MEASURE AND INTEGRATION
(DM22)
(MSC MATHEMATICS)

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LESSON-0 - APPENDIX

INTRODUCTION :

The aim of this lesson is to help the reader in reviewing some important results of analysis that are to be used in the subsequent lessons. These include some definitions and results on sequences of real numbers.

Sequences of Real numbers

Definition 1 : Let $<x_n>$ be a sequence of real numbers. We say that a real number l is a limit of the sequence $<x_n>$ if for each $\in > 0$ there is a positive integer N such that $|x_n - l| < \in$, for all $n \geq N$. It can be easily verified that a sequence of real numbers can have at most one limit. If l is a limit of the sequence $<x_n>$ then we write $l = \lim x_n$.

Defnition 2 : A sequence $<x_n>$ of real numbers is called a Cauchy sequence if given $\in > 0$, there is a positive integer N such that for all $n \geq N$ and for all $m \geq N$, we have $|x_n - x_m| < \in$.

The Cauchy criterion states that a sequence of real numbers converges if and only if it is a Cauchy sequence.

We extend the notion of limit of a sequence of real numbers to include the values ∞ and $-\infty$.

Definition 3 : We say that ∞ is a limit of the sequence $<x_n>$ if for each real number Δ there is a positive integer N such that for all $n \geq N$ we have $x_n > \Delta$.

If ∞ is a limit of the sequence $<x_n>$ we write $\lim x_n = \infty$

Definition 4 : We say that $-\infty$ is a limit of the sequence $<x_n>$ if for each real number Δ there is a positive integer N such that for all $n \geq N$ we have $x_n < \Delta$.

If $-\infty$ is a limit of the sequence $<x_n>$ we write $\lim x_n = -\infty$.

A sequence is called convergent if it has a limit.

Remark 5 : In most of analysis we restrict ourselves to limits of sequences of real numbers which are real numbers. But here we find it more convenient to allow $\pm\infty$ as limits in good standing.

If $l = \lim x_n$, we often write $x_n \to l$.

We know that if $<x_n>$ is a sequence and $<n_k>$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < .....$, then $< x_{n_k} >$ is called a subsequence of $<x_n>$.

If a sequence $<x_n>$ converges and $\lim x_n = l$ then every subsequence $< x_{n_k} >$ of $<x_n>$ converges and $\lim x_{n_k} = l$.

If $<x_n>$ is not convergent then we consider the convergent subsequences of $<x_n>$.

Definition 6 : A real number l is called a cluster point of the sequence $<x_n>$ if, given $\in > 0$ and given a positive integer N there is an integer $n \geq N$ such that $| x_n - l | < \in$.

Definition 7 : We say that ∞ is a cluster point of $<x_n>$ if, given a real number Δ and given a positive integer N there is an integer $n \geq N$ such that $x_n \geq \Delta$.

Definition 8 : We say that $-\infty$ is a cluster point of $<x_n>$ if, given a real number Δ and given a positive integer N there is an integer $n \geq N$ such that $x_n \leq \Delta$.

Consider the sequence $<x_n>$, where $x_n = (-1)^n$, $n = 1,2,.......$

$<x_n>$ is not convergent but 1, -1 are cluster points of $<x_n>$.

It can be verified that l is a cluster point of $<x_n>$ if and only there is a subsequence $< x_{n_k} >$ that converges to l.

Definition 9 : Let $<x_n>$ be a sequence of real numbers. We define limit superior of $<x_n>$ by

$$\overline{\lim}\, x_n = \inf_n \left(\sup_{k \geq n} x_k \right) \inf_n y_n, \text{ where } y_n = \sup_{k \geq n} x_k = \sup \{x_n, x_{n+1}, x_{n+2}, \cdots\}$$

Definition 10 : Let $<x_n>$ be a sequence of real numbers. We define limit inferior of $<x_n>$ by

$$\underline{\lim}\, x_n = \sup_n \inf_{k \geq n} x_k = \sup_n z_n \text{ where } z_n = \inf_{k \geq n} x_k = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$$

Result 11 : A real number l is the limit superior of the sequence $<x_n>$ if and only if.

1. Given $\in > 0$, there exists a positive integer k such that $x_n < l + \in$ for all $n \geq k$

and 2. Given $\in > 0$ and given a positive integer n there is an integer N such that $N \geq n$ and $l - \in < x_N$

Proof: Suppose that l is a real number and $l = \overline{\lim}\, \overline{x_n}$.

Let $y_n = \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}$, $n = 1, 2, \ldots$ Now $l = \inf_n y_n$.

Let $\in > 0$. Now $l + \in$ is not a lower bound for $\{ y_1, y_2, \ldots y_n, \ldots\}$.

So, we get a positive integer k such that $y_k < l + \in$.

Therefore $x_n < l + \in$ for all $n \geq k$. Let n be a positive integer. Now $l - \in < y_n$. If $x_k \leq l - \in$ for all $k \geq n$ then $y_n \leq l - \in$, a contradiction.

Therefore, there is a positive integer $p > n$ such that $x_p > l - \in$. Conversly suppose that conditions 1 and 2 hold.

Suppose that $s = \overline{\lim}\, x_n$.

We prove that $s = l$. On the contrary suppose that $s \neq l$.

Case I : Suppose that $l < s$. we get a $\in > 0$ such that $l < l + \in < s$. By condition 1, we get a positive integer k such that $x_n < l + \in$ for all $n \geq k$. So $y_k \leq l + \in < s$, a contradiction to the fact that $s < y_n$ for all $n = 1, 2, \ldots$

Case II : Suppose that $s \leq l$. we get a $\in > 0$ such that $s < l - \in < l$. let p be a positive integer. By condition 2, we get a positive integer q such that $q \geq p$ and $l - \in < x_q$. So $l - \in < y_q$. Since $y_1 > y_2 > y_3 \ldots$, $y_q \leq y_p$. Therefore, $l - \in < y_p$ and that

$$s < l - \in < \inf_n y_n = \overline{\lim}\, x_n = s, \text{ a contradiction.}$$

From case I & case II we get that, $s = l$.

Result 12 : The extended real number ∞ is the limit superior of $<x_n>$ if and only if given a real number Δ and a positive integer n there is a $k \geq n$ such that $x_k > \Delta$.

Proof : Suppose that $\infty = \overline{\lim}\, x_n$. Let $y_n = \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}$

Now $\infty = \overline{\lim}\, x_n = \inf_n y_n$. So $y_n = \infty$ for all $n = 1, 2, \ldots$

Let Δ be a real number and n be a positive integer.
Since $\infty = y_n$, there is a positive integer $k \geq n$ such that $x_k > \Delta$.

Conversely suppose that given a real number Δ and a positive number n there is a $k \geq n$ such that $x_k > \Delta$. By our assumption $y_n = \infty$ for all $n = 1, 2, \ldots$ Therefore $\overline{\lim}\, x_n = \inf y_n = \infty$.

Result 13 : The extended real number $-\infty$ is the limit superior of $<x_n>$ if and only if $-\infty = \overline{\lim}\, x_n$

Proof : Let $y_n = \sup \{x_n, x_{n+1}, x_{n+2}, \ldots\}$, $n = 1, 2, \ldots$

Let $-\infty = \overline{\lim}\, x_n = \inf_n y_n$. Let Δ be a real number. Since $-\infty = \inf_n y_n$, Δ is not a lower bound for $\{y_1, y_2, \ldots\}$. We get a positive integer N such that $y_N < \Delta$. So $x_k < \Delta$ for all $k \geq N$. Therefore $\lim x_n = -\infty$.

Conversely suppose that $\lim x_n = -\infty$ we have $y_1 \geq y_2 \geq y_3 \ldots$

Let Δ be a real number. Since $\lim x_n = -\infty$, we get a positive integer N such that $x_n < \Delta$ for all $n \geq N$. So $y_N < \Delta$. Therefore, no real number is a lower bound for $\{y_1, y_2, \ldots y_n, \ldots\}$ and that

$-\infty = \inf_{n} y_n = \overline{\lim}\, x_n$.

Similarly we get the following.

Result 14 : A real number l is the limit inferior of the sequence $<x_n>$ if and only if.

1. Given $\in > 0$, there exists a positive integer N such that $x_n > l - \in$ for all $n \geq N$.
2. Given $\in > 0$ and given a positive integer n, there exists an integer $k \geq n$ such that $x_k < l + \in$.

Result 15 : The extended real number ∞ is the limit inferior of the sequence $<x_n>$ if and only if $\infty = \lim x_n$.

Result 16 : The extended real number $-\infty$ is the limit inferior of the sequence $<x_n>$ if and only if given a real number Δ and a positive integer n, there is an integer $k \geq n$ such that $x_k < \Delta$.

Result 17 : $\overline{\lim}\, x_n$ and $\underline{\lim}\, x_n$ are the largest and smallest cluster points of the sequence $<x_n>$.

Proof: Let $l = \overline{\lim}\, x_n$.

Case I : Suppose that l is a real number. By result 11 (conditions 1 & 2) l is a cluster point of $<x_n>$. Let p be a cluster point of the sequence $<x_n>$ and $l < p$. If $p = \infty$ then it contradicts condition 2 in result 11. Therefore p is a real number. We get a $\in > 0$ such that $l + \in < p - \in < p$. By condition 2 of result 11, we get a positive integer k such that $x_n < l + \in < p - \in$, for all $n \geq k$. This is a contradiction to our assumption that p is a real number and is a cluster point of $<x_n>$. Therefore $p \leq l$.

Case II : Suppose that $l = \infty$. By result 12, ∞ is a cluster point of $<x_n>$. Hence $l = \infty$ is the largest cluster point of $<x_n>$.

Case III: Suppose that $l = -\infty$. By result 13, $-\infty = \lim x_n$. So $-\infty$ is the only cluster point of $\langle x_n \rangle$. Hence $l = -\infty$ is the largest cluster point of $\langle x_n \rangle$.

Similarly we get the $\underline{\lim}\, x_n$ is the smallest cluster point of $\langle x_n \rangle$.

Result 18 : $\underline{\lim}\, x_n \le \overline{\lim}\, x_n$, where $\langle x_n \rangle$ is a sequence of real number.

Proof : Proof follows from result 17.

Result 19 : Let $\langle x_n \rangle$ be a sequence of real numbers. Then $\underline{\lim}\, x_n = \overline{\lim}\, x_n = l$ if and only if $l = \lim x_n$.

Proof: Suppose that $\underline{\lim}\, x_n = \overline{\lim}\, x_n = l$. Let $\in > 0$.

Case I : Suppose that l is a real number. By condition 1 of result 11 & result 14. We get a positive integer N such that $|l - x_n| < \in$ for all $n \geq N$. So $\lim\limits_{n} x_n = l$.

Case II: Suppose $l = \infty$. By result 15, $\lim x_n = \infty$

Case III : Suppose $l = -\infty$. By result 13, $\lim x_n = -\infty$

Conversely suppose that $l = \lim x_n$. l is the only cluster point of $\langle x_n \rangle$. Therefore by result 17, $\overline{\lim}\, x_n = l = \underline{\lim}\, x_n$

Definition 20 : We say that a sequence (or series) $\langle x_n \rangle$ is summable to the real number s or has a sum s if the sequence $\langle s_n \rangle$ defined by $s_n = \sum\limits_{k=1}^{n} x_k$ has s as a limit. In this case we write $s = \sum\limits_{k=1}^{n} x_k$.

Definition 21 : Let $< x_n >$ and $< y_n >$ be sequences of real numbers.

Then $\underline{\lim}\, x_n + \underline{\lim}\, y_n \le \underline{\lim}\, (x_n + y_n) \le \overline{\lim}\, x_n + \underline{\lim}\, y_n$

$$\le \overline{\lim}\, (x_n + y_n) \le \overline{\lim}\, x_n + \overline{\lim}\, y_n,$$

provided no sum is of the form $\infty + -\infty$.

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LESSON-1 : ALGEBRA OF SETS, THE EXTENDED REAL NUMBERS, BOREL SETS

1.1. Introduction :

The present lesson is devoted to a review and systematization of those results which will be useful later. We define algebra and σ-algebra of subsets of an arbitrary set X and study some of their properties. The axiom of choice and infinite direct products are also explained Partial ordering and linear ordering are defined. Moreover, the maximal principle and the related concepts are discussed.

The extended real numbers and their use in defining Sups and Infs for all subsets S of real numbers are discussed. We introduce a class of sets in IR called Borel sets and also establish some of its properties. The limit superior and limit inferior of a sequence of real numbers are defined and some of their properties are studied.

1.2 Algebra of Sets :

We define algebra of sets and σ-algebra of subsets of an arbitrary set X.

1.2.1 Definition :

A non-empty collection $\mathcal{A}$ of subsets of X is called an algebra of sets or a Boolean algebra if

(i) $A \in \mathcal{A}, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ and

(ii) $A \in \mathcal{A} \Rightarrow \tilde{A} \in \mathcal{A}$.

Where $\tilde{A}$ is the complement of A in X, that is $\tilde{A} = X - A$.

1.2.2. Examples :

(1) Let X be any non-empty set. The collections $\{\phi, X\}$ and $\mathcal{P}(X) = \{E : E \subseteq X\}$ are trivial the power set of X, are trial examples of algebras of subsets of X.

(2) Let X be any non-empty set. Let $\mathcal{A} = \{A \subseteq X\}$ either X-A is finite. Then $\mathcal{A}$ is an algebra of subsets of X. In case X is a finite set then $\mathcal{A} = \mathcal{P}(X)$ and hence it is clearly an algebra of subsets of X. Suppose X is not finite, clearly $\phi, X \in \mathcal{A}$ and if $A \in \mathcal{A}$, $\tilde{A} \in \mathcal{A}$. Finally suppose $A, B \in \mathcal{A}$

If both A and B are finite then AUB is finite and hence AUB $\in \mathcal{A}$. If $\tilde{A}$ and $\tilde{B}$ are finite then $\widetilde{A \cup B} = \tilde{A} \cap \tilde{B}$ is finite and hence AUB $\in \mathcal{A}$. If either $\tilde{A}$ is finite or $\tilde{B}$ is finite then obviously $\widetilde{A \cup B} = \tilde{A} \cap \tilde{B}$ is finite and hence, AUB $\in \mathcal{A}$.

1.2.3. Self Assessment Question :

Let $\mathcal{A}$ be an algebra of subsets of X

(i) Then $\phi, X \in \mathcal{A}$

(ii) $A \in \mathcal{A}, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$.

(iii) If $A_1, A_2, \ldots, A_n$ are sets in $\mathcal{A}$. Then $A_1 \cup A_2 \cup \ldots A_n$ is again in $\mathcal{A}$ similarly $A_1 \cap A_2 \cap \cdots \cap A_n$ is in $\mathcal{A}$

1.2.4. Proposition :

Given any collection $\mathscr{C}$ of subsets of X there is a smallest algebra $\mathcal{A}$ which contains $\mathscr{C}$; that is there is an algebra $\mathcal{A}$ containing $\mathscr{C}$ and such that if $\mathscr{B}$ is any algebra containing $\mathscr{C}$ then $\mathscr{B}$ contains $\mathcal{A}$.

Proof: Let $\mathscr{C}$ be a collection of subsets of X. Let $\mathscr{F} = \{ \mathscr{B} / \mathscr{B}$ is an algebra of subsets of X and $\mathscr{C} \subseteq \mathscr{B}\}$. We know that, $\mathscr{P}(x)$ is an algebra subsets of X (1.2.2.) and clearly $\mathscr{C} \subseteq \mathscr{P}(x)$. Hence $\mathscr{P}(x) \in \mathscr{F}$, thus $\mathscr{F}$ is non-empty. Let $\mathcal{A} = \cap \{\mathscr{B} : \mathscr{B} \in \mathscr{F}\}$. We will show that $\mathcal{A}$ is the smallest algebra of subsets of X containing $\mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{B}$, for all $\mathscr{B} \in \mathscr{F}$. ϕ, X $\in \mathcal{A}$. Let A, B $\in \mathcal{A}$ then for each $\mathscr{B} \in \mathscr{F}$, we have $A \in \mathscr{B}$ and $B \in \mathscr{B}$. Since $\mathscr{B}$ is an algebra, $A \cup B$ belongs to $\mathscr{B}$ Since this is true for every $\mathscr{B} \in \mathscr{F}$, we have $A \cup B$ is in $\bigcap_{\mathscr{B} \in \mathscr{F}} \mathscr{B}$. Similarly, we see that if $A \in \mathcal{A}$, then $\tilde{A} \in \mathcal{A}$. Hence $\mathcal{A}$ is an algebra of subsets of X. If $\mathscr{B}$ is an algebra containing $\mathscr{C}$ then from the definition of $\mathcal{A}$ it follows that $\mathscr{B}_0 \subseteq \mathcal{A}$. Hence $\mathcal{A}$ is the smallest algebra of subsets of X containing $\mathscr{C}$.

1.2.5. Self Assessment Question :

(i) Let, $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of algebras of subsets of a set X and let $\mathcal{A} = \bigcap_{\alpha \in \Delta} \mathcal{A}_\alpha$ show that $\mathcal{A}$ is also an algebra of subsets of X.

(ii) Let $\{\mathcal{A}_n\}_{n\geq 1}$ be a sequence of algebras of subsets of a set X. Under what circumstances can you conclude that $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ is also an algebra ?

1.2.6. Definition : Let $\mathscr{C}$ be a collection of subsets of X. The smallest algebra of subsets of X containing $\mathscr{C}$ is called the algebra generated by $\mathscr{C}$.

1.2.7. Proposition : Let $\mathcal{A}$ be an algebra of subsets and $\{A_n\}$ a sequence of sets in $\mathcal{A}$. Then there is a sequence $\{B_n\}$ of sets in $\mathcal{A}$ such that $B_n \cap B_m = \phi$ for $n \neq m$ and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$.

Proof : Let $B_1 = A_1$ and for each integer $n > 1$, $B_n = A_n - (A_1 \cup A_2 \cup \ldots\ldots \cup A_{n-1}) = A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots\ldots \cap, \tilde{A}_{n-1}$. Since $\mathcal{A}$ is an algebra and $A_1, A_2, \ldots., A_n \in \mathcal{A}$, we have that $\tilde{A}_1, \tilde{A}_2, \ldots\ldots \tilde{A}_{n-1} \in \mathcal{A}$. Therefore $B_n = A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots\ldots \cap \tilde{A}_{n-1} \in \mathcal{A}$. Clearly $B_m \subseteq A_m$ for all m = 1,2,3,....... Let m and n be positive integers and m < n.

Now $B_m \cap B_n \subseteq A_m \cap B_n$

$$= A_m \cap A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots\ldots \cap \tilde{A}_m \ldots \cap \tilde{A}_{n-1}.$$

$$= (A_m \cap \tilde{A}_m) \cap (A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots\ldots \cap \tilde{A}_{m-1} \cap \tilde{A}_{m+1} \cap \ldots \cap \tilde{A}_{n-1})$$

$$= \phi \cap (A_n \cap \tilde{A}_1 \cap \tilde{A}_2 \cap \ldots\ldots \cap \tilde{A}_{m-1} \cap \tilde{A}_{m+1} \cap \ldots \cap \tilde{A}_{n-1}) = \phi.$$

Therefore $B_m \cap B_n = \phi$ for $m \neq n$.

Since $B_m \subseteq A_m$, $\bigcup_{m=1}^{\infty} B_m \subseteq \bigcup_{m=1}^{\infty} A_m$ .... (1)

Let $x \in \bigcup_{m=1}^{\infty} A_m$ for some positive integer k, $x \in A_k$.

Let n be the smallest positive integer such that $x \in A_n$.

So $x \in A_n$ and $x \notin \bigcup_{i=1}^{n-1} A_i$. Therefore $x \in B_n$ and that $x \in \bigcup_{m=1}^{\infty} B_m$

Thus, $\bigcup_{m=1}^{\infty} A_m \subseteq \bigcup_{m=1}^{\infty} B_m$ .... (2)

From (1) & (2) $\bigcup_{m=1}^{\infty} B_m = \bigcup_{m=1}^{\infty} A_m$.

Note : In the proof of proposition 1.2.7 we can also observe that $\bigcup_{m=1}^{n} B_m = \bigcup_{m=1}^{n} A_m$ for all n=1,2...

1.2.8 Definition : An algebra $\mathcal{A}$ of sets is called a σ-algebra or a Borel field if every union of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$. That if $\{A_n\}$ is a sequence of sets then $\bigcup_{n=1}^{\infty} A_n$ must again be in $\mathcal{A}$.

1.2.9. Self Assessment Question :

Let $\mathcal{A}$ be a σ-algebra of sets. Then prove that the intersection of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$.

1.2.10 Proposition : Given any collection $\mathcal{C}$ of subsets of X, there is a smallest σ-algebra of sets of X that contains $\mathcal{C}$, that is, there is a σ-algebra $\mathcal{A}$ of subsets of X, containing $\mathcal{C}$ such that if $\mathcal{B}$ is any σ-algebra of subsets of X, containing $\mathcal{C}$, then $\mathcal{A} \subseteq \mathcal{B}$.

Proof : Let $\mathcal{C}$ be a collection of subsets of X.

Let $\mathcal{F} = \{\mathcal{B} / \mathcal{B}$ is a σ-algebra of subsets of X and $\mathcal{C} \subseteq \mathcal{B}\}$.

Let D be the collection of all subsets of X. Clearly D is a σ-algebra containing $\mathcal{C}$. So D $\in \mathcal{F}$ and that $\mathcal{F}$ is non empty.

Let $\mathcal{A} = \bigcap_{B \in F} \mathcal{B}$. Since $\mathcal{C} \in \mathcal{B}$ for all $\mathcal{B} \in \mathcal{F}$, $\mathcal{C} \subseteq \bigcap_{\mathcal{B} \in \mathcal{F}} \mathcal{B} = \mathcal{A}$.

Since $\phi, x \in \mathscr{B}$ for all $\mathscr{B} \in \mathscr{F}$, $\phi, X \in \bigcap_{\mathscr{B} \in \mathscr{F}} \mathscr{B} = \mathcal{A}$.

Therefore $\mathcal{A}$ is a non-empty collection of sets of X containing $\mathscr{C}$. Let $A \in \mathcal{A}$.

1. $A \in \mathscr{B}$ for all $\mathscr{B} \in F$. Since $\mathscr{B} \in F$ is a σ-algebra $\tilde{A} \in \mathscr{B}$ for all $\mathscr{B} \in F$. Therefore $\tilde{A} \in \bigcap_{\mathscr{B} \in \mathscr{F}} \mathscr{B} = \mathcal{A}$.

2. Let $\{A_n\}$ be a sequence of sets in $\mathcal{A}$ now $\{A_n\}$ is a sequence of sets in $\mathscr{B}$ for all $\mathscr{B} \in \mathscr{F}$. So $\bigcup_n A_n \in \mathscr{B}$ for all $\mathscr{B} \in \mathscr{F}$ Therefore $\bigcup_n A_n \in \bigcap_{\mathscr{B} \in \mathscr{F}} \mathscr{B} = \mathcal{A}$.

So $\mathcal{A}$ is a σ-algebra of subsets of X containing $\mathscr{C}$.

Let $\mathscr{B}'$ be a σ-algebra of subsets of X containing $\mathscr{C}$.

By the definition of $\mathscr{F}$, $\mathscr{B}' \in F$. Therefore $\mathcal{A} = \bigcap_{\mathscr{B} \in \mathscr{F}} \mathscr{B} \subseteq \mathscr{B}'$.

Hence $\mathcal{A}$ is the smallest σ-algebra of subsets of X containing $\mathscr{C}$.

1.2.11 Proposition : Let $\mathscr{C}$ be a collection of subsets of X. The smallest σ-algebra $\mathscr{C}$ is called the σ-algebra generated by $\mathscr{C}$.

1.2.12 Proposition : If $\mathcal{A}$ is the algebra generated by a collection $\mathscr{C}$ of subsets of X then $\mathcal{A}$ and $\mathscr{C}$ generate the same σ-algebra.

Proof : Let $\mathscr{C}$ be a collection of subsets of X and $\mathcal{A}$ the algebra generated by $\mathscr{C}$. Let $\mathscr{B}_1$ & $\mathscr{B}_2$ be the σ-algebras generated by $\mathscr{C}$ and $\mathcal{A}$ respectively.

As $\mathscr{C} \subseteq \mathscr{B}_1$, $\mathscr{B}_1$ is an algebra containing $\mathscr{C}$. So $\mathcal{A} \subseteq \mathscr{B}_1$.

Therefore $\mathscr{B}_1 \subseteq, \mathscr{B}_2$ as $\mathscr{B}_2$ is the smallest σ-algebra containing $\mathcal{A}$.

Since $\mathcal{A} \subseteq \mathscr{B}_1$ of $\mathscr{C} \subseteq \mathscr{B}_2$.

Therefore $\mathscr{B}_1 \subseteq, \mathscr{B}_2$ as $\mathscr{B}_1$ is the smallest σ-algebra containing $\mathscr{C}$.

Hence $\mathscr{B}_1 = \mathscr{B}_2$. So $\mathcal{A}$ and $\mathscr{C}$ generate the same σ-algebra.

1.2.13. Proposition : Let $\mathscr{C}$ be a collection of sets and E an element in the σ-algebra generated by $\mathscr{C}$ Then there is a countable sub collection $\mathscr{C}_0 \subseteq \mathscr{C}$ such that E is an element of the σ-algebra $\mathcal{A}_0$ generated by $\mathscr{C}_0$.

Proof :

Let $\mathscr{C}$ be a collection of sets and E an element in the σ-algebra generated by $\mathscr{C}$.

Let $\mathscr{F} = \{\mathcal{A}_\alpha / \alpha \in \Delta\}$ be the collection of all σ-algebras generated by countable subsets of $\mathscr{C}$.

Let $\mathcal{A}^1 = \bigcup_{\alpha \in \Delta} \mathcal{A}_\alpha$. Let $\mathcal{A}$ be the σ-algebra generated by $\mathscr{C}$.

We claim that $\mathcal{A}^1$ is also a σ-algebra of sets and $\mathscr{C} \subset \mathcal{A} \subset \mathcal{A}^1$.

Obviously $\mathcal{A}^1$ is a non-empty collection of sets.

1. Let $\{A_n\}$ be a countable collection of sets in $\mathcal{A}'$. we may assume that $A_n \in \mathcal{A}_{\alpha_n}$ $\alpha_n \in \Delta$ for all n. Suppose that $\mathcal{A}_{\alpha_n}$ be the σ-algebra generated by a countable collection $\{B_{kn}\}_k$, $B_{kn} \in \mathscr{C}$. Now $\{B_{kn}\}_{k,n}$ is also a countable collection of sets in $\mathscr{C}$. Let $\mathcal{A}_{\mathscr{B}}$ be the σ-algebra generated by $\{B_{kn}\}_{k,n}$. Now $\mathcal{A}_{\mathscr{B}} \in \mathscr{F}$ and $\mathcal{A}_{\alpha_n} \subseteq \mathcal{A}_{\mathscr{B}}$. Since $\mathcal{A}_{\mathscr{B}}$ is a σ-algebra, $\bigcup_n \mathcal{A}_{\mathscr{B}} \in \mathcal{A}_{\mathscr{B}} \subseteq \mathcal{A}'$.

1.2.14 Proposition :

Let $\mathscr{C} = \{X_\lambda\}$ be a collection of sets indexed by a (nonempty) set Δ. We define the direct product $\underset{\lambda}{X} X_\lambda$ to be the collection of all sets $\{X_\lambda\}$ induced by Δ and having the property that $x_\lambda \in X_\lambda$.

If one of the X_λ is empty then $\underset{\lambda}{X} X_\lambda$ is also empty. The axiom of choice is equivalent to the converse statement : if none of the X_λ are empty, then $\underset{\lambda}{X} X_\lambda$ is not empty.

1.2.15 Self Assessment Question :

Let $f : X \to Y$ be a mapping onto Y then there is a mapping $g : Y \to X$ such that fog is the identity map on Y.

1.3. PARTIAL ORDERINGS AND THE MAXIMAL PRINCIPLE

Let X be a nonempty set. A subset R of XxX is called a relation on X. Let $x, y \in X$ and R is a relation on X. Then we write xRy if $(x,y) \in R$.

1.3.1. Definition :

Let R be a relation on a set X.

1. R is said to be reflexive on X if for all $x \in X$ we have xRx
2. R is said to be antisymmetric on X if xRy and yRx imply $x = y$ for all $x, y \in X$.
3. R is said to be transitive on X if xRy and yRz imply xRz for all x, y and $z \in X$.

1.3.2. Definition :

A relation $\prec$ is said to be a partial ordering of a set X if it is transitive and antisymmetric.

So $\leq$ is a partial ordering on the real numbers, where $\leq$ is the natural order on the real numbers i.e., if x and y are real numbers then $x \leq y$ if and only if $y - x$. is non negative.

Let X be a non-empty set and $\mathscr{P}(X)$ be the set of all subets of X. Now $\leq$ is partial ordering on $\mathscr{P}(X)$, where for any A, B $\in \mathscr{P}(X)$, $A \leq B$ if and only if A is a subset of B.

1.3.3. Definition : A partial ordering $\prec$ on a set X is said to be a linear ordering (or simple ordering) of X if for any two elements x and y in X we have either $x \prec y$ or $y \prec x$.

The partial order $\leq$ on the set of real numbers seen above is a linear ordering.

The partial ordering $\leq$ on $\mathscr{P}(X)$, X is a non-empty set containing more than one element seen above is not a linear ordering.

1.3.4. Definition : Let $\prec$ be a partial order on X. We say that $\prec$ is a reflexive partial order if $x \prec x$ for all $x \in X$. $\prec$ is calld a strict partial order if for no $x \in X$, $x \prec x$.

So $\prec$ is a strict partial order on the set of real numbers and $\leq$ is a reflexive partial order on the set of real numbers, where for any two real numbers x and y, $x < y$ if and only if y - x is a positive real number.

1.3.5. Definition : Let $\prec$ be a partial order on X. Let E be a subset of X. We say that an element $a \in E$ is the first element in E or the smallest element in E if, when ever $x \in E$ and $x \neq a$, we have $a \prec x$.

An element $a \in E$ is called the last (or largest) element in E if, when ever $x \in E$ and $x \neq a$, we have $x \prec a$.

An element $a \in E$ is called a minimal element in E if there is no $x \in E$ with $x \neq a$ and $x \prec a$.

An element $a \in E$ is called a maximal element in E if there is no $x \in E$ with $x \neq a$ and $a \prec x$.

Remark : Let $\prec$ be a partial order on X and E be a subset of X. If $a \in E$ is the smallest element in E then a is a minimal element of E. Also if $b \in E$ is the largest element in E then b is a maximal element of E. If $\prec$ is a linear order on X and $a \in X$ is a minimal element of X then a is the least element of X.

The following principle is equivalent to the axiom of choice and is often more conveninent to apply.

1.3.7. Hausdorff Maximal Principle : Let $\prec$ be a partial ordering on a set X. Then there is a maximal linearly ordered subset S of X, that is, a subset S of X which is linearly ordered by $\prec$ and has the property that if $S \subseteq T \subseteq X$ and T is linearly ordered by $\prec$, then $S = T$.

1.3.8. Proposition :

Let $\prec$ be a partial order on X. Then there is a unique strict partial order $\prec$ and a unique reflexive partial order $\leq$ on X such that for $x \neq y$ we have $x \prec y \Leftrightarrow x<y \Leftrightarrow x \leq y$.

Proof: Let $\prec$ be a partial order on X.

1. For $x,y \in X$, define $x < y$ if and only if $x \neq y$ and $x \prec y$.

   (a) By our definition for no $x \in X$, $x < x$.

   (b) Let $x,y \in X$ and $x < y$, $y < x$.
   So, $x \neq y$ and $x \prec y$ and $y \prec x$.
   But $x \prec y$ and $y \prec x \Rightarrow x = y$, a contradiction to $x \neq y$.
   Therefore $x < y$ and $y < x$ can't happen simulteniously .

   (c) Let $z, x, y \in X$ and $x<y$ and $y<z$. So $x \prec y$ and $y \prec z$ and $x \neq y, y \neq z$. Now $x \prec y$ & $y \prec z \Rightarrow x \prec z$. If $x = z$ then we get that $x \prec y$ and $y \prec x$ and that $x = y$ a contradiction. So $x \neq z$. Therefore $x < z$.

Hence $<$ is a strict partial order on X such that for $x \neq y$ we have $x \prec y \Leftrightarrow x<y$.

Let $<'$ be a strict partial order on X such that for $x \neq y$ we have $x \prec y \Leftrightarrow x<^{1}y$.

1. Let $x,y \in X$ and $x \neq y$. $x<^{1}y. \Leftrightarrow x \prec y \Leftrightarrow x < y$.
   Therefore the strict partial orders $<'$ and $<$ are same.
   Hence $<$ is the unique strict partial order on X such that for $x \neq y$ we have $x \prec y \Leftrightarrow x<y$.

2. For $x,y \in X$ define $x \leq y$ if and only if either $x = y$ or $x \prec y$.

   (a) From the definition $x \leq x$ for all $x \in X$

   (b) Let $x,y \in X$ and $x \leq y$, $y \leq x$.
   now either $x = y$ or $x \prec y$ and $y \prec x$.
   But $x \prec y$ of $y \prec x \Rightarrow x = y$, therefore $x = y$.

   (c) Let $z, x, y \in X$ and $x \leq y$ and $y \leq z$ .Now either $x = y$ & $y = z$ or $x = y$ & $y \prec z$ or $x \prec y$ & $y = z$ or $x \prec y$ & $y \prec z$. In all the cases we get that $x \leq z$.

Therefore $\leq$ is a reflexive partial order on X such that for $x \neq y, x \leq y \Leftrightarrow x \prec y$.

Suppose that $\leq'$ is a reflexive partial order on X such that for $x \neq y, x \leq' y \Leftrightarrow x \prec y$.

If $x \in X$ then clearly $x \leq x$ & $x \leq' x$.

Let $x, y \in X$ and $x \neq y$, $x \leq' y \Leftrightarrow x \prec y \Leftrightarrow x \leq y$.

Therefore the reflexive partial order $\leq$ & $\leq'$ are the same.

Hence $\leq$ is the unique reflexive partial order on X such that for $x \neq y$ we have $x \leq y \Leftrightarrow$ y.

1.4. THE EXTENDED REAL NUMBERS

1.4.1 Definition :

The set of extended real numbers consists of the set of real numbers IR and two symbols, $+\infty$ and $-\infty$.

We extend the definition of < to the extended real numbers by postulating $-\infty < \infty$ and $-\infty < x < \infty$, for each real number x.

We define
$$x + \infty = \infty, \quad x + -\infty = -\infty$$
$$x.\infty = \infty, \quad \text{if } x > 0$$
$$x.-\infty = -\infty, \text{ if } x > 0 \text{ for all real numbers } x$$

and set
$$\infty + \infty = \infty, \quad -\infty + -\infty = -\infty$$
$$\infty.(\pm\infty) = \pm\infty, \quad -\infty(\pm\infty) = \pm\infty.$$

The operation $\infty + -\infty$ is left undefined, but we shall adopt the arbitrary convention that $0.\infty = 0$.

One use of extended real numbers is in the expression "sup S". Let S be a non empty set of real numbers which has an upper bound. We define supS to be the least upper bound of S. We know that supS always exists and is a real number.

Suppose now that S is a non empty set of real numbers which has no upper bound. Then we write $\sup S = \infty$. If S is empty, we define $\sup S = -\infty$.

Therefore if S is a subset of real numbers then sup S is the smallest extended real number which is greater than or equal to each element of S.

Let S be a set of real numbers.

If S is nonempty and has a lower bound, we define inf S to be the greatest lower bound of S. We know that inf S exists and is a real number.

If S is nonempty and has no lower bound, we write $\inf S = -\infty$. If S is empty, we define $\inf S = \infty$.

So one advantage of the extended real numbers is that it enables us to speak of sup S and inf S for all subsets S of the real numbers.

1.4.2. Definition : A function whose values are in the set of extended real numbers is called an extended real valued function.

1.4.3. Result : Show that $\inf E \leq \sup E$ if and only if $E \neq \phi$.

Proof :

Let E be a set of real numbers.

Suppose that $\inf E < \sup E$. We claim that $E \neq \phi$.

On the contrary suppose that $E = \phi$. Now $\sup E = -\infty < \infty = \inf E$.

A contradiction to our assumption that $\inf E \leq \sup E$.

Therefore $E \neq \phi$. Now suppose that $E \neq \phi$. Let $a \in E$.

Clearly $\inf E \leq a$ and $a \leq \sup E$

Therefore $\inf E \leq \sup E$.

1.5 BOREL SETS

We know that the intersection of any collection of closed subsets of real numbers is closed and the union of any finite collection of closed subsets of real numbers is closed. But the union of a countable collection of closed subsets of real numbers need not be closed. For example, the set of rational numbers is the union of a countable collection of closed sets each of which contains exactly one rational number. So, we are interested in σ-algebra of sets that contain all of the closed sets.

1.5.1. Definition : The collection $\mathscr{B}$ of Borel sets is the smallest σ-algebra of sets of real numbers which contains all of the open subsets of real numbers.

1.5.2. Definition : The collection $\mathscr{B}$ of Borel sets is the smallest σ-algebra which contains all of the closed subsets of real numbers.

Proof :

We know that the collection $\mathscr{B}$ of Borel sets is the smallest σ-algebra which contains all of the open sets .

Let $\mathscr{B}_1$ be the σ-algebra generated by the collection of all closed sets .

Let O be an open set. Now $\tilde{O}$ is a closed set. So $\tilde{O} \in \mathscr{B}_1$.

Since $\mathscr{B}_1$ is a σ-algebra, $O = \approx{O} \in \mathscr{B}_1$.

So $\mathscr{B}_1$ contains the collection of all open sets.

Therefore $\mathscr{B} \subseteq \mathscr{B}_1$, as $\mathscr{B}$ is the σ-algebra generated by the collection of all open sets. Let F be a closed set. Now $\tilde{F}$ is an open set.

So $\tilde{F} \in \mathscr{B}$, Since $\mathscr{B}$ is σ-algebra, $F = \tilde{\tilde{F}} \in \mathscr{B}$.

So $\mathscr{B}$ contains the collection of all closed sets.

Therefore $\mathscr{B}_1 \subseteq \mathscr{B}$. Hence $\mathscr{B} = \mathscr{B}_1$.

1.5.3. Self-Assessment Question :

The collection of $\mathscr{B}$ of Borel sets is the smallest σ-algebra which contains all the open intervals.

1.5.4. Definition : A set which is a countable union of closed sets is called an F_σ -set.

Clearly each F_σ -set is in β , the collection of all Borel sets. Obviously each closed set is an F_σ -set . So ϕ &R are F_σ -sets.

Since $(a,b) = \sum_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right]$, $a<b$ we have that the open interval (a,b) is an F_σ.

Now $(-\infty, b) = \bigcup_{n=1}^{\infty}(a-n, b)$ is an F_σ and $(a,\infty) = \bigcup_{n=1}^{\infty}(a, b+n)$ is an F_σ as a countable union of sets in F_σ is again in F_σ.

Therefore each open interval is an F_σ.

Since every non empty open set is a countable union of open intervals we get the each open set is an F_σ.

1.5.5. Definition :

We say that a set is a G_δ if it is the intersection of a countable collection of open sets.

Therefore the complement of an F_σ is a G_δ and conversely.

We also consider sets of type $F_{\sigma\delta}$, which are the intersections of countable collections of sets each of which is an F_σ.

Similarly, we can construct the closses $G_{\delta\sigma}$, $F_{\sigma\delta\sigma}$, , etc.

A set of type $G_{\delta\sigma}$ is the union of a countable collection of sets each of which is a G_δ

A set of type $F_{\sigma\delta\sigma}$ is the union of a countable collection of sets each of which is a $F_{\sigma\delta}$, **Thus the** classes in two sequences

$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, \cdots$
$G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, \cdots$

are all classes of Borel sets. However, not every Borel set belongs to one of these classes.

1.6. Answers to SAQ's :

1.2.5. : See Proof of 1.2.4.

1.2.9 : Let $\{A_n\}$ be a sequence in $\mathcal{A}$. Since $A_n \in \mathcal{A}$, we have that $\tilde{A}_n \in \mathcal{A}$.

Therefore $\bigcup_n \tilde{A}_n \in \mathcal{A}$. So $\left(\widetilde{\bigcup_n A_n}\right) \in \mathcal{A}$.

But $\left(\widetilde{\bigcup_n A_n}\right) = \bigcap_n (\tilde{\tilde{A}}_n) = \bigcap_n A_n \in \mathcal{A}$. Hence $\bigcap_n A_n \in \mathcal{A}$.

1.2.15 : Let $f: X \rightarrow Y$ be a mapping onto Y. For each $y \in Y$, let $A_y = f^{-1}(\{y\}) = \{x \in X / f(x) = y\}$.

Let $\mathcal{A} = \{A_y / y \in Y\}$. Since f is onto Y, A_y is non empty for each $y \in Y$. Therefore, by axiom of choice, $\underset{y \in Y}{X} A_y$ is nonempty.

Let $\{a_y\} \in \underset{y \in Y}{X} A_y$. Define $g: Y \rightarrow X$ by $g(y) = a_y$ for all $y \in Y$.

Clearly g is a mapping. fog is a mapping of Y into Y.
$(fog)(y) = f(g(y)) = f(a_y) = y$, for all $y \in Y$

Therefore fog is the identity map on Y.

We have that the collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra which contains all open sets. Let $\mathcal{B}'$ be the σ-algebra generated by the collection of all open intervals. Since each open interval is an open set, $\mathcal{B}$ is a σ-algebra containing all the open intervals. Therefore $\mathcal{B}' \subseteq \mathcal{B}$. It is clear that $\phi \in \mathcal{B}'$. Since each nonempty open set is a countable union of open intervals, $\mathcal{B}'$ contains the collection of all nonempty open sets. Therefore $\mathcal{B}'$ contains the collection of all open sets. So $\mathcal{B} \subseteq \mathcal{B}'$. Hence $\mathcal{B} = \mathcal{B}'$.

1.7. Model Examination Questions :

1. Define an algebra of sets. If $\mathcal{A}$ is an algebra of sets and $<A_i>$ is a sequence of sets in $\mathcal{A}$, then prove that there is a sequence $<B_i>$ of sets in $\mathcal{A}$ such that $B_n \cap B_m = \phi$ for $n \neq m$ and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i .$$

2. Define a σ-algebra of sets. Given any collection $\mathscr{C}$ of subsets of a set X, there is a smallest σ-algebra that contains $\mathscr{C}$.

3. State Hausdorff maximal principle. If $\prec$ is a partial order on X, then prove that there is a unique strict partial order $<$ and a unique reflexive partial order $\leq$ on X such that for $x \neq y$ we have $x \prec y \Leftrightarrow x < y \Leftrightarrow x \leq y$.

4. Define the collection $\mathscr{B}$ of Borel sets. Also define a G_δ – set and F_σ – set of real numbers. Prove that the collection $\mathscr{B}$ of Borel sets is the smallest σ-algebra that contains all open intervals.

1.8. Exercises :

Given an example of an algebra of sets which is not a σ-algebra.

[**Hint :** Let N be the set of positive integers. $\mathcal{A} = \{A \subseteq N / \text{either } A \text{ is finite or } N-A \text{ is finite}\}$

$\mathcal{A}$ is an algebra of sets as seen in 1.7 but not a σ-algebra.]

1.9 : Reference Book :- Real Analysis, H.L. Royden

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LESSON :2 LEBESGUE OUTER MEASURE AND IT'S PROPERTIES

2.1 INTRODUCTION :

The concept of Lebesgue measure which is basic to the theory of Lebesgue Integration arose in an attempt to assign the notion of length in $\mathbb{R}$ to more general sets than the finite intervals.

Mathematically we want to define the notion of a length in $\mathbb{R}$ to a large class of sets such that this class contains the intervals in $\mathbb{R}$ in such a way that the definition give back the familiar notion of length to the intervals. The notion thus defined is called the measure of the set.

Lebesgue outer measure (hereafter called outer measure) of an arbitrary set of real numbers is introduced and measurability of a set is defined via this outer measure. A number of properties of the outer measure viz countable sub-addivity, translation invariance are established and proved that for any interval I in $\mathbb{R}$ $m^*(I)$ is same as the length of I.

2.2. Set Functions :

An extended real valued function defined on a class ς of sets is called a set function. We consider the set functions defined on a class of subsets of the real number system $\mathbb{R}$. We would like to construct a set function m that assigns to each set E in some collection $\mathscr{M}$ of sets of real numbers a non-negative extended real number mE called the measure of E. We should like m to have the following properties.

(i) mE is defined for each set E of real numbers $\mathscr{M} = \mathscr{P}(\mathbb{R})$,

(ii) For an interval I, $mI = l(I)$

(iii) If {En} is a sequence of disjoint sets (for which m is defined), $m\left(\bigcup_n E_n\right) = \sum_n m(E_n)$

(iv) m is translation invariant, that is, if E is a set for which m is defined and if E+y is the set $\{ x + y : x \in E \}$ then m (E+y) = mE.

However, it is known that there are no set functions satisfying all the above four conditions. Consequently, one of these properties must be weakened, and it is most useful to retain the last three properties and to weaken the fIRst condition so that mE to be defined for as many sets as possible and will find it convenient to require the family $\mathscr{M}$ of sets for which m is defined to be a σ-alegebra (Definition 2.2.7).

Definition 2.2.1 :

Let m be a set function defined on a σ-alegebra $\mathcal{A}$. Then m is said to be countably additive if $m(A) \geq 0$ for all $A \in \mathcal{A}$ and for each sequence $\{A_n\}$ of pairwise disjoint members of $\mathcal{A}$.

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$$

Definition 2.2.2. :

A set function m defined on a σ-alegebra $\mathcal{A}$ is said to be countably subadditive if $m(A) \geq 0$ for all $A \in \mathcal{A}$ and for any sequence $\{A_i\}$ in $\mathcal{A}$.

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i)$$

Definition 2.2.3 :

A set function m defined on a σ-alegebra $\mathcal{A}$ is said to be finitely additive if $m(A) \geq 0$ for all $A \in \mathcal{A}$ and for each paIR A, B of disjoint members of $\mathcal{A}$, $m(AUB) = m(A) + m(B)$.

2.2.4 : Self Assessment Question :

Show that any countably additive set function defined on a σ-alegebra is countably sub additive and also finitely additive.

2.2.5 : Self Assessment Question :

Give an example of a countably sub-additive set function defined on a σ-alegebra m is not countably additive

Theorem 2.2.6 :

Let m be a countably additive set function defined on a σ – algebra $\mathcal{A}$. Then the following hold

(i) If $A \subseteq B$ and A and B $\in \mathcal{A}$, then $m(A) \leq m(B)$

(ii) If there is a set $A \in \mathcal{A}$ such that $m(A) < \infty$ then $m(\phi)=0$.

Proof:

(i) Let $A, B \in \mathcal{A}$ and $A \subseteq B$. Then $B\text{-}A = B \cap X\text{-}A \in \mathcal{A}$ and $A \cup (B\text{-}A) = B$ and $A \cap (B\text{-}A) = \phi$
By the countably additivity of m, we have $m(B) = m(A) + m(B\text{-}A) \geq m(A)$ since, $m(B\text{-}A) \geq 0$

(ii) Let $A \in \mathcal{A}$ and $m(A) < \infty$ then $m(A) = m(A) + m(\phi) + m(\phi) + \ldots\ldots$
(consider $A = \bigcup_i A_i$ where $A_1 = A$, $A_i = \phi$ for $i > 1$) Since $m(\phi) \geq 0$ it follows that $m(\phi) = 0$.

2.2.7. Definition :

A non negative extended real valued countably additive set function defined on a σ-algebra is called a measure.

2.2.8. Example :

Let X be any set and $\mathcal{A} = \mathcal{P}(x)$, the class of all subsets of X. Define for any $A \in \mathcal{A}$

$$m(A) = \begin{cases} +\infty & \text{if A is infinite} \\ |A| & \text{if A is finite} \end{cases}$$

Where $|A|$ is the number of elements in A. Then m is a countably additive set function defined on $\mathcal{A}$ and m is a measure.

2.3. LEBESGUE OUTER MEASURE

We shall construct a set function satisfying almost all properties mentioned in the begining of the previous section. We begin with the following.

2.3.1. Definition :

The length of a finite interval I, with end points a,b with $a < b$ is defined to be $l(I) = b\text{-}a$ and if I is an infinite interval then $l(I) = \infty$. Thus $l(I)$ has the following properties.

(i) $l(I) > 0$
(ii) $l(I \cup J) \leq l(I) + l(J)$ if I, J and $I \cup J$ are intervals
(iii) If $I \subseteq J$ then $l(I) \leq l(J)$
(iv) $l(I+\alpha) = l(I)$ for every $\alpha \in \mathbb{R}$.

2.3.2. Definition :

For any set A of real numbers, define

$$m^*(A) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and each } I_n \text{ is an open interval} \right\}$$

The set function m* defined on $\mathscr{P}(\mathbb{R})$ is called the Lebesgue Outer Measure. For any $A \subseteq \mathbb{R}$ m*(A) is called the outer measure of A..

2.3.3. Remark :

(1) A countable collection $\{I_n\}$ of open intervals is said to be a cover for the set E if $E \subseteq \bigcup_{n=1}^{\infty} I_n$

(2) If A is a set of real numbers, {(-n,n)} is a countable collection of open intervals that covers A as $A \subseteq \mathbb{R} = \bigcup_{n=1}^{\infty}(-n,n)$ and that such a collection always exists. since lengths of the intervals and positive real numbers the sum of the lengths of the intervals $\sum_n l(I_n)$ is uniquely defined independently of the order of the terms. Thus we define the outer measure m*(A) of A to be the infimum of all such sums.

(3) For notational convenience we need only deal with countable coverings of A, the finite case is included since we may take $I_n = \phi$ except for a finite number of integers n.

We now obtain some elementary properties of the outer measure.

2.3.4. Theorem : a) Non-negativity : $m^*(A) \geq 0$ for all $A \subseteq \mathbb{R}$
b) Monotonicity : If $A \subseteq B$ then $m^*(A) \leq m^*(B)$
c) $m^*(\phi) = 0$
d) $m^*(\{a\}) = 0$ for any $a \in \mathbb{R}$
e) Translation invariance : $m^*(A+x) = m^*(A)$ for any $x \in \mathbb{R}$.

Proof :

(a) For any set A, $m^*(A) \geq 0$ since the outer measure is the infimum of a set of non-negative numbers.

(b) If $A \subseteq B$ then every cover $\{I_n\}$ of B is a cover for A so that for such cover

$m^*(A) \leq \sum_{n=1}^{\infty} l(I_n)$. That is, $m^*(A)$ is a lower bound for the set

$\left\{\sum_{n=1}^{\infty} \ell(I_n) : B \subseteq \sum_{n=1}^{\infty} (I_n)\right\}$. Therefore, $m^*(A) \leq \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n\right\} = m^*(B)$

(c) By (a) $m^*(\phi) \geq 0$ and for any $\in > 0$ the interval $I = (a, a+\in)$ is a cover for ϕ and so $m^*(\phi) \leq l(I) = \in$ showing $m^*(\phi) < \in$ for each $\in > 0$ Hence, $m^*(\phi) = 0$

(d) For any $\in > 0$ then open interval $I = \left(x - \frac{\in}{2}, x + \frac{\in}{2}\right)$ is a cover for $\{x\}$ so that $m^*(\{x\}) \leq l(I) = \in$. This together with (a) give $m^*(3x\}) = 0$

(e) For each $\in > 0$ there exists a collection $\{I_n\}$ such that $A \subseteq \cup I_n$ and $m^*(A) \geq \sum \ell(I_n) - \in$
But clearly $A + x \subseteq \cup(I_n + x)$. So, for each $\in$,
$m^*(A+x) \leq \sum l(I_n+x) = \sum l(I_n) \leq m^*(A) + \in$. So, $m^*(A+x) \leq m^*(A)$. But, $A = (A+x) - x$ so we have $m^*(A) \leq m^*(A+x)$. Thus, $m^*(A) = m^*(A+x)\ \forall x \in \mathbb{R}$.

We now prove that the outer measure m^* is an extension of the set function l (length) i.e., $m^*(I) = \ell(I)$ for any interval I.

We now prove that the outer measure m^* is an extention of l i..e,$m^*(I)=l(I)$ for all intervals I.

2.3.5. Proposition : The outer measure of an interval is its length

Proof : Let I be an interval.

Case I :

Suppose that I is a closed and finite interval.

Now $I = [a,b]$ for some real numbers a and b, $a<b$.

Let $\in > 0$. $[a,b] \subseteq (a - \in, b + \in)$. So $m^*([a,b]) \leq l((a-\in, b+\in)) = (b-a) + 2\in$. Since $\in > 0$ is arbitrary, $m^*([a,b]) \leq b-a$.

We prove now that $m^*([a,b]) \geq b-a$. This is equivalent to showing that if $\{I_n\}$ is a countable collection of open intervals with $[a,b] \subseteq \bigcup_n I_n$ then $b-a \leq \sum_n \ell(I_n)$. Let $\{I_n\}$ be a countable collection of open intervals with $[a,b] \subseteq \bigcup_n I_n$. Since $[a,b]$ is a closed and bounded subset of real numbers, by Heine-Borel Theorem there exists a finite subcollection of $\{I_n\}$, say $I_1, I_2, \ldots, I_m$ which covers $[a,b]$.

Now $\sum_{j=1}^{m} \ell(I_j) \leq \sum_n \ell(I_n)$ and $[a,b] \subseteq \bigcup_{j=1}^{m} I_j$. As $a \in [a,b] \subseteq \bigcup_{j=1}^{m} I_j$

We get a $1 \leq k \leq m$ such that $a \in I_k$. Let $I_k = (a_1, b_1)$.

So $a_1 < a < b_1$. Now $b < b_1$ or $b_1 \leq b$.

If $b < b_1$ then $a_1 < a < b < b_1$ and that $b-a \leq b_1 - a_1 \leq \sum_{j=1}^{m} \ell(I_j) \leq \sum_n \ell(I_n)$.

Suppose that $b_1 \leq b$. Now $b_1 \in [a,b]$ and $b_1 \notin (a_1, b_1)$. Therefore we get a $1 \leq p \leq m$ with $p \neq k$ such that $b_1 \in I_p$. Let $I_p = (a_2, b_2)$. So $a_2 < b_1 < b_2$. We get a sequence $(a_1,b_1), (a_2,b_2), \ldots (a_k,b_k)$ from the collection $\{I_1, I_2, \ldots, I_m\}$ such that $a_i < b_{i-1} < b_i$, $i = 2,3,\ldots k$. Since $\{I_1, I_2, \ldots I_m\}$ is a finite collection, our process terminates with some interval (a_k,b_k). But it terminates with (a_k,b_k) only if $b \in (a_k,b_k)$. Suppose that the process terminates with (a_k,b_k). Now $a_k < b < b_k$.

$$\sum_n \ell(I_n) \geq \sum_{j=1}^{m} \ell(I_j) \geq \sum_{i=1}^{k} \ell((a_i,b_i)) = (b_k-a_k) + (b_{k-1}-a_{k-1}) + \ldots + (b_1-a_1)$$

$$= b_k - (a_k - a_{k-1}) - (a_{k-1} - b_{k-2}) \ldots - (a_2 - b_1) - a_1$$

$$> b_k - a_1, \text{ since } a_i < b_{i-1}, i = 2,3,\ldots k$$

$$> b - a, \text{ since } b_k > b \text{ and } a > a_1$$

Therefore $\sum_n \ell(I_n) \geq b-a$

So $b-a \leq \inf\left\{ \sum_n \ell(I_n) \,/\, \{I_n\} \text{ is a countable collection of open intervals with } [a,b] \subseteq \bigcup_n I_n \right\}$

$= m^*([a,b])$.

Therefore $m^*([a,b]) = b-a$.

Case II :

Suppose that I is a finite interval.

So I = [a,b) or (a,b] or (a,b) or [a,b], where a and b are real numbers and a < b.

Let $\in > 0$. Let $J = \left[a + \frac{\in}{4}, b - \frac{\in}{4}\right]$.

Now $J \subseteq I$ and $l(J) = (b-a) - \frac{\in}{2} = l(I) - \frac{\in}{2}$

Therefore $l(I) - \in < l(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = m^*([a,b]) = b - a = l(I)$.

So, $l(I) - \in < m^*(I) \leq l(I)$. Since $\in > 0$ is arbitrary

We get that $m^*(I) = l(I)$.

Case III :

Suppose that I is an infinite interval.

So $I = (-\infty, a)$ or $(-\infty, a]$ or $(a, \infty]$ or $[a, \infty)$ or $(-\infty, \infty)$ where a is a real number. Let Δ be a positive real number.

We get a closed interval $J \subseteq I$ with $l(J) = \Delta$.

Now $\Delta = l(J) = m^*(J) \leq m^*(I)$. Since $\Delta > 0$ is arbitrary,

$m^*(I) = \infty$. But $l(I) = \infty$. Therefore $m^*(I) = \infty = l(I)$.

From case I, case II and case III we get that the outer measure of an interval is its length.

We prove now that the outer measure m* is countably subadditive.

2.3.6. Proposition :

Let $\{A_n\}$ be a countable collection of sets of real numbers. Then m $m^*\left(\bigcup_n I_n\right) \leq \sum_n m^*(A_n)$

Proof:

If one of the sets A_n has infinite outer measure the inequality holds trivially.

So, we assume that $m^*(A_n)$ is finite for all n. Let $\in>0$.

Since $m^*(A_n)$ is finite from the definition of $m^*(A_n)$, $m^*(A_n)+\frac{\in}{2^n}$ is not a lower bound for $\left\{\sum_m \ell(J_m) / \{J_n\} \text{ is countable collection of open intervals with } A_n \subseteq \bigcup_m J_m\right\}$ and that we get a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A_n \subseteq \bigcup_m I_{n,i}$ and $\sum_i \ell(I_n,i) < m^*(A_n)+\frac{\in}{2^n}$.

Since union of a countable number of countable collections is countable, $\{I_{n,i}\}_{n,i}$ is a countable collection of open intervals.

Also $\bigcup_n A_n \subseteq \bigcup_{n,i} I_{n,i}$

So
$$m^*\left(\bigcup_n A_n\right) \le \sum_{n,i} \ell(I_{n,i}) = \sum_n \sum_i \ell(I_{n,i})$$
$$\le \sum_n \left(m^*(A_n)+\frac{\in}{2^n}\right)$$
$$= \sum_n m^*(A_n)+\in\left(\sum_n \frac{1}{2^n}\right)$$
$$\le \sum_n \left(m^*(A_n)+\in\cdot 1\right) \quad \left(\text{Since } \sum_{n=1}^{\infty}\frac{1}{2^n}=1\right)$$

Therefore $m^*\left(\bigcup_n A_n\right) \le \sum_n \left(m^*(A_n)+\in\right)$

Since $\in>0$ is arbitrary, we get that $m^*\left(\bigcup_n A_n\right) \le \sum_n \left(m^*(A_n)\right)$.

2.3.7. Corollary : If A is a countable subset of real numbers then $m^*(A) = 0$

Proof : Let A be a countable subset of real numbers then $A = \bigcup_{a \in A} \{a\}$

Since m* is countable subadditive, $m^*(A) = m^*\left(\bigcup_{a \in A} \{a\}\right)$

$$\leq \sum_{a \in A} m^*\{a\}$$

$$= 0$$

So $m^*(A) \leq 0$. But $m^*(A) \geq 0$. Therefore $m^*(A) = 0$.

2.3.8. Self Assessment Question :

Show that any non-empty interval is uncountable.

2.3.9. Proposition :

Let A be a set of real numbers and $\in > 0$. Then there is an openset O such that $A \subseteq O$ and $m^*(O) \leq m^*(A) + \in$ and there is a G_δ-set G such that $A \subseteq G$ and $m^*(A) = m^*(G)$.

Proof :

Let A be a set of real numbers and $\in > 0$.

Case I :

Suppose that $m^*(A) = \infty$. Take $O = R$

Now $A \subseteq O$ and $m^*(O) = \infty = m^*(A) + \in$.

Also O is a G_δ-set and $m^*(A) = m^*(O) = \infty$.

Case II :

Suppose that $m^*(A)$ is finite.

Then there exists a countable collection $\{I_n\}$ of open intervals such that $A \subseteq \bigcup_n I_n$ and $\sum_n \ell(I_n) < m^*(A) + \in$.

Put $O = \bigcup_n I_n$ Now O is an open set and $A \subseteq O$.

Now $m^*(O) = m^*\left(\bigcup_n I_n\right) \le \sum_n \left(m^*(I_n)\right) = \sum_n \ell(I_n) < m^*(A) + \in$

Thus for each $\in > 0$ there is an open set O such that $A \subseteq O$ and $m^*(O) \le m^*(A) + \in$.

Therefore for each positive integer n, there exists an open set O_n such that $A \subseteq O_n$ and $m^*(O_n) \le m^*(A) + 1/n$.

Put $G = \bigcap_{n=1}^{\infty} O_n$. clearly G is a G_δ-set and $A \subseteq G$.

So $m^*(A) \le m^*(G)$.

Since $G \subseteq O_n$, $m^*(G) < m^*(O_n) < m^*(A) + 1/n$ for all n = 1,2,.....

Therefore $m^*(G) \le m^*(A)$. Hence $m^*(A) = m^*(G)$, where G is a G_δ-set.

Answers to SAQs :

2.2.4. :

Let $\mathcal{A}$ be a σ-algebra of subsets of a set X and m be a countably additive set function defined on $\mathcal{A}$. Let $\{A_n\}$ be a sequence of sets in $\mathcal{A}$. Then by **1.2.7,** there exists a disjoint sequence $\{B_n\}$ in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ and $B_n \subseteq A_n$ for all n. Now we have, $m\left(\bigcup_{n=1}^{\infty} A_n\right) =$

$= m\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} m(B_n) \le \sum_{n=1}^{\infty} m(A_n)$ since m is countably additive and by **2.3.4.(b)**

Let A, B $\in \mathcal{A}$ and $A \cap B = \phi$. Put $A_n = \phi$ for $n > 2$, $A_1 = A, A_2 = B$.

Thus, $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n) = m(A) + m(B)$. But, $\bigcup_{n=1}^{\infty} A_n = A \cup B$ hence $m(A \cup B) =$ $m(A) + m(B)$. i.e. m is finitely additive.

2.2.5. :

Let $\mathcal{A} = \mathcal{P}(\mathbb{R})$ the class of all subsets of $\mathbb{R}$: For any A $\in \mathcal{A}$, define $m(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \end{cases}$ then m is countably subadditive but not countably additive since $1 = m(z+) = m\left(\bigcup_n \{n\}\right)$ and $\sum_n m(\{n\}) = \infty$.

2.3.8 :

If I = [a,b] is an interval m*(I) = b-a by Theorem 2.3.5. So that m*(I) > 0. Hence I is uncountable (For if I is countable tnen m*(I) = 0 by corollary 2.3.7}.

2.5. Model Examination Questions :

1. Define the concepts of a countably additive set function and of a countably subadditive set function and prove that every countably additive set function is countably subadditive. Is the converse true? Justify your answer.

2. If m is finitely additive set function defined on $\mathcal{A}$ and m(B) < ∞ then m(B-A) = m(B)-m(A) for every A,B $\in \mathcal{A}$ with $A \subseteq B$.

3. Define outer measure of a set and prove that the outer measure of any interval is its length.

4. Prove that the outer measure m* is countably subadditive.

5. Show that m* is translation invariant.

2.6. Exercises :

1. Show that if $m^*(A)=0$ then $m^*(A \cup B)=m^*(B)$ for any set B.

2. For any subset A of $\mathbb{R}$, prove that there is a G_δ-set G such that $A \subseteq G$ and $m^*(A)=m^*(G)$

3. Show that every finite set has outer measure zero.

4. For any sets A and E of $\mathbb{R}$, prove that $m^*(A) \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$ where $\tilde{E}$ is the complement of E in $\mathbb{R}$.

5. Let A be the set of rational numbers between 0 and 1 and let $\{In\}$ be a finite collection of open intervals covering A. Then, $\sum \ell(\ell_n) \geq 1$.

2.7. : Reference Book : Real Analysis - H.L. Royden

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LESSON - 3

MEASURABLE SETS AND LEBESGUE MEASURE

3.1 Introduction:

The Lebesgue outer measure m^* which is defined on the collection of all subsets of real numbers is not countably additive (4.11.2). However, if we restrict m^* to a class of subsets of real numbers, namely the Lebesgue measurable sets then m^* is countably additive on this class. It is proved that the collection of all Lebesgue measurable sets is a σ-algebra containing all open and closed subsets of real numbers. It is shown that the Lebesgue measure m, which is the restriction of the outer measure m^* to the collection of all Lebesgue measurable sets is countably additive and translation invariant.

Note: All the sets considered are subsets of real numbers unless otherwise stated. We are going to adopt here the definition of measurability due to caratheodory which is motivated by the following consequence of countable sub-additivity of the outer measure.

3.2 Remark : Given a set E, for any set A, $A = (A \cap E) \cup (A \cap \tilde{E})$ implies

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap \tilde{E})$

3.3 Definition : A set E is said to be measurable or Lebesgue measurable if for every set A, we have $m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$ where $\tilde{E}$ is the complement of E in $\mathfrak{R}$.

3.4 Remark : (i) The definition of measurability says that the measurable sets are those which split every set into two pieces that are additive with respect to the outer measure.

(ii) A set E is measurable iff for every set A, $m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$ (in view of the definition 3.3 and remark 3.2)

3.5 Lemma : (i) ϕ and $\mathfrak{R}$ are measurable

(ii) If E is measurable so is $\tilde{E}$

(iii) If $m^*(E) = 0$ then E is measurable.

Proof : -

(i) For any set A, we have $A \cap \phi = \phi$, $A \cap \tilde{\phi} = A \cap \mathfrak{R} = A$ and $A \cap \mathfrak{R} = A$, $A \cap \tilde{\mathfrak{R}} = \phi$

so that $m^*(A \cap \phi) + m^*(A \cap \tilde{\phi}) = m^*(\phi) + m^*(A) = m^*(A)$ and $m^*(A \cap \mathfrak{R}) + m^*(A \cap \tilde{\mathfrak{R}})$

$= m^*(A) + m^*(\phi) = m^*(A)$. Hence ϕ and $\mathfrak{R}$ are measurable.

(ii) If E is measurable then $m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$ for every set A. That is,

$m^*(A) = m^*(A \cap \tilde{\tilde{E}}) + (A \cap \tilde{E})$ for every set A, showing $\tilde{E}$ is measurable.

(iii) Suppose $m^*(E)=0$. If A is any set then $A \cap E \subseteq E$ and $A \cap \tilde{E} \subseteq A$. Since m^* is

monotone, $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap \tilde{E}) \leq m^*(A)$ so that, $m^*(A \cap E) +$

$m^*(A \cap \tilde{E}) \leq m^*(A)$. Hence by remark 3.4 (ii), E is measurable.

3.6 Lemma: Let E be a measurable set. Then show that for each y, the set $E + y = \{x+y / x \in E\}$ is measurable and the measures are the same.

Proof : Suppose E is a measurable set. Then for any set A, it is easy to see that

$A \cap (E+y) = (A-y) \cap E$. Also $(\widetilde{E+y}) = \tilde{E} + y$.

Therefore, $m^*(A \cap (E+y)) + m^*(A \cap (\widetilde{E+y}))$

$= m^*((A-y) \cap E) + m^*(A \cap (\tilde{E} + y))$

$= m^*((A-y) \cap E) + m^*((A-y) \cap \tilde{E})$

$= m^*(A-y)$, by the measurability of E

$= m^*(A)$ since m^* is translation invariant

Thus, E+y is measurable.

Hence, $m(E+y) = m^*(E+y) = m^*(E) = m(E)$

3.7 Lemma : If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.

Proof : Suppose E_1 and E_2 are measurable sets. Let A be a set of real numbers. We have, $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap \tilde{E}_1)$ and therefore

$$m^*(A \cap (E_1 \cup E_2) + m(A \cap E_2 \cup E_2) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$= m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1 \cap E_2) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$= m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) \text{ (since } E_2 \text{ is measurable)}$$

$$= m^*(A) \text{ (since } E_1 \text{ is measurable)}$$

Thus, $E_1 \cup E_2$ is measurable.

3.8 Corollary : The family $\mathfrak{M}$ of all measurable sets is an algebra of sets.

Proof : Let $\mathfrak{M}$ be the family of all measurable sets. Since ϕ and $\mathfrak{R}$ are measurable sets $\mathfrak{M}$ is a non-empty collection of sets. If $E_1 \in \mathfrak{M}$ $E_2 \in \mathfrak{M}$ then $E_1 \cup E_2 \in \mathfrak{M}$ by Lemma 3.5. If $E \in \mathfrak{M}$ then $\tilde{E} \in \mathfrak{M}$ by lemma 3.5 (ii) Hence $\mathfrak{M}$ is an algebra of sets.

3.9 Lemma : Let A be a set and $E_1, E_2, \ldots, E_n$ be a finite sequence of disjoint measurable sets. Then, $m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_i)$

Proof : We shall use induction on n. It is trivial for n=1. Let n>1 and assume that the result holds for n-1. Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Since E_i's are pair-wise disjoint, we have $E_i \cap E_n = \phi$ for all i<n and hence $E_i \subseteq \tilde{E}_n$ for all i<n and

$$(\bigcup_{i=1}^{n} E_i) \cap \tilde{E}_n = \bigcup_{i=1}^{n} E_i \cap \tilde{E}_n = \bigcup_{i=1}^{n-1} E_i \text{ and } (\bigcup_{i=1}^{n} E_i) \cap E_n = E_n.$$

Since E_n is measurable we have

$$m^*(A \cap \bigcup_{i=1}^{n} E_i) = m^*(A \cap (\bigcup_{i=1}^{n} E_i) \cap E_n) + m^*(A \cap (\bigcup_{i=1}^{n} E_i) \cap \tilde{E}_n)$$

$$= m^*(A\cap E_n)+m^*(A\cap \bigcup_{i=1}^{n-1} E_i)$$

$$= m^*(A\cap E_n)+ \sum_{i=1}^{n-1} m^*(A\cap E_i) \text{ (by the induction hypothesis)}.$$

$$= \sum_{i=1}^{n} m^* (A\cap E_i)$$

Hence the result.

3.10 Theorem : The collection $\mathfrak{M}$ of measurable sets is a σ- algebra; that is, the complement of a measurable set is measurable and the union of a countable collection of measurable sets is measurable.

Proof : We have already observed that $\mathfrak{M}$ is an algebra of sets and so we have only to prove that if a set E is the union of a countable collection of measurable sets it is measurable. By proposition---------, such an E must be the union of a sequence $\{E_n\}$ of pair-wise disjoint measurable sets ie, $E = \bigcup_{n=1}^{\infty} E_n$, where E_n are measurable sets and $E_n\cap E_m = \phi$ for $n\neq m$. Let $F_n = \bigcup_{i=1}^{n} E_i$ since E_i are measurable, F_n is measurable $\forall$ n=1,2,..., also, since $F_n \subseteq E$ we have $\tilde{E} \subseteq \tilde{F}_n$ for all n=1,2....,

Let A be a set of real numbers. Now $A\cap \tilde{E} \subseteq A\cap \tilde{F}_n$ since m^* is montone we have

$m^*(A\cap \tilde{E}) \leq m^*(A\cap \tilde{F}_n)$.

Since F_n is measurable,

$$m^*(A) = m^*(A\cap F_n)+m^*(A\cap \tilde{F}_n)$$

$$\geq m^*(A\cap F_n)+m^*(A\cap \tilde{E})$$

$$= m^*(A\cap \bigcup_{i=1}^{n} E_i) +m^*(A\cap \tilde{E})$$

$$= \sum_{i=1}^{n} m^*(A\cap E_i) +m^*(A\cap \tilde{E}), \text{ (by lemma 3.8).}$$

Therefore, $m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$ for all n=1,2,3.....,

Hence, $m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$

$\geq m^*(A \cap E) + m^*(A \cap \tilde{E})$

since, $m^*(A \cap E) = m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = m^*(\bigcup_{i=1}^{\infty} A \cap E_i) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i)$.

Hence E is measurable.

Therefore $\mathfrak{M}$ is a σ-algebra.

3.11 Lemma : The interval (a,∞) is measurable.

Proof : Let A be any set, write $A_1 = A \cap (a,\infty)$ and $A_2 = A_2 = A \cap (-\infty, a]$. If we prove $m^*(A) \geq m^*(A_1) + m^*(A_2)$-------(1) it follows that (a,∞) is measurable. Now (1) is trivial if $m^*(A) = +\infty$. Assume, $m^*(A) < \infty$. Let $\epsilon > 0$ we will show that $m^*(A_1) + m^*(A_2) \leq m^*(A) + \epsilon$. By the definition of $m^*(A)$, we get a countable collection $\{I_n\}$ of open intervals such that $A \subseteq \bigcup_n I_n$ and $\sum_n l(I_n) \leq m^*(A) + \epsilon$ ------(2)

Put, $I_n^{I} = I_n \cap (a,\infty)$ and $I_n^{II} = I_n \cap (-\infty, a]$. Then I_n^{I} and I_n^{II} are either empty or intervals, with $I_n^{I} \cup I_n^{II} = I_n$ and $I_n^{I} \cap I_n^{II} = \phi$. Therefore, $l(I_n) = l(I_n^{I}) + l(I_n^{II}) = m^*(I_n^{I}) + m^*(I_n^{II})$. Now since $A_1 \subseteq \bigcup_{n=1}^{\infty} I_n^{I}$, $A_2 \subseteq \bigcup_{n=1}^{\infty} I_n^{II}$ we have,

$m^*(A_1) \leq \sum_{n=1}^{\infty} l(I_n^{I}) = \sum_{n=1}^{\infty} m^*(I_n^{I})$ and

$m^*(A_2) \leq \sum_{n=1}^{\infty} l(I_n^{II}) = \sum_{n=1}^{\infty} m^*(I_n^{II})$ so that.

$m^*(A_1) + m^*(A_2) \leq \sum_{n=1}^{\infty} \{m^*(I_n^{I}) + m^*(I_n^{II})\} = \sum_{n=1}^{\infty} l(I_n)$ ------(3)

Therefore (2) and (3) give, $m^*(A_1) + m^*(A_2) \leq m^*(A) + \epsilon$.

Since $\epsilon > o$ is arbitrary we get the inequality in ..... (1).

Thus $(a,\infty) \in \mathfrak{M}$ for all $a \in \mathfrak{R}$.

3.12 Theorem : Every Borel set is measurable. In particular each open set and each closed set is measurable.

Proof : For any $a \in \mathfrak{R}$, (a,∞) is measurable and hence its complement $(-\infty,a]$ is also measurable. Now for any $b \in \mathfrak{R}$, $(-\infty,b) = \bigcup_{n=1}^{\infty} [-\infty, b - \frac{1}{n}]$ and since $[-\infty, b - \frac{1}{n}]$ is measurable so is $(-\infty,b)$. Since $(a,b) = (-\infty,b) \cap (a,\infty)$, it follows that each open interval is measurable. Also since any open set is a countable union of open intervals we get that every open set is measurable. Since the collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra containing the class of all open sets and since the collection $\mathfrak{M}$ of all measurable sets is a σ-algebra containing all the open sets, it follows that $\mathcal{B}$ is contained is $\mathfrak{M}$, that is every Borel set is measurable. Since open sets and closed sets are borel sets, they must be measurable.

3.13 Definition : For any measurable set E its Lebesgue measure m(E) is defined to be the outer measure of the set E. That is if $E \in \mathfrak{M}$ then, $m(E) = m^*(E)$.

Thus the Lebesgue measure m is the set function obtained by restricting the outer measure m^* to the class $\mathfrak{M}$ of measurable sets. Note that.

(i) The Lebesgue measure m is a non-negative set function defined on the σ-algebra $\mathfrak{M}$ of measurable sets.

(ii) $m(I) = m^*(I) = l(I)$ for all intervals.

(iii) $m(E+y) = m^*(E+y) = m^*(E) = m(E)$ for all measurable sets E and for all real numbers y. Recall that if E is measurable then E+y is measurable for any real number y. We now prove that m is countably additive; so that m is a non-negative extended real valued countably additive set function defined on a σ-algebra, thus m is a measure (Definition 2.2.7) and this m is called the Lebesgue measure.

3.14 Proposition : Let $\{E_n\}$ be a sequence of measurable sets. Then $m(\bigcup_n E_n) \leq \sum_n m(E_n)$. If the sets E_n are pair wise disjoint then, $m(\bigcup_n E_n) = \sum_n m(E_n)$.

Proof : let $\{E_n\}$ be a sequence of measurable sets. Now $\bigcup_n E_n$ is measurable. Hence

$m(\bigcup_n E_n) = m^*(\bigcup_n E_n) \leq \sum_n m^*(E_n)$ (Since m^* is countably sub-additive)

$$= \sum_n m(E_n)$$

Therefore, $m(\bigcup_n E_n) \leq \sum_n m(E_n)$

Now suppose that E_n's are pairwise disjoint. From Lemma 3.9 (by taking $A=\mathfrak{R}$), we have

$m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} E_i$ for all positive intergers n and

Therefore $m(\bigcup_{i=1}^{\infty} E_i) \geq m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(E_i)$ for all n.

Implies that $m(\bigcup_{i=1}^{\infty} E_i) \geq \sum_{i=1}^{\infty} m(E_i)$

Also $m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$ (as seen above by the countable subadditivity of m^* and hence of m).

Therefore, $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$. Hence, m is countably additive.

3.15 Proposition : Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{n+1} \subseteq E_n$ for each n.

Let $m(E_1)$ be finite. Then $m(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

Proof : Let $E = \bigcap_{n=1}^{\infty} E_n$ and let $F_n = E_n - E_{n+1}$, n=1,2,.....

We claim that, $E_1 - E = \bigcup_{i=1}^{\infty} F_n$ and F_n are pair-wise disjoint.

Let $x \in E_1 - E$, now $x \in E_1$ and $x \notin E = \bigcap_{n=1}^{\infty} E_n$.

4

For some positive integer i, $x \notin E_i$. Let j be the least positive integer such that $x \notin E_j$. So $x \in E_{j-1} - E_j = F_{j-1}$ and that $x \in \bigcup_{i=1}^{\infty} F_n$.

Therefore $E_1 - E \subseteq \bigcup_{i=1}^{\infty} F_n$. Let $x \in \bigcup_{i=1}^{\infty} F_n$. $x \in F_n$ for some positive integer n. So $x \in E_n$ and $x \notin E_{n+1}$. Therefore $x \in E_1 - E$ and that $\bigcup_{n=1}^{\infty} F_n \subseteq E_1 - E$.

So $E_1 - E = \bigcup_{n=1}^{\infty} F_n$. Let n and m be positive integers and $n \neq m$. Without loss of generality suppose that $n<m$. So $n+1 \leq m$ and that

$$E_m \subseteq E_{n+1} \Rightarrow \tilde{E}_{n+1} \cap E_m = \phi. \; F_n \cap F_m = (E_n - E_{n+1}) \cap (E_m - E_{m+1})$$

$$= E_n \cap \tilde{E}_{n+1} \cap E_m \cap \tilde{E}_{m+1} = \phi$$

Therefore F_n are pair-wise disjoint measurable sets. Since $E_1 - E = \bigcup_{n=1}^{\infty} F_n$, we have $m(E_1 - E) = m(\bigcup_{n=1}^{\infty} F_n)$. Since F_n are pair-wise disjoint, $m(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} m(F_n)$.

Therefore $m(E_1 - E) = \sum_{n=1}^{\infty} m(F_n)$---------(1)

We have $E_1 = E \cup (E_1 - E)$ and $E \cap (E_1 - E) = \phi$.

So $m(E_1) = m(E \cup (E_1 - E)) = m(E) + m(E_1 - E)$----(2)

Since $m(E_1) < \infty$ measures of all the subsets E, $E_1 - E$, F_n, E_n, $n=1,2,\ldots\ldots$ of E_1 are finite. So $m(E_1 - E) = m(E_1) - m(E)$ by (2)

Since $E_n = E_{n+1} \cup (E_n - E_{n+1}) = E_{n+1} \cup F_n$ amd $E_{n+1} \cap F_n = \phi$,

$m(E_n) = m(E_{n+1} \cup F_n) = m(E_{n+1}) + m(F_n)$

By the above argument, $m(F_n) = m(E_n) - m(E_{n+1})$.

Therefore, from (1) $m(E_1) - m(E) = \sum_{n=1}^{\infty} (m(E_n) - m(E_{n+1}))$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (m(E_i) - m(E_{i+1})) = \lim_{n\to\infty} (m(E_1) - m(E_{n+1}))$$

Since $m(E_1) < \infty$, $-m(E) = -\lim_{n\to\infty} m(E_{n+1})$ *ie*, $m(E) = \lim_{n\to\infty} m(E_{n+1})$

ie, $m(E)=\lim_{n\to\infty} m(E_n)$.

Therefore, $m(\bigcap_{n=1}^{\infty} E_n)=\lim_{n\to\infty} m(E_n)$.

3.16 Proposition: Let E be a set of real numbers. Then the following five statements are equivalent.

1. E is measurable.
2. Given $\epsilon>0$, there is an open set $O \supseteq E$ with $m^*(O-E)<\epsilon$.
3. Given $\epsilon>0$, there is a closed set $F\subseteq E$ with $m^*(E-F)<\epsilon$.
4. There is a G in G_δ with $E \subseteq G$ and $m^*(G-E) = 0$.
5. There is a F in F_σ with $F \subseteq E$ and $m^*(E-F)=0$

   If $m^*(E)$ is finite, the above statements are equivalent to

6. Given $\epsilon>0$, there is a finite union U of open intervals such that $m^*(U\Delta E)<\epsilon$

Proof : Let E be a set of real numbers. Let $\epsilon>0$, we prove that $1\Rightarrow2\Rightarrow3\Rightarrow4\Rightarrow1$ and $1\Rightarrow3\Rightarrow5\Rightarrow1$.

$1\Rightarrow2$ Assume that E is measurable.

Case I : Suppose that $m(E)<\infty$. By proposition 2.17 there is an open set O such that $E\subseteq O$ and $m^*(O)< m^*(E)+\epsilon$. Since E is measurable, $m^*(O) = m^*(O\cap E)+m^*(O\cap \tilde{E})$ $= m^*(E) + m^*(O-E)$. Since $m^*(E)< \infty$, $m^*(O-E) = m^*(O) - m^*(E)<\epsilon$

Case II : Suppose that $m(E) = \infty$.

Let $I_n=(-n,n)$. I_n is a finite interval of length 2n.

Now $R= \bigcup_{n=1}^{\infty} I_n$. $E=E\cap R=E\cap(\bigcup_{n=1}^{\infty} I_n)= \bigcup_{n=1}^{\infty}(E \cap I_n)$.

Let $E_n=E\cap I_n$. n=1,2,……now $E=\bigcup_{n=1}^{\infty} E_n$.

Since $E_n\subseteq I_n$, $m^*(E_n) \le m^*(I_n)= \ell(I_n)=2n<\infty$. Also since E & I_n are measurable, E_n is also measurable. Therefore by case I, we get an open set O_n such that $E_n\subseteq O_n$ and

$m^*(O_n-E_n)< \epsilon/2^n$. Let $O=\bigcup_{n=1}^{\infty} O_n$. Clearly O is an open set and $E\subseteq O$ as $E=\bigcup_{n=1}^{\infty} E_n$ and $E_n\subseteq O_n$.

$$O-E=\bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (O_n-E_n).$$

Therefore, $m^*(O-E) \le m^*(\bigcup_{n=1}^{\infty} (O_n-E_n)) \le \bigcup_{n=1}^{\infty} m^*(O_n-E_n) < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon$.

<u>2⇒4</u> Assume 2, we get an open set O_n such that $E\subseteq O_n$ and $m^*(O_n-E)<\frac{1}{n}$ for all $n=1,2,\ldots$. Let $G=\bigcap_{n=1}^{\infty} O_n$. G is a G_δ-set and $E\subseteq G$.

Since $G-E\subseteq O_n-E$, $m^*(G-E)\le m^*(O_n-E)<\frac{1}{n}$ for all $n=1,2,\ldots$.

Therefore, $m^*(G-E)\le 0$. But $m^*(G-E)\ge 0$.

Hence $m^*(G-E)=0$.

<u>4⇒1</u> Assume (4) so there is a G_δ -set G such that $E\subseteq G$ and $m^*(G-E)=0$. By Lemma 3.5, G–E is measurable. G is measurable as G is a Borel set. $E = G-(G-E)$ as $E\subseteq G$. Therefore, E is measurable.

<u>1⇒3</u> Assume (1). E is measurable. So $\tilde{E}$ is also measurable. Since 1⇒2, we get an open set O such that $\tilde{E}\subseteq O$ and $m^*(O-\tilde{E})<\epsilon$. $O-\tilde{E}=O\cap E=E\cap\tilde{\tilde{O}}=E-\tilde{O}$.

Let $F=\tilde{O}$. Since O is open $\tilde{O}=F$ is a closed set.

$\tilde{E}\subseteq 0$ implies $\tilde{O}\subseteq\tilde{\tilde{E}}=E$ ie $F\subseteq E$. so $m^*(E-F)=m^*(E-\tilde{O})=m^*(0-\tilde{E})<\epsilon$.

<u>3⇒5</u> Assume (3). For each positive integer n, we get a closed set $F_n\subseteq E$ such that $m^*(E-F_n)<\frac{1}{n}$. Let $F=\bigcup_{n=1}^{\infty} F_n$. F is a F_σ -set and $F\subseteq E$. Since $E-F\subseteq E-F_n$, $m^*(E-F)\le m^*(E-F_n)<\frac{1}{n}$ for all $n=1,2,\ldots$. Therefore, $m^*(E-F)\le 0$. But $m^*(E-F)\ge 0$.

Hence, $m^*(E-F)=0$.

5⇒1 Assume(5). There is a F_σ - set F such that $F\subseteq E$ and $m^*(E-F) = 0$. By Lemma 3.5, E–F is measurable. Since F is a Borel set it is measurable. $E=F\cup(E-F)$ as $F\subseteq E$. Therefore E is measurable.

Therefore statements 1,2,3,4 and 5 are equivalent.

Suppose now that $m^*(E)<\infty$. We prove that 1⇒6⇒1.

1⇒6 We have that E is a measurable set of finite measure. Since 1⇒2, we get an open set O such that $E\subseteq O$ and $m^*(O-E)< \frac{\epsilon}{2}$. Since O and E are measurable sets and

$O = E\cup(O-E)$ and $E\cap(O-E) = \phi$, $m^*(O)= m^*(E)+ m^*(O-E)$.

Since $m^*(E)< \infty$ and $m^*(O-E)<\epsilon/2$, $m^*(O)< \infty$.

Without loss of generality we may assume that O is nonempty. We get a countable collection $\{I_n\}$ of disjoint open internals such that $O=\bigcup_n I_n$. Since each I_n is measurable, and I_n's are disjoint, $\sum_n l(I_n) = \sum_n m^*(I_n) = m^*(\bigcup_n I_n) = m^*(O)< \infty$.

Case I Suppose that the collection $\{I_n\}$ is finite consisting of $I_1, I_2\ldots.., I_k$.

Let $V=\bigcup_{i=1}^{k} I$; now V=O. So $E\Delta V = (E-V) \cup(V-E)=(E-O) \cup(O-E)=O-E$.

Therefore $m^*(E\Delta V)= m^*(O-E)< \epsilon/2<\epsilon$.

Case I Suppose that the countable collection $\{I_n\}$ is not finite. Now $O= \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n)$

$= m^*(O)< \infty$. We get an integer N such that $\sum_{n=N+1}^{\infty} l(I_n)< \epsilon/2$.

Let, $V = \bigcup_{n=1}^{N} I_n$. now $E\Delta V=(E-V) \cup(V-E) \subseteq(O-V) \cup(O-E)$.

$O-V = \bigcup_{n=1}^{\infty} I_n- \bigcup_{n=1}^{N} I_n = \bigcup_{n=N+1}^{\infty} I_n$ as I_n are pairwise disjoint. Since I_n are pairwise disjoint measurable sets, $m^*(\bigcup_{n=N+1}^{\infty} I_n) = \sum_{n=N+1}^{\infty} m^*(I_n) = \sum_{n=N+1}^{\infty} l(I_n)< \epsilon/2$.

So $m^*(O-V)= m^*(\bigcup_{n=N+1}^{\infty} I_n)< \epsilon/2$.

Therefore, $m^* (E\Delta V) \leq m^*((O-V)\cup (O-E)) \leq m^*(O-V)+ m^* (O-E)< \epsilon/2+\epsilon/2=\epsilon$.

6⇒1 Assume (6). We have that $m^*(E) < \infty$. There is a finite union V of open intervals $I_1, I_2, \ldots I_n$ such that $m^*(V\Delta E) < \epsilon/3$. We get an open set G such that $E-V \subseteq G$ and $m^*(G) \le m^*(E-V) + \epsilon/3$. Let $O = G \cup V$. O is an open set as G and V are open sets.

$E \subseteq (E-V) \cup V \subseteq G \cup V = 0$.

$O-E = (G\cup V) - E = (G-E) \cup (V-E)$.

Therefore $m^*(O-E) \le m^*(G-E) + m^*(V-E)$

$$\le m^*(G) + m^*(V-E)$$

$$\le m^*(E-V) + \epsilon/3 + m^*(V-E)$$

$$\le m^*(E\Delta V) + \epsilon/3 + m^*(E\Delta V)$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \text{ (Since } E-V \text{ and } V-E \text{ are subsets of } E\Delta V)$$

$$= \epsilon$$

So we have proved 6⇒12. Since 2⇒1, we have that 6⇒1, this completes the proof.

3.17 **Self Assessment Question:**

Show that if E is a measurable set, then each translate E+y of E is measurable.

3.18 **Self Assessment Question:**

Show that if E_1 and E_2 are measurable then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

3.19 **Self Assessment Question:**

Show that the condition $m(E_1) < \infty$ is necessary in proposition 3.15 by giving a decreasing sequence $\{E_n\}$ of measurable sets with $\phi = \bigcap_{n=1}^{\infty} E_n$ and $m(E_n) = \infty$ for each n.

3.20 **Self Assessment Question:**

Let $\{E_n\}$ be a sequence of disjoint measurable sets and A be any set. Then prove that, $m^*(A \cap \bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m^*(A \cap E_n)$.

3.21 **Self Assessment Question:**

Show that the cantor ternary set has measure zero.

3.22 **Self Assessment Question:**

Let E be a set of real numbers and $m^*(E) < \infty$. Prove that E is measurable if and only if for each $\in > 0$, there is a finite union U of pair-wise disjoint open intervals such that $m^*(U \Delta E) < \in$.

3.23 ANSWERS TO SAQ'S.

3.17 (See Lemma 3.6 for alternate proof)

Let E be a measurable set and y be a real number. Let $\in > 0$. By proposition 3.15, there exists an open set O such that $E \subseteq O$ and $m^*(O-E) < \in$.

Since O is an open set O+y is also an open set and $E+y \subseteq O+y$.

Now, $(O+y) - (E+y) = (O-E)+y$.

$m^*((O+y) - (E+y)) = m^*((O-E)+y) = m^*(O-E)$ as m^* is translation invarient. So,

$m^*((O+y) - (E+y)) = m^*(O-E) < \in$

Therefore, again by proposition 3.15, E+y is measurable.

3.18 Let E_1 and E_2 be measurable sets.

Since E_1 is measurable, $m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap \tilde{E}_1)$

Since E_2, $E_2 \cap E_1$, $E_2 \cap \tilde{E}_1$ are measurable, $m(E_2) = m(E_2 \cap E_1) + m(E_2 \cap \tilde{E}_1)$

Similarly as E_1 is measurable, $m(E_1 \cup E_2) = m(E_1 \cup E_2) \cap E_1) + m((E_1 \cup E_2) \cap \tilde{E}_1)$

$= m(E_1) + m(E_2 \cap \tilde{E}_1)$

as $(E_1 \cup E_2) \cap E_1 = E_1$ and $(E_1 \cup E_2) \cap \tilde{E}_1 = E_2 \cap \tilde{E}_1$

Therefore, $m(E_1 \cup E_2) \cap m(E_1 \cap E_2) = m(E_1) + m(E_2 \cap \tilde{E}_1) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

3.19 Let $E_n=(n, \infty)$, $n=1,2,\ldots\ldots$

Now $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots\ldots$ and $\bigcap_{n=1}^{\infty} E_n = \phi$

Each E_n is measurable. $m(\bigcap_{n=1}^{\infty} E_n) = m(\phi) = 0$

Since $m(E_n)=m(n, \infty)=\infty$ for all $n=1,2,\ldots..$

$\lim_{n\to\infty} m(E_n) = \infty \neq m(\bigcap_{n=1}^{\infty} E_n) = 0$

Therefore, the condition $m(E_1)<\infty$ is necessary in proposition 3.15

3.20 Let $\{E_n\}$ be an infinite sequence of disjoint measurable sets and A any set. Since

$A \cap (\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} (A \cap E_n)$

$m^*(A \cap (\bigcup_{n=1}^{\infty} E_n)) = m^*(\bigcup_{n=1}^{\infty} (A \cap E_n)) \leq \sum_{n=1}^{\infty} m^*(A \cap E_i)$ by the countable sub-additivity of m^* ---------(1)

By Lemma 3.9, $m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_n)$, for all $n=1,2,\ldots.$

Now $m^*(A \cap (\bigcup_{i=1}^{\infty} E_i)) \geq m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_i)$, for all $n=1,2,\ldots.$

Therefore, $m^*(A \cap (\bigcup_{n=1}^{\infty} E_n)) \geq \sum_{n=1}^{\infty} m^*(A \cap E_n)$ ------(2)

From (1) & (2), $m^*(A \cap (\bigcup_{n=1}^{\infty} E_n)) = \sum_{n=1}^{\infty} m^*(A \cap E_n)$

3.21 Let $C_0=[0,1]$. Delete from C_0 the open interval $(1/3,2/3)$ which is its middle third and now write, $C_1=[0,1/3] \cup [2/3,1]$ Now delete from C_1, the open intervals $(1/9,2/9)$ and $(7/9,8/9)$ which are the middle thirds of $[0,1/3]$ and $[2/3,1]$ respectively and write, $C_2=[0,1/9] \cup [2/9,1/3] \cup [2/3, 7/9] \cup [8/9,1]$. Continuing this process, we get a sequence $\{C_n\}_{n=1}^{\infty}$ of sets where C_{n+1} is obtained from C_n by deleting the open middle third of each of the 2^n disjoint closed intervals of which

C_n is composed of. The cantor set C is defined by $C=\bigcap_{n=1}^{\infty} C_n$. Each C_n is composed of 2^n disjoint closed intervals each of length $\frac{1}{3^n}$.

Therefore, C_n is measurable and $m(C_n)= 2^n.\frac{1}{3^n}=\left(\frac{2}{3}\right)^n$, n=1,2,3,......

Now, $C=\bigcap_{n=1}^{\infty} C_n$ is also measurable and $m(C) \leq m(C_n)=(2/3)^n$ for all n=1,2,..... Hence, $m(C)\leq 0$. Therefore $m(C)=0$.

Note : It is known that the cantor set C is measurable. So, cantor set C is an example of an uncountable set of measure zero.

3.22 This follows from proposition 3.15

3.24 MODEL EXAMINATION QUESTIONS.

1. If E_1 and E_2 are measurable, then show that $E_1 \cup E_2$ is measurable
2. Show that the interval (a, ∞) is measurable, where a is a real number
3. Prove that every Borel set is measurable.
4. Prove that the collection $\mathfrak{M}$ of all Lebesgue measurable sets is a σ-algebra.
5. Prove that the Lebesgue measure m is countably additive.
6. Let $<E_n>$ be an infinite decreasing sequence of measurable sets, that is, a sequence $E_{n+1} \subseteq E_n$ for each n. Let $m(E_1)$ be finite. Then, prove that $m(\bigcap_{n=1}^{\infty} E_n)= \lim_{n\to\infty} m(E_n)$.
7. For a set E of real numbers, prove that the following are equivalent
    1. E is measurable.
    2. Given $\epsilon>0$ there is an open set O such that $E\subseteq O$ and $m^*(O-E)< \epsilon$
    3. There is a G in G_δ with $E \subseteq G$, $m^*(G\sim E)=0$

3.25 EXERCISES

1. Prove that for any set A there exists a measurable set E containing A and such that $m^*(A)=m(E)$.

2. If $\{E_n\}$ is an increasing sequence of measurable sets then prove that $m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$

3. If E is a measurable set. Prove that E+y is also measurable

4. Prove that the set of all rational numbers and the set of all irrational numbers are Borel sets and hence Lebesque measurable.

5. Show that there exists uncountable sets of zero measure.

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LESSON-4 : A NON-MEASURABLE SET

4.1 INTRODUCTION :

In this lesson we consider the relation between the class $\mathcal{M}$ of Lebesgue measurable sets and the class $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$. Using Lemma 4.4, we show in theorem 4.5. that $\mathcal{M} \neq \mathcal{P}(\mathbb{R})$. Since the characteristic function χ_A is measurable if and only if, A is measurable we have the corresponding relation between the two classes of measurable functions and the class of all real-vlaued functions. Now, we are going to show the existence of a non-measurable set.

4.2. Definition :

If x and y are real numbers in [0,1), we define the sum modulo 1 of x and y to be x+y, if x+y<1 and to be x+y-1 if x+y≥1. Let us denote the sum modulo 1 of x and y by $x \overset{\circ}{+} y$. If $E \subseteq [0,1)$ and $y \in [0,1)$ then the translate modulo 1 of E by y denoted by $E \overset{\circ}{+} y$ is defined as $E \overset{\circ}{+} y = \{x \overset{\circ}{+} y / x \in E\}$

4.3. Remark :

$\overset{\circ}{+}$ is a commutative and associative operation taking pairs of numbers in [0,1) into numbers in [0,1).

4.4. Lemma :

Let E be a measurable set of real numbers and $E \subseteq [0,1)$. Then for each $y \in [0,1)$ the set $E \overset{\circ}{+} y$ is measurable and $m(E \overset{\circ}{+} y) = m(E)$.

Proof :

Let E be a measurable set of real numbers and $E \subseteq [0,1)$ and $y \in [0,1)$.

$[0,1) = [0,1\text{-}y) \cup [1\text{-}y,1)$ & $[0,1\text{-}y) \cap [1\text{-}y,1) = \phi$.

$E = E \cap [0,1) = (E \cap [0,1\text{-}y)) \cup (E \cap [1\text{-}y,1))$

Let $E_1 = E \cap [0,1\text{-}y)$ and $E_2 = E \cap [1\text{-}y,1))$

Now $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$. Also E_1 & E_2 are measurable.

So $m(E) = m(E_1 \cup E_2) = m(E_1) + m(E_2)$.

$E \dot{+} y = E_1 + y$ and that $E_1 \dot{+} y$ is measurable.

$m(E_1 \dot{+} y) = m(E_1 \dot{+} y) = m(E_1)$ as m is translation invariant

$E_2 \dot{+} y = E_2 + (y - 1)$ and that $E_2 \dot{+} y$ is measurable.

$m(E_2 \dot{+} y) = m(E_2 + (y-1)) = m(E_2)$ as m is translation invariant.

Since $E = E_1 \cup E_2$, $E \dot{+} y = (E_1 \dot{+} y) \cup (E_2 \dot{+} y)$. We see now that

$(E_1 \dot{+} y) \cap (E_2 \dot{+} y) = \phi$. Let $x \in (E_1 \dot{+} y) \cap (E_2 \dot{+} y)$.

$x = a \dot{+} y = b \dot{+} y$, $a \in E_1$, $b \in E_2$

$a + y = a \dot{+} y = b \dot{+} y = b + (y - 1)$. Therefore $b - a = 1$

A contradiction to the fact that $b - a < 1$.

Therefore $(E_1 \dot{+} y) \cap (E_2 \dot{+} y) = \phi$ So $m(E \dot{+} y) = m(E_1 \dot{+} y) + m(E_2 \dot{+} y)$

$= m(E_1) + m(E_2) = m(E)$

Hence the Lemma.

4.5 : Theorem : There is a non-measurable set of real numbers

Proof: Let $x, y \in [0,1)$. Define $x \sim y$ if and only if $x - y$ is a rational number. Clearly $\sim$ is an equivalence relation on $[0,1)$. This equivalence relation partitions $[0,1)$ into disjoint equivalence classes and any two elements of the same class differ by a rational number while any two elements of different classes differ by an irrational number. By axiom of choice there is a set P which contains exactly one element from each equivalence class. Since all the rational numbers present in $[0,1)$ are countably infinite they can be written in the form of a sequence $\{r_n\}_{n=0}^{\infty}$ with $r_0 = 0$ and $r_n \neq r_m$ for $n \neq m$. Define $P_n = P \dot{+} r_n$, $n = 0,1,23, \ldots\ldots$

Now $P_0 = P$. We claim that $P_n \cap P_m = \phi$ for $n \neq m$ and $\bigcup_{n=0}^{\infty} P_n = [0,1)$

Let $x \in [0,1)$ Now $x \sim y$ for some $y \in P$. $x - y$ is a rational number.

Case I :

Suppose that $x - y \geq 0$ and now $x - y = r_i$ for some integer $i \geq 0$.

So $x = y + r_i = y \dotplus r_i \in P \dotplus r_i = P_i$. Therefore $x \in \bigcup_{n=1}^{\infty} P_n$

Case II :

Suppose that $x - y < 0$. Now $y - x > 0$. Let $y - x = r_j$ for some integer $j \geq 0$.

$1 - r_j = r_k$ for some integer $k \geq 0$

$y - r_j = x \Rightarrow y + 1 - r_j = 1 + x \Rightarrow y \dotplus (1 - r_j) = (1 + x) - 1 = x \Rightarrow x = y \dotplus r_k \in P \dotplus r_k = P_k$

Therefore $x \in \bigcup_{n=0}^{\infty} P_n$.

From case I & case II we get that $[0,1) \subseteq x \in \bigcup_{n=0}^{\infty} P_n$. Obviously $\bigcup_{n=0}^{\infty} P_n \subseteq [0,1)$.

Therefore $\bigcup_{n=0}^{\infty} P_n = [0,1)$.

Let $x \in P_n \cap P_m$, whre n and m are non negative integers

Now $x = a \dotplus r_n = b \dotplus r_m$, where $a, b \in P$.

$a \dotplus r_n = a + r_n$ or $a + r_n - 1$ and $b \dotplus r_m = b + r_m$ or $b + r_m - 1$

As $a \dotplus r_n = b \dotplus r_m$ we get that $a - b$ is a rational number.

So a and b belong to the same equivalence class, i.e, $a \sim b$.

Since $a, b \in P$ and $a \sim b$ by our construction of P, $a = b$.

Now $x = a \dotplus r_n = a \dotplus r_m$

Suppose that $a \dotplus r_n = a + r_n - 1$ and $a \dotplus r_n = a + r_m$.

Now $a + r_n - 1 = a + r_m$. So $r_n - 1 = r_m$ i.e, $r_m + 1 = r_n$, a

Contradiction to the fact that $0 \leq r_n < 1$.

Similarly we arrive at a contradiction if $a \dotplus r_n = a + r_n$ and $a \dotplus r_m = a + r_m - 1$

Therefore either $a \dotplus r_n = a + r_n$ and $a \dotplus r_m = a + r_m$

or $a \dotplus r_n = a + r_n - 1$ and $a \dotplus r_m = a + r_m - 1$.

In either case we get that $r_n = r_m$ i.e, $n = m$

Therefore P_n are pairwise disjoint.

Now suppose that P is measurable.

By Lemma 4.4, each P_n is measurable and $m(P) = m(P_n)$ for all n. Since $[0,1) = \bigcup_{n=0}^{\infty} P_n$ and P_n are disjoint, we have $1 = m([0,1)) = m\left(\bigcup_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} m(P_n) = \sum_{n=0}^{\infty} m(P)$. Now either $m(P) = 0$ or $m(P) > 0$.

If $m(P) = 0$ then $1 = \sum_{n=0}^{\infty} m(P) = 0$, a contradiction

If $m(P) > 0$ then $1 = \sum_{n=0}^{\infty} m(P) = \infty$, a contradiction

Therefore P is not measurable.

Thus P is a non-measurable subset of real numbers contained in $[0,1)$.

4.6. Self Assessment Question :

Show that if E is measurable and $E \subseteq P$ then $m(E) = 0$, where P is the set defined in Theorem 4.5.

4.7. Self Assessment Question :

Show that if A is any set with $m^*(A) > 0$ then there is a non-measurable set E contained in A.

4.8. Self Assessment Question ;

Give an example of a sequence of sets $\{E_n\}$ with $E_n \supseteq E_{n+1}$, $m^*(E_n) < \infty$ and $m^*\left(\bigcap_{n=1}^{\infty} E_n\right) < \lim_n m^*(E_n)$.

4.9 Answers to SAQs :

4.6. Let $\{r_n\}_{n=0}^{\infty}$ be the sequence considered in Theorem 4.4.

Suppose that E is measurable set and $E \subseteq P$, P is defined in theorem 4.5. From Theorem 4.4 we have that $P_n = P \dotplus r_n$ for all $n = 0,1,2,.....$ and $\bigcup_{n=0}^{\infty} P_n = [0,1)$ and P_n are pairwise disjoint. As $E \subseteq P$, $E_n = E \dotplus r_n \subseteq P \dotplus r_n = P_n$.

Since P_n are disjoint, E_n are disjoint and $\bigcup_{n=0}^{\infty} E_n \subseteq \bigcup_{n=0}^{\infty} P_n = [0,1)$

Since E is measurable, by lemma 4.4. E_n is measurable and $m(E_n) = m(E)$ for all $n = 0,1,2,..$

Since $\bigcup_{n=0}^{\infty} E_n \subseteq [0,1)$, $m\left(\bigcup_{n=0}^{\infty} E_n\right) \le m([0,1)) = 1$

So $m\left(\bigcup_{n=0}^{\infty} E_n\right) = \sum_{n=0}^{\infty} m(E_n) = \sum_{n=0}^{\infty} m(E) \le 1$

Therefore $m(E) = 0$.

4.7 Let A be a set of real numbers and $m^*(A) > 0$.

Case I :

Suppose that $A \subseteq [0,1)$

Let $\{P_n\}_{n=0}^{\infty}$ be the sequence of sets defined in theorem 4.5.

Let $B_n = A \cap P_n$ for all n = 0,1,2,......

Since $\bigcup_{n=0}^{\infty} P_n = [0,1)$ and $A \subseteq [0,1)$,

$$A = A \cap [0,1) = A \cap \left(\bigcup_{n=0}^{\infty} P_n\right) = \bigcup_{n=0}^{\infty} (A \cap P_n) = \bigcup_{n=0}^{\infty} B_n .$$

Suppose that B_n is measurable for all n = 0,1,2,....

So $B_n \dotplus x$ is measurable for all $x \in [0,1)$ and n = 0,1,2,.....

Clearly $x \dotplus y \dotplus (1-y) = x$ for all $x,y \in [0,1)$

$B_n \dotplus (1-r_n) \subseteq P \dotplus (1-r_n) = (P \dotplus r_n) \dotplus (1-r_n) = P$, for all n = 0,1,2,.....

By the above problem, $m(B_n \dotplus (1-r_n)) = 0$ for all n = 0,1,2,.....

Since $B_n = B_n \dotplus (1-r_n) \dotplus r_n$, by lemma 4.4. $m(B_n) = 0$, for all n

Therefore $m^*(B_n) = 0$ for all n = 0,1,2,,...... (1)

Since $A = \bigcup_{n=0}^{\infty} B_n$, we have, $0 < m^*(A) = m^*\left(\bigcup_{n=0}^{\infty} B_n\right) \le \sum_{n=0}^{\infty} m^*(B_n) = 0$ from (1) a contradiction. Therefore for some non-negative integer n, $B_n \subseteq A$ is not measurable.

Case II :

Suppose that $A \not\subseteq [0,1)$

IR is the union of the disjont intervals [n, n+1), n is an integer.

Therefore $A = \cup A_n$, where $A_n = A_n \cap [n, n+1)$ and n is an integer.

Now, $0 < m^*(A) = m^*\left(\bigcup_n A_n\right) \le \sum_n m^*(A_n)$.

So for some integer n, $m^*(A_n) > 0$. Now, $B_n = A_n + (-n) \subseteq [0,1)$. Since m* is translation invariant, $0 < m^*(A_n) = m^*(A_n + (-n)) = m^*(B_n)$. Therefore by case I, B_n contains a non-measurable set E. Now, $E + n \subseteq A_n \subseteq A$.

If $E + n$ is measurable then E is measurable. Since E is not measurable, $E + n$ is not measurable. Thus $E + n$ is the required non-measurable subset of A.

4.8. Let P and P_n have the meaning as in the proof of theorem 4.5. We know that $P_n \cap P_m = \phi$ for $n \neq m$ and $m^*(P_n) = m^*(P_m) \neq 0$ for every n,m.

If $E_n = \bigcup_{m=n}^{\infty} P_m$ then $E_n \supseteq E_{n+1}$. Clearly $P_n \subseteq E_n$

So that, $0 < m^*(P) = m^*(P_n) \le m^*(E_n) < \infty$ and therefore

$\lim_{n\to\infty} m^*(E_n) > 0$. Also, $\bigcap_{n=1}^{\infty} E_n = \phi$ gives $m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$,

Proving the result.

4.10 Model Examination Questions :

4.10.1. Let $E \subseteq [0,1)$ be a measurable set. Then for each $y \in [0,1)$. Show that $E \dotplus y$ is measurable and $m(E \dotplus y) = mE$.

4.10.2. Give an example of a set which is not Lebesgue measurable.

4.11. Exercises :

1. Show that $\dotplus$ is a commutative and associative operation taking numbers in [0,1) into numbers in [0,1)

2. Give an example where $\{E_i\}$ is a disjoint sequence of sets and $m^*(\cup E_i) < \sum m^* E_i$.

4.12. Reference Book : Real Analysis - H.L.Royden.

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LESSON – 5

MEASURABLE FUNCTIONS

5.1 Introduction: In this lesson we introduce the concept of a Lebesgue measurable function and study certain properties of Lebesgue measurable functions. The class of Lebesgue measurable functions which includes the class of continuous functions as a proper subclass play an important role in the Lebesgue theory of integration. Further we prove the Littlewood's Second principle which states that every measurable function is nearly continuous. (Proposition 5.20).

5.2 Proposition: Let f be an extended real valued function whose domain D is measurable. Then the following statements are equivalent.

1. For each real number α, the set $\{x \in D / f(x) > \alpha\}$ is measurable.
2. For each real number α, the set $\{x \in D / f(x) \geq \alpha\}$ is measurable.
3. For each real number α, the set $\{x \in D / f(x) < \alpha\}$ is measurable.
4. For each real number α, the set $\{x \in D / f(x) \leq \alpha\}$is measurable. These statements imply
5. For each extended real number α, the set $\{x \in D / f(x) = \alpha\}$is measurable.

Proof: Let f be an extended real valued function whose domain D is measurable. We prove that $1 \Leftrightarrow 4$, $2 \Leftrightarrow 3$ and $1 \Leftrightarrow 2$.

Let α be a real number.

$1 \Rightarrow 4$ We assume (1). So $\{x \in D / f(x) > \alpha\}$ is measurable.

Clearly $\{x \in D / f(x) \leq \alpha\} = D - \{x \in D / f(x) > \alpha\}$

Since D and $\{x \in D / f(x) > \alpha\}$ = A one measurable, $\{x \in D / f(x) \leq \alpha\}$ is measurable.

$4 \Rightarrow 1$ We assume (4). So $\{x \in D / f(x) \leq \alpha\}$ is measurable.

Now $\{x \in D / f(x) > \alpha\} = D - \{x \in D / f(x) \leq \alpha\}$.

Since D and $\{x \in D / f(x) \leq \alpha\}$ are measurable, $\{x \in D / f(x) > \alpha\}$ is measurable.

$2 \Rightarrow 3$. We assume (2). So $\{x \in D / f(x) \geq \alpha\}$ is measurable.

Now $\{x \in D / f(x) < \alpha\} = D - \{x \in D / f(x) \geq \alpha\}$

Since D and $\{x \in D \{f(x) \geq \alpha\}$ are measurable, $\{x \in D / f(x) < \alpha\}$ is measurable.

Similarly we get $3 \Rightarrow 2$

$1 \Rightarrow 2$. We assume 1. $\{x \in D / f(x) > \alpha\}$ is measurable for all real numbers α. Clearly

$$\{x \in D / f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in D / f(x) > \alpha - \frac{1}{n}\}.$$

Since $\{x \in D / f(x) > \alpha - \frac{1}{n}\}$ is measurable for each n = 1,2,........,

$\bigcap_{n=1}^{\infty} \{x \in D / f(x) > \alpha - \frac{1}{n}\}$ is measurable as countable intersection of measurable sets is measurable. Therefore $\{x \in D / f(x) \geq \alpha\}$ is measurable.

$2 \Rightarrow 1$. We assume 2. $\{x \in D / f(x) \geq \alpha\}$ is measurable for all real numbers α.

Clearly $\{x \in D / f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in D / f(x) \geq \alpha + \frac{1}{n}\}$.

Since $\{x \in D / f(x) \geq \alpha + \frac{1}{n}\}$ is measurable for each n = 1,2, ...

$\bigcup_{n=1}^{\infty} \{x \in D / f(x) \geq \alpha + \frac{1}{n}\}$ is measurable as countable union of measurable sets is measurable. Therefore $\{x \in D / f(x) > \alpha\}$ is measurable. Now $1 \Leftrightarrow 2$, $2 \Leftrightarrow 3$ and $1 \Leftrightarrow 4$. Therefore the first four statements are equivalent.

We prove now that the first four statements imply 5^{th} statement. Assume the first four statements.

Case I Suppose that α is a real number.

Now $\{x \in D/f(x) = \alpha\} = \{x \in D/f(x) \geq \alpha\} \cap \{x \in D/f(x) \leq \alpha\}$

By our assumption $\{x \in D/ f(x) \geq \alpha\}$, $\{x \in D / f(x) \leq \alpha\}$ are measurable.

So $\{x \in D / f(x) \geq \alpha\} \cap \{x \in D / f(x) \leq \alpha\}$ is measurable.

Therefore $\{x \in D / f(x) = \alpha\}$ is measurable.

Case II Suppose that $\alpha = \infty$

Now $\{x \in D / f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in D / f(x) \geq n\}$

Since $\{x \in D / f(x) \geq n\}$ is measurable for each $n = 1,2,\ldots\ldots$

$\bigcap_{n=1}^{\infty} \{x \in D / f(x) \geq n\}$ is measurable, as countable intersection of measurable sets is measurable. Therefore $\{x \in D / f(x) = \infty\}$ is measurable.

Case III: Suppose that $\alpha = -\infty$

Now $\{x \in D / f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in D / f(x) \leq -n\}$

Since $\{x \in D / f(x) \leq -n\}$ is measurable for each $n = 1,2, \ldots.$

$\bigcap_{n=1}^{\infty} \{x \in D / f(x) \leq -n\}$ is also measurable. Therefore $\{x \in D / f(x) = -\infty\}$ is measurable.

This completes the proof.

5.3 Self Assessment Question: Show that the statement 5 in the proposition 5.2 need not imply any one of the first four statements.

5.4 Definition: An extended real valued function f is said to be measurable or Lebesgue measurable if its domain is measurable and if it satisfies one of the first four statements of proposition 5.2.

5.5 Example: A continuous real valued function with measurable domain D is measurable since for any real number α , the set $\{x \in D / f(x) > \alpha\} = f^{-1}(\alpha, \infty)$ being the inverse image of open set (α,∞) is open in D and hence measurable.

Note that the converse is not true. For example the function f defined on IR by $f(x) = 0$ if $x \in (0,1)$ and $f(x) = 1$ otherwise, is measurable but not continuous. Thus the class of continuous functions is a proper subclass of the class of measurable functions.

5.6 Self Assessment Question:

If f is a measurable function and E is a measurable subset of the domain of f then the function obtained by restricting f to E is also measurable.

5.7 Self Assessment Question:

Prove that every constant function defined on a measurable set is a measurable function.

5.8 Proposition: Let c be a constant and f and g two measurable real valued functions defined on the same domain. Then the functions f+c, cf, f+g, f – g and fg are also measurable.

Proof: Suppose that c is a constant and f and g are two measurable real valued functions defined on the same domain D.

Let α be a real number.

1. We show that f + c is a measurable function.

   $\{x \in D / (f + c)(x) < \alpha\} = \{x \in D / f(x) + c < \alpha\} = \{x \in D / f(x) < \alpha - c\}$

   Since f is measurable, $\{x \in D / f(x) < \alpha - c\}$ is measurable.

   So $\{x \in D / (f+c)(x) < \alpha\}$ is measurable. Therefore f + c is measurable.

2. We show that cf is measurable.

   Case I Suppose that c = 0

   Now cf = 0, a constant function, and hence measurable by SAQ 5.7

   Case II Suppose that c > 0

   $\{x \in D/ (cf)(x) > \alpha\} = \{x \in D/ cf(x) > \alpha\} = \{x \in D / f(x) > \frac{\alpha}{c}\}$

   since f is measurable, $\{x \in D/ f(x) > \frac{\alpha}{c}\}$ is measurable. Therefore

   $\{x \in D / (cf)(x) > \alpha\}$ is measurable. Hence cf is measurable.

   Case III Suppose that c < 0.

   $\{x \in D/ (cf)(x) > \alpha\} = \{x \in D / c.f(x) > \alpha\} = \{x \in D / f(x) < \frac{\alpha}{c}\}$

   Since f is measurable $\{x \in D/ f(x) < \frac{\alpha}{c}\}$ is measurable.

   So $\{x \in D/ (cf)(x) > \alpha\}$ is measurable. Therefore cf is measurable.

3. We show that f + g is measurable.

   $\{x \in D / (f+g)(x) < \alpha\} = \{x \in D / f(x) + g(x) < \alpha\} = \{x \in D / f(x) < \alpha - g(x)\}$

   Since between any two real numbers there is a rational number, given

   $x \in \{x \in D / f(x) < \alpha - g(x)\}$, we get a rational number r such that $f(x) < r < \alpha - g(x)$.

   Therefore $\{x \in D / (f+g)(x) < \alpha\} = \bigcup_r (\{x \in D / f(x) < r\} \cap \{x \in D / g(x) < \alpha - r\})$

   Since rational numbers are countable and $\{x \in D / f(x) > r\}$ and $\{x \in D / g(x) < \alpha - r\}$ are measurable (as f and g are measurable),

   $\bigcup_r (\{x \in D / f(x) < r\} \cap \{x \in D / g(x) < \alpha - r\})$ is a countable union of measurable sets and hence measurable. Therefore $\{x \in D / (f+g)(x) < \alpha\}$ is measurable and that f + g is measurable.

4. We show that f – g is measurable.

   f – g = f + (-1) g. Since g is measurable, (-1)g is measurable by (2). Since f and (-1)g is measurable, f+ (-1)g is measurable by (3).

   Therefore f – g is measurable.

5. We show now that fg is measurable.

   First we prove that f^2 is measurable.

   If $\alpha \geq 0$ then $\{x \in D / f^2(x) > \alpha\} = \{x \in D / f(x) > \sqrt{\alpha}\} \cup \{x \in D / f(x) < -\sqrt{\alpha}\}$

   and as $\{\{x \in D / f(x) > \sqrt{\alpha}\}$ and $\{x \in D / f(x) < -\sqrt{\alpha}\}$ are measurable (as f is measurable), $\{x \in D / f^2(x) > \alpha\}$ is measurable.

   If $\alpha < 0$ then $\{x \in D / f^2(x) > \alpha\} = D$ and that $\{x \in D / f^2(x) > \alpha\}$ is measurable.

   Therefore $\{x \in D / f^2(x) > \alpha\}$ is measurable and that f^2 is measurable.

   now $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$

   Since f and g are measurable f^2, g^2 are measurable as seen above and f+g is measurable by (3). Since f+g is measurable as seen above $(f+g)^2$ is measurable.

   $(f+g)^2 - f^2$ is measurable by (4). Also $(f+g)^2 - f^2 - g^2$ is measurable again by (4).

Now

$\frac{1}{2}[(f+g)^2 - f^2 - g^2]$ is measurable by (2) i.e. fg is measurable.

5.9 Note: In proposition 7.6 we considered the sum and product of two real valued measurable functions defined on the same domain and not the sum and product of two extended real valued measurable functions defined on the same domain. Let f and g be extended real valued functions defined on the same domain D. f+g is not defined at a point $x \in D$ where $f(x) = \infty$ and $g(x) = -\infty$ or $f(x) = -\infty$ and $g(x) = \infty$ as $\infty + -\infty$ is undefined. If we take the same value for f+g at such points in D then we see that f+g is measurable. Also if the set of all such points in D is a set of measure zero then f+g is measurable, whatever values we take for f+g at these points in D. However fg is always measurable.

5.10 Theorem: Let $\{f_n\}$ be a sequence of measurable functions with the same domain of definition. Then the functions $\sup\{f_1, f_2, \ldots, f_n\}$, $\inf\{f_1, f_2, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\underline{\lim} f_n$ and $\overline{\lim} f_n$ are all measurable.

Proof: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions with the same domain of definition D. Let α be a real number.

1. Let $h = \sup\{f_1, f_2, \ldots, f_n\}$. Now $h(x) = \sup\{f_1(x), f_2(x), \ldots, f_n(x)\}$ for all $x \in D$.

We claim that $\{x \in D / h(x) > \alpha\} = \bigcup_{i=1}^{n}\{x \in D / f_i(x) > \alpha\}$.

Let $x \in D$ and $h(x) > \alpha$. Now $h(x) = f_j(x)$ for some $1 \le j \le n$.

Therefore $f_j(x) > \alpha$ and that $x \in \{x \in D / f_j(x) > \alpha\} \subseteq \bigcup_{i=1}^{n}\{x \in D / f_i(x) > \alpha\}$

So $\{x \in D / h(x) > \alpha\} \subseteq \bigcup_{i=1}^{n}\{x \in D / f_i(x) > \alpha\}$

Let $x \in \bigcup_{i=1}^{n}\{x \in D / f_i(x) > \alpha\}$. Now for some $1 \le k \le n$, $f_k(x) > \alpha$.

Since $h(x) \geq f_k(x) > \alpha$, $x \in \{x \in D / h(x) > \alpha\}$

Therefore $\bigcup_{i=1}^{n} \{x \in D / f_i(x) > \alpha\} \subseteq \{x \in D / h(x) > \alpha\}$

Hence $\{x \in D / h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x \in D / f_i(x) > \alpha\}$.

Since f_i is measurable, $\{x \in D / f_i(x) > \alpha\}$ is measurable for $i = 1, 2, \ldots .n$ and that $\bigcup_{i=1}^{n} \{x \in D / f_i(x) > \alpha\}$ is measurable. So $\{x \in D / h(x) > \alpha\}$ is measurable. Hence h is measurable.

2. Let $g = \inf\{f_1, f_2, \ldots f_n\}$. Now $g(x) = \inf\{f_1(x), f_2(x), \ldots f_n(x)\}$ for all $x \in D$.
Now $g = \inf\{f_1, f_2, \ldots f_n\} = -\sup\{-f_1, -f_2, \ldots, -f_n\}$
Since f_i is measurable, $-f_i = (-1) f_i$ is also measurable for $i = 1,2, \ldots .n$. Therefore by (1), $\sup\{-f_1, -f_2, \ldots, -f_n\}$ is measurable and that $-\sup\{-f_1, -f_2, \ldots, -f_n\}$ is also measurable. Therefore g is measurable.

3. Let $h = \sup\{f_1, f_2, \ldots, f_n, \ldots\} = \sup\limits_n f_n$

now $h(x) = \sup\{f_1(x), f_2(x), \ldots, f_n(x), \ldots\} = \sup\limits_n f_n(x)$, for all $x \in D$.

We claim that $\{x \in D / h(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in D / f_n(x) > \alpha\}$.

Let $x \in D$ and $h(x) > \alpha$. Suppose that $f_n(x) \leq \alpha$ for all $n = 1,2, \ldots\ldots$ now $\sup\limits_n f_n(x) \leq \alpha$

i.e. $h(x) \leq \alpha$, a contradiction. So for some positive integer m, $f_m(x) > \alpha$.

So $x \in \bigcup_{n=1}^{\infty} \{x \in D / f_n(x) > \alpha\}$. Therefore $\{x \in D / h(x) > \alpha\} \subseteq \bigcup_{n=1}^{\infty} \{x \in D / f_n(x) > \alpha\}$ (1)

Let $x \in \bigcup_{n=1}^{\infty} \{x \in D / f_n(x) > \alpha\}$. For some positive integer k, $f_k(x) > \alpha$. Since $h(x) \geq f_k(x) > \alpha$,

$x \in \{x \in D / h(x) > \alpha\}$. Therefore $\bigcup_{n=1}^{\infty} \{x \in D / f_n(x) > \alpha\} \subseteq \{x \in D / h(x) > \alpha\}$ – (2)

From (1) & (2) $\{x \in D / h(x) > \alpha\} = \bigcup_{n=1}^{\infty}\{x \in D / f_n(x) > \alpha\}$. Since f_n is measurable,

$\{x \in D / f_n(x) > \alpha\}$ is measurable for all n = 1,2, ... Therefore $\bigcup_{n=1}^{\infty}\{x \in D / f_n(x) > \alpha\}$ is measurable. So $\{x \in D / h(x) > \alpha\}$ is measurable. Hence h is measurable.

4. Let $l = \inf\{f_1, f_2, \ldots, f_n, \ldots\} = \inf_n f_n$.

now $l(x) = \inf\{f_1(x), f_2(x), \ldots, f_n(x), \ldots\} = \inf_n f_n(x)$.

$l = \inf\{f_1, f_2, \ldots, f_n, \ldots\} = -\sup\{-f_1, -f_2, \ldots, -f_n, \ldots\}$.

Since f_n is measurable, $-f_n$ is measurable for all n = 1,2,.... So, by (3) $\sup\{-f_1, -f_2, \ldots, -f_n \ldots\}$ is measurable. Now $-\sup\{-f_1, -f_2, \ldots, -f_n, \ldots\}$ is measurable that is l is measurable.

5. Let $p = \overline{\lim_n} f_n$. $\overline{\lim_n} f_n = \inf_n \sup_{k\geq n} f_k$.

Let $g_n = \sup_{k\geq n} f_k = \sup\{f_n, f_{n+1}, \ldots\}$, n = 1,2, ...

Now $p = \inf_n g_n$. Since f_n is measurable for all n = 1,2, ... By (3) g_n is measurable for all n = 1,2, ... therefore by (4) p is measurable.

6. Let $q = \underline{\lim_n} f_n$. $\underline{\lim_n} f_n = \sup_n \inf_{k\geq n} f_k$

Let $h_n = \inf_{k\geq n} f_k = \inf\{f_n, f_{n+1}, \ldots\}$, n = 1,2 .....

Now $q = \sup_n h_n$. Since f_n is measurable for all n = 1,2,... by (4) h_n is measurable for all

n = 1,2,..... Therefore by (3) q is measurable.

5.11 Result: Let $\{f_n\}$ be a sequence of measurable functions defined on the same domain D. Let $\lim_{n\to\infty} f_n = f$ i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in D$. Then f is a measurable function on D.

Proof: Since $\lim_{n\to\infty} f_n = f$, $f = \overline{\lim_{n\to\infty}} f_n = \overline{\lim_n} f_n$

By theorem 5.10, $\overline{\lim_n} f_n$ is measurable as each f_n is measurable. Therefore f is measurable.

5.12 Definition: A property is said to hold almost every where (abbreviated a.e.) if the set of points where it fails to hold is a set of measure zero.

For example: (I) Let f and g are extended real valued functions with the same domain D where D is a set of real numbers. Then we say that f = g a.e. if $m(\{x \in D \,/\, f(x) \neq g(x)\}) = 0$

ii) Let $\{f_n\}$ be a sequence of extended real valued functions defined on the same domain D,D a set of real numbers. Then we say f_n converges to g almost every where if there is a set E of measure zero such that $f_n(x)$ converges to g(x) for each x not in E.

iii) If f and g are functions with the same domain and $\{x: f(x) > g(x)\}$ has measure zero then we say that $f \leq g$ a.e.

iv) Let f be a function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases}$$

then $f(x) = 0$ a.e., since, $m\{x: f(x) \neq 0\} = m\{x: x \text{ is rational}\} = 0$

One consequence of equality a.e is the following:

5.13 Proposition: If f is a measurable function and f = g a.e., then g is measurable.

Proof: Let f be a measurable function and f = g a.e. Let D be the domain of f and g and let $E = \{x \in D : f(x) \neq g(x)\}$ Since f = g a.e, m (E) = 0. Let α be a real number. Now,

$\{x \in D : g(x) > \alpha\} = [\{x \in D : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}] - \{x \in E : g(x) \leq \alpha\}$.. (1)

Now the measurability of f implies that $\{x: f(x) > \alpha\}$ is a measurable set. Also since each one of the sets $\{x \in E: g(x) > \alpha\}$ and $\{x \in E : g(x) \leq \alpha\}$, being subsets of E of measure zero, is of measure zero. Hence (1) shows that $\{x \in D : g(x) > \alpha\}$ is a measurable set for each real α, proving g is measurable.

5.14 Definition: Let A be a set of real numbers. The characteristic function χ_A of the set A is a real valued function defined on the set of real numbers by $\chi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$

5.15 Result: Let A be a set of real numbers, then χ_A is measurable if and only if A is measurable.

Proof: Let A be a set of real numbers. Suppose that χ_A is measurable. Now

$\{x \in R / \chi_A(x) \geq \frac{1}{2}\} = A$ is measurable.

Suppose now that A is measurable. Let α be a real number.

$$\text{Now } \{x \in R / \chi_A(x) > \alpha\} = \begin{cases} R, \text{ if } \alpha < 0 \\ A, \text{ if } 0 \leq \alpha < 1 \\ \phi, \text{ if } 1 \leq \alpha \end{cases}$$

Since R, A and ϕ are measurable, χ_A is measurable.

Note: If A is a non measurable subset of R then by the above remark, χ_A is not measurable. So the existence of a non measurable set implies the existence of a non-measurable function.

5.16 Definition: A real valued function ϕ is called simple if it is measurable and assumes only a finite number of values.

Let ϕ be a simple function. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the distinct values assumed by ϕ. Let $A_i = \{x / \phi(x) = \alpha_i\}$, $1 \leq i \leq n$. Since ϕ is measurable, A_i is measurable for all $1 \leq i \leq n$.

As $\alpha_1, \alpha_2, \ldots, \alpha_n$ are distinct $A_1, A_2, \ldots, A_n$ are pair wise disjoint. Clearly $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$.

5.17 Result: the sum, product and difference of two simple functions is simple.

Proof: Let ϕ_1, ϕ_2 be two simple functions defined on E. since ϕ_1, ϕ_2 are measurable, real valued functions $\phi_1+\phi_2$, $\phi_1-\phi_2$ and $\phi_1\phi_2$ also assumes finite number of values. Therefore $\phi_1+\phi_2$, $\phi_1-\phi_2$ and $\phi_1\phi_2$ are simple.

5.18 Definition: A real valued function ϕ defined on [a, b] is said to be a step function if there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that for each i, ϕ assumes only one value on (x_i, x_{i+1}), $i = 0, 1, \ldots, n-1$.

5.19 Result: A step function is measurable and hence simple.

Proof: Let ϕ be a step function on [a,b]. We get a partition $a = x_0 < x_1, < ..... < x_n = b$ such that $\phi = a_i$ on (x_{i-1}, x_i) for $i = 1,2,.....n$ for some real numbers $a_1, a_2, ..., a_n$.

Let $E_i = (x_{i-1}, x_i)$, $i = 1,2,....n$. Let $\Psi = \sum_{i=1}^{n} a_i \chi_{E_i}$. Since E_i are measurable, Ψ is measurable.

Also $\phi = \Psi$ except possibly at the partition points $x_0, x_1, ..., x_n$. Since a countable set of real numbers has measure 0, $m(\{x_0, x_1,...., x_n\}) = 0$. Therefore $\phi = \Psi$ a.e., since Ψ is measurable by proposition 5.13, ϕ is measurable. Also, since ϕ assumes only a finite number of values, we get that ϕ is simple.

5.20 Theorem: Let f be a measurable function defined on an internal [a, b], and assume that f takes the values $\pm\infty$ only on a set of measure zero. Then, given $\in > 0$, we can find a step function g and a continuous function h such that $|f - g| < \in$ and $|f - h| < \in$ except on a set of measure less than $\in$; that is, $m(\{x: |f(x) - g(x)| \geq \in\}) < \in$ and $m(\{x / |f(x) - h(x)| \geq \in\}) < \in$. If in addition f is bounded and $m \leq f \leq M$, then we may choose the functions g and h so that $m \leq g \leq M$ and $m \leq h \leq M$.

Proof: With out loss of generality, we may assume that f is real valued.

Case 1 Suppose that $f = \chi_E$ and E is a measurable subset of [a, b]. By proposition 3.16 and 3.22, we get pairwise disjoint open intervals $I_2, I_2, ..., I_n$ such that

$m^*\left((\bigcup_{i=1}^{n} I_i) \Delta E\right) < \in$. Now $\mathbb{R} - \left(\bigcup_{i=1}^{n} I_i\right) = \bigcup_{i=1}^{m} J_i$, where $J_1, J_2, ..., J_m$ are pair wise disjoint intervals.

Let $t = \sum_{i=1}^{n} \chi_{I_i} + \sum_{i=1}^{m} s_i \chi_{J_i}$, where $s_i = 0$ for all $1 \leq j \leq m$.

Let g be the restriction of t to [a, b].

Now $\{x \in [a,b] / |f(x) - g(x)| \geq \in\} = \phi$, if $\in > 1$ and for $0 < \in \leq 1$,

$$\{ x \in [a,b] / |f(x) - g(x)| \geq \in\} \leq \left(\bigcup_{i=1}^{n} I_i\right) \Delta E$$

So, $m^*(\{x \in [a,b] / |f(x) - g(x)| \geq \in\}) < \in$. Next, choose a continuous function h on [a, b] such that.

$m^*(\{x \in [a,b] / |f(x) - h(x)| \geq \in\}) < \in$.

Case 2: Suppose that f is a simple function on [a, b] and $f = \sum_{i=1}^{n} a_i \chi_{E_i}$, where $a_1, a_2, \ldots, a_n$ are real numbers and $E_1, E_2, \ldots, E_n$ are pair wise disjoint measurable sets with $\bigcup_{i=1}^{n} E_i = [a, b]$. As seen in Case I, for each $1 \leq i \leq n$ We get a step function g_i and a continuous function h_i on [a, b] such that $m^*(\{x \in [a,b] / |a_i \chi_{E_i}(x) - g_i(x)| \geq \frac{\in}{n}\}) < \frac{\in}{n}$ and $m^*(\{x \in [a, b] / |a_i \chi_{E_i}(x) - h_i(x)| \geq \frac{\in}{n}\}) < \frac{\in}{n}$

Let $g = \sum_{i=1}^{n} g_i$ and $h = \sum_{i=1}^{n} h_i$.

g and h have the required properties.

Case 3: Suppose that f is a bounded measurable function on [a, b]

We get a simple function t on [a, b] such that $|f(x) - t(x)| < \in/2$ for all $x \in [a, b]$. The existence of such a simple function is proved later in Lesson 7, proposition 8. By case 2, we get a step function g and a continuous function h on [a, b] such that

$m^*(\{x \in [a, b] / |t(x) - g(x)| \geq \frac{\in}{2}\}) < \frac{\in}{2}$ and

$m^*(\{x \in [a, b] / |t(x) - h(x)| \geq \frac{\in}{2}\}) < \frac{\in}{2}$. For $x \in [a, b]$, $|f(x) - g(x)| \geq \in \Rightarrow$

$|f(x) - g(x)| \geq \in/2$ and $|f(x) - h(x)| \geq \in \Rightarrow |t(x) - h(x)| \geq \in/2$. Now g and h have the required properties.

Case 4: Suppose that f is an arbitrary measurable function. For each positive integer n, let $E_n = \{x \in [a, b] / |f(x)| \geq n\}$. Clearly E_n's are measurable sets, $E_n \supseteq E_{n+1}$ for all n and $\bigcap_{n=1}^{\infty} E_n = \phi$. So, $m^*(E_n) \to 0$ as $n \to \infty$, by proposition 3.15. We get a N such that

$m^*(E_N) < \frac{\in}{2}$. Let $\bar{f}$ be the function defined by $\bar{f}(x) = \begin{cases} f(x) \text{ if } x \notin E_N \\ N \text{ if } x \in E_N \end{cases}$

Now $\bar{f}$ is a bounded measurable function on [a,b].

So, by case 3, we get a step function g and a continuous function h on [a, b] such that

$m^*(\{x \in [a, b] / |\bar{f}(x) - g(x)| \geq \frac{\epsilon}{2}\}) < \frac{\epsilon}{2}$

and $m^*(\{x \in [a, b] / |\bar{f}(x) - h(x)| \geq \frac{\epsilon}{2}\}) < \frac{\epsilon}{2}$

Let $E = \{x \in [a, b] / |f(x) - g(x)| \geq \frac{\epsilon}{2}\}$

and $F = \{x \in [a, b] / |\bar{f}(x) - g(x)| \geq \frac{\epsilon}{2}\}$

Now $m^*(F) < \frac{\epsilon}{2}$.

so $\{x \in [a, b] / |f(x) - g(x)| \geq \epsilon\} \subseteq E \subseteq E_N \cup (E_N^1 \cap E) = E_N \cup (E_N^1 \cap F)$ where E_N^1 is the complement of E_N in [a, b].

So, we have $m^*(\{x \in [a, b] / |f(x) - g(x)| \geq \epsilon\}) \leq m^*(E_N) + m^*(F) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Similarly, it can be shown that $m^*(\{x \in [a, b] / |f(x) - h(x)| \geq \epsilon\} < \epsilon$

This completes the proof of the theorem.

5.21 Self Assessment Question:

Let D be a dense set of real numbers and let f be an extended real valued function on the set of real numbers such that $\{x / f(x) > \alpha\}$ is measurable for each $\alpha \in D$ then prove that f is measurable.

5.22 Self Assessment Question:

Give an example of a function f such that |f| is measurable but f is not.

5.23 Self Assessment Question:

Let D and E be measurable sets and f a function with domain $D \cup E$. Show that F is measurable iff its restrictions to D and E are measurable.

5.24 Self Assessment Question:

Let f be a function with measurable domain D. Show that f is measurable iff the function g defined by g(x) = f(x) for $x \in D$ and g(x) = 0 for $x \notin D$ is measurable.

5.25 Self Assessment Question:

Let f be an extended real valued function with measurable domain D and let

$D_1 = \{x / f(x) = \infty\}$ and $D_2 = \{x / f(x) = -\infty\}$. Then f is measurable iff D_1 and D_2 are measurable and the restriction of f to $D - (D_1 \cup D_2)$ is measurable.

5.26 Self Assessment Question:

Prove that the product of two measurable extended real valued functions is measurable.

5.27 Self Assessment Question:

If f and g are measurable extended real valued functions and α a fixed number, then f+g is measurable. If we defined f+g be α whenever it is of the form $\infty + -\infty$ or $-\infty + \infty$

5.28 Self Assessment Question:

Let f and g be measurable extended real valued functions that are finite atmost everywhere. Then f+g is measurable no matter how it is defined at points where it has the form $\infty + -\infty$ or $-\infty + \infty$.

5.29 Answers to Self Assessment Questions:

5.3 Let E be a non-measurable set contained in [0, 1). Define a real valued function f on the measurable set [0, 1) by $f(x) = \begin{cases} x \text{ if } x \in E \\ -x \text{ if } x \notin E \end{cases}$

Let α be a real number then, $\{x \in [0, 1): f(x) = \alpha\} \subseteq \{\alpha, -\alpha\}$ and hence

$\{x \in [0, 1) : f(x) = \alpha\}$ is measurable, since any finite set is measurable.

Also, $\{x \in [0, 1) : f(x) = \infty\} = \phi$, $\{x \in [0, 1) : f(x) = -\infty\} = \phi$ and hence measurable.

Thus f satisfies statement 5.

Now either $0 \in E$ or $0 \notin E$

If $0 \in E$, then $\{x \in [0, 1) : f(x) \geq 0\} = E$ is not measurable.

If $0 \notin E$ then $\{x \in [0, 1) : f(x) > 0\} = E$ is not measurable.

Therefore statement 5 does not imply any one of the first four statements in.

5.6 Let f be a measurable function with measurable domain D. Let E be a measurable subset of D. Let f/E be the restriction of f to E. Let α be a real number. Then.

$\{x \in E : f/E > \alpha\} = E \cap \{x \in D : f(x) > \alpha\}$ is measurable. Therefore, f/E is measurable.

5.7 Suppose f(x) = c for all $x \in E$, where E is a measurable set. Then for any real number α, we have, $\{x: f(x) > \alpha\} = \begin{cases} \phi \text{ if } \alpha \geq c \\ E \text{ if } \alpha < c \end{cases}$

Here ϕ and E are measurable sets and hence f is measurable.

5.21. Let D be a dense set of real numbers. Suppose that f is an extended real valued function on $\mathbb{R}$ such that $\{x: f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Let r be a real number. For each positive integer n there is an $\alpha_n \in D$ such that $r - \frac{1}{n} < \alpha_n < r$, since D is dense in $\mathbb{R}$. Now we have $\{x: f(x) < r\} = \bigcup_{n=1}^{\infty}\{x : f(x) < \alpha_n\}$. But, by hypothesis each set $\{x: f(x) < \alpha_n\}$ is measurable and hence $\{x: f(x) < r\}$ is measurable. Thus f is measurable.

5.22 Let E be a non-measurable set. Then, $\chi_E - \frac{1}{2}$ is not measurable, but $|\chi_E(x) - \frac{1}{2}| = \frac{1}{2}$ for all x so $|\chi_E - \frac{1}{2}|$ is measurable.

5.23 Let D and E be measurable set and f a function with domain $D \cup E$. by SAQ 5.6 the restrictions of f to D and E are measurable. Conversely suppose that the restrictions f_1 and f_2 of f to D and E are measurable. Let α be a real number, then

$\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f_1(x) > \alpha\} \cup \{x \in E : f_2(x) > \alpha\}$

Since f_1 and f_2 are measurable, $\{x \in E : f_1(x) > \alpha\}$ and $\{x \in D : f_2(x) > \alpha\}$ are measurable and hence $\{x \in D \cup E : f(x) > \alpha\}$ is measurable. Therefore f is measurable.

5.24 Let f be a function with measurable domain D. Now $\tilde{D}$ is also measurable and g is a function defined on $D \cup \tilde{D}$ by g(x) = f(x) for all $x \in D$ and g(x) = 0 for all $x \in \tilde{D}$. Suppose that f is measurable. The restriction of g to D is f and hence measurable. The restriction of g to $\tilde{D}$ is the constant function 0 and is measurable. Therefore g is measurable by 5.23. Conversely suppose that g is measurable since the restriction of g to D is f, we have f is measurable.

5.25 Let f be an extended real valued function with measurable domain D and let $D_1 = \{x: f(x) = \infty\}$ and $D_2 = \{x: f(x) = -\infty\}$. Suppose that f is measurable. By proposition 5.2, D_1 and D_2 are measurable. Also since $D_1 \cup D_2$ is measurable $D - (D_1 \cup D_2)$ is measurable and that the restriction of $f / D \sim (D_1 \cup D_2)$ is also measurable. Conversely suppose that D_1 and D_2 are measurable and the restriction f_1 of f to $D - (D_1 \cup D_2)$ is measurable. Let α be a real number. Then $\{x \in D : f(x) > \alpha\} = \{x \in D - (D_1 \cup D_2) : f_1(x) > \alpha\} \cup D_1$. Since f_1 is measurable, $\{x \in D \sim (D_1 \cup D_2) : f_1(x) > \alpha\}$ is measurable and that $\{x \in D : f(x) > \alpha\}$ is measurable, since D_1 is also measurable. Therefore, f is measurable.

5.26 Let f and g be measurable extended real valued functions defined on D.

Put $A_1 = \{x \in D : f(x) = \infty\}$ and $A_2 = \{x \in D : f(x) = -\infty\}$

$B_1 = \{x \in D : g(x) = \infty\}$ and $B_2 = \{x \in D = g(x) = -\infty\}$

$C_1 = \{x \in D : (fg)(x) = \infty\}$ and $C_2 = \{x \in D : (fg)(x) = -\infty\}$

Then $C_1 = \{x \in D : (fg)(x) = \infty\}$

$= \{x \in D : f(x)\, g(x) = \infty\}$

$= A_1 \cap \{x \in D : g(x) > 0\} \cup (B_1 \cap \{x \in D: f(x) > 0\}) \cup (A_2 \cap \{x \in D : g(x) < 0\}) \cup (B_2 \cap (x \in D : f(x) < 0\})$

Since f and g are measurable all the sets involved on the r.h.s. are measurable and hence C_1 is measurable.

$C_2 = \{x \in D : (fg)(x) = -\infty\}$

$= \{x \in D : f(x)\, g(x) = -\infty\}$

$= B_2 \cap \{x \in D : f(x) > \alpha\} \cup (A_1 \cap \{x \in D : g(x) < 0\}) \cup (B_1 \cap \{x \in D : f(x) < 0\}) \cup A_2 \cap \{x \in D : g(x) > \alpha\}$

Since f and g are measurable all the sets involved on the r.h.s are measurable and hence C_2 is measurable.

Let $E = [(A_1 \cup A_2) \cap \{x \in D / g(x) = 0\}] \cup [(B_1 \cup B_2) \cap \{x \in D / f(x) = 0\}]$

Since f and g are measurable all the sets involved on the right hand side of the above equality are measurable and hence E is measurable.

6

By our convention that $0.\infty = 0 = 0.(-\infty)$,

$(fg)(x) = f(x)\, g(x) = 0$ for all $x \in E$. Therefore $fg/E = 0$ is measurable since C_1, C_2 and E is measurable, $C_1 \cup C_2 \cup E$ is measurable and that $D - (C_1 \cup C_2 \cup E)$ is measurable.

Since f and g are measurable, the restrictions of f and g to $D - (C_1 \cup C_2 \cup E)$ are measurable. Moreover the restrictions of f and g to $D - (C_1 \cup C_2 \cup E)$ are also real valued functions and that by proposition 5.8, the restriction of fg to $D - (C_1 \cup C_2 \cup E)$ is measurable.

Now $(D - (C_1 \cup C_2 \cup E)) \cup E = D - (C_1 \cup C_2)$

Since the restriction of fg to $D - (C_1 \cup C_2 \cup E)$ and E are measurable by SAQ 5.6, the restriction of fg to $D - (C_1 \cup C_2)$ is measurable. As C_1 & C_2 are measurable fg is measurable.

5.27 Let f and g be measurable extended real valued functions defined on a common domain D and let α be a fixed number.

Let $A_1 = \{x \in D : f(x) = \infty\}$ and $A_2 = \{x \in D : f(x) = -\infty\}$

$B_1 = \{x \in D : g(x) = \infty\}$ and $B_2 = \{x \in D : g(x) = -\infty\}$

$C_1 = \{x \in D : (f+g)(x) = \infty\}$ and $C_2 = \{x \in D : (f+g)(x) = -\infty\}$

Now, $C_1 = \{x \in D : (f+g)(x) = \infty\} = (x \in D : f(x) + g(x) = \infty\}$

$= A_1 \cap \{x \in D : g(x) \neq -\infty\} \cup (B_1 \cap \{x \in D : f(x) \neq -\infty\})$

$= (A_1 - B_2) \cup (B_1 - A_2)$

Also $C_2 = \{x \in D : (f+g)(x) = -\infty\} = \{x \in D : f(x) + g(x) = -\infty\} =$

$= (A_2 \cap \{x \in D / g(x) \neq \infty\}) \cup (B_2 \cap \{x \in D : f(x) \neq \infty\})$

$= (A_2 - B_1) \cup (B_2 - A_1)$

since f and g are measurable, we get that A_1, A_2, B_1 and B_2 are measurable and hence C_2 is measurable. Put, $E = (A_1 \cap B_2) \cup (A_2 \cap B_1)$. Clearly, E is measurable.

Now $C_1 \cup C_2 \cup E$ is measurable. Thus $D - (C_1 \cup C_2 \cup E)$ is measurable and f an g are real valued functions on $D \sim (C_1 \cup C_2 \cup E)$. Therefore by proposition 5.8, f + g is a measurable function on $D \sim (C_1 \cup C_2 \cup E)$. Now, define $f + g = \alpha$ on E. Clearly, $f + g$ is measurable on E being a constant function on E. Therefore $f + g$ is measurable on $(D \sim (C_1 \cup C_2 \cup E)) \cup E = D - (C_1 \cup C_2)$. Since C_1 and C_2 are measurable, we have by SAQ 5.25, $f + g$ is measurable on D.

5.28 Let f and g be measurable extended real valued functions that are finite almost every where.

Let D be the common domain of f and g

Let $A = \{x \in D / f(x) = \pm\infty\}$ and $B = \{x \in D / g(x) = \pm\infty\}$

By our assumption $m(A) = m(B) = 0$

Therefore $m(A \cup B) = 0$. So every subset of $A \cup B$ is measurable. Now $D - (A \cup B)$ is a measurable subset of D and f, g are real valued measurable functions on $D - (A \cup B)$.

Therefore f+g is a measurable function on $D - (A \cup B)$ by proposition 7.6.

Let a be a real number. Suppose that f+g is defined arbitrarily on $A \cup B$.

$\{x \in D / f{+}g(x) > a\} = \{x \in D - (A \cup B) / \{f{+}g)(x) > a\} \cup \{x \in A \cup B / (f{+}g)(x) > a\}$

So f+g is measurable on $D - (A \cup B)$ $\{x \in D - (A \cup B) / (f + g)(x) > a\}$ is measurable.

Also as $\{x \in A \cup B / (f{+}g)(x) > a\}$ is a subset of $A \cup B$, $\{x \in A \cup B / f + g(x) > a\}$ is measurable. Therefore $f + g$ is measurable.

5.30 Model Examination Questions

5.30.1 Show that the sum and product of two simple functions are simple.

Show that

$$\chi_{A \cap B} = \chi_A \cdot \chi_B$$

$$\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$

$$\chi_{\tilde{A}} = 1 - \chi_A$$

5.30.2 Let c be a constant and f, g be measurable real valued functions defined on the measurable set E. Then prove that $f + g$, cf, fg are also measurable.

5.30.3. If f is measurable and f = g a.e then prove that g is also measurable.

5.30.4. Let <f_n> be a sequence of measurable functions (with the same domain of definition).then prove that the functions sup $\{f_1, f_2, - f_n\}$ and inf $\{f_1, f_2, \ldots, f_n\}$ then $\overline{\lim}\, f_n$ and $\underline{\lim}\, f_n$ are all measurable.

5.31 Exercises:

1. Give an example of a measurable function which is not continuous.
2. Show that constant functions are measurable.
3. Let $\{f_i\}$be a sequence of measurable functions converging a.e. to f. Show that f is measurable.
4. Show that the set of points on which a sequence of measurable functions $\{f_n\}$ converges, is measurable.
5. If f and g are measurable functions defined on E then prove that $\{x \in E / f(x) = g(x)\}$ and $\{x \in E / f(x) \neq g(x)\}$ are measurable.
6. If f is a measurable function then prove that f^+, f^- and $|f|$ are measurable.
7. Let f be a continuous function and g a measurable function show that the composite f o g is measurable.
8. Show that monotone functions are measurable.
9. Give an example of a function f such that $|f|$ is measurable but f is not.
10. Prove that for any non negative measurable function f defined on a measurable set E there is a sequence $\{\phi_n\}$ of simple functions such that $\phi_1 \leq \phi_2 \leq \phi_3 \leq \ldots\ldots \leq f$ and $\lim_{n\to\infty} \phi_n (x) = f(x)$ on E.
11. Let f be a measurable function on [a, b] . Given $\in > 0$ and M, show that there is a simple function ϕ such that $|f(x) - \phi (x)| < \in$ except where $|f(x)| \geq M$.

Reference: Real Analysis – H.L. Royden

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LESSON – 6

LITTLEWOOD'S THREE PRINCIPLES

6.1 Introduction: There are three important principles, identified by J.E. Littlewood which roughly say that measurable sets are 'nearly' finite union of open intervals, measurable functions are 'nearly' continuous functions and convergent sequences of measurable functions are 'nearly' uniformly convergent. Various forms of the first principle are given by proposition 3.16 one version of the second principle is given by theorem 5.20 another version by Lusin's theorem. The following proposition gives one version of the third principle. A slightly stronger form is given by Egoroff's theorem.

6.2 Proposition: Let E be a measurable set of finite measure and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x) \to f(x)$. Then given $\in > 0$ and $\delta > 0$, there is a measurable set $A \subseteq E$ with $m(A) < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \in$.

Proof: Let E be a measurable set of finite measure and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x) \to f(x)$.

Let $\in > 0$

Put $G_n = \{x \in E : |f_n(x) - f(x)| \geq \in\}$, $n = 1, 2, \ldots\ldots$

Since each f_n is measurable and $\{f_n\}$ converges point wise to f by result 5.10, f is measurable. So $f_n - f$ is a measurable function and that $|f_n - f|$ is measurable for all $n = 1, 2, \ldots$. So G_n is a measurable set for $n = 1, 2, \ldots\ldots$

Put $E_n =$ $= \{x \in E \mid |f_m(x) - f(x)| \geq \in$, for some $m \geq n\}$

Each E_n is measurable since each G_m is measurable.

Since $E_{n+1} \subseteq E_n$ for all n = 1,2, ... $\{E_n\}_{n=1}^{\infty}$ is a decreasing sequence of measurable sets.

We claim that, $\bigcap_{n=1}^{\infty} E_n = \phi$.

Suppose $x \in \bigcap_{n=1}^{\infty} E_n$. Since $\lim_{n\to\infty} f_n(x) = f(x)$, we get a positive integer k such that

$|f_n(x) - f(x)| < \in$ for all $n \geq k$. Now $x \notin E_k$. This is a contradiction to $x \in \bigcap_{n=1}^{\infty} E_n$.

Therefore $\bigcap_{n=1}^{\infty} E_n = \phi$. Since $E_1 \subseteq E$, $m(E_1) \leq m(E) < \infty$. Therefore by proposition 5.14,

$\lim_{n\to\infty} m(E_n) = m(\bigcap_{n=1}^{\infty} E_n) = m(\phi) = 0$. Let $\delta > 0$.

We get a positive integer N such that $m(E_n) < \delta$ for all $n \geq N$. Let $A = E_N$. Now $m(A) = m(E_N) < \delta$. For all $x \notin A = E_N$, $|f_n(x) - f(x)| < \in$ for all $n \geq N$. This completes the proof.

6.3 Proposition: Let E be a measurable set of finite measure and $\{f_n\}$ a sequence of measurable functions that converges to a real valued function f a.e. on E. Then, given $\in > 0$ and $\delta > 0$ there is a set $A \subseteq E$ with $m(A) < \delta$ and an N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \in$.

6.4 Definition: Let $\{f_n\}$ be a sequence of extended real valued function defined on a set E and f also be an extended real valued function defined on E. We say that $\{f_n\}$ converges point wise to f on E if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$. We say that $\{f_n\}$ converges point wise to f a.e. on E if there is subset B of E with m(B) = 0 such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E - B$.

Proof of Proposition 6.3: Let E be a measurable set of finite measure and $\{f_n\}$ a sequence of measurable functions that converges to a real valued function f a.e. on E. We get a subset $B \subseteq E$ such that m(B) = 0 and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E - B$. Let $\in > 0$

and $\delta > 0$. By proposition 6.2, there is a measurable set $A \subseteq (E \setminus B)$ with $m(A) < \delta$ and a positive integer N such that for all $x \in (E \setminus B) \setminus A$ and for all $n \geq N$, $|f_n(x) - f(x)| < \in$. Since A and B are measurable, $A \cup B$ is a measurable subset of E and $m(A \cup B) \leq m(A) + m(B) < \delta$ we have $(E \setminus B) \setminus A = E \setminus (A \cup B)$. Therefore $m(A \cup B) < \delta$ and for all $x \notin A \cup B$ and all $n \geq N$, $|f_n(x) - f(x)| < \in$.

6.5 SAQ: Give an example to show that we must require $m(E) < \infty$ in proposition 6.2.

Let $E = [1, \infty)$ $m(E) = \infty$. Let $f_n = \chi_{[n, n+1)}$, $n = 1,2,\ldots$. Let $x \in E$. We get a positive integer k such that $k \leq x < k + 1$ now $0 = f_{k+1}(x) = f_{k+2}(x) = \ldots\ldots$

Therefore $\lim_{n \to \infty} f_n(x) = 0$. Let $f = 0$

So $\{f_n\}$ converges point wise to f on $[1, \infty)$.

Suppose that proposition 6.2 is true for E and $\{f_n\}$.

Let $\in = \frac{1}{2}$ and $\delta = \frac{1}{2}$. We get a measurable set $A \subseteq E$ such that $m(A) < \frac{1}{2}$ and a positive integer N such that for all $x \notin A$ and for all $n \geq N$, $|f_n(x) - f(x)| < \frac{1}{2}$.

So, for all $x \notin A$, $|f_N(x) - f(x)| < \frac{1}{2}$.

$|f_N(x) - f(x)| < \frac{1}{2} \Rightarrow \chi_{[N, N+1)}(x) < \frac{1}{2} \Rightarrow x \notin [N, N+1)$

Therefore $x \notin A$ implies $x \notin [N, N+1)$ i.e. $x \in [N, N+1)$ implies $x \in A$. i.e. $[N,N+1) \subseteq A$.

So $1 = m([N, N+1)) \leq m(A) < \frac{1}{2}$, a contradiction.

Therefore the proposition 6.2 is not true for E and $\{f_n\}$ as m(E) not finite.

Therefore $m(E) < \infty$ is necessary in proposition 6.2.

6.6 Theorem: (Egoroff's theorem) If $\{f_n\}$is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure, then given $\in > 0$, there is a subset $A \subseteq E$ with $m(A) < \in$ such that $\{f_n\}$ converges to f uniformly on $E - A$.

Proof: Let $\{f_n\}$ be a sequence of measurable functions that converge to a real valued function f a.e. on a measurable set E of finite measure. Let $\in > 0$. By proposition 6.4, for each positive integer n we get a measurable subset A_n of E with $m(A_n) < \frac{\in}{2^n}$ and a positive integer K_n such that for all $x \notin A_n$ and $m \geq K_n$, $|f_m(x) - f(x)| < \frac{1}{n}$.

Let $A = \bigcup_{n=1}^{\infty} A_n$. A is measurable as each A_n is measurable. Now $A \subseteq E$ and $m(A) \leq \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} \frac{\in}{2^n} = \in$.

We see now that $\{f_n\}$ converges uniformly to f on E – A. Let $\delta > 0$. We get a positive integer n such that $\frac{1}{n} < \delta$. For $x \in E - A \subseteq E - A_n$ and $m \geq K_n$, $|f_m(x) - f(x)| < \frac{1}{n} < \delta$.

Therefore $\{f_n\}$ converges uniformly to f on E – A.

6.7 Theorem: (LUSIN'S THEOREM) Let f be a measurable real – valued function on an interval [a,b]. Then given $\delta>0$, there is a continuous function ϕ on [a, b] such that $m(\{x / f(x) \neq \phi(x)\}) < \delta$.

Proof: Let f be a measurable real valued function on an interval [a, b]. Let $\delta > 0$. By proposition 5.20 for each positive integer n there is a continuous function h_n and a measurable subset A_n of [a, b] such that $|h_n(x) - f(x)| < \frac{\delta}{2^{n+1}}$ for all $x \notin A_n$ and

$m(A_n) < \frac{\delta}{2^{n+1}}$. Let $E = \bigcap_{n=1}^{\infty} \tilde{A}_n$. As A_n is measurable, $\tilde{A}_n$ is measurable and that E is measurable, where $\tilde{A}_n = [a,b] - A_n$. Now $E \subseteq [a, b]$ and that $m(E) \leq m([a,b]) = b-a < \infty$. For $x \in E$, $x \in \tilde{A}_n$ for all n = 1,2, ... and that $|h_n(x) - f(x)| < \frac{\delta}{2^{n+1}}$ for all n = 1,2, ... So for each $n \in E$, $\lim_{n\to\infty} h_n(x) = f(x)$. By example 5.5, each h_n is measurable. By Egoroff's theorem, there is a measurable set $A \subseteq E$ with $m(A) < \frac{\delta}{4}$ and $\{h_n\}$ converges uniformly to f on E – A. Since E – A is measurable by proposition 3.16, there is a closed set

$F \subseteq E - A$ such that $m((E - A) - F) < \frac{\delta}{4}$. Since the sequence $\{h_n\}$ of continuous functions on $E - A$ and converges uniformly to f on $E - A$, f is continuous on $E - A$. As $F \subseteq E - A$, f is continuous on F. Since $F \subseteq [a, b]$ and f is continuous on the closed set F, f can be extended to a continuous function g on [a, b]. So $f(x) = g(x)$ for all $x \in F$ and g is continuous on [a, b]. Now $\{x \in [a,b] \mid f(x) \neq g(x)\} \subseteq \tilde{F} \subseteq (\tilde{E} \cup A) \cup ((E - A) - F)$.

For $x \in \tilde{F}$, if $x \in E - A$ then $x \in (E - A) - F$ and if $x \notin E - A$ then $x \in \widetilde{E - A} = \widetilde{E \cap \tilde{A}} = \tilde{E} \cup A$.

So $m(\{x \in [a, b] / f(x) \neq g(x)\}) \leq m(\tilde{F}) \leq m(\tilde{E}) + m(A) + m((E\text{-}A)\text{-}F)$

$m(\tilde{E}) \left(\widetilde{\bigcap_{n=1}^{\infty} \tilde{A}_n}\right) = m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n) < \sum_{n=1}^{\infty} \frac{\delta}{2^{n+1}} = \frac{\delta}{2}$. Also $m(A) < \frac{\delta}{4}$ and $m((E \setminus A) \setminus F) < \delta/4$. Therefore $m(\{x \in [a, b] \mid f(x) \neq g(x)\}) < \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4} = \delta$.

Hence g is a continuous function on [a, b] such that $f(x) = g(x)$ except on a set of measure less than δ.

6.8 Model Examination Questions:

6.8.1 Let E be a measurable set of finite measure, and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x) \to f(x)$. Then, prove that given $\in > 0$ and $\delta > 0$, there is a measurable set $A \subseteq E$ with $m(A) < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \in$.

6.8.2. State and prove Egoroff's theorem

6.8.3. State and prove Lusin's theorem

6.8.4. Show that the condition m(E) is finite is necessary in Egoroffs theorem.

Reference Book: Real Analysis – H.L. Royden

Lessor Writer: R. SRINIVASA RAO

LESSON 7 : LEBESGUE INTEGRAL - DEFINITIONS AND ELEMENTARY PROPERTIES

INTRODUCTION :

The Riemann integral, inspite of its utility in finding areas and volumes, has its own limitations and shortcomings to mention a few, integrability of $|f|$ does not necessarily guarantees integrability of f. Convergence does not necessarily have compatibility with integral. More precisely point wise convergence of a sequence of Riemann integrable functions does not imply integrability of the limit function, even so the limit of the sequence of integrals may not exist and even if this limit exists the limit of the sequence of integral may not be the integral of the limit.

A more useful integration theory that overcomes these drawbacks is developed by Henry Lebesgue, after whom the integral is appropriately named.

This lesson and the next one are devoted to a systematic development of the Lebesgue integral. In contrast to the Riemann partitions of the domain we divide the range into disjoint measurable sets and define the integral in terms of the measures of sets. As such we start with the definition of the in. gral of a characteristic function, extend this to finite linear combinations of such functions. These are called simple functions - and then take up the integral of a bounded measurable functions establish its linearity among other things.

We use the integral of a bounded measurable function to define the integral of a non negative measurable function. We define integrability of a non negative measurable function and extend this to arbitrary measurable functions. We finally show among others that this integral possess linearity properties and monotonicity.

Integral of a simple function :

Definitions : If $E \subseteq IR$ the characteristic function χ_E of E is defined by :

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$$

A finite linear combination of characteristic functions of measurable sets is called a simple function.

Remarks :(1). ϕ is a simple function if and only if there exist finitely many measurable sets $E_1, \ldots E_n$

and numbers $a_1, a_2, \ldots, a_n$ such that , $\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$

It is clear that the range of a simple function is finite. However the above representation is not necessarily unique since the ruler function $\chi_R = \chi_{(-\infty,0)\cap R} + \chi_{(0,\infty)\cap Q}$.

If ϕ is a simple function with non zero range $\{a_1, \ldots, a_n\}$ and $A_i = \{x / \phi(x) = a_i\}$ then each A_i is measurable and $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x), \forall x$

This representation of ϕ is called the canonical representation of ϕ.

(2) If E is measurable χ_E is a measurable function because if $\alpha \in IR$,

$$\{x / \chi_E(x) > \alpha\} = \begin{cases} \phi \text{ if } \alpha \geq 1 \\ IR \text{ if } \alpha < 0 \text{ and} \\ E \text{ if } 0 \leq \alpha < 1 \end{cases}$$

Since a finite linear combination of measurable functions is measurable, every simple function is measurable.

(3) If ϕ is a simple function and E is a measurable set $\phi\,\chi_E$ is a measurable function because

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

$$\Rightarrow \quad \phi . \chi_E = \sum_{i=1}^{n} a_i \chi_{E_i \cap E}$$

3. Definition : Let ϕ be simple function which vanishes outside a set of finite measure. If the canonical representation of ϕ is given by $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$

We define the integral of ϕ by : $\int \phi(x)\,dx = \sum_{i=1}^{n} a_i m(A_i)$

This integral is some times denoted by $\int \phi$. If E is any measurable set we define the integral of ϕ over E by $\int_E \phi = \int \phi . \chi_E$

4. Example : The Lebesgue integral of $\chi_Q \int \chi_Q = 0$ since m(Q) = 0.

5. Lemma : Let ϕ be a simple function which vanishes outside a set of finite measure.

If $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ where each E_i is a measurable set of finite measure and $E_i \cap E_j = \phi$ if $i \neq j$ then

$$\int \phi = \sum_{i=1}^{n} a_i\, m(E_i)$$

Proof : The representation for ϕ in the statement is not necessarily the canonical representation as some a_i's could be equal. However this can be reduced to the canonical representation as follows. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the distinct non zero members of $\{a_1, \ldots, a_n\}$ so that $1 \leq r \leq n$. Each α_j may repeat a number of times. For each j let $A_j = \{x / \phi(x) = \alpha_j\}$. Then A_j is the union of those E_i for which $a_i = \alpha_j$. Hence A_j is measurable and $\phi = \sum_{j=1}^{r} \alpha_j \chi_{A_j}$ is the canonical representation of ϕ.

$$\text{Hence } \int \phi = \sum_{j=1}^{r} \alpha_j \chi_{A_j} = \sum_{j=1}^{r} \alpha_j m(A_j)$$

$$= \sum_{j=1}^{r} \alpha_j \, m\Big(\bigcup_{a_i=\alpha_j} E_i\Big)$$

$$= \sum_{j=1}^{r} \alpha_j \sum_{a_i=\alpha_j} m(E_i)$$

$$= \sum_{j=1}^{r} \sum_{a_i = \alpha_j} \alpha_j m(E_i) = \sum_{i=1}^{n} a_i m(E_i)$$

6. Proposition :

Let ϕ, be simple functions each of which vanishes outside a set of finite measure. Then for any a, b :

(i) $\int(a\phi + b\psi) = a\int\phi + b\int\psi$ and

(ii) $\int\phi \geq \int\psi$ if $\phi(x) \geq \psi(x)$ a.e.

Proof :

(i) Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $= \sum_{j=1}^{m} b_j \chi_{B_j}$ be the canonical representations of ϕ *and* ψ.

and let $A_0 = \{x / \phi(x) = 0\}$ and $B_0 = \{x / \quad (x) = 0\}$. For each pair (i,j) where $1 \leq i \leq n$ and $1 \leq j \leq m$ write $E_{ij} = A_i \cap B_j$. The set $\{E_{ij} / \forall i \leq i \leq n, 1 \leq j \leq m\}$ are measurable, pairwise disjoint and $\phi(x) = 0$ for $x \in E_{0,j}$ as well as $E_{i,0}$

$$A_i = A_i \cap \left(\bigcup_{j=0}^{m} B_j \right) = \bigcup_{j=0}^{m} E_{ij} \text{ and } \phi = \sum_{i=1}^{n} a_i \chi_{A_i} = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{E_{ij}}$$

By lemma 4 $\int\phi = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i m(E_{ij})$

Similarly, $\int\psi = \sum_{j=1}^{m} \sum_{i=1}^{n} b_j m(E_{ij})$

Since $a\phi + b\psi = \sum_{i=1}^{n} \sum_{j=1}^{m} (aa_i + bb_j) \chi_{E_{ij}}$, $\int a\phi + b\psi = \sum_{i=1}^{n} \sum_{j=1}^{m} (aa_i + bb_j) m(E_{ij})$

Hence $\int a\phi + b\psi = a\int\phi + b\int\psi$.

(ii) We first prove that f> o a.e then $\int f$ =0 and deduce the general case from (i). Now assume that f is a simple function with canonical representation $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $f \geq 0$ a.e., $a_i \geq 0$ for $1 \leq i \leq r$ and $a_i < 0$ if $i > r$.

Then $m(A_i) = 0$ since $\{ x / f(x) < 0\} = \bigcup_{i=r+1}^{n} A_i$ has measure zero.

Hence $\int f = \sum_{i=1}^{n} a_i m(A_i) \geq 0$ since $a_i \geq 0$ for $1 \leq i \leq r$.

In the general case $\phi - \psi$ is a simple function and $\phi - \psi > 0$ a.e.,

Hence $\int \phi - \int \psi = \int \phi - \psi \geq 0$. This implies that $\int \phi \geq \int \psi$.

7. If $E_1, E_2, \ldots, E_n$ are measurable sets such that $m\left(\bigcup_{i=1}^{n} E_i\right) < \infty$ and $a_1, \ldots a_n$ are real numbers and $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ then $\int \phi = \sum_{i=1}^{n} a_i m_{E_i}$.

Proof : Follows from Proposition 6.

Remark : From this corollary it follows that when ϕ is simple. $\int \phi$ is independent of the representation of ϕ.

8. Proposition :

Let f be defined and bounded on a measurable set E with finite measure. Then the following are equivalent

(1) f is measurable

(2) $\inf_{f \leq \psi} \int_E \psi(x)\, dx = \sup_{\phi \leq f} \int_E \phi(x)\, dx$.........* where ϕ and ψ are simple functions

Proof : Denote the lhs of (*) by A and the rhs by B. If ϕ *and* ψ are simple functions such that $\phi(x) \leq f(x) \leq$ (x) on E, $\phi \chi_E \leq \chi_E$, hence $\int_E \phi = \int \phi \chi_E \leq \int \psi \chi_E = \int_E \psi$

This being true for every $\phi, \psi \ni \phi \leq f \leq$. $\int_E \psi$ is an upper bound for the set

$\{\int_E \phi(x)\,dx \leq f(x)$ on E$\}$ so that $B \leq \int_E \psi$ $\forall$ * simple function $\geq f$ on E.

Now taking the infimum on we get $B \leq A$.

We now prove that (1) $\Rightarrow$(2).

Since f is bounded, $\exists$ a $M > 0 \ni -M \leq f(x) \leq M$ for $x \in E$. Fix a positive integer n and write

$$E_k = \{x / x \in E \text{ and } \frac{(k-1)M}{n} \leq f(x) \leq \frac{kM}{n}\}, (-n \leq k \leq n),$$

$$\phi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\chi_{E_k}(x)$$

and $\quad {}_n(x) = \frac{M}{n} \sum_{k=-n}^{n} \frac{k}{n} \chi_{E_k}(x)$ clearly ϕ_n, ψ_n are simple functions

satisfying $\phi_n(x) \leq f(x) \leq {}_n(x)$ on E. Hence

$$\int_E \phi_n(x)\,dx = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\,m(E_k) \leq B \leq A \leq \int_E \psi_n = \frac{M}{n} \sum_{k=-n}^{n} km(E_k)$$

$$\Rightarrow \quad 0 \leq B - A < \frac{M}{n} \sum_{k=-n}^{n} m(E_k) = \frac{M}{n} m(E)$$

This being true $\forall$ positive integer n it follows that B = A.

To prove (2) $\Rightarrow$ (1).

Since B = A for every positive integer n. $\exists$ si... functions ϕ_n, ${}_n$ such that $\phi_n(x)$ · f(x) ·

$\forall$ x $\in$E and A - $\frac{1}{2n}$ < $\int_E \phi_n(x)dx \leq \int_E \psi_n(x)dx$ < A + $\frac{1}{2n}$.

So that $\int_E (\psi_n - \phi_n)(x)\,dx < \frac{1}{n}$ ..... (1)

Write * (x) = $\inf_n \psi_n(x)$ and $\varphi^*(x) = \sup_n \phi_n(x)$ (x $\in$ E)

Since each ϕ_n, $_n$ are measurable ϕ * and * are measurable .

More over ϕ * (x) $\leq$ f(x) $\leq$ * (x) for x $\in$E.

If we show that ϕ * (x) = * (x) for almost all x in E it will then follow that f = ϕ * = * a.e. on E. From the measurability of ϕ * we can then conclude that f is measurable.

To this end, write Δ = { x $\in$E / ϕ *(x) < * (x) }

and $\forall$ positive integer k, Δ_k = { x $\in$E / ϕ *(x) < * (x) - 1/k}

and $\forall$ positive integers n, k $\Delta_k^{(n)}$ = { x $\in$E / ϕ_n(x) < $_n$(x) - 1/k}

clearly $\Delta = \bigcup_{k=1}^{\infty} \Delta_k$ so m(Δ) $\leq \sum_{k=1}^{\infty} m(\Delta_k)$ ..... (2)

Also $\Delta_k \leq \bigcap_{n=1}^{\infty} \Delta_k^{(n)}$ since x $\in \Delta_k \Rightarrow \forall$ positive integer n,

$\phi_n(x) \leq \sup_n \phi_n(x) \leq \phi$ * (x) < * (x) - $\frac{1}{k} \leq$ $_n$(x) - $\frac{1}{k}$

So that $x \in \Delta_k^{(n)}$ $\forall$ n and hence x $\in \bigcap_{n=1}^{\infty} \Delta_k^{(n)}$

We now show that m(Δ_k) = 0 $\forall$ k. Since $_n$(x) - ϕ_n(x) > 1/k for x $\in \Delta_k^{(n)}$

$\int_E \chi_{\Delta_k^{(n)}}(x)dx \leq \int_E (\psi_n - \phi_n)(x)\,dx < \frac{1}{n}$ by (*)

Hence $0 \le \frac{1}{k} m(\Delta_k^{(n)}) < \frac{1}{n} \ \forall n$

$$\Rightarrow \quad 0 \le \frac{1}{k} m(\Delta_k) \le \frac{1}{k} m(\Delta_k^{(n)}) \le \frac{1}{n} \ \forall \text{ n.}$$

$$\Rightarrow \quad 0 \le m(\Delta_k) \le \frac{k}{n} \ \forall \text{ n.} \qquad \text{Since } \lim_{n} \frac{1}{n} = 0$$

it follows that $m(\Delta_k) = 0$.

From (2) we get $m(\Delta) = 0$.

Hence $\phi^* = f = \quad ^*$ a.e., on E and hence f is measurable.

Remark : Since f is bounded $\exists$ a $m_0 \in$ IR $\ni m \le f(x)$ for all $x \in E$. If is any simple function such that $f \le$ then $m_r <$ so, $m_0\, m(E) < \int_E \psi$. This shows that lhs of $*$ in the above proposition is a real number. Likewise the rhs is also a real number. It is thus clear that if f is a bounded measurable function. Then the equal value in ($*$) is a real number.

9. Integral of a bounded measurable function :

Definition : Let f be a bounded measurable function defined on a measurable set E with finite measure. We define the Lebesgue integral of f over E by,

$$\int_E f(x)dx \quad = \inf \{ \int_E \psi(x)dx \ / \quad \text{simple function} \ni f(x) \le \quad (x) \text{ for all } x \in E\}$$

This integral is some times denoted by $\int_E f$.

If E = [a, b] we write $\int_{[a,b]} f = \int_a^b f$. If f is a bounded measurable function (defined on IR) which vanishes outside a set E of finite measure we write $\int f$ for $\int_E f$.

Remark : When f is a simple function vanishing outside a set E which is measurable and has finite measure then according to the above definition $\int_E f = \int f = \int f.\chi_E$.

Since f = f. χ_E on E, $\int_E f$ is a member of the set whose infimum is defined as the integral. For every other the condition f ≤ on E implies that $\int_E f \le \int_E \psi$ hence $\int_E f$ is the infimum of the rhs in the above definition and is real. Thus this definition is consistent with the defnition of the integral for a simple function that vanishes outside a measurable set of finite measure.

Comparision with the Riemann Integral :-

10. **Theorem :** Let f be a bounded function defined on [a,b]. If f is Riemann integrable on [a,b] then f is measurable and the Riemann integral $R\int_a^b f(x)dx$ = the Lebesgue integral $\int_{[a,b]} f(x)dx$.

Proof : By definition f is Riemann integrable iff

$\inf_p U(p,f) = \sup_p L(p,f)$, where the infimum and supremum are taken over all partitions

$p = \{a = x_0 < x_1 < \ldots\ldots < x_n = b\}$: $L(p,f) = \sum_{i=1}^{n} m_i \Delta x_i$ and $U(p,f) = \sum_{i=1}^{n} M_i \Delta x_i$.

Where m_i = g. l. b. $\{ f(x) / x_{i-1} \le x \le x_i \}$ amd M_i = lub $\{ f(x) / x_{i-1} \le x \le x_i \}$ and $\Delta x_i = x_i - x_{i-1}$

For such a partition p,

Write $\phi_p = \sum_{i=1}^{n} m \chi_{(x_{i-1}, x_i)}$ and $p = \sum_{i=1}^{n} M_i \chi_{(x_{i-1}, x_i)}$

and at each x_i, let $\phi_p(x_i) = \psi_p(x_i) = f(x_i)$.

Then $\phi_p \le f \le \psi_p$ and ϕ_p ψ_p are simple functions.

Also $\int_E \phi_p(x)dx$ = L(p,f) and U(p,f) = ∪(p,f) so that

$$L(p,f) = \int_E \phi_p(x)dx \le \sup_{\phi \le f\ [a,b]} \int \phi(x)dx \le \inf_{\ge f^*[a,b]} \int \psi(x)\,dx \le U(p,f)$$

Since this is true for every partition p and f is Riemann integrable on [a,b] we get

$$\sup_{<[a,b]} \int \psi(x)\,dx \le \inf_{\psi \ge [a,b]} \int \psi(x)\,dx \le R\int_a^b f(x)\,dx$$

Hence the three quantities in the above inequality become equalities. This implies that f is measurable and

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dx = R\int_a^b f(x)dx$$

Linearity of the Integral for bounded measurable functions :

11. Proposition :

If f and g are bounded measurable functions defined on a measurable set E of finite measure then

(i) $\int_E f+g = \int_E f + \int_E g$ and

(ii) $\forall\ a \in IR$, $\int_E af = a\int_E f$

Proof :

(i) if $_1$, $_2$ are simple functions such that f(x) ≤ $_1$(x) on E and g(x) ≤ $_2$(x) on E then (f + g)(x) ≤ ($_1$+ $_2$) (x) on E. Since $_1$+ $_2$ is a simple function it follows that :

$$\int_E f+g \le \int_E (\psi_1+\psi_2) = \int_E \psi_1 + \int_E \psi_2 \text{ .... by 5}$$

Fixing first $_2$ and taking the infimum for all $_1$≥ f we get : $\int_E (f+g) = \int_E f + \int_E \psi_2$

Now taking the infimum for all $_2 \geq g$ we get $\int_E (f+g) = \int_E f + \int_E g$ ...(1)

If ϕ_1, ϕ_2 are simple functions such that $\phi_1(x) \leq f(x)$ on E and $\phi_2(x) \leq g(x)$ on E then $\phi_1 + \phi_2$ is a simple function and $(\phi_1 + \phi_2)(x) \leq (f+g)(x)$ on E so that, as above :

$$\int_E \phi_1 + \int_E \phi_2 \leq \int_E (f+g)$$

As above taking the supremum for all $\phi_1 \leq f$ on E, keeping ϕ_2 fixed and then taking the supremum for all $\phi_2 \leq g$ on E we get $\int_E f + \int_E g \leq \int_E (f+g)$ ... (2)

From (1) and (2) we get $\int_E (f+g) = \int_E f + \int_E g$.

Proof : (ii) If a = 0, lhs = rhs = 0

If a > 0 then $\psi \geq af \Leftrightarrow \frac{1}{a}\psi f$. Also ψ is a simple function iff $a\psi$ is simple

$$\int_E af = \inf_{af \leq \psi} \int_E \psi = \inf_{f \leq \frac{\psi}{a}} \int_E \psi = \inf_{f \leq \psi_1} \int_E a\psi_1 \quad (\psi, \psi_1, \text{simple function})$$

$$= \inf_{f \leq \psi_1} a \int_E \psi_1 = \inf_{f \leq \psi_1} \int_E \psi_1 = a \int_E f$$

Let a = -1 If we write -A for the set {-x/ x ∈ A} we have inf(-A) = - sup A and sup (-A) = -inf A Also ϕ is a simple function iff $-\phi$ is a simple function.

$$\text{Thus } \int_E -f = \inf_{-f \leq \psi} \int_E \psi = \inf_{-\psi \leq f} \int_E \psi$$

$$= \inf_{\phi \leq f} \int_E -\phi = \inf_{\phi \leq f} -\int_E \phi \text{ by (5)}$$

$$= -\sup_{\phi \leq f} \int_E \phi = -\int_E f$$

If $a < 0$, $-a > 0$ so $\int_E af = -\int_E (-a)(-f) = -a\int_E (-f) = a\int_E f$

Proposition : If A and B are disjoint measurable sets of finite measure and f is defined, bounded and measurable on A U B.

Proof : If ψ is a simple function such that $f(x) \leq \psi(x)$ on AUB then $\int_{A\cup B} f \leq \int_{A\cup B} \psi$

For any simple function g,

$$\int_{A\cup B} g = \int g\,\chi_{(A\cup B)} = \int g(\chi_A + \chi_B) = \int g\,\chi_A + \int g\,\chi_B = \int_A g + \int_B g$$

Hence $\forall\ \psi \geq f$, $\int_{A\cup B} \psi = \int_A \psi + \int_B \psi \geq \int_A f + \int_B f$

So that, $\int_{A\cup B} f = \inf_{\psi\geq f} \int_{A\cup B} \psi \geq \int_A f + \int_B f$ (1)

Replacing f by -f we get , $-\int_{A\cup B} f = \int_{A\cup B} -f \geq \int_A -f + \int_B -f$ so that

$$\int_{A\cup B} f \geq \int_A f + \int_B f \qquad (2)$$

combining (1) and (2) we get equality

Proposition : Let f and g be bounded measurable functions defined on a measurable set E with finite measure. Show that :

(i) If $f \geq g$ a.e., on E $\int_E f \geq \int_E g$

(ii) If $f = g$ a.e., on E $\int_E f = \int_E g$

(iii) $\left|\int_E f\right| \leq \int_E |f|$

(iv) If $A \leq f(x) \leq B$ a.e. on E then $A\, m(E) \leq \int_E f \leq B\, m(E)$

Proof :

(i) $f \geq g \Leftrightarrow f - g \geq 0$. Since $\int_E f - g = \int_E f - \int_E g$ it is enough to prove the result when $g = 0$.

In this case $f \geq 0$ a.e. on E. Hence any simple function $\psi \geq f_1$, satisfies $\psi \geq 0$ a.e. on E. Hence by $\int_E \psi \geq 0$.

$$\text{Hence } \int_E f = \inf_{\psi \geq f} \int_E \psi \geq 0$$

(ii) $f = g$ a.e. $\Leftrightarrow f \geq g$ a.e and $g \geq f$ a.e.

Now the equality of the integrals is a consequence of (i) above.

(iii) $-|f(x)| \leq f(x) \leq |f(x)| \ \forall\, x \in E$. Also f and $|f|$ are simultaneously measurable. Hence (i) and

$$-\int_E |f| \leq \int_E f \leq \int_E |f|$$. This implies that $\left|\int_E f\right| \leq \int_E |f|$

(iv) Since $\int_E dx = m(E)$, by (i) it follows that $A\, m(E) \leq \int_E f(x)dx \leq B\, m(E)$.

The integral of a non negative function :

12. Definition : If f is a non negative measurable function defined on a measurable set E. We define the integral of f over E by $\int_E f = \sup_{h \leq f} \int_E h$

where the supremum is taken over all bounded measurable functions h which vanish outside a set of finite measure.

Remark : The integral of a bounded measurable function defined on a measurable set with finite measure is finite. However that the integral of a non negative measurable function is finite. It is possible that this value is ∞.

Linearity of the integral :

Theorem : If f and g are non negative measurable functions defined on a measurable set E then

(i) $\int_E f+g = \int_E f + \int_E g$ and

(ii) $\int_E Cf = C\int_E f$ if $C > 0$

Proof (i):

If h and k are bounded measurable functions on E such that $\{x / h(x) \neq 0\}$ and $\{x / k(x) \neq 0\}$ have finite measure and $h(x) \leq f(x)$ and $g(x) \leq k(x)$ $\forall$ $x \in E$ then :

$\{x / h(x) + k(x) \neq 0\} \subseteq \{x / h(x) \neq 0\} \cup \{x / k(x) \neq 0\}$ hence has finite measure and $h(x) + k(x) \leq f(x) + g(x)$ $\forall$ $x \in E$.

Hence $\int_E h + \int_E k = \int_E (h+k) \leq \int_E (f+g)$

Since this is true for every such h and k, we have : $\int_E f + \int_E g \leq \int_E (f+g)$.

To prove the reverse inequality let l be a bounded measurable function such that $\{x / l(x) \neq 0\}$ has finite measure and $l(x) \leq (f+g)(x)$ $\forall$ $x \in E$. .... (1)

write $h(x) = \min\{f(x), l(x)\}$ clearly h(x) is measurable.

If $A \leq l(x) \leq B$. Then $\min\{A,0\} \leq h(x) \leq B$ $\forall$ $x \in E$ So that h is bounded. Since $l(x) \geq h(x)$ $\forall$ x, $h(x) \neq 0 \Rightarrow l(x) \neq 0$, hence $\{x / h(x) \neq 0\} \subseteq \{x / l(x) \neq 0\}$. Since this second set on the rhs has finite measure, the set on the left has finite measure.

If $k(x) = l(x) - h(x)$, k(x) is a bounded measurable function mE $h(x) \leq f(x)$.

Since $h + k = l$, and $h(x) = \min\{f(x), l(x)\}$

$$\begin{aligned} k = l - h &= l - \min\{f, l\} \\ &= \max\{l - f, 0\} \\ &\leq \max\{g, 0\} = g \;(\; g \geq 0) \end{aligned}$$

Since h and k are bounded measurable functions vanishing outside sets with finite measure and satisfy $h \leq f$ & $h \leq g$.

$$\int_E l = \int_E h+k = \int_E h + \int_E k \le \int_E f + \int_E g$$

Since this is true for every such l, it follows that $\int_E (f+g) \le \int_E f + \int_E g$ ..... (2)

From (1) and (2) we get $\int_E (f+g) = \int_E f + \int_E g$.

Proof (ii) :

By Definition $\int_E f = \sup_{h \le Cf} \int_E h$ where the supremum is taken over all bounded measurable functions h such that { x / h(x) $\neq 0$ } has finite measure since $c > 0$, $\frac{1}{c}$ h(x) = 0 $\Leftrightarrow$ h(x) = 0. Hence { x / $\frac{1}{c}$ h(x) $\neq 0$ }has finite measure. Further h(x) $\le$ c f(x) if and only if $\frac{h(x)}{c} \le$ f(x) hence

$$\sup_{h \le Cf} \int_E h = \sup_{h/C \le f} \int_E h = \sup_{h_1 \le f} \int_E C h_1$$

$$= \sup_{h_1 \le f} C \int_E h_1$$

$$= C \sup_{h_1 \le f} \int_E h_1 = C \int_E f$$

Where the supremum is taken over all bounded measurable functions $\ni$ h1(x) $\le$ f(x) $\forall$ x $\in$ E and {x / h1(x) $\neq 0$} has finite measure. Hence $\int_E Cf = C \int_E f$

The general Lebesgue integral

For any real number a, we define the

postive part a^+ and negative part a^- by :

$$a^+ = a \vee 0 = \max\{a,0\} \text{ and } a = \max\{-a,0\}.$$

Clearly $a^- = (-a)^+$, $a^+ \geq 0$, $a^- \geq 0$, $a = a^+ - a^-$ and $|a| = a^+ - a^-$

Remark : For any real numbers a, b $(a+b)^+ \leq a^+ + b^+$ and $(a+b)^- \leq a^- + b^-$.

13. Definition : A non negative measurable function f defined on a measurable set E is said to be integrable on E if $\int_E f < \infty$.

Remark :

If f is a nonnegative measurable function defined on a measurable set E then $A = \{x / x \in E$ and $f(x) = \infty\}$ has measurable zero (See SAQ) unless m(A) = 0.

14. Definition : If f is defined and measurable on a measurable set E we say that f is integrable over E if the functions f^+, f^-. defined by $f^+(x) = f(x)+$ and $f^-(x) = f(x)-$ are integrable over E. In this we call $\int_E f^+ - \int_E f^-$ the integral of f over E and write $\int_E f(x)\,dx = \int_E f = \int_E f^+ - \int_E f^-$

Linearity of the integral :

15. Proposition :

If f and g are defined and integrable over a measurable set E. Then (i) f + g is integrable and $\int_E (f+g) = \int_E f + \int_E g$ and (ii) For every $c \in \mathbb{R}$, cf is integrable and $\int_E cf = c\int_E f$

Proof of (i) : f + g is defined on a subset A of E at each point x of which f(x) + g(x) is not of the form $\infty - \infty$ or $-\infty + \infty$. Since f is integrable the set $A(f) = \{x \in E / f(x) = \pm\infty\}$ and likewise $A(g) = \{x \in E / g(x) = \pm\infty\}$ are both of measure zero. Hence $A_0 = A(f) \cup B(f)$ has measure zero. Clearly $A \subseteq E, A_0$. Since $m(A_0) = 0$ the integrability is not affected by assigning any constant value on A_0. Then we may assume without loss of generality that f + g is defined on E itself.

For every $x \in E$, $f(x) \leq f^+(x)$ and $g(x) \leq g^+(x)$ so that $(f+g)(x) \leq (f^+ + g^+)(x)$. Since $0 \leq (f^+ + g^+)(x)$ it follows that $(f+g)^+(x) \leq (f^+ + g^+)(x)\ \forall\ x \in E$. so that $(f+g)^+ \leq f^+ + g^+$ replacing f by -f and g by -g we get

$$(f+g)^- = (-f-g)^+ \leq (-f)^+ + (-g)^+ = f^- + g^-$$

Since f^+, f^-, g^+ and g^- are integrable $(f+g)^+$ and $(f+g)^-$ are integrable.

Since $(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-$,
$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$.

Since all the functions on lhs and rhs are non negative by 11

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+$$

Since all quantiti on lhs as well as rhs are real numbers we get :

$$\int_E (f+g) = \int_E (f+g)^+ - \int_E (f+g)^-$$

$$= \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$= \int_E f + \int_E g$$

Proof of (ii) : If $C = 0$, $Cf = 0 =$ so $Cf^+ = Cf^- = (Cf)^+ = (Cf)^-$. Hence if f is integrable so is Cf and $\int_E Cf = C\int_E f = 0$. If $C > 0\ \forall\ x \in E$, $(Cf)^+(x) = Cf^+(x)$ and $(Cf)^-(x) = Cf^-(x)$. Hence if f is integrable over E so are f^+ and f^- and therefore $(Cf)^+$ and $(Cf)^-$ are integrable. This implies that Cf is integrable over E and $\int_E Cf = \int_E (Cf)^+ - \int_E (Cf)^- = C\int_E f^+ - C\int_E f^-$

$$= C\left(\int_E f^+ - \int_E f^-\right) = C\int_E f$$

Since $(-f)^+ = f^-$ and $(-f)^- = f^+$ integrability of f implies that of $(-f)^+$ and $(-f)^-$ so that -f is integrable over E.

Further, $\int_E -f = \int_E (-f)^+ - \int_E (-f)^- = \int_E f^- - \int_E f^+ = C \int_E f^+ - \int_E f^- = -\int_E f$

Finally if $C < 0$, $-C > 0$. So $Cf = (-C)(-f)$ is integrable and

$$\int_E Cf = \int_E (-C)(-f)^{\cdot} = (-C)\int_E -f = C\int_E f$$

16. Corollary : If A and B are disjoint measurable sets in E, E is measurable and f is integrable on E then $\int_{A\cup B} f = \int_A f + \int_B f$.

Proof : $f\chi_{A\cup B}(x) = f(x)$ if $x \in A \cup B$

$= 0$ if $x \notin A \cup B$

Since f^+ and f^- are integrable on $A \cup B$ (being a subset of E) $(f\chi_{A\cup B})^+$ and $(f\chi_{A\cup B})^-$ are integrable on $A \cup B$ and hence on E. Hence $f\chi_{A\cup B}$ is integrable on E. Similarly $f\chi_A$, $f\chi_B$ are integrable on E.

$$\int_{A\cup B} f = \int_E f\chi_{A\cup B} = \int_E f(\chi_A + \chi_B)$$

$$= \int_E f\chi_A + \int_E f\chi_B$$

$$= \int_A f + \int_B f$$

17. Corollary : If f and g are integrable on E and $f \le g$ a.e. on E then $\int_E f \le \int_E g$

Proof : If $g \ge 0$ a.e. on E and $A = \{x \in E / g(x) < 0\}$, $m(A) = 0$. On E-A, g is integrable and $\int_{E-A} g > 0$. Since the value of g on a set of measure zero in E does not alter the value of the integral on E it follows that $\int_E g \ge 0$.

If $g \ge f$ a.e. on E, $g - f \ge 0$ a.e. on E, hence $\int_E g - \int_E f = \int_E g - f \ge 0$ by **(14)**

Hence $\int_E f \le \int_E g$.

Short Answer Questions with solutions :

1. If f is a nonnegative measurable function on a (measurable) set E and $F \subset E$ is measurable then $\int_F f \leq \int_E f$.

Solution :

Define $g(x) = \chi_F(x) \cdot f(x)$ for $x \in E$, so that $g(x) < f(x)\ \forall x \in E$. Then g is a non negative and measurable. Since $g(x) < f(x)\ x \in E$, $\int_E g \leq \int_E f$

So that $\int_F f = \int_E f\chi_F = \int_E g \leq \int f_F$

2. If f is integrable over E then $\{x / x \in E \text{ and } f(x) \notin R\}$ has measure zero.

Solution :

Let $A = \{x / x \in E \text{ and } f(x) = +\infty\}$ and $B = \{x / x \in E \text{ and } f(x) = -\infty\}$. Clearly A and B are measurable. If $m(A) > 0$ $\int_E f^+ \geq \int_A f^+$.

Since $m(A) > 0$, and $f^+(x) \geq n$ on A for every positive integer n, $\int_A f^+ \geq n\, m(A)\ \forall$ positive integer n.

So $\int_A f^+ = \infty$ hence $\int_E f^+ = \infty$. But this contradicts integrability of f on E. Hence m(A)=0

Similarly we can show that m(B) = 0.

Hence $\{x / f(x) \notin R, x \in E\}$ has measure zero.

If f and g are measurable and $|f(x)| \leq |g(x)|$ a.e and g is integrable then f is integrable.

3. If f and g are non negative measurable functions defined on a measurable set E.

(i) $f \geq g$ a.e. on E $\Rightarrow \int_E f \geq \int_E g$

(ii) $f = g$ a.e. on E $\Rightarrow \int_E f = \int_E g$

(i) Let h be a bounded measurable function such that $h(x) \leq g(x)\ \forall\ x \in E$. Then $h(x) \leq f(x)$ a.e. on E. Let $A = \{x / h(x) > f(x)\}$. Then $m(A) = 0$. Define $h_1 = h\,\chi_{E-A}$. Then h_1 is a bounded measurable function and $h_1 \leq f$ on E so that $\int_E h_1 \leq \int_E f$.

$$\text{Thus } \int_E f \geq \int_E h_1 = \int_{E-A} h_1 + \int_A h_1 = \int_{E-A} h + \int_A h \quad \left(\int_A h_1 = \int_A h = 0\right)$$

$$= \int_A h$$

This being true for every such h, we get $\int_E g \leq \int_E f$

(ii) follows from (i)

4. If m(E) = 0, then (i) $\int_E f = 0\ \forall$ non negative measurable function on E, (ii) $\int_E f = 0$ for every integrable function f on E.

(i) If $f = \chi_A$ then $\int_E f = \int_E \chi_A = m(E \cap A) = 0$. If f is a simple function and

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}, \text{ then } \int_E f = \sum_{i=1}^{n} a_i\, m(A_i \cap E) = 0.$$

Since the integral of a bounded measurable function is the supremum of the integrals over simple functions, it follows that $\int_E f = 0$ when f is a bounded measurable function. Again, by definition $\int_E f$, when f is nonnegative is the supremum of the integrals of bounded measurable functions so that $\int_E f = 0$.

(ii) If f is integrable so are f^+ and f^- from (i) $\int_E f^+ = \int_E f^- = 0$,

so $\int_E f = \int_E f^+ - \int_E f^- = 0$.

5. If f and g are measurable $|f(x)| \leq |g(x)|$ a.e. and g is integrable then f is integrable.

Solution : Since the set $\{ x / |f(x)| > |g(x)| \}$ has measure zero, and the integral over a set of measure zero is zero, we may assume without loss of generality that $|f(x)| \leq |g(x)|$ every where. Since $|f| = f^+$ and f^- and f^+ and f^- are nonnegative $f^+ \leq |f|$ and $f^- \leq |f|$ so that $f^+ \leq |g|$ and $f^- \leq |g|$. Hence f^+ and f^- are integrable so that f is integrable.

6. If f is integrable then show that $\int f(x)\,dx = \int f(x+t)\,dx$

Solution : If f is the characteristic function of a set E, $f = \chi_E$ then $\int f(x)\,dx = m(E)$ and $\int f(x+t) = m(E-t)$ $\quad x + t \in E \Leftrightarrow x \in E - t$.

Since $m(E) = m(E-t)$, l.h.s. = r.h.s.when $f = \chi_E$. Consequently equality holds when f is a simple function vanishing outside a set of finite measure. When f is a bounded measurable function $\int f = \sup_{\phi \leq f} \int \phi$. Where the supremum is taken over all simple functions ϕ vanishing outside a set of finite measure and for such ϕ, $\int \phi(x)\,dx = \int \phi(x+t)\,dx$.

Hence $\int f = \int f(x+t)$.

If f is a non negative measurable function,

$\int f = \sup \{ \int g \,/\, g \leq f, g$ bounded measurable function vanishing outside a set of finite measure$\}$
$= \int f(x+t)$.

Since $\int f = \int f^+ - \int f^-$ where f is any integrable function it now follows that

$$\int f = \int f^+(x) - \int f^-(x) = \int f^+(x+t) - \int f^-(x+t) = \int f(x+t)$$

22. **Model Examination Questions :**

1. Show that a bounded real valued function f on IR is measurable iff $\sup_{\phi \le f} \int \phi = \inf_{\psi \ge f} \int \psi$ where the integrals are taken over all simple functions ϕ, ψ vanishing outside a set of finite, measure and satisfying $\phi \le f \le \psi$.

2. Show that a measurable function, f is integrable iff $|f|$ is integrable.

3. If f is a nonnegative measurable function and $\int f = 0$ then $f = 0$ a.e.

4. If $f = g$ a.e. and f is measurable show that f is integrable iff g is .

5. If f is integrable on A and B show that f is integrable on $A \cup B$ and $\int_{A \cup B} f = \int_A f + \int_B f$.

23. **Exercises :**

1. If f is a non negative measurable function defined on a measurable set E and A, B are measurable subsets of E such that $A \cap B = \phi$ show that $\int_{A \cup B} f = \int_A f + \int_B f$.

2. If $f \ge 0$ on E and $\int_E f = 0$ show that $f = 0$ a.e. on E.

   Hint : $\forall n$ let $A_n = \{ x / x \in E \text{ and } f(x) > 1/n \}$ and $A = \{ x / x \in E, f(x) > 0 \}$
   Show that $A = \cup A_n$ and $m(A_n) = 0 \ \forall n$

3. Show that for the characteristic function χ_A, $\int \chi_A < \infty$ is finite iff $\overline{A} / A^0$ has measure zero.

4. If f and g are measurable, $0 \le f \le g$ and g is integrable, Is f integrable ? If f and g are measurable and $|f| \le g$ and g is integrable is f integrable ? Justify your answer.

5. If f is integrable E show that $\{ x \in E / |f(x)| = \infty \}$ is measurable and has measure zero.

6. Show that $\left|\int f\right| = \int |f|$ iff either $f \ge 0$ a.e. or $f \le 0$ a.e.

7. Show that f is integrable if and only if $|f|$ is integrable.

8. Let ϕ be a simple function, which vanishes outside a set of finite measure. Show that ϕ is integrable in the sense of definition 14.

   If $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ show that $\int \phi^+ - \int \phi^- = \sum_{i=1}^{n} a_i m(E_i)$.

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LESSON 8 : LEBESGUE INTEGRAL - CONVERGENCE THEOREMS

INTRODUCTION :

As mentioned in Lesson in the Riemnn integral does not send the integral of point wise convergent sequences of measurable functions to the integral of the pointwise limit. There are three possibilities. Either $\lim_n f_n$ may not be Riemann integrable when $\{f_n\}$ converges pointwise or $\lim f_n$ may be Riemann integrable but $\lim_n \int f_n \neq \int \lim f_n$ or $\lim_n \int f_n = \infty$. However in the case of lebesgue integrable several convergence theorems are available unlike the Riemann integral. In this lesson we present these convegence theorems as their consequences. We also prove uniform continuity of the integral of a non negative integrable function.

2 a **Example:** Let $\{r_1, r_2, \ldots\ldots\ldots r_n \ldots\ldots\ldots\}$ be enumeration of the set of rational numbers in $[0,1]$. Define $f(n) = \chi_{A_n}(x)$ where $A_n = \{r_1, \ldots r_n\}$ $f_n(x)$ is zero except at $r_1, \ldots\ldots r_n$. So f_n has a finite number of discontinuities and hence f_n is Riemann integrable and $\int_0^1 f_n = 0$.

Further $\lim_n f_n(x) = 0$ if x is an irrational number while $f_n(r_k) = 1$ for $n \geq k$ so that $\lim_n f_n(r_k) = 1$.

Hence $f(x) = \lim_n f_n(x) = 1$ if x is rational and 0 if x is irrational clearly f is not Riemann integrable.

b **Example:** Let $f_n(x) = n\,x\,(1-x^2)^n$ if $0 \leq x \leq 1$.

For each n, f_n is continuous so Riemann integrable. Further

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{n}{2} y^n\,dy = \frac{n}{2n+2} \text{ so } \lim_n f_n(x) = \frac{1}{2}$$

since $0 \leq x \leq 1$ $0 \leq (1-x^4)^n \leq 1$. So if $0 < x < 1$, $(1-x^2)^n < \dfrac{1}{(1+x^2)^n} < \dfrac{1}{nx^2}$

Hence $n\,x\,(1-x^2)^n < \dfrac{1}{x(1+x^2)^n}$ $\lim_n \dfrac{1}{(1+x^2)^n} = 0$ So $\lim_n f_n(x) = 0$

if $0 < x < 1$. If $x = 0$ or 1, $f_n(0) = f_n(1) = 0$ $\forall$ n So $\lim_n f_n(0) = \lim_n f_n(1) = 0$

Thus $\lim_{n} f_n(x) = 0$ in [0,1].

This example shows that $\lim_{n} f_n(x)$ is Riemann integrable but

$$\lim_{n} \int_0^1 f_n \neq \int_0^1 (\lim_{n} f_n)$$

C **Example:** Let $f_n(x) = n^2 x (1-x^2)^n \quad 0 \leq x \leq 1$.

As above f_n is continuous for every n so Riemann integrable and $\int_0^1 f_n(x)dx = \frac{n^2}{2n+2}$ So that

$$\lim_{n} \int_0^1 f_n(x)dx = +\infty$$

However $\lim_{n} f_n(x) = 0$. So $\int_0^1 \lim_{n} f_n(x) = 0$.

3. Bounded Convergence Theorem:

Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and x

If $f(x) = \lim_{n} f_n(x)$ for each x in E, then $\int_E f = \lim_{n} \int_E f_n$

We make use of Littlewoods third principle for $\{f_n\}$. Given $\varepsilon > 0$ there is a subset A of E ∋

$m(A) < \frac{\varepsilon}{4m}$ and a positive integer N such that for $n \geq N$ and $x \in E \setminus A$.

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2m(E)}$$

$$\text{so that } \int_{E-A} |f_n - f| \leq \frac{\varepsilon}{2m(E)} m(E \setminus A) < \frac{\varepsilon}{2} \quad \text{....... 1}$$

Since $|f_n(x)| \leq M \ \forall x \in E$ and $f(x) = \lim_{n} f_n(x)$,

$|f(x)| = \lim_n |f_n(x)| \leq M \;\forall x \in E$ and in particular for $x \in A$.

Hence $\int_A |f_n - f| \leq \int_A \{|f_n| + |f|\} = \int_A |f_n| + \int_A |f| \leq Mm(A) + Mm(A) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$ 2

Now for $n \geq N$

$$\left|\int_E f_n - \int_E f\right| = \int_E (f_n - f) \leq \int_E |f_n - f| = \int_{E-A} |f_n - f| + \int_A |f_n - f| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ from(1) and (2)}$$

Hence $\int_E f = \lim_n \int_E f_n$

Remark: That Bounded convergence therem does not hold for Riemann integral is evident from example 2 a.

4 **On Fatou's Lemma :**

It is evident that under pointwise convergence of Riemann integrable functions, the limit function even though Riemann integrable the sequence of integrals may not converge to the integral of the limit. Bounded convergence theorem serves a limited purpose only. When compared to the wide scope of Lebesgue integral. The sequence considered in Example 2c shuts down the doors for any weaker form the result as $\lim_n f_n(x) = \infty$ In the case of Lebesgue integral for nonnegative measurable functions the first positive result is Fatou's Lemmna which is equivalent to Monotone convergence theorem which will be proved so on. As this lemma involves limit inferiors in place of limits we first develop the neessary machinery in this regard. Let us first recall the definition of the limit inferior of a sequence $\{a_n\}$of real numbers. By definition limit inferior a_n, or lim inf a_n or $\underline{\lim}_n a_n$ is by definition $\inf_n s_n$ where $s_n = \sup_{k \geq n} a_k$ and limit superior a_n or lim sup a_n or $\overline{\lim}_n a_n$ is $\sup_n s_n$ where $s_n = \inf_{k \geq n} a_k$ where $\{a_n\}$ is a bounded sequence $\underline{\lim}_s a_n$, $\overline{\lim}\, a_n$ are real numbers otherwise they can be $\pm\infty$. An equivalent way defining these terms is through the cluster points. A cluster point is in effect a subsequencial limits. More precisely l is a cluster point of $\{a_n\}$ iff there is a subsequence $\{a_{n_k}\}$ of $\{a_n\} \ni \lim a_{n_k} = l$. The infimum of the set of cluster points of $\{a_n\}$ is $\underline{\lim}_n a_n$ while the supremum is $\overline{\lim}_n a_n$. We use the following properties of the limit inf in Fatou's Lemma.

4.2 **Lemma:** Let $\{a_n\}$ be a sequence of nonnegative real members. Then

(i) $\lim a_n = a \Leftrightarrow \underline{\lim}_n a_n = \overline{\lim}_n a_n = a$

(ii) $a_n \le b_n \Rightarrow \underline{\lim}_n a_n \le \underline{\lim}_n b_n$

Proof : The first statement is clear since every subsequence of $\{a_n\}$ has limit a when $\lim a_n = a$. The second statement is a consequence of the definition itself as $a_n \le b_n \ \forall n$

$$\Rightarrow \sup_{k \ge n} a_k \le \sup_{k \ge n} b_k$$

$$\Rightarrow \inf_{n \ge 1} \sup_{k \ge n} a_k \le \inf_{n \ge 1} \sup_{k \ge n} b_k$$

$$\Rightarrow \underline{\lim}_n a_n \le \underline{\lim}_n b_n$$

The proof for limit superiors is similar.

4.3 **Some other properties of $\underline{\lim}$ and $\overline{\lim}$**

1. $\underline{\lim}\, x_n + \overline{\lim}\, y_n \le \overline{\lim}\,(x_n+y_n) \le \overline{\lim}\, x_n + \overline{\lim}\, y_n$ whenever lhs and rhs are not of the form $\infty - \infty$

2. If $\lim x_n = l$ then $\underline{\lim}\,(x_n+y_n) = \lim x_n + \underline{\lim}\, y_n$ and $\overline{\lim}\,(x_n+y_n) = \lim x_n + \overline{\lim}\, y_n$

3. $\underline{\lim}\,(\alpha\, x_n) = \alpha\, \underline{\lim}\, x_n$ if $\alpha \ge 0$
$= \alpha\, \overline{\lim}\, x_n$ if $\alpha < 0$

4. $\overline{\lim}(\alpha x_n) = \alpha\, \overline{\lim}\, x_n$ if $\alpha \geq 0$

$= \alpha\, \underline{\lim}\, x_n$ if $\alpha < 0$

5. **Fatou's Lemma:**

If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E then $\int_E f \leq \underline{\lim} \int_E f_n$

Proof: Let $A = \{x / x \in E \ni \lim_n f_n(x) \neq f(x)\}$. Then $m(A) = 0$. For any nonnegative measurable function g, $\int_A g = 0$. Hence it is enough to show that $\int_{E-A} f \leq \underline{\lim} \int_{E-A} f_n$ Thus we may assume that $A = \phi$ and $f_n(x) \to f(x)$ for every $x \in E$.

Let h be a bounded measurable function which vanishes outside a set E' of finite measure and satisfies $h(x) \leq f(x)$. We may assume that $h(x) \geq 0\ \forall\, x$.

Write $h_n(x) = \min\{h(x), f_n(x)\}$ for $x \in E$ for $n \geq 1$. Each h_n is a bounded measurable function such that $h_n(x) \leq f_n(x)$ and $\{x \in E / h_n(x) \neq 0\}$ has finite measure.

Since $h_n \to h$ on and E' and $h(x) = 0$ on $E \setminus E'$, by bounded convergence theorem.

$$\int_E h = \int_{E'} h = \lim_n \int_{E'} h_n = \underline{\lim}_n \int_{E'} h_n \leq \underline{\lim}_n \int_{E'} f_n = \underline{\lim}_n \int_E f_n$$

Hence $\int_E f = \sup_h \int_E h \leq \underline{\lim}_n \int_E f_n$.

This completes the Proof:

Remark: It is quite possible that strict inequality holds in Fatou's lemma. For example let $A_n = [n, n+1)$ and $f_n = \chi_{A_n}$ for $n \geq 1$. Clearly $\underline{\lim}_n f_n(x) = 0$ for every x. However for every n, $\int f_n = m(A_n) = 1$. So

$$\int \lim_n f_n \leq \underline{\lim}_n \int f_n$$

6. **Monotone Convergence theorem:**

Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions and let $f = \lim_n f_n$

then $\int f = \lim_n \int f_n$

Proof: By Fatou's lemma $\int f \le \underline{\lim} \int f_n$ .... (1)

For each n, $f_n \le f \Rightarrow \int f_n \le \int f$

$\Rightarrow \overline{\lim} \int f_n \le \int f$ .... (2)

From (1) and (2) we have

$$\int f \le \underline{\lim} \int f_n \le \overline{\lim} \int f_n \le \int f$$

Hence $\int f = \lim \int f_n$

7. **Corollary:** Let $\{u_n\}$ be a sequence of nonnegative measurable functions and $f = \sum_{n=1}^{\infty} u_n$.

Then $\int f = \sum_{n=1}^{\infty} \int u_n$

Proof: Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} u_n$, defined by $s_n = u_1 + \ldots\ldots + u_n$.

Then $\{s_n\}$ is monontically increasing and $\lim s_n = \sum_{n=1}^{\infty} u_n$.

Also $\int s_n = \int u_1 + \int u_2 + \ldots\ldots + \int u_n$

Hence by Monotone convergence theorem $\int \sum_{n=1}^{\infty} u_n = \lim_n \int s_n = \sum_{n=1}^{\infty} \int u_n$

8.Corollary: Let f be a nonnegative measurable function and $\{E_n\}$ be a disjoint sequence of measurable sets and $E = \bigcup_n E_n$. Then $\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$

Proof: Let $f_n = f\chi_{E_n}$. Then $\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f\chi_{E_n} = f.\sum_{n=1}^{\infty} \chi_{E_n} : f\chi_E$

By corollary(7) $\int_E f = \int_E f\chi_E = \sum_{n=1}^{\infty} \int_{E_n} f$

9.Theorem:

Let f be a nonnegative integrable function over a set E. Then given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subseteq E$ with $m(A) < \delta$, $\int_A f < \varepsilon$

Proof: Suppose f is bounded. Then $\exists$ a $M > 0 \ni |f(x)| \leq M \ \forall x \in E$. If $\varepsilon > 0$ and $A \subset E$ satisfies $m(A) < \frac{\varepsilon}{M}$, $\int_A f \leq M.\, m(A) < \varepsilon$.

Suppose f is unbounded. For each positive integer n let $f_n(x) = \min\{f(x), n\}$ for $x \in E$. clearly each f_n is measurable and $0 \leq f_n(x) \leq n$.

If $x \in E$ and $f(x) = \infty$, $f_n(x) = n \ \forall n$ so $f_n(x) < f_{n+1}(x)$ and $\sup_n f_n(x) = \infty = f(x)$

If $0 \leq f(x) < \infty$, there is a positive integer N such that $N \leq f(x) < N+1$

so $f_n(x) = \min\{f(x), n\} = n$ if $n \leq N$

$= f(x)$ if $n > N$

$\Rightarrow$ $f_n(x) \leq f_{n+1}(x) \ \forall n$ and $f_n(x) = f(x)$ for $n > N$

$\Rightarrow$ $f(x) = \lim_n f_n(x) = \sup_n f_n(x)$

Thus $\{f_n\}$ is monotonically increasing on E and $\lim_n f_n(x) = f(x)$ on E. By Monotone convergence theorem.

$$\int_E f = \lim_n \int_E f_n .$$

Given $\varepsilon > 0$ there exists a positive integer N such that

$$\int_E f - \int_E f_n = \left| \int_E f_n - \int_E f \right| < \frac{\varepsilon}{2} \text{ for } n \geq N_0$$

$$\Rightarrow \int_E f - \int_E f_{N_0} < \frac{\varepsilon}{2} \qquad \ldots\ldots \quad (1)$$

Choose $0 < \delta < \frac{\varepsilon}{2N_0}$ If $m(A) < \delta$

$$\int_A f = \int_A (f - f_{N_o} + f_{N_o})$$

$$= \int_A (f - f_{N_o}) + \int_A f_{N_o}$$

$$< \frac{\varepsilon}{2} + N_0 m(A) \qquad \text{by (1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \epsilon$$

Lebesque Convergence Theorem:

Let g be integrable over a measurable set E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n(x)| \leq g(x)$ on E and for almost all x in E. We have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \int_E \lim f_n$$

Proof: Since the set $A = \{x / x \in E, \lim_n f_n(x) \neq f(x)\}$ has measure zero and the integral of a function over a set of measure zero is zero, we may assume that $f_n(x) \to f(x)$ on E. Since g is integrable and $|f_n(x)| \leq g(x)$ for all x,each f_n is integrable, hence the set $\{x / |f_n(x)| = \infty\}$ and $\{x / |g(x)| = \infty\}$ have measure zero. As such we may remove these sets as well while considering integration. Thus we may assume without loss of generality that all the sets mentioned above are empty so that $(g \pm f_n)(x)$ is defined for all $x \in E$.

Since $|f_n(x)| \leq g(x) < \infty$ on E, $|f(x)| \leq g(x)$ on E so that $g(x) - f(x) \geq 0$ and $g(x) - f_n(x) \geq 0$ $\forall$ $x \in E$ and $n \geq 1$. Since each of these functions is measurable and $g - f = \lim_n g - f_n$ by Fatou's lemma

$$\int_E (g - f) \leq \underline{\lim} \int_E (g - f_n)$$

$$\Rightarrow \int_E g - \int_E f \leq \int_E g - \overline{\lim} \int_E f_n$$

$$\Rightarrow \int_E f \geq \overline{\lim} \int_E f_n \qquad \text{.... (1)}$$

Also $g(x) + f(x) \geq 0$, $g(x) + f_n(x) \geq 0$ $\forall$ n and $x \in E$. Since $\lim_n (g+f_n) = (g+f)$, as above we get

$$\int_E g + f \leq \underline{\lim} \int_E (g + f_n) = \int_E g + \underline{\lim} \int_E f_n$$

Hence $$\int_E f \leq \underline{\lim} \int_E f_n \qquad \text{.... (2)}$$

from (1) and (2) we get $\int_E f = \lim_n \int_E f_n$

11. A generalisation of Labesque convergence theorem:

If $\{g_n\}$ is a sequence of integrable functions which converge a.e. to an integrable function g and $\{f_n\}$ is a sequence of measurable functions such that $|f_n| \leq g_n$ and $\{f_n\}$ converges to f a.e and if $\int g = \lim \int g_n$ then $\int f = \lim \int f_n$

Proof: We make use of the following facts on lim inf and lim sup.

Fact: If $\lim s_n = b$, $\underline{\lim}(a_n + b_n) = \underline{\lim}\, a_n + b$ and $\overline{\lim}(a_n + b_n) = \overline{\lim}\, a_n + b$.

Since g_n is integrable and $|f_n| \le g_n$, we may assume that g_n & f_n are real valued functions. Since $-g_n \le f_n \le g_n$, $0 \le g_n + f_n$ and $0 \le g_n - f_n$ since $\lim g_n = g$ and $\lim f_n = f$ (we may assume that the convergence is everywhere) $\lim g_n + f_n = g + f$ and $\lim g_n - f_n = 0$. We apply Fatou's lemma and get

$$\int g + f \le \underline{\lim} \int (g_n + f_n) \text{ and } \int g - f = \underline{\lim} \int g_n - f_n . \le \overline{\lim} \int g_n - f_n$$

$$\Rightarrow \int g + \int f \le \underline{\lim} \int g_n + \int f_n = \int g + \underline{\lim} \int f_n \Rightarrow \int f \le \underline{\lim} \int f_n$$

Also $\int g - \int f \le \overline{\lim} \int g_n - \int f_n = \int g - \overline{\lim} \int f_n \Rightarrow \overline{\lim} \int f_n \le \int f$

Hence $\int f \le \underline{\lim} \int f_n \le \overline{\lim} \int f_n \le \int f$

$$\Rightarrow \int f = \lim \int f_n .$$

12. Lebesque's theorem on Riemann integrability

A necessary and sufficient condition for a bounded function to be Riemann integrable is that the variation between the upper and lower sums can be made arbitrarily small. As a consequence it follows that a continuous function is Riemann integrable. However some discontinuous functions are also Riemann integrable. Lebesgue settled the relationship between continuity and Riemann integrability of a bounded function by proving that a bounded real valued function f on [a,b] is Riemann integrable if and only if f is continuous almost every where. We present one proof of this result here. We first introduce some fundamental notions prove a few results in the form of problems with solutions or short answer questions with solutions and prove Lebesgue's theorem using these results. To distinguish Riemann integration we put (R) before the integral sign. In what follows $f : [a,b] \to R$ is a bounded function.

For $\delta > 0$ and $x \in [a,b]$, $I_\delta(x)$ $(= I_\delta)$, stands for $(x - \delta, x + \delta) \cap [a,b]$

12.1 Definitions:

Semi continuity : f is said to be <u>lower semicontinuous</u> at $x \in [a,b]$ if for every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $f(x) - \varepsilon < f(y)$ for $y \in I_\delta(x)$.

f is upper semi continuous at x if for $\varepsilon > 0$ there corresponds $\delta > 0$ such that $f(y) < f(x) + \varepsilon$ for $y \in I_\delta(x)$.

f is lower (upper) semi continuous if f is lower (upper) semicontinuous at every point.

Envelope : For $\delta > 0$ and $x \in [a,b]$ write

$$f_\delta(x) = \inf\{f(y) / y \in I_\delta(x)\}$$
$$f^{(\delta)}(x) = \sup\{f(y) / y \in I_\delta(x)\}$$

$$f_*(x) = \inf_{\delta < 0} f_\delta(x)$$

$$f^*(x) = \inf_{\delta < 0} f^{(\delta)}(x)$$

f_* is called the lower envelope and f^*, the upper envelope of f.

12.2 Some immediate consequence of the definitions:

(i) f is continuous at x if and only if f is both lower and upper semicontinous at x.

(ii) f is upper semicontinuous if and only if -f is lower semicontinuous.

(iii) $\forall x \in [a,b]$ and $\delta_1 > \delta_2 > 0$ $f_{\delta_2}(x) \leq f_{\delta_1}(x) \leq f^{(\delta_1)}(x) \leq f^{(\delta_2)}(x)$

(iv) if $f(x) \geq g(x)$ $\forall x \in [a,b]$ $f^*(x) \geq g^*(x)$

(v) $f_*(x) \leq f(x) \leq f^*(x)$

(vi) $(-f)_*(x) = -(f^*(x))$ and $(-f^*)(x) = -(f_*(x))$

The following results are needed in the proof of Lebesgue's theorem. We state the results here and supply proofs eleswhere.

12.2. A Result: f * is upper semicontinuouss and f_* is lower semicontinuous and f is continuous at x iff $f^*(x) = f_*(x)$.

12.2. B Result: There is a sequence of step functions $\{\phi_n\}$ such that for every $n \geq 1$ & x

$$\phi_n(x) > \phi_{n+1}(x) \text{ and } \lim_n \phi_n(x) = f^*(x)$$

If $m \leq f(x) \leq M$ for all $x \in [a,b]$ we may choose $\phi_n \ni m \leq \phi_n(x) \leq M$ for all n and x.

Remark: It is not in general true that a bounded semi continuous function is Riemann integrable. However such a function is measurable so that $\int_{[a,b]} f$ exists as a real number. In the case of f^*. measurability follows from (A). Since for each n, the step function ϕ_n is measurable.

12.3 Lemma: If f [a,b] → R is bounded

$$(R)\ \overline{\int_a^b} f(x)dx = \int_{[a,b]} f^*(x)\,dx \text{ and } (R)\ \underline{\int_a^b} f(x)dx = \int_{[a,b]} f_*$$

We prove this lemma in two steps.

Step 1: Let $P = \{a = x_0 < x_1 \ldots < x_n = b\}$ be any partition of [a,b]

$M_i = \sup \{f(x) / x_{i-1} \leq x \leq x_i\}$

Define $\phi(x) = M_j$ in $I_j = (x_{j-1}, x_j]$.

Then $\phi(x) \geq f^*(x)$ for all x in [a,b] (by iv).

Hence $\int_{[a,b]} \phi \geq \int_{[a,b]} f^*$

But $\int_{[a,b]} \phi = \int_{[a,b]} \sum_{j=1}^{n} M_j \chi_{I_j} = \sum_{j=1}^{n} M_j (x_j - x_{j-1}) = U(P,f)$

For any partition P, $U(P,f) \geq \int_{[a,b]} f^*$ ...... (1)

Hence $(R)\int_{a}^{\overline{b}} f(x)dx \geq \int_{[a,b]} f^*$

Step 2: From (12.2B) there is a decreasing sequence of step functions $\{\phi_n\}$ such that $\lim_n \phi_n(x) = f^*(x)$. If $m \leq f(x) \leq M$ for all x, we may choose $\{\phi_n\} \ni m \leq \phi_n(x) \leq M$ for all n and $x \in [a,b]$. Hence by the bounded covergence theorem.

$$\int_{[a,b]} f^x = \lim_n \int_{[a,b]} \phi_n$$

Since ϕ_n is Riemann integrable,

$$\int_{[a,b]} \phi_n = (R)\int_a^b \phi_n = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = U(P_n f)$$

Where P_n is the partition that defines the step function ϕ_n and M_i is the lub of $\phi_n(x)$ in the interval of P_n. In fact $\phi_n(x) = M_i$ on this i th interval...

$$\text{Thus } \int_{[a,b]} f^* = \lim_n \int_{[a,b]} \phi_n = \lim_n U(P_n, f) > (R)\int_a^{\overline{b}} f(x)dx \quad \text{..... (2)}$$

From (1) and (2) we have $\int_{[a,b]} f^* = (R)\int_a^{\overline{b}} f(x)dx$

12.4 Main Theorem:

f is Riemann integrable if and only if the set of points in [a,b] at which f is discontinuous has measure zero.

Proof: Since f is discontinuous at if x iff $f^*(x) \neq f_*(x)$ the set in the above statement is precisely $E = \{x / x \in [a,b] \text{ and } f^*(x) \neq f(x)\}$.

If m(E) = 0 $\int_{[a,b]} f^* = \int_{[a,b]} f_*$ hence by the lemma

$$(R)\ \overline{\int_a^b} f = \int_{[a,b]} f^* = \int_{[a,b]} f_* = (R)\ \underline{\int_a^b} f \qquad \text{..... (1)}$$

So that f is Riemmann integrable conversely if f is Riemann integrable then again (1) holds.

Thus $\int_{[a,b]} (f^* - f) = 0$. Since $f^* - f \geq 0$ it follows that $f^* = f_*$ a.e. in [a,b]

Thus f is continuous a.e. in [a,b]

13. Short Answer Questions with Solutions:

SAQ-1: Let f be a nonnegative measurable function and $f_n(x) = \min\{f(x), n\}$ Then $\lim_n \int f_n = \int f$

Solution: If $f(x) \leq n$, $f_n(x) = f(x) = f_{n+1}(x)$

If $n < f(x)$, $f_n(x) = n < n+1$ so $f_n(x) \leq f(x)$

$\Rightarrow f_n(x) \leq \min\{f(x), n+1)\} = f_{n+1}(x)$

If $f(x) = +\infty$ $f_n(x) = n$ $\forall n$ so $\lim_n f_n(x) = \infty = f(x)$

If $f(x) < +\infty$ $\exists$ N $\ni N \leq f(x) < N+1$ so $f_n(x) = f(x)$ for $n \geq N+1$ and $\lim_n f_n(x) = f(x)$

By monotone convergence theorem $\int f = \lim_n \int f_n$

SAQ-2 : Monotone convergence theorem for monotonically decreasing (integrable) functions:

Let $\{f_n\}$ be a sequence of nonnegative measurable functions such that $0 \leq f_{n+1}(x) \leq f_n(x)$

$\forall$ n and x and $\lim_n f_n(x)$ = f(x). If for some k, $\int f_k < \infty$ then $\lim_n \int f_n = \int f$

Proof: For $n \geq k$, $0 \leq \int f_n \leq \int f_k < \infty$. Since $\{f_n\}$ is decreasing, $\{f_k - f_n\}$ is increasing from the integrability of f_k & f_n ($n \geq k$) we may assume that $f_k(x) < \infty$ for all x.

Hence by the monotone convergence theorem $\lim_n \int f_k - f_n = \int f_k - f = \int f_k - \int f$.

Hence $\lim_n \int f_n = \lim_n \int f_k - (f_k - f_n) = \lim_n \int f_k - \lim_n \int f_k - f_n = \int f_k - \int f_k + \int f = \int f$

SAQ-3 : An Application of monotone convergence theorem and Fatou's lemma:

Let $\{f_n\}$ be a sequence of nonnegative measurable functions and $\lim_n f_n(x) = f(x)$.

If $f_n(x) \leq f(x) \forall$ n and x then $\lim_n \int f_n = \int f$

Proof: Write $g_n(x) = \max\{f_1(x), \ldots\ldots f_n(x)\}$. For every n g_n is a nonnegative measurable function and $\forall$ x,

$$f_n(x) \leq g_n(x) \leq g_{n+1}(x) \leq f(x)$$

Hence $f(x) = \lim_n f_n(x) \leq \underline{\lim}\, g_n(x) \leq \overline{\lim}\, g_n(x) \leq f(x)$ so that $\lim_n g_n(x) = f$

By monotone convergence theorem $\int f(x) = \lim_n \int g_n(x)$

Also $\lim g_n(x) = \overline{\lim}\, g_n(x) \geq \overline{\lim_n}\, f_n(x) \geq \underline{\lim_n}\, f_n(x) \geq f(x)$ by Fatous lemma. Hence $f(x) = \lim_n f_n(x)$

Let $\{f_n\}$ be a sequence of nonnegative measurable functions defined on IR and $f = \lim_n f_n$ suppose

$\int f < \infty$. Then for each measurable set E $\int_E f = \lim_n \int_E f_n$.

Since $\int f < \infty$, $\{x / f(x) = \infty\}$ has measure zero. Since $\lim \int f_n < \infty$, $\int f_n < \infty$ for sufficiently large n, so that we may assume that $\forall$ n, $\int f_n < \infty$ and as above $\{x / f_n(x) = \infty\}$ has measure zero. Since sets of measure zero do not contribute to the integral, we can also assume that $0 \leq f_n(x) < \infty$ and $0 \leq f(x) < \infty$ for all x.

Now define $g_n = f_n \chi_E$. Since E is measurable g_n is measurable & $0 \leq g_n \leq f_n$. Also $\lim_n g_n = f\chi_E$ and $\lim_n (f_n - g_n) = \lim_n f_n \chi_{E'} = f\chi_{E'}$.

By Fatou's lemma $\int_E f = \int f\chi_E \leq \underline{\lim}_n \int g_n = \underline{\lim}_n \int_E f_n$ and ..... (1)

$\int f\chi_{E'} \leq \underline{\lim} \int (f_n - g_n) = \lim \int f_n - \overline{\lim} \int g_n$

$= \int f - \overline{\lim} \int_E f_n$

$\Rightarrow \overline{\lim}_n \int_E f_n \leq \int f - \int f\chi_{E'} = \int (f - f\chi_{E'}) = \int f\chi_E = \int_E f$ ..... (2)

From (1) and (2) we get $\int_E f = \lim_n \int_E f_n$

14. Model Examination Questions

1. State and prove Lesbegue's dominated convergence Theorems. Show that this does not hold good for Riemann integrable.

2. State and prove Fatou's lemma. Show that Fathou's lemma has no analogue for Riemann integral.

3. State and prove monotone convergence Theorem. Give an example to show that the result does not hold for decreasing sequences.

4. State and prove Lebesgue's bounded convergence theorem

5. Show that a nonnegative measurable function is the limit of an increasing sequence of simple functions. Deduce that if f is a nonnegative measurable function,

$\int f = \sup\{\int\phi / \phi \text{ simple}, \phi \leq f\}$.

6. If $\{f_n\}$ is a sequence of integrable functions such that $\sum_{n=1}^{\infty}\int|f_n| < \infty$ then show that

$f = \sum_{n=1}^{\infty} f_n$ converges a.e., f is integrable and $\sum_{n=1}^{\infty}\int f_n = \int f$

15. Exercises

1. Let $\{f_n\}$ be a sequence of measurable functions, $|f_n| \leq g(x)\ \forall x$ and $\lim_n f_n = f$ a.e. show that

$\lim_n \int|f_n - f| = 0$. Hint Apply Lebesgue convergence theorem to $\{f_n - f\}$

2. If $0 < x < 1$ and x is rational write $f(x) = 0$. If $0 < x < 1$ and x is irrational let n be the smallest

integer $\geq \frac{1}{x}$ and $f(x) = \left[\frac{1}{x}\right]^{-1}$. show that $\int_{(0,1)} f = \infty$.

3. Let $\{f_n\}$ be a sequence of integrable functions such that $\sum\int|f_n| < \infty$ show that the series

$\sum f_n$ converges a.e., its sum f is integrable and $\int f = \sum_n \int f_n$.

4. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n(x) \leq f_{n+1}(x)\ \forall\ x$ and $f(x) =$

$\lim_n f_n(x)$ Show that $\int f = \lim_n \int f_n$.

Hint: Show that $\{x / f_1(x) = \infty\}$ has measure zero. Apply monotone convergence theorem to $\{f_n - f_1\}$.

5. If f and g are measurable, $|f(x)| \leq |g(x)|$ a.e. and g is integrable show that f is integrable.

6. Let $f_n = -n \chi_{[0,1]} + n \chi_{[1,2]}$

(a) If $0 \leq x < 1$ $f_n(x) = -n$
If $1 < x \leq 2$ $f_n(x) = n$ and $f_n(x) = 0$ otherwise

(b) each f_n is integrable

(c) $\int f_n = 0 \ \forall n$

(d) $\int \liminf x_n$ does not exist.

7. let $\{f_n\}$ be a sequence of integrable functions $\ni f_n(x) \leq f_{n+1}(x)$ $\forall x$ & n show that $\int f = \lim_n \int f_n$.

8. Let F be a non negative integrable function. Show that the functions $F(x) = \int_{-\infty}^{x} f(t)$ is uniformly continuous. **Hint:** Apply Theorem 9

9. Let g be an integrable function over E; $\{f_n\}$ a sequence of measurable functions on E such that $|f_n(x)| \leq g(x)$ a.e. on E. Show that $\int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E \overline{\lim} f_n$.

Hint: Apply Fatou's lemma to $\{g + f_n\}$ and $\{g-f_n\}$ appropriately.

10. Let $f_1 = \chi_{[1/2,1]}$ $f_2 = \chi_{[1/4,1]}$ $f_3 = \chi_{[0,3/4]}$ $f_4 = 1 - f_1$, $f_5 = f_1$, $f_6 = f_2$ and in general $f_{4n+k} = f_k$ for $n \geq 1$ and $1 \leq k \leq 3$.

For this sequence $\{f_n\}$ show that all the inequalities in 9 above are strict.

11. $\{f_n\}$ be a sequence of integrable functions such that $f_1(x) \leq f_2(x) \leq \ldots\ldots \leq f_n(x) < f_{n+1}(x) \leq \ldots\ldots$ and let $f(x) = \lim_n f(x)$. Show that $\lim_n \int f_n = \int f$.

Hint: We may assume $-\infty < f_1(x) < \infty$ for all x . Apply monotone convergence theorem to $\{f_n - f_1\}$.

12. Let $\{f_n\}$ be a sequence of integrable functions, g an integrable function and $f_n(x) \geq g(x)$ for all x. Show that $\int \underline{\lim}_n f_n(x) \leq \underline{\lim}_n \int f_n(x)$.

Hint: Apply Fatou's lemma to $\{f_n - g\}$ appropriately.

13. Show that if f is integrable then

(a) $\int f = \lim_{n} \int_{-n}^{n} f$

(b) $\forall\, t \in R\ \lim_{n} \int_{-n+t}^{n+t} f\, dx = \lim_{n} \int_{-n}^{n} f\,(x+t)\, d\,x$

(c) $\int f\,(x+t)\, dx = \int f\,(x)\, dx \ \forall\, t \in R.$

14. Let f: $IR \to IR$ be such that f is Riemann integrable in [a,b] for every a,b such that $-\infty < a < b < \infty$. If $\lim_{b \to \infty} \int_a^b f(x)\,dx$ exists in IR. We say that $\int_a^{\infty} f(x)dx$ exists as an improper Riemann integral. If $\lim_{a \to \infty} \int_a^b f(x)\,dx$ exists in IR we say that $\int_{-\infty}^{a} f(x)dx$ exists as an improper Riemann integral. If for some a $\in$ R $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$ exists as improper Riemann integrals. we say that the improper Riemann integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists. and write $\int_{-\infty}^{\infty} f(x)\,dx =$ $\int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$,

(a) Show that if $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{\infty} f(x)\,dx$ exists as improper Riemann integrals for some a then these integral exist for every a and $\int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$ is independent of the choice of a

(b) Show that if $\int_{-\infty}^{\infty} f(x)\,dx$ exists then $\int_{-\infty}^{\infty} f(x)\,dx$ exists.

(c) Show that for the function f defined on IR by $f(x) = \frac{\sin x}{x}$ if $x > 0$ and 0 for $x \leq 0$

$\int_{-\infty}^{\infty} f(x)\,dx$ eixsts but $\int_{-\infty}^{\infty} |f(x)|\,dx$ does not exist.

(d) Show that the function f in (c) above is not Lebesgue integrable (even though the improper Riemann integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists.

15. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. If f is integrable show that $\lim_n \int |f_n - f| = 0$ iff $\lim_n \int |f_n| - \int |f|$.

16. If f is integrable over [a,b] and $\varepsilon > 0$ show that (A) there is a step function h and (B) a continuous function g such that g vanishes outside a set of finite measure and

$$\int_{[a,b]} |f - h| < \varepsilon \quad \text{and} \quad \int_{[a,b]} |f - g| < \varepsilon$$

17. Hence or otherwise show that the conclusion in exercise 16 holds good for arbitrary measurable sets E.

18. Prove Riemann Lebesgue lemma: If f is integrable, $\lim_n \int f(x) \cos n\,x = 0$.

19 (a) Let f be a nonnegative measurable. Show that there is an increasing sequence $\{\phi_n\}$ of simple functions such that $\lim_n \phi_n = f$.

(b) Deduce that $\int f = \sup \int \phi$ where ϕ is simple and $\phi \leq f$

20. Prove 12.2 A

21. Prove 12.2 B

16. Problem :

Let f be any integrable function defined on [a,b]. Then for every $\varepsilon > 0$ there exists a step function h such $\int_{[a,b]} |f - h| < \varepsilon$

Solution: If f is integrable so are f^+ and f^-. Since f= $f^+ - f^-$, if we are able to prove the existence of h for nonnegative integrable functions, we will get step forms h_1, h_2 ∋ $\int_{[a,b]} |f^+ - h_1| < \frac{\varepsilon}{2}$ and $\int_{[a,b]} |f^- - h_2| < \frac{\varepsilon}{2}$ and the function h = h_1-h_2 satisfies the required conditions. Thus we may assume that f(x) ≥ 0 for all x. By definition $\int_E f = \sup_{\phi \le f} \int_E \phi$ where E = [a.b] and ϕ runs over all simple functions ≤ f on E. Since $\int_E f < \infty$, given $\in > 0$ there is a simple function $\phi \le$ f vanishing outside E ∋ $\int_E f - \frac{\varepsilon}{2} < \int_E \phi$

so that $\int_E |f - \phi| = \int_E f - \phi = \int_E f - \int_E \phi < \varepsilon / 2$ ..... (1)

It is thus enough to prove the existence of a step function h such that $\int_{[a,b]} |\phi - h| < \frac{\varepsilon}{2}$.

Since f ≥ 0 we may assume that ϕ is nonnegative.

Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$; where Each E_i is measurable, $E_i \cap E_j = \phi$ if i ≠ j and $\bigcup_{i=1}^{n} E_1 = E = [a.b]$.

Let $M_1 = \max_i a_i$ so that $M_1 \ge 0$. Since $m(E_i) < \infty$, by Littlewoods first principle there is a finite union of open intervals, which we denote by V_i ∋ $m(E_i \Delta V_i) < \frac{\varepsilon}{2n(M+1)}$

(Recall $E_i \Delta V_i = E_i - V_i \cup V_i - E_i$)

$$(\chi_{E_i} - \chi_{V_i})(x) = \begin{cases} 1 \text{ if } x \in E_i - V_i \\ -1 \text{ if } x \in V_i - E_i \\ \phi \text{ if } x \in (E_i \cap V_i) \cup (E'_i \cap V'_i) \end{cases}$$

Hence $\int |\chi_{E_i} - \chi_{V_i}| = \int \chi_{E_i \Delta V_i} = m(E_i \Delta v_i) < \frac{\varepsilon}{2n(M+1)}$ Set $h = \sum_{i=1}^{n} a_i \chi_{v_i}$. Since each V_i is a finite union of open intervals, h is a step function which vanishes outside $\bigcup_{i=1}^{n} V_i$

$$\int_E |\phi - h| = \int_E \left| \sum_{i=1}^{n} a_i \left(\chi_{E_i} - \chi_{V_i} \right) \right|$$

$$\leq \sum_{i=1}^{n} a_i \int_E \left| \chi_{E_i} - \chi_{V_i} \right|$$

$$< \sum_{i=1}^{n} a_i \frac{\varepsilon}{2n(M+1)}$$

$$< \frac{\varepsilon}{2} \qquad a_i \leq M \ \forall i$$

This completes the proof.

17. Problem:

Let f be a bounded measurable function defined on [a,b]. If $\varepsilon > 0$ there exists a continuous functions g such that g vanishes outside a finite interval (not necessarily [a,b]) and

$$\int_{[a,b]} |f - g| < \varepsilon$$

Solution: Choose a step function h vanishing outside a finite interval not necessarily [a,b] $\ni$ $\int_{[a,b]} |f - h| < \varepsilon$. It is enough to find a continuous function g $\ni$ $\int_{[a,b]} |f - g| < \frac{\varepsilon}{2}$ and g vanishes outside a finite interval.

Let h = $\sum_{i=1}^{n} a_i \chi_{E_i}$ Where each E_j an open interval say $E_j = (c_j, d_j)$ Let $0 < \eta < \min\{2(d_j - c_j), \varepsilon\}$.

For $x \in E_j$ Set $g_j(x) = 1$ if $x \in \left(c_j + \frac{\eta}{4}, d_j - \frac{\eta}{4} \right)$

$$= 0 \text{ if } x \notin \left(c_j + \frac{\eta}{4}, d_j - \frac{\eta}{4}\right)$$

For $x \notin E_j$ set $g_j(x) = 0$. Clearly g_j is continuous on $\left(c_j + \frac{\eta}{4}, d_j - \frac{\eta}{4}\right), \left(c_j, c_j + \frac{\eta}{4}\right),$ $\left(d_j - \frac{\eta}{4}, d_j\right)$. At the points c_j, $c_j + \frac{\eta}{4}, d_j - \frac{\eta}{4}$, d_j all the one sided limits are zero so that g_j is continuous for every j. Further $\int \left|\chi_{E_i} - g_i\right| < \frac{\varepsilon}{2}$ Let $g = \sum_{j=1}^{n} a_j g_j$. Then g is continuous, vanishes outside a finite interval and $\int_{[a,b]} |h-g| = \int_{[a,b]} \left|\sum a_i \left(\chi_{E_i} - g_i\right)\right|$

$$\le \int_{[a,b]} \sum a_i \left|\chi_{E_i} - g_i\right|$$

$$\le \frac{\varepsilon}{2} \sum |a_i|$$

Since ε is arbitrary the result follows.

18. Problem : State and prove Riemann Lebesgue lemma

Riemann Lebesgue Lemma :

If f is integrable, $\lim_{n} \int f(x) \cos nx \, dx = 0$

Proof: If f(x) = C on (a,b) and 0 outside (ab) then

$$\int f(x) \cos nx \, dx = C. \int_a^b \cos nx \, dx = \frac{c}{n} (\sin nb - \sin na)$$

So $\left|\int f(x)\cos nx\, dx\right| \le \frac{2c}{n}$. Since $\lim \frac{1}{n} = 0$, $\lim_{n} \int f(x) \cos nx\, dx = 0 = 0$.

If f is a step function vanishing outside an interval it follows that $\lim_{n} \left|\int f(x)\cos nx\, dx\right| = 0$.

If f is a simple function vanishing outside a set of finite measure $\forall\, \varepsilon > 0$ there is a step function ϕ such that $\int |f - \phi| < \varepsilon$. Hence $\lim_{n} \left|\int f(x)\cos nx\, dx\right| = 0$.

If f is a nonnegative measurable function and $\int f < \infty$ we may assume that $0 \le f(x) < \infty$ for all x. For every $\varepsilon > 0$ there is a simple function ϕ vanising outside a set of finite measure such that $\int |f - \phi| < \varepsilon$. Hence $\left|\int f(x)\cos nx - \phi(x)\cos nx\right| < \varepsilon$. Since $\lim_{n} \int \phi(x)\cos nx = 0$ it follows that

$\lim_{n} \int f(x)\cos nx = 0$

19. Problem :

Let f be a nonnegative measurable function. Show that (a) there is as sequence $\{\phi_n\}$ of simple functions such that $\phi_n(x) \le \phi_{n+1}(x)\ \forall x$ and $\lim_{n} \phi_n(x) = f(x)$. Deduce that (b) $\int f = \sup \int \phi$ where ϕ is simple and $\phi \le f$.

Proof (a). For each positive integers n and k ∋ $1 \le k \le n2^n$, write $E_{n_k} = f^{-1}\left(I_{n_k}\right)$ where

$I_{n_k} = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ and $F_n = f^{-1}(J_n)$ where $J_n = (n, \infty)$.

Put $\phi_n = \sum_{k=1}^{n_2} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n}$.

Since f is measurable E_{n_k} for $1 \leq k < n2^n$ and F_n are measurable so that ϕ_n is a mesurable $\forall$ n. If $f(x) > n+1$, then $f(x) > n$ so $\phi_{n+1}(x) = n+1 > n$ $\phi_n(x)$. If $n \leq f(x) \leq n+1$ then $\phi_n(x) = n$ while $\phi_{n+1}(x)$ is the left end point of the interval $\left(\frac{k-1}{2^{n+1}}, \frac{k}{2^{n+1}}\right)$ that contains f(x). However $f(x) \geq n$.

So this left end point $\geq n$ so $\phi_{n+1}(x) \geq n = \phi_n(x)$.

If $f(x) < n$ and $\frac{k-1}{2^n} < f(x) < \frac{k}{2^n}$ then $\frac{2k-2}{2^{n+1}} < f(x) \leq \frac{2k-1}{2^{n+1}}$ or $\frac{2k-1}{2^{n+1}} < f(x) \leq \frac{2k}{2^{n+1}}$ so that $\phi_n(x) \leq \phi_{n+1}(x)$ If $0 < f(x) < \infty$ then $\exists$ a positive integer N $\ni$ for $n \geq N$ and some $k \ni 1 \leq k \leq 2^n$.

$\frac{k-1}{2^n} < f(x) < \frac{k}{2^n}$

$$\Rightarrow \quad |f(x) - \varphi_n(x)| < \frac{1}{2^n}$$

$$\Rightarrow \quad \lim_n \phi_n(x) = f(x).$$

If $f(x) = \infty$, $\phi_n(x) = n \ \forall$ n so $\lim_n \phi_n(x) = f(x)$.

(b) Clearly $\int\phi \leq \int f$ for a simple function $\phi \leq f$.,

If $\{\phi_n\}$ is a sequence of simple functions $\ni \phi_n(x) \leq \phi_n(x) \ \forall$ x & $\lim_n \phi_n = f$ then $\lim_n \int\phi_n = \int f$ by the Monotone convergence theorem. Thus if $\alpha < \int f$ there is a simple function $\phi \leq f$ such that $\alpha < \int\phi$. Hence $\int f = \int\phi$ where ϕ is simple function and $\phi \leq f$.

20 Problem :

Proof of 12.2 A: f^* is upper semicontinuous and f_* is lower semi continuous. Further f^* is continuous at x if and only if $f^*(x) = f_*(x)$.

Proof: It is clear from the definition that for every $\delta > 0$.

$$f_*(x) \le f_\delta(x) \le f(x) \le f^{(\delta)}(x) \le f^*(x) \qquad \ldots\ldots (1)$$

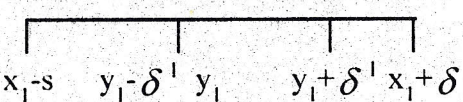
Fix $x \in [a,b]$. If $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0 \ni$ for $y \in I_{\delta_1}(x)$. and $z \in I_{\delta_2}(x)$.

$$f_*(x) - \frac{\epsilon}{2} < f_{\delta_1}(x) \le f(y) \text{ and } f_z(z) \le f^{(\delta_2)}(x) < f^*(x) + \frac{\varepsilon}{2}.$$

If $\delta = \min\{\delta_1, \delta_2\}$ then for $y \in I_\delta(x)$

$$f_*(x) - \frac{\varepsilon}{2} < f_\delta \le f(y) \le f^\delta(x) < f^*(x) + \frac{\varepsilon}{2}.$$

For a fixed $y \in I_\delta(x)$, we choose $\delta' > 0 \ni I_{\delta'}(y) \le I_\delta(x)$



Clearly $f_*(x) - \frac{\varepsilon}{2} < f_\delta(x) \le f_{\delta'}(y) \le f^{(\delta')}{}_y \le f^{(\delta)}(x) < f^*(x) + \frac{\varepsilon}{2}.$

$$\Rightarrow \quad f_*(x) - \frac{\varepsilon}{2} \le f_*(y) \le f^*(x) - \frac{\varepsilon}{2}. \text{ for } f \in I_\delta(x) \qquad \ldots\ldots (2)$$

$\Rightarrow$ f_* is lower semicontinuous at x and f^* is upper semicontinous at x. In particular when $f^*(x) = f(x)$ (2) tells that $\forall \in > 0, \exists a \delta > 0 \ni$

for $y \in I_\delta(x)$ $f_*(x) - \frac{\varepsilon}{2} < f(z) < f^*(x) + \frac{\varepsilon}{2}$ so that $|f(z) - f(x)| < \in \forall z \in I_\delta(x)$

This implies continuity of f at x.conversely if f is continuous at x $\forall \in > 0\ \exists a \delta > 0 \ni$ for

$y \in I_\delta(x)$ $f(x) - \frac{\varepsilon}{2} < f(y) < f(x) + \frac{\varepsilon}{2}$

$$\Rightarrow \quad f(x) - \frac{\varepsilon}{2} \le f_\delta(x) \le f(x) < f^{(\delta)}(x) \le f(x) + \frac{\varepsilon}{2}$$

$\Rightarrow \quad f(x) - \frac{\varepsilon}{2} \leq f_{\delta}(x) \leq f_*(x) \leq f^*(x) \leq f^{(\delta)}(x) < f(x) + \frac{\varepsilon}{2}$

$\Rightarrow \quad \left|f^*(x) - f_*(x)\right| < \varepsilon$ and this is true $\forall \in > 0$

$\Rightarrow \quad f^*(x) = f_*(x)$.

21. Problem : Prove 12.2 B

Proof of 12.2 B: Given a bounded function f: [a,b] → IR, there exists a decreasing sequence $\{\phi_n\}$ of step functions such that $f^*(x) = \lim_n \phi_n(x)$. If $m \leq f(x) \leq M$ then we can choose $\{\phi_n\} \ni m\ \phi_n(x) \leq M$ for all x.

Proof: For each n let P_n be the partitiion $P_n : \left\{a = x_{n,0} < x_{n,1} < x_{n_{2^n}} = b\right\}$ such that

$x_{n,j} - x_{n,j-1} = \frac{1}{2^n}\ \forall j$, write $I_{n_j} = \left[x_{n_{j-1}}, x_{n_j}\right]$ for $j < 2^n$ and $I_{n_{2^n}} = \left[x_{n_{2^n-1}}, b\right]$ let

$M_{n,j} = \sup\left\{f(x)\ x \in I_{n_j}\right\}$ $(1 \leq j \leq 2^n)$.

Define $\phi_n = \sum_{j=1}^{2^n} M_{n_j} I_{n_j}$. Clearly ϕ_n is a step function and $\phi_n(x) \geq f(x)$ for all n and x.

Since P_{n+1} is obtained by adding the mid points of $\left[x_{n_{j-1}}, x_{n_j}\right]$ to P_n $b \neq x \in I_{n_j}$

$\Rightarrow$ either $x \in \left[x_{n_{j-1}}, \frac{x_{n_{j-1}} + x_{n_j}}{2}\right]$ or $\left[\frac{x_{n_{j-1}} + x_{n_j}}{2}, x_{n_{j-1}}\right]$ so that $\phi_{n+1}(x) < \phi_n(x)$.

When x = b we get equality. Then $\{\phi_n\}$ is a decreasing sequence. Clearly if $m \leq f(x) \leq M$, $m \leq \phi_n(x) \leq M$.

More over $\phi_n(x) \geq f(x) \ \forall$ n and x. Hence $\phi_n(x) \geq f^*(x)$

Further if $\epsilon > 0$, $f^*(x) + \frac{\varepsilon}{2} > f^*(x) \Rightarrow f^*(x) + \frac{\varepsilon}{2} > f(y)$ for every $y \in I(\delta)(x)$ for some

$\delta > 0$. If $\frac{1}{2^{N+1}} < \frac{\delta}{2}$ for some j, $x \in I_{n_\delta} \subseteq I_\delta(x)$ then $f^*(x) + \varepsilon > M_{N_j} > \phi_N(x)$.

Hence for $n \geq N$, $f^*(x) + \varepsilon > \phi_N(x) \geq \phi_n(x) \geq f^*(x) > f^*(x) - \varepsilon$

Thus $\left|\phi_n(x) - f^*(x)\right| < \varepsilon$ for $n \geq N$. Here $\lim_n \phi_n(x) = f^*(x)$

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LESSON-9 : LEBESGUE'S THEOREM ON DIFFERENTIATION OF A MONOTONE FUNCTION

9.1 INTRODUCTION:

We have discussed the limitations of Riemann's theory of integration to some extent in the earlier lessons. Another important aspect that deserves discussion is about integration versus differentiation.

Is the integral an anti derivative and vice versa ?

When does $\int_a^b f^1(x)\,dx = f(b) - f(a)$?

When does $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$?

In the first case 'f' must be Riemann Integrable while the second equality holds for continuous functions. Unfortunately the classes of continuous functions and Riemann integrable functions are very small.

The purpose of the next three lessons, 9, 10 and 11 is to enlarge the classes of functions for which integration and differentiation are mutually reciprocal process.

It will be shown that the second relation holds more generally almost everywhere. The first question, as we see in the sequel., though cannot be fully answered in the present context we will characterize certain classes of functions that include the above classes and for which the equality holds.

In this lesson '9' we learn the famous Lebesgue theorem on differentiation of a monotone function which makes use of Vitali's lemma.

9.2 Vitali's lemma :

If E is a set of finite outer measure and $\mathcal{G}$ is a collection of intervals that covers E in the sense of Vitali, then given $\varepsilon>0$ there is a finite disjoint collection of intervals in $\mathcal{G}$ such that

$$m^*(E \setminus \bigcup_{i=1}^{N} < \varepsilon).$$

Proof : We divide the proof into a number of steps.

Step 1 : The intervals in $\mathcal{G}$ may be assumed to be closed.

Reason: If I is any interval Let $\bar{I}$ be the closure of I and let $\bar{\mathcal{G}} = \{ \bar{I} / I \in \mathcal{G} \}$. Since $\bar{I} \setminus I$ is finite, $m^*(E \setminus \bigcup_{i=1}^{N} \bar{I}) = m^*(E \setminus \bigcup_{i=1}^{N} I_i)$ for any finite collection $\{ I_1, ..., I_n \}$. Thus if we prove for $\bar{\mathcal{G}}$ then the conclusion holds good for $\mathcal{G}$ as well. As such we may assume that $\mathcal{G}$ consists of closed intervals.

Step 2 : If no finite sub family of $\mathcal{G}$ covers E, there is a sequence $\{I_n\}$ of pairwise disjoint intervals in $\mathcal{G}$ such that $\sum_{n=1}^{\infty} l(I_n) < \infty$.

Proof :

Since $m^*(E) < \infty$ $\exists$ an open set $O \supset E \ni m^*(O \setminus E) < \infty$ so that $m^*(O) \leq m^*(E) + m^*(O \setminus E) < \infty$. If $x \in E \subset O$ $\exists$ $r > O$ such that $(x-r, x+r) \subset O$. If $I \in \mathcal{G}$, $x \in I$ and $l(I) < \frac{r}{2}$ then $x \in I \subset (x-r, x+r) \subset O$. Thus we may assume that each I in $\mathcal{G}$ is a subset of O.

If some finite sub family of $\mathcal{G}$ covers E, then $E \subset \bigcup_{i=1}^{N} I_i$ for some $I_1, ..., I_N$ in $\mathcal{G}$ so that $m^*(E \setminus \bigcup_{i=1}^{N} I_i) = m^*(\phi) = 0 < \varepsilon \; \forall \; \varepsilon > 0$. Assume now that no finite sub family in $\mathcal{G}$ covers E.

Choose $I_1 \in \mathcal{G}$. Since $E \not\subset I_1$, $\exists \; x \in E \setminus I_1$. Let $I_1 = [\alpha, \beta]$. Since $x \notin I_1$, either $x < \alpha$ or $x > \beta$. In the former case we choose $\varepsilon \ni 0 < \varepsilon < \alpha - x$ where as we choose $\varepsilon \ni 0 < \varepsilon < x - \beta$ in the other case. In either case $\mathcal{G}$ contains a I such that $x \in I$, $l(I) < \frac{\varepsilon}{2}$ and $I \cap I_1 = \phi$.

Hence $\mathcal{G}_1 = \{ I / I \in \mathcal{G}, I \cap I_1 = \phi \}$ is non empty.

Let $k_1 = \sup \{ l(I) / I \in \mathcal{G}_1 \}$.

Since $m(O) < \infty$ and $I \in \mathcal{G} \Rightarrow I \subseteq O$, $0 < k_1 < \infty$.

Define a sequence $\{I_n\}$ of intervals $\mathcal{G}$ inductively as follows :

Let $\mathcal{G}_n = \{ I / I \in \mathcal{G}, I \cap (\bigcup_{i=1}^{N} I_i) = \phi \}$ and $k_n = \sup \{ l(I) / I \in \mathcal{G}_n \}$. Choose $I_{n+1} \in \mathcal{G}_1$ so that $k_n < 2 l(I_{n+1})$.

Since $I_n \subset O$ for every n and $\{I_n\}$ is a sequence of pairwise disjoint intervals, $\sum_{n=1}^{\infty} l(I_n) =$

$$\sum_{n=1}^{\infty} m(I_n) = m(\bigcup_{i=1}^{\infty} I_n) \leq m(O) < \infty .$$

Hence $\lim_n m(I_n) = 0$ and $\forall\, \varepsilon > 0\ \exists$ a $N \in \mathbb{N} \ni \sum_{n=N}^{\infty} l(I_n) < \frac{\varepsilon}{5}$.

Step : $\{I_1, I_2, ..., I_n\}$ has the required property.

Proof :

Write $R = E \setminus \bigcup_{i=1}^{N} I_i$. If $x \in R$, there is a $I \in \mathcal{G} \ni x \in I$ and $I \cap (\bigcup_{j=1}^{N} I_j) = \phi$.

If $I \cap I_j = \phi$. If $I \cap I_j = \phi\ \forall j$, then $I \in \mathcal{G}_n$. Hence $l(I) \leq k_n \leq 2 l(I_{n+1})$.

Since $\lim l(I_n) = 0$, $l(I) = 0$. Since I is a closed interval (with different end points) this is not possible. Thus there is at least one n with $I \cap I_n \neq \phi$. Let n be the smallest integer with this property. Since $I \cap \bigcup_{i=1}^{N} I_i = \phi$, $n > N$.

Let I_n = [a,.b] and C = a+b /2. Since x $\in$ I, and I $\cap I_n = \phi$, x < a or x > b . In either case :

$$|x-c| \leq l(i)+\frac{l(I_n)}{2}$$

$$\leq k_{n-1}+\frac{l(I_n)}{2}$$

$$\leq 2l(I_n)+\frac{l(I_n)}{2} = \frac{5}{2}l(I_n).$$ Write r = 5 $l(I_n)$

Let J_n = [c-r, c+r]. Then $I_n \subseteq J_n$. Hence R = $E \setminus \bigcup_{i=1}^{N} I_i \subseteq \bigcup_{i=N+1}^{\infty} J_i$.

Hence $m^*(R) \leq \bigcup_{i=N+1}^{\infty} l(J_i) \leq 5 \bigcup_{i=N+1}^{\infty} l(I_i) < \varepsilon$.

This completes the proof.

9.3. Dini Derivatives :

Let us recall that sup E = ∞ if E is unbounded above and inf E = - ∞ if E is unbounded below. Thus every subset of R possesses supremum and infimum in the extended real number system.

Definitions : Let f : (a, b) $\rightarrow$ R be a function. We define

$$\lim_{x\to a+} \sup f(x) = \inf_{\delta>0} \sup_{0<h<\delta} f(x+h)$$ and denote this by $\overline{\lim_{x\to a+}} f(x)$

$$\lim_{x\to a+} \inf f(x) = \sup_{\delta>0} \inf_{0<h<\delta} f(x+h)$$ and denote this by $\underline{\lim}_{x\to a+} f(x)$.

and define in a similar way $\lim_{x\to b^-} \sup f(x)$ f(x) and $\lim_{x\to b\_} \inf f(x)$. If a < x < b the four Dini's of f at x are defined as follows :

$D^+ f(x)$ = upper right Dini derivative of f at x $= \lim_{h\to 0^+} \sup \dfrac{f(x+h)-f(x)}{h}$

$D_+ f(x)$ = lower right Dini derivative of f at x $= \lim_{h\to 0^+} \inf \dfrac{f(x+h)-f(x)}{h}$

$D^- f(x)$ = upper left Dini derivative of f at x $= \lim_{h\to 0_-} \sup \dfrac{f(x+h)-f(x)}{h}$

$D_- f(x)$ = lower left Dini derivative of f at x $= \lim_{h\to 0_-} \inf \dfrac{f(x+h)-f(x)}{h}$

We say that f has right Dini derivative at x if $D_+f(x) = D^+f(x)$ and in the case denote the equal value by $f'_+(x)$.

We say that f has left Dini derivative at x if $D_-f(x) = D^-f(x)$ and in the case denote the equal value by $f'_-(x)$.

We say that f has derivative (f is differentiable at x with derivative $f'(x)$) if $f'_+(x) = f'_-(x)$ and this equal value is denoted by $f'(x)$.

Remarks :

(1) We do not exclude the possibility that any of the above derivatives is $+\infty$, $-\infty$

(2) $D_-f(x) \le D^-f(x)$ and $D_+f(x) \le D^+f(x)$.

(3) $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ exists in IR, i.e., f is differentiable at x in the usual sense if and only if all the four Dini derivatives are equal and the equal value is in IR. In this case this equal value is the limit considered above.

(4) Some authors define the left Dini Derivatives as $D_- f(x) = \overline{\lim_{h\to 0-}} \dfrac{f(x)-f(x-h)}{h}$ and

$$D^-f(x) = \lim_{h\to 0-} \frac{f(x)-f(x-h)}{h}.$$

But $\underline{\lim}_{h \to 0-} \frac{f(x)-f(x-h)}{h} = \underline{\lim}_{-h \to 0-} \frac{f(x)-f(x+c-h)}{-h}$

$= \underline{\lim}_{h \to 0+} \frac{f(x+h)-f(x)}{h}$

$= D_{-}f(x)$

and similarly $\overline{\lim}_{h \to 0-} \frac{f(x)-f(x-h)}{h} = D^{-} f(x)$.

This equivalence allows us to make use of three notions as per our convenience.

9.4. Example : Let $f(x) = \chi_Q$. Find D_+f, D^+f, D_-f and D^-f at $x \in IR$. For $x \in R$, $h \neq 0$ we find $f_x(h) = \frac{f(x+h)-f(x)}{h}$ in various cases.

When x, x+h both belong to Q or both to Q^c, $f_x(h) = 0$.

When $x \in Q$ and $x + h \in Q^c$ $f_x(h) = -\frac{1}{h}$ where as

When $x \in Q^C$ and $x + h \in Q$ $f_x(h) = \frac{1}{h}$

If $x \in Q$ and $\delta > 0$ $\sup_{0<h<\delta} f_x(h) = 0$ and $\inf_{0<h<\delta} f_x(h) = -\infty$

In this case $D^+f(x) = 0$ and $D_+f(x) = -\infty$.

Similarly when $x \in Q^C$ and $\delta > 0$ $\sup_{0<h<\delta} f_x(h) = \infty$, and $\inf_{0<h<\delta} = 0$. So that $D^+f(x) = \infty$ and $D_+f(x) = 0$:

In a similar way we can prove the following :

$$D^{-} f(x) = \begin{cases} \infty & \text{if } x \in Q \\ 0 & \text{if } x \in Q^{C}. \end{cases}$$

and

$$D_{-} f(x) = \begin{cases} 0 & \text{if } x \in Q \\ -\infty & \text{if } x \in Q^{C}.. \end{cases}$$

9.5. Example : $f(x) = |x|$ if $x \in R$.

Solution : If $x > 0$ and $h > 0$, $\dfrac{f(x+h)-f(x)}{h} = 1$.

If $x < 0$ and $0 < h < -x$, $h + x < 0$ so $\dfrac{f(x+h)-f(x)}{h} = -1$

Hence if $x > 0$ $\overline{\lim_{h\to 0+}} \dfrac{f(x+h)-f(x)}{h} = \underline{\lim_{h\to 0+}} \dfrac{f(x+h)-f(x)}{h} = 1.$

and if $x < 0$ $\overline{\lim_{h\to 0+}} \dfrac{f(x+h)-f(x)}{h} = \underline{\lim_{h\to 0+}} \dfrac{f(x+h)-f(x)}{h} = -1$

So that $D_{-}f(x) = D^{-}f(x) = D_{+}f(x) = D^{+}f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$

Since $\dfrac{f(h)-f(0)}{h} = \dfrac{|h|}{h} \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0. \end{cases}$

$D^{+} f(0) = \overline{\lim_{h\to 0+}} \dfrac{f(h)-f(0)}{h} = 1 = D^{+}f(0)$ while $D^{-} f(0) = \overline{\lim_{h\to 0}} \dfrac{f(h)-f(0)}{h} = -1 = D_{-} f(0).$

9.6. Lebesgue's theorem on the derivative of a monotone function :

Let f be an increasing real valued function on the interval [a,b]. Then

(i) f is differentiable a.e. in [a,b]

(ii) f' is measurable and

(iii) $\int_a^b f'(x)\,dx \le f(b) - f(a)$

Proof of (i) : We divide this proof into four steps.

Step 1 : It is enough if we show that for any rationals u, υ $u > \upsilon$ the set $E_{u,\upsilon} = \{x / x \in [a,b]$ and $D^+f(x) > u > \upsilon > D_-f(x)\}$ has measure zero.

Proof : By definition f is differentiable at x if and only if all the four Dini derivatives $D^+f(x), D^-f(x), D_+f(x), D_-f(x)$ are equal. The set $E_-^+ = \{x / D^+f(x) > D_-f(x)\} = \bigcup_{u > \upsilon} E_{u,\upsilon}$ $u \in Q$, $\upsilon \in Q$.

If we prove that $m(E_{u,\upsilon}) = 0$ then since u, υ vary over the countable set Q, we will get $m(E) = 0$.

Likewise it follows that the set $E_+^- = \{x / D^+f(x) < D_-f(x)\}$, and other sets with various possibilities for the Dini derivatives are of measure zero.

If E is the set of all $x \in [a,b]$ such that f is not differentiable at x, then E is the union of sets of the type E_-^+, E_+^- and as each of these sets has measure zero, $m(E) = 0$.

Step 2 : If u, υ are rational numbers $\ni u > \upsilon$ and $m^*(E_{u,\upsilon}) = s$ then there is a finite disjoint collection of intervals $I_1, \ldots, I_N$ where $I_j = [x_j - h_j, x_j]$ such that

(a) $\sum_{j=1}^{N} f(x_j) - f(x_j - k_j) < \upsilon\,(s + \in)$ and

(b) $m^*(E_{u,\upsilon} \setminus \bigcup_{j=1}^{N} I_i) < \in$.

Proof : Let O be an open set $\ni O \supset E_{u,\upsilon}$ and $m^*(O \setminus E_{u,\upsilon}) < \epsilon$.

so that $m^*(O) < m^*(E_{u,\upsilon}) + \varepsilon = s + \varepsilon$.

$x \in E_{u,\upsilon} \Rightarrow D_{-}f(x) < \upsilon$

$$\Rightarrow \sup_{\delta>0} \inf_{0<h<\delta} \frac{f(x)-f(x-h)}{h} < \upsilon.$$

$\Rightarrow \forall \delta > 0, \exists h > 0 \ni 0 < h < \delta$ and $f(x) - f(x-h) < \upsilon h$.

Since O is open and $x \in O$ the above h may be choosen so that $[x-h, x] \leq O$.

The collection $\varepsilon : \{([\alpha,\beta] / [\alpha,\beta] \subset O$ and $f(\beta) - f(\alpha) < \upsilon(\beta - \alpha)\}$ is therefore a Vitali covering of $E_{u,\upsilon}$. Finite disjoint collection $\{I_1, ... I_N\}$ in the above Vitali cover such that

$$m^*(E_{u,\upsilon} \setminus \bigcup_{i=1}^{N} I_i) < \epsilon.$$

Let $I_j = [x_j - h_j, x_j]$ and $A = E_{u,\upsilon} \cap \bigcup_{i=1}^{N} I_i$ and $B = E_{u,\upsilon} - \bigcup_{i=1}^{N} I_i$

Since $E_{u,\upsilon} = A \cup B$, $s = m^*(E_{u,\upsilon}) < m^*(A) + m^*(B) < m^*(A) + \epsilon$. So that $m^*(A) > s - \varepsilon$.

Further ,

$$\sum_{j=1}^{N} [f(x_j) - f(x_j - h_j)] < \sum_{j=1}^{N} \upsilon h_j.$$

$$= \upsilon \sum_{j=1}^{N} h_j.$$

$$= \upsilon m^* \left(\sum_{j=1}^{N} I_j \right)$$

$$\leq \upsilon m^* (O)$$

$$< \upsilon (s + \varepsilon)$$

This completes the proof of step 2.

Step 3 : There is a finite disjoint collection of closed intervals $J_1, \ldots, J_M, J_r = [y_r, y_r + k_r]$ such that each $J_r \subseteq I_i$ for some i , $m^* (A - \bigcup_{r=1}^{M} J_r) < \in$ and $\sum_{r=1}^{M} [f(y_r + k_r) - f(y_r)] > u(s - 2 \in)$.

Proof : If $y \in A$, for every $\delta > 0$ there corresponds a k such that $0 < k < \delta$ and $f(y+k) - f(y) > ku$. The collection : $\{ [\alpha, \beta] / [\alpha, \beta] \} \subseteq I_i$, for some i $1 \leq i \leq N$ and $f(\beta) - f(\alpha)$ xx> $u(\beta - \alpha)$ } is a Vitali covering of A. So $\exists$ a finite disjoint sub collection $\{J_1, \ldots J_M\}$. Where $J_r = [y_r, y_r + k_r]$ say V such that $m^* (A \setminus \bigcup_{r=1}^{M} J_r) < \varepsilon$.

Both the collections $\{I_1, \ldots, I_N\}$ and $\{J_1, \ldots, J_M\}$ are pairwise disjoint and each J_r is contained in some I_i.

If $B_1 = A \setminus \bigcup_{r=1}^{M} J_r$ and $B_2 = A \cap \bigcup_{r=1}^{M} J_r$, then

$$s - \varepsilon < m^*(A) = m^* (B_1 \cup B_2) < m^*(B_1) + m^*(B_2) < m^*(B_2) + \varepsilon .$$

Hence
$$\sum_{r=1}^{M} f(y_r + k_r) - f(y_r) \geq \sum_{r=1}^{M} u k_r$$

$$= u \sum_{r=1}^{M} m^* (J_r)$$

$$\geq u m^* (B_2)$$

$$> u (s - 2\varepsilon)$$

This completes the proof of step 3.

Step 4 : $m^*(E_{u,\upsilon}) = 0$

Proof : If among the J_i, in step 3, $J_{n_1}, \ldots, J_{n_r}$ are the J_i that are contained in some I_n of step 2 where $1 \leq n \leq N$ then by the monotonicity of f (f is increasing) we have

$$f(x_n) - f(x_n - h_n) \geq \sum_{j=1}^{r} \left\{ f(y_{n_j} + k_{n_j}) - f(y_{n_j}) \right\}$$

So that $\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) \geq \sum_{n=1}^{N} \sum_{j=1}^{r} \left\{ f(y_{n_j} + k_{n_j}) - f(y_{n_j}) \right\}$

$$= \sum_{n=1}^{M} f(y_j + k_j) - f(y_j)$$

This from steps (2) and (3) we get $\upsilon(s+\varepsilon) > u/(s-2\varepsilon)$. Since this is true $\forall\, \varepsilon > 0$ it follows that $s = m^*(E_{u,\upsilon}) = 0$. From step 1 it nows follows that f is differentiable a.e. proof of (i) is complete.

Proof of (ii) :

Since f is differentiable a.e. on [a,b] the function $g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is defined for almost all x in [a,b]. Further f is differentiable whenever g(x) is real. In this case $f'(x) = g(x)$.

Now extend f to $[a, \infty)$ by setting $f(x) = f(b)$ for $x \geq b$ and define $g_n(x) = \frac{f(x+1/n) - f(x)}{1/n}$

Since f is monotonically increasing, f is continuous a.e. hence measurable and hence g_n is measurable $\forall$ n.

Since $g(x) = \lim_{n} g_n(x)$, g is measurable.

Since $g(x) = f^1(x)$ a.e.. , f^1 is measurable.

This completes the proof of (ii).

Proof of (iii) : By Fatou's lemma $\int_a^b g \le \underline{\lim}_{n} \int_a^b g_n$

$$= \underline{\lim}_{n} \int_a^b \frac{f(x+1/n)-f(x)}{1/n}$$

$$= \underline{\lim}_{n} \left\{ f(b) - n \int_a^{a+1/n} f(x) \right\}$$

(f(x) = f(b) if x ≥ b)

$$\le f(b) - f(a) \left\{ \left(a+\frac{1}{n} - a\right) n \right\}$$

$$= f(b) - f(a).$$

Since $f' = g$ a.e. we have $\int_a^b f' = \int_a^b g \le f(b) - f(a)$

This completes the proof (iii) and hence the theorem.

n.7. Short Answer Questions with solutions :

SAQ - 1 : If f is continuous on [a,b] and one of the Dini Derivatives is non negative show that f is monotonically increasing on [a,b]

Solution : For definiteness assume that $D^+f(x) \ge 0$ in [a,b]. If f(b) < f(a) choose $\varepsilon \ni < \varepsilon < \frac{f(a)-f(b)}{b-a}$ and define $g(x) = f(x) - f(a) + \varepsilon(x-a)$.

Since f is continuous, so is g hence the set { x / a ≤ x ≤ b, g(x) = 0 } is bounded and closed so that C = max { x / a ≤ x ≤ b, g(x) = 0 } exists so that if c < x < b, g(x) < 0 because if 0 < g(x), then g(b) < 0 < g(x) , so that by the intermediate value property of g, there would exist a d $\in$ (x,b) such that g(d) = 0 contradicting the maximality of g. Also g(x) $\neq$ 0, because x > c. Thus for 0 < h < b-a g(c+h) < 0 so that :

$$0 > \frac{g(c+h)}{h} = \frac{g(c+h)-g(c)}{h} = \frac{f(c+h)-f(c)}{h} + \varepsilon$$

This implies that $0 \geq D^+ f(c) + \varepsilon \Rightarrow 0 > -\varepsilon \geq D^+ f(c)$ contradicting the hypothesis. Thus $f(b) \geq f(a)$ As this holds good for every $x > y$ in $[a,b]$, f is monotonically increasing.

SAQ -2 : If f has a local maximum at $c \in (a,b)$ then show that $D^+ f(c) \leq 0 \leq D_- f(c)$.

Solution : $\exists$ a $\delta > 0 \ni |h| < \delta \Rightarrow f(c+h) \leq f(c)$

$$\Rightarrow \frac{f(c+h)-f(c)}{h} \leq 0 \leq \frac{f(c-h)-f(c)}{-h} \quad \text{if } 0 < h < \delta$$

$$\Rightarrow D^+ f(c) \leq 0 \leq D_- f(c).$$

SAQ - 3 : The union of an arbitrary collection of intervals is measurable.

Solution : Let $\{ I_\alpha \setminus \alpha \in \Delta \}$ be an arbitrary collection of intervals and $E = \bigcup_{\alpha \in \Delta} I_\alpha$. First assume that $m^*(E) < \infty$. It is easy to verify that the collection $\mathcal{G} = \{ I / I$ is closed interval contained in some $I_\alpha \}$ is a Vitali cover for E. Since $m^*(E) < \infty$ there is a finite sub collection $\{I_{n_1}, \ldots\ldots, I_{n_{k_n}})$ in $\mathcal{G}$ such that $m^*(E \setminus \bigcup_{j=1}^{k_n} I_{n_j}) < \frac{1}{n}$. The union of these intervals

$E_n = \bigcup_{j=1}^{k_n} I_{n_j}$ is measurable hence $E_0 = \bigcup_{n=1}^{\infty} E_n$ is measurable .

$$m^*(E \setminus E_0) = m^*(E \setminus \bigcup_{n=1}^{\infty} E_n) = m^*(\bigcap_{n=1}^{\infty} E \setminus E_n) \leq m^*(E - E_n) \leq \frac{1}{n} \ \forall n.$$

Hence $m^*(E \setminus E_0) = 0$. Since E_0 is measurable and $E \setminus E_0$ is measurable, E is measurable.

If $m^*(E) = \infty$ then we consider the sets $E_n = E \cap (-n,n)$ and the intervals $J_\alpha^{(n)} = I_\alpha \cap (-n,n)$

Clearly $E_n = \bigcup_{\alpha \in \Delta} J_\alpha^{(n)}$ and $m^*(E_n) < \infty$. Hence E_n is measurable, since $E = \bigcup_{n=1}^{\infty} E_n$ it follows that E is measurable.

SAQ 4 Show by means of an example that $D^+(f+g) \neq D^+f + D^+g$ in general.

Solution Let f(x) = x, for $x \in Q$ and f(x) = -x for $x \in Q^C$ and g(x) = -x.

$$D^+ f(0) = \inf_{\delta>0} \sup_{0<h<\delta} \frac{f(h)-f(0)}{h}$$

$$= \inf_{\delta>0} \sup \{ 1, -1 \} = 1$$

$$D^+ g(0) = \inf_{\delta>0} \sup_{0<h<\delta} \frac{g(h)-g(0)}{h}$$

$$= \inf_{\delta>0} \sup \{ 1, -1 \} = 1$$

So that $(D^+ f + D^+ g)(0) = 2$.

SAQ 5 If f assumes its maximum at C then $D^+ f(C) \leq 0 \leq D_- f(C)$.

Solution By definition $D^+(f)(C) = \inf_{\delta>0} \sup_{0<h<\delta} \frac{f(x+h)-f(x)}{h} \leq 0$ $\quad f(x+h) \leq f(x$

for sufficiently small h. Similarly one can show that $D_- f(C) \geq 0$.

SAQ 6 If $f^1(x)$ exists then $D^+(f+g) = D^+(f) + D^+(g)$

Solution Since $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f^1(x)$,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)-hf^1(x)}{h} = 0$$

Write $f(x+h) - f(x) - hf^1(x) = \phi(h)$

Then $f(x+h) - f(x) = hf^1(x) + \phi(h)$ and $\lim_{h\to 0} \frac{\phi(h)}{h} = 0$.

$$D^{+}(f+g)(x) = \inf_{\delta>0} \sup_{0<h<\delta} \frac{(f+g)(x+h)-f(+g)(x)}{h}$$

$$= \inf_{\delta>0} \sup_{0<h<\delta} \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= \inf_{\delta>0} \sup_{0<h<\delta} \frac{f^{1}(x)+\phi(h)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= f^{1}(x) + \inf_{\delta>0} \sup_{0<h<\delta} \frac{\phi(h)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$= f^{1}(x) + \inf_{\delta>0} \sup_{0<h<\delta} \frac{g(x+h)-g(x)}{h} \quad \left(\lim \frac{\phi(h)}{h} = 0 \right)$$

$$= f^{1}(x) + D^{+} g(x)$$

$$= D^{+} f(x) + D^{+} g(x).$$

9.8 Model Examination Questions :

1. Define $D^{+} f(x)$, $D_{+}f(x)$, $D^{-} f(x)$ and $D_{-} f(x)$ and show that $D^{+} f(x) \geq D_{+}f(x)$ and $D^{-} f(x) \geq D_{-} f(x)$.

2. Let f be a monotically increasing function on [a,b]. If $u > \upsilon$ $E_{u.\upsilon} = \{ x / D^{+}f(x) > u > \upsilon\ D\_f(x)\}$, $\mathcal{g} = ([x-h,x] \}$ show that $\mathcal{g}$ is a vitali cover $E_{u\upsilon}$.

3. If f assumes maximum at C show that $D^{+} f(C) \leq 0 \leq D\_f(C)$.

4. Show that if f is a function of bounded variation on [a,b] then f is the difference of two monotonically increasing functions.

9-9 Exercises :

1. Show that a collection intervals $\mathcal{g}$ is a Vitali cover of E if and only if $\bar{\mathcal{g}}$ is a Vitali cover of E.

2. Show that the collections of ξ in step 2 and in step 3 of 9. 6 are Vitali covers.

3. Let $f(x) = a\chi_{[0,1]} + b\chi_{[1,2]}$ where $b > a > 0$ and $F(x) = \int_a^x f(t)dt$ show that F is continuous but not differentiable at x = 1.

4. Let $f(x) = \begin{cases} 1/q \text{ if } x = p/q \text{ in the simplest from } (f(0) = 1) \\ 0 \text{ if } x \text{ is rational} \end{cases}$

and $F(x) = \int_0^x f(t)dt$. Discuss the validity of $f'(x) = f(x)$.

5. If f is continuous on [a,b] and one of its derivatives say $D^+ f \geq 0$ on [a,b] show that $f(b) \geq f(a)$.
Hint: First prove this for g when g is continuous and $D^+g \geq \varepsilon > 0$.
Then consider $g = f + x\,\varepsilon$

6. Lebesgue Point : $x \in [a,b]$ is called a Lebesgue point of f if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} \{f(t) - f(x)\}\, dt = 0$$

If $F(x) = \int_a^x f(t)dt$, show that $f'(x) = f(x)$ at every Lebesgue point x of f.

7. Show that every point of continuity of an integrable function is a Lebesgue point.

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Lesson - 10

CONVERGENCE IN MEASURE

10.1 Introduction

We have proved several results concerning the equality of

$$\lim_{n} \int f_n \text{ and } \int f$$

where f_n is a sequence of measurable functions such that

$$f_n \to f \text{ a.e.}$$

Those results hold under a weaker assumption regarding the convergence of f_n to f. We deal with the topic in this lesson.

The methods of proving the results are more or less the same as in the previous case.

Suppose that E is a measurable set, f and f_n for each n in $\mathbb{N}$ are measurable functions on E.

10.2 Definition : We say that the sequence $\langle f_n \rangle$ converges to f in measure if given any $\varepsilon > 0$ we can find an $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have

$$m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon .$$

10.3 Remark : It is to be remembered that we have made "f is measurable" a part of the hypothesis in the definition 12.1.

As soon as we define limit of a sequence it is the custom to define Cauchy sequence. Here we follow the practice.

10.4 Definition : Suppose f_n for each $n \in \mathbb{N}$ is a measurable function defined on a measurable set E. We say that

$\langle f_n \rangle$ is a Cauchy sequence in measure, if given any $\varepsilon > 0$, we can find an $n(\varepsilon) \in \mathbb{N}$ such that for all k and ℓ in $\mathbb{N}$, $k, \ell \geq n(\varepsilon)$

$$m\{x : |f_k(x) - f_\ell(x)| \geq \varepsilon\} < \varepsilon .$$

10.5 Proposition : Suppose $\langle f_n \rangle$ is a Cauchy sequence in measure and a subsequence

$$\{f_n\}_k$$

converges to a measurable function g in measure then $\langle f_n \rangle$ converges to g in measure.

Proof : Let $\varepsilon > 0$.

Since $\langle f_n \rangle$ is a Cauchy sequence in measure there is a $k_1 = k_1(\varepsilon) \in \mathbb{N}$ such that for all p, q in $\mathbb{N}$, $p, q \geq k_1$

$$m\left\{x : \left|f_p(x) - f_q(x)\right| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}.$$

Let $f_{n_k} = g_k$. Since $\langle g_k \rangle$ converges to g in measure, there is a $k_2 = k_2(\varepsilon) \in \mathbb{N}$ such that for all $k \geq k_2$

$$m\left\{x : \left|g_k(x) - g(x)\right| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}$$

Set $k = k_1 + k_2$, $n(\varepsilon) = n_k = n_{k_1 + k_2}$.

Suppose $n \geq n(\varepsilon)$. We note that $k \leq n_k$ and so

$$m\left\{x : \left|f_n(x) - f_{n_k}(x)\right| \geq \frac{\varepsilon}{2}\right\} < \frac{\varepsilon}{2}.$$

Since

$$\{x : \left|f_n(x) - g(x)\right| \geq \varepsilon\}$$

$$\subseteq \left\{x : \left|f_n(x) - f_{n_k}(x)\right| \geq \frac{\varepsilon}{2}\right\} \cup \left\{x : \left|f_{n_k}(x) - g(x)\right| \geq \frac{\varepsilon}{2}\right\}$$

and $g_k = f_{n_k}$ we have

$$m\{x : \left|f_n(x) - g(x)\right| \geq \varepsilon\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ proposition is proved.

10.6 Proposition : If $\langle f_n \rangle$ is a Cauchy sequence in measure, then there is a measurable function f such that $\langle f_n \rangle$ converges to f in measure.

Proof : We prove first that there is a subsequence

$$\langle f_{n_k} \rangle$$

which converges in measure to some measurable function g.

Corresponding to each $k \in \mathbb{N}$, there is some $n\left(\frac{1}{k}\right) \in \mathbb{N}$ such that for all $p, q \in \mathbb{N}$

$p, q \geq n\left(\frac{1}{k}\right)$

$$m\left\{x : \left|f_p(x) - f_q(x)\right| \geq \frac{1}{2^k}\right\} < \frac{1}{2^k}.$$

We set

$$n_k = n(1) + \cdots\cdots + n\left(\frac{1}{k}\right).$$

For each $k \in \mathbb{N}$ we define

$$g_k = f_{n_k},$$

and

$$F_k = \left\{x \in E : \left|g_{k+1}(x) - g_k(x)\right| \geq \frac{1}{2^k}\right\}$$

and then set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} F_k$$

since $g_k = f_{n_k}$ and $n_k > n\left(\frac{1}{k}\right)$ we have

$$m(F_k) = m\left\{x \in E : \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right| \geq \frac{1}{2^k}\right\} < \frac{1}{2^k}.$$

Therefore $m\left(\bigcup_{k=n}^{\infty} F_k\right) \leq \sum_{k=n}^{a} m(F_k) \leq \sum_{k=n}^{a} \frac{1}{2^k} = \frac{1}{2^{n-1}}$

and since $F \subseteq \bigcup_{k=n}^{\infty} F_k$

for each $n \in \mathbb{N}$ we have $m(F) = 0$

Now suppose $x \notin F$. Then there is an $n \in \mathbb{N}$ such that

$$x \notin \bigcup_{k=n}^{\infty} F_k$$

therefore $x \notin F_p$ for $p \geq n$.

Suppose now $n < r < s$, then we have

$$|g_s(x) - g_r(x)| \leq |g_s(x) - g_{s-1}(x)| + \cdots\cdots + |g_{r+1}(x) - g_r(x)|$$

$$< \frac{1}{2^{s-1}} + \cdots\cdots + \frac{1}{2^r} \text{ since } x \notin F_p \text{ for } p \geq n$$

$$= \frac{1}{2^{r-1}}\left(1 - \frac{1}{2^{s-r}}\right)$$

$$< \frac{1}{2^{r-1}}$$

This shows that the sequence of real numbers

$$\langle g_k(x) \rangle$$

is a Cauchy sequence and hence converges to a real number. We denote it by $g(x)$.

Thus we have proved that the sequence of measurable functions $\langle g_k \rangle$

converges to g on $E \setminus F$; and $m(F)=0$. Therefore g is a measurable function and we have proved $\langle g_k \rangle$ converges to g a.e.

We ow show that $\langle g_k \rangle$ converges to g in measure.

Suppose $x \notin \bigcup_{k=p}^{\infty} F_k$.

Then $x \notin F_{p+r-1}$ for $r \in \mathbb{N}$ and so

$$|g_{p+r-1}(x) - g_{p+r}(x)| < \frac{1}{2^{p+r-1}}.$$

The series

$$\sum_{r=1}^{\infty}\left(g_{p+r-1}(x)-g_{p+r}(x)\right)$$

converges absolutely. Thus

$$g_p(x)-g(x)=\sum_{r=1}^{\infty}\left(g_{p+r-1}(x)-g_{p+r}(x)\right)$$

and
$$\left|g_p(x)-g(x)\right|\le\sum_{r=1}^{\infty}\left|g_{p+r-1}(x)-g_{p+r}(x)\right|$$

$$<\sum_{r=1}^{\infty}\frac{1}{2^{p+r-1}}=\frac{1}{2^{p-1}}.$$

therefore

$$\left\{x\in E:\left|g_p(x)-g(x)\right|\ge\frac{1}{2^{p-1}}\right\}\subseteq\bigcup_{k=p}^{\infty}F_k$$

and so

$$m\left\{x\in E:\left|g_p(x)-g(x)\right|\ge\frac{1}{2^{p-1}}\right\}<\frac{1}{2^{p-1}}.\text{ -------------- }(*)$$

Suppose $\varepsilon>0$ is given. Choose a $k(\varepsilon)\in\mathbb{N}$ such that

$$\frac{1}{2^{k(\varepsilon)-1}}<\varepsilon.$$

If $K\ge k(\varepsilon)$, then $\dfrac{1}{2^{K-1}}\le\dfrac{1}{2^{k(\varepsilon)-1}}<\varepsilon$ and so by $(*)$ for $K\ge k(\varepsilon)$

$$m\left\{x\in E:\left|g_k(x)-g(x)\right|\ge\varepsilon\right\}<\varepsilon.$$

This proves that

$\langle f_{n_k}\rangle$ converges to g in measure.

Now by proposition 10.5 we conclude that

$\langle f_n\rangle$ converges to g in measure.

10.7 Notation : Suppose E is a set, f and f_n for $n \in \mathbb{N}$ are real valued functions on E and $\varepsilon > 0$. In this lesson some subsets of E occur repeatedly. These are defined in terms of f, f_n and ε. We introduce the following notation.

$$B(n, \varepsilon) = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$$

$$A(n, \varepsilon) = \bigcup_{m \geq n} B(m, \varepsilon)$$

$$= \{x \in E : |f_m(x) - f(x)| \geq \varepsilon \text{ for some } m \geq n\}$$

$$A(\varepsilon) = \bigcap_n A(n, \varepsilon)$$

$$= \{x \in E : \text{for each } m \in \mathbb{N} \text{ we can find a } k_m \geq m \text{ such that } |f_{k_m}(x) - f(x)| \geq \varepsilon\}$$

these sets have the following properties

$A(n, \varepsilon) \supseteq A(n+1, \varepsilon)$; if $0 < \varepsilon < \delta$, then

$B(n, \delta) \subseteq B(n, \varepsilon)$

10.8 Lemma : Suppose for some x in E

$$\lim_n f_n(x) = f(x).$$

Then, if $\varepsilon > 0$ $x \notin A(\varepsilon) = \bigcap_n A(n, \varepsilon)$;

hence if f_n converge to f pointwise

$$\bigcap_n A(n, \varepsilon) = \phi.$$

Proof : We can find an $n(\varepsilon, x) \in \mathbb{N}$ such that $|f_m(x) - f(x)| < \varepsilon$

if $m \geq n(\varepsilon, x)$. This means that

$$x \notin B(n, \varepsilon) \text{ for } n \geq n(\varepsilon, x);$$

that is $x \notin A(n, \varepsilon)$ for $n \geq n(\varepsilon, x)$

and so $x \notin \bigcap_n A(n, \varepsilon)$

f_n converges to f pointwise means that for each $x \in E$

$$\lim_{n} f_n(x) = f(x).$$

Therefore in that case for each $\varepsilon > 0$

$$\bigcap A(n, \varepsilon) = \phi.$$

10.9 Proposition : Suppose E is a measurable set of finite measure, f and f_n for $n \in \mathbb{N}$ are measurable functions on E such that

$\langle f_n \rangle \to f$ almost everywhere.

Then $\langle f_n \rangle$ converges to f in measure.

Proof : Suppose $\varepsilon > 0$. With the notation introduced above we have

$$A(n, \varepsilon) \supseteq A(n+1, \varepsilon)$$

and $\quad A(\varepsilon) = \bigcap_{n} A(n, \varepsilon) = \phi$ by 12.8,

and $\quad m(A(n, \varepsilon)) \leq m(E) < +\infty$

Therefore

$$\lim_{n \to \infty} m(A(n, \varepsilon)) = 0$$

Since $B(n, \varepsilon) \subseteq A(n, \varepsilon)$

We obtain that

$$\lim_{n \to \infty} m(B(n, \varepsilon)) = 0$$

This implies that we can find $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have

$$m(B(n, \varepsilon)) < \varepsilon$$

i.e. $\quad m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon$

for all $n \geq n(\varepsilon)$. The proposition is proved.

10.10 Proposition : Suppose $\int |f_n - f| \, dm \to 0$ as $n \to \infty$. Then $\langle f_n \rangle$ converges to f in measure.

Proof : Suppse $\varepsilon > 0$. Then we note that on the one hand

$$\int_{B(n,\varepsilon)} |f_n - f| dm \leq \int |f_n - f| dm$$

and on the other hand

$$\int_{B(n,\varepsilon)} |f_n - f| dm \geq \varepsilon \, m(B(n,\varepsilon))$$

We choose $n(\varepsilon^2) \in \mathbb{N}$ such that

$$\int |f_n - f| dm < \varepsilon^2$$

for all $n \geq n(\varepsilon^2)$. Then

$$m(B(n, \varepsilon)) \cdot \varepsilon \leq \int |f_n - f| dm < \varepsilon^2$$

i.e. $m(B(n,\varepsilon)) < \varepsilon$

if $n \geq n(\varepsilon^2)$. The proposition is proved.

10.11 Example : We now give an example of a sequence

$$f_n : [0, 1] \to \mathbb{R}$$

of measurable functions such that

1) $\langle f_n \rangle$ converges to the function 0 in measure

2) there is no x in $[0, 1]$ such that $\langle f_n(x) \rangle$ converges.

Definition of f_n. Suppose n is a positive integer and

$$n = k + 2^r, \; 0 \leq k < 2^r.$$

We note that r is the unique positive integer such that

$$2^r \leq n < 2^{r+1}.$$

We define $f_n = 1$ on $\left[\frac{k}{2^r}, \frac{k+1}{2^r}\right]$

and $f_n = 0$ on the complement.

f_n is clearly a measurable function.

Now we fix some number x in $[0, 1]$ and try to determine

$$f_n(x)$$

for various n. Suppose r, k, n are as above. We write the "binary expansion" for x

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots\cdots + \frac{a_p}{2^p} + \cdots\cdots$$

where a_p is either 0 or 1. For calculating

$$f_n(x)$$

we have to determine if

$$x \in \left[\frac{k}{2^r}, \frac{k+1}{2^r}\right].$$

The condition

$$\frac{k}{2^r} \le x \le \frac{k+1}{2^r}$$

is equivalent to

$$k \le 2^r x \le (k+1).$$

We have $2^r x = 2^{r-1} a_1 + \cdots\cdots + a_r + \frac{a_{r+1}}{2} + \frac{a_{r+2}}{2^2} + \cdots\cdots$

since $a_p = 0$ or 1 we have

$$0 \le a_1 2^{r-1} + \cdots\cdots + a_r \le 2^{r-1} + \cdots\cdots + 1 = 2^r - 1$$

and $$0 \le \frac{a_{r+1}}{2} + \frac{a_{r+2}}{2} + \cdots\cdots \le 1.$$

Suppose we set

$$k = a_1 2^{r-1} + \cdots\cdots + a_r.$$

Then we have

$$\frac{k}{2^r} \le x \le \frac{k+1}{2^r}$$

Therefore only for

$$n=(k-1)+2^r,\ k+2^r,\ (k+1)+2^r$$ it is possible that

$$f_n(x)=1;$$

for $n=k+2^r$ we certainly have

$$f_n(x)=1.$$

For all $n=p+2^r$

$$0 \le p \le 2^r-1,\quad p \ne k-1,\ k,\ k+1$$

we have

$$f_n(x)=0$$

we note that $n \ge 2^r$.

Therefore given any positive integer p we can find a positive integer.

$$n \ge 2^p$$

such that $f_n(x)=1$.

Suppose for the same p we set

$$n=r+2^p$$

$0 \le r \le 2^{p-1}$ and $r \ne k-1,\ k,\ k+1$.

Then it is clear that

$$x \notin \left[\frac{r}{2^p}, \frac{r+1}{2^p}\right]$$

and hence $f_n(x)=0$.

Thus there are infinitely many integers n such that $f_n(x)=1$ and also infinitely many integers q such that $f_q(x)=0$.

Therefore the sequence $\langle f_n(x) \rangle$ does not converge to any limit.

we now show that $\langle f_n \rangle$ converges to zero in measure.

Suppose r is a natural number and n in $\mathbb{N}$ is $\geq 2^r$. If s is a positive integer such that

$$2^s \leq n < 2^{s+1}$$

we have $r \leq s$; and if we set $k = n - 2^s$

f_n is 1 on $\left[\frac{k}{2^s}, \frac{k+1}{2^s}\right]$

f_n is 0 on on the complement.

So,

$$m\{x \in [0,1] : f_n(x) > 0\} = \frac{1}{2^s} \leq \frac{1}{2^r}.$$

Let $\varepsilon > 0$ choose $n(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{2^{n(\varepsilon)}} < \varepsilon$$

Then for $n \geq n(\varepsilon)$ we have

$$m\left\{x \in [0, 1] : \underset{n}{f}(x) \geq \varepsilon\right\} = \begin{cases} 0 & \text{if } 1 < \varepsilon \\ < \varepsilon & \text{if } \varepsilon \leq 1 \end{cases}.$$

Therefore $\langle f_n \rangle$ converges to zero in measure.

We now prove that a sequence $\langle f_n \rangle$ converging to f in measure has a subsequence $\langle f_{n_k} \rangle$ that converges to f almost every where.

10.12 Proposition : Suppose E is a measurable set and $\langle f_n \rangle$ is a sequence of measurable functions on E, f is a measurable function on E and $\langle f_n \rangle$ converges to f in measure. Then there is a subsequence $\langle f_{k_n} \rangle$ of $\langle f_n \rangle$ such that

$\langle f_{k_n} \rangle$ converges to f a.e.

Proof : Case 1 : Suppose f_n and f satisfy the condition

$$m\left\{x \in E : \left|f_n(x) - f(x)\right| \geq \frac{1}{2^n}\right\} < \frac{1}{2^n} \text{ for all } n \in \mathbb{N}.$$

In the notation we have introduced the condition is

$$m\left(B\left(n, \frac{1}{2^n}\right)\right) < \frac{1}{2^n}$$

If x does not belong to $B\left(n, \frac{1}{2^n}\right)$ we have

$$\left|f_n(x) - f(x)\right| < \frac{1}{2^n}.$$

Therefore outside the set $\bigcup_{k=n}^{\infty} B\left(m, \frac{1}{2^m}\right)$

we have $\left|f_{n+k}(x) - f(x)\right| < \frac{1}{2^{n+k}}$.

Let us write for convenience

$$E_n = \bigcup_{m=n}^{\infty} B\left(m, \frac{1}{2^m}\right)$$

Then we have

$$E_n \supseteq E_{n+1}.$$

What we have proved is that outside

$$E_0 = \bigcap E_n$$

the sequence f_n converges to f

We have

$$m(E_n) \leq \sum_{k=0}^{\infty} m\left(B\left(n+k, \frac{1}{2^{n+k}}\right)\right)$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{2^{n+k}}$$

$$= \frac{1}{2^{n-1}}.$$

So it follows that

$$m(E_0) = \lim_n m(E_n) = 0$$

Thus $\langle f_n \rangle$ converges to f a.e.

Case 2 : $\langle f_n \rangle$ converges to f in measure otherwise arbitrary.

We choose the sequence $\langle f_{k_n} \rangle$ by induction on n.

For each n we write

$$\varepsilon_n = \frac{1}{2^n}$$

Since $\langle f_n \rangle \to f$ in measure, there is an n_1 in $\mathbb{N}$ such that

$$m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon_1\} = B(n_1, \varepsilon_1).$$

We set $k_1 = n_1$.

There is an n_2 in $\mathbb{N}$ such that for all $n \geq n_2$

$$m(B(n, \varepsilon_2)) < \varepsilon_2.$$

We set

$$k_2 = 1 + \max\{n_2, k_1\}.$$

Suppose we have defined positive integers

$$k_1, k_2, \dots\dots\dots\dots, k_p$$

such that

1) $k_1 < k_2 < \dots\dots\dots < k_p$ and

2) $m(B(n, \varepsilon_r)) < \varepsilon_r$ for $n \geq k_r$, $r = 1, \dots\dots\dots, p$.

10.13 Theorem Bounded Convergence Theorem for Convergence in Measure :

Suppose E is a set of finite measure

$$f_n : E \to \mathbb{R}$$

is a measurable function for each $n \in \mathbb{N}$. Suppose the sequence $\langle f_n \rangle$ is such that

(1) $\langle f_n \rangle$ converges in measure to a measurable function f and

(2) There is an M in $\mathbb{R}$ such that $|f_n| \leq M$ for all n.

Then we have

$$\lim_{n\to\infty} \int f_n = \int \lim_n f_n .$$

Proof : Suppose $\varepsilon > 0$. Then since $\langle f_n \rangle$ converges to f in measure, there is a positive integer $n(\varepsilon)$ such that the set

$$B(n, \varepsilon)$$

has measure less than ε for all $n \geq n(\varepsilon)$. Corresponding to ε_{p+1} there is an $n_{p+1} \in \mathbb{N}$ such that for all $n \geq n_{p+1}$

$$m\left(B\left(n, \varepsilon_{p+1}\right)\right) < \varepsilon_{p+1} .$$

We set $k_{p+1} = 1 + \max\left\{n_{p+1}, k_p\right\}$.

It is clear that $k_1, \ldots\ldots, k_{p+1}$ satisfy conditions (1) and (2) above for $p+1$.

Thus we have constructed a subsequence

$$\left\langle f_{k_n} \right\rangle .$$

We claim that

$$\left\langle f_{k_n} \right\rangle \to f \text{ a.e.}$$

By our construction

$$B\left(k_n, \frac{1}{2^n}\right) \text{ has measure } < \frac{1}{2^n} .$$

The sequence

$g_n = f_{k_n}$ satisfies the following conditions

1) $\langle g_n \rangle \to f$ in measure and

2) $m\left\{x \in E : \left|g_n(x) - f(x)\right| \geq \frac{1}{2^n}\right\} < \frac{1}{2^n}$.

So by what we have proved in case 1

$$\langle g_n \rangle \to f \text{ a.e.}$$

we write

$$E(f, M+\varepsilon) = \{x \in E : |f(x)| \geq M+\varepsilon\}$$

Thus we have

$$m(E(f, M+\varepsilon)) < \varepsilon.$$

To see this choose any $n \in \mathbb{N}$. If $x \in E(f, M+\varepsilon)$ we have

$$-M \leq f_n(x) \leq M \text{ and}$$

$$f(x) \leq -(M+\varepsilon) \text{ or } f(x) \geq M+\varepsilon.$$

Therefore $f(x) - f_n(x) \geq (M+\varepsilon) - M = \varepsilon$ if $f(x) \geq M+\varepsilon$

$$f_n(x) - f(x) \geq (-M) - (-(M+\varepsilon)) = \varepsilon \text{ if } f(x) \leq -(M+\varepsilon)$$

Thus $\left|f_n(x) - f(x)\right| \geq \varepsilon$

and so we have $E(f, M+\varepsilon) \subseteq B(n, \varepsilon)$;

Consequently $m(E(f, M+\varepsilon)) < \varepsilon$.

If $0 < \varepsilon' < \varepsilon$, then

$$E(f, M+\varepsilon) \subseteq E(f, M+\varepsilon'), \text{ and}$$

$$m\left(E\left(f, M+\frac{1}{n}\right)\right) < \frac{1}{n}. \text{ It follows that}$$

$$|f| \leq M \text{ a.e.}$$

Let $F = \{x \in E : |f| > M\}$

We may set $g(x) = f(x)$ if $x \in E$ and $|f(x)| \leq M$

$$g(x) = 0 \qquad \text{if } x \in E \text{ and } |f(x)| > M$$

Then it is easy to see that

1) $\langle f_n \rangle$ converges to g in measure

2) $|g| \le M$ and

3) $\int f = \int g$.

So it is enough to prove that

$$\lim_n \int f_n = \int g$$

we have $\int_E f_n - g = \int_{B(n,\varepsilon)} f_n - g + \int_{E \setminus B(n,\varepsilon)} f_n - g$

Therefore

$$\left| \int_E f_n - f \right| \le \int_{B(n,\varepsilon)} |f_n - f| + \int_{E \setminus B(n,\varepsilon)} |f_n - f|$$

$$\le 2M \cdot \varepsilon + \varepsilon m(E \setminus B(n,\varepsilon))$$

$$\le \varepsilon(2M + mE)$$

This proves that

$$\lim_n \int f_n = \int f$$

10.14 Fatou's Lemma for Convergence in Measure : Suppose E is a measurable set, f and f_n for each n in $\mathbb{N}$ are measurable functions on E such that

1) f_n are non-negative

and 2) $\langle f_n \rangle$ converges to f in measure.

Then $\int f \le \underline{\lim}_n \int f_n$.

Prof : Suppose g is a simple function such that it vanishes outside a set of finite measure and

$$g \le f.$$

We define $g_n(a) = \min\{g(x), f_n(x)\}$

Suppose x is such that

$$g(x)-g_n(x)\neq 0.$$

By definition $g_n(x)\leq g(x)$

and $g_n(x)$ is either $g(x)$ or $f_n(x)$. If $g_n(x)\neq g(x)$ it follows that

$$g_n(x)=f_n(x) \text{ and } f_n(x)<g(x)$$

Therefore we have

$$g(x)=f_n(x)<g(x)\leq f(x).$$

Therefore

$$g(x)-g_n(x)\leq f(x)-f_n(x) \text{ and so}$$

$$\{x\in E:|g(x)-g_n(x)|\geq\varepsilon\}\subseteq\{x\in E:|f(x)-f_n(x)|\geq\varepsilon\}$$

This implies that the sequence $\langle g_n\rangle$ converges to g in measure. By bounded convergence theorem we have

$$\int g=\lim_n \int g_n$$

Now, $g_n\leq f_n$ and hence $\int g_n\leq\int f_n$

And so

$$\int g=\underline{\lim}_n \int g_n \leq \underline{\lim}_n \int f_n$$

This inequality being true for all g we have

$$\int f=\sup_g\ \{\int g : g \text{ simple, } g\leq f \text{ and } g \text{ is zero outside a set of finite measure}\}$$

$$\leq \underline{\lim}_n \int f_n$$

10.15 Lebesgue Convergence Theorem for Convergence in Measure : Suppose E is a measurable set, f_n is a sequence of measurable functions on E such that

1) $\langle f_n\rangle$ converges in measure to f

2) There is a sequence of integrable functions g_n on E such that

$|f_n| \leq g_n$

3) $\langle g_n \rangle$ converges to g in measure and

$$\lim_n \int g_n = \int g.$$

Then $\lim_n \int f_n = \int f$.

i.e. $\lim_n \int f_n$ exists and is equal to $\int \lim_n f_n$

Proof : Consider the sequence of functions $h_n = g_n - f_n$.

Since $|f_n| \leq g_n$ we obtain $h_n \geq 0$. The sequence $\langle h_n \rangle$ converges to $(g - f)$ in measure.

Therefore

$$\int (g - f) = \int \lim_n h_n \leq \underline{\lim}_n \left\{ \int g_n - \int f_n \right\}.$$

Let us set

$$b_n = \int g_n, \quad a_n = \int f_n$$

$$b = \int g, \quad a = \int f$$

We have assumed that

$$\lim_{n \to \infty} b_n = b.$$

Therefore $\dfrac{\lim}{n}\left\{ \int g_n - \int f_n \right\} = \dfrac{\lim}{n}(b_n - a_n)$

$$= \lim_n b_n + \frac{\lim}{n}(-a_n)$$

$$= b + \left(- \overline{\lim_n}\, a_n \right)$$

Thus we have

$$\int g - \int f \leq b - \overline{\lim_n}\, a_n.$$

That is

$$\overline{\lim_{n}}\ a_n \leq \int f = a$$

By considering the sequence

$$k_n = g_n + f_n$$

We obtain $a = \int f \leq \underline{\lim_{n}}\ a_n$

This implies that

$$\lim_{n \to \infty} a_n = a.$$

10.16 Short Answer Questions

10.16.1 : Find $\underline{\lim_{n}}$ of $\{a_n\}\ \{b_n\}\ \{c_n\}$ where

(i) $a_n = 1 + (-1)^n$, $b_n = 1 + (-1)^{n+1}$ and $c_n = a_n + b_n$

(ii) $a_n = -n$, $b_n = n$ and $c_n = a_n + b_n$

(iii) $a_n = 1 - n$, $b_n = n$ and $c_n = a_n + b_n$.

Solution : (i) $a_{2n} = 2$ and $a_{2n-1} = 0$ for $n \geq 1$. $\underline{\lim_{n}}\ a_n = 0$

$b_{2n} = a_{2n-1}$ So $\underline{\lim_{n}}\ b_n = 0$.

$c_n = 2$ for all n so $\underline{\lim_{n}}\ c_n = 2$

10.16.2 SAQ : Find $\underline{\lim_{n}}\ f_n$ where

$f_n(x) = x - n$ if n is even

$= nx$ if n is odd.

Solution : $\lim_{n} f_n(x) = -\infty \ \forall \ x$.

10.16.3 SAQ : Show that if $\{f_n\}$ converges to f in measure and $\{g_n\}$ converges to g in measure then

(i) $\{f_n + g_n\}$ converges to $f+g$ in measure.

and (ii) $\{a\, f_n\}$ converges to $a\, f$ in measure $\forall \ a \in \mathbb{R}$.

Solution : (i) Let $\in > 0$. $\exists \ n_1, n_2$ in $\mathbb{N}$ such that

$$m\left\{x : \left|f_n(x) - f(x)\right| \geq \frac{\in}{2}\right\} < \frac{\in}{2} \text{ for } n \geq n_1 \text{ and}$$

$$m\left\{x : \left|g_n(x) - g(x)\right| \geq \frac{\in}{2}\right\} < \frac{\in}{2} \text{ for } n \geq n_2$$

If $n \geq n_1 + n_2$ then

$$m\left\{x : \left|(f_n + g_n)(x) - (f+g)(x)\right| \geq \in\right\} < \in$$

(ii) We may assume $a \neq 0$. For $\in > 0$.

$$\left\{x : \left|a\, f_n(x) - a\, f(x)\right| \geq \in\right\}$$

$$= \left\{x : \left|f_n(x) - f(x)\right| \geq \frac{\in}{|a|}\right\}$$

10.16.4 SAQ : If $f_n(x) = f(x) = x$ and $g_n(x) = \frac{1}{n} \forall x \in \mathbb{R}$ and $n \in \mathbb{N}$ then

$f_n \to f$ in measure

$g_n \to 0$ in measure

but $\{f_n\, g_n\}$ does not converge to 0 in measure.

Solution : $\left\{x : \left|f_n\, g_n(x)\right| \geq \in\right\}$

$$= \left\{ x : \frac{|x|}{n} \geq \in \right\}$$

$$= \{ x : |x| \geq n \in \} = (-\infty, -n \in] \cup [n \in, \infty)$$

10.17 Model Examination Questions

1. Define convergence in measure.

   Does convergence in measure imply convergence a.e. ? Justify your answer.

2. Let $\{f_n\}$ be a sequence of measurable functions which converges to f in measure. Show that there is a subsequence of $\{f_n\}$ which converges to f a.e .

3. Show that $\{f_n\}$ converges to f in measure and $\{g_n\}$ converges to g in measure imply that $\{f_n + g_n\}$ converges to $f + g$ in measure.

10.18 Exercises

1. If $\{f_n\}$ converges to f in measure show that $\{|f_n|\}$ converges to $|f|$ in measure.

2. Show that a sequence $\{f_n\}$ of measurable functions defined on a set E of finite measure converges in measure to f if and only if every subsequence of $\{f_n\}$ has a subsequence which converges to f in measure.

10.19 Reference Book

Real Analysis - H.L. Royden

Lesson Writer :

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LESSON 11 : FUNCTIONS OF BOUNDED VARIATION

INTRODUCTION :

The Vector Space generated by the class of Monotone functions has a special role in Lebesgue Theory. Functions in this class are precisely functions of bounded variation. In this lesson we study some properties of functions of bounded variation. Their utility in the present context will be unravelled in the next lesson.

Let us recall that for any real number a, $a^+ = \max\{a,0\}$, $a^- = \max\{-a,0\}$ so that $|a| = a^+ + a^-$ If f: $[a,b] \to R$ is a function for any partition $\eta = \{a = x_0 < x_1 < .... x_n\}$ write $P = P_f(\eta) =$

$$\sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+, \ \eta = n_f(\eta) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^- \text{ and } t = t_f(\eta) = P(\eta) = P_f(\eta) + n_f(\eta).$$

Let 'S' be the set of all partitions of [a,b]. Notation: We write $P_a^b = P = \sup \{P_f(\eta)/\eta \in S\}$

$N_a^b = N = \sup \{n_f(\eta)/\eta \in S\}$ and $T_a^b = T = \sup \{t_f(\eta)/\eta \in S\}$.

11.1 Definition: $f : [a,b] \to R$ is said to be of bounded variation if $T < \infty$. T is called the total variation of f on [a,b], which P is called the positive variation and N is called the negative variation. The set of all functions of bounded variation on [a,b] is denoted by BV [a,b] or simply BV. When there is no confusion we write $P(\eta)$ for $P_f(\eta)$ and so on.

11.2 Proposition: If f is of bounded variation on [a,b], then $T_a^b = P_a^b = N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$.

Proof: For any partition $\eta = \{a = x_0 < ...... x_n = b\}$, $P(\eta) = \sum_{i=1}^{n} [f(x_i) - f(x_{j-1})]^+$, $n(\eta) =$

$$\sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^-$$

$$\Rightarrow \quad P(\eta) - n(\eta) = \sum_{i=1}^{n} \Delta_i^+ - \Delta_i^- \quad \text{where } \Delta_i = f(x_i) - f(x_{i-1})$$

$$= \sum_{i=1}^{n} \Delta_i = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$= f(b) - f(a)$

$\Rightarrow$ $P(\eta) = n(\eta) + f(b) - f(a) \leq N + f(b) - f(a)$. Since this is true $\forall \eta$, it follows that

$$P = \sup_{\eta} P(\eta) \leq N + f(b) - f(a)$$

$\Rightarrow$ $P - N \leq f(b) - f(a)$ Since $(-a)^+ = \max\{-a,0) = a^-$ and $(-a)^- = \max\{a,0\} = a^+$ for any

$\eta = \{a = x_0 < \ldots\ldots x_n = b\}$, write $\Delta_i = f(x_i) - f(x_{i-1})$. Then $P_{-f}(\eta) = \sum_{n^-=1}^{n} (-\Delta_i)^+ = \sum_{i=1}^{n} \Delta_i^-$

$= n_f(\eta)$ and hence so that $n_{(-f)}(\eta) = P_f(\eta)$

$n_f(\eta) \leq P_f(\eta) + f(a) - f(b)$,

hence $N \leq P + f(a) - f(b)$ so that

$N - P \leq f(a) - f(b)$.

Hence $P - N = f(b) - f(a)$

Hence $t(\eta) = P(\eta) + n(\eta) = P(\eta) + P(\eta) - \{ f(b) - f(a) \}$

$= 2P(\eta) + f(a) - f(b)$

$= 2P(\eta) + N - P$;

$T \geq 2P(\eta) + N - P$.

$\Rightarrow$ $T \geq 2P + N - P = P + N$ ..... (i)

Moreover $\forall \eta \in S$, $t(\eta) = P(\eta) + n(\eta) \leq P + N$.

Hence $T = \sup_{\eta \in S} t(\eta) \leq P + N$ .... (ii)

From (i) and (ii) T = P + N.

11.3 Theorem: A function f [a,b] $\rightarrow$ R is of bounded variation on [a,b] if and only if f is the difference of two monotonically increasing functions on [a,b].

Proof: If g and h are monotonically increasing on [a,b] $\forall$ partition $\eta = \{a = x_0 < x_1 < .... x_n = b\}$,

$$t_{g-h}(\eta) = \sum_{i=1}^{n} \left|(g-h)(x_i)-(g-h)(x_{i-1})\right|$$

$$\leq \sum_{i=1}^{n} \left|g(x_i)-g(x_{i-1})\right| + \sum_{i=1}^{n} \left|h(x_i)-h(x_{i-1})\right|$$

$$= \sum_{i=1}^{n} g(x_i)-g(x_{i-1}) + \sum_{i=1}^{n} h(x_i)-h(x_{i-1})$$

$$= [g(b) - g(a)] + [h(b)] - h(a)]$$

This is true $\forall \eta$. Hence $T_{g-h} = \sup_{\eta} t_{g-h}(\eta) \leq g(b) - g(a) + h(b) - h(a) < \infty$. Hence g-h $\in$ BV on [a,b]. Conversely suppose f is of bounded variation on [a,b]. Then $T_a^b f < \infty$. For a <x $\leq$ b define

$g(x) = P_a^b$, $h(x) = N_a^b$ and g (a) = h (a) = 0.

If $\eta_1 = \{a = x_0 < x_2 < .... < x_n = y_1\}$, $\eta_2 = \{a = x_0 < .... < x_n < x_{n+1} = y_2\}$ where $a < y_1 < y_2 < b$,

$P(\eta_1) = \sum_{i=1}^{n} [f(x_i)- f(x_{i-1})]^+ \leq \sum_{i=1}^{n} [f(x_i)- f(x_{i-1})]^+$ $P(\eta_2) \leq P_a^{y_2}$ This is true $\forall \eta_1$

so $P_a^{y_1} \leq P_a^{y_2}$ by $n(\eta_1) \leq n(\eta_2)$ so that $N_a^{y_1} \leq N_a^{y_2}$ and $t(\eta_1) \leq -(\eta_2)$ so that

$T_a^{y_1} \leq T_a^{y_2} \leq T_a^b$ since f$\in$B V, $T_a^b < \infty$ so that $0 \leq P_a^x \leq T_a^x \leq T_a^b < \infty$ and $0 \leq N_a^x \leq T_a^x \leq T_a^b < \infty$.

Hence g and h are real valued monotonically increasing functions on [a,b]. Also f(x) - f(a) =

$P_a^x - N_a^x$ = g(x) - h(x) $\forall$ x $\in$ [a,b] so that f(x) = g(x) - (h(x) - f(a)).

Since g(x) and h(x)-h(a) are monotonically increasing, the request follows.

Let $\eta = \{a = c_0 < c_1 - \delta_1 < c_1 + \delta_1 < c_2 - \delta_2 < \ldots\ldots < c_k - \delta_k < c_k + \delta_k < b\}$

Then $t(\eta) \geq \sum_{i=1}^{k} [f(c_i + \delta_i) - f(c_i - \delta_i)] > \frac{k}{n}$.

$\Rightarrow \quad T \geq t(\eta) > \frac{k}{n}$.

$\Rightarrow \quad k < n\,T$

Thus, the number of elements in $S_n < n$ T, hence S_n is finite for every n. This complete the proof.

11.4 Example: If f is integrable on [a,b] then the function F defined by

$F(x) = \int_a^x f(t)\,dt$ for $a < x \leq b$ and F(a) = 0 is a continuous function of bounded variation on [a,b].

Solution: Continuity : since f is integrable on [a,b], by 2.9 given $\varepsilon > 0 \ \exists$ a $\delta > 0 \ni m(A) < \delta$ and

$A \subseteq [a,b] \Rightarrow \left| \int_A f \right| < \epsilon$. Hence if $|x-y| < \delta$, $|F(x) - F(y)| = \left| \int_x^y f(t)\,dt \right| < \epsilon$

(when x > y $\int_x^y f(t)\,dt = -\int_y^x f(t)\,dt$). Hence F is infact uniformly continuous on [a,b]

For any partition P = $\{a = x_0 < x_1 < \ldots < x_n = b\}$ and for $1 \leq i \leq n$ $F(x_i) - f(x_{i-1})$ | =

$\left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right| \leq \int_a^b |f(t)|dt$ since f is integrable, so is |f| hence t (P) $\leq \int_a^b |f(t)|dt \ \forall P$. This implies that f is of bounded variation on [a,b] and $T = \sup_{P} t(P) \leq \int_a^b |f(t)|dt$.

11.5 Theorem;

If f: [a,b] $\rightarrow$ R is a function of BV then f has atmost a countable number of discontinuities on [a,b]

Proof: Since f $\in$ B V on [a,b], there exist monotonically increasing functions g and h such that f = g-h on [a,b]. It is thus enough to show that a monotonically increasing function can have atmost countable number of discontinuties. Thus we may assume that f is monotonically increasing.

If $a < c < b$, and $a < x < c < y < b$, $f(a) \leq f(x) \leq f(c) \leq f(y) \leq f(b)$. Hence $\{f(x)\ a \leq x < c\}$ is bounded above by f(c) while $\{f(y)\ c < y \leq b\}$ is bounded below. Let A = lub $\{f(x)\ a \leq x < c\}$. Clearly $A \leq f(c)$. We show that f (c-) = A. If $\varepsilon > 0$, $A - \varepsilon < A$ So $\exists\ x_1 \ni a \leq x_1 < c$ and $f(x_1) > A - \varepsilon$. Since f is monotonically increasing, $x_1 < x < c \Rightarrow f(x_1) \leq f(x)$ so that $A - \varepsilon < f(x_1) \leq f(x) \leq A < A + \varepsilon$. Thus $|f(x) - A| < \varepsilon$ if $x_i < x < c$. This implies that f(c-) = A.

We similarly show that f(c+) = B.
= inf { f(x) / c < x < b}. Thus f (c-) $\leq$ f(c) $\leq$ f (c+).

Since f is continous iff f(c-) = f(c+) = f(c), f is discontinuous at c if and only if f(c-) < f(c+). In other words the set of discontinuities of f is precisely the set $S = \{c / a < c < b \text{ and } f(c+) - f(c-) > 0\}$ together with possibly a and / or b. It is therefore enough to show that S is atmost countable. If $S_n = \{c / a < c < b \text{ and } f(c+) - f(c^-) > 1/n\}$ then $S = \bigcup_{n=1}^{\infty} S_n$. Hence it sufficies to show that for every positive integer n, S_n is atmost countable. We show that S_n is finite.

To show that S_n is finite let $c_1, c_2, \ldots c_k$ be any k elements of S_n and $a \leq c_1 < c_2 < \ldots < c_n$. $\forall i$ choose $\delta_i > 0 \ni$

$f(c_i+h) - f(c_i-h) > 1/n$ for $c_i < h \leq c_i + \delta_i$ and also such that $a < c_1 - \delta_1 - < c_1 < c_1 + s_1 < c_2 - \delta_2 \leq c_2 < c_2 + \delta_2 < \ldots < c_n - \delta_n < c_n$

11.6 SHORT ANSWER QUESTIONS WITH SOLUTIONS:

SAQ.1. Define $f(x) = x \sin \frac{1}{x}$ if $0 < x \le 1$ and f(0)

show that f is not of bounded variation [0,i].

For each n Let P_n be the partition

$$P_n : \{0 < \frac{1}{(4n+1)\pi/2} < \frac{1}{4n\pi/2} < \frac{1}{(4n-1)\pi/2} < \ldots\ldots < \frac{1}{3\pi/2} < \frac{1}{2\pi/2} < \frac{1}{\pi/2} < 1\}$$

$$t(P_n) = \left|f\left(\frac{1}{(4n+1)\pi/2}\right) - f(0)\right| + \left|f\left(\frac{1}{(4n+1)\pi/2}\right) - f((4n+1)\pi/2)\right| + \ldots\ldots \left|f\left(\frac{1}{\pi/2}\right) - f(1)\right|$$

$$\Rightarrow \left|\frac{1}{(4n+1)\pi/2} - 0\right| + \left|\frac{0}{4} - \frac{1}{(4n+1)\pi/2}\right| + \ldots\ldots + \left|\frac{1}{\pi/2} - f(1)\right|$$

$$> \frac{2}{\pi}\left\{\frac{1}{(4n+1)} + \frac{1}{(4n+1)} + \frac{1}{(4n-1)} + \frac{1}{(4n-1)} + \ldots \frac{1}{3} + \frac{1}{3}\right\} = \frac{4}{\pi}\left\{\frac{1}{4n+1} + \frac{1}{4n-1} + \frac{1}{4n+3} + \ldots\ldots + \frac{1}{3}\right\}$$

Since the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots$ is divegent, the sequence $\{s_n\}$ where $s_n = 1/3 + 1/5 + \ldots\ldots \frac{1}{4n+1}$ is divergent. Hence $\{t(P_n)\}$ diverges to $+\infty$. Hence f is not of bounded variation on [0,1]

SAQ.2. Define $f(x) = x^2 \sin 1/x$ for $0 < |x| \le 1$ and $f(0) = 0$. Show that f is of bounded variation on [-1, 1].

Solution: If $0 < |x| \le 1$ $f^1(x) = 2x \sin 1/x . \cos 1/x$. So $|f^1(x)| \le 3$ since $|\sin 1/x| \le 1$ and $|\cos 1/x| \le 1$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin 1/x = 0.$$

Hence $|f'(x)| \le 3 \ \forall x \in [0,1]$; If $\eta = \{-1 = x_0 < x_1 < \ldots\ldots < x_n = 1\}$ is any partition of [-1,1] for every i, $\exists t_i \in (x_{i-1}, x_i) \ni f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1})$ so that $|f(x_i) - f(x_{i-1})|$

$= |f'(t_i)||(x_i - x_{i-1})| \le 3(x_i - x_{i-1})$.

Hence $t(\eta) = \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \le 3 \sum_{i=1}^{n} (x_i - x_{i-1}) = 3 \times 2 = 6$.

This being true for every partition η of [-1,1], it follows that f is of bounded variation on [-1,1].

SAQ.3. Let f be a function of bounded variation on [a,b] and $V = V_f$ be the total variation of f defined by $V_f(a) = 0$ and $V_f(x) = var_f[a,x] = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| / P = \{a = x_0 < .... < x_n = b\}$ show that V_f is cdontinuous at x if f is continuous at x. Let $a \le c < b$. We show that $V_f(c+) = V_f(c)$.

Solution: Given $\varepsilon > 0 \ \exists$ a $\delta > 0 \ni |f(x) - f(c)| < \frac{\varepsilon}{2}$ if $|x-c| < \delta$ and $x \in [a,b]$. Since $V_f[c, b] - \frac{\varepsilon}{2}$ is not an upper bound of the collection of sum $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$ of partitions $c = x_0 < ..... < x_n = b$, $\exists$ such a partition $c = x_0^0 < x_1^0 < ...... < x_n^0 = b \ni$

$$\sum_{i=1}^{n} |f(x_i^0) - f(x_{i-1}^0)| > V_f[c,b] - \frac{\in}{2}$$ Choose $x'_1 \in (c,x) \ni c\ x_1^1 < C + S$

$|f(x_1) - f(c) \le |f(x_1) - f(x'_1) + |f(x'_1) - f(c)|$

$\le |f(x_1) - f(x'_1) + \frac{\varepsilon}{2}$

Let $Q = \{c = x_0^0 < x_1^0 < ...... < x_n^0 = b\} = PV\{x_1^1\}$.

$|f(x_1^1) - f(x_0^0)| + |f(x_1^0) - f(x_1^1)| + |f(x_2^0) - f(x_1^0)| + ... + |f(x_n^0) - f(x_{n-1}^0)|$

$$\geq |f(x_1^0) - f(c)| + \sum_{i=2}^{n} | f(_i^0) - (x_{i-1}^0) |.$$

$$> V_f [c,b] - \frac{\varepsilon}{2}$$

$\Rightarrow$ $V_f [c,b] - \frac{\varepsilon}{2} < |f(x'_1) - f(c)| + V_f [x'_1 b]$

$\Rightarrow$ $V_f [c,b] - f(x'_1) < \frac{\varepsilon}{2}$

This being true $\forall\ x'_1$ in $(c, c + \delta)$ it follows that $V_f(c+) = V_f(c)$.

The proof for $V_f(c-) = V_f(c)$

Hence $V_f(c+) = V(c) < V_F(c+) = V(c)$

Hence V_f is right continuous at c.

Left continuity follows from continuity of f at c and infinum property

Then $\lim_{x \to c} V_f(x) = V_f(c)$

Exercises:

1. If f and g are functions of BV on [a,b] show that f + g, and α f where α is any real number, are functions of BV on [a,b] show also that

   (a) T_a^b (f + g) $\leq T_a^b$ (f) + T_a^b g

   (b) T_a^b (α f) = $|\alpha|$ T_a^b (f.)

2. Show that if f $\in$ BV on [a,b], f is bounded, Deduce that if f, g $\in$ BV on [a,b] f g $\in$ BV on [a,b].

3. If $f \in BV$ on $[a,b]$ and $a < c < b$ show that $f \in BV$ on $[a,c]$ as well as $[c,b]$. Show also that $T_a^c(f) + T_c^b(f) = T_a^b(f)$.

4. If $f \in BV$ on $[a,b]$ show that $g(x) = T_a^x$ and $h(x) = T_a^x - f(x)$ are monotonically increasing and $g - h = f$.

5. Let $\{f_n\}$ be a sequence of function on $[a,b]$ and $\forall\, x \in [a,b]$, $f(x) = \lim_n f_n(x)$ show that $T_a^b(f) \leq \underline{\lim}\, T_a^b(f_n)$

6. Let $\{x_n\}$ be a countable set in (a,b), and $\sum_{n=1}^{\infty} C_n$ be a convergent series of positive terms.

Define $f(x) = \sum_{x_n < x} C_n$ Show that f is monotonically increasing on $[a,b]$ and f is discontinuous at x if f $x = x_n$ for some n.

7. Show that if $f \in BV$ on $[a,b]$ f is continous on $[a,b]$ whenever then $g(x) = T_a^x(f)$ is continuous

8. Define f on $[0,1]$ by $f(x) = x^p \sin 1/x$ for $x > 0$ and $f(0) = 0$ show $f \in BV$ on $[0,1]$ if $P \geq 2$.

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LESSON 12 : ABSOLUTE CONTINUITY

INTRODUCTION : In this lesson we characterize the class of functions which satisfy the fundamental theorem of calculus for Lebesgue integral. We define an absolutely continuous function on a closed interval [a,b] study its properties and show that the indefinite integral of an integrable function is absolutely continuous. We further show that if f is absolutely continuous on [a,b], f is differentiable almost everywhere and is the integral of its derivative.

2. DEFINITION: A real valued function f defined on [a,b] is said to be absolutely continuous on [a,b] if given $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^{n} |f(x'_i) - f(x_i)| < \varepsilon$ for every finite collection of nonover lappyingintervals $\{(x_i, x'_i) \; 1 \leq i \leq n\}$ with $\sum_{i=1}^{n} (x'_i - x_i) < \delta$.

Remark: An absc'ıtely continuous function is uniformly continuous.

3. Proposition: If f is absolutely continuous on [a,b] then f is of bounded variation on [a,b].

Proof: Since f is absolutely continuous on [a,b], there exists a positive number δ such that $\sum_{i=1}^{n} |f(x_i) - f(x'_i)| < 1$ for every choice of a finite collection $\{(x_i, x'_i)| \; 1 \leq i \leq n\}$ of nonoverlapping interval in [a,b], such that $|x'_i - x_i| < \delta$. Let k be the largest integer less than $1 + \frac{b-a}{\delta}$ and $a = y_0 < y_1 < \ldots < y_k = b$ be such that $y_j - y_{j-1} = \frac{b-a}{k}$ $\forall$ j and $P = \{a = x_0 < \ldots < x_n = b\}$ be any partition of [a,b]. Let $x_r, x_{r+1}, \ldots, x_{s-1}, x_s$ be those points that lie in (y_{i-1}, y_i). $y_{i-1} \leq x_r < x_{r+1} < \ldots x_s \leq y_i$ then $(y_{i-1}, x_r), (x_r, x_{r+1}) \ldots (x_{r-1}, y_i)$ are nonoverlapping and the sum of these intervals $= y_i - y_{i-1} < \delta$.

Hence $\delta_i = |f(x_i) - f(y_{i-1})| + |f(x_{r+1}) - f(x_r)| + \dots + |f(y_i) - f(s_{i-1}) < 1$.

As there are K such intervals, $\sum_{i=1}^{k} s_i < K$.

But $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{k} s_i < K$ Hence $t(P) < K \ \forall$ P.

This implies that f is of bounded variation on [a,b].

4 Theorem: If f is absolutely continuous on [a,b] and $f'(x) = 0$ a.e in [a,b] then f is constant.

Proof: We show that $f(c) = f(a) \ \forall c \in [a,b]$. Since $f'(x) = 0$ a.e. in [a,c] (where $a < c \leq b$) the set $F = \{x/a \leq x \leq c \text{ and } f'(x) \neq 0\}$ has measure zero so that $E = [a,c] \setminus F = \{x/a \leq x \leq c \text{ and } f'(x) = 0\}$ has measure c-a. Let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. Choose $\delta > 0$ corresponding to ε that satisfies the condition for absolute continuity.

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = f'(x) = 0 \text{ if } x \in E$$

$\Rightarrow \quad \exists\, h > 0 \ni$ is $|f(x+h) - f(x)| < h\,\eta$.

The collection [x,y] such that $|f(x) - f(y)| < \eta\,(y - x)$ is a vitalicover of E. By vitali's lemma there

is a finite subcollection $\{[x_1,y_1], \dots [x_n,y_n]\}$ such that $m^*\left(E \setminus \bigcup_{j=1}^{n} [x_j - y_j]\right) < \delta$

we may assume that $y_0 = a \leq x_1 < y_1 < x_2 < y_2 < \dots < x_n < y_n \leq c = x_n$.

$$c-a = x_1 - y_0 + x_2 - y_1 + \dots + x_{n+1} - y_n + \sum_{i=1}^{n} (y_i - x_i)$$

$$= \sum_{i=0}^{n} (x_{i+1} - y_i) + \sum_{i=0}^{n} (y_i - x_i)$$

$= m^*(E \setminus \bigcup_{i=1}^{n} (x_i, y_i) < \delta$

By the definition of absolute continuity $\sum_{i=0}^{n} |f(x_{i+1}) - f(y_i)| < \varepsilon$. Moreover $\sum_{i=1}^{n} |f(y_i) - f(x_i)|$

$< \sum_{i=1}^{n} \eta\,(y_i - x_i) < \eta\,(c-a)$. Hence $|f(c) - f(a)| \le \sum_{i=0}^{n} |f(y_{i+1}) - f(x_i)| + \sum_{i=1}^{n} |f(y_i) - f(x_i)|$

$< \varepsilon + \eta\,(b-a)$

Since this is true and $\forall\ \varepsilon > 0$ and $\eta > 0$ it follows that $|f(c) - f(a)| = 0$ so that $f(c) = f(a)$. Since this is true $\forall$ c in [a,b] it follows that $f(c) = f(a)$, $\forall$ c in [a,b]

5 Theorem: f [a,b] $\rightarrow$ R is absolutely continuous if and only if there is a F: [a,b] $\rightarrow$ R such that

$\int_a^x F(t)dt = f(x)$ a.e. in [a,b]

Proof: Assume that f is absolutely continuous. Then f is of bounded variation.
Hence $\exists$ monotonically increasing functions f_1 and f_2 such that $f(x) = f_1(x) - f_2(x)\ \forall x$.
Since f_1 and f_2 are differentiable a.e., f is differentiable a.e. in [a,b] and $f'(x) = f'_1(x) - f'_2(x)$ whenever r.h.s. exists. Hence f' is measurable and $|f'(x)| < f'_1(x) + f'_2(x)$

$\Rightarrow \int_a^b |f'(x)|\,dx \le \int_a^b f_1'(x)\,dx + \int_a^b f_2'(x)\,dx \le f_1(b) - f_1(a) + f_2(b) - f_2(a)$ Let $g(x) = \int_a^x f(t)dt$. g is absolutely continuous in [a,b]. Since f and g are absolutely continuous, f-g is absolutely continuous Hence $(f-g)'(x) = f'(x) - g'(x) = f'(x) - f'(x) = 0$ a.e. By (Th: 4) f - g is a constant function.

$\Rightarrow \quad f(x) = f(a) + g(x) = f(a) + \int_a^x f(t)dt.$

Conversely assume that $f(x) = \int_a^x F(t)dt$. Then given $\varepsilon > 0\ \exists$ a $\delta > 0 \ni \int_A F(t)dt < \varepsilon$ whenever $m(A) < \delta$.

In particular for any collection of nonoverlapping intering intervals $(x_1, y_1) \ldots (x_n, y_n) \ni$

$$\sum_{i=1}^{n} (y_i - x_i) < \delta$$

$$\int_{U(x_i, y_i)} F(t)\,dt < \varepsilon$$

$$\Rightarrow \sum_{i=1}^{n} \int_{x_i}^{y_i'} F(t)dt < \varepsilon$$

$$\Rightarrow \sum_{i=1}^{n} | f(y_i) - f(x_i) | < \varepsilon$$ Hence f is absolutely continuous.

6. Corollary: If f is absolutely continuous [a,b] f is the indefinite integral of its derivative.

$$f(x) = \int_a^x f'(t)dt.$$

Proof: Since f is absolutely continuous on [a,b], $\exists$ a F : [a,b] $\rightarrow$ R $\ni$

$$f(x) = \int_a^x F(t)dt.$$

$$\Rightarrow \quad f'(x) = F(x) \text{ a.e.}$$

$$\Rightarrow \quad f(x) = \int_a^x f'(t)dt.$$

7. Lemma : If f is integrable on [a,b] and $\int_a^x f = 0 \ \forall \ x \in [a,b]$ then f(x) = 0 a.e. in [a,b]

Proof: Let A = {x / f(x) > 0, x $\in$ [a,b]} and B = {x/f(x) < 0, x $\in$ [a,b]}. Suppose m (A) > 0

Then $\exists$ a closed set F $\subseteq$ A $\ni$ m (F) > 0. The set V = (a,b) \ F is then open. Since $0 = \int_a^b f =$

$\int_V f + \int_A f$, $\int_V f = -\int_A f$ since m (A) > 0 & f (x) > 0 on A. $\int_A f > 0$ so $\int_V f \neq 0$. Since V is open, there is a sequence (a_n,b_n) of pairwise disjoint open intervals such that $V = \bigcup_{n=1}^{\infty} (a_n,b_n)$ so that

$$\int_V f = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f$$

$$0 \neq \int_V f = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f \Rightarrow \int_{a_n}^{b_n} f \neq 0. \text{ for some n}$$

$$\Rightarrow \int_a^{a_n} f + \int_a^{b_n} f \neq 0 \text{ for some n}$$

$$\Rightarrow \int_a^{a_n} f \neq 0 \text{ or } \int_a^{b_n} f \neq 0.$$ But this contradicts the hypothesis. This Proves the lemma.

Lemma: If f is of bounded and measurable on [a,b] and $F(x) = \int_a^x f(t)\,dt + F(a)$ then F is differentiable a.e. and $f'(x) = f(x)$ a.e. on [a,b].

Proof: Let $P = \{a = x_0 < \ldots\ldots < x_n = b\}$ be any partition of [a,b].

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} dt\right| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|dt = \int_a^b |f(t)|dt$$. (remember that $|f|$ is integrable!). This being true ∀ partition of [a,b], it follows that F is of bounded variation. Hence F is of bounded variation on [a,b]. Since F is the difference of two monotonically increasing functions by Lebesgue's Theorem F is differentiable a.e in [a,b]

For each positive integer let $f_n(x) = \dfrac{F(x+1/n)-F(x)}{1/n}$

clearly $f_n(x) = n \displaystyle\int_x^{x+1/n} f(t)dt$

8. Since f is bounded, $\exists\ K > 0 \ni |f(x)| \leq K\ \forall\ x \in [a,b]$

Hence $|f_n(x)| \leq n \int_x^{x+1/n} f(t)dt \leq K$. Clearly $f'(x) = \lim_n f_n(x)$. By the bounded convergence theorem it follows that $\forall c \in [a,b]$.

$$\int_a^c f'(x)dx = \lim_n \int_a^c f_n(x)dx = \lim_n \int_a^c \frac{F(x+1/n)-F(x)}{1/n}dx$$

$$= \lim_n n\left\{\int_a^c F(x+1/n)\,dx \int_a^c F(x)\,dx\right\}$$

$$= \lim_n n\left\{\int_{a+1/n}^{c+1/n} F(t)dt - \int_a^c F(t)dt\right\}$$

$$= \lim_n n\left\{\int_a^{c+1/n} F(t)dt - \int_a^{a+1/n} F(t)dt\right\}$$

$$= \lim_n \{f_n(c) - f_n(a)\}$$

$$= F(c) - F(a)$$

$= \int_a^c f(x)\,dx$ Hence $\int_a^c \{f'(x)-f(x)\}dx=0\ \forall C\in[a,b]$. Hence $f'(x) = \int_a^x |f(t)|dt + F(a)$ a.e.

9. Theorem: Let f be an integrable function on [a,b] and suppose that $F(x) = F(a) + \int_a^x |f(t)|dt$.

Then F is differentiable a.e. in [a,b] and $F'(x) = f(x)$ whenever lhs exists.

Proof: We may assume that $f(x) \geq 0$ for all x, because once we prove the result for nonnegative f, the general follows by applying this to each of f^+ and f^-.

Under this assumption write $f_n(x) = \min\{f(x) \text{ and } x\}$

i.e. $f_n(x) = \begin{cases} f(x) \text{ if } f(x) < n \\ n \text{ if } f(x) > n \end{cases}$

Each f_n is bounded, measurable, $o \le f_n(x) \le f(x)\ \forall\ x$ ..... (1)

and $f(x) = \lim_n f_n(x)$. Since $(f - f_n)(x) \ge 0 \forall x \in [a,b]$

$G_n(x) = \int_a^x (f - f_n)(t)dt$ is an increasing function of x. Hence G_n is differentiable a.e in [a,b]

$G'_n(x) \ge 0$ whenever the derivative exists.

Since f_n is bounded and measurable by (1) $F_n(x) = \int_a^x f_n(t)dt$ is differentiable a.e. and $F_n'(x) = f_n(x)$

whenever the derivative exists.

$$F_n(x) + G_n(x) = \int_a^x f(t)dt = F(x) - F(a)$$

$$\Rightarrow \quad F_n'(x) + G_n'(x) = f'(x)$$

$$\Rightarrow \quad f'(x) \ge F_n'(x) = f_n(x) \text{ a.e. in } [a,b]$$

$$\Rightarrow \quad f'(x) > \lim f_n(x) = f(x) \text{ a.e. in } [a, b]$$

$$\Rightarrow \quad \int_a^b f'(x) = F(b) - F(a) = \int_a^b f(x)$$

$$\Rightarrow \quad \int_a^b \{ f'(x) - f(x) \} = 0,$$

since $F'(x) - f(x) \ge 0$ a..e., if follows that $F'(x) - f(x) = 0$ a.e.,

$$\Rightarrow \quad F'(x) = f(x) \text{ a.e. in } [a,b]$$

10. Short Answer Questions with solutions:

Q.1 Let $f(x) = x \sin 1/x$ if $x \neq 0$ and $f(0) = 0$. Show that

(a) f is continuous in [0,1]
(b) f is absolutely continuous in $[\varepsilon, 1]$ $\forall \varepsilon \ni 0 < \varepsilon < 1$
(c) f is not absolutely continuous in [0,1]

Solution: $|f(x) - f(0)| = |x \sin 1/x| \leq |x| < \varepsilon$ if $|x| < \varepsilon$. This proves (a).

If $0 < \varepsilon < x \leq 1$ $|f'(x)| = |\sin 1/x - 1/x \cos 1/x| \leq 1 + i/\varepsilon = k$ (say).

If $\in < x < y \leq 1 \exists c \ni \left|\frac{f(x)-f(y)}{x-y}\right| \leq |f^1(c)| \leq k$ and $x < c < y$.

So that $|f(x) - f(y)| \leq k|x-y|$. Thus if $\eta > 0$. if $\{(x_i, y_i) | 1 \leq i \leq n\}$ is any finite collection of nonoverlapping intervals in $[\varepsilon, 1]$ such that $\sum_{i=1}^{n} (y_i - x_i) < \frac{1}{k} \sum_{i=1}^{n} |f(y_i) - f(x_i)| < k. \sum_{i=1}^{n} (y_i - x_i) < \eta$.
Thus f is absolutely continuous in $[\varepsilon, 1]$. This proves (b). To prove (c) we recall that an absolutely continuous functions of bounded variation which f is not of bounded variation.

Q.2 If $f: [a,b] \rightarrow R$ is absolutely continuous monotone $E \subset [a,b]$ and $m(E) = 0$ then $m(f(E)) = 0$

Proof: Assume that f is monotonically increasing in [a,b]. Since f is absolutely continuous, given $\varepsilon > 0$ there exists an $\delta > 0 \ni$ for every finite collection of nonoverlapping intervals

$\{(x_i, y_i)\} \ni \Sigma(y_i - x_i) < \delta$. $\sum_i |f(x_i) - f(y_i)| < \varepsilon$. Since $m(E) = 0$ $\exists$ a sequence of disjoint open intervals (c_k, d_k) in $[a,b] \ni E \subset \bigcup_{k=1}^{\infty} (c_k, d_k)$ and $\sum_{k=1}^{\infty} (d_k - c_k) < \delta$ since f is continuous on $[c_k, d_k]$

$\exists \alpha_k, \beta_k$ in $[c_k, d_k] \ni \forall$ x in $[c_k, d_k]$ $f(\alpha_k) \leq f(x) \leq f(\beta_k)$ so that $f([c_k, d_k]) = [f(\alpha_k), f(\beta_k)]$

Since $\sum_{k=1}^{\infty} (\beta_k - \alpha_k) \leq \sum_{k=1}^{\infty} (d_k - c_k) < \delta$, $\forall$ positive integer n, $\sum_{k=1}^{n} \beta_k - \alpha_k < \delta$ so that

$$\sum_{k=1}^{n} |f(\alpha_k) - f(\beta_k)| < \varepsilon.$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} |f(\alpha_k) - f(\beta_k)| < \varepsilon$$

Since $f(E) \leq \underset{k}{U} f([c_k, d_k]) = \underset{k}{U}[f(\alpha_k), f(\beta_k)]$

$$m(f(E)) < \sum_{k-1}^{\infty} |f(\beta_k) - f(\alpha_k)| < \varepsilon.$$

This is true $\forall\ \varepsilon > 0$. So $m(f(E)) = 0$

Q.3 If f is monotonically increasing function defined on [a,b] there is an absolutely continuous function f_1 and a "singular" function f_2 such that $f(x) = f_1(x) + f_2(x)$ for $x \in [a,b]$.

By definition a real valued function defined on [a,b] is singular if its derivative is zero almost every where.

Proof: Since f is monotone, f is differentiable a.e. Further $f_1(x) = \int_a^x |f(t)| dt$ is absolutely continuous.

Clearly $f_2 = f - f_1$ is singular

Q.4 A function $f : [a,b] \to R$ is said to satisfy a Lipschitz condition if there a $M > 0$ such that $|f(x) - f(y)| \leq M |x-y|$ for all x,y in [a,b].

If f satisfies Lipschitz condition on [a,b] then f is absolutely continuous

Proof: Let $\varepsilon > 0$ and $\{(x_i, y_i)\}$ $1 \leq i \leq n$ be any finite collection of nonoverlapping intervals such that $\sum_{i=1}^{n} (y_i - x_i) < \frac{\varepsilon}{M}$

$\Rightarrow \sum_{i=1}^{n} |f(x_i) - f(y_i)| \leq \sum_{i=1}^{n} M(y_i - x_i) < \varepsilon$. Hence f is absolutely continuous.

Short Answer Question with Solutions

Q.1 If f: [a,b] $\to$ R is absolutely continuous in [c,b] $\forall$ c in [a,b], continuous at a and is of bounded variation in [a,b] then f is absolutely continuous in [a,b].

Proof: Since f is of bouinded variation and continuous at a, the total variation function V_f is continuous at a. Hence given $\varepsilon > 0 \ \exists$ a $\delta > 0$ such that $|V_f(x) - V_f(a)| < \frac{\varepsilon}{2}$ if $a \le x \le a + \delta = c < b$.

Since f is absolutely continuous in [c,b] $\exists$ a $\delta^1 > 0$ for every finite collection of nonoverlapping intervals (x_j, y_j) in [c,b] $\ni \Sigma (y_j - x_j) < \delta^1$, $\Sigma \mid f(y_j) - f(x_j) \mid < \frac{\varepsilon}{2}$. Let $\{I_j\ 1 \le j \le n\}$ be any nonoverlapping finite collection of intervals in [a,b] with $\Sigma l(I_j) < \delta^1$ where $l(I_j)$ is the length of I_j.

Let $I_1, \ldots, I_r$ be the intervals in [a,c] and $I_{r+1}, \ldots, I_n$ be from [c,b]. Write $I_j = (\alpha_j, \beta_j)$ and assume that $a \le \alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \ldots \le \alpha_r \le \beta_r \le c$. Then

$$|f(\alpha_1) - f(a)| + |f(\beta_1) - f(\alpha_1)| + |f(\alpha_2) - f(\beta_1)| + \ldots + |f(c) - f(\beta_r)| \le V_f(c) < \frac{\varepsilon}{2}$$

Hence $\sum_{i=1}^{r} |f(\beta_r) - f(\alpha_r)| < \frac{\varepsilon}{2}$

Consequently $\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon$

This implies that f is absolutely continuous in [a,b]

11. Model Examination Questions

1) Define absolute continuity. Show that if f is absolutely continuous on [a,b] then f is a function of bounded variation on [a,b]

2) Show that if f is absolutely continuous on [a,b] and $f'(x) = 0$ a.e. on [a,b] then f is a constant function.

3) Show that if f is absolutely continuous on [a,b] then f(x) = $\int_a^b f'(x)dx$ a.e.

4) If f is integrable on [a,b] show that $\int_a f(t)dt$ is absolutely continuous.

12. Exercises

Let g be a monotone increasing absolutely continuous function on [a,b] and g(a) = c, g(b) = d.

1) Show that for any open set O $\subset$ [c,d] m (O) = $\int_{g^{-1}(O)} g^1(x)dt$

2) Let H = $\{x / g^1(x) \neq 0\}$. If E $\subset$ [c,d] has measure zero show that $g^{-1}(E) \cap H$ has measure zero.

3) If E $\subset$ [c,d] is measurable show that $g^{-1}(E) \cap H$ is measurable

4) Show that if f' (x) is bounded on [a,b] then f is absolutely continuous on [a,b].

5.a) If f and g are absolutely continuous and gof is defined on f show that gof is not necessarily absolutely continuous.
Hint : f(x) = $\sqrt{x}$ g(x) = x^2 Sin $\pi / 2x$

b) Show that if in addition the function g is monotonically increasing then fog is absolutely continuous.

6) Write f(x) = x^a sin 1/x for x $\neq$ 0 and f(0) = 0.
For what values of a is f an absolutely continuous function ?

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Lesson - 13

THE CLASSICAL BANACH SPACES - I

13.1 Introduction

In the earlier lessons we have dealt with the Lebesgue measure and integral on the real line $\mathbb{R}$. Several results including a number of convergence theorems for the integral have been proved. We use these results and study special classes of measurable functions namely L^p spaces where $1 \leq p \leq \infty$. These spaces are also known as the classical Banach spaces. The L^p spaces of functions defined on $[0, 1]$ and their analogues namely ℓ^p spaces for any exponent p such that $1 < p < \infty$ were introduced by F. Riesz in 1910 - 1913.

In this lesson we define "essentially bounded" functions and show that these functions can be split into disjoint equivalence classes which form a vector space. Likewise the class $\mathcal{L}^p$ of all measurable functions f on $[0, 1]$ such that $|f|^p$ is integrable can also be divided into disjoint equivalence classes which form a vector space.

We introduce the concept of norm on a vector space and show that the above vector spaces become normed vector spaces with appropriate norms. We denote these normed vector space by $L^p\ (1 \leq p \leq \infty)$.

In what follows all the functions under consideration are extended real valued measurable functions on $[0, 1]$.

13.2 Definition : We call a measurable function f on $[0, 1]$ essentially bounded if there is a real number $M > 0$ such that the set

$$E_M(f) = \{x / x \in [0, 1] \text{ and } |f(x)| > M\}$$

has measure zero. Any such M is called an essential bound of f. The infimum of the set of essential bounds of f is called the essential supremum of f and is denoted by $||f||_\infty$.

$$||f||_\infty = \inf\left\{ M/M > 0,\ m\left(\{x/|f(x)| > M\}\right) = 0\right\}.$$

We write $\mathscr{L}^{\infty} = \{ f / f$ is essentially bounded on [0, 1] }.

Let us recall that the definition of $f+g$ when f and g are real valued functions is given by $(f+g)(x) = f(x)+g(x) \forall\, x \in [0, 1]$. When f and g are extended real valued functions $f(x)+g(x)$ is not necessarily defined for all x. We fix any number α and define $(f+g)(x) = \alpha$ whenever $f(x)+g(x)$ is not defined. With this definition it is known that $f+g$ is measurable whenever f and g are measurable. In this lesson we fix this α to be zero so that $(f+g)(x) = 0$ whenever f and g are measurable functions and $f(x)+g(x)$ is not defined. With this in mind, we prove the following.

13.3 Proposition : Suppose $f \in \mathscr{L}^{\infty}$, $g \in \mathscr{L}^{\infty}$ $\exists\, a \in \mathbb{R}$. Then

(i) $|f(x)| \le ||f||_{\infty}$ a.e. in $[0, 1]$

(ii) $f+g \in \mathscr{L}^{\alpha}$ and $||f+g||_{\infty} \le ||f||_{\infty} + ||g||_{\infty}$ and

(iii) $a f \in \mathscr{L}^{\infty}$ and $||a\, f||_{\infty} = |a| \cdot ||f||_{\infty}$.

By the definition of essential supremum, for every positive integer n there exists a $M_n > 0$ such that $M_n < \|f\|_{\infty} + \frac{1}{n}$ and $m(E_n) = 0$ where $E_n = \{x \in [0,1] / |f(x)| > M_n\}$.

Since $E = \{x \in [0, 1] / |f(x)| > \|f\|_{\infty}\} = \bigcup_{n=1}^{\infty} E_n$, it follows that $0 \le m(E) \le \sum_{n=1}^{\infty} m(E_n) = 0$ so that $m(E) = 0$. This proves (i).

To prove (ii) let $E_f = \{x \in [0, 1] / |f(x)| > \|f\|_{\infty}\}$ and

$$E_g = \{x \in [0, 1] / |g(x)| > \|g\|_{\infty}\}.$$

By (i) $m(E_f) = m(E_g) = 0$.

If $x \notin E_f \cup E_g$, $|f(x)| \le \|f\|_{\infty}$ & $|g(x)| \le \|g\|_{\infty}$

So that $|(f+g)(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$.

Thus $E=\{x \in [0,1]/|(f+g)(x)| > \|f\|_\infty + \|g\|_\infty\} \subseteq E_f \cup E_g$ so that $m(E)=0$, hence $f+g \in \mathscr{L}^\infty$ and $\|f\|_\infty + \|g\|_\infty$ is an essential bound for $f+g$. So that $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. This completes the proof of (ii). we now prove (iii). If $a=0$, LHS = RHS = 0.

Assume that $a \neq 0$. Then $|(af)(x)| \ge M \Leftrightarrow |f(x)| \ge \frac{M}{a}$.

Hence M is an essential bound of af if and only if $\frac{M}{|a|}$ is an essential bound of f.

Then $\|af\|_\infty = \inf\{M/M \text{ is an essential bound of } af\}$

$= \inf\{|a|M' / M' \text{ is an essential bound of } f\}$

$=|a|\,\|f\|_\infty$. This completes the proof of (iii).

13.4 Corollary :

The space $\mathscr{L}^\infty$ is a vector space and the essential supremum satisfies the following properties :

(i) $\|f\|_\infty \ge 0$ for $f \in \mathscr{L}^\infty$ and $\|f\|_\infty = 0$ if and only if $f(x) = 0 \cdot$a.

(ii) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

(iii) $\|af\|_\infty = |a|\,\|f\|_\infty$.

Proof : In view of proposition 13.3 it is enough to prove (i). It is clear that $\|f\|_\infty = 0$.

If $\|f\|_\infty = 0$, for every positive integer n, there is a $M_n \ni 0 < M_n < \frac{1}{n}$ and $m(\{x|f(x)| > M_n\}) = 0$ so that

$$m\left(\left\{x/|f(x)| > \frac{1}{n}\right\}\right) = 0.$$

Write $E=\{x/f(x)\neq 0\}$ and $E_n = \left\{x/|f(x)|>\frac{1}{n}\right\}$.

Then $E = \bigcup_{n\geq 1} E_n$ so that $0\leq m(E)\leq \sum_n m(E_n) = 0$

Hence $m(E) = 0$, so that $f(x) = 0$ a.e.

Conversely suppose $f(x) = 0$ a.e.

Then $\forall \epsilon>0 \; m(\{x/|f(x)|>\epsilon\}) = 0$.

Hence every positive real number is an essential bound for f. Hence f is essentially bounded and

$$\|f\|_\infty = \inf\{\epsilon/\epsilon>0\} = 0.$$

13.5 Proposition : The relation ~ defined on $\mathscr{L}^\infty$ by $f \sim g$ iff

$f(x) = g(x)$ a.e. is an equivalence relation :

$N=\{f/f\in\mathscr{L}^\infty \ni f\sim 0\}$ is a linear subspace of $\mathscr{L}^\infty$ and $f\sim g \Rightarrow \|f\|_\infty = \|g\|_\infty$

Proof : That ~ is an equivalence relation is clear.

$f_1 \sim g_1,\ f_2\sim g_2 \Rightarrow f_1(x)=g_1(x)$ a.e. and $f_2(x) = g_2(x)$ a.e.

so $(f_1 + f_2)(x) = (g_1+g_2)(x)$ a.e. Hence $f_1+g_1 \sim f_2+g_2$

For $a\in\mathbb{R}$, $a\,f_1(x) = a\,g_1(x)$ a.e. so $af_1 \sim ag_1$

In particular $f\in N,\ g\in N \Rightarrow f\sim 0,\ g\sim 0 \Rightarrow f+g\sim 0$ and $\forall a\in\mathbb{R}\ (af)\sim 0$ so that $f+g\in N$ and $af\in N$. Hence N is a linear subspace of $\mathscr{L}^\infty$.

If $f\sim g$ then $f(x) = g(x)$ a.e. so that $(f-g)(x) = 0\ a\cdot e$ hence $\|f-g\|_\infty = 0$.

$$\Rightarrow \|f\|_\infty = \|f-g+g\|_\infty \leq \|f-g\|_\infty + \|g\|_\infty$$

$$\Rightarrow \|f\|_\infty - \|g\|_\infty \le \|f-g\|_\infty = 0 \Rightarrow \|f\|_\infty \le \|g\|_\infty$$

By symmetry $\|g\|_\infty \le \|f\|_\infty$. Hence $\|f\|_\infty = \|g\|_\infty$.

13.6 The Space $\mathscr{L}^\infty$

Consider the space $\mathscr{L}^\infty$. By 13.4 the mapping $f \to \|f\|_\infty$ has the following properties.

(i) $\|f\|_\infty \ge 0 \ \forall \ f \in \mathscr{L}^\infty, \|f\|_\infty = 0$ if and only if $f=0$ a.e.

(ii) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty \ \forall \ f \in \mathscr{L}^\infty$ and $g \in \mathscr{L}^\infty$ and

(iii) $\|a\ f\|_\infty = |a|\ \|f\|_\infty \ \forall \ f. \in \mathscr{L}^\infty$ and $a \in \mathbb{R}$.

From proposition 13.4 it follows that $N=\left\{f \in \mathscr{L}^\infty / \|f\|_\infty = 0\right\}$ is a linear subspace of $\mathscr{L}^\infty$ and the map

$\|f+N\| = \|f\|_\infty$ defines a norm on the quotient space $\mathscr{L}^\infty / N$.

We denote this quotient space by L^∞. Elements of L^∞ are equivalence classes of essentially bounded functions f formed by the equivalence relation defined in 13.5 by $f \sim g$ iff $f = g$ a.e.

Since $\|f+N\| = \|g\|_\infty \ \forall \ g \in f+N,$ we may identify the coset $f+N$ with any $g \in f+N$ keeping in mind that $\|f\|_\infty = \|g\|_\infty$ whenever $f \sim g$.

We thus treat L^∞ itself as the space of all essential bounded functions.

13.7 The Space $\mathscr{L}^p$ $(1 \le p < \infty)$

If $0<p<\infty$ $\mathscr{L}^p$ stands for the space of all measurable functions f on $[0, 1]$ such that $\int_0^1 |f|^p < \infty$.

$$\mathscr{L}^\infty = \left\{ f:[0,1] \to \mathbb{R} \cup \{\pm\infty\} \Big/ f \text{ is measurable and } \int_0^1 |f|^p < \infty \right\}$$

We write $\|f\|_p = \left\{ \int_0^1 |f|^p \right\}^{1/p}$ for $f \in \mathscr{L}^p$.

If f and g belong to $\mathscr{L}^p$, the sets

$$A = \{x / f(x) = \pm\infty\} \text{ and } B = \{x / g(x) = \pm\infty\}$$

have measure zero so that the set of x for whcih $f(x)+g(x)$ is not defined, being a subset of $A \cup B$ has measure zero. We define

$$(f+g)(x) = \begin{cases} f(x)+g(x) & \text{if this is not of the form } \infty - \infty \text{ or } -\infty + \infty \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f+g$ is measurable on $[0, 1]$. With this definition for $f + g$ we have the following.

13.8 Proposition : If $0 < p < \infty$ $\mathscr{L}^p$ is a vector space over $\mathbb{R}$.

Proof : It is enough if we show that

(i) $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^p \Rightarrow f+g \in \mathscr{L}^p$ and

(ii) $f \in \mathscr{L}^p$ and $a \in \mathbb{R} \Rightarrow af \in \mathscr{L}^p$

because the other conditions can be verified in a routine way.
Since (ii) is clear, we verify (i) only.

For $0 \le x \le 1$, $|(f+g)(x)| \le |f(x)| + |g(x)|$

$$\le 2\max\{|f(x)|, |g(x)|\}$$

$$\Rightarrow |(f+g)(x)|^p \le 2^p \max\{|f(x)|^p, |g(x)|^p\}$$

$$\le 2^p \{|f(x)|^p + |g(x)|^p\}$$

$$\Rightarrow \int_0^1 |(f+g)(x)|^p \le 2^p \left\{ \int_0^1 |f(x)|^p + \int_0^1 |g(x)|^p \right\}$$

$\Rightarrow f+g \in \mathscr{L}^p$. This proves (i)

13.9 Proposition

Let α, β be nonnegative real numbers and suppose $0<\lambda<1$. Then

$$\alpha^{\lambda}\,\beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\beta$$

with equality if and only if $\alpha=\beta$.

Proof : If $\alpha\beta = 0$ LHS $=0\leq\lambda\alpha = \lambda\alpha + (1-\lambda)\beta =$ RHS. Assume that $\alpha\beta\neq 0$.

Now define $\phi:[0, \infty) \to \mathbb{R}$ by

$$\phi(t) = \left(1-t^{\lambda}\right) - \lambda(1-t).$$

ϕ is differentiable and $\phi'(t) = \lambda\left(1-t^{\lambda-1}\right)$

Since $\lambda<1,\ 0<t<1 \Rightarrow t^{\lambda-1} = \left(\frac{1}{t}\right)^{1-\lambda} > 1$ so that $\phi'(t)<0$

and if $t>1$, $t^{\lambda-1}<1$ so that $\phi'(t)>0$.

So $\phi'(t)$ is increasing in $(1, \infty)$ and decreasing in $(0, 1)$. By the continuity of ϕ at 1 it follows that

$$\phi(t)<\phi(1)=0 \text{ for all } t\neq 1.$$

Hence $(1-\lambda)+\lambda t \geq t^{\lambda}$ with of equality if and only if $t=1$.

If $\beta\neq 0$ put $t=\frac{\alpha}{\beta}$. Then

$$(1-\lambda)+\lambda\frac{\alpha}{\beta} \geq \left(\frac{\alpha}{\beta}\right)^{\lambda} \Rightarrow \alpha^{\lambda}\,\beta^{1-\lambda} \leq (\lambda\alpha) + (1-\lambda)\beta$$

Equality occurs if and only if $\alpha\beta = 0$ or $\alpha=\beta\neq 0$

13.10 Holder's inequality

Let $1\leq p<\infty$:

Write $q = \begin{cases} \infty & \text{if } p = 1 \\ \text{and} & \\ \dfrac{p}{p-1} & \text{if } p \neq 1 \end{cases}$

Clearly $\frac{1}{p} + \frac{1}{q} = 1$ if $1 < p < \infty$.

If $f \in \mathscr{L}^p$, $g \in \mathscr{L}^q$ then $f\,g \in \mathscr{L}^1$ and $\int_0^1 |f\,g| \le \|f\|_p \, \|g\|_q$.

Proof : If $p = 1$, $q = \infty$ so $|f(x)| \le \|f\|_\infty$ a.e.

Since $f \in \mathscr{L}^1$, and $|f\,g|(x) \le |f(x)| \, \|g\|_\infty$ a.e., $f\,g \in \mathscr{L}^1$ and

$$\int_0^1 |f\,g| \le \int_0^1 |f| \|g\|_\infty = \|g\|_\infty \, \|f\|_1 .$$

Now let $1 < p < \infty$. Then $\frac{1}{p} + \frac{1}{q} = 1$. So that $1 < q < \infty$.

First assume that $\|f\|_p = \|g\|_q = 1$.

For $0 \le t \le 1$, $\left(|f(t)|^p\right)^{\frac{1}{p}} \left(|g(t)|^q\right)^{1-\frac{1}{p}} \le \frac{1}{p}|f(t)|^p + \left(1 - \frac{1}{p}\right)|g(t)|^q$

$$\Rightarrow |f(t)|\,|g(t)| \le \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$$

$$\Rightarrow \int_0^1 |f\,g| \le \frac{1}{p}\int_0^1 |f(t)|^p + \frac{1}{q}\int_0^1 |g(t)|^q$$

$$= \frac{1}{p}\left(\|f\|_p\right)^p + \frac{1}{q}\left(\|g\|_q\right)^q.$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

In the general case if $\|f\|_p = 0,\ \int_0^1 |f(t)|^p = 0$ so that

$f(t)=0$ a.e., hence LHS = 0 = RHS.

Similarly if $\|g\|_q = 0$ LHS = 0 = RHS

Now suppose that $\|f\|_p \neq 0 \neq \|g\|_q$. Then $\left\| \frac{f}{\|f_p\|} \right\|_p = \left\| \frac{g}{\|g_q\|} \right\|_q = 1$

Hence $\int_0^1 \left| \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} \right| \leq 1$.

$$\Rightarrow \int_0^1 |f\,g| \leq \|f\|_p \, \|g\|_q .$$

That $f\,g \in L'$ is clear from this inequality.

13.11 Equality in Holder's inequality

We prove that if $1<p<\infty$ then $\int_0^1 |f\,g| = \|f\|_p \, \|g\|_q$ if and only if there exist real numbers α, β such that $\alpha |f(t)|^p = \beta |g(t)|^q$ a.e.

Proof : $\int_0^1 |f\,g| = \|f\|_p \, \|g\|_q = 0$

$\Leftrightarrow \|f\|_p = 0$ or $\|g\|_q = 0$

$\|f\|_p = 0 \Leftrightarrow f = 0$ a.e.

$$\Leftrightarrow |f(t)|^p = 0 \, |g(t)|^q \text{ a.e.}$$

Similarly when $\|g\|_q = 0$, $\int_0^1 |f(t)\, g(t)| = 0 \Leftrightarrow |g(t)|^q = 0 |f(t)|^p$ a.e.

Now assume that $\|f\|_p \, \|g\|_q \neq 0$. Further consider the case $\|f\|_p = \|g\|_q = 1$.

In this case $\int_0^1 |f\ g| = 1 \Leftrightarrow \int_0^1 \left((1 - |f\, g|)\right) = 0$

Assume that $1 < p < \infty$

$$\Leftrightarrow \int_0^1 \left\{ \frac{1}{p} f + \left(1 - \frac{1}{p}\right) g - |f(t)\, g(t)| \right\} = 0$$

since $|f(t)\, g(t)| \leq \frac{1}{p} |f(t)|^p + \frac{1}{q} |g(t)|^q \ \forall\, t \in [0, 1]$

the above equality occurs if and only if

$$|f(t)\, g(t)| = \frac{1}{p} |f(t)|^p + \frac{1}{q} |g(t)|^q \text{ a.e.}$$

This holds if and only if $|f(t)|^q = |g(t)|^p$ a.e.

In the general case we replace f by $\dfrac{f}{\|f\|_p}$ and g by $\dfrac{g}{\|g\|_q}$

so that in this case equality occurs if and only if

$$\frac{|f(t)|^p}{\left(\|f\|_p\right)^p} = \frac{|g(t)|^q}{\left(\|g\|_q\right)^q} \quad \text{a.e.}$$

if and only if $\|g\|_q^q \, |f(t)|^p = \|f\|_p^p \, |g(t)|^q$ a.e.

13.12 Minkowski's inequality :

If $1 \le p \le \infty$ $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^p$ then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

Proof : If $p=1$, $\|f+g\|_1 = \int_0^1 |(f+g)(t)| \le \int_0^1 |f(t)| + \int_0^1 |g(t)| = \|f\|_1 + \|g\|_1$

If $p=\infty$, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$ from 13.4

Now assume that $1<p<\infty$. If $\|f+g\|_p = 0$ clearly LHS $\le$ RHS

Assume that $\|f+g\|_p \ne 0$. For $0 \le t \le 1$

$$|(f+g)(t)|^p = \left(|f(t)+g(t)|\right)^{p-1} |f(t)+g(t)|$$

$$\le \left(|f(t)+g(t)|\right)^{p-1} \left(|f(t)|+|g(t)|\right)$$

$$= \left(|f(t)+g(t)|\right)^{p-1} |f(t)| + |f(t)+g(t)|^{p-1} |g(t)|$$

so that $\int_0^1 |(f+g)(t)|^p \le \int_0^1 |(f+g)(t)|^{p-1} |f(t)| + \int_0^1 |(f+g)(t)|^{p-1} |g(t)|$

We apply Holder's inequality to the integrals on the RHS.

$$\int_0^1 |(f+g)(t)|^{p-1} |f(t)| \le \|f\|_p \left\|(f+g)^{p-1}\right\|_q$$

and $\int_0^1 |(f+g)(t)|^{p-1} |g(t)| \le \|g\|_p \left\|(f+g)^{p-1}\right\|_q$

$$\left\|(f+g)^{p-1}\right\|_q = \left\{\int_0^1 \left(|f(t)+g(t)|^{p-1}\right)^q\right\}^{\frac{1}{q}}$$

$$= \left\{\int_0^1 |f(t)+g(t)|^{(p-1)q}\right\}^{\frac{1}{q}}$$

$$= \left\{\int_0^1 |f(t)+g(t)|^p\right\}^{\frac{1}{q}} = \left(\|f+g\|_p\right)^{\frac{p}{q}}$$

Hence $\left\{\|f+g\|_p\right\}^p \le \left(\|f\|_p + \|g\|_p\right)\left\{\|f+g\|_p\right\}^{p/q}$

Since $\|f+g\|_p \ne 0$, $\|f+g\|_p^{p\left(1-\frac{1}{q}\right)} \le \|f\|_p + \|g\|_p$

Since $1-\frac{1}{q} = \frac{1}{p}$ we now get $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

13.13. A Second Proof of Minkowski's Inequality :

The following proof of Minkowski's inequality does not make use of Holder's inequality. However we require the notion of a convex function.

13.13.1 Definition : A function $\phi:[a, b]\to\mathbb{R}$ is said to be convex if

$$\phi(\lambda x + (1-\lambda)y) \le \lambda\phi(x)+(1-\lambda)\phi(y)$$

for all x, y in $[a, b]$ and $0\le\lambda\le 1$.

Proposition : If ϕ is twice differentiable in $[a,b]$ and $\phi''(x)>0$ in $[a, b]$ then $\phi(x)$ is "strictly convex" i.e.

$$\phi(\lambda x+(1-\lambda)y)<\lambda\phi(x)+(1-\lambda)\phi(y)$$

when $x, y \in [a, b]$ and $0<\lambda<1$

Consequence : If $1<p<\infty$ x^p is strictly convex in $[0, 1]$.

We now prove Minkowski's inequality : We consider the case $1<p<\infty$.

Assume that $f \in \mathscr{L}^p$, $g \in \mathscr{L}^p$.

If $\|f\|_p = 0$ or $\|g\|_p = 0$ then the corresponding function is zero almost everywhere and hence equality occurs in Minkowski's inequality.

Now assume that $a=\|f\|_p \neq 0 \neq \|g\|_p = b$. Write $f_0 = \frac{f}{a}$ and $g_0 = \frac{g}{b}$. Then $f_0 \in \mathscr{L}^p$, $g_0 \in \mathscr{L}^p$ and $\|f_0\|_p = \|g_0\|_p = 1$.

For $x \in [0, 1]$.

$$|(f+g)(x)|^p \leq (|f(x)|+|g(x)|)^p$$

$$=(a|f_0(x)|+b|g_0(x)|)^p$$

$$=(a+b)^p\left(\frac{a|f_0(x)|}{a+b}+\frac{b|g_0(x)|}{a+b}\right)^p$$

$$\leq(a+b)^p\left(\frac{a}{a+b}|f_0(x)|+\frac{b}{a+b}|g_0(x)|^p\right)\left(\because \frac{a}{a+b}+\frac{b}{a+b}=1\right)$$

(and x^p is convex)

Hence $$\|f+g\|_p^p \leq (a+b)^p\left\{\frac{a}{a+b}\cdot\int_0^1|f_0(x)|^p+\frac{b}{a+b}\cdot\int_0^1|g_0(x)|^p\right\}$$

$$=(a+b)^p\left\{\frac{a}{a+b}(\|f_0\|_p)^p+\frac{b}{a+b}(\|g_0\|_p)^p\right\}$$

$$=(a+b)^p \text{ since } \|f_0\|_p = \|g_0\|_p = 1$$

This completes proof of Minkowski's inequality.

13.13.2 Proposition : If $1 \le p \le \infty$ $f \to \|f\|_p$ defined for $f \in \mathscr{L}^p$ satisfies the following properties.

(i) $\|f\|_p \ge 0 \ \forall f \in \mathscr{L}^p$ and $\|f\|_p = 0$ if and only if $f(x)=0$ a.e.

(ii) $\|f+g\|_p \le \|f\|_p + \|g\|_p \ \forall f \in \mathscr{L}^p$ and $g \in \mathscr{L}^p$.

(iii) $\|af\|_p = |a| \, \|f\|_p \ \forall f \in \mathscr{L}^p \ \& \ a \in \mathbb{R}$

Proof : (ii) is Minkowski's inequality. The first part of (i) and (iii) are clear. Moreover

$$\|f\|_p = 0 \Leftrightarrow \int_0^1 |f(t)|^p = 0$$

$$\Leftrightarrow |f(t)|^p = 0 \text{ a.e.}$$

$$\Leftrightarrow f(t) = 0 \text{ a.e.}$$

This completes the proof of the proposition.

13.13.3 Proposition : The relation ~ defined on $\mathscr{L}^p$ by $f \sim g$ iff $f(x)=g(x)$ a.e. is an equivalence relation.

$$N = \{f / f \in \mathscr{L}^p \text{ and } f \sim 0\}$$

is a linear subspace of $\mathscr{L}^p$. Further $f \sim g \Rightarrow \|f\|_p = \|g\|_p$

Proof : As in 13.5.

At this stage we recall the definition of norm on a vector space over the field of real numbers $\mathbb{R}$. A norm on a vector space X is a real valued function assigning to each x in X, $\|x\|$ called norm x satisfies.

(i) $\|x\| \ge 0$ for every x in X with equality if and only if $x=0$

(ii) $\|x+y\| \le \|x\| + \|y\| \ \forall\ x, y$ in X and

(iii) $\|\alpha x\| = |\alpha|\ \|x\|\ \forall\ x$ in X and $\alpha \in \mathbb{R}$.

We now state the following proposition in the form of SAQ.

13.13.4 SAQ 1 : Let X be a linear (= vector) space over $\mathbb{R}$. Suppose $\phi : X \to \mathbb{R}$ satisfies the following conditions.

(i) $\phi(x) \ge 0$ for every $x \in X$

(ii) $\phi(x+y) \le \phi(x) + \phi(y)$ for every x, y in X and

(iii) $\phi(ax) = |a|\,\phi(x)\ \forall\ x$ in X and $a \in \mathbb{R}$.

Then $N = \{x \in X / \phi(x) = 0\}$ is a linear subspace of X, $\phi(x) = \phi(y)$ iff $x - y \in N$ and

$\|x + N\| = \phi(x)$ is a well defined function on the quotient space X/N which is a norm.

13.14 The Space $L^p\ (1 \le p < \infty)$

Consider the space $\mathcal{L}^p$. By $f \to \|f\|_p$ has the following properties.

$\|f\|_p \ge 0\ \forall\ f \in \mathcal{L}^p$ with equality if and only $f(x) = 0$ a.e.

$\|f+g\|_p = \|f\|_p + \|g\|_p\ \forall\ f \in \mathcal{L}^p$ and $g \in \mathcal{L}^p$ and

$\|a f\|_p = |a|\ \|f\|_p\ \forall\ f \in \mathcal{L}^p$ and $a \in \mathbb{R}$.

From proposition 13.13.3 it follows that $N = \left\{ f \in \mathcal{L}^p / \|f\|_p = 0 \right\}$ is a linear subspace of $\mathcal{L}^p$ and the map

$$\|f + N\| = \|f\|_p \text{ defines a norm on the quotient space } \mathcal{L}^p/N$$

We denote this quotient space by L^p. Elements of L^p are equivalence classes of functions

f such that $\int_0^1 |f(t)|^p < \infty$ formed by the equivalence relation defined in 13.13.3 by $f \sim g$ iff $f(x) = g(x)$ a.e.

Since $\|f + N\| = \|g\|_p \; \forall \; g \in f + N$, we may identify the coset $f + N$ with any $g \in f + N$, keeping in mind that $\|f\|_p = \|g\|_p$ whenever $f \sim g$.

We thus treat L^p itself as the space of all measurable functions f such that $\int_0^1 |f(t)|^p < \infty$.

13.15 The Sequence spaces $\ell^p \; (1 \le p \le \infty)$:

In analogy to the classical Banach spaces L^p we have the sequence spaces $\ell^p \; (1 \le p \le \infty)$. The spaace ℓ^∞ consists precisely of all bounded sequences and the space $\ell^p \; (1 \le p < \infty)$ consists of all those sequences $\{x_n\}$ of real numbers which satisfy.

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

We define for $x = \{x_n\}$

$$\|x\|_\infty = \sup_n |x_n| \text{ and}$$

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

Proofs of the following results are available in the study material for "paper I Topology and functional analysis - Functional analysis lesson 3".

Result 1 : ℓ^p is a vector space for $1 \le p \le \infty$

2 : **Holders inequality** : for any sequences $\{x_n\}$ in ℓ^p and $\{y_n\}$ in ℓ^q when $1 < p < \infty$

and $\frac{1}{p}+\frac{1}{q}=1$ or $p=1$ and $q=\infty$

$$\sum_{n=1}^{\infty}|x_n y_n| \leq \|x\|_p \ \|y\|_q$$

Result 3 : **Minkowski's inequality :** For any sequence $\{x_n\}$ and $\{y_n\}$ in ℓ^p $(1 \leq p \leq \infty)$

$$\|\{x_n + y_n\}\|_p \leq \|\{x_n\}\|_p + \|\{y_n\}\|_p .$$

13.16 : Let X be a linear space over $\mathbb{R}$. Suppose $\phi : X \to \mathbb{R}$ satisfies the following conditions

(i) $\phi(x) \geq 0$ for every $x \in X$, $\phi(0)=0$

(ii) $\phi(x+y) \leq \phi(x)+\phi(y)$ $\forall$ x, y in X and

(iii) $\phi(ax) = |a| \ \phi(x)$ $\forall$ $x \in X$ and $a \in \mathbb{R}$.

Then $N = \{x / x \in X \text{ and } \phi(x)=0\}$ is a linear subspace of X.

$\phi(x) = \phi(y)$ iff $x - y \in N$

and $\|x+N\| = \phi(x)$ is a well defined function on the quotient space X/N which is a norm.

Proof :(i) N is a linear subspace of X : Clearly $\phi(0)=0$ so $0 \in N$.

$$x \in N,\ y \in N \Rightarrow \phi(x) = \phi(y) = 0$$

$$\Rightarrow 0 \leq \phi(x+y) \leq \phi(x)+\phi(y) = 0$$

$$\Rightarrow x+y \in N .$$

This shows that N is a linear subspace of X.

(ii) $x+N \to \phi(x)$ is well defined on X/N

$$x+N = y+N \Rightarrow x-y \in N \Rightarrow \phi(x-y) = 0$$

since $\phi(x) = \phi(x-y+y) \le \phi(x-y)+\phi(y)$,

$$\therefore \phi(x)-\phi(y) \le \phi(x-y).$$

Interchanging x and y, $\phi(y) - \phi(x) \le \phi(y-x) = \phi(x-y)$

so that $|\phi(x)-\phi(y)| \le \phi(x-y)=0$

Hence $0 \le |\phi(x)-\phi(y)| \le 0$ so that $\phi(x)=\phi(y)$.

(iii) If we define $\|x+N\| = \phi(x)$ for $x \in X$, $\| \quad \|$ satisfies the properties of a norm on the quotient space X/N

Clearly $\|x+N\|$ is non - negative. If $x+N=N$, $x \in N$ so that $\|x+N\| = \phi(x)=0$: If $\phi(x)=0$, $x \in N$ so $\|x+N\| = \phi(x) = 0$.

Thus $\|x+N\| \ge 0 \ \forall \ x \in X$ and $\|x+N\| = 0 \Leftrightarrow x+N = N$.

Since $\phi(x+y) \le \phi(x)+\phi(y)$

$$\|(x+N)+(y+N)\| = \|(x+y)+N\|$$

$$= \phi(x+y) \le \phi(x)+\phi(y)$$

$$= \|x+N\| + \|y+N\|$$

Finally $\|a(x+N)\| = \|ax+N\|$

$$= \phi(ax) = |a|\phi(x)$$

$$= |a| \, \|x+N\|$$

Thus $x+N \to \|x+N\|$ defines a norm on the quotient space X/N.

13.17 SAQ 2 : Suppose f is a bounded measurable function $[0, 1]$.

Then $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$

Solution : Write $M = \|f\|_\infty$ and for $0 < \in < M$.

$$E_\in = \{x \in [0, 1] / M - \in \le f(x) \le M\}$$

Then by the definition of M, $m(E_\in) > 0$.

Also $\|f\|_p^p = \int\limits_0^1 |f(x)|^p$

$$\ge \int\limits_{E_\in} |f(x)|^p$$

$$\ge (M - \in)\, m(E_\in)$$

so that $(M - \in)\, \{m(E_\in)\}^{\frac{1}{p}} \le \|f\|_p$

This implies $M - \in \le \lim\limits_{p\to\infty} \|f\|_p$.

It is clear that $\|f\|_p \le M \;\forall\; p \ge 1$

It now follows that $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$

13.18 SAQ 3 : Suppose $\{f_n\}$ is a sequence of elements in L^p such that $\{f_n\}$ converges to f where $f \in L^p$. Then $\{f_n\}$ converges to f in L^p i.e. $\|f_n - f\|_p \to 0$ as $n \to \infty$, if and only if $\|f_n\|_p \to \|f\|_p$.

Solution : We have $\left| \|f_n\|_p - \|f\|_p \right| \le \|f_n - f\|_p$.

This implies that $\|f_n\|_p \to \|f\|_p$ if $\|f_n - f\|_p \to 0$

Conversely suppose $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$

Clearly $|f_n \to f| \le |f_n| + |f|$

set $g_n = 2\left(|f_n|^p + |f|^p\right)$.

Then $|f_n - f|^p \le 2\left(|f_n|^p + |f|^p\right) = g_n$

Since $f_n \to f$ we get $\|f_n - f\|_p \to 0$

13.19 Model Examination Questions

1. Show that L^p is a vector space if $1 \le p < \infty$

2. Show that if f is measurable, $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$

3. Show that if $f \in L^1$ and $g \in L^\infty$ then

$$\int |f\, g| \le \|f\|_1 \; \|g\|_\infty$$

4. Show that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ if $1 \le p \le \infty$ and $f \in L^p$ and $g \in L^p$.

13.20 Exercises

1. Show that if f is bounded on $[0, 1]$ and measurable, f is essentially bounded and $\|f\|_\infty = \sup\{|f(x)| / 0 \le x \le 1\}$.

2. Let $f(x) = 0$ if x is irrational

   $= n$ if $x = x_n$ where $\{x_n\}$ is a sequence which is an enumeration of the set Q of rationals.

   Show that f is measurable, essentially bounded but not bounded.

3. Show that if $f \in \mathscr{L}^p$ $(1 \ge p < \infty)$ then $f \in \mathscr{L}^\infty$.

4. Define $f(x)=\left(\frac{1}{n}\right)^{\frac{1}{p}}$ if $\frac{1}{n+1}<x\leq\frac{1}{n}$ (n positive integer) and

$f(0)=0$.

Show that f is bounded, measurable and $f\notin\mathscr{L}^p$ where $1\leq p<\infty$

5. Show that if $f''(x)>0$ then $f(x)$ is strictly convex in $[a, b]$.

6. Show that Minkowski does not hold good when $0<p<1$.

7. Show that $\mathscr{L}^p\leq\mathscr{L}^q$ is $p>q\geq 1$

If f is measurable on $[0, 1]$ show that $\|f\|_p\leq\|f\|_q$ if $1\leq p\leq q$.

8. If $1\leq p\leq\infty$ show by an example that $\|f\|_p=\|g\|_q$ does not imply $f\sim g$

9. Prove proposition 13.13.3

13.21 REFERENCE BOOK

Real Analysis - Royden

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Lesson - 14

THE CLASSICAL BANACH SPACES - II

14.1 INTRODUCTION

In this lesson we continue the study of L^p spaces. We prove that for $1 \le p \le \infty$, L^p is a Banach space. This is Riesz - Fischer theorem. We then obtain a one - one correspondence between the bounded linear functions on $L^p\,(1 \le p < \infty)$ and the elements of L^q where p and q are "conjugate pairs" i.e. $\frac{1}{p}+\frac{1}{q}=1$ if $1<p<\infty$ and $q=\infty$ when $p=1$. In fact this one - one correspondence is linear norm preserving and onto. This is the famous Riesz - Representation theorem for L^p spaces. The analogues for ℓ^p spaces introduced in lesson 13 are also valid.

We begin with some fundamentals of normed linear spaces which are essential to establish Riesz - Fisher theorem and Riesz - Representation theorem.

Definitions : Let X be a normed linear space. A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if for every $\in > 0$ there corresponds a positive integer $N_\in$ such that $\|x_n - x_m\| < \in$ whenever $n \ge N_\in$ and $m \ge N_\in$; (equivalently $n > m \ge N_\in$).

$\{x_n\}$ is said to be convergent in X if there is a $x \in X$ such that $\lim\limits_{n}\|x_n - x\| = 0$. i.e. for every positive number $\in$ there corresponds a positive integer $N_\in$ such that

$$\|x_n - x\| < \in \text{ whenever } n \ge N_\in$$

In this case x is uniquely fixed and is called the limit of $\{x_n\}$ and is denoted by $\lim\limits_{n} x_n$.

X is said to be a Banach space if every Cauchy sequence in X converges in X.

A bounded linear functional on a normed linear space X is a function $F : X \to \mathbb{R}$ which is linear i.e. $F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)\ \forall\ x, y$ in X and α, β in $\mathbb{R}$ and for which there is a $M > 0$ such that

$$|F(x)| \le M\|x\| \text{ for all } x \in X.$$

If $F : X \to \mathbb{R}$ is a bounded linear functional the set $\left\{ \frac{F(x)}{\|x\|} \Big/ 0 \ne x \in X \right\}$ is bounded above.

The supremum of this set is denoted by $\|F\|$ and is called the norm of F.

14.2 Definition :

Let X be a normed linear space. A series $\sum_{n=1}^{\infty} f_n$ in X is said to be summable to a sum s if $s \in X$ and the sequence of partial sums $\{s_n\}$ defined by $s_n = f_1 + \cdots + f_n$ converges to s; that is,

$$\lim_n \|s_n - s\| = 0$$

In this case we write $s = \sum_{n=1}^{\infty} f_n$.

The series is said to be absolutely summable if $\sum_{n=1}^{\infty} \|f_n\| < \infty$

In the case of real numbers absolute summability implies summability because of this obvious reason that the real line $\mathbb{R}$ is complete. However it is not necessarily true in a normed linear space that absolute summability implies summability. Then implication is valid if and only if the normed linear space is complete.

14.3 Theorem : A normed linear space X is complete if and only if every absolutely summable series is summable.

Proof : Assume that X is complete and let $\sum_{n=1}^{\infty} \|x_n\| < \infty$ where $x_n \in X \ \forall\ n$. If $\in > 0\ \exists$ a positive integer $N_\in$ such that

$$\sum_{k=n+1}^{m} \|x_k\| < \in \text{ if } m > n \geq N_\in .$$

By the triangle inequality,

$$\left\| \sum_{k=n+1}^{m} x_k \right\| \leq \sum_{k=n+1}^{m} \|x_k\| < \in \text{ for } m > n \geq N_\in .$$

If $s_n = x_1 + \cdots + x_n$ then for $m > n \geq N_\in$

$$\|s_m - s_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| < \in.$$

Hence $\{s_n\}$ is a Cauchy sequence in X. Since X is complete, $\{s_n\}$ converges to some s in X. Hence $\sum_{n=1}^{\infty} x_n = \lim_n s_n = s$. This shows that every absolutely summable series is summable in X.

Conversely assume that every absolutely summable series in X is summable in X. To show that X is complete, let $\{x_n\}$ be any Cauchy sequence in X. We find a subsequence $\{x_{n_k}\}$ such that $\left\|x_{n_k} - x_{n_{k-1}}\right\| < \frac{1}{2^{k-1}}$ $\forall\ k > 1$.

When $k=1$, we take $\in = 1$. Then $\exists$ a positive integer n_1 such that $\|x_n - x_m\| < 1$ for $n > m \geq n_1$

so that $\left\|x_n - x_{n_1}\right\| < 1$ for $n > n_1$.

Similarly $\exists$ a positive $n_2 > n_1 \ni \left\|x_n - x_{n_2}\right\| < \frac{1}{2}$ for $n \geq n_2$.

Since $n_2 > n_1$, we have $\left\|x_{n_2} - x_{n_1}\right\| < 1$.

Assume that $n_1, n_2, \ldots, n_{k-1}$ in $\mathbb{N}$ are chosen so that

$n_1 < n_2 < \ldots < n_{k-1}$ and

$$\left\|x_{n_j} - x_{n_{j-1}}\right\| < \frac{1}{2^{j-1}} \text{ for } j = 1, 2, \ldots, k.$$

Since $\{x_n\}$ is a Cauchy sequence in X there exists a positive integer $n_k > n_{k-1}$ such that

$$\|x_n - x_m\| < \frac{1}{2^k} \text{ for } n \geq n_k$$

and a positive integer $n_{k+1} > n_k$ such that

$$\|x_n - x_m\| < \frac{1}{2^{k+1}} \text{ for } n \geq n_{k+1}.$$

Since $n_{k+1} > n_k$ we have

$$\left\| x_{nk+1} - x_{nk} \right\| < \frac{1}{2^k}.$$

By induction there is a sequence $\{x_n\}$ of positive integers such that

$$\left\| x_{n_k} - x_{n_{k-1}} \right\| < \frac{1}{2^{k-1}} \text{ for all } n_k.$$

We show that $\left\{ x_{n_k} \right\}$ converges in X.

Since the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ is convergent given $\in > 0$ there is a $k_0 \in \mathbb{N} \ni$

$$\sum_{k=s}^{r} \frac{1}{2^k} < \in \text{ for } r > s \geq k_0$$

Hence $\sum_{k=s}^{r} \left\| x_{n_{k+1}} - x_{n_k} \right\| \leq \sum_{k=s}^{r} \frac{1}{2^k} < \in$ for $r > s > k_0$

Write $y_k = x_{n_{k+1}} - x_{n_k}$.

The series $\sum_{k=1}^{\infty} \left\| y_{n_k} \right\|$ satisfies Cauchy criterion. Hence $\sum_{k=1}^{\infty} \|y_k\|$ converges. By hypothesis $\sum_{k=1}^{a} y_k$ converges in X.

If $s_k = y_1 + y_2 + \cdots + y_k$ then $s_k = x_{n_{k+1}} - x_{n_1}$.

Hence the sequence $\left\{ x_{n_{k+1}} - x_{n_k} \right\}$ converges in X.

This implies that $\left\{ x_{n_{k+1}} \right\}$ and hence $\left\{ x_{n_k} \right\}$ converges in X. Thus $\{x_n\}$ has a convergent subsequence in X. It now follows from that $\{x_n\}$ converges in X. Since every Cauchy sequence in X converges in X, X is complete.

14.4 Definition : A linear functional on a linear (vector) space X is a transformation $f : X \to \mathbb{R}$ which satisfies

$f(x+y)=f(x)+f(y)$ for all x, y in X and $f(ax)=af(x)$ for all x in X and $a \in \mathbb{R}$.

A linear functional f on a normed linear space X is said to be bounded if there is a real number $M>0$ such that $|f(x)| \le M\|x\|$ for all $x \in X$.

If $f: X \to \mathbb{R}$ is a bounded linear functional, then

the set $\left\{ \frac{|f(x)|}{\|x\|} \Big/ 0 \neq x \in X \right\}$ is bounded above.

The least upper bound of this set is called the norm of f and is denoted by $\|f\|$.

$$\|f\| = \sup\left\{ \frac{|f(x)|}{\|x\|} \Big/ 0 \neq x \in X \right\}.$$

14.5 SAQ : Let X be a normed vector space and $\{x_n\}$ be a Cauchy sequences in X. If some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges then $\{x_n\}$ converges.

Proof : Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\lim\limits_{n_k} x_{n_k} = x$. If $\in > 0$ there exist positive integers N_1, N_2 such that

$$\|x_n - x_m\| < \frac{\in}{2} \text{ if } n > m \ge N_1 \text{ and}$$

$$\left\|x_{n_k} - x\right\| < \frac{\in}{2} \text{ if } n_k \ge N_2.$$

Let $N_\in = \max\{N_1, N_2\}$, $x \ge N_\in$ and n_k be a fixed integer $> N_\in$. Then $\left\|x_n - x_{n_k}\right\| < \frac{\in}{2}$ since $n \ge N_1 \le n_k \ge N_2$. Also $\left\|x_{n_k} - x\right\| < \frac{\in}{2}$ since $n_k \ge N_2$.

Hence for $n \ge N_\in$ $\|x_n - x\| \le \left\|x_n - x_{n_k}\right\| + \left\|x_{n_k} - x\right\| < \frac{\in}{2} + \frac{\in}{2} = \in$.

This shows that $\lim\limits_{n} x_n = x$.

14.6 Definition : A series $\sum\limits_{i=1}^{a} x_i$ in a normed linear space X is said to be summable to a sums in

X if the sequence $\{s_n\}$ of partial sums defined by $s_n = \sum_{i=1}^{n} x_i$ converges to s in X. If this is the case we write $s = \sum_{n=1}^{\infty} x_n$.

$\sum_{n=1}^{\infty} x_n$ is said to be absolutely summable if the series of nonnegative terms $\sum_{n=1}^{\infty} \|x_n\|$ is convergent.

14.7 Riesz - Fischer Theorem : If $1 \le p < \infty$ L^p is complete.

Proof : In view of 13.5 it is enough to show that every absolutely summable series in L^p is summable in L^p. Suppose $f_n \in L^p$ for $n \ge 1$ and $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$.

Define $g_n(x) = |f_1(x)| + \cdots + |f_n(x)|$ for $x \in [0,1]$ and $n \ge 1$.

Clearly $0 \le g_n(x) \le g_{n+1}(x)$.

By Minkowski's inequality

$$\|g_n\|_p \le \|f_1\|_p + \cdots + \|f_n\|_p \le M \ \forall\ n.$$

so that $\left(\|g_n\|_p\right)^p \le M^p \ \forall\ n$.

Since $\{g_n(x)\}$ is monotonically increasing, $\lim_n g_n(x) = g(x)$ exists and is measurable.

By Fatou's lemma

$$\left(\|g\|_p\right)^p = \int_0^1 (g(x))^p \le \liminf_n \int_0^1 (g_n(x))^p \le M^p$$

Hence g^p is integrable. We set

$$E = \{x / x \in [0, 1] \text{ and } g(x) = \infty\} \text{ and}$$

$$E_n = \{x / x \in [0, 1] \text{ and } |f_n(x)| = \infty\}$$

Since g^p and $|f_n|^p$ are integrable, $m(E) = m(E_n) = 0$.

Suppose $x \notin E$. Then $0 \leq |f_n(x)| < \infty \, \forall n$ and

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty$$

Therefore the series $\sum_{n=1}^{\infty} f_n(x)$ converges tc a real number.

Define $\overline{f_n}(x) = \begin{cases} f_n(x) \text{ if } x \notin E \\ 0 \quad \text{ if } x \in E \end{cases}$

and write $\overline{f}(x) = \sum_{n=1}^{\infty} \overline{f}_n(x)$ and $\overline{s}_n(x) = \overline{f_1}(x) = \overline{f_1}(x) + \cdots + \overline{f_n}(x)$

Then $\int_0^1 |f_n(x)|^p = \int_0^1 |\overline{f_n}(x)|^p$.

We show that $\left\| \overline{s_n}(x) - \overline{f}(x) \right\|_p \to 0$ as $n \to \infty$

It is clear that $\sum_{n=1}^{\infty} \overline{f_n}(x) = \overline{f}(x)$ and that

$$|\overline{s}_n(x) - \overline{f}(x)| \leq |\overline{f}_1(x)| + \cdots + |\overline{f_n}(x)| + |\overline{f}(x)|$$

$$\leq g(x) + g(x) = 2g(x)$$

Hence $|\overline{s}_n - f|^p \leq (2g)^p$.

By the Lebesgue convergence Theorem 8.10 we obtain

$$\lim_n \left\| \sum_{k=1}^{n} f_k - f \right\|_p = 0$$

This completes the proof.

14.8 Proposition : Let $f \in L^p \, (1 \leq p < \infty)$. For each positive integer n and $x \in [0,1]$ define

$$f_n(x) = \begin{cases} n \text{ if } f(x) \geq n \\ f(x) \text{ if } |f(x)| \leq n \\ -n \text{ if } f(x) \leq -n \end{cases}$$

Then f_n is measurable, $|f_n(x)| \leq n \; \forall \; x \in [0, 1]$ and $n \geq 1$ and $\lim_n \|f_n - f\|_p = 0$.

Proof : Measurability of f_n is clear.

If $E = \{x / x \in [0, 1] \text{ and } f(x) = \pm\infty\}$ then since $f \in L^p$, $m(E) = 0$ and also

$$|f_n(x) - f(x)| \to 0 \text{ for } x \notin E$$

Further $|(f - f_n)(x)|^p \leq |f(x)|^p$ for $x \notin E$

Hence by Lebesgue's bounded convergence theorem

$$\lim_n \int_0^1 |f_n - f_n|^p = 0$$

So, $\|f_n - f\|_p \to 0$ as $n \to \infty$

14.9 Proposition : If $1 \leq p \leq \infty$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$ when $1 < p < \infty$ while $q = \infty$ when $p = 1$ and $q = 1$ when $p = \infty$, each function $g \in L^q$ defines a bounded linear functional F on. L^p defined by

$$F(f) = \int_0^1 f g$$

We have $\|F\| = \|g\|_q$.

Proof : If $f \in L^p$ and $g \in L^q$ then clearly $F(f) \in \mathbb{R}$. By the linearity of the integral we have for f_1, f_2 in L^p and α_1, α_2 in $\mathbb{R}$.

$$F(\alpha_1 f_1 + \alpha_2 f_2) = \int_0^1 (\alpha_1 f_1 + \alpha_2 f_2) g = \alpha_1 \int_0^1 f_1 g + \alpha_2 \int_0^1 f_2 g.$$

$$=\alpha_1 F(f_1)+\alpha_2 F(f_2).$$

Hence F is linear.

We now prove that F is bounded and $\|F\| \le \|g\|_q$.

By Holder's inequality, for $f \in L^p$ and $g \in L^q$

$$\left|\int_0^1 f\, g\right| \le \int_0^1 |f\, g| \le \|f\|_p \cdot \|g\|_q$$

so that $\dfrac{|F(f)|}{\|f\|_p} \le \|g\|_q \;\forall$ non zero f in L^p.

This shows that F is bounded and $\|F\| \le \|g\|_q$.

We now prove that $\|F\| = \|g\|_q$.

If $a \in \mathbb{R}$ write $\operatorname{sgn} a = 1, 0$ or -1 according as a is positive, zero or negative. Clearly $|\operatorname{sgn} a| = |a|$.

If $1 < p < \infty$ write $f_0 = |g|^{q/p} \operatorname{sgn} g$

Then $|f_0|^p = |g|^q\, |\operatorname{sgn} g|^p = |g|^q$

and $f_0 g = |g|^{q/p} (g \operatorname{sgn} g) = |g|^{q/p}\, |g|$ (since $g \operatorname{sgn} g = |g|$).

$$= |g|^{\frac{q}{p}+1} = |g|^q \quad \left(\text{since } \frac{q}{p}+1 = q\right)$$

Hence $f_0 \in L^p$, $\|f_0\|_p = \left(\|g\|_q\right)^{q/p}$ so that

$$F(f_0) = \int_0^1 f_0\, g = \int_0^1 |g|^q$$

$$= \|g\|_q^q = \|f_0\|_p^p.$$

$$= \|g\|_q \|f_0\|_p$$

Hence $\dfrac{|F(f_0)|}{\|f_0\|_p} = \|g\|_q$

Thus $\|F\| = \|g\|_q$.

14.16 Proposition : Let g be an integrable function on $[0, 1]$ and assume that there is a real number $M > 0$ such that

$$\left|\int_0^1 f\, g\right| \leq M\|f\|_p$$

for all bounded measurable functions f. Then $g \in L^q$ and $\|g\|_q \leq M$.

Proof : First assume that $1 < p < \infty$.

Define for each positive integer n and $x \in [0,1]$

$$g_n(x) = \begin{cases} g(x) \text{ if } |g(x)| \leq n \\ 0 \text{ if } |g(x)| > n \end{cases}$$

and $f_n(x) = |g_n(x)|^{q/p} \operatorname{sgn}(g_n(x))$.

Then $\int_0^1 |f_n(x)|^p = \int_0^1 |g_n(x)|^q$ so that $\left(\|f_n\|_p\right)^p = \left(\|g_n\|_q\right)^q$ and hence

$$\|f_n\|_p = \left(\|g_n\|_q\right)^{q/p} < \infty.$$

Also $|g_n(x)|^q = |g_n(x)|^{q/p+1} = |g_n(x)|^{\frac{q}{p}} |g_n(x)|$

$$= |g_n|^{\frac{q}{p}} (\operatorname{sgn} g_n(x))\, g_n(x) = f_n(x)\, g_n(x)$$

$$= f_n(x)\, g(x).$$

Hence $\left(\|g_n\|_q\right)^q = \int_0^1 |g_n(x)|^q = \int_0^1 f_n(x)\, g(x).$

$$= \left|\int_0^1 f_n(x)\, g(x)\right|$$

$$\leq M \|f_n\|_p .$$

$$= M\left(\|g_n\|_q^{\frac{q}{p}}\right)$$

$$\Rightarrow \|g_n\|_q = \left(\|g_n\|_q\right)^{q-\frac{q}{p}} \leq M .$$

$$\Rightarrow \int_0^1 |g_n(x)|^q \leq M^q .$$

Since $g_n(x) \to g(x)$ a.e., by Fatou's lemma.

$$\int_0^1 |g(x)|^q \leq \lim\inf \int_0^1 |g_n(x)|^q \leq M^q$$

Hence $g \in L^q$ and $\|g\|_q \leq M$.

When $p=1\ \forall \in>0$ let $E=\{x/0\leq x \leq 1 \text{ and } |g(x)|>M+\in\}$ and define

$$f(x)=\begin{cases}\text{sgn } g(x) \text{ if } x\in E\\ 0 \quad \text{otherwise}\end{cases}$$

Then $\int_0^1 |f(x)| = \int_E 1 = m(E_\in)$.

$$\int_0^1 f\ g = \int_E g(x) \text{ sgn } g(x) = \int_E |g| \geq (M+\in)\ m(E)$$

while $\left|\int_0^1 f\,g\right| \le M\|f\|_1 = m(E)M$

Then $m(E)M \ge m(E)(M+\in)$

This holds only when $m(E)=0$.

Hence $\|g\|_\infty \le M+\in$.

Since this is true for every $\in>0$ it follows that $\|g\|_\infty \le M$.

14.11 Proposition : Let $f \in L^p$, $1\le p<\infty$ and $\in>0$. Then there is a step function g and there is a continuous function L on $[0, 1]$ such that

$$\|f-g\|_p < \in \text{ and } \|f-h\|_p < \in.$$

Proof : $\forall\ n\ge 1$ By Proposition 14.8 we can find a f_n in $L^p \ni$

$$-n\le f_n(x)\le n \text{ and } \|f_n - f\|_p < \frac{\in}{2}.$$

Also there exist g, h defined on $[0, 1]$ and $E\subseteq[0, 1] \ni -n \le g(x) \le n, -n\le h(x)\le n$ on $[0, 1]$

$$|f_n(x) - g(x)| < \frac{1}{2^{\frac{1}{p}}}\cdot\frac{\in}{2} \text{ and } |f_n(x)-h(x)| < \frac{1}{2^{\frac{1}{p}}}\frac{\in}{2}\ \forall\ x\notin E \text{ and}$$

$$m(E)<\frac{1}{2}\left(\frac{\in}{2\cdot 2\,n}\right)^p$$

By Minkowski's inequality

$$\|f-g\|_p \le \|f-f_n\|_p + \|f_n-g\|_p$$

Also $\|f_n - g\|_p^p = \int_0^1 |f_n(x) - g(x)|^p$

$$= \int_{[0,1]\setminus E} |f_n(x)-g(x)|^p + \int_E |f_n(x)-g(x)|^p$$

Since $|f_n - g| \le 2n$,

$$\int_E |f_n(x)-g(x)|^p \le (2n)^p m(E)$$

$$<(2n)^p \frac{1}{2}\left(\frac{\epsilon}{2\cdot 2n}\right)^p = \frac{1}{2}\left(\frac{\epsilon}{2}\right)^p.$$

Also

$$\int_{[0,1]\setminus E} |f_n - g|^p \le \frac{1}{2}\left(\frac{\epsilon}{2}\right)^p$$

Hence $\|f_n - g\|_p^p < \frac{1}{2}\left(\frac{\epsilon}{2}\right)^p + \frac{1}{2}\left(\frac{\epsilon}{2}\right)^p = \left(\frac{\epsilon}{2}\right)^p$

$$\Rightarrow \|f-g\|_p \le \|f-f_n\|_p + \|f_n - g\|_p$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The argument for $\|f-h\|_p < \epsilon$ is parallel.

14.12 : Riesz Representation Theorem : Let F be a bounded linear functional on $L^p\ (1 \le p < \infty)$.

Then there is a function g in L^q such that

$$F(f) = \int_0^1 f\, g.$$

We also have $\|F\| = \|g\|_q$.

Proof : For $0 \le s \le 1$. Let χ_s be the characteristic function of $[0, s]$. We show that the function Φ defined by

$$\Phi(s) = F(\chi_s)$$

is absolutely continuous. Then Φ is the indefinite integral of some $g : \Phi(s) = \int_0^s g(t)$.

We show that this g has the required properties.

Step 1 : Absolute continuity of Φ.

Let $\{I_j \mid 1 \le j \le n\}$ be a finite collection of nonoverlapping subintervals of $[0, 1]$, $I_j = (s_j, s'_j)$

$$\chi_j = \chi_{s'_j} - \chi_{s_j}$$

and $$f = \sum_{j=1}^{n} \chi_j \operatorname{sgn} F(\chi_j)$$

Then $$F(f) = \sum_{j=1}^{n} \left|F(\chi_j)\right| = \sum_{j=1}^{n} \left|F(\chi_{s'_j}) - F(\chi_{s_j})\right|$$

$$= \sum_{j=1}^{n} \left|\Phi(s'_j) - \Phi(s_j)\right|$$

$$\|f\|_p^p = \int_0^1 |f|^p$$

$$= \sum_{j=1}^{n} \int_0^1 |\chi_j|^p = \sum_{j=1}^{n} (s_{j'} - s_j)$$

Now if $\in > 0$ choose $\delta \ni 0 < S < \frac{\in^p}{(1+\|F\|)^p}$.

Then for every choice of $\{I_j / 1 \le j \le n\}$ with $\sum_{j=1}^{n} (s'_j - s_j) < \delta$

$$\sum_{j=1}^{n} \left|\Phi(s'_j) - \Phi(s_j)\right| = F(f)$$

$$\leq \|F\| \|f\|_p$$

$$< \|F\| \cdot S^{1/p}$$

$$\leq \frac{\|F\|}{1+\|F\|} \cdot \in$$

$$< \in.$$

since this holds for every positive integer and for every choice of nonoverlapping intervals $\{I_j / 1 \leq j \leq n\}$ it follows that Φ is absolutely continuous.

Let g be an integrable function on $[0, 1]$ such that $\Phi(s) = \int_0^s g$.

Step 2 : $F(f) = \int_0^1 f\, g$ for every bounded measurable function f.

If f is a step function, $\exists$ finitely many s_i, say $s_1, \cdots, s_n$ in $[0, 1]$ and $\alpha_1, \cdots, \alpha_n$ such that $0 \backslash= s_1 < s_2 < \cdots < s_n = 1$

$$f(x) = \alpha_i \text{ in } (s_{i-1}, s_i) \text{ for } 1 \leq i \leq n.$$

so that $f = \sum_{i=1}^{n} \alpha_i \chi_{(s_{i-1},\, s_i)}$ for $x \notin \{s_1, \cdots, s_n\}$

$$= \sum_{i=1}^{n} \alpha_i \left(\chi_{s_i} - \chi_{s_{i-1}}\right) \text{ for } x \notin \{s_1, \ldots, s_n\}$$

$$\Rightarrow F(f) = \sum_{i=1}^{n} \alpha_i \left\{F\left(\chi_{s_i}\right) - F\left(\chi_{s_{i-1}}\right)\right\}$$

$$= \sum_{i=1}^{n} \alpha_i \left\{\Phi(s_i) - \Phi(s_{i-1})\right\}$$

$$= \sum_{i=1}^{n} \alpha_i \int_0^1 g\left(\chi_{s_i}\right) - \int_0^1 g\left(\chi_{s_{i-1}}\right)$$

$$= \int_0^1 g\left(\sum_{i=1}^{n} \alpha_i \left(\chi_{s_i} - \chi_{s_{i-1}}\right)\right)$$

$$= \int_0^1 g\, f$$

Let f be a bounded measurable function on $[0, 1]$, and $-M \le f(x) \le M \,\forall\, x \in [0, 1]$.

Then there is a sequence $\{f_k\}$ of step functions on $[0, 1]$ such that

$-M \le f_k(x) \le M \;\forall\; x \in [0, 1]$ and $\lim_k f_k(x) = f(x)$ a.e. on $[0, 1]$.

Clearly $|f(x) - f_k(x)|^p \le (2M)^p \;\forall\; x \in [0, 1]$.

Hence by the Bounded convergence theorem

$$\lim_k \|f - f_k\|_p = 0.$$

Since F is continuous and linear, it follows that

$$|F(f) - F(f_k)| = |F(f - f_k)|$$

$$\le \|F\| \cdot \|f - f_k\|_p$$

and hence $\lim_k F(f_k) = F(f)$.

Since f_k is a step function, $F(f_k) = \int_0^1 f_k\, g$.

Hence $F(f) = \lim_k \int_0^1 f_k\, g$

Again since $\lim_K f_K = f$ a.e., $\lim_k f_k\, g = f\, g$ a.e.

Also $|f_k\, g| = |f_k||g| \le 2M|g|$.

Hence by the Bounded convergence theorem

$$\lim_k \int_0^1 f_k g = \int_0^1 f g$$

Hence $F(f) = \int_0^1 f g$.

This completes the proof of step 2.

Step 3 : $F(f) = \int_0^1 f g \ \forall \ f \in L^p$.

Given $\in > 0$ there is a step function $f_0 \in L^p$ such that $\|f - f_0\|_p < \in$.

Since f_0 is bounded, by step 2

$$F(f_0) = \int_0^1 f_0 \, g.$$

Therefore $\left| F(f) - \int_0^1 f g \right|$

$$= \left| F(f) - F(f_0) + F(f_0) - \int_0^1 f g \right|$$

$$\leq \left| F(f - f_0) \right| + \left| \int_0^1 (f_0 - f) g \right|$$

$$\leq \|F\| \, \|f - f_0\|_p + \|f - f_0\|_p \|g\|_q$$

$$< \|F\| \in + \|g\|_q \in.$$

$$= \left(\|F\| + \|g\|_q \right) \in.$$

Since $\in > 0$ is arbitrary it follows that

$$F(f) = \int_0^1 f g.$$

Step 4 : $\|F\| = \|g\|_q$.

From the above equation it is clear that

$$|F(f)| = \left|\int_0^1 f\,g\right| \le \|f\|_p \, \|g\|_q \quad \forall\, f \in L^p$$

Hence $\|F\| \le \|g\|_q$.

As $\left|\int_0^1 f\,g\right| = |F(f)| \le \|F\|\,\|f\|_p \quad \forall\, f \in L^p$

by proposition 14.16 $\|g\|_q \le \|F\|$.

Hence $\|F\| = \|g\|_q$.

This completes the proof.

14.13 P approximants : Let $f \in L'$ and $P: 0 = \mathscr{E}_0 < \mathscr{E}_1 < \cdots < \mathscr{E}_m = 1$ be any partition of $[0, 1]$.

Write $\Delta_j = \mathscr{E}_{j+1} - \mathscr{E}_j$, $\Delta(P) = \Delta = \max\{\Delta_0, \ldots \Delta_{m-1}\}$

$$\alpha_j = \int_{\mathscr{E}_j}^{\mathscr{E}_{j+1}} f \quad \text{for } 0 \le j \le m-1$$

and $T_P(f)(x) = \dfrac{1}{\Delta_j}\,\alpha_j$ if $x \in [\mathscr{E}_j, \mathscr{E}_{j+1}]$ and $0 \le j \le m-1$ and $T_P(f)(\mathscr{E}_m) = \dfrac{1}{\Delta_{m-1}}\,\alpha_{m-1}$

The function $T_P(f)$ is called the P approximant of f in the mean.

The following properties of T_P can be easily verified.

1) $T_P(f)$ is a step function $\forall\, f \in L'$

2) $T_P(f) \in L^r$ and $\|T_P(f)\|_r \le \|f\|_r$ for $1 \le r$.

3) $T_P(f+g) = T_P(f) + T_P(g)$ for $f \in L'$ and $g \in L'$

4) $T_P(a f) = a\,T_P(f)$ for $f \in L'$ and $a \in \mathbb{R}$.

14.14 Proposition : For $r \geq 1$ $f \in L^r$ and $\in > 0$ there exists a positive number $\delta(E)$ such that for every partition $P : O = \mathscr{E}_0 < \mathscr{E}_1 < \cdots < \mathscr{E}_m = 1$ with $\Delta(P) < \delta(\in)$

$$\|T_P(f) - f\|_r < \in .$$

Proof : Choose a step function g on $[0, 1]$ such that $\|g - f\|_r < \frac{\in}{3}$

Let $M = \sup\{g(x)/x \in [0, 1]\}$.

Since g is a step function, there is a partition

$$P_0 : 0 = \mathscr{E}_0 < \mathscr{E}_0 \cdots < \mathscr{E}_m = 1$$

such that g is constant on $(\mathscr{E}_j, \mathscr{E}_{j+1})$ for $a \leq j \leq m-1$.

Let $\delta(\in) = \left(\dfrac{\in}{3(2M \cdot m)}\right)^r$

Let $P : 0 = t_0 < t_1 < \cdots < t_n = 1$

be any partition of $[0, 1]$ such that $\Delta(P) < \delta(\in)$.

Define $f_n(x) = \begin{cases} n & \text{if } f(x) > n \\ f(x) & \text{if } |f(x)| \leq n \\ -n & \text{if } f(x) < -n \end{cases}$

Since $f \in L^r$, $f \in L'$, and $f_n \in f'$. Also $g \in L'$.

Since g is constant on $(\mathscr{E}_K, \mathscr{E}_{K+1})$,

g is constant on (t_j, t_{j+1}) if $(t_j, t_{j+1}) \subseteq (\mathscr{E}_K, \mathscr{E}_{K+1})$ for some K and hence continuous.

Therefore if $T_P(g)$ is not continuous on (t_j, t_{j+1}), this interval must contain some $\mathscr{E}_K$ where $1 \leq K \leq m-1$. So there are atmost $(m-1)$ intervals (t_j, t_{j+1}) such that $T_P(g)$ is not continuous on (t_j, t_{j+1}). Write

$$\Delta_j = t_{j+1} - t_j \text{ and } \Delta = \max\{\Delta_0, \cdots, \Delta_{n-1}\}.$$

There $\|T_P(g)-g\|_r \le m\cdot 2M\,\Delta^{\frac{1}{r}}$ and $\Delta<\delta(\in)$.

Thus

$$\|T_P(f)-f\|_r = \|T_P(f)-T_P(g)+T_P(g)-g+g-f\|_r$$

$$\le \|T_P f - T_P(g)\|_r + \|T_P(g)-g\|_r + \|g-f\|_r$$

$$\le \|T_P(f-g)\|_r + m2M\,\Delta^{\frac{1}{r}} + \|f-g\|_r$$

$$\le 2\|f-g\|_r + 2Mm\,\Delta^{\frac{1}{r}}$$

$$< \frac{\in}{3} + \frac{2\in}{3} = \in.$$

This completes the proof.

14.15 Corollary : The P approximants f in L "converges" to f in measure.

14.16 The ℓ^p spaces : Proofs of the following results are available in Lesson 3 of the study material for Functional Analysis of paper I - Topology and Functional Analysis for M.Sc. final mathematics under Distance education mode. As such we merely state the results without proofs.

14.16.1 Result : The space ℓ^p $(1\le p\le\infty)$ is a Banach space with respect to $\| \quad \|_p$.

14.16.2 Result : There is a one - one correspondence between the bounded linear functionals on ℓ^p $(1\le p\le\infty)$ and elements of ℓ^q where p and q are "conjugate pairs". This correspondence is linear, norm preserving and onto.

14.17 The Spaces c_0 and c : The sequence space c consists of all convergent sequences of real numbers. The space c_0 consists of all sequences which converge to 0. Clearly c is a linear space and c_0 is a linear subspace of c. These spaces are Banach spaces with respect to the ℓ^∞ - norm defined by $\|\{x_n\}\|_\infty = \sup_n |x_n|$.

There is a one - one correspondence between the bounded linear functionals on c_0 (as well as c) and the sequences in ℓ^∞. This correspondence is linear, norm preserving and onto.

As the proofs are available in lesson 3 of the reading materials on Functional Analysis of paper - I : Topology and Functional Analysis for M.Sc. final under Distance education, mode we omit the details.

14.18 SAQ : Let X be a normed linear space and X' be the collection of a bounded linear functionals on X.

Then X' is a linear space with respect to the pointwise operations defined by

$$(F+G)(x) = F(x)+G(x).$$

and $$(\alpha F)(x) = \alpha(F(x))$$

for F, G in X', $x \in X$ and $\alpha \in \mathbb{R}$.

Solution : (a) Linearity of $F+G$:

For x, y in X and α, β in $\mathbb{R}$

$$(F+G)(\alpha x+\beta y) = F(\alpha x+\beta y) + G(\alpha x+\beta y) \quad \text{(defn of } F+G)$$

$$= (\alpha F(x) +\beta F(y)) + (\alpha G(x) + \beta G(y)) \quad \text{(Linearity of } F, G)$$

$$= \alpha(F+G)(x) +\beta(F+G)(y) \quad \text{(definition } F+G)$$

(b) Linearity of υF :

$$(\gamma F)(\alpha x + \beta y) = \gamma(F(\alpha x + \beta y)) \quad \text{(definition of } \gamma F)$$

$$= \gamma\{\alpha F(x)+\beta F(y)\} \quad \text{(distributive law)}$$

$$= \alpha(\gamma F)x + \beta(\gamma F)(y) \quad \text{(commutaitivity of multiplication in } \mathbb{R} \text{ and definition of } \gamma F)$$

(c) Boundness of $K+G$:

For $x \in X$ $|(F+G)(x)| = |F(x)+G(x)|$

$$\leq |F(x)| + |G(x)|$$

$$\Rightarrow \text{for } 0 \neq x \in X, \ \frac{|(F+G)(x)|}{\|x\|} \leq \frac{F(x)}{\|x\|} + \frac{|G(x)|}{\|x\|} \leq \|F\| + \|G\|$$

$$\Rightarrow |(F+G)(x)| \leq (\|F\|+\|G\|)\,\|x\| \;\forall\; x \in X$$

$\Rightarrow F+G$ is bounded and $\|F+G\| \leq \|F\| + \|G\|$.

(d) boundedness of γF:

For $0 \neq x \in X \dfrac{|(\gamma F)(x)|}{\|x\|} = |\gamma| \dfrac{|F(x)|}{\|x\|}$

$$\Rightarrow \|\gamma F\| = \sup_{0 \neq x \in X} \left\{ \frac{|(\gamma F)(x)|}{\|x\|} \right\}$$

$$= \sup_{0 \neq x \in X} \frac{|\gamma|\,|F(x)|}{\|x\|}$$

$$= |\gamma| \sup_{0 \neq x \in X} \frac{|F(x)|}{\|x\|}$$

$$= |\gamma|\,\|F\| < \infty$$

Hence $\|\gamma F\| = |\gamma|\|F\|$

Thus X' is closed under pointwise addition and scalar multiplication. Since X' is a nonempty subset of the vector space $\mathscr{F}$ of all functions from X into $\mathbb{R}$ with pointwise operations X' is a linear subspace of $\mathscr{F}$ and hence is a vector space by itself.

14.19 SAQ : If X is a normed linear space then $F \rightarrow \|F\|$ defines a norm on X'.

Solution : In view of (c) and (d) of SAQ it is enough to show that $\|F\| \geq 0$ and $\|F\| = 0$ if and only if $F = 0$.

Clearly $\|F\| = \sup\left\{ \dfrac{|F(x)|}{\|x\|} \Big/ 0 \neq x \in X \right\} \geq 0$ and $\|\,0\,\| = 0$.

If $F \neq 0$, $F(x) \neq 0$ for some $x(\neq 0)$ in X so that $\|F\| \geq \dfrac{|F(x)|}{\|x\|} > 0$.

14.20 Model Examination Questions :

1. Prove that L^{∞} is complete.

2. If $1<p<\infty$ and $\frac{1}{p}+\frac{1}{q}=1$ show that each g in L^q defines a bounded linear functional $F_g : L^p \to \mathbb{R}$ by $F_g(f) = \int_0^1 f\, g$ and that $\|F\| = \|g\|_q$.

3. Let g be integrable on $[0, 1]$ and suppose that there is a positive real number M such that

$$\left|\int_0^1 f\, g\right| \le M\|f\|_p$$

for all bounded measurable functions f. Show that $g \in L^q$ and $\|g\|_q \le M$.

4. Let $1 \le p < \infty$; F a bounded linear functional on L^p and for $0 \le s \le 1$ $\Phi(s) = F(\chi_s)$ where χ_s is the characteristic function of $[0, s]$. Show that Φ is absolutely continuous.

5. Let $\{f_n\}$ be a sequence in L^p, $1 \le p < \infty$, which converges almost everywhere to a function in L^p. Show that $\{f_n\}$ converges to f in L^p if and only if $\|f_n\|_p \to \|f\|_p$.

14.21 Exercises :

1. Let c_{00} be the space of all sequence $\{x_n\}$ such that $x_n \neq 0$ for atmost finitely many n.

   (a) Show that c_{00} is a linear space and $c_{0\,0} \subseteq c_0 \subseteq c$.

   (b) Show that c_{00} is a normed linear space with respect to the ℓ^{∞} norm.

   (c) Let $\{x^{(n)}\}$ be the sequence in c_{00} where for each n; $x^{(n)}$ is the sequence of numbers with $\frac{1}{n^2}$ in the n^{th} place and zero elsewhere. Show that the series $\sum_{n=1}^{\infty} x^{(n)}$ is absolutely summable, but not summable in c_{00}.

2. Prove that every convergent sequence in a normed linear space is Cauchy sequence.

3. Let $\mathbb{C} = C[0\ 1]$ be the space of all continuous functions on $[0, 1]$. Show that C is a Banach space where the norm is defined by

$$\|f\| = \|f\|_{\infty} = \sup\{|f(x)|/0 \le x \le 1\}.$$

4. Show that the $g \in L^q$ in Riesz Representation theorem is unique.

5. If X is a normed vector space show taht X' is a Banach space (see lesson 3 of Reading material on Functional Analysis of Paper I Topology and Functional Analysis)

REFERENCE BOOK

Real Analysis - Royden

Lesson Writer :

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Lesson - 15

ABSTRACT MEASURE AND MEASURABLE FUNCTIONS

15.1 INTRODUCTION

We consider general spaces and generalize many of the results of lesson 1. In this lesson you would learn how to define a measure on an abstract set and some simple and far reaching properties of such a measure. The concept of a complete measure space is introduced.

Recall from lesson 3 Theorem 3.10 that the class of all (Lebesgue) measurable sets is a $\sigma-$ algebra. Motivated by this observation, we would prefer to define measure on a $\sigma-$ algebra of subsets of X. We recall the definition of a $\sigma-$ algebra of sets.

15.2 MEASURABLE SPACES AND MEASURE SPACES

15.2.1 Definition : Let X be a non-empty set; $\mathscr{A}$ be a non-empty collection of subsets of X satisfying the conditions

(a) $A \in \mathscr{A} \Rightarrow \tilde{A} \in \mathscr{A}$ ($\tilde{A}$: complement of A)

(b) for any countable collection $\{A_n\}$ of members of $\mathscr{A}$ their union $\bigcup_{n=1}^{\infty} A_n$ is also a member of $\mathscr{A}$.

As we remarked earlier the domain of a measure is going to be a σ - algebra.

15.2.2 Definition : If X is any non-empty set and $\mathscr{A}$ is a σ - algebra of subsets of X then $(X, \mathscr{A})$ is called a measurable space.

In this case the members of $\mathscr{A}$ are called measurable sets relative to $\mathscr{A}$ in the measurable space. We now define a measure on a measurable space.

15.2.3 Definition : Suppose $(X, \mathscr{A})$ is a measurable space. An extended real - valued set function μ defined on $\mathscr{A}$

(that is $\mu: \mathscr{A} \rightarrow [\infty, \infty]$) is called a measure if it satisfies the conditions.

(i) $\mu(\phi) = 0$

(ii) Non-negativity : $\mu(A) \geq 0$ for all $A \in \mathscr{A}$ and

(iii) countable additivity : For any sequence $\{A_n\}$ of pairwise disjoint sets in $\mathscr{A}$ we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

15.2.4 Definition : If $(X, \mathscr{A})$ is a measurable space and μ is a measure on it then $(X, \mathscr{A}, \mu)$ is called a measure space.

15.2.5 Example :

(i) $\mathbb{R}$ is the real line, m is the lebesgue measure on $\mathbb{R}$. $\mathscr{M}$ is the family of all lebesgue measurable subsets of $\mathbb{R}$. Then $(\mathbb{R}, \mathscr{M}, m)$, is a measure space.

(ii) On $\mathbb{R}$, $\mathscr{B}$ is the σ - algebra of Borel subsets of $\mathbb{R}$. Then $(\mathbb{R}, \mathscr{B})$ is a measurable space and $(\mathbb{R}, \mathscr{B}, m)$ is a measure space.

Note that the measure space in (i) and (ii) are different though the measure in both the cases is the same 'm'.

15.2.6 Example : X is an uncountable set.

$$\mathscr{B} = \{A \subseteq X : A \text{ is countable or } X - A \text{ is countable}\}$$

Define μ on $\mathscr{B}$ by

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } \tilde{A} \text{ is countable} \end{cases}$$

Then $(X, \mathscr{B}, \mu)$ is a measure space.

Solution : We know that $\mathscr{B}$ is a σ - algebra of sets. Clearly μ is non-negative and $\mu(\phi) = 0$. Let $\{A_i\}$ be a countable disjoint collection of sets from $\mathscr{B}$. If each A_i is countable, $\bigcup A_i$ is countable and hence by the definition of μ, $\mu\left(\bigcup_i A_i\right) = 0 = \sum_i \mu(A_i)$.

Suppose $\widetilde{A_{i_0}}$ is countable for some i_0, then $\mu\left(A_{i_0}\right) = 1$. Since $A_{i_0} \cap A_j = \phi \ \forall \ j \neq i_0$, we have, $A_j \subseteq \widetilde{A_{i_0}}$ for every $j \neq i_0$. Thus, A_j is countable for every $j \neq i_0$ implies $\mu\left(A_j\right) = 0$ for every $j \neq i_0$. Also, $\widetilde{\bigcup_i A_i} \subseteq \widetilde{A_{i_0}}$, implies $\widetilde{\bigcup_i A_i}$ is countable. Hence, $\mu\left(\bigcup_i A_i\right) = 1$ and

$\sum \mu(A_i) = 1 = \mu\left(\bigcup_i A_i\right)$. Therefore, μ is a measure on $\mathscr{B}$. Hence $(X, \mathscr{B}, \mu)$ is a measure space.

15.2.7 Example : Let X be a set. and for any $A \in \mathscr{P}(X)$ define,

$$\mu(A) = \begin{cases} \text{the number of elements in } A \text{ if } A \text{ is finite} \\ \infty \qquad\qquad\qquad\qquad\qquad \text{if } A \text{ is infinite} \end{cases}$$

Then, $(X, \mathscr{P}(X), \mu)$ is a measure space. μ is called the counting measure on X.

Clearly μ is non-negative and $\mu(\phi) = 0$. Let $\{A_n\}$ be a countable disjoint sequence of sets. If A_n is infinite for at least one n_0 then $\mu\left(A_{n_0}\right) = \infty$ and $\mu\left(\bigcup_n A_n\right) = \infty$. Therefore, $\mu\left(\bigcup_n A_n\right) = \infty = \sum_n \mu(A_n)$. If each A_n is finite say with m_n elements then $\mu(A_n) = m_n$ for $n=1,2,3,\ldots\ldots\ldots$ by the definition of μ, and since A_n's are disjoint, we have $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m_n = \sum_{n=1}^{\infty} \mu(A_n)$. Therefore, μ is a measure on $\mathscr{P}(X)$ and hence $(X, \mathscr{P}(X), \mu)$ is a measure space.

15.2.8 Example : Let X be a set, $x_0 \in X$, $A \subseteq X$.

Define, $$\mu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

Then $(X, \mathscr{P}(X), \mu)$ is a measure space. This μ is called Dirac measure.

Clearly μ is non-negative. $\mu(\phi) = 0$ since $x_0 \notin \phi$. Let $\{E_i\}$ be a countable disjoint sequence of sets. If $x_0 \in \bigcup_i E_i$ then $\mu\left(\bigcup_i E_i\right) = 1$. Since, $x_0 \in \bigcup_i E_i$ and E_i's are disjoint, $x_0 \in E_{i_0}$ for some unique i_0. Then $\mu\left(E_{i_0}\right) = 1$ and $\mu(E_i) = 0$ for every $i \neq i_0$. Hence, $\mu\left(\bigcup_i E_i\right) = 1 = \sum_i \mu(E_i)$. If

$x_0 \notin \bigcup_i E_i$ then $x_0 \notin E_i$ for every i and hence,

$\mu\left(\bigcup_i E_i\right) = 0 = \sum_i \mu(E_i)$. Therefore μ is a measure and $(X, \mathscr{P}(X), \mu)$ is a measure space.

We continue with the derivation of properties of the measure μ defined in 15.2.3.

15.2.9 Self Assessment Question :

If $(X, \mathscr{A}, \mu)$ is a measure space and $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets in $\mathscr{A}$. Prove that

$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$. (This property is called the finite additivity of μ).

15.2.10 Theorem : Suppose $(X, \mathscr{A}, \mu)$ is a measure space.

(i) If $A \in \mathscr{A}$, $B \in \mathscr{A}$ and $A \subseteq B$ then $\mu(A) \leq \mu(B)$.

(ii) If $A_i \in \mathscr{A}$ for $i = 1, 2, 3, \ldots\ldots\ldots\ldots$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Proof : (i) Since $B = A \cup (B - A)$ is a disjoint union of $A \in \mathscr{A}$ and $B - A \in \mathscr{A}$ we have, by SAQ 15.2.9, that, $\mu(B) = \mu(A) + \mu(B - A)$. Since μ is non-negative, $\mu(B - A) \geq 0$. Therefore $\mu(B) \geq \mu(A)$.

(ii) Given $A_i \in \mathscr{A}$ for $i = 1, 2, 3, \ldots\ldots\ldots$ while, $B_i = A_i - \bigcup_{k=1}^{i-1} A_k$. Then $B_i \subseteq A_i$ for $i = 1, 2, 3, \ldots\ldots\ldots$ and $\{B_i\}$ is a pairwise disjoint family of sets in $\mathscr{A}$ such that $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$. Therefore, by countable additivity of μ, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$$

$$\leq \sum_{i=1}^{\infty} \mu(A_i)$$

since $\mu(B_i) \leq \mu(A_i)$.

15.2.11 Remark : The property proved in (i) of Theorem 15.2.10 is called the monotonicity of μ while that in (ii) is called the countable subadditivity of μ.

15.2.12 Theorem : Suppose $(X, \mathscr{A}, \mu)$ is a measure space. If $\{A_i\}$ is a decreasing sequence of sets in $\mathscr{A}$ (i.e., $A_{i+1} \subseteq A_i$ for $i = 1, 2, 3, \ldots\ldots$) with $\mu(A_1) < \infty$ then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n)$$

Proof : Let $A = \bigcap_{i=1}^{\infty} A_i$ and $B_i = A_i - A_{i+1}$ for $i \geq 1$. Then $B_i \in \mathscr{A}$ for $i \geq 1$, $B_i \cap B_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = A_1 - A$ (verify !). Therefore, by (iii) of definition we have,

$$\mu(A_1 - A) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \text{ ------------------ (1)}$$

Since, $A \subseteq A_{i+1} \subseteq A_i \subseteq A_1$, we get $\mu(A) \leq \mu(A_{i+1}) \leq \mu(A_i) \leq \mu(A_1) < \infty$

Therefore $\mu(A_1) = \mu(A \cup (A_1 \setminus A)) = \mu(A) + \mu(A_1 - A)$ and

$$\mu(A_i) = \mu(A_{i+1} \cup (A_i \setminus A_{i+1})) = \mu(A_{i+1}) + \mu(A_i \setminus A_{i+1})$$

respectively give

$$\mu(A_1 - A) = \mu(A_1) - \mu(A) \text{ ------------------ (2)}$$

and $\mu(A_i \setminus A_{i+1}) = \mu(A_i) - \mu(A_{i+1})$ ----------------(3)

Now we get from (1), (2) and (3) that

$$\mu(A_1) - \mu(A) = \sum_{i=1}^{\infty} [\mu(A_i) - \mu(A_{i+1})]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} \left(\mu(A_i) - \mu(A_{i+1})\right)$$

$$= \lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right)$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n).$$

This implies $\mu(A) = \lim_{n\to\infty} \mu(A_n)$, since $\mu(A_1) < \infty$.

15.2.13 Self Assessment Question :

Suppose $(X, \mathscr{A}, \mu)$ is a measure space. If $\{A_i\}$ is a sequence of sets in $\mathscr{A}$ such that $A_i \subseteq A_{i+1}$ for $i = 1, 2, 3, \ldots\ldots$ then, prove that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_i \mu(A_i)$.

15.2.14 Theorem : Suppose $(X, \mathscr{A}, \mu)$ is a measure space and $E_1, E_2 \in \mathscr{A}$. Then, $\mu(E_1 \Delta E_2) = 0$ implies that $\mu(E_1) = \mu(E_2)$.

(The symmetric difference $E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$).

Proof : From the hypothesis we have

$$0 = \mu(E_1 \Delta E_2) = \mu(E_1 \setminus E_2) + \mu(E_2 \setminus E_1) \text{ ------------ (1)}$$

Since μ is non-negative $\mu(E_1 \setminus E_2) \geq 0$ and $\mu(E_2 \setminus E_1) \geq 0$ and hence from (1) $\mu(E_1 - E_2) = 0$ and $\mu(E_2 \setminus E_1) = 0$. We know that, $E_1 = (E_1 \setminus E_2) \cup (E_1 \cap E_2)$, $(E_1 \setminus E_2) \cap (E_1 \cap E_2) = \phi$

$$E_2 = (E_2 \setminus E_1) \cup (E_1 \cap E_2),\ (E_2 \setminus E_1) \cap (E_1 \cap E_2) = \phi$$

Thus,

$$\mu(E_1) = \mu(E_1 \setminus E_2) + \mu(E_1 \cap E_2)$$

$$= \mu(E_1 \cap E_2)$$

and similarly, $\mu(E_2) = \mu(E_1 \cap E_2)$. Thus we have $\mu(E_1) = \mu(E_2)$.

15.3 Classification of Measure

15.3.1 Definition : A measure μ on the measurable space $(X, \mathscr{A})$ is said to be

(i) finite if $\mu(X)<\infty$

(ii) $\sigma-$ finite if there is a sequence $\{A_n\}$ in $\mathscr{A}$ such that $\mu(A_n)<\infty$ for every n and $X=\bigcup_{n=1}^{\infty} A_n$.

For example the Lebesgue measure on $[0, 1]$ is a finite measure while the Lebesgue measure on $(\mathbb{R}, m)$ is $\sigma-$ finite since $\mathbb{R}=\bigcup_{n=1}^{\infty}(-n, n)$ and $m((-n, n)) = 2n<\infty$ for every n.

15.3.2 Definition : A measure space $(X, \mathscr{A}, \mu)$ is said to be complete if $A\in\mathscr{A}$, $\mu(A)=0$ and $B\subseteq A$ imply $B\in\mathscr{A}$. In otherwords, a measure space $(X, \mathscr{A}, \mu)$ is complete if $\mathscr{A}$ contains all subsets of sets of measure zero.

15.3.3 Examples :

(i) $(\mathbb{R}, \mathscr{M}, m)$ is a complete measure space

(ii) Any measure on $(X, \mathscr{P}(X))$ is complete

(iii) Let $X\neq\phi$ and X contains more than one element. Let $\mathscr{A}=\{\phi, X\}$. Define $\mu(\phi) = 0 = \mu(X)$. Clearly μ is not complete.

(iv) Let $X=\{a,b,c\}$, $\mathscr{A}=\{\phi,\{a\}, \{b,c\}, X\}$ and $\mu:\mathscr{A}\to\mathbb{R}$ is defined by $\mu(\phi)=\mu(\{b, c\}) = 0$ $\mu(\{a\}) = \mu(X)=1$. Then $(X, \mathscr{A}, \mu)$ is a measure space which is not complete since $\{b, c\}\in\mathscr{A}$ is of zero measure but has subsets $\{b\}, \{c\}$ neither of which lies in $\mathscr{A}$.

Note that the Lebesgue measure is complete while Lebesgue measure restricted to the $\sigma-$ algebra of Borel sets is not compete. Thus not all measure spaces are complete. However, every non-complete measure space is included in a comple. measure space. This process is called completion.

15.3.4 Theorem (completion) : If $(X, \mathscr{B}, \mu)$ is a measure space, then we can find a complete measure space $(X, \mathscr{B}_0, \mu_0)$ such that

(i) $\mathscr{B} \subseteq \mathscr{B}_0$

(ii) $E \in \mathscr{B} \Rightarrow \mu(E) = \mu_0(E)$

(iii) $E \in \mathscr{B}_0 \Leftrightarrow E = A \cup B$, where $B \in \mathscr{B}, A \subseteq C, C \in \mathscr{B}$ with $\mu(C)=0$.

Proof : Let $(X, \mathscr{B}, \mu)$ is a measure space. Write

$$\mathscr{B}_0 = \{E/E = A \cup B,\ A \subseteq C,\ C \in \mathscr{B} \text{ with } \mu(C) = 0 \text{ and } B \in \mathscr{B}\}$$

We leave proving $\mathscr{B}_0$ is a $\sigma-$ algebra and $\mathscr{B} \subseteq \mathscr{B}_0$ as a simple exercise for you. Now observe the following. If $E = A_1 \cup B_1 = A_2 \cup B_2$ where $B_1, B_2 \in \mathscr{B}, A_i \subseteq C_i$, $C_i \in \mathscr{B}$ with $\mu(C_i)=0$ for $i=1, 2$. Then $B_1 \subseteq E = A_2 \cup B_2 \subseteq C_2 \cup B_2$

$$\Rightarrow \mu(B_1) \leq \mu(C_2 \cup B_2) \leq \mu(C_2) + \mu(B_2) = \mu(B_2)$$

Since $\mu(C_2)=0$. Hence, $\mu(B_1) \leq \mu(B_2)$. Similarly we can prove that $\mu(B_2) \leq \mu(B_1)$. Therefore, $\mu(B_1) = \mu(B_2)$.

Now define μ_0 on $\mathscr{B}_0$ as follows

If $E \in \mathscr{B}_0$ then $E = A \cup B,\ B \in \mathscr{B},\ A \subseteq C,\ C \in \mathscr{B},\ \mu(C) = 0$

Define, $\mu_0(E) = \mu(B)$.

By the above observation, μ_0 is well defined. Since μ is non-negative, μ_0 is non-negative, and $\mu_0(\phi) = \mu(\phi)=0$. Now let $\{E_n\}$ be a disjoint sequence of sets in $\mathscr{B}_0$. We will show that

$$\mu_0\left(\bigcup_n E_n\right) = \sum_n \mu_0(E_n).$$

Now, $E_n \in \mathscr{B}_0$ implies $E_n = A_n \cup B_n$, $A_n \subseteq C_n$, $C_n \in \mathscr{B}$, $\mu(C_n) = 0$ and $B_n \in \mathscr{B}$ for every n. Then,

$$\mu_0\left(\bigcup_n E_n\right) = \mu_0\left(\left(\bigcup_n A_n\right) \cup \left(\bigcup_n B_n\right)\right)$$

$$= \mu\left(\bigcup_n B_n\right)$$

$$= \sum_n \mu(B_n)$$

$$= \sum_n \mu_0(E_n). \text{ (since, } \mu_0(E_n) = \mu(B_n) \forall n).$$

Hence, $(X, \mathscr{B}_0, \mu_0)$ is a measure space. Next we show that μ_0 is complete. Let $E \in \mathscr{B}_0$, $\mu_0(E)=0$ and $F \subseteq E$. Since $E \in \mathscr{B}_0$ we have $E = A \cup B$, $A \subseteq C$, $C \in \mathscr{B}$, $\mu(C)=0$ and $B \in \mathscr{B}$. Now by the definition of μ_0, $\mu_0(E) = \mu(B) = 0$.

Now, $F \subseteq E = A \cup B \subseteq C \cup B$ and

$\mu(C \cup B) \leq \mu(C) + \mu(B) = 0$. Clearly, $C \cup B \in \mathscr{B}$. Now, $F = F \cup \phi$, $F \subseteq C \cup B$, $C \cup B \in \mathscr{B}$, $\mu(C \cup B) = 0$, and $\phi \in \mathscr{B}$. Therefore $F \in \mathscr{B}_0$. Hence, μ_0 is complete and $(X, \mathscr{B}_0, \mu_0)$ is a complete measure space.

Note : The measure space $(X, \mathscr{B}_0, \mu_0)$ has the following property. If $(X, \mathscr{B}_1, \mu_1)$ is any complete measure space such that

(i) $\mathscr{B} \subseteq \mathscr{B}_1$ (ii) $\mu(B) = \mu_1(B)$ for every $B \in \mathscr{B}$ then $\mathscr{B}_0 \subseteq \mathscr{B}_1$ and $\mu_1 = \mu_0$ on $\mathscr{B}_0$.

15.3.5 Definition : The complete measure space $(X, \mathscr{B}_0, \mu_0)$ given in the above proposition is called the completion of the given measure space $(X, \mathscr{B}, \mu)$.

15.3.6 Self Assessment Question : Let $(X, \mathscr{B}, \mu)$ be a measure space. Let $Y \subseteq X$, $Y \in \mathscr{B}$, define $\mathscr{B}_Y = \{A \in \mathscr{B} : A \subseteq Y\}$ and $\mu_Y(A) = \mu(A)$. Then $(Y, \mathscr{B}_Y, \mu_Y)$ is a measure space. μ_Y is called the restriction of μ to Y.

15.4 MEASURABLE FUNCTIONS :

In this section we extend the concept of a Lebesgue measurable function (introduced in lesson) to a measurable function in an abstract measurable space.

15.4.1 Definition : Let $(X, \mathscr{B})$ be a measurable space and f an extended real valued function defined on X. We say that f is measurable if for every real number α, the set $\{x : f(x) > \alpha\} \in \mathscr{B}$.

15.4.2 Proposition : Suppose $(X, \mathscr{B})$ is a measurable space and f is an extended real valued function defined on X. Then the following statements are equivalent.

(a) For every $\alpha \in \mathbb{R}$ the set $\{x : f(x) > \alpha\} \in \mathscr{B}$

(b) For every $\alpha \in \mathbb{R}$ the set $\{x : f(x) \geq \alpha\} \in \mathscr{B}$

(c) For every $\alpha \in \mathbb{R}$ the set $\{x : f(x) < \alpha\} \in \mathscr{B}$

(d) For every $\alpha \in \mathbb{R}$ the set $\{x : f(x) \leq \alpha\} \in \mathscr{B}$.

Proof : We prove that $(a) \Rightarrow (b)$. $(b) \Rightarrow (c)$. $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ from which the theorem follows.

Assume (a) : For any $\alpha \in \mathbb{R}$ we have

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > \alpha - \frac{1}{n}\right\}.$$

Now, by (a), $\left\{x : f(x) > \alpha - \frac{1}{n}\right\}$ is in $\mathscr{B}$ for each $n \geq 1$ and hence $\{x : f(x) \geq \alpha\} \in \mathscr{B}$, since $\mathscr{B}$ is a $\sigma-$ algebra. Thus $(a) \Rightarrow (b)$.

Assume (b) : For any $\alpha \in \mathbb{R}$, we have $\{x : f(x) < \alpha\} = X - \{x : f(x) \geq \alpha\}$

Since, $X \in \mathscr{B}$ and by (b), $\{x : f(x) \geq \alpha\} \in \mathscr{B}$, we get $\{x : f(x) < \alpha\} \in \mathscr{B}$ proving that $(b) \Rightarrow (c)$.

Assume (c) : For any $\alpha \in \mathbb{R}$ we have

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) < \alpha + \frac{1}{n}\right\}$$

Now, by (c), each set on the right is in $\mathscr{B}$ so that

$\{x : f(x) \leq \alpha\} \in \mathscr{B}$, showing $(c) \Rightarrow (d)$.

Finally assume (d) : For any $\alpha \in \mathbb{R}$

$$\{x : f(x) > \alpha\} = X - \{x : f(x) \leq \alpha\}$$

shows $\{x : f(x) > \alpha\} \in \mathscr{B}$ for any $\alpha \in \mathbb{R}$. Hence, $(d) \Rightarrow (a)$.

Remark : Note that an extended real-valued function defined on a measurable space $(X, \mathcal{B})$ is measurable if any one of the statements in the above propositon holds.

15.4.3 Example : (i) If f is measurable, then $\{x : f(x) = \alpha\}$ is measurable for each extended real number α (ii) the characteristic function χ_A is measurable if, and only if $A \in \mathcal{B}$; (iii) the constant functions are measurable.

15.4.4 Self Assessment Question :

Prove that every extended real-valued constant function is measurable.

15.4.5 Theorem : If f and g are measurable real valued functions and c is any constant then $f+c,\ cf,\ f+g,\ f-g,\ f^2$ and fg are measurable functions.

Proof : (i) If $c = +\infty$ (or) $-\infty$ then $f + c = +\infty$ or $-\infty$ so that $f + c$ is measurable by SAQ 15.4.4.

Therefore assume $c \in \mathbb{R}$ and $c \neq 0$. Then for any $\alpha \in \mathbb{R}$

We have,

$\{x : (f+c)(x) > \alpha\} = \{x : f(x) + c > \alpha\} = \{x : f(x) > \alpha - c\}$ and by the measurability of f the set on the right is in $\mathcal{B}$ proving $f + c$ is measurable.

(ii) If $c = +\infty, -\infty$ or 0 then cf is a constant function so that cf is measurable by SAQ 15.4.4. Therefore assume $c \in \mathbb{R}$ and $c \neq 0$. Then for any $\alpha \in \mathbb{R}$, we have,

$$\{x : (cf)(x) > \alpha\} = \{x : cf(x) > \alpha\}$$

$$= \begin{cases} \{x : f(x) > \alpha/c\} & \text{if } c > 0 \\ \{x : f(x) < \alpha/c\} & \text{if } c < 0 \end{cases}$$

since the set on the right lies in $\mathcal{B}$ by the measurability of f we get that $\{x : (cf)(x) > \alpha\}$ is in $\mathcal{B}$ for each $\alpha \in \mathbb{R}$. Proving that cf is measurable.

(iii) If $f(x) + g(x) < \alpha$ then $f(x) < \alpha - g(x)$ and therefore we can find a rational number r such that $f(x) < r < \alpha - g(x)$. Therefore for any $\alpha \in \mathbb{R}$

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_r \{\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}\}$$

where the union is over rational numbers r. Since $\{x : f(x) < r\} \in \mathcal{B}$ and $\{x : g(x) < \alpha - r\} \in \mathcal{B}$ their intersection is in $\mathcal{B}$. Thus the set on the right is the union of a countable family from $\mathscr{B}$ and hence lies in $\mathscr{B}$. That is, $\{x : f(x)+g(x) < \alpha\} \in \mathscr{B}$ for any $\alpha \in \mathbb{R}$, proving that $f+g$ is measurable.

(iv) Since $f-g = f+(-1)g$, it is measurable by (ii) and (iii)

(v) To prove the measruability of f^2, note that for any $\alpha \in \mathbb{R}$

$$\{x : f^2(x) > \alpha\} = \begin{cases} X \text{ if } \alpha \le 0 \\ \{x : |f(x)| > \sqrt{\alpha}\} \text{ if } \alpha > 0 \end{cases}$$

$$= \begin{cases} X \text{ if } \alpha \le 0 \\ \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\} \text{ if } \alpha > 0 \end{cases}$$

In either case, by the measurability of f, the sets on the right are in $\mathscr{B}$. Hence, f^2 is measurable.

(vi) Now, since

$$fg = \frac{1}{4}\left\{(f+g)^2 - (f-g)^2\right\}$$ it follows that fg is also measurable.

15.4.6 Definition : If f and g are extended real valued functions defined on X then $f \vee g$ and $f \wedge g$ are defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

and $$(f \wedge g)(x) = \min\{f(x), g(x)\}$$

for any $x \in X$.

Note that in the case $g(x) = 0$ for all x, then

$$(f \vee 0)(x) = \begin{cases} f(x) \text{ if } f(x) \ge 0 \\ 0 \quad \text{ if } f(x) < 0 \end{cases}$$

while

$$(f \wedge 0)(x) = \begin{cases} 0 \quad \text{ if } f(x) \ge 0 \\ f(x) \text{ if } f(x) < 0 \end{cases}$$

15.4.7 Self Assessment Question : If f and g are measurable show that $f \vee g$ and $f \wedge g$ are also measurable.

15.4.8 Theorem : If $\{f_n\}$ is a sequence of measurable functions on X then (i) $\sup\limits_{1\le i\le n} f_i$ is measurable for each n (ii) $\inf\limits_{1\le i\le n} f_i$ is measurable for each n (iii) $\sup\limits_{n} f_n$ is measurable (iv) $\inf\limits_{n} f_n$ is measurable (v) $\limsup f_n$ is measurable (vi) $\liminf f_n$ is measurable.

Proof : (i) Since $\left\{x : \sup\limits_{1\le i\le n} f_i(x) > \alpha\right\} = \bigcup\limits_{i=1}^{n}\{x : f_i(x) > \alpha\}$, we have $\sup\limits_{1\le i\le n} f_i$ is measurable.

(ii) $\inf\limits_{1\le i\le n} f_i = -\sup\limits_{1\le i\le n}(-f_i)$ and so is measurable.

(iii) $\{x : \sup f_n(x) > \alpha\} = \bigcup\limits_{n=1}^{\infty}\{x : f_n(x) > \alpha\}$, so $\sup f_n$ is measurable.

(iv) $\inf\limits_{n} f_n = -\sup\limits_{n}(-f_n)$ and so is measurable

(v) $\limsup f_n = \inf\left(\sup\limits_{i\ge n} f_i\right)$ is measurable by (iii) and (iv)

(vi) $\liminf f_n = -\limsup(-f_n)$ and so is measurable.

15.4.9 Self Assessment Question : Prove that a subset E of a measurable space $(X, \mathscr{A})$ is measurable if and only if its characteristic function χ_E is measurable on X.

We have the following result showing that the result 19 of lesson 8 holds in any measurable space.

15.4.10 Theorem : Let f be a non-negative measurable function on X. Then there is a sequence $\{\phi_n\}$ of simply functions such that

(a) $0 \le \phi_1 \le \phi_2 \le \cdots \le f$ and

(b) $\lim\limits_{n\to\infty} \phi_n(x) = f(x)$ for all $x \in X$.

Proof : For $n = 1, 2, 3, \cdots$ and $1 \le i \le n2^n$ define

$$E_{n,\,i} = \left\{ x \in X : \frac{i-1}{2^n} \le f(x) < \frac{i}{2^n} \right\} \text{ and}$$

$$F_n = \{ x \in X : f(x) \ge n \}$$

Then for any n and i, we have $E_{n,i} \in \mathscr{A}$ and $E_n \in \mathscr{A}$ (by the measurability of f). Now define

$$\phi_n(x) = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}}(x) + n\chi_{F_n}(x)$$

Then $\{\phi_n\}$ is the required sequence satisfying (a) and (b) (The proof given in Theorem...... of lesson.... holds here also).

15.4.11 Definition : Suppose $(X, \mathscr{A}, \mu)$ is a measure space. A property is said to hold almost, every where (or briefly a.e.) if the set of points where it fails to hold is a set of measure zero.

For example, we say that $f = g$ a.e. if f and g have the same domain and $\mu\{x : f(x) \ne g(x)\} = 0$ we say that f_n converges to g almost every where if there is a set E of measure zero such that $f_n(x)$ converges to $g(x)$ for each x not in E.

One consequence of equality a.e. is the following.

15.4.12 Theorem : Suppose $(X, \mathscr{A}, \mu)$ is a complete measurable space and f is measurable on X. If $g = f$ a.e. then g is also measurable on X.

Proof : Write $E = \{x : g(x) > \alpha\}$, $E_1 = \{x : f(x) > \alpha\}$

$E_2 = \{x : f(x) \ne g(x)\}$. Then E_1 and E_2 are measurable and, as μ is complete, so is $E \cap E_2$. So, $E = (E_1 - E_2) \cup (E \cap E_2)$ is measurable.

For example if $\{f_i\}$ is a sequence of measurable functions converging a.e. to f then f is measurable, since, $f = \lim\sup f_i$ a.e. the result follows by the above theorem.

15.5 ANSWERS TO SAQs

15.2.9 SAQ : If $A_1, A_2, \cdots\cdots, A_n$ are pairwise disjoint members of $\mathscr{A}$ where $(X, \mathscr{A}, \mu)$ is a measure space, define $A_k = \phi$ for $k \ge n$. We get by countable additivity of μ, that

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = \sum_{k=1}^{n} \mu(A_k)$$

since $\mu(A_K) = 0$ for $K \geq n$.

15.2.13 SAQ : Write $F_1 = E_1, F_i = E_i - E_{i-1}$ for $i > 1$. Then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$ and the sets F_i are measurable and disjoint.

Hence, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(F_i) = \lim_n \sum_{i=1}^{n} \mu(F_i) = \lim \mu\left(\bigcup_{i=1}^{n} F_i\right)$

$$= \lim \mu\left(\bigcup_{i=1}^{n} E_i\right) = \lim_n \mu(E_n)$$

15.3.6 : It is easy to verify that $\mathcal{B}_Y$ is a $\sigma-$ algebra of subsets of Y. Now, $\mu_Y(\phi) = \mu(\phi) = 0$ and the countable additivity of μ_Y is inherited from the countable additivity of μ.

15.4.4 : If $f(x) = -\infty$ for all x then $\{x : f(x) > \alpha\} = \phi$, for any $\alpha \in \mathbb{R}$ shows that f is measurable.

If $f(x) = +\infty$ for all x then $\{x : f(x) > \alpha\} = X$ for any $\alpha \in \mathbb{R}$ proving the measurability of f.

If $f(x) = c$ for all $x \in X$ (where $c \in \mathbb{R}$) then

$$\{x : f(x) > \alpha\} = \begin{cases} \phi & \text{if } c < \alpha \\ X & \text{if } c > \alpha \end{cases}$$

and in either case the set on the right is measurable. Therefore f is measurable.

15.4.7 : For any real number α, we have

$$\{x : (f \vee g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\} \text{ and}$$

$$\{x : (f \wedge g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cap \{x : g(x) > \alpha\}$$

Now since $\{x : f(x) > \alpha\}$ and $\{x : g(x) > \alpha\}$ are both measurable we get that the sets $\{x : (f \vee g)(x) > \alpha\}$ and $\{x : (f \wedge g)(x) > \alpha\}$ are measruable, proving that $f \vee g$ and $f \wedge g$ are measurable.

15.4.9 : Proceed as in lesson 5

15.6 SAMPLE EXAMINATION QUESTIONS :

(1) If $(X, \mathscr{A}, \mu)$ is a measure space and $\{A_n\}$ is a sequence in $\mathscr{A}$ such that $A_{n+1} \subseteq A_n$ for $n = 1, 2, 3, \ldots\ldots$ with $\mu(A_1) < \infty$ then show that $\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$.

(2) If f and g are measurable functions and c is any constant prove that $f + c,\ c f,\ f + g,\ f^2,\ fg$ are also measurable.

15.7 EXERCISES :

1. Show that 15.2.12 will hold if $\mu(E_i)$ is finite for some i, and the result will not hold generally in the absence of such a finiteness condition.

2. (a) Let $(X, \mathscr{B})$ be a measurable space (a) If μ and υ are measures defined on $\mathscr{B}$ then the set function λ defined on $\mathscr{B}$ by $\lambda E = \mu E + \upsilon E$ is also a measure we denote λ by $\mu + \upsilon$.

   (b) If μ and υ are measures on $\mathscr{B}$ and $\mu \geq \upsilon$ then there is a measure λ on $\mathscr{B}$ such that $\mu = \upsilon + \lambda$.

   (c) If υ is $\sigma-$ finite, the measure λ in (b) is unique.

3. If $\{A_i\}$ is a sequence of sets from $\mathscr{B}$. Prove that $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right)$

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Lesson - 16

INTEGRATION WITH RESPECT TO AN ABSTRACT MEASURE

16.1 INTRODUCTION

We attempt to generalize the concept of Lebesgue integral to the integral with respect to an abstract measure. The striking feature of the Lebesgue measure is that its completeness - which an abstract measure may fail to possess. Therefore we begin the integration with respect to a complete measure. We shall proceed with the process of integration, when $\mathbb{R}$ is replaced by any set X, the σ – algebra of Lebesgue measurable sets by a σ – algebra $\mathscr{A}$ of subsets of X and m the Lebesgue measure by a measure μ on $\mathscr{A}$. We shall define the integral for the class of functions of the type $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$, which serve as building blocks for our integral. Then keeping in mind the limiting property the integral should have, we will extend it to a larger class of functions called measurable functions. In the sequel $(X, \mathscr{A}, \mu)$ stands for a complete measure space unless otherwise mentioned.

16.2 INTEGRAL OF A SIMPLE FUNCTION

16.2.1 Definition : Suppose E is a measurable set and ϕ is a non-negative simple function given by

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$$ for any x, we define the integral of ϕ with respect to μ, denoted by $\int_E \phi \, d\mu$ by $\int_E \phi \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i \cap E)$.

16.2.2 Theorem : If α and β are non-negative numbers ϕ and ψ are simple functions then

$$\int_E (\alpha\phi + \beta\psi) \, d\mu = \alpha \int_E \phi \, d\mu + \beta \int_E \psi \, d\mu$$

Proof : Suppose $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ and $\psi(x) = \sum_{j=1}^{m} b_j \chi_{B_j}(x)$ for every x. Then, $\alpha\phi + \beta\psi$

takes the values $\alpha a_i + \beta b_j \ (1 \le i \le n, 1 \le j \le m)$ on $E_{ij} = A_i \cap B_j$ and hence,

$$(\alpha\phi + \beta\psi)(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha a_i + \beta b_j) \chi_{E_{ij}}(x).$$

Therefore,

$$\int_E (\alpha\phi + \beta\psi)\, d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha a_i + \beta b_j) \mu(E_{ij} \cap E)$$

$$\int_E (\alpha\phi + \beta\psi)\, d\mu = \alpha \sum_{i=1}^{n} \sum_{j=1}^{m} a_i\, \mu(E_{ij} \cap E) + \beta \sum_{i=1}^{n} \sum_{j=1}^{m} b_j\, \mu(E_{ij} \cap E) \text{ ------------ (1)}$$

Now, $\bigcup_{j=1}^{m} (E_{ij} \cap E) = \bigcup_{j=1}^{m} (A_i \cap B_j \cap E) = (A_i \cap E) \cap \left(\bigcup_{j=1}^{m} B_j \right)$

$= (A_i \cap E) \cap X = A_i \cap (E \cap X) = A_i \cap E$ and

$(E_{ij} \cap E) \cap (E_{ik} \cap E) = \phi$ for $j \ne k$ imply that

$\mu(A_i \cap E) = \sum_{j=1}^{m} \mu(E_{ij} \cap E)$, by the countable additivity of μ. Therefore the first term on the right of (1) is

$$\alpha \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} \mu(E_{ij} \cap E) \right) = \alpha \sum_{i=1}^{n} a_i\, \mu(A_i \cap E) = \alpha \int_E \phi\, d\mu$$

Similarly we can show that the second term on the right of (1) is $\beta \int_E \psi\, d\mu$. Hence the theorem is proved.

16.3 INTEGRAL OF A NON-NEGATIVE MEASURABLE FUNCTION

We use the integral of a simple function to define the integral of a non-negative measurable function.

16.3.1 Definition : Suppose f is a non-negative extended real-valued measurable function on the measure space $(X, \mathscr{A}, \mu)$ and $E \in \mathscr{A}$. The integral of f with respect to μ, denoted by

$\int_E f\,d\mu$ is defined as the supremum of the integrals $\int_E \phi\,d\mu$ where ϕ ranges over all simple functions satisfying $0 \le \phi \le f$.

That is, $\int_E f\,d\mu = \sup_{0\le\phi\le f} \int_E \phi\,d\mu$.

Note : The supremum may be $+\infty$. Hence the integral can be $+\infty$.

16.3.2 Self Assessment Question : If f and g are non-negative measurable functions on X such that $f \ge g$ on X then prove that

$$\int_E f\,d\mu \ge \int_E g\,d\mu \text{ and } \int_E cf\,d\mu = c\int_E f\,d\mu \text{ if } c \ge 0.$$

To prove some other linearlity (of addition) properties we need a few convergence theorems.

16.3.3 Theorem (Fatou's lemma) : Let $\{f_n\}$ be a sequence of non-negative measurable functions that converge almost every where on a set to a function f. Then

$$\int_E f \le \underline{\lim} \int_E f_n$$

Proof : First we recall that convergence a.e. on E means convergence, pointwise on E except on a set of measure zero. Let $F \subseteq E$, $F \in \mathscr{A}$, $\mu(F)=0$ be such that $f_n(x) \to f(x)$ for all $x \in E \setminus F$.

Then $\int_E f\,d\mu = \int_{E-F} f\,d\mu + \int_F f\,d\mu$

Let ϕ be a simple function such that $0 \le \phi \le f$ and let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$. Now for $x \in A_i$,

$f(x) \ge a_i$. $F \cap A_i \in \mathscr{A}$ and $\mu(F \cap A_i) = 0$. Hence, $\sum_{i=1}^{n} a_i\,\mu(F \cap A_i) = 0$. Thus,

$\int_F f\,d\mu = \sup\{0\} = 0$. Similar argument yields that $\int_F f_n\,d\mu = 0$ for each n. This shows that in the inequality, the integrals need be taken over $E \setminus F$ alone, not necesarily on the whole of E. Thus we can assume without lossof generality that $f_n \to f$ on the whole of E. Also in view of Definition 16.3.1, it is enough to prove that for each simple function with $0 \le \phi \le f$ the inequality

$$\int_E \phi\,d\mu \le \underline{\lim}_n \int_E f_n\,d\mu \text{ -------------- (1) holds}$$

Let ϕ be a simple function such that $0 \le \phi \le f$ and $\phi = \sum_{i=1}^{n} c_i \chi_{E_i}$

Case 1 : $\int_E \phi \, d\mu = \infty$

In this case there is a measurable subset A of E such that $\mu(A) = \infty$ and $\phi(x) > 0$ for every $x \in A$; Since $\int_E \phi \, d\mu = \infty$ we have by the definition $\sum_{i=1}^{n} c_i \mu(E \cap E_i) = \infty$; hence, $c_i \mu(E \cap E_i) = \infty$ for some i. Now $c_i > 0$ and hence $\mu(E \cap E_i) = \infty$. Put, $A = E \cap E_i$. Clearly A is a measurable subset of E and $\mu(A) = \infty$. For $x \in A$, $\phi(x) = c_i > \frac{c_i}{2} = a(\text{say}) > 0$. Thus there is a positive number a such that $0 < a < \phi(x)$ for all $x \in A$. We define for each n,

$A_n = \{x \in E : f_k(x) > a \text{ for all } k \ge n\}$. Then $\{A_n\}$ is an increasing sequence of measurable sets. Therefore $\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$ --------------- (2)

But, since $\phi \le f = \lim_n f_n$ we get $A \subseteq \bigcup_{n=1}^{\infty} A_n$. In fact if $x \in A$ then, $\lim_n f_n(x) \ge \phi(x) > a$ gives $f_n(x) > a$ for all $n \ge n_0$ where n_0 is some integer, so that $x \in A_{n_0}$ showing $x \in \bigcup_{n=1}^{\infty} A_n$.

Therefore, $\mu(A) \le \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$ which gives, $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \infty$. Hene, by (2), $\lim_{n \to \infty} \mu(A_n) = \infty$.

Now, since $\int_E f_n \, d\mu \ge \int_{A_n} f_n \, d\mu > a \int_{A_n} d\mu = a \mu(A_n)$.

$$> a \, \underline{\lim_n} \, \mu(A_n) = \infty$$

We get, $\underline{\lim}_{n \to \infty} \int_E f_n \, d\mu$ so that $\underline{\lim}_n \int_E f_n \, d\mu = \infty$. Therefore, $\int_E \phi \, d\mu = \underline{\lim}_n \int_E f_n \, d\mu$ in this case. So (1) holds.

Case (ii) : Suppose $\int\limits_E \phi \, d\mu < \infty$

Now $\int\limits_E \phi \, d\mu = \sum\limits_{i=1}^{n} c_i \, \mu(E \cap E_i) < \infty$ implies $c_i \, \mu(E \cap E_i) < \infty$ for all i, Now $c_i \neq 0$ gives $\mu(E \cap E_i) < \infty$. If we take $A = E \cap \left(\bigcup\limits_{i=1}^{n} E_i \right)$ then A is a measurable subset of E, $\phi(x) = 0$ for every $x \notin A$ and $\mu(A) = \mu\left(\bigcup\limits_{n=1}^{n} E \cap E_i \right) \leq \sum\limits_{i=1}^{n} \mu(E \cap E_i) < \infty$. Thus if $\int\limits_E \phi < \infty$,, then the set $A = \{x \in E : \phi(x) > 0\}$ is a measurable set of finite measure. Let M be the maximum of ϕ, that is $\phi(x) \leq M$ for every $x \in E$ and if $0 < \epsilon < 1$ write $A_n = \{x \in E : f_k(x) > (1-\epsilon)\phi(x) \text{ for every } k \geq n\}$.

Then the sets A_n are measurable, $A_n \subseteq A_{n+1}$ for each n and $\bigcup\limits_{n=1}^{\infty} A_n \supseteq A$. Therefore, $\{A - A_n\}$ is a decreasing sequence of sets, $\bigcap\limits_{n=1}^{\infty} (A - A_n) = \phi$. Since $\mu(A) < \infty$ we have by theorem 15.2.12

$$\mu\left(\bigcap_{n=1}^{\infty} (A - A_n) \right) = \lim_n \mu(A \setminus A_n).$$

But $\mu\left(\bigcap\limits_{n=1}^{\infty} A - A_n \right) = \mu(\phi) = 0$. Thus, $\lim\limits_n \mu(A \setminus A_n) = 0$. Hence there exists a +ve integer N such that $\mu(A \setminus A_n) < \epsilon$ for every $n \geq N$

Thus for $n \geq N$,

$$\int_E f_n \, d\mu \geq \int_{A_n} f_n \, d\mu > \int_{A_n} (1-\epsilon)\phi \, d\mu$$

$$> (1-\epsilon) \int_E \phi \, d\mu - \int_{A-A_n} \phi \, d\mu$$

$$\geq \int_E \phi \, d\mu - \epsilon \int_E \phi \, d\mu - M\mu(A \setminus A_n)$$

$$\geq \int_E \phi \, d\mu - \in \int_E \phi \, d\mu \; - M \in \text{(since, } \mu(A \setminus A_n) < \in \text{)}$$

$$\geq \int_E \phi \, d\mu - \in \left(M + \int_E \phi \, d\mu \right).$$

Since, $M + \int_E \phi \, d\mu$ is a finite number, we get

$$\int_E f_n \, d\mu > \int_E \phi \, d\mu - \in \text{ for all } n \geq N.$$

Proving $\underline{\lim}_n \int_E f_n \, d\mu \geq \int_E \phi \, d\mu$. Thus (1) is proved in this case.

16.3.4 Monotone Convergence Theorem : Let $\{f_n\}$ be a sequence of measruable functions which converge almost every where to a function f and suppose that $f_n \leq f$ for every n. Then

$$\int f = \lim_n \int f_n$$

Proof : Since $f_n \leq f$ for all n, we have,

$$\int_E f_n \, d\mu \leq \int_E f \, d\mu$$

so that $\overline{\lim}_n \int f_n \, d\mu \leq \int f \, d\mu$' ---------- (1)

By Fatou's lemma,

$$\int f \, d\mu \leq \underline{\lim}_n \int f_n \, d\mu \text{ ----------- (2)}$$

Now from (1) and (2) the theorem follows.

16.3.5 Theorem :

(i) If f and g are non-negative measurable functions and α and β are non-negative real numbers then

$$\int_E (\alpha f + \beta g) \, d\mu = \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu$$

(ii) If f is a non-negative measurable function then $\int_E f \, d\mu \geq 0$ with equality if and only if $f = 0$ a.e.

Proof : To prove (i) let $\{\phi_n\}$ and $\{\psi_n\}$ be increasing sequences of simple functions which converge to f and g. Then $\{\alpha\phi_n+\beta\psi_n\}$ is an increasing sequence of non-negative simple functions converging to $\alpha f+\beta g$ and therefore by the Monotone convergence theorem we get

$$\int_E(\alpha f+\beta g)\,d\mu=\lim_{n\to\infty}\int_E(\alpha\phi_n+\beta\psi_n)\,d\mu$$

$$=\lim_{n\to\infty}\left\{\alpha\int_E\phi_n\,d\mu+\beta\int_E\psi_n\,d\mu\right\}$$

$$=\alpha\lim_{n\to\infty}\int_E\phi_n\,d\mu+\beta\lim_{n\to\infty}\int_E\psi_n\,d\mu$$

$$=\alpha.\int_E f\,d\mu+\beta\int_E g\,d\mu$$

(ii) Obviously, $\int_E f\,d\mu\ge 0$, since $f\ge 0$. Now if $\int_E f\,d\mu=0$, let $A_n=\left\{x\in E: f(x)\ge\frac{1}{n}\right\}$ for $n=1,2,3,\ldots\ldots.$ Then A_n is measurable and $f\ge\frac{1}{n}\chi_{A_n}$ so that $0=\int_E f\,d\mu\ge\frac{1}{n}\mu(A_n)$, proving $\mu(A_n)=0$ for each n. Therefore, if $A=\{x\in E: f(x)>0\}$ then $A=\bigcup_{n=1}^{\infty}A_n$, so that

$\mu(A)=\mu\left(\bigcup_{n=1}^{\infty}A_n\right)\le\sum_{n=1}^{\infty}\mu(A_n)=0$ giving $\mu(A)=0$. Thus, $f=0$ a.e. on E.

16.3.6 Corollary : Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. Then

$$\int_E\sum_{n=1}^{\infty}f_n=\sum_{n=1}^{\infty}\int_E f_n.$$

Proof : Let $u_n=\sum_{k=1}^{n}f_k$ for $n=1,2,3,\ldots\ldots\ldots$ Then $\{u_n\}$ is a sequence of non-negative measurable functions with $\lim_{n\to\infty}s_n=\sum_{n=1}^{\infty}f_n$. Therefore, by monotone convergence theorem,

$$\int_E \left(\sum_{n=1}^{\infty} f_n \right) d\mu = \lim_{n\to\infty} \int_E u_n \, d\mu$$

But $\int_E u_n \, d\mu = \int_E \left(\sum_{k=1}^{n} f_k \right) d\mu = \sum_{k=1}^{n} \int_E f_k \, d\mu$

so that $\lim_{n\to\infty} \int_E u_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu$, proving the result.

16.3.7 Definition : A non-negative function f is said to be integrable over a measurable set E with respect to μ if it is measurable and

$$\int_E f d\mu < \infty.$$

We now have a new result which shows how integrals can be used to construct new measures with a special continuity property.

16.3.8 Theorem : Let $(X, \mathscr{A}, \mu)$ be a measure space and f a non-negative measurable function. Then $\phi(E) = \int_E f \, d\mu$ is a measure on the measurable space $(X, \mathscr{A})$. If in addition, $\int f d\mu < \infty$ then for every $\in > 0$, there exists $\delta > 0$ such that, if $A \in \mathscr{A}$ and $\mu(A) < \delta$, then $\phi(A) < \in$.

Proof : Clearly $\phi(\phi) = \int_\phi f \, d\mu = 0$. Since the integral of a non-negative measurable function is non-negative, ϕ is a non-negative set function. If $\{E_n\}$ is a sequence of disjoint sets of $\mathscr{A}$,

$$\phi\left(\bigcup_{n=1}^{\infty} E_n \right) = \int_{\bigcup_{n=1}^{\infty} E_n} f \, d\mu = \sum_{n=1}^{\infty} \int \chi_{E_n} f \, d\mu \qquad \text{(by 16.3.6)}$$

$$= \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu = \sum_{n=1}^{\infty} \phi(E_n)$$

Thus ϕ is a measure on the measurable space $(X, \mathscr{A})$. Write $f_n = \min(f, n)$. Then f_n is measurable, $f_n \uparrow f$ and $\lim_n \int f_n \, d\mu = \int f \, d\mu$ by Theorem 16.3.4, s if $\int f \, d\mu < \infty$ i.e. f is integrable on $(X, \mathscr{A}, \mu)$, then for each $\in > 0$ there exists N such that

$$\int f\, d\mu < \int f_N\, d\mu + \in/2$$

If $A \in \mathscr{A}$ and $\mu(A) < \in/2N$ we have

$\int_A f_N\, d\mu < \in/2$. So take $\delta = \in/2N$ to get

$$\int_A f\, d\mu = \int_A (f - f_N)\, d\mu + \int_A f_N\, d\mu$$

$$\leq \int (f - f_N)\, d\mu + \in/2 < \in.$$

16.4 INTEGRAL OF AN ARBITRARY MEASURABLE FUNCTION

We now intrioudce the notion of integrability with respect to μ of a general measurable function f defined on a measure space $(X, \mathscr{A}, \mu)$.

16.4.1 Definition : An arbitrary function f is said to be integrable if both f^+ and f^- are integrable. In this case we define

$$\int_E f = \int_E f^+ - \int_E f^-$$

Some of the properties of the integral are contained in the following proposition.

16.4.2 Proposition : If f and g are integrable functions and E is a measurable set, then

(i) $\int_E (c_1 f + c_2 g)\, d\mu = c_1 \int_E f\, d\mu + c_2 \int_E g\, d\mu$

(ii) If $|h| \leq |f|$ and h is measurable then h is integrable.

(iii) If $f \geq g$ a.e. then $\int f \geq \int g$

Proof : We first prove the following

$$\int_E (c f)\, d\mu = c \int_E f\, d\mu \text{ ---------------- (1)}$$

and $$\int_E (f + g)\, d\mu = \int_E f\, d\mu + \int_E g\, d\mu \text{ ---------------- (2)}$$

where c is any real number and f, g are as in the hypothesis. First note that

$$(cf)^+ = \begin{cases} cf^+ \text{ if } c \geq 0 \\ -cf^- \text{ if } c < 0 \end{cases}$$

$$(cf)^- = \begin{cases} cf^- \text{ if } c \geq 0 \\ -c\, f^+ \text{ if } c < 0 \end{cases}$$

Therefore, when $c \geq 0$, we have by Theorem 16.3.5, that

$$\int_E (cf)\, d\mu = \int_E (cf)^+ d\mu - \int_E (cf)^- d\mu$$

$$= \int_E cf^+ d\mu - \int_E cf^- d\mu$$

$$= c \int_E f^+ d\mu - c \int_E f^- d\mu$$

$$= c\left(\int_E f^+ d\mu - \int_E f^- d\mu \right)$$

$$= c \int_E f\, d\mu$$

Again if $c < 0$

$$\int_E (cf)\, d\mu = \int_E (cf)^+ d\mu - \int_E (cf)^- d\mu$$

$$= \int_E -cf^- d\mu - \int_E -cf^+ d\mu$$

$$= (-c) \int_E f^- d\mu - (-c) \int_E f^+ d\mu$$

$$= c\left(\int_E f^+ d\mu - \int_E f^- d\mu \right)$$

$$= c \int_E f\, d\mu$$

Thus (1) is proved.

First note that if $f = f_1 - f_2$ where f_1 and f_2 are non-negative then

$$\int_E f\,d\mu = \int_E f_1\,d\mu - \int_E f_2\,d\mu \text{ ---------------- (3)}$$

Now, $f+g = \left(f^+ + g^+\right) - \left(f^- + g^-\right)$, where

$f^+ + g^+ \geq 0$ and $f^- + g^- \geq 0$, so that by (3)

$$\int_E (f+g)\,d\mu = \int_E \left(f^+ + g^+\right)d\mu - \int_E \left(f^- + g^-\right)d\mu$$

and now using Theorem 16.3.5, we get

$$\int_E (f+g)\,d\mu = \left(\int_E f^+\,d\mu + \int_E g^+\,d\mu\right) - \left(\int_E f^-\,d\mu + \int_E g^-\,d\mu\right)$$

$$= \left(\int_E f^+\,d\mu - \int_E f^-\,d\mu\right) + \left(\int_E g^+\,d\mu - \int_E g^-\,d\mu\right)$$

$$= \int_E f\,d\mu + \int_E g\,d\mu$$

This proves (2). Now consider

$$\int_E (\alpha f + \beta g)\,d\mu = \int_E (\alpha f)\,d\mu + \int_E (\beta g)\,d\mu$$

$$= \alpha \int_E f\,d\mu + \beta \int_E g\,d\mu$$

by (2) and (1) established in the thoerem.

(ii) If h is measurable and $|h| \leq |f|$ then

$$\int_E |h|\,d\mu \leq \int_E |f|\,d\mu < \infty$$

shows $|h|$ and hence h is integrable.

(iii) Let $f \geq g$ a.e. then $f - g \geq 0$ a.e. so that by Theorem 16.3.5 (ii), $\int_E (f-g)\,d\mu \geq 0$. Now by proposition 16.4.2, this gives $\int_E f\,d\mu - \int_E g\,d\mu \geq 0$, proving the result.

16.4.3 Self Assessment Question : If f is measurable on E then show that f is integrable on E if and only if $|f|$ is integrable on E and in this case.

$$\left|\int_E f\,d\mu\right| \leq \int_E |f|\,d\mu$$

16.4.4 Self Assesment Question : If $f \geq 0$ is integrable over $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ are pairwise disjoint measurable sets prove that

$$\int_E f\,d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\,d\mu$$

16.4.5 Lebersgue Convergence Theorem : Let g be integrable over E, and suppose that $\{f_n\}$ is a sequence of measurable functions such that on E

$$|f_n(x)| \leq g(x)$$

and such that almost every where on E

$$f_n(x) \to f(x).$$

Then $\int_E f = \lim_{n\to\infty} \int_E f_n$

Proof : Since $|f_n(x)| \leq g(x)$ on E we have, $-g(x) \leq f_n(x) \leq g(x)$ for all $x \in E$ so that $\{g+f_n\}$ and $\{g-f_n\}$ are sequences of non-negative measurable functions respectively converging to $g+f$ and $g-f$ almost every where on E. Therefore, by Fatou's lemma, we get

$$\int_E (g+f)\,d\mu \leq \underline{\lim}_n \int_E (g+f_n)\,d\mu$$

and $\int_E (g-f)\,d\mu \leq \underline{\lim}_n \int_E (g-f_n)\,d\mu$

Which can be written by Theorem 16.3.5, as

$$\int_E g\,d\mu + \int_E f\,d\mu \le \int_E g\,d\mu + \underline{\lim}_n \int_E f_n\,d\mu$$

and $$\int_E g\,d\mu - \int_E f\,d\mu \le \int_E g\,d\mu - \overline{\lim}_n \int_E f_n\,d\mu$$

Since g is integrable, $\int_E g\,d\mu < \infty$, so that the above inequalities give

$$\int_E f\,d\mu \le \underline{\lim}_n \int_E f_n\,d\mu$$

and $$\int_E f\,d\mu \ge \overline{\lim}_n \int_E f_n\,d\mu$$

from which the theorem follows.

16.5 ANSWERS TO SAQs

16.3.2 : If f and g are non-negative measurable functions on X such that $f \ge g$ on X, then any simple function $\phi \le g$ also satisfies $\phi \le f$. Hence

$$\sup\left\{ \int_E \phi : 0 \le \phi \le g \right\} \le \sup\left\{ \int_E \phi : 0 \le \phi \le f \right\}$$

That is, $\int_E g\,d\mu \le \int_E f\,d\mu$.

Again if $c \ge 0$, f is a non-negative measurable function on X, then

$\int_E c\,f\,d\mu = c \int_E f\,d\mu$ is very obvious.

16.4.3 : Let f be measurable on E.

If f is integrable then f^+ and f^- are both integrable by definition, so that $f^+ + f^- = |f|$ is also integrable. If $|f|$ is integrable then since $f^+ \le |f|$ and $f^- \le |f|$, we get f^+ and f^- are both integrable so that $f^+ - f^- = f$ is integrable.

Since $-|f| \leq f \leq |f|$ we get

$$-\int_E |f| \, d\mu \leq \int_E f \, d\mu \leq \int_E |f| \, d\mu$$ proving the inequality.

16.4.4 : Give $f \geq 0$ measurable function and $E = \bigcup_{n=1}^{\infty} E_n$ where $\{E_n\}$ is a sequence of disjoint measurable sets.

Than $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$

$$= \int_X f \left(\sum_{n=1}^{\infty} \chi_{E_n} \right) d\mu$$

$$= \int_X \sum_{n=1}^{\infty} \left(f \chi_{E_n} \right) d\mu$$

$$= \sum_{n=1}^{\infty} \left(\int_X f \chi_{E_n} \right) d\mu$$

$$= \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu$$

Proving the result.

16.6 MODEL EXAMINATION QUESTIONS

16.6.1 : Define the integral of a non-negative measurable function f with respect to a measure μ. State and prove Fatou's lemma.

16.6.2 : State and prove Lebesgue convergence theorem.

16.7 EXERCISES

16.7.1 : Show that if f is integrable, then the set $\{x : f(x) \neq 0\}$ is of $\sigma -$ finite measure.

16.7.2 : Let f be integrable, then $\left|\int f\, d\mu\right| \leq \int |f|\, d\mu$ with equality if, and only if, $f \geq 0$ a.e. or $f \leq 0$ a.e.

16.7.3 : (a) Let $(X, \mathscr{B}, \mu)$ be a measure space and g a non-negative measurable function on X. Let $\upsilon(E) = \int_E g\, d\mu$. Show that υ is a measure on $\mathscr{B}$.

(b) Let f be a non-negative measurable function on X. Then

$$\int f\, d\upsilon = \int f\, g\, d\mu.$$

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Lesson - 17

SIGNED MEASURES

17.1 INTRODUCTION

The aim of this lesson is to discuss the properties of set functions which are countably additive but are not necessarily non-negative or even real-valued. Such set functions arise naturally. For example, if we consider a linear combination of finite measures, it need not be a measure, i.e. it need not be non-negative (of course, it will be countably additive). Another way in which such set functions can arise is when we integrate an integrable function : $\upsilon(E) = \int_E f\, d\mu$.

We have seen in theorem 16.3.8 that if f is a non-negative measurable function on the measure space $(X, \mathscr{A}, \mu)$ then the set function ϕ defined on $\mathscr{A}$ by $\phi(E) = \int_E f\, d\mu$ is a measure. If f is any measurable function whose integral with respect to μ exists, then $\upsilon(E) = \int_E f\, d\mu$ is a set function on $\mathscr{A}$ which is countably additive and which behaves in most respects like a measure. This suggests extending the definition of a measure to allow negative values. This is done in Definition 17.2.1 The Hahn and Jordan decomposition show how in the study of such measures we may keep to the non-negative measures already discussed. We will prove that the set X on which signed measure is defined can be partitioned by means of the measure (Theorem 17.3.4) and each such measure can be written as the difference of two non-negative measures (Theorem 17.4.3).

17.2 SIGNED MEASURES

When you allow a measure to have both positive and negative values you are likely to get a $\infty - \infty$ situation. Hence, we should take care of such situation while defining the signed measure.

17.2.1 Definition : A set function υ defined on a measurable space $(X, \mathscr{A})$ is said to be a signed measure if the values of υ are extended real numbers and

(i) υ takes atmost one of the values ∞ and $-\infty$

(ii) $\upsilon(\phi) = 0$ and

(iii) $\upsilon\left(\bigcup_{n=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \upsilon(E_i)$ if $E_i \cap E_j = \phi$ for $i \neq j$, where the equality is taken to mean that the series on the right converges absolutely if $\upsilon(\bigcup E_i)$ is finite and that it diverges to $+\infty$ or $-\infty$ as the case may be.

Note (i) : In the above definition means that if $\upsilon(A)=+\infty$ for some $A \in \mathscr{A}$ then for no $B \in \mathscr{A}$, $\upsilon(B)=-\infty$ (if $\upsilon(A)=-\infty$ for some $A \in \mathscr{A}$ then for no $B \in \mathscr{A}$ $\upsilon(B)=+\infty$).

17.2.2 Example : Every measure is a signed measure, since it never takes the value $-\infty$, it takes the value 0 at ϕ and it is countably additive.

The converse is not true, for example, if μ is a measure on a measurable space $(X, \mathscr{A})$ and if we define $\upsilon(E) = -\mu(E) \forall\, E \in \mathscr{A}$, then υ is a signed measure but not a measure.

17.2.3 Example : If υ is a finite measure and μ is a measure on the measurable space $(X, \mathscr{A})$ then for any real number α the set function $\upsilon - \alpha\mu$ is a signed measure. In fact if $\mu(E)=\infty$ for some E then $(\upsilon-\alpha\mu)(E) = -\infty$ or $+\infty$ according as $\alpha \geq 0$ or $\alpha < 0$; $(\upsilon-\alpha\mu)(\phi)=0$ and

$$(\upsilon-\alpha\mu)\left(\bigcup_{n=1}^{\infty} E_n\right) = \upsilon\left(\bigcup_{n=1}^{\infty} E_n\right) - \alpha\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} (\upsilon-\alpha\mu)(E_n)$$ whenever $\{E_n\}$ are pair wise disjoint measurable sets.

17.2.4 Example : Let $(X, \mathscr{A}, \mu)$ be a measure space and f be any measurable function on $\mathscr{A}$. Define, $\phi(E) = \int_E f\, d\mu$ where $\int f\, d\mu$ is defined, then ϕ is a signed measure. We have either $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$ so (i) of Definition 17.2.1 follows. (ii) is trivial (since $\phi(\phi) = \int_\phi f\, d\mu = 0$). Suppose $\{E_i\}$ is a pairwise disjoint sequence in $\mathscr{A}$ and for $E \in \mathscr{A}$ write

$\phi^+(E) = \int_E f^+ d\mu,\ \phi^-(E) = \int_E f^- d\mu$, so that by Theorem 16.3.8 ϕ^+ and ϕ^- are measures.

$$\text{Then, } \phi\left(\bigcup_{i=1}^{\infty} E_i\right) = \phi^+\left(\bigcup_{i=1}^{\infty} E_i\right) - \phi^-\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{\infty} \phi^+(E_i) - \sum_{i=1}^{\infty} \phi^-(E_i) = \sum_{i=1}^{\infty} \phi(E_i)$$

as we cannot get $\infty - \infty$ at any stage.

17.2.5 Self Assessment Question : If μ_1 and μ_2 are measures on $(X, \mathscr{A})$ such that atleast one of them is finite then show that $\upsilon(E) = \mu_1(E) - \mu_2(E) \forall E \in \mathscr{A}$ is a signed measure on $\mathscr{A}$

We shall show that the method described in example 17.2.5 is the only way of constructing signed measures (see theorem 17.3)

17.2.6 Self Assessment Question : Let υ be a signed measure on $(X, \mathscr{A})$. Then prove the following :

(i) If $A, B \in \mathscr{A}$ and $A \cap B = \phi$ then $v(A \cup B) = \upsilon(A) + \upsilon(B)$.

(ii) If $A \in \mathscr{A}$ with $|\upsilon(A)| < \infty$ and $B \in \mathscr{A}$ with $B \subseteq A$ then $|\upsilon(B)| < +\infty$ and $\upsilon(A \setminus B) = \upsilon(A) - \upsilon(B)$.

(iii) υ is finite iff $|\upsilon(A)| < +\infty \ \forall A \in \mathscr{A}$

17.3 SETS ASSOCIATED WITH A SIGNED MEASURE

If υ is a signed measure, it is difficult to handle it as it is. We wish to describe it interms of non-negative measures and use the knowlede of such measures in studying signed measures. A step in this direction is to classify subsets of X in relation to υ.

17.3.1 Definition : Suppose υ is a signed measure on a measurable space $(X, \mathscr{A})$. A set $E \in \mathscr{A}$ is said to be a

(i) Positive set with respect to υ if $\upsilon(A) \geq 0$ for every measurable subset A of E.

(ii) Negative set with respect to υ if $\upsilon(A) \leq 0$ for every measurable subset A of E.

(iii) Null set with respect to υ or a υ - null set if it is both a positive and a negative set with respect to υ.

Clearly A is a negative set with respect to υ it is a positive set with respect to - υ.

For example the empty set is a positive set with respect to any signed measure. It may be noted that E is a υ - null set if and only if $E \in \mathscr{A}$ and $\upsilon(A) = 0$ for all $A \in \mathscr{A}$ with $A \subseteq E$. The reader should carefully note the distinction between a null set and a set of measure zero : While every null set must have measure zero, a set of measure zero may well be a union of two sets whose measures are not zero but are negatives of each other.

We have the following Lemmas concerning positive sets. Similar statements hold, of course, for negative sets.

17.3.2 Lemma : Every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive.

Proof : The first statement is trivially true by the definition of a positive set. To prove the second statement, let A be the union of a sequence $\{A_n\}$ of positive sets. If E is any measurable subset of A, write

$$E_n = E \cap A_n \cap \tilde{A}_{n-1} \cap .... \tilde{A}_1$$

Then E_n is a measurable subset of A_n and so $\upsilon(E_n) \geq 0$. Since the E_n are disjoint and $E = \bigcup E_n$, we have,

$$\upsilon(E) = \sum_{n=1}^{\infty} \upsilon(E_n) \geq 0$$

Thus A is a positive set.

17.3.3 Lemma : Let E be a measurable set such that $0 < \upsilon(E) < \infty$. Then there is a positive set A contained in E with $\upsilon(A) > 0$.

Proof : If E contains no set of negative υ - measure then E is a positive set and $A = E$ gives the result. If E is not a positive set then there exists a measurable subset. B of E and $\upsilon(B) < 0$. Then we can find a natural number n wi h $\upsilon(B) < -\frac{1}{n}$. Let n_1 be the smallest such integer and E_1 a corresponding measurable subset of E with $\upsilon(E_1) < -\frac{1}{n_1}$; we now consider $E \setminus E_1$; if this is not a positive set, as earlier we find the smallest positive integer n_2 such that there is a measurable set $E_2 \subseteq E \setminus E_1$ such that $\upsilon(E_2) < -\frac{1}{n_2}$. Continue this process.

Having chosen $n_1, n_2, ..., n_{k-1}$ and measurable subsets $E_1, E_2, ..., E_{k-1}$, we choose the smallest positive integer $n_k > n_{k-1}$ and a measurable subset E_k of $E - \bigcup_{j=1}^{k-1} E_j$ with $\upsilon(E_k) < -\frac{1}{n_k}$.

If the process stops at n_k, say, then $A=E-\bigcup_{j=1}^{k} E_j$ is a positive set. Also, $\upsilon(A)>0$. In fact if $\upsilon(A)=0$ then, $\upsilon(E)=\upsilon\left(\bigcup_{j=1}^{k} E_j\right)=\sum_{j=1}^{k}\upsilon(E_j)<0$ a contradiction to the hypothesis. Then A is the required set in this case.

If the process continues indefinitely, we shall show that $A=E-\bigcup_{k=1}^{\infty} E_k$ is a positive set satisfying the inequality $\upsilon(A)>0$ and $A\subseteq E$.

Now, $E=A\cup\left(\bigcup_{k=1}^{\infty} E_k\right)$ and these sets are pairwise disjoint. Thus we have,

$$\upsilon(E)=\upsilon(A)+\upsilon\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$=\upsilon(A)+\sum_{k=1}^{\infty}\upsilon(E_k) \text{ -------------- (1)}$$

Since $\upsilon(E)<\infty$, the series on the right hand side of the above equality is absolutely convergent. Hence $\sum_{k=1}^{\infty}\frac{1}{n_k}$ is convergent and we have $n_k\to\infty$ as $k\to\infty$. $\exists$ a +ve integer $k_0 \ni n_k>1$ for $k>k_0$. If $B\in\mathscr{A}$, $B\subseteq A$ and $k>k_0$ then $B\subseteq E-\bigcup_{j=1}^{k} E_j$ so that $\upsilon(B)\geq-\frac{1}{(n_k-1)}$, by the definition of n_k. Since this inequality holds for all $k>k_0$, so letting $k\to\infty$ we have $\upsilon(B)\geq 0$ and so A is a positive set. Also since $\upsilon(E_k)\leq 0\,\forall\, k$ and $\upsilon(E)>0$, we have $\upsilon(A)>0$ as required.

17.3.4 Hahn Decomposition Theorem : Let υ be a signed measure on the measurable space $(X,\mathscr{A})$. Then there is a positive set A and a negative set B such that $X=A\cup B$ and $A\cap B=\phi$.

Proof : Let υ be a signed measure on $(X, \mathscr{A})$. We may assume that $\upsilon(A)<+\infty$ for every $A \in \mathscr{A}$ since υ cannot assume $+\infty$ and $-\infty$ on $\mathscr{A}$. Let $X = \sup\{\upsilon(A) : A \text{ is a positive set}\}$. Since empty set is positive set we get $\lambda \geq 0$. Let $\{A_i\}$ be a sequence of positive sets such that $\lambda = \lim_{i \to \infty} \upsilon(A_i)$ and put $A = \bigcup_{i=1}^{\infty} A_i$.

By lemma 17.3.2 the set A is itself a positive set, and so $\lambda \geq \upsilon(A)$. But $A \sim A_i \subseteq A$ and so $\upsilon(A \sim A_i) \geq 0$. Thus

$$\upsilon(A) = \upsilon(A_i) + \upsilon(A \sim A_i) \geq \upsilon(A_i)$$

Hence $\upsilon(A) \geq \lambda$, and so $\upsilon(A) = \lambda$ and $\lambda < \infty$.

Let $B = \tilde{A}$, the complement of A. If E is a positive subset of B then $E \cap A = \phi$ and $E \cup A$ is a positive set so that $\lambda \geq \upsilon(E \cup A) = \upsilon(E) + \upsilon(A) = \upsilon(E) + \lambda$, hence $\upsilon(E) = 0$, since $0 \leq \lambda < \infty$. Thus B contains no positive subsets of positive measure and hence no subsets of positive measure by lemma 17.3.3 consequently B is a negative set.

17.3.5 Definition : Suppose υ is a signed measure on a measurable space $(X, \mathscr{A})$. If there is a positive set A and a negative set B such that $A \cup B = X$ and $A \cap B = \phi$ then the pair $\{A, B\}$ is called a Hahn decomposition of X with respect to υ.

Theorem 17.3.4 shows that a Hahn decomposition of X always exists that a Hahn decomposition need not be unique, follows from the next example.

17.3.6 Example : Let A and B be as in the theorem. Let $N \in \mathscr{A}$ be a υ - nullset. Then $(A \setminus N)$, $B \cup N$ is also a Hahn decomposition of X. Furhter if A_1, B_1 and A_2, B_2 are two Hahn decompositions of X with respect to υ, then $\upsilon(A_1 \Delta A_2) = \upsilon(B_1 \Delta B_2) = 0$ and for every $E \in \mathscr{A}, \upsilon(E \cap A_1) = \upsilon(E \cap A_2), \upsilon(E \cap B_1) = \upsilon(E \cap B_2)$.

17.4 THE JORDAN DECOMPOSITION

We now use the Hahn decomposition to obtain a decomposition of a signed measure into the difference of measures.

17.4.1 Definition : Two measures υ_1 and υ_2 on a measurable $(X, \mathscr{A})$ are said to be mutually singular if there exist disjoint measurable sets A and B with $X = A \cup B$, $\upsilon_1(A)=0=\upsilon_2(B)$. In this cas we write $\upsilon_1 \perp \upsilon_2$.

17.4.2 Example : Let μ be a measure and let the measures υ_1, υ_2 be given by $\upsilon_1(E) = \mu(A \cap E), \upsilon_2(E) = \mu(B \cap E)$ where $\mu(A \cap B)=0$ and $E, A, B \in \mathscr{A}$. Then $\upsilon_1 \perp \upsilon_2$ since $\upsilon_1(B) = \mu(A \cap B)=0$, $\upsilon_2(A)=\mu(B \cap A)=0$.

17.4.3 (Jordan decomposition) Theorem : For every signed measure υ defined on a measurable space $(X, \mathscr{A})$ there exists a unique pair of mutually singular measures υ^+ and υ^- on $(X, \mathscr{A})$ such that $\upsilon = \upsilon^+ - \upsilon^-$

Proof : Let $\{A, B\}$ be a Hahn decomposition of X with respect to υ and define υ^+ and υ^- by $\upsilon^+(E) = \upsilon(E \cap A), \upsilon^-(E) = -\upsilon(E \cap B)$ for every $E \in \mathscr{A}$. Then υ^+ and υ^- are measures by example 17.3.2 and $\upsilon^+(B) = \upsilon^-(A) = 0$. So $\upsilon^+ \perp \upsilon^-$. Also for $E \in \mathscr{A}$, $\upsilon(E) = \upsilon(E \cap A) + \upsilon(E \cap B) = \upsilon^+(E) - \upsilon^-(E)$. So, $\upsilon = \upsilon^+ - \upsilon^-$ and the proof will be complete when we show that the decomposition is unique. Let $\upsilon = \upsilon_1 - \upsilon_2$ is any decompsition of υ into mutually singular measures. Then there exists disjoint measurable sets C and D with $X = C \cup D$, $\upsilon_1(D)=0=\upsilon_2(C)$.

We have the following

$$E \subseteq C \text{ implies } \upsilon(E) = \upsilon_1(E) - \upsilon_2(E) = \upsilon_1(E)$$

$$E \subseteq D \text{ implies } \upsilon(E) = \upsilon_1(E) - \upsilon_2(E) = -\upsilon_2(E)$$

Also, if $E \subseteq A$ then $\upsilon(E) = \upsilon^+(E)$ and if $E \subseteq B$ then $\upsilon(E) = -\upsilon^-(E)$.

Then, $\upsilon(B \cap C) = -\upsilon^-(B \cap C) \le 0$

also $\upsilon(B \cap C) = \upsilon_1(B \cap C) \ge 0$

hence, $\upsilon(B \cap C) = 0$. Thus we have,

$\upsilon(B \cap C) = 0 = \upsilon^-(B \cap C) = \upsilon_1(B \cap C)$. Also we get

$\upsilon(A \cap D) = 0 = \upsilon^+(A \cap D) = \upsilon_2(A \cap D)$.

Now for any $E \in \mathscr{A}$,

$$\upsilon^+(E) = \upsilon^+(E \cap A) + \upsilon^+(E \cap B)$$

$$= \upsilon^+(E \cap A \cap C) + \upsilon^+(E \cap A \cap D) \quad (\because \upsilon^+(E \cap B) = 0)$$

$$= \upsilon^+(E \cap A \cap C) = \upsilon(E \cap A \cap C) \quad (\because E \cap A \cap C \subseteq A)$$

$$\upsilon_1(E) = \upsilon_1(E \cap C) + \upsilon_1(E \cap D)$$

$$= \upsilon_1(E \cap C \cap A) + \upsilon_1(E \cap C \cap B) \quad (\upsilon_1(E \cap D) = 0 \text{ since } \upsilon_1(D) = 0)$$

$$= \upsilon_1(E \cap C \cap A) \quad (\text{since } \upsilon_1(B \cap C \cap E) = 0)$$

$$= \upsilon(E \cap C \cap A)$$

$\therefore \upsilon^+(E) = \upsilon_1(E)$ hence $\upsilon^+ = \upsilon_1$ similarly we can show that $\upsilon^- = \upsilon_2$.

Thus the decomposition is unique.

17.4.4 Definition : The decomposition of υ given in the above theorem is called the Jordan decomposition of υ. The measures υ^+ and υ^- are called the positive variation and negative variation of υ.

Note that since υ assumes atmost one of the values $+\infty$ and $-\infty$, either υ^+ or υ^- is finite. If both υ^+ and υ^- are finite, υ is said to be a finite signed measure.

17.4.5 Definition : For any signed measure υ on a measurable space $(X, \mathscr{A})$, its total variation or absolute value $|\upsilon|$ is the measure defined by

$$|\upsilon|(E) = \upsilon^+(E) + \upsilon^-(E) \text{ for each } E \in \mathscr{A}$$

17.4.6 Self Assessment Question : Show that the Hahn decomposition is unique except for null sets

17.4.7 Self Assessment Question : Let $(X, \mathscr{A}, \mu)$ be a measure space and let $\int f\, d\mu$ exist. Define υ by $\upsilon(E) = \int_E f\, d\mu$ for $E \in \mathscr{A}$. Find a Hahn decomposition and the Jordan decomposition with respect to υ.

17.5 ANSWERS TO SELF ASSESSMENT QUESTIONS

17.2.5 : Clearly $\upsilon(\phi)=0$. Let $E=\bigcup_{n=1}^{\infty} E_n$ where E_n are pairwise disjoint elements of $\mathcal{A}$. Then

$\mu_i(E)=\sum_{n=1}^{\infty}\mu_i(E_n)$, $i=1,2$. Suppose $\mu_1(A)<+\infty\ \forall\ A\in\mathcal{A}$. In case $\mu_2(E)<\infty$ also, then

the series $\sum_{n=1}^{\infty}(\mu_1(E_n)-\mu_2(E_n))$ is absolutely convergent to $\mu_1(E)-\mu_2(E)$. Hence

$$\upsilon(E)=\mu_1(E)-\mu_2(E)=\sum_{n=1}^{\infty}\mu_1(E_n)-\sum_{n=1}^{\infty}\mu_2(E_n)$$

$$=\sum_{n=1}^{\infty}(\mu_1(E_n)-\mu_2(E_n))$$

$$=\sum_{n=1}^{\infty}\upsilon(E_n)$$

In case $\mu_2(E)=+\infty$ or $-\infty$, clearly the series $\sum_{n=1}^{\infty}(\mu_1(E_n)-\mu_2(E_n))$ is divergent to $-\mu_2(E)=\mu_1(E)-\mu_2(E)$. Thus υ is a signed measure. If both μ_1, μ_2 are finite measures then $|\upsilon(X)|\le|\mu_1(X)|+|\mu_2(X)|<\infty$ i.e. υ is a finite signed measure.

17.2.6 : The proof of (i) is obvious. To prove (ii) let $A\in\mathcal{A}$ and $|\upsilon(A)|<\infty$. If $B\in\mathcal{A}$ and $B\subseteq A$ then $A=(A-B)\cup B$ and we have

$$\upsilon(A)=\upsilon(A-B)+\upsilon(B).$$

Since $|\upsilon(A)|<\infty$ and υ can take atmost one of the values $+\infty$ or $-\infty$, we get $|\upsilon(A\setminus B)|<\infty$ and $|\upsilon(B)|<\infty$. Further, $\upsilon(A\setminus B)=\upsilon(A)-\upsilon(B)$. (iii) follows from (ii).

17.4.6 : Let $\{A, B\}$ and $\{A_1, B_1\}$ be Hahn decompositions of X. Then we have

$X = A \cup B$, $X = A_1 \cup B_1$ where A and A_1 are positive sets, B and B_1 are negative sets.

Consider $A \sim A_1 = A \cap (X \setminus A_1) = A \cap B_1$ and

A is positive set and B_1 is a negative set follows that $A \sim A_1$ is both positive and negative i.e. a null set. Similarly we can show that $A_1 \sim A$ is a null set. Hence, we have $A \Delta A_1$ is a null set. Also, $B \Delta B_1$ is a null set.

17.4.7 : From example 17.2.4, υ is a signed measure. Let $A = \{x : f(x) \geq 0\}$, $B = \{x : f(x) < 0\}$. Then, A, B form a Hahn decomposition, while υ^+ and υ^- given by $\upsilon^+(E) = \int_E f^+ \, d\mu$, $\upsilon^-(E) = \int_E f^- \, d\mu$ from the Jordan decomposition.

17.5 MODEL EXAMINATION QUESTIONS

17.5.1 : Define a signed measure on a measurable space show that every integrable function f on a measurable space $(X, \mathscr{A}, \mu)$ defines a signed measure on X.

17.5.2 : Define a positive set, negative set and a null set with respect to a signed measure. Prove that the union of countable collection of positive sets is also a positive set.

17.5.3 : If υ is a signed measure on a measurable space $(X, \mathscr{A})$. Prove that every $E \in \mathscr{A}$ with $0 < \upsilon(E) < \infty$ contains a positive set A with $\upsilon(A) < 0$.

17.5.4 : State and prove the Hahn decomposition theorem. Show that Hahn decomposition is unique except for null sets.

17.5.5 : Prove that every signed measure on a measurable space can be written as a difference of mutually singular measures on the space.

17.6 EXERCISES

17.6.1 : Show that if E is any measurable set. Then $-\upsilon^- E \leq \upsilon E \leq \upsilon^+ E$ and

$$|\upsilon(E)| \leq |\upsilon|(E).$$

17.6.2 : Show that the Hahn decomposition is unique except for null sets.

17.6.3 : Show that if υ_1 and υ_2 are any two finite signed measures, then so is $\alpha\upsilon_1 + \beta\upsilon_2$, where α and β are real numbers.. Show that $|\alpha\upsilon| = |\alpha|\,|\upsilon|$.

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Lesson - 18

THE RADON - NIKODYM THEOREM

18.1 Introduction : In this lesson the absolute continuity of a measure with respect to another is defined on a measurable space $(X, \mathscr{A})$ and such measures are characterized when the space is $\sigma-$ finite. Integrating a non-negative function over the sets of a $\sigma-$algebra produces a new measure from the original one and in the Radon Nykodym theorem we show that any new measure continuous in a certain way can be formed in this manner. This gives rise to the derivative of one measure with respect to another.

18.2 Definition : If μ and υ are measures on a measurable space $(X, \mathscr{A})$ such that $\upsilon(A)=0$ for each set A for which $\mu(A)=0$ then υ is said to be absolutely continuous with respect to μ and we write $\upsilon \ll \mu$ in this case.

18.3 Example : If f is a non-negative integrable function on the measure space $(X, \mathscr{A}, \mu)$ define υ by $\upsilon(A)= \int_A f\, d\mu$, for each $A \in \mathscr{A}$. Then, υ is a measure and $\upsilon \ll \mu$, since $\mu(A)=0$ implies $\upsilon(A)=0$.

We need the following lemmas to prove the main theorem. These lemmas show that given a family $\mathscr{B}$ of measurable sets, a measurable function can be derived from this family.

18.4 Lemma : Suppse that to each α in a countable set D of real numbers there is assigned a set B_α such that $B_\alpha \subseteq B_\beta$ for $\alpha < \beta$. Then there is a unique measurable extended real valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$.

Proof : For each $x \in X$ define

$$f(x) = \inf\{\alpha \in D : x \in B_\alpha\}$$

where $\inf \phi = \infty$. If $x \in B_\alpha$, then $f(x) \leq \alpha$. If $x \notin B_\alpha$, then $x \notin B_\beta$ for each $\beta < \alpha$. If $f(x) < \alpha$ then by the definition of f there exists a $\delta \in D \ni x \in B_\delta$ and $\delta < \alpha$. Now, $\delta < \alpha$ implies $B_\delta \subseteq B_\alpha$ and $x \in B_\delta \subseteq B_\alpha$ i.e. $x \in B_\alpha$ a contradiction. Therefore, $f(x) \geq \alpha$ on $X - B_\alpha$. Now we will show that f is measurable. Let α be any real number. Then, the set $\{x/f(x)<\alpha\} = \bigcup_{\beta<\alpha} B_\beta$.

If $f(x)<\alpha$, then there is a $\beta<\alpha$ and $x \in B_\beta \subseteq \bigcup_{\beta<\alpha} B_\beta$. Now if $x \in \bigcup_{\beta<\alpha} B_\beta$ then $x \in B_\beta$ for some $\beta<\alpha$. Hence, $f(x) \le \beta<\alpha$. Thus, $f(x)<\alpha$. Therefore,

$$\{x: f(x)<\alpha\} = \bigcup_{\beta<\alpha} B_\beta$$

since, $\bigcup_{\beta<\alpha} B_\beta$ is a measurable set we get f is a measurable function.

Note : If the set D in Lemma 18.4 is dense in $\mathbb{R}$, then f in the lemma is uniquely determined.

18.6 Lemma : Suppose that for each α in a countable set D of real numbers there is assigned a set B_α in $\mathscr{B}$ such that $\mu(B_\alpha \sim B_\beta)=0$ for $\alpha<\beta$. Then there is a measurable function f such that $f \le \alpha$ a.e. on B_α and $f \ge \alpha$ a.e. on $X \sim B_\alpha$.

Proof : Write, $C = \bigcup_{\alpha<\beta} (B_\alpha \sim B_\beta)$

Now, $0 \le \mu(C) \le \sum_{\alpha<\beta} \mu(B_\alpha \sim B_\beta) = 0$. Hence, $\mu(C)=0$.

For each $\alpha \in D$, put, $B'_\alpha = B_\alpha \cup C$.

If $\alpha<\beta$ consider,

$$\begin{aligned} B'_\alpha \sim B'_\beta &= (B_\alpha \cup C) \sim (B_\beta \cup C) \\ &= (B_\alpha \sim B_\beta) \sim C \\ &= \phi \end{aligned}$$

Hence, $B'_\alpha \subseteq B'_\beta$ for $\alpha<\beta$. Therefore by lemma 18.5 there is a measurable function f such that $f(x) \le \alpha$ on B'_α and $f(x) \ge \alpha$ on $X - B'_\alpha$. Therefore, $f \le \alpha$ on B_α and $f \ge \alpha$ on $X \sim (B_\alpha \cup C)$. Hence, $f \le \alpha$ a.e. on B_α and $f \ge \alpha$ on $X \sim B_\alpha$ except for $x \in C$ and $\mu(C)=0$. Therefore, we have, $f \le \alpha$ a.e. on B_α and $f \ge \alpha$ a.e. on $X - B_\alpha$.

The following theorem, the Radon-Nikodym theorem, characterizes absolutely continuous measures υ on a $\sigma-$finite measure space $(X, \mathscr{A}, \mu)$. In fact every measure $\upsilon \ll \mu$ is of the form given the example 18.3.

18.7 Theorem (Randon - Nikodym) : Let $(X, \mathscr{B}, \mu)$ be $\sigma-$ finite measure space, and let υ **be a** measure defined on $\mathscr{B}$ which is absolutely continuous with respet to μ. Then there is **a** non-negative measurable function f such that for all E in $\mathscr{B}$ we have

$$\upsilon(E) = \int_E f \, d\mu$$

The function f is unique in the sense that if g is also a non-negative measurable function such that $\upsilon(E) = \int_E g \, d\mu, \ E \in \mathscr{B}$ then $f = g$ a.e. (μ).

Proof :

Step - 1 : Suppose that the result has been proved for finite measures. Then in the general case we have $X = \bigcup_n X_n$ and $\mu(X_n) < \infty$ **for every** n. We can assume that X_n's are disjoint.

Write $\mathscr{B}_n = \{E \cap X_n / E \in \mathscr{B}\}$

Then $(X_n, \mathscr{B}_n, \mu)$ is a measure space, and $\mu(X_n) < \infty$, for every n. By assumption, there exists a non-negative measurable function f_n on X_n such that $\upsilon(E) = \int_E f_n \, d\mu \ \forall \ E \in \mathscr{B}_n$.

Define f on X by, $f(x) = f_n(x)$ if $x \in X_n$. Clearly f is non-negative, since each f_n is non-negative. Let α be a real number, now $\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x / f_n(x) > \alpha\}$. Therefore f is measurable.

Every E in $\mathscr{B}$ can be written as $E = \bigcup_n E_n$ where $E_n = E \cap X_n \in \mathscr{B}_n$ for every n. Therefore

$$\int_E f \, d\mu = \int_{\bigcup_n E_n} f \, d\mu = \sum_n \int_{E_n} f = \sum_n \upsilon(E_n) = \upsilon(E).$$ So the general case follows.

Step - 2 : *So we need to show that for finite measure such a function* f exists. Thus without loss of generality we can assume that $\mu(X) < \infty$. Now for each rational

α, $(\upsilon-\alpha\mu)(E)=\upsilon(E)-\alpha\mu(E)$. Since μ is finite, we have, $\upsilon-\alpha\mu$ is a signed measure. Therefore by Hahn decomposition theorem, there exists $\{A_\alpha, B_\alpha\}$ such that $X=A_\alpha \cup B_\alpha$, $A_\alpha \cap B_\alpha=\phi$, A_α is a positive set with respect to $\upsilon-\alpha\mu$ and B_α is a negative set with respect to $\upsilon-\alpha\mu$. For $\alpha=0$, take $A_0=X$ and $B_0=\phi$.

Step - 3 : We will show that $\mu(B_\alpha-B_\beta)=0$ if $\alpha<\beta$.

Consider $B_\alpha \sim B_\beta = B_\alpha \cap \tilde{B}_\beta = B_\alpha \cap A_\beta$ is a subset of B_α which is a negative set with respect to $\upsilon-\alpha\mu$ and hence, $(\upsilon-\alpha\mu)(B_\alpha-B_\beta)\le 0$. Also $B_\alpha \sim B_\beta$ is a subset of A_β which is a positive set with respect to $\upsilon-\beta\mu$ and hence $(\upsilon-\beta\mu)(B_\alpha \sim B_\beta)\ge 0$. Now if $\beta>\alpha$, we have,

$$\upsilon(B_\alpha-B_\beta)\le \alpha\mu(B_\alpha \sim B_\beta)$$

$$\le \beta\mu(B_\alpha \sim B_\beta)$$

$$\le \upsilon(B_\alpha \sim B_\beta)$$

Hence, $\alpha\mu(B_\alpha \sim B_\beta)=\beta\mu(B_\alpha \sim B_\beta)$, implies that $\mu(B_\alpha \sim B_\beta)=0$. Thus, there exists a measurable function f such that; $f\ge\alpha$ a.e. on A_α and $f\le\alpha$ a.e. on $X-A_\alpha=B_\alpha$ by lemma 18.5. In particular $f\ge 0$ a.e. on X. Also without loss of generality we can assume that $f\ge 0$ on X.

Step - 4 : Let E be a measurable set and let N be a positive integer. For $k\ge 0$,

$$\text{Put } E_k = E\cap\left(B_{\frac{k+1}{N}} \sim B_{\frac{k}{N}}\right) = E\cap B_{\frac{k+1}{N}} \cap A_{\frac{k}{N}}$$

$$E_\infty = E - \bigcup_{k=0}^{\infty} B_{\frac{k}{N}}.$$

Then $\{E_k\}$ are pairwise disjoint, each of them is disjoint from E_∞ and $E=E_\infty \cup \bigcup_{k=0}^{\infty} E_k$,

so that

$$\upsilon(E) = \upsilon(E_\infty) + \sum_{k=0}^{\infty} \upsilon(E_k) \text{ ---------- (1)}$$

Also, $E_k \subseteq B_{\frac{k+1}{N}} \cap A_{\frac{k}{N}}$ gives $\left(\upsilon - \frac{k+1}{N}\mu\right)(E_k) \le 0$ and $\left(\upsilon - \frac{k}{N}\mu\right)(E_k) \ge 0$ which imply

$$\frac{k}{N}\mu(E_k) \le \upsilon(E_k) \le \frac{k+1}{N}\mu(E_k) \text{ ---------- (2)}$$

Since $E_k \subseteq B_{\frac{k+1}{N}} - B_{\frac{k}{N}} = B_{\frac{k+1}{N}} \cap A_{\frac{k}{N}}$ we have by the choice of the measurable function f that

$$\frac{k}{N} \le f(x) \le \frac{k+1}{N} \text{ for } x \in E_k \text{ so that}$$

$$\frac{K}{N}\mu(E_k) \le \int_{E_k} f\,d\mu \le \frac{k+1}{N}\mu(E_k) \text{ ---------- (3)}$$

From (2) and (3) we have

$$\upsilon(E_k) - \frac{1}{N}\mu(E_k) \le \int_{E_k} f\,d\mu \le \upsilon(E_k) + \frac{1}{N}\mu(E_k) \text{ --------- (4)}$$

Step - 5 : $\upsilon(E_\infty) = \int_{E_\infty} f\,d\mu$ ---------- (5)

If $x \in E_\infty$ then $x \notin B_{\frac{k}{N}}$ for every K implies, $x \in A_{\frac{k}{N}}$ for every k and hence $f(x) \ge \frac{k}{N}$ a.e. for every k. Hence, $f(x) = \infty$ a.e. on E_∞.

If $\mu(E_\infty) > 0$ then $\left(\text{v} - \frac{k}{N}\mu\right)(E_\infty) \ge 0$ for every k. Since E_∞ is a subset of $A_{\frac{k}{N}}$ for every k. Hence,

$$\upsilon(E_\infty) \ge \frac{k}{N}\mu(E_\infty) \text{ for every } K, \text{ implies } \upsilon(E_\infty) = \infty, \text{ since } \mu(E_\infty) > 0.$$

Therefore, in this case, we get $\int_{E_\infty} f\,d\mu = \upsilon(E_\infty)$.

If $\mu(E_\infty)=0$ then $\upsilon(E_\infty)=0$, since $\upsilon \ll \mu$. Hence, $\upsilon(E_\infty)=\int_{E_\infty} f\,d\mu=0$.

Step - 6 : From (1), (4) and (5), we get

$$\upsilon(E)-\frac{1}{N}\mu(E)\le\upsilon(E_\infty)+\sum_{k=0}^{\infty}\upsilon(E_k)-\frac{1}{N}\sum_{k=0}^{\infty}\mu(E_k)$$

$$\le \int_{E_\infty} f\,d\mu+\sum_{k=0}^{\infty}\int_{E_k} f\,d\mu$$

$$\le\upsilon(E_\infty)+\sum_{k=0}^{\infty}\upsilon(E_k)+\frac{1}{N}\sum_{k=0}^{\infty}\mu(E_k)$$

$$\le\upsilon(E)+\frac{1}{N}\mu(E)$$

Hence, $\upsilon(E)\le\int_E f\,d\mu\le\upsilon(E)$ since $\mu(E)<\infty$ and N is arbitrary. Therefore,

$$\upsilon(E)=\int_E f\,d\mu$$

Step - 7 Uniqueness of f: Suppose there exists a non-negative measurable function g such that $\upsilon(E)=\int_E g\,d\mu$ for every $E\in\mathscr{B}$. Then for each $E\in\mathscr{B}$, we have,

$\int_E (f-g)\,d\mu=\int_E f\,d\mu-\int_E g\,d\mu=0$, proving $f-g=0$ a.e. $[\mu]$ i.e $f=g$ a.e. $[\mu]$.

18.8 Definition : Suppose $(X,\mathscr{B},\mu)$ is a $\sigma-$ finite measure space and υ is an absolutely continuous measure with respect to μ. The function f obtained by Radon-Nikodym theorem such that

$\upsilon(E)=\int_E f\,d\mu$ for every $E\in\mathscr{B}$, is called the Radon-Nikodym derivative of υ

with respect to μ and is denoted by $\left[\frac{d\upsilon}{d\mu}\right]$. Thus, if $\upsilon\ll\mu$ then $\upsilon(E)=\int_E\left[\frac{d\upsilon}{d\mu}\right]d\mu$ for all $E\in\mathscr{B}$.

18.9 Self Assessment Question : If υ_1 and υ_2 are $\sigma-$ finite measures on $(X,\mathcal{B})$ and $\upsilon_1 \ll \mu$, $\upsilon_2 \ll \mu$ then

$$\frac{d(\upsilon_1+\upsilon_2)}{d\mu} = \frac{d\upsilon_1}{d\mu} + \frac{d\upsilon_2}{d\mu}\,[\mu].$$

The following theorem shows that every $\sigma-$ finite measure on a measure space $(X, \mathcal{B}, \mu)$ can be written uniquely as a sum of two measures one of them is singular with respect to μ and the other is absolutely continuous with respect to μ.

18.10 Lebesgue Decomposition Theorem : Suppose $(X,\mathcal{B},\mu)$ is a $\sigma-$ finite measure space and υ is a $\sigma-$ finite measure on $\mathcal{B}$. Then $\upsilon=\upsilon_0+\upsilon_1$, where υ_0 is singular with respect to μ $(\upsilon_0 \perp \mu)$ and υ_1 is absolutely continuous with respect to $\mu(\upsilon_1 \ll \mu)$. The measures υ_0 and υ_1 are unique.

Proof : Since μ and υ are $\sigma-$ finite measures so is the measure $\lambda=\mu+\upsilon$. Also, $\mu \ll \lambda$ and $\upsilon \ll \lambda$. Therefore by the Radon-Nikodym theorem there exist non-negative measurable functions f and g such that

$$\mu(E) = \int_E f\, d\lambda \text{ and } \upsilon(E)= \int_E g\, d\lambda.$$

Now if, $A=\{x: f(x)>0\}$ and $B=\{x: f(x)=0\}$ then A and B are disjoint measurable sets with $A \cup B = X$ and $\mu(B)=0$. Define, υ_0 and υ_1 both on $\mathcal{B}$ by $\upsilon_0(E)=\upsilon(E\cap B)$ and

$$\upsilon_1(E)=\upsilon(E\cap A) \text{ for any } E\in\mathcal{B}.$$

Then, $\upsilon_0(A)=\upsilon(A\cap B)=\upsilon(\phi)=0$ and $\mu(B)=0$ imply that υ_0 is mutually singular with μ. i.e. $\upsilon_0 \perp \mu$. We will now show that $\upsilon_1 \ll \mu$. Let $E\in\mathcal{B}$ be such that $\mu(E)=0$. Then $\int_E f\, d\mu=0$ showing $f=0$ a.e. $[\lambda]$ since f is non-negative. Since $f(x)>0$ for $x\in A\cap E$ we have $\lambda(A\cap E)=0$ showing $\upsilon(A\cap E)=0$ (since $\upsilon \ll \lambda$). This gives $\upsilon_1(E)=0$. Hence, $\upsilon_1 \ll \mu$. Also for any $E\in\mathcal{B}$ we have

$$\upsilon_0(E)+\upsilon_1(E) = \upsilon(E\cap B)+\upsilon(E\cap A)$$

$$= \upsilon(E \cap (A \cup B))$$

$$= \upsilon(E \cap X) = \upsilon(E).$$

showing that, $\upsilon = \upsilon_0 + \upsilon_1$. Thus every $\sigma-$ finite measure on $\mathscr{B}$ can be written as $\upsilon = \upsilon_0 + \upsilon_1$ where $\upsilon_0 \perp \mu$ and $\upsilon_1 \ll \mu$.

To prove the uniqueness of the decomposition, suppose $\upsilon = \upsilon_0 + \upsilon_1 = \upsilon_0' + \upsilon_1'$ where $\upsilon_0 \perp \mu, \upsilon_1 \ll \mu, \upsilon_0' \perp \mu$ and $\upsilon_1' \ll \mu$. Then there exists sets A, B, A', B' such that

$$A \cap B = \phi = A' \cap B',\ A \cup B = X = A' \cup B' \text{ and } \upsilon_0(B) = \mu(A) = \upsilon_0'(B') = \mu(A') = 0.$$

Now for any $E \in \mathscr{B}$ we have

$$E = (E \cap B \cap B') \cup (E \cap A' \cap B) \cup (E \cap A \cap A') \cup (E \cap A \cap B')$$

since, $\mu(E \cap A' \cap B) = \mu(E \cap A \cap A') = \mu(E \cap A \cap B') = 0$

we get, $\upsilon_1(E \cap A' \cap B) = \upsilon_1(E \cap A \cap A') = \upsilon_1(E \cap A \cap B') = 0$ and

$$\upsilon_1'(E \cap A' \cap B) = \upsilon_1'(E \cap A \cap A') = \upsilon_1'(E \cap A \cap B') = 0$$

since $\upsilon_1 \ll \mu$ and $\upsilon_1' \ll \mu$.

Therefore for any $E \in \mathscr{B}$, we get

$$(\upsilon_1' - \upsilon_1)(E) = (\upsilon_1' - \upsilon_1)(E \cap B \cap B')$$

$$= (\upsilon_0 - \upsilon_0')(E \cap B \cap B')$$

$$= 0, \text{ since } \upsilon_0(B) = 0 \text{ and } \upsilon_0'(B') = 0$$

Therefore, $\upsilon_1'(E) = \upsilon_1(E)$ i.e. $\upsilon_1' = \upsilon_1$ implies $\upsilon_0'(E) = \upsilon_0(E)$. i.e. $\upsilon_0' = \upsilon_0$, proving the uniqueness of υ_0 and υ_1.

18.11 Self Assessment Question :

(a) Show that if υ is a signed measure such that $\upsilon \perp \mu$ and $\upsilon \ll \mu$, then $\upsilon = 0$.

(b) Show that if υ_1 and υ_2 are singular with respect to μ, then so is $c_1 \upsilon_1 + c_2 \upsilon_2$.

(c) Show that if υ_1 and υ_2 are absolutely continuous with respect to μ so is $c_1\upsilon_1 + c_2\upsilon_2$.

(d) Prove the uniqueness assertion in the Lebesgue decomposition.

18.12 Answers to SAQs :

18.9 : Clearly $\upsilon_1 + \upsilon_2$ is a $\sigma-$ finite measure and $\upsilon_1 + \upsilon_2 \ll \mu$. For $E \in \mathcal{B}$,

$$(\upsilon_1 + \upsilon_2)(E) = \upsilon_1(E) + \upsilon_2(E) = \int_E \frac{d\upsilon_1}{d\mu} d\mu + \int_E \frac{d\upsilon_2}{d\mu} d\mu$$

so the uniqueness of $d(\upsilon_1 + \upsilon_2)/d\mu$ gives the result.

18.11 (a) : Since $\upsilon \perp \mu$ there exist disjoint sets A and B such that $X = A \cup B$, $\upsilon(A) = 0 = \mu(B)$. Since, $\upsilon \ll \mu$, we have $\upsilon(B) = 0$, we will show that $\upsilon = 0$. Let $E \in \mathcal{B}$, then $E = (E \cap A) \cup (E \cap B)$ and hence $\upsilon(E) = \upsilon(E \cap A) + \upsilon(E \cap B) = 0$. Therefore $\upsilon = 0$.

(b) : $\upsilon_1 \perp \mu$ there exists disjoint sets A and B such that $X = A \cup B$, $\upsilon_1(A) = 0 = \mu(B)$.

$\upsilon_2 \perp \mu$ there exists disoint sets A' and B' such that $X = A' \cup B'$, $\upsilon_2(A') = 0 = \mu(B')$. Now, $c_1\upsilon_1 + c_2\upsilon_2$ is a measure.

Now, $X = (A \cap A') \cup (B \cap B')$, where

$$(c_1\upsilon_1 + c_2\upsilon_2)(A \cap A') = 0 \text{ and } \mu(B \cap B') = 0$$

Therefore, $(c_1\upsilon_1 + c_2\upsilon_2) \perp \mu$

(c) : is obvious.

(d) : Suppose $\upsilon = \upsilon_0 + \upsilon_1 = \upsilon_0' + \upsilon_1'$ where $\upsilon_0 \perp \mu, \upsilon_0' \perp \mu, \upsilon_1 \ll \mu, \upsilon_1' \ll \mu$. By (b) and (c) we have, $(\upsilon_0 - \upsilon_0') \perp \mu$ and $(\upsilon_1 - \upsilon_1') \ll \mu$.

By (a) we have, $\upsilon_0 - \upsilon_0' = 0$

and $\upsilon_1 - \upsilon_1' = 0$

Therefore, $\upsilon_0 = \upsilon_0', \ \upsilon_1 = \upsilon_1'$

18.13 Model Examination Questions

18.13.1 : State and prove the Radon-Nikodym theorem.

18.13.2 : Prove that every measure υ on a finite measure space $(X, \mathcal{B}, \mu)$ can be decomposed as $\upsilon = \upsilon_0 + \upsilon_1$ where $\upsilon_0 \perp \mu$ and $\upsilon_1 \ll \mu$.

18.14 Exercises

18.14.1 : Show that the follwoing conditions on the signed measures μ and υ on $(X, \mathcal{B})$ are equivalent : (i) $\upsilon \ll \mu$ (ii) $|\upsilon| \ll |\mu|$ (iii) $\upsilon^+ \ll \mu$ and $\upsilon^- \ll \mu$

18.14.2 : Show that if μ and υ are measures such that $\upsilon \ll \mu$ and $\upsilon \perp \mu$ then υ is identically zero.

18.14.3 : Show that the condition μ $\sigma-$ finite is necessary in the Radon - Nikodym theorem.

REFERENCE BOOK

1. Real Analysis - H.L. Royden

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Lesson - 19

OUTER MEASURE AND THE EXTENSION THEOREM

19.1 INTRODUCTION

In this lesson we first consider some of the ways in which a measure can be defined on a $\sigma-$algebra. In the case of Lebesgue measure we defined measure for open sets and used this to define outer measure, from which we obtain the notion of measurable set and Lebesgue measure. Such a procedure is feasible in general. In the first section we discuss the process of deriving a measure from an outer measure, and in the second section, we start with a measure on an algebra of sets and extend it to the smallest $\sigma-$algebra containing it - this extension is called Cartheodory's measure.

Finally, we start with a "semi-algebra" of sets and a non-negative set function, we consider the possibilities of extending it to a measure on the smallest algebra containing it.

19.2 OUTER MEASURE AND MEASURABILITY

The purpose of this section is to introduce the concept of an outer measure μ^* on the class of all subsets of a given set X and there by obtain a class $\mathscr{B}$ of subsets (called the class of μ^*- measurable sets) which is a $\sigma-$algebra of subsets of X.

19.2.1 Definition : By an outer measure μ^* we mean a nonnegative extended real-valued set function defined on all subsets of a set X and having the following properties :

(i) $\mu^*(\phi)=0$

(ii) $A\subseteq B \Rightarrow \mu^*(A)\leq \mu^*(B)$

(iii) $E\subseteq \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^*(E)\leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

The second property is called monotonicity and the third countable subadditivity.

19.2.2 Example : The Lebesgue outer measure m^* defined in lesson 1, is an outer measure on the class of all subsets of $\mathbb{R}$. Since $m^*(\phi)=0$; $A\subseteq B\Rightarrow m^*(A)\leq m^*(B)$ and m^* is countably subadditive.

19.2.3 Self Assessment Question : Show that the outer measure μ^* satisfies the finite subadditive property.

19.2.4 Self Assessment Question : Show that condition (iii) in the definition 19.2.1 can be replaced by the following condition

$$E=\bigcup_{i=1}^{\infty} E_i,\ E_i \text{ disjoint} \Rightarrow \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$$

19.2.5 Definition : An outer measure μ^* is called finite if $\mu^*(X)<\infty$. In view of monotonicity of μ^* we have $\mu^*(A)<\infty$ for every $A\subseteq X$ if μ^* is finite.

We know that the Lebesgue outer measure is not countably additive but it is countably additive on the class of all measurable sets. Analogously the outer measure μ^* defined on $\mathcal{P}(X)$ need not even finitely additive. So we have to identify some subclass S of $\mathcal{P}(X)$ such that μ^* restricted to S will be countably additive. This is the class S which we call the class of μ^*-measurable sets of X. A set $E\subseteq X$ is in S if we use it as a knife to cut any subset Y of X into two parts, $Y\cap E$ and $Y\cap E^c$, then their sizes $\mu^*(Y\cap E)$ and $\mu^*(Y\cap E^c)$ add up to give the size $\mu^*(Y)$ of Y. This motivate our next definition

19.2.6 Definition : A set $E\subseteq X$ is said to be measurable with respect to μ^* or μ^*-measurable if for every set $A\subseteq X$ we have

$$\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap \tilde{E})$$

For example,

(i) the empty set ϕ is μ^*-measurable,

since for any $A\subseteq X,\ \mu^*(A\cap\phi)+\mu^*(A\cap\tilde{\phi})=\mu^*(\phi)+\mu^*(A)=\mu^*(A)$

(ii) If E is such that $\mu^*(E)=0$ then E is μ^* - measurable. In fact, if A is any set then $A\cap E\subseteq E$ implies $\mu^*(A\cap E)\leq\mu^*(E)=0$ so that $\mu^*(A\cap E)=0$ and $A\cap\tilde{E}\subseteq A$ gives $\mu^*(A\cap\tilde{E})\leq\mu^*(A)$. Therefore

$$\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})\leq\mu^*(A).$$ Again since,

$A=(A\cap E)\cup(A\cap\tilde{E})$, we get by the finite subadditivity that $\mu^*(A)\leq\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})$

Hence, $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})$ for every $A\subseteq X$, proving that E is μ^*-measurable.

19.2.7 Remark : The finite subadditivity of μ^* gives

$\mu^*(A)=\mu^*\left[(A\cap E)\cup(A\cap\tilde{E})\right]\le\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})$ for every set A. Therefore, the μ^*-measurability of E follows iff $\mu^*(A)\ge\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})$ holds for every $A\subseteq X$. Again if $\mu^*(A)=\infty$, this is obvious. Hence to establish the μ^*-measurability of a set E, it is enough to prove the inequality $\mu^*(A)\ge\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})$ for every set $A\subseteq X$ with $\mu^*(A)<\infty$.

19.2.8 Theorem : The class $\mathscr{B}$ of μ^*-measurable sets is a $\sigma-$algebra. If $\bar{\mu}$ is μ^* restricted to $\mathscr{B}$ then $\bar{\mu}$ is a complete measure on $\mathscr{B}$.

Proof : The empty set ϕ is μ^*-measurable.

Also since the condition for μ^*-measurability of E is $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap\tilde{E})$ is symmetric in E and $\tilde{E}$ it follows that $\tilde{E}$ is μ^*-measurable whenever E is. We will show that if $E_1, E_2\in\mathscr{B}$ then $E_1\cup E_2\in\mathscr{B}$. Let $E_1,E_2\in\mathscr{B}$, since E_2 is μ^*-measurable we have

$$\mu^*(A)=\mu^*(A\cap E_2)+\mu^*(A\cap\tilde{E}_2)$$

which the measurability of E_1 gives

$\mu^*(A\cap\tilde{E}_2)=\mu^*(A\cap\tilde{E}_2\cap E_1)+\mu^*(A\cap\tilde{E}_2\cap\tilde{E}_1)$; and combining these two we get

$$\mu^*(A) = \mu^*(A\cap E_2)+\mu^*(A\cap\tilde{E}_2\cap E_1)+\mu^*(A\cap\tilde{E}_2\cap\tilde{E}_1) \text{ -------------- (1)}$$

But since, $A\cap(E_1\cup E_2) = (A\cap E_2)\cup(A\cap\tilde{E}_2\cap E_1)$ we have by the subadditivity that

$$\mu^*(A\cap(E_1\cup E_2))\le \mu^*(A\cap E_2)+\mu^*(A\cap\tilde{E}_2\cap E_1) \text{ ------------------- (2)}$$

Then (1) and (2) imply that

$\mu^*(A)\ge \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap\widetilde{E_1\cup E_2})$, proving that $E_1\cup E_2$ is μ^*-measurable.

Thus $\mathscr{B}$ is an algebra of sets of X. By induction the union of any finite number of measurable sets is measurable. Assume that $E=\cup E_i$, where $\{E_i\}$ is a disjoint sequence of measurable sets; and put

$$G_n = \bigcup_{i=1}^{n} E_i.$$

Then, G_n is measurable, and

$\mu^*(A) = \mu^*(A \cap G_n) + \mu^*(A \cap \tilde{G}_n) \geq \mu^*(A \cap G_n) + \mu^*(A \cap \tilde{E})$, since, $\tilde{E} \subseteq \tilde{G}_n$. Now $G_n \cap E_n = E_n$ and $G_n \cap \tilde{E}_n = G_{n-1}$ and by the measurability of E_n we have,

$$\mu^*(A \cap G_n) = \mu^*(A \cap G_n \cap E_n) + \mu^*(A \cap G_n \cap \tilde{E}_n)$$

$$= \mu^*(A \cap E_n) + \mu^*(A \cap G_{n-1})$$

But by induction we can prove that $\mu^*(A \cap G_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$ and so

$$\mu^*(A) \geq \mu^*(A \cap \tilde{E}) + \sum_{i=1}^{\infty} \mu^*(A \cap E_i)$$

$$\geq \mu^*(A \cap \tilde{E}) + \mu^*(A \cap E), \text{ since,}$$

$A \cap E \subseteq \bigcup_{i=1}^{\infty} A \cap E_i$. Thus E is measurable. Since the union of any sequence of sets in an algebra can be replaced by a disjoint union of sets in the algebra, it follows that $\mathcal{B}$ is a σ – algebra.

Given that $\bar{\mu}$ is the restriction of μ^* to $\mathcal{B}$ that is $\bar{\mu}(E) = \mu^*(E)$ for all $E \in \mathcal{B}$. To prove that $\bar{\mu}$ is a measure, first we shɔw that it is finitely additive. Let E_1 and E_2 be disjoint measurable sets. Then the measurability of E_2 implies that

$$\bar{\mu}(E_1 \cup E_2) = \mu^*(E_1 \cup E_2)$$

$$= \mu^*((E_1 \cup E_2) \cap E_2) + \mu^*((E_1 \cup E_2) \cap \tilde{E}_2)$$

$$= \mu^* E_2 + \mu^* E_1$$

$$= \bar{\mu} E_2 + \bar{\mu} E_1$$

Now by induction the finite additivity of $\bar{\mu}$ follows.

Suppose $E = \bigcup_{i=1}^{\infty} E_i$ where $E_i \in \mathcal{B}$, $E_i \cap E_j = \phi$ for $i \neq j$. Then for any $n \geq 1$,

$$\bar{\mu}(E) \geq \bar{\mu}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \bar{\mu}(E_i) \text{ which gives } \sum_{i=1}^{\infty} \bar{\mu}(E_i) \leq \bar{\mu}(E)$$

Also, $\bar{\mu}(E) = \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i) = \sum_{i=1}^{\infty} \bar{\mu}(E_i)$

Therefore, $\bar{\mu}(E) = \bar{\mu}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \bar{\mu}(E_i)$ i.e. $\bar{\mu}$ is countably additive. Further, $\bar{\mu}(\phi) = \mu^*(\phi) = 0$. Clearly $\bar{\mu}$ is non-negative, since μ^* is non-negative.

Therefore $\bar{\mu}$ is a measure on $\mathcal{B}$.

To prove that the measure $\bar{\mu}$ is complete, recall that every set of outer measure zero is μ^*- measurable. (19.2.6 Ex(ii)). Suppose, $E \in \mathcal{B}$ with $\bar{\mu}(E) = 0$ and $F \subseteq E$. Now $\bar{\mu}(E) = \mu^*(E)$ and $F \subseteq E$ implies $\mu^*(F) \leq \mu^*(E)$. Therefore, $\mu^*(F) = 0$ which gives F is μ^*- measurable. Thus $\bar{\mu}$ is complete.

19.2.9 Self Assessment Question :

(a) If $\{E_i\}$ is a sequence of disjoint measurable sets and $E = \bigcup E_i$. Then for any set A we have $\mu^*(A \cap E) = \sum \mu^*(A \cap E_i)$.

(b) Show that the outer measure μ^* is countably additive on $\mathcal{B}$.

19.3 THE EXTENSION THEOREM

19.3.1 : In this section we start from the definition of a measure μ on an algebra $\mathcal{A}$ of subsets of X and define an outer measure μ^*, called the outer measure induced by μ. We denote the class of μ^*- measurable sets by $\mathcal{B}$ and prove that the restriction $\bar{\mu}$ of μ^* to $\mathcal{B}$ is an extension of μ

to a $\sigma-$ algebra containing $\mathscr{A}$ that is $\bar{\mu}=\mu^*/\mathscr{B}$. Further we show that starting with a set function defined on a semi-algebra it is possible to define a measure on an algebra.

19.3.2 Definition : A non-negative extended real-valued set function μ defined on an algebra $\mathscr{A}$ of subsets of X is said to be a measure on $\mathscr{A}$ if

(i) $\mu(\phi)=0$ and

(ii) for any sequence $\{A_i\}$ of pairwise disjoint sets in A whose union is also a member of $\mathscr{A}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right)=\sum_{i=1}^{\infty}\mu(A_i).$$

The process by which we construct μ^* from μ is analogous to that by which we constructed Lebesgue outer measure from the lengths of the intervals.

19.3.3 Definition : Suppose μ is a measure on an algebra $\mathscr{A}$ of subsets of X. For any $E\subseteq X$, define

$$\mu^*(E)=\inf\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A},\ E\subseteq\bigcup_{i=1}^{\infty}A_i\right\}$$

The set function μ^* is called the outer measure induced by μ.

19.3.4 Remark :

(i) Given any $E\subseteq X$, there exist at least one covering $\{A_i\}_{i\geq 1}$ of E by elements of $\mathscr{A}$, namely $\{X\}$.

(ii) The set function $\mu^*(E)$ can take the value $+\infty$ for some sets E.

19.3.5 Self Assessment Question : Show that

$$\mu^*(E)=\inf\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A},\ A_i\cap A_j=\phi \text{ for } i\neq j \text{ and } \bigcup_{i=1}^{\infty}A_i\supseteq E\right\}$$

19.3.6 Lemma : Let μ be a measure on an algebra $\mathscr{A}$. If $A\in\mathscr{A}$ and if $\{A_i\}$ is any sequence of sets in $\mathscr{A}$ such that $A\subseteq\bigcup_{i=1}^{\infty}A_i$, then, $\mu(A)\leq\sum_{i=1}^{\infty}\mu(A_i)$.

Proof : Put, $B_n = A \cap A_n \cap \tilde{A}_{n-1} \cap \cdots \cap \tilde{A}_1$. Then $B_n \in \mathscr{A}$ and $B_n \subseteq A_n$. But A is the disjoint union of the sequence $\{B_n\}$, and so by countable additivity $\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

19.3.7 Corollary : If $A \in \mathscr{A}$ then $\mu^*(A) = \mu(A)$.

Proof : If $A \in \mathscr{A}$ then $\{A\}$ covers A and hence by the definition of μ^*, we have, $\mu^*(A) \leq \mu(A)$.

But by the above lemma, $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$, if $\{A_i\}$ is any sequence of sets in $\mathscr{A}$ such that $A \subseteq \bigcup_{i=1}^{\infty} A_i$. Again by the definition of μ^* we have, $\mu(A) \leq \mu^*(A)$. Therefore, $\mu(A) = \mu^*(A)$.

Note : 19.3.7 shows that μ^* is an extension of μ.

19.3.8 Lemma : The set function μ^* is an outer measure.

Proof : Clearly μ^* is a non-negative set function defined on the class of all subsets of X.

Since $\phi \in \mathscr{A}$ by 19.3.7, we get $\mu^*(\phi) = \mu^*(\phi) = 0$. Again if $A \subseteq B \subseteq X$ then any sequence $\{A_i\}$ of $\mathscr{A}$ with $B \subseteq \bigcup_{n=1}^{\infty} A_i$ is also a sequence for which $A \subseteq \bigcup_{n=1}^{\infty} A_i$ and therefore $\mu^*(A) \leq \mu^*(B)$, proving the monotonicity of μ^*. It remains only to prove the countable subadditivity of μ^*. Let $E \subseteq \bigcup_{i=1}^{\infty} E_i$. If $\mu^*(E_i) = \infty$ for some i then the inequality.

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i) \quad \text{----------} \quad (1)$$

is trivial. Therefore assume $\mu^*(E_i) < \infty$ for each i. Then by the definition of $\mu^*(E_i)$, to each $\epsilon > 0$ we can find a sequence $\{A_{ij}\}_{j=1,2,3,\ldots}$ in $\mathscr{A}$, such that $E_i \subseteq \bigcup_{j=1}^{\infty} A_{ij}$ and

$$\sum_{j=1}^{\infty} \mu(A_{ij}) < \mu^*(E_i) + \in/2^i \text{ ------------- (2)}$$

Now, $E \subseteq \bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} A_{ij}$, where $A_{ij} \in \mathscr{A}$ for all i and j so that by the definition of $\mu^*(E)$ and (2) we have

$$\mu^*(E) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(A_{ij})$$

$$< \sum_{i=1}^{\infty} \left(\mu^*(E_i) + \frac{\in}{2^i} \right)$$

$$= \sum_{i=1}^{\infty} \mu^*(E_i) + \in .$$

Since $\in > 0$ is arbitrary, we get (1) in this case also.

Thus, μ^* is an outer measure.

19.3.9 Lemma : If $A \in \mathscr{A}$, then A is measurable with respect to μ^*.

Proof : Let E be an arbitrary set of finite outer measure and $\in$ a positive number. Then there is a sequence $\{A_i\}$ in $\mathscr{A}$ such that $E \subseteq \bigcup_i A_i$ and

$$\sum \mu(A_i) < \mu^*(E) + \in \text{ ---------- (1)}$$

Now, $A_i = A_i \cap X = A_i \cap (A \cup \tilde{A}) = (A_i \cap A) \cup (A_i \cap \tilde{A})$. By the additivity of μ on $\mathscr{A}$, we have

$$\mu(A_i) = \mu(A_i \cap A) + \mu(A_i \cap \tilde{A})$$

From which we get,

$$\sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i \cap A) + \sum_{i=1}^{\infty} \mu(A_i \cap \tilde{A}) \text{ ----------- (2)}$$

But $E \cap A \subseteq \bigcup_{i=1}^{\infty} A_i \cap A$, $E \cap \tilde{A} \subseteq \bigcup_{i=1}^{\infty} A_i \cap \tilde{A}$ and the countable subadditivity of μ^* imply that

$$\mu^*(E \cap A) \leq \sum_{i=1}^{\infty} \mu(A_i \cap A) \text{ and } \mu^*(E \cap \tilde{A}) \leq \sum_{i=1}^{\infty} \mu(A_i \cap \tilde{A}) \text{ -------- (3)}$$

Combining (1), (2) and (3) we have

$$\mu^*(E) + \in \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A})$$

and since $\in > 0$ is arbitrary, we get

$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap \tilde{A})$ thus proving that A is μ^* - measurable.

19.3.10 Definition : If μ is a measure on an algebra $\mathscr{A}$ of subsets of X and μ^* is the set function on the class of all subsets of X constructed in the above theorem then μ^* is called the outer measure induced by μ.

19.3.11 Definition : Given an algebra $\mathscr{A}$ of sets, the class of all those sets which are unions of countable collection of sets of $\mathscr{A}$ is denoted by $\mathscr{A}_\sigma$ and the class of all those sets which are intersections of countable collections of sets in $\mathscr{A}_\sigma$ is denoted by $\mathscr{A}_{\sigma\delta}$.

19.3.12 Proposition : Let μ be a measure on an algebra $\mathscr{A}$, μ^* the outer measure induced by μ, and E any set. Then for $\in > 0$, there is a set $A \in \mathscr{A}_\sigma$ with $E \subseteq A$ and

$$\mu^*(A) \leq \mu^*(E) + \in.$$

There is also a set $B \in \mathscr{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$.

Proof : By the definition of μ^*, there is a sequence $\{A_i\}$ from $\mathscr{A}$ such that $E \subseteq \bigcup A_i$ and

$$\sum_{i=1}^{\infty} \mu(A_i) \leq \mu^* E + \in \text{ ---------- (1)}$$

Put $A = \bigcup_{i=1}^{\infty} A_i$. Then, $\mu^*(A) \leq \sum \mu^*(A_i) = \sum \mu(A_i)$ ---------- (2)

From (1) and (2), we get that $A \in \mathcal{A}_\sigma$, $E \subseteq A$ and $\mu^*(A) \le \mu^*(E) + \epsilon$, proving the first part. Now by the first part of the proposition, to each positive integer n, there is a set $A_n \in \mathcal{A}_\sigma$ such that $E \subseteq A_n$ and $\mu^*(A_n) < \mu^*(E) + \frac{1}{n}$. Let, $B = \bigcap_{n=1}^{\infty} A_n$ then, $B \in \mathcal{A}_{\sigma\delta}$ and $E \subseteq B$. Also since $B \subseteq A_n$ for every n, we get

$\mu^*(B) \le \mu^*(A_n) < \mu^*(E) + \frac{1}{n}$ for every n, showing $\mu^*(B) \le \mu^*(E)$. But since, $E \subseteq B$ we also have $\mu^*(E) \le \mu^*(B)$. Thus, $\mu^*(E) = \mu^*(B)$.

The following proposition gives the structure of the measurable sets in the $\sigma-$ finite case.

19.3.13 Proposition : Let μ be a $\sigma-$ finite measure on an algebra $\mathcal{A}$, and let μ^* be the outer measure generated by μ. A set E is μ^* measurable if and only if E is the proper difference $A \sim B$ of a set A in $\mathcal{A}_{\sigma\delta}$ and a set B with $\mu^*(B) = 0$. Each set B with $\mu^*(B) = 0$ is contained in set C in $\mathcal{A}_{\sigma\delta}$ with $\mu^*(C) = 0$.

Proof : Suppose $E = A \sim B$ where $A \in \mathcal{A}_{\sigma\delta}$ and $\mu^*(B) = 0$. Now each set in $\mathcal{A}_{\sigma\delta}$ must be measurable since the measurable sets form a $\sigma-$algebra, while each set of μ^* - measure zero must be measurable since $\bar{\mu}$ is complete. Therefore, E is μ^* - measurable.

Conversely suppose that E is μ^* - measurable. Since μ is $\sigma-$ finite, there exists a sequence $\{X_i\}$ of sets from $\mathcal{A}$ such that $X = \bigcup_i X_i$ and $\mu(X_i) < \infty$ for every i. We can assume that X_i's are disjoint. Now, $X_i \in \mathcal{A} \Rightarrow \mu^*(X_i) = \mu(X_i) < \infty$ for every i. Now, $E = \bigcup_i (E \cap X_i) = \bigcup_i E_i$ where $E_i = E \cap X_i$ and $E_i \in \mathcal{B}$ for every i, since $E \in \mathcal{B}$ and $X_i \in \mathcal{A} \subseteq \mathcal{B}$.

Also, $\bar{\mu}(E_i) = \mu^*(E_i) \le \mu^*(X_i) < \infty$ for every i. Therefore by the above proposition, to each positive integer 'n' there exists a set A_{ni} in $\mathcal{A}_\tau$ such that $E_i \subseteq A_{ni}$ and

$\mu^*\left(A_{ni}\right) \le \mu^*\left(E_i\right) + \frac{1}{n2^i}$. Put $A_n = \bigcup_i A_{n,i}$ then $E = \bigcup_i E_i \subseteq \bigcup_i A_{ni} = A_n$ for every n implies $E \subseteq \bigcap_n A_n = A$ (say). Then, $A_n \in \mathcal{A}_\sigma$ for every n, implies, $A \in \mathcal{A}_{\sigma\delta}$. Now we will show that $\mu^*(A_n \sim E) \le \frac{1}{n}$ for every n.

$$A_n \sim E \subseteq \bigcup_{i=1}^{\infty}\left(A_{ni} \sim E_i\right) \text{ and hence } \mu^*(A_n \sim E) \le \sum_{i=1}^{\infty} \mu^*\left(A_{ni} \sim E_i\right)$$

$$\le \sum_{i=1}^{\infty} \frac{1}{n2^i} = \frac{1}{n}$$

Now, $A \sim E \subseteq A_n - E$.

Hence, $\mu^*(A \sim E) \le \mu^*(A_n - E) \le \frac{1}{n}$ for each n.

Therefore, $\mu^*(A \sim E) = 0$. Thus,

$E = A \sim (A \sim E)$ where $A \in \mathcal{A}_{\sigma\delta}$ and $E \subseteq A$ and $\mu^*(A \sim E) = 0$.

Suppose $\mu^*(B) = 0$. Then B is μ^* - measurable. So by the above proof there is a set C in $\mathcal{A}_{\sigma\delta}$ such that $B \subseteq C$ and $\mu^*(C - B) = 0$.

Now, C, $B \cap C$ and $C \cap \tilde{B}$ are all μ^* - measurable and hence,

$\mu^*(C) = \bar{\mu}(C) = \bar{\mu}(C \cap B) + \bar{\mu}\left(C \cap \tilde{B}\right)$ where $\bar{\mu}\left(C \cap \tilde{B}\right) = \mu^*(C - B) = 0$ and $\bar{\mu}(B \cap C) \le \bar{\mu}(B) = \mu^*(B) = 0$. Thus, $\mu^*(C) = 0$.

Therefore every set B with $\mu^*(B) = 0$ is contained in a set C in $\mathcal{A}_{\sigma\delta}$ with $\mu^*(C) = 0$. We summarize the results of this section in the following theorem.

19.3.14 Theorem (Caratheodory) : Let μ be a measure on an algebra $\mathcal{A}$ and μ^* the outer measure induced by μ. Then the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a $\sigma-$ algebra containing $\mathcal{A}$. If μ is finite (or $\sigma-$ finite) so is $\bar{\mu}$. If μ is $\sigma-$ finite, then

$\overline{\mu}$ is the only measure on the smallest $\sigma-$algebra containing $\mathscr{A}$ which is an extension of μ.

Proof : Let μ be a measure on an algebra $\mathscr{A}$ and μ^* the outer measure induced by μ. Let $\overline{\mu}$ be the restriction of μ^* to the μ^*- measurable sets. Now by corollary 19.3.7, we have, $\mu^*(A)=\mu(A)$ for each $A \in \mathscr{A}$. But, $A \in \mathscr{A} \Rightarrow \mu^*(A) = \overline{\mu}(A)$ since any $A \in \mathscr{A}$ is μ^*-measurable and $\overline{\mu}$ is the restriction of μ^* to the μ^* - measurable sets. Hence, $\overline{\mu}(A) = \mu(A)$ for every $A \in \mathscr{A}$. Thus $\overline{\mu}$ is an extension of μ to a $\sigma-$algebra of μ^* - measurable sets containing $\mathscr{A}$. If μ is finite then $\mu(X)<\infty$ and since, $X \in \mathscr{A}$ we have, $\overline{\mu}(X)=\mu(X)<\infty$ and hence $\overline{\mu}$ is finite. Similarly we can show that if μ is $\sigma-$ finite so is $\overline{\mu}$.

<u>**Uniqueness :**</u> Suppose μ is $\sigma-$ finite. Let $\mathscr{B}$ be the smallest $\sigma-$ algebra containing $\mathscr{A}$ and $\tilde{\mu}$ is a measure on $\mathscr{B}$ such that $\tilde{\mu}/\mathscr{A} = \mu$. Then we have to show that $\overline{\mu} = \tilde{\mu}$. First we will show that $\overline{\mu}$ and $\tilde{\mu}$ agree on $\mathscr{A}_\sigma$. Let $A \in \mathscr{A}_\sigma$ then $A=\bigcup_i A_i$, $A_i \in \mathscr{A}$ and $A_i \cap A_j = \phi$ for $i \neq j$. Consider

$$\overline{\mu}(A) = \overline{\mu}\left(\bigcup_i A_i\right) = \sum_i \overline{\mu}(A_i) = \sum_i \mu(A_i) \quad \text{(since, } \overline{\mu}/\mathscr{A} = \mu\text{)}$$

$$= \sum_i \tilde{\mu}(A_i) \quad \text{(since } \tilde{\mu}/\mathscr{A} = \mu\text{)}$$

$$= \tilde{\mu}\left(\bigcup_i A_i\right) \quad \text{(since } \tilde{\mu} \text{ is a measure)}$$

$$= \tilde{\mu}(A).$$

Next we will show that $\overline{\mu}$ and $\tilde{\mu}$ agree on $\mathscr{B}$. We first show that

$\overline{\mu}(B) = \tilde{\mu}(B)\ \forall B \in \mathscr{B}$ such that $\mu^*(B)<\infty$. Now, let $B \in \mathscr{B}$ such that $\mu^*(B)<\infty$. Now we will show that $\tilde{\mu}(B) \leq \overline{\mu}(B) = \mu^*(B)$. Let $\in > 0$ then there exists $A \in \mathscr{A}_\sigma$ such that $B \subseteq A$ and $\mu^*(A) \leq \mu^*(B) + \in$.

Since $B \subseteq A$, $\tilde{\mu}(B) \leq \tilde{\mu}(A) = \overline{\mu}(A) = \mu^*(A) \leq \mu^*(B) + \in$.

20 Hence, $\tilde{\mu}(B) \leq \mu^*(B) + \in\ \forall \in > 0$. Therefore,

$\tilde{\mu}(B) \leq \mu^*(B)$ for each B in $\mathscr{B}$.

Since $\mathscr{B}$ is the smallest $\sigma-$algebra containing $\mathscr{A}$. We have $\mathscr{B} \subseteq$ the $\sigma-$ algebra of μ^* - measurable sets.

Therefore $B \in \mathscr{B} \subseteq \sigma-$algebra of μ^*-measurable sets implies, $\mu^*(B) = \bar{\mu}(B)$. Therefore,

$$\tilde{\mu}(B) \leq \mu^*(B) = \bar{\mu}(B). \text{ for each } B \in \mathscr{B}.$$

Now we will show that, $\bar{\mu}(B) \leq \tilde{\mu}(B) + \in \ \forall \in > 0$ and for each $B \in \mathscr{B}$ such that $\mu^*(B) < \infty$.

Let $\in > 0$ then there exists $A \in \mathscr{A}_\sigma$ with $B \subseteq A$ and

$$\mu^*(A) \leq \mu^*(B) + \in.$$

Now, $\bar{\mu}(B) \leq \bar{\mu}(A) = \tilde{\mu}(A)$

$$= \tilde{\mu}(B) + \tilde{\mu}(A-B)$$

$$\leq \tilde{\mu}(B) + \mu^*(A-B)$$

$$< \tilde{\mu}(B) + \in$$

since, $\mu^*(B) < \infty$ we have, $\mu^*(A-B) = \mu^*(A) - \mu^*(B) < \in$. Therefore,

$\bar{\mu}(B) \leq \tilde{\mu}(B)$. Hence, $\bar{\mu}(B) = \tilde{\mu}(B)$ for every B in $\mathscr{B}$ with $\mu^*(B) < \infty$.

Since μ is $\sigma-$ finite, there exists a disjoint sequence $\{X_i\}$ in $\mathscr{A}$ such that $X = \bigcup_i X_i$ and $\mu(X_i) < \infty$ for every i. Let $B \in \mathscr{B}$ then $B = B \cap X = B \cap \left(\bigcup_i X_i \right) = \bigcup_i (B \cap X_i)$. Now, $B \in \mathscr{B}$ and $X_i \in \mathscr{A} \subseteq \mathscr{B}$ and hence $B \cap X_i \in \mathscr{B}$. Therefore, since $\tilde{\mu}$ and $\bar{\mu}$ are measures on $\mathscr{B}$, we have,

$$\tilde{\mu}(B) = \sum_i \tilde{\mu}(B \cap X_i) \text{ and}$$

$$\bar{\mu}(B) = \sum_i \bar{\mu}(B \cap X_i)$$

Now, $\mu^*(B \cap X_i) \leq \mu^*(X_i) = \mu(X_i) < \infty$, since $X_i \in \mathscr{A}$

$\mu^*(X_i) = \mu(X_i)$. Therefore by the above proof we have $\tilde{\mu}(B \cap X_i) = \bar{\mu}(B \cap X_i)$ for every i.

Therefore, $\tilde{\mu}(B) = \bar{\mu}(B)$ thus, $\bar{\mu} = \tilde{\mu}$.

19.3.15 Definition : Let X be any set. A collection $\mathscr{C}$ of subsets of X is called a semialgebra of sets if the intersection of any two sets in $\mathscr{C}$ is in $\mathscr{C}$ and the complement of any set in $\mathscr{C}$ is a finite disjoint union of sets in $\mathscr{C}$.

19.3.16 Example :

(i) The collection of all intervals forms a semi-algebra of sets of $\mathbb{R}$.

(ii) If $\mathscr{A}$ is an algebra of subsets of X then $\mathscr{A}$ is a semi-algebra of sets but the converse is not true. For example $X = \{1, 2\}$ and $\mathscr{C} = \{\phi, \{1\}, \{2\}\}$. Clearly $\mathscr{C}$ is a semi-algebra of sets but not an algebra of sets. Since $\{1,2\} \notin \mathscr{C}$.

19.3.17 Self Assessment Question :

If $\mathscr{C}$ is a semi algebra of sets then

$$\mathscr{A} = \{\phi\} \cup \left\{ A : A = \bigcup_{i=1}^{n} C_i,\ C_i \in \mathscr{C},\ C_i\text{'s are disjoint} \right\}.$$

Then show that $\mathscr{A}$ is an algebra of sets. This algebra $\mathscr{A}$ is called the algebra generated by $\mathscr{C}$ i.e. $\mathscr{A}$ is the smallest algebra containing $\mathscr{C}$.

The following proposition gives conditions under which a non-negative set function defined on a semi-algebra can be extended to a measure on an algebra.

19.3.18 Proposition : Let $\mathscr{C}$ be a semi algebra of sets and μ a nonnegative set function defined on $\mathscr{C}$ with $\mu(\phi) = 0 (\text{if } \phi \in \mathscr{C})$. Then μ has a unique extension to a measure on the algebra $\mathscr{A}$ generated by $\mathscr{C}$ if the following conditions are satisfied.

(i) If a set C in $\mathscr{C}$ is the union of a finite disjoint collection $\{C_i\}$ of sets in $\mathscr{C}$, then $\mu C = \sum_i \mu C_i$.

(ii) If a set C in $\mathscr{C}$ is the union of a countable disjoint collection $\{C_i\}$ of sets in $\mathscr{C}$, then $\mu C \leq \sum \mu C_i$.

Proof : Let $\mathscr{C}$ be a semi-algebra of sets and let $\mathscr{A}$ be the algebra generated by $\mathscr{C}$. Then,

$$\mathscr{A} = \{\phi\} \cup \left\{ A : A = \bigcup_{i=1}^{n} C_i,\ C_i \in \mathscr{C},\ C_i\text{'s are disjoint} \right\}$$

Define $\tilde{\mu}$ on A as follows :

Let $A \in \mathscr{A}$ then $A = \bigcup_{i=1}^{n} C_i,\ C_i \in \mathscr{C}, C_i \cap C_j = \phi\ \forall\ i \neq j$.

$$\tilde{\mu}(A) = \sum_{i=1}^{n} \mu(C_i)$$

$\tilde{\mu}$ is well defined.

Suppose $A = \bigcup_{j=1}^{m} D_j,\ D_j \in \mathscr{C}, D_i \cap D_j = \phi\ \forall\ i \neq j$. $C_i \subseteq A = \bigcup_{j=1}^{m} D_j$ implies

$C_i = C_i \cap \bigcup_{j=1}^{m} D_j = \bigcup_{j=1}^{m} C_i \cap D_j$, where, $C_i \cap D_j \in \mathscr{C}$. Therefore by (1) in the statement,

$\mu(C_i) = \sum_{j=1}^{m} \mu\left(C_i \cap D_j\right)$ and hence

$\sum_{i=1}^{n} \mu(C_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mu\left(C_i \cap D_j\right)$.

Similarly we can prove that

$$\sum_{j=1}^{m} \mu\left(D_j\right) = \sum_{j=1}^{m} \sum_{i=1}^{n} \mu\left(C_i \cap D_j\right)$$

Therefore, $\sum_{i=1}^{n} \mu(C_i) = \sum_{j=1}^{m} \mu\left(D_j\right)$. Thus $\tilde{\mu}$ is well-defined.

$\tilde{\mu}$ is a measure :

If $A \in \mathscr{C},\ \tilde{\mu}(A) = \mu(A)$. Hence, $\tilde{\mu}/\mathscr{C} = \mu$ and $\tilde{\mu}(\phi) = \mu(\phi) = 0$ (since $\phi \in \mathscr{C}$).

Now we will show that $\tilde{\mu}$ is finitely additive.

Let $A, B \in \mathscr{A}$ such that $A \cap B = \phi$

Suppose, $A = \bigcup_{i=1}^{n} C_i,\ B = \bigcup_{j=1}^{m} D_j,\ C_i, D_j \in \mathscr{C}$ and pairwise disjoint. Consider,

$$A \cup B = \left(\bigcup_{i=1}^{n} C_i \right) \cup \left(\bigcup_{j=1}^{m} D_j \right), \text{ then,}$$

$$\tilde{\mu}(A \cup B) = \sum_{i=1}^{n} \mu(C_i) + \sum_{j=1}^{m} \mu(D_j) = \tilde{\mu}(A) + \tilde{\mu}(B).$$

By induction we can prove that $\tilde{\mu}$ is finitely additive. Now we will show that $\tilde{\mu}$ is monotone.

Suppose $A \subseteq B$, then $B = A \cup (B \setminus A)$, since $\tilde{\mu}$ is finitely additive

$$\tilde{\mu}(B) = \tilde{\mu}(A) + \tilde{\mu}(B \setminus A)$$

$$\geq \tilde{\mu}(A), \text{ since, } \tilde{\mu}(B \setminus A) \geq 0.$$

Therefore, $\tilde{\mu}$ is monotone.

Suppose $\{A_i\}$ is a sequence of disjoint sets in $\mathscr{A}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$. Put, $A = \bigcup_{i=1}^{\infty} A_i$.

Now for every n, $\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{\infty} A_i \Rightarrow \tilde{\mu}\left(\bigcup_{i=1}^{n} A_i \right) \leq \tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i \right)$, hence

$$\sum_{i=1}^{n} \tilde{\mu}(A_i) \leq \tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i \right)$$

Therefore, $\tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i \right) \geq \sum_{i=1}^{n} \tilde{\mu}(A_i)$, for every n, letting $n \to \infty$ we have, $\tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i \right) \geq \sum_{i=1}^{\infty} \tilde{\mu}(A_i)$.

To prove the other in equality, since $A = \bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$ we can write, $A = C_1 \cup C_2 \cup \cdots \cup C_n$, $C_i \in \mathscr{C}$, $C_i \cap C_j = \phi \, \forall i \neq j$. and $A_k \in \mathscr{A} \Rightarrow A_k = \bigcup_{j=1}^{n_k} A_{kj}$, $A_{kj} \in \mathscr{C}$, A_{kj}'s disjoint.

Consider $C_i = C_i \cap A = C_i \wedge \left(\bigcup_k A_k \right)$

$$= \bigcup_k (C_i \cap A_k)$$

$$= \bigcup_k \left(C_i \cap \bigcup_{j=1}^{n_k} A_{kj} \right)$$

$$= \bigcup_k \left(\bigcup_{j=1}^{n_k} (C_i \cap A_{kj}) \right)$$

For each i, $\left\{ C_i \cap A_{kj} \middle/ \begin{matrix} 1 \le k \le \infty \\ 1 \le j \le n_k \end{matrix} \right\}$ is a sequence of disjoint sets from $\mathscr{C}$. Since μ satisfies condition (2) in the statement, we have,

$\mu(C_i) \le \sum_{k,j} \mu(A_{kj} \cap C_i)$. Therefore,

$$\tilde{\mu}\left(\bigcup_i A_i \right) = \tilde{\mu}(A) = \sum_{i=1}^{n} \mu(C_i) \le \sum_{i=1}^{n} \sum_{k,j} \mu(A_{kj} \cap C_i)$$

$$= \sum_{k,j} \sum_{i=1}^{n} \mu(A_{kj} \cap C_i) \text{ -------- (1).}$$

Now, $A_{kj} \cap C_1, A_{kj} \cap C_2, \cdots\cdots, A_{kj} \cap C_n$ is a finite collection of sets in $\mathscr{C}$ and $\bigcup_{i=1}^{n} (A_j \cap C_i) = A_{kj}$. Hence by condition (1) in the statement we have,

$$\mu(A_{kj}) = \sum_{i=1}^{n} \mu(A_{kj} \wedge C_i)$$

Therefore, $\tilde{\mu}(A) \le \sum_{k_1 s} \mu(A_{kj}) = \sum_k \sum_{j=1}^{n_k} \mu(A_{kj}) = \sum_k \tilde{\mu}(A_k)$.

Thus, $\tilde{\mu}$ is a measure on the algebra $\mathscr{A}$.

To prove the unicity of $\tilde{\mu}$, let μ' be a measure on $\mathscr{A}$ such that $\mu'/\mathscr{C} = \mu$. Now, let $A \in \mathscr{A}$ then, $A = \bigcup_{j=1}^{n} C_j, C_j \in \mathscr{C}$.

$$\tilde{\mu}(A) = \sum_{j=1}^{n} \mu(C_j) = \sum_{i=1}^{n} \mu'(C_j) = \mu'\left(\bigcup_{j=1}^{n} C_j\right) = \mu'(A)$$

Hence, $\tilde{\mu} = \mu'$

Therefore, $\tilde{\mu}$ is unique.

19.4 ANSWERS TO SAQs

19.2.3 : If $E \subseteq \bigcup_{i=1}^{n} E_i$ then $E \subseteq \bigcup_{i=1}^{\infty} E_i$ where $E_{n+1} = E_{n+2} = E_{n+3} = \cdots\cdots = \phi$, so that, by (iii) of Definition 19.2.1,

$$\mu^*(E) \le \sum_{i=1}^{\infty} \mu^*(E_i) = \sum_{i=1}^{n} \mu^*(E_i) + \sum_{i=n+1}^{\infty} \mu^*(E_i)$$

$$= \sum_{i=1}^{n} \mu^*(E_i).$$

inview of (i) of definition 19.2.1 (note that if $i > n$ then $\mu^*(E_i) = \mu^*(\phi) = 0$).

19.2.4 : We have to prove that μ^* is an outer measure if and only if (i) $\mu^*(\phi) = 0$ (ii) $A \subseteq B \Rightarrow \mu^*(A) \le \mu^*(B)$ and (iii) if $E = \bigcup_{i=1}^{\infty} E_i$ and $E_i \cap E_j = \phi$ for $i \ne j$ implies $\mu^*(E) \le \sum_{i=1}^{\infty} \mu^*(E_i)$.

The necessity is trivial. Suppose that μ^* is an extended real valued function such that (i), (ii) and (iii) in the above satisfied. We have to prove (iii) in 19.2.1. Let $E \subseteq \bigcup_{i=1}^{\infty} E_i$. Then define,

$F_1 = E_1$ and $F_i = E_i - (E_1 \cup E_2 \cup \cdots \cup E_{i-1})$ for $i \geq 2$ so that $F_i \cap F_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$. Now since $E \subseteq \bigcup_{i=1}^{\infty} F_i$, we get by (iii) of the hypothesis that $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(F_i)$ and since $F_i \subseteq E_i$ we have by the monotonicity of μ^* that $\mu^*(F_i) \leq \mu^*(E_i)$ so that $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$. Hence μ^* is an outer measure.

19.2.9 (a) : We first show that, $\mu^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$. Put, $A_n = \bigcup_{i=1}^{n} E_i$, for $n = 1, 2, 3, \ldots\ldots$

Since $\mathcal{B}$ is a $\sigma-$algebra we have $A_n \in \mathcal{B}$ for each $n \geq 1$. The measurability of E_n gives

$$\mu^*(A \cap A_n) = \mu^*(A \cap A_n \cap E_n) + \mu^*\left(A \cap A_n \cap \tilde{E}_n\right)$$

$$= \mu^*(A \cap E_n) + \mu^*(A \cap A_{n-1})$$

since, $A_n \cap E_n = E_n$ and $A_n \cap \tilde{E}_n = A_{n-1}$. Now

$$\mu^*(A \cap A_n) = \mu^*(A \cap E_n) + \mu^*(A \cap A_{n-1})$$

$$= \mu^*(A \cap E_n) + \mu^*(A \cap E_{n-1}) + \mu^*(A \cap A_{n-2})$$

= ----------------------------------

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$$= \mu^*(A \cap E_n) + \mu^*(A \cap E_{n-1}) + \cdots\cdots + \mu^*(A \cap E_1)$$

Therefore, $\mu^*(A \cap A_n) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$. Now, $A \cap A_n = A \cap \bigcup_{i=1}^{n} E_i \subseteq A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)$ for every n and hence, $\mu^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \geq \mu^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$ for every n. Letting $n \to \infty$ we get,

$$\mu^*\left(A\cap\bigcup_{i=1}^{\infty}E_i\right)\geq\sum_{i=1}^{\infty}\mu^*(A\cap E_i) \text{ ------------ (1)}$$

$$\mu^*\left(A\cap\bigcup_{i=1}^{\infty}E_i\right)=\mu^*\left(\bigcup_{i=1}^{\infty}A\cap E_i\right)\leq\sum_{i=1}^{\infty}\mu^*(A\cap E_i) \text{ -------- (2)}$$

From (1) and (2) we get

$$\mu^*\left(A\cap\bigcup_{i=1}^{\infty}E_i\right)=\sum_{i=1}^{\infty}\mu^*(A\cap E_i)$$

(b) Put $A=X$ in (a) then we get

$\mu^*\left(\bigcup_{i=1}^{\infty}E_i\right)=\sum_{i=1}^{\infty}\mu^*(E_i)$ proving that μ^* is countably additive on $\mathscr{B}$, the $\sigma-$ algebra of μ^*-measurable sets.

19.3.5 : $$\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A},A_i\cap A_j=\phi \text{ for } i\neq j \text{ and } \bigcup_{i=1}^{\infty}A_i\supseteq E\right\}$$

$$\subseteq\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A},\text{and } \bigcup_{i=1}^{\infty}A_i\supseteq E\right\}$$

Hence, $\inf\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A} \text{ and } \bigcup_{i=1}^{\infty}A_i\supseteq E\right\}\leq$

$$\inf\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A},\ A_i\cap A_j=\phi \text{ for } i\neq j \text{ and } \bigcup_{i=1}^{\infty}A_i\supseteq E\right\}$$

That is, $\mu^*(A)\leq\inf\left\{\sum_{i=1}^{\infty}\mu(A_i):A_i\in\mathscr{A},\ A_i\cap A_j=\phi \text{ for } i\neq j \text{ and } \bigcup_{i=1}^{\infty}A_i\supseteq E\right\}$

Let $A_i \in \mathscr{A}$ and $\bigcup_{i=1}^{\infty} A_i \supseteq E$. Then we know that there exists $B_i \in \mathscr{A}$ such that $B_i \cap B_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$, $B_i \subseteq A_i\ \forall\ i$. Then, $\bigcup_{i=1}^{\infty} B_i \supseteq E$ and hence,

$$\sum_{i=1}^{\infty} \mu(B_i) \geq \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathscr{A},\ A_i \cap A_j = \phi \text{ for } i \neq j,\ \bigcup_{i=1}^{\infty} A_i \supseteq E\right\}$$

But, $\sum_{i=1}^{\infty} \mu(A_i) \geq \sum_{i=1}^{\infty} \mu(B_i)$.

Therefore, $\sum_{i=1}^{\infty} \mu(A_i) \geq \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathscr{A},\ A_i \cap A_j = \phi \text{ for } i \neq j \text{ and } \bigcup_{i=1}^{\infty} A_i \supseteq E\right\}$

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathscr{A},\ \bigcup_{i=1}^{\infty} A_i \supseteq E\right\}$$

$$\geq \inf\left\{\sum_{i=1}^{\infty} \mu(A_i) : A_i \in \mathscr{A},\ A_i \cap A_j = \phi \text{ for } i \neq j \text{ and } \bigcup_{i=1}^{\infty} A_i \supseteq E\right\}$$

Therefore we get equality.

19.3.17 : $\phi \in \mathscr{A}$. Suppose $E \in \mathscr{A}$, $F \in \mathscr{A}$ with $E = \bigcup_{i=1}^{n} C_i$ and $F = \bigcup_{j=1}^{m} D_j$ where C_i, D_j are sets in $\mathscr{A}$ such that $C_i \cap C_j = \phi$ for $i \neq j$, $D_j \cap D_s = \phi$ for $j \neq s$ then

$E \cap F = \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} (C_i \cap D_j)$ showing $E \cap F \in \mathscr{A}$, since $C_i \cap D_j \in \mathscr{C}$ for any i and j.

Also if $E \in \mathscr{A}$ and $E = \bigcup_{i=1}^{n} E_i$ with $E_i \cap E_j = \phi$ for $i \neq j$, $E_i \in \mathscr{C}$ then $\widetilde{E} = \bigcap_{i=1}^{n} \widetilde{E}_i$. Now each $\widetilde{E}_i$ is a union of a finite collection from $\mathscr{C}$ implies that $\widetilde{E}$ also has this property. That is $\widetilde{E} \in \mathscr{A}$ whenever $E \in \mathscr{A}$.

Thus $\mathscr{A}$ is an algebra.

19.5 MODEL EXAMINATION QUESTIONS :

19.5.1 : Define an outer measure μ^* on the class of all subsets of a set X and μ^*-measurability of a set prove that the class $\mathscr{B}$ of all μ^*-measurable sets is a $\sigma-$ algebra of subsets of X . Also prove that the restriction $\bar{\mu}$ of μ^* to $\mathscr{B}$ is a complete measure on $\mathscr{B}$.

19.5.2 : Define a measure on an algebra $\mathscr{A}$ of subsets of X . Prove that it induces an outer measure on the class of all subsets of X .

19.5.3 : State and prove Caratheodary's theorem.

REFERENCE BOOK

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