \subseteq C$ and hence $C_i \subseteq A \cup B$. $(C_i \cap A) \cap (C_i \cap B) \subseteq A_1 \cap B_1 = \phi$. Since C_i is connected by Lemma 12.5, either $C_i \cap A = \phi$ or $C_i \cap B = \phi$. Thus $C_i \subseteq A$ or $C_i \subseteq B$. Since $\cap C_i \neq \phi$ we have either all the C_i are contained in A or all the C_i are contained in B. Thus $C = \cup C_i \subseteq A$ or $C = \cup C_i \subseteq B$. Hence $C \subseteq A$ or $C \subseteq B$. Thus $C \cap B = \phi$ or $C \cap A = \phi$

Hence C is connected.

12. 36. Theorem. Let A be a connected subspace of a topological space X. Let B be a subspace of X such that $A \subseteq B \subseteq \overline{A}$. Then B is connected; in particular $\overline{A}$ is connected.

Proof. Assume that B is disconnected. Then $\exists$ open sets G and H of X such that

$B \subseteq G \cup H$, $G_1 = B \cap G \neq \phi$, $H_1 = B \cap H \neq \phi$ and $G_1 \cap H_1 = \phi$

Since $A \subseteq B \subseteq G \cup H$ and A is connected, either $A \subseteq H$ and $A \cap G = \phi$ or $A \subseteq G$ and $A \cap H = \phi$. Suppose $A \cap G = \phi$. Then $\overline{A} \cap G = \phi$. If $A \cap H = \phi$ then $\overline{A} \cap H = \phi$. Since $B \subseteq \overline{A}$ we get that $B \cap G = \phi$ or $B \cap H = \phi$ which is a contradiction. Thus B is connected.

12. 37. Theorem. Let X be a topological space. Then we have the following

(i) Each point of X is contained in exactly one component of X.

(ii) Each connected subspace of X is contained in a component of X.

(iii) Each component of X is closed in X

(iv) A connected subspace of X which is both open and closed is a component of X.

Proof: (i) Let $x \in X$. Let $A = \{ C \subseteq X / x \in C$ and C is connected subspace of X.

The $A \neq \phi$ since $\{ x \} \in A$ and $x \in \bigcap_{C \in A} C$

$\therefore$ By theorem 12.35, $C_x = \bigcup_{C \in A} C$ is connected.

If D is any connected subspace of $X \ni C_x \subseteq D$ then $x \in D$. So, D is in the class A.

Hence $D \subseteq C_x$

$\therefore$ $D = C_x$.

Thus C_x is a component of X. If E is any component of $X \ni x \in E$ then $E \subseteq C_x$.

Since E is a component and C_x is connected we have $C_x = E$.

(ii) Let C be a connected subspace of X. If $x \in C$ then $C \subseteq C_x$.

(iii) Let C be a component. Since C is connected, by theoren 12.36, $\overline{C}$ is connected $C \subseteq \overline{C}$ and C is a component $\Rightarrow C = \overline{C}$

$\Rightarrow$ C is closed.

(iv) Let C be a connected subspace which is both open and closed in X. By (ii) $\exists$ a component $E \ni C \subseteq E$. Then C is open and closed in E also. Since E is connected we have $C = \phi$ or $C = E$. Since C is a subspace, we have $C \neq \phi$, Hence $C = E$ is a component.

12. 38. **SAQ.** Prove that a topological space X is connected if, and only if, X has no non-empty proper subset which is both open and closed.

12. 39. SAQ. If the product $\prod X_i$ is connected prove that each X_i is connected

12. 40. SAQ. Is a component of X open in X?

12. 41. SAQ. Prove that the components of a space form a partition of X. If there are only a finite number of components of a space X, prove that each component is open.

12. 42. Answers to SAQs

12. 38. SAQ. Suppose X is connected. If A is a non-empty proper subset of X which is both open and closed then $X = A \cup (X \setminus A)$ would form a disconnection of X.

If $X = A \cup B$ is a disconnection of X then A (and also B) is a non-empty proper subset of X which is both open and closed in X.

12. 39. SAQ. Hint. Use theorem 12.11 with the projection mapping $p_i : X \to X_i$.

12. 40. SAQ. See example 12.34. Singleton sets are not open in Q, since if $\{ r \} \subseteq Q$ is open in Q then $\{ r \} \supseteq Q \cap (a, b)$ for some open internal (a, b) in $\mathbb{R}$. But $Q \cap (a, b)$ has infinitely many points.

12. 41. SAQ. Let X be a topological space, Each $x \in X$ belongs to a unique component C_x

Then $X = \bigcup_{x \in X} C_x$. If $C_x \cap C_y \neq \phi$ then

$C = C_x \cup C_y$ is connected $C_x \subseteq C$ and $C_y \subseteq C$ imply $C = C_x = C_y$. Thus the components of X form a partition of X.

Let $C_1, C_2, \ldots, C_n$ be the only distinct components of X. The $X = \bigcup_{i=1}^{n} C_i$ and each C_i being a component, is closed.

For each i, $D_i = \bigcup_{\substack{j=1 \\ j \neq i}}^{n} C_j$ is closed and hence

$C_i = X - D_i$ is open.

12. 42. Exercises

12. 43. Exercise. Let $\{C_i\}$ be a non-empty class of connected subspaces of a topological space X such that $C_i \cap C_j \neq \phi$ for all i and j. Prove that $\cup C_i$ is also connected. (Hint: proof of theorem 12.35).

12. 44. Exercise. Let $A_1, A_2, \ldots, A_n \ldots$ be a sequence of connected subspaces of a topological space X such that $A_n \cap A_{n+1} \neq \phi$ for n = 1, 2, ... Prove that $\bigcup_{n=1}^{\infty} A_n$ is connected. (Hint let $B_n = \bigcup_1^n A_k$. Then $\bigcup_1^{\infty} A_n = \bigcup_1^{\infty} B_n$. Use induction to prove B_n is connected and use theorem 12.35 with the class $\{B_n\}$.

12.45. Exercise. Use theorem 12.35 to prove that $X \times Y$ is connected if X and Y are connected.

12. 46. Exercise. Prove that an open subspace of the complex plane is connected if, and only if, any two points in it can be joined by a polygonal line.

12.47. MODEL EXAMINATION QUESTIONS

12.48. Define a connected space and prove that a topological space X is connected iff there is no continuous function from X onto the discrete two point space {0, 1}.

12.49. Prove that the product of any non-empty class of connected spaces is connected.

12.50. Describe connected subsets of the real line $\mathbb{R}$.

12.51. Prove that the continuous image of a connected space is also a connected space.

12.52. Prove that $\mathbb{R}^n$ and C^n are connected.

12.53. Define a component of a topological space. What are the components of Z, the set of all integers, as a subspace of the real line $\mathbb{R}$ with the usual topology?

12.55. If $A \subseteq B \subseteq \bar{A}$ for subspaces A and B of a topological space X and A is connected, show that B is also connected.

12.56. Prove that the components of a topological space X are closed subsets of X. Can we prove that components of X are also open subsets of X ? Justify your answer.

Lesson Writer: V.S. Lal

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SUPPLEMENT TO LESSON 12

FUNCTIONAL ANALYSIS

PROJECTIONS

Let us recall that by a projection P on a Banach space N we mean a $P \in B(N, N)$ such that $P^2 = P$. In 6.7 it was proved that $P \in B(N, N)$ is a projection if and only if B = Range P $\oplus$ Null space P. The norm on a Hillbert space has special properties because of the special features of the inner product. In what follows H stands for a complex Hilbert space.

12.18 Theorem: If P is a projection on H with range M and null space N then $M \perp N$ if and only if P is self adjoint. In this case $N = M^{\perp}$

Proof: If $x \in H$, $P(x - P(x)) = P(x) - P^2(x) = P(x) - P(x) = 0$

so that $x - P(x) \in N$. Also $x \in M \cap N \Rightarrow P(x) = 0$ and $x = P(x)$ so that $x = P(x) = 0$. Thus every vector x in H can be uniquely written as $x = y + z$ where $y \in M$ and $z \in N$.

Now if $M \perp N$ then $(y, z) = 0$ so that $z \perp y$.

Hence $(P^*(x), x) = (x, P(x)) = (y + z, P(x)) = (y, P(x)) + (z, P(x))$

$= (y, y)$ ($\because P x = y$ and $(z, y) = 0$)

Also $(P(x), x) = (y, y + z) = (y, y)$

so that $(P^*(x), x) = (P(x), x)$, hence $((P^* - P)(x), x) = 0$ for every x.

This implies that $P^* - P = 0$ by 11.7 so that $P^* = P$ i.e. P is self adjoint.

Conversely suppose that $P^* = P$. If $x \in M$ and $y \in N$ then $P x = x$ and $P^* y = Py = 0$ ($\because$ t $\in N \Rightarrow py = 0$) so that

$(x, y) = (P x, y) = (x, P^* y) = (x, 0) = 0.$

This implies that $x \perp y$ if $x \in M$ and $y \in N$ so that $M \perp N$.

We now show that if $M \perp N$ then $N = M^{\perp}$.

Clearly $N \subset M^{\perp}$. If $N \neq M^{\perp}$, N is a proper linear closed subspace of the Hulbert space $M^{\perp}$. So there exists nonzero $z \in M^{\perp}$ which is orthogonal to N. Since $z \perp M$ and N and $H = M \oplus N$ it follows that $z \perp H$, there by implying that $z = 0$ which contradicts that $z \neq 0$.

Thus $N = M^{\perp}$.

12.19 Definition: If $P \in B(H)$ and $P^2 = P = P^*$ then P is called a projection or perpendicular projection on H. We also say, in this case, that P is a projection on $M = \{P(x) \mid x \in H\}$.

12.20 Proposition: Let P be a projection on M. then

(i) I - P is a projection.

(ii) $\| P(x) \| = \|x\| \Leftrightarrow x \in M$

(iii) P is a positive operator on H.

Proof: (i) $P^2 = P \Leftrightarrow (I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$.

$\Rightarrow$ I - P is a projection.

To prove (ii), suppose P is a projection on M.

If $x \in M$ then $P(x) = x$ so $\|P(x)\| = \|x\|$.

For any $x \in H$, $x = P(x) + (I - P)(x)$. Since $(P(x), (I - P)(x)) = 0$,

$$\|x\|^2 = \|P(x)\|^2 + \|(I - P)x\|^2 \quad \text{.............. (1)}$$

So $\|P(x)\| = \|x\| \Rightarrow \|(I - P)(x)\| = 0 \Rightarrow x \in M$.

From (1) above it is clear that $\|Px\| \leq \|x\|$ so that $\|P\| \leq 1$.

$$(\because \|P\| = \sup_x \frac{\|Px\|}{\|x\|})$$

Moreover for every $x \in H$,

$(Px, x) = (P^2x, x) = (Px, P^*x) = (Px, Px)$ $(\because P = P^*) = \|Px\|^2 \geq 0$

This yields (iii)

12.21 Definition: Let $T \in B(H)$. A closed linear subspace M of H is said to be invariant under T if $T(M) \subseteq M$. If both M and $M^\perp$ are invariant under T, we say that M reduces T or T is reduced by M.

12.22 Theorem: A closed linear subspace M of H is invariant under $T \in B(H)$ if and only if $M^\perp$ is invariant under T^*.

Proof: Since $M^{\perp\perp} = M$ and $T^{**} = T$ it is enough to prove that if M is invariant under T then $M^\perp$ is invariant under T^*.

Assume that M is invariant under T and let $y \in M^\perp$. If $x \in M$, then $(Tx, y) = 0$ so $(x, T^*y) = 0$ hence $T^*y \in M^\perp$.

12.23 Theorem: A closed linear subspace M of H reduces an operator T if and only if M is invariant under T and T^*.

Proof: M reduces T iff M and $M^{\perp}$ are invariant under T

if and only if M is invariant under T and T^*.

12.24 Theorem: If P is a projection on a closed linear subspace M of H then M is invariant under $T \in B(H)$ if and only if TP = PTP.

Proof: If $x \in H$, then $P(x) \in M$. If M is invariant under T then $T(M) \subseteq M$ so that $TP(x) \in M$, hence $P(TP)(x) = TP(x)$. ($\because x \in M \Rightarrow P(x) = x$) and hence PTP = TP.

Conversely if PTP = TP, then for $x \in M$, Px = x so that TP(x) = PT P(x) hence T(x) = PT (x) = PTP(x). This implies that M is invariant under T.

12.25 Theorem: If P is a projection on a closed linear subspace M of H then M reduces $T \in B(H)$ if and only if TP = PT.

Proof: M reduces T if and only if M is invariant under T and T^*, if and only if TP = PTP and $T^*P = PT^*P$

if and only if TP = PTP and PT = PTP

if and only if TP = PTP = PT.

Since $P^2 = P$, $TP = PT \Rightarrow PTP = P^2T = PT$

and also $TP^2 = (TP)P = PTP$ so that PTP = TP = PT.

12.26 Theorem: If P and Q are projections on closed linear subspaces M and N of H then $M \perp N$ if and only if PQ = 0 if and only if QP = 0

Proof: It is clear that

$PQ = 0 \Leftrightarrow (PQ)^* = 0 \Leftrightarrow Q^* P^* = 0 \Leftrightarrow QP = 0$

If $M \perp N$ then $N \subseteq M^{\perp}$ so that $\forall x \in H$, $PQ(x) = P(Q(x)) = 0$ as $Q(x) \in M^{\perp}$. Conversely if $PQ = 0$, then for $x \in N$, $P(x) = PQ(x) = 0$ so that $N \subseteq M^{\perp}$ and hence $M \subseteq N^{\perp}$. This gives $M \perp N$.

12.27 Theorem: If $P_1, P_2, \ldots\ldots, P_n$ are projections on closed linear subspaces $M_1, \ldots\ldots, M_n$ of H respectively then $P = P_1 + \ldots\ldots + P_n$ is a projection if and only if $P_i P_j = 0$ whenever $i \neq j$, (i.e. $P_i \perp P_j$ if $i \neq j$). In this case P is a projection on $M = M_1 + \ldots\ldots + M_n$.

Proof: Since $P_i^* = P_i \; \forall\, i$, $P^* = P$. Thus P is a projection if and only if $P^2 = P$.

If $P_i P_j = 0$ when $i \neq j$,

$$P^2 = POP = \sum_{i,j} P_i P_j = \sum_i P_i^2 = \sum_i P_i = P$$

Conversely suppose that $P^2 = P$. To show that $P_i P_j = 0$ whenever $i \neq j$ it is enough to prove that $M_i \perp M_j$ whenever $i \neq j$.

Let $x \in M_i$ and $j \neq i$. Then $P_i(x) = x$.

$$\Rightarrow \|x\|^2 = \|P_i(x)\|^2 \leq \sum_{k=1}^{n} (P_k\, x, x) = \left(\sum_{k=1}^{n} P_k\, x, x\right)$$

$$= (P\ x, x) = \|P(x)\|^2 \leq \|x\|^2$$

$$\text{Hence } \|P_i(x)\|^2 = \sum_{k=1}^{n} + \| P_k(x) \|^2$$

Hence $P_j(x) = 0$ for $r \neq i$. This implies that $M_i \subseteq M_j^{\perp}$ if $i \neq j$.

Consequently $M_i \perp M_j^{\perp}$ if $i \neq j$ and hence $P_i P_j = 0$ if $i \neq j$.

We now prove that P is the projection on M.

If $x \in M_i$, $P_j x = 0$ for $j \neq i$ so $px = p_i x$, hence $||Px|| = ||x||$ thus $x \in M_i \Rightarrow x \in$ Range P so that $M \subseteq$ Range P.

Conversely if $x \in$ Range P,

$x = Px = P_1 x + \ldots\ldots + P_n x \in M_1 + \ldots\ldots. + M_n = M.$

Thus Range P = M

This completes the proof.

12.28 SAQ: If P and Q are projections on closed linear subsapces M and N of H prove that PQ is a projection if and only if PQ = QP. In this case show that PQ is the proejction on $M \cap N$.

Answers to SAQ's:

12.29 SAQ: Under the hypothesis of SAQ prove that $P \leq Q \Leftrightarrow ||P(x)|| \leq ||Q(x)|| \Leftrightarrow PQ = P \Leftrightarrow QP = P$

SAQ 12.28 Suppose M, N are linear subspaces of H and P is a projection on M while Q is a projection on N. If PQ = QP, $(PQ)^2 = PQ\,PQ = P(PQ)\,Q = P^2Q^2 = PQ$. Furth + $(PQ)^* = Q^*P^* = QP = PQ$. So PQ is a projection.

Since M, N are closed, $M \cap N$ is closed.

If $x \in M \cap N$, Px = x and Qx = x so QPx = Qx = x.

Since PQ = QP, $x \in$ Range PQ

If $x \in$ Range PQ, PQ x = x $\Rightarrow$ Px = P(PQx) = PQx = x.

$\Rightarrow x \in$ Range P = M. By symmetry $x \in$ N.

$\Rightarrow x \in M \cap N$.

Thus $M \cap N$ = Range PQ.

SAQ 12.29: Let P, Q be projections on closed linear subspaces M and N of H.

Then $P \leq Q \Leftrightarrow \forall\, x \in H, (Px, x) \leq (Qx, x)$

$\Rightarrow \|P(x)\|^2 = (P(x), P(x)) = (P^2(x), x) = (P(x), x) \leq (Qx, x) = \|Qx\|^2$

$\Rightarrow \|P\,x\| \leq \|Qx\|$

The reverse inequalities also hold so that $\|P(x)\| \leq \|Q(x)\| \Rightarrow P \leq Q$.

$PQ = P \Rightarrow PQ$ is a projection $\Rightarrow PQ = QP \Rightarrow QP = P$.

Interchanging P, Q we get $QP = P \Rightarrow PQ = P$

Thus $PQ = P \Leftrightarrow QP = P$

$PQ = P \Rightarrow (P(x), x) = \|P(x)\|^2 = \|PQ(x)\|^2 \leq \|Qx\|^2 = (Qx, x)$

Thus $P\,Q = P \Rightarrow P \leq Q$.

Conversely suppose $P \leq Q$. Then $\forall\, x\ \|P(x)\| = \|Q(x)\|$

$\Rightarrow N^{\perp} \subseteq M^{\perp}$. For every $x \in H$, $x - Q(x) \in N^{\perp}$

$\Rightarrow P(x - Q(x)) = 0\ \forall\, x$

$\Rightarrow P(x) = PQ(x)\ \forall\, x$

$\Rightarrow PQ = P$

Thus $P \leq Q \Leftrightarrow PQ = P$.

12.30 Model Examination Questions:

1. Let P be a projection on a Hilbert space H with range M and null space N. Show that P is self adjoint if and only if $M \perp N$.

2. Show that a linear subsapce M of a Hilbert space H is invariant under an operator T if and only if $M^{\perp}$ is invariant under T^*

3. Show that if P is a projection on a closed linear subspace M of H then M reduces $T \in B(H) \Leftrightarrow TP = PT$.

4. If P and Q are projections on closed linear subspaces M and N show that $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$.

12.31 Exercises:

1. Show that the projections on H form a complete lattice with respect to the natural ordering of self adjoint operators.

2. If P, Q are projections on closed linear subspaces M, N of H show that $M \subseteq N \Leftrightarrow PQ = P \Leftrightarrow QP = P$.

3. Under the hypothesis of 2 above

   Prove that P - Q is a projection if and only if $Q \leq P$.

LESSON NO -13

APPENDIX

The following famous results, without which an introduction to topology is not complete and some of which are useful in Functional Analysis as well are presented here under. Some equivalent forms of Zorn's lemma are also stated without proof.

1. Cantor's intersection theorem
2. Baire category theorem
3. Schoroder – Bernstein theorem
4. For any set A there is no bijection from A onto P(A)
5. Equivalent forms of Zorn's lemma

13.1 Cantors Intersection Theorem:

In lesson 9 we proved that a metric space is compact if and only if it is complete and totally bounded. We characerised compactness by X is compact if and onlyif every class of closed subsets of X with finite intersection property has non-empty intersection. Suppose we are given a countable class $\{E_n\}_{n \in \mathbb{N}}$ of closed subsets of X. then the class of sets $F_n = E_1 \cap \ldots\ldots \cap E_n$ has the following properties.

F_n is closed, $F_n \supseteq F_{n+1}$

If the given class $\{E_n\}_{n \in \mathbb{N}}$ has finite intersection property then

$F_n \neq \phi$

So in this case the compactness of X implies $\cap F_n \neq \phi$

That is

A decreasing sequence of non-empty closed subsets of X has non-empty intersection. We consider this property and prove two theorems. These are used extensively in several branches.

We assume that the space X is a complete metric space.

Example 1: Let $X = \mathbb{R}$ and let $E_n = (-\infty, -n] \cup [n, \infty)$ for all positive integers n. Then E_n is closed and $E_n \supseteq E_{n+1}$. We have $\bigcap_{n=1}^{\infty} E_n = \phi$. This follows from the Archimedean property. Given any α in $\mathbb{R}$ there is a natural number n such that $-n < \alpha < n$

The E_n's are closed decreasing and non-empty but not bounded while X is complete.

Example 2: We define

$$E_n = \{\alpha \in Q : \alpha > 0, 2 - \frac{1}{n} \leq \alpha^2 \leq 2 + \frac{1}{n}\}$$

Then we clearly have

$$E_n \supseteq E_{n+1}$$

If there is a rational number p / q in all E_n then taking limit in the inequality

$$2 - \frac{1}{n} \leq (p/q)^2 \leq 2 + \frac{1}{n}$$

We obtain $2 = \left(\frac{p}{q}\right)^2$

This means that $\sqrt{2}$ is a rational number. We know as part of the construction of real number that

1) $\sqrt{2}$ is irrational and
2) The set E_n is non-empty.

We claim that the set E_n is a closed subset of Q^+. To see this, consider the map

$f : Q^+ \to Q^+$ defined by $f(x) = x^2$

f is continuous and E_n is the inverse image of the closed set. $[2 - \frac{1}{n}, 2 + \frac{1}{n}]$

and so E_n is a closed subset of Q^+. The sets are all bounded above by 2. If $2 < \frac{p}{q}$ then

$4 < \frac{p^2}{q^2}$. We shall now prove that $E_n \neq \phi$.

We suppose $\delta > 0$ and consider the interval $(2 - \delta, 2 + \delta)$

We choose a natural number n and consider the numbers $x_K = 1 + \frac{k}{n}$, $k = 0, 1, \ldots\ldots, n$.

We have

$$x_{k+1}^2 - x_k^2 = (x_{k+1} + x_k)(X_{k+1} - x_k)$$

$$= (2 + \frac{2k+1}{n})\frac{1}{n}$$

$$< 4\frac{1}{n} = \frac{4}{n}.$$

The numbers x_k^2 satisfy

$$1 = x_0^2 < x_1^2 < .... < x_{n-1}^2 < x_n^2 = 4.$$

We now suppose $\delta < \frac{1}{2}$. Then there is an integer k such that

$$x_k^2 \leq 2 - \delta < x_{k+1}^2$$

Suppose $2 + \delta \leq x_{k+1}^2$, then $2\delta = (2+\delta) - (2-\delta) \leq x_{k+1}^2 - x_k^2 < \frac{4}{n}$

By Archimedes' Axiom we can find a natural number m such that

$$m > \frac{2}{\delta} \text{ i.e. } \delta > \frac{2}{m}.$$

If we use such an m in the place of n above we have $x_k^2 \leq 2 - \delta < x_{k+1}^2 < 2 + \delta$.

From this we may conclude $E_n \neq \phi$.

Therefore we have a decreasing sequence of bounded non-empty closed sets E_n in Q^+ such that $\bigcap_{n=1}^{\infty} E_n = \phi$. Here we note that Q^+ is not complete.

There is a condition on the sets E_n which ensures that $\bigcap_{n=1}^{\infty} E_n$ is non-empty when X is a complete metric space. To state the condition we need some preliminaries.

13.2 Definition: Suppose (A, d) is a metric space and B is a non-empty subset of A. We define the diameter of B to be

$$\underset{x,y \in B}{\text{l.u.b}}\ d(x, y)$$

We denote it by d (B)

13.3 Remarks: We have defined d(B) only when B is a non-empty subset. So each time we talk of the diameter of a set B we have to be certain that B is non-empty.

The numbers d(x,y) are all non-negative and $0 = d(x,x)$ is certainly one of them. Therefore $d(B) \geq 0$.

If there are two distinct elements x and y in B then

$d(x,y) > 0$

therefore $d(B) > 0$

It follows that $d(B) = 0$

if and only if B consists of a single point we now state.

13.4 Main Theorem: (Cantor's intersection Theorem). Suppose X is a complete metric space. If $\{E_n\}_{n\in\mathbb{N}}$ is a sequence of closed subsets of X such that

i) $E_n \neq \phi$

ii) $E_n \supseteq E_{n+1}$ and

iii) $d(E_n) \to 0$ as $n \to \infty$

then $E = \cap E_n \neq \phi$, and consists of a single point.

Proof: For each positive integer n we choose an **element x_n** arbitrarily from the non-empty set E_n. Thus we obtain a sequence $\{x_n\}$, $x_n \in E_n$.

Since we have assumed that $E_n \supseteq E_{n+1}$

We have $x_{n+p} \in E_{n+p} \subseteq E_n$ for all $p \in \mathbb{N}$.

We have assumed that $d(E_n) \to 0$ as $n \to \infty$. Therefore given any $\varepsilon > 0$ we can choose a positive integer $n(\varepsilon)$ such that $d(E_n) < \varepsilon$ if $n \geq n(\varepsilon)$; therefore $d(x_m, x_n) < \varepsilon$ if $n, m \geq n(\varepsilon)$. This means that $\{x_n\}$ is a Cauchy sequence. Since X is complete the sequence tends to a limit say z. All the x_n are in E_1. Since E_1 is closed. $z \in E_1$.

Choose any n. The sequence $y_m = x_{n+m}$ is a subsequence of $\{x_k\}$

Since $\{x_k\}$ tends to z, the sequence $\{y_m\}$ also tends to z. Since $y_m \in E_n$ for all m and E_n is closed we obtain

$z \in E_n$

It follows that

$$z \in E_n$$

It follows that

$$z \in \cap E_n$$

Suppose z' is also in E then $d(z, z^1) \leq d(E_n)$ for all n. Since $d(E_n) \to 0$ we must have $d(z, z^1) = 0$ that is $z = z^1$. The theorem is proved.

There is another theorem we prove in this lesson. It is extensively used . It is known as Baire category theorem. Cantor's intersection theorem is used in its proof. For the statement of the theorem we need some preliminaries.

13.5 Definition: Let X be a topological space. A subset A of X is said to be nowhere dense in X if the closure $\overline{A}$ of A in the space X has empty interior : Int $(\overline{A}) = \phi$.

13.6 Remarks: It is important to remember that the closure of E is taken in X. Suppose we simply consider E as a subset of itself with the topology induced from X then we have

$\overline{E} = E$ and Int $\overline{E} = E$

As an illustrative example we consider

$E = Q$ and $X = \mathbb{R}$.

Then $\overline{Q}$ in $\mathbb{R}$ is $\mathbb{R}$ and $\overline{Q}$ in Q is Q.

Int Q in $\mathbb{R}$ is the empt set and Int Q in Q is Q itself.

Exercise: If E and F are nowhere dense in X then $E \cup F$ is nowhere dense in X.

13.7 Remark: It might happen that

Int $E = \phi$ and Int $F = \phi$ and Int $(E \cup F) \neq \phi$

For example $E = Q$, $F = Q' = \mathbb{R} \setminus Q$ and $X = \mathbb{R}$

So one should remember that for a set E to be nowhere dense it must satisfy

Int $\overline{E} = \phi$, Where the closure is taken as a subset of X.

13.8 Baire Category Theorem:

Suppose X is a complete metric space and E_n, $n \in \mathbb{N}$ is a sequence of nowhere dense subsets of X. Then $\bigcup_{n=1}^{\infty} E_n \underset{\neq}{\subset} X$.

Remark: If E is nowhere dense in X then $\overline{E}$ is nowhere dense in X. Therefore we could have stated the above theorem as : If E_n is a sequence of nowhere dense subsets of X. then

$$\bigcup_n \overline{E}_n \underset{\neq}{\subset} X.$$

Exercise: Suppose the above theorem is assumed. When $E_n \subseteq E_{n+1}$, and E_n' are closed. The above theorem may be deduced from it.

Proof: We denote the closure of E_n by F_n. $F_n = \overline{E_n}$; Int $F_n = \phi$. F_n is closed. We have assumed that Int $F_1 = \phi$. This implies that $F_1 \underset{\neq}{\subset} X$. So there is a point x_1 in X which is not in F_1. F_1 is a closed subset of X and therefore there is a $\delta_1 > 0$. Such that

$$S_{\delta_1}(x_1) \cap F_1 = \phi$$

For this δ_1 we have $S_{\delta_1/2}[x_1] \subseteq S_{\delta_1}(x_1)$ and $S_{\delta_1/2}[x_1]$ is a closed subset of X. F_2 is a closed subset of X with Int $F_2 = \phi$ and $S_{\delta_1/2}(x_1)$ is a non-empty open subset of X. Therefore $S_{\delta_1/2}(x_1) \underset{\neq}{\subset} F_2$.

There is an x_2 in $S_{\delta_1/2}(x_1)$, not in F_2 and since F_2 is closed, there is a $\delta_2 > 0$ such that

$$\delta_2 \le \frac{1}{2}\delta_1,$$

$$S_{\delta_2}(x_2) \subseteq S_{\delta_1/2}(x_1)$$

$S_{\delta_2}(x_2) \cap F_2 = \phi$. We then have

$S_{\delta_1/2}[x_2] \subseteq S_{\delta_2}(x_2)$ and $S_{\delta_2/2}[x_2]$ is a closed subset of X.

Inductively we choose x_n and δ_n in the above manner. We thus obtain a sequence of points x_n in X and a sequence of positive numbers δ_n with the following properties:

i) $S_{\delta_n}(x_n) \cap F_n = \phi$

ii) $\delta_{n+1} \leq \frac{1}{2}\delta_n$

iii) $S_{\delta_{n+1}}(x_{n+1}) \subseteq S_{\frac{\delta n}{2}}(x_n)$

Suppose we set $X_n = S_{\frac{\delta n}{2}}[x_n]$ then we have

$$X_{n+1} = S_{\delta_{n+1/2}}[x_{n+1}] \subseteq S_{\delta_{n+1}}(x_{n+1})$$

$$\subseteq S_{\frac{\delta n}{2}}(x_n)$$

$$\subseteq S_{\frac{\delta n}{2}}[x_n]$$

$$= X_n$$

Also d $(S_\delta[x]) \leq 2\,\delta$, and so d $(x_n) \leq 2\,\frac{\delta_n}{2} = \delta_n$; since by our choice $\delta_{n+1} \leq \frac{\delta n}{2}$. We obtain d $(X_n) \rightarrow 0$ as $n \rightarrow \infty$. The E_n are closed and X is complete. Therefore by Cantor's intersection theorem we obtain an x_0 in the intersection $\cap X_n$. Since x_0 is an element of X_n and $X_n = S_{\frac{\delta n}{2}}[x_n] \cap F_n = \phi$, x_0 is not an element of $\bigcup_{n=1}^{\infty} F_n$. Thus the theorem is proved.

13.9 Remarks: To understand the proof the student is advised to study the following example. The topological space X is $\mathbb{R}^2$. The sets E_n are $E_1 = \{(0,y) : y \in \mathbb{R}\}$

$E_n = \{(\frac{1}{2^{n-1}}, y) : y \in \mathbb{R}\}$. The points x_n are $(\frac{1}{2^{n-1}}, 0)$ and the positive numbers $\delta_n = \frac{1}{2^n}$

with these choices we have

$S_{\delta_n}(x_n) \cap E_n = \phi$,

Int $E_n = \phi$.

$\cap S_{\delta_n}(x_n) = \phi$

$\cap S_{\delta_n}[x_n] = (0, 0) \in E_1$.

13.10 Schroder – Bernstien theorem:

This famous theorem which is useful in proving that any two bases in a vector space over a field have the same cardinality can be proved by using the set of natural numbers and inductive definition. A clear proof which makes the idea transparent is given in the reading material on paper I Algebra (C.D.E) prepared by Acharya Nagarjuna University. We now present a proof of this theorem due to Littlewood, which does not involve the set of natural numbers.

13.11 Theorem: Suppose A is a set, T: A $\to$ A is a one-to-one map and $A_1 \subseteq A$ is such that

$T(A) \subseteq A_1 \subseteq A$.

Then there is a bijection $S: A \to A_1$.

Proof: For the proof we introduce the notion of a suitable set as defined by Littlewood. A subset X of A is called a suitable set of $A_1 - T(x) \subseteq A - X$.

For example ϕ and $A - A_1$ are suitable subsets of A.

Claim: Suppose $\{X_i\}_{i \in I}$ is a class of suitable subsets of A and $X = \bigcup_{i \in I} X_i$. Then X is a suitable set.

Proof of the Claim $A_1 \setminus T(x) = A_1 \setminus \bigcup_i T(X_i)$

$= \bigcap_i A_1 \setminus T(x_i)$

$\subseteq \bigcap_i (A \setminus x_i)$

$= A \setminus \left(\bigcup_i x_i\right)$

$= A \setminus X$.

Our claim is proved. We denote the union of all suitable sets by H.

If B is any suitable subset of A we have $B \subseteq H$. We have

$A_1 \setminus T(H) \subseteq A \setminus H$.

By taking complements in A we have

$H \subseteq A \setminus (A_1 \setminus T(H))$

$= (A \setminus A_1) \cup T(H)$ since $T(H) \subseteq T(A) \not\subseteq A_1$

$= E \cup T(H)$ where we have set $E = A \setminus A_1$

Write $K = E \cup T(H)$

Since $H \subseteq K$

$A_1 \setminus T(K) \subset A_1 \setminus T(H)$

$= (A \setminus E) \setminus T(H)$

$= A \setminus (E \cup T(H)) \because E \cap T(H) = \phi$

$= A \setminus K$.

Thus K is a suitable set and so we have

$K \subseteq H \subseteq K$;

therefore

$$H = E \cup T(H),$$

$A \setminus H = A \setminus (E \cup T(H))$

$= (A \setminus E) \cap (A \setminus T(H))$

$= A_1 \cap (A \setminus T(H))$

$= A_1 \setminus T(H)$ since $T(H) \subseteq A_1$

We define $\phi : A \to A_1$ by

$$\phi(x) = \begin{cases} T(x) & \text{if } x \in H \\ x & \text{if } x \in A \setminus H \end{cases}$$

It is easily verified that $\phi : A \to A_1$ is a bijection.

13.12 Theorem: For any set A, there is no bijection from A onto P(A).

Proof: Suppose, if possible, there is a bijection

$\lambda : A \to P(A)$

Write $B = \{a \in A : a \notin \lambda(a)\}$ Since λ is a bijection and $B \in P(A)$, there is a unique $b \in A$ such that $\lambda(b) = B$.

The element b of A and the subset B of A have the following properties.

$b \in B \Leftrightarrow b \in \lambda(b) \Leftrightarrow b \notin B$.

The last equivalence is by the definition of the set B. The assertion $b \in B \Leftrightarrow b \notin B$ is not valid. Since the construction of B is based on the bijection λ, it follows that there is no such λ. Here there is a question. Suppose $B = \phi$. Then what do we get? What we obtain is for each

$a \in \lambda(a)$

since $\{a\} \subseteq A$, there is some b_0 in A such that

$$\lambda(b_0) = \{a\};$$

since by our assumption

$$b_0 \in \{a\}$$

it follows that

$$b_0 = a$$

so the map

$$\lambda : A \rightarrow P(A)$$

is given by

$$\lambda(a) = \{a\}$$

It follows that if $a_1, a_2 \in A$ and $a_1 \neq a_2$ then there is no a in A s.t.

$$\lambda(a) = \{a_1, a_2] \subseteq A.$$

So we must have

$$A = \{a\}.$$

Here we use the fact that the empty set ϕ is counted as a set and is also a subset of every set. Thus in the map

$$\lambda(a) = \{a\}$$

the empty subset ϕ of A is missing in the image.

13.13 Some equivalent forms of Zorn's lemma:

In the preliminaries we introduced Zorn's lemma which comes to our rescue while trying to establish "existence" without bothering about the "form". Many a time Zorn's lemma may fail to do the needful in which case other equivalent forms of this famour lemma. Just as Zorn's lemma cannot be proved with the existing set theory and without using any or the equivalent forms, none of these equivalent forms do not have any proof based on set theory minus any of the equivalent forms. Even though the proof for their equivalence is simple and straight forward, it requires however, that the reader is sufficiently advanced in conceptual thinking. As such we mention here (without proof) a few equivalent forms of Zorn's lemma.

1. **Axiom of choice:** If A is a nonempty set and $\{A_i\}$ $i \in I\}$ is a nonempty family of nonempty sets in A then there is a map $\phi : I \to A$ such that $\phi(i) \in A_i$ $\forall i \in I$.

13.14 Definition: A class A of subsets of a set A is said to be of finite character if the following holds:

A subset X of A is in A if and only if every finite subset of X is in A.

Hausdorff's maximality Principle: If A is a class of subsets of a set A of finite character then A has a maximal element : that is there is an $X \subseteq A$ such that (1) $X \in A$ and

(2) there is no $Y \subseteq A$ such that (i) $Y \in A$ and (ii) $X \subsetneq Y$.

13.15 Definition: A partially ordered set $(X, \leq)$ is said to be well ordered if each nonempty subset of X has a first element : that is $\exists x \in X$ such that $x \leq y$ for every $y \in X$.

Well ordering Principle: Given any nonempty set X, there is a partial order on X, say $\leq$, such that $(X, \leq)$ is a well ordered set.

Lesson Writer: V.J. LAL