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M. Sc. (STATISTICS) – FINAL YEAR :: PAPER – II : OPERATIONS RESEARCH

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Lesson – 1

Nature and Scope of Operations Research

1.0 OBJECTIVE :

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STRUCTURE

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1.1 ORIGIN AND DEVELOPMENT

The term, *Operations Research*, was first coined in 1940 by McClosky and Trefthen of the United Kingdom. This subject came into existence in the military context. During World War II, military management called on scientists from various disciplines and organised them into teams, to assist in solving strategic and tactical problems, i.e., to discuss, evolve and suggest ways and means to improve the execution of various military projects. Their suggestions showed remarkable improvements. This new approach to systematic and scientific study of the operations of the system was called the ***Operations Research*** or ***Operational Research*** (abbreviated as O.R.).

Following the end of the World War, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex execution-type problems. During the year 1950, O.R. achieved recognition as a subject worthy of academic study in the universities. Since then, the subject has been gaining more and more applications in Economics, Management, Public Administration, Behavioral Sciences, Social work, Mathematics, Commerce and Engineering.

In India, Operations Research came into existence in 1949 with the opening of an O.R. unit at the Regional Research Laboratory at Hyderabad. At the same time, another group was set up in the Defense Science Laboratory which devoted itself to the problems of stores, purchase and planning. In 1953, a O.R. unit was established in the Indian Statistical Institute, Calcutta, for the application of O.R. methods in national planning and survey. O.R. Society of India was formed in 1957. It became a member of the International Federation of O.R. Societies in 1959.

Prof. Mahalonobis was the first person to use O.R. Techniques in Planning in India. He formulated the Second Five-Year Plan with the help of O.R. techniques to forecast the trends of demand, availability of resources and schemes that are useful for development of our country's economy.

Currently O.R. is a popular subject in almost every management institute and is gaining tremendous popularity in almost every industrial establishment.

1.2 NATURE AND MEANING

Literally the word “Operation” can be defined as the action that is applied to a problem or hypothesis and the word “Research” is an organized procedure to investigate or analyze the facts about the same.

Meaning of Operations Research can be interpreted in many ways subject to the problem to which it is applied. Some of them are given below :

1. OR is a scientific method of providing an analytical and objective basis for decisions.
2. Or is the art giving “bad answers” to problems which otherwise have “worse answers”.
3. OR can be considered as an attempt to study those operations of modern society which involves organizations of men or industries.
4. OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so .as to provide control on operations with optimum solution to the problems.
5. OR is an aid for the executive in making his decisions by providing him the needed quantitative information based on scientific method of analysis.

1.3 DEFINITIONS

Many definitions of OR have been suggested from time to time. Some of the definitions suggested are given below.

1. OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. **-Morse & Kimball**
2. OR, in the most general sense, can be characterized as the application of scientific methods, tools and techniques to problems involving the

operations of systems so as to provide those in control of the operations with optimum solutions to the problems.

-Churchman, Ackoff, Arnoff

3. Operations research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive when he tries to achieve a thorough going rationality in dealing with his decision problems. **- Miller and Starr**

4. Operations research is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. **- C. Kittel**

5. Operations research is the systematic, method-oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision-making. **-E.L. Arnoff and M.J. Netzorg**

6. Operations research is the application of scientific methods to problems arising from operations involving integrated systems of men, machines and materials. It normally utilizes the knowledge and skill of an interdisciplinary research team to provide the managers of such systems with optimum operating solutions. **-Fabrycky and Torgersen**

7. Operations research is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man-machine systems; and operations research workers are actively engaged in applying this knowledge to practical problems in business, government and society. **-Operations Research Society of America**

8. Operations research utilizes the planned approach (updated scientific method) and an interdisciplinary team in order to represent complex functional relationships as mathematical models for the purpose of providing a quantitative basis for decision-making and uncovering new problems for quantitative analysis. **-Thierauf and Klekamp**

The most comprehensive and modern definition of operations research can be summarized as below :

9. O.R. is the application of modern methods of mathematical science to complex problems involving management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurement of factors such as chance and risk to predict and compare the outcomes of alternative decisions, strategies or controls. **-J.O.R. Society, U.K**

1.4 FEATURES OF OR

The features of OR approach to any decision and control problems can be considered under the following methodologies :

1.4.1 Interdisciplinary Approach

Interdisciplinary teamwork is essential because while attempting to solve a complex management problem one person may not have the complete knowledge of all aspects (such as economic, social, political, psychological, engineering etc.) of it. This means we should not expect a desirable solution to managerial problems by a single person. Therefore, a team of individuals specializing in different fields like mathematics, statistics, economics, engineering, computer science, psychology, etc. can work together so that each aspect of the problem could be analyzed by a particular specialist in that field in order to arrive at an appropriate and desirable solution of the problem. However, certain problems may be analyzed even by one individual also.

1.4.2 Methodological Approach

The scientific method consists of observing and defining the problem; formulating and testing the hypothesis; and analyzing the results of the test. The data so obtained are then used to decide whether the hypothesis should be accepted or not. If the hypothesis is accepted, then the results should be implemented. Otherwise an alternative hypothesis has to be formulated.

1.4.3 Wholistic Approach

While arriving at a decision, an operations research team examines the relative importance of all conflicting and multiple objectives and validity of the claims of various departments of the organization from the perspective of the whole organization.

1.4.4 Scientific Approach

The most important feature of operations research is the use of the scientific method and building of decision models. There are three phases of the scientific method on which OR practice is based.

a) **Judgment Phase :** This phase includes (i) identification of the real-life problem, (ii) selection of an appropriate objective and the various variables related to this objective, (iii) application of the appropriate scale of measurement, i.e., deciding the measures of effectiveness, and (iv) formulation of an appropriate model of the problem, so that a solution at the decision-maker's goals can be sought.

b) **Research Phase :** This phase is the largest and longest among other two phases. However, the remaining two are also equally important as they provide the basis for a scientific method. This phase utilizes : (i) observations and data collection for a better understanding of the problem, (ii) formulation of hypothesis and models, (iii) observation and experimentation to test the hypothesis on the basis of additional data, (iv) analysis of the available information and verification of the hypothesis using pre-established measures of desirability, (v) predictions of various results from the hypothesis and (iv) generalization of the result and consideration of alternative methods.

c) **Action Phase :** This phase consists of making recommendations for implementation. One must be aware of the environment in which the problem occurred, objective, assumption, and omissions of the model of the problem.

c) **Action Phase :** This phase consists of making recommendations for implementation. One must be aware of the environment in which the problem occurred, objective, assumption, and omissions of the model of the problem.

1.5 DIFFERENT MODELS

We can solve both simple and complex problems of real life easily by concentrating on some portion or some key features instead of on every detail of a system. This method of maintaining only the essential elements of the system, which may be constructed in various forms by establishing relationship among specified variables and parameters of the system, is called a model. Models do not, and cannot, represent every aspect of system because of the in numerable and changing characteristics of the real life problems to be represented. For example, to study the flow of material through a factory a scaled diagram on paper showing the factory floor, position of equipment, tools, and workers can be constructed. It would not be necessary to give such details as the color of machines, the heights of the workers, or the temperature of the building. But, for a model to be effective, it must be representative of those aspects of the system that are being investigated and have a major impact on the decision situation.

A model, allows to examine the behavioral changes of a system without' disturbing the on going operations.

A model should be as simple as possible so as to give the desired result. However, over simplifying the problem can lead to a poor decision.

1.5.1 Classification of O.R. Models

Models can be classified with respect to various features. Some of them are discussed below.

1. **Physical models :** Physical models are useful only in design problems because they are easy to observe, build and describe. For example, in

   the aircraft industry, scale models of the proposed new aircraft are build and tested in wind tunnels to record the stresses experienced by the air frame.

   Some of the physical properties and characteristics of the system they represent are taken into consideration in these models. It is an idealized form or a scaled version of the system. Examples are blue prints of a home, globes, photographs, drawings, atom etc.

   Some models represent a system by the set of properties different from that of the original system and does not resemble physically. After the problem is solved, the solution is re-interpreted in terms of the original system. For example, Graphs of time series, stock-market changes, frequency curves, etc. are be used to represent quantitative relationship between any two properties.

2. **Verbal Models :** These models describes a situation in written or spoken language. Written sentences, books, etc. are examples of a verbal model.

   Descriptive models simply describe some aspects of a situation based on observation survey, questionnaire results or other available data of a situation and do not predict or recommend.

   Examples of such models are : (i) Organization chart, (ii) Plant layout diagram, (iii) Block diagram representing an algorithm etc.

3. **Mathematical Models :** These models involve the use of mathematical-symbols, letters, numbers and mathematical operators (+, -, ÷, x) to represent relationship among various variables of the system to describe its properties or behavior.

4. **Predictive models :** They relate dependent and independent variables and permit trying out, 'what if questions.

For example, S = *a* + *b*A + cI is a model that describes how the sale (S) of a product changes with a change in advertising expenditure (A) and disposable personal income (I).

5. **Optimization models :** These models provide the "best" or "optimal" solution to problems subject to certain limitations on the use of resources.

   For example, in mathematical programming, models are formulated for optimizing the given objective function, subject to restrictions on resources in the context of the problem under consideration and non-negativity of variables.

6. **Static models :** Static models present a system at some specified time and do not account for changes over time. For example, an inventory model can be developed and solved to determine economic order quantity for the next period assuming that the demand in planning period would remain the same as that for today.

7. **Dynamic models :** In a dynamic model, time is considered as one of the variables and admit the impact of changes generated by time in the selection of the optimal courses of action. Dynamic programming is an example of a dynamic model.

8. **Deterministic models :** If all the parameters, constants and functional relationships are assumed to be known with certainty when decisions are taken, then the model is said to be deterministic. Linear programming models are examples of deterministic models.

9. **Probabilistic (stochastic) models :** Models in which at least one parameter or decision variable is a random variable, are called probabilistic (or stochastic) models. However, it is possible to predict a pattern of values of the variable by their probability distributions. Insurance against risk of fire, accidents, sickness, etc. are examples where the pattern of events is studied in the-form of a probability distribution.

1.6 GENERAL METHODS FOR SOLVING OR MODELS

In general, the following three methods are used for solving OR models. In all these methods, values of decision variables are obtained that optimize the given objective function.

1.6.1 Analytical (or Deductive) Method

In this method, optimization techniques such as calculus, finite difference and graphs are used for solving an OR model. In this case a general optimal solution can be obtained in a non-iterative manner.

1.6.2 Numerical (or Iterative) Method

When analytical methods fail to obtain the solution of a particular problem due to its complexity in terms of constraints or number of variables, then a numerical (or iterative) method is used to get the solution. In this method instead of solving the problem directly a general (algorithm) is applied to obtain a specific numerical solution. These methods are specific and different for different problems.

1.6.3 Monte Carlo Method (Simulation Method)

In simulation method, a model or situation to represent the real model or environment and studies are made. In this method random samples of random variables are drawn to know what is happening to the system for a selected period of time under different conditions. The random samples from a probability distribution that represents the real life system and from this probability distribution, the value of the desired random variable can be estimated. For example pilot training, Rocket launching are preferred with simulations first.

1.7 SUMMARY :

In this lesson we discussed about Origin and development of Operations Research, meaning and different definitions of Operations Research, different approaches to handle problems of Operations Research, a number of models classified based on various criterion and general methods to solve them.

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Lesson – 2

Linear Programming Problem - Formulation

2.0 OBJECTIVE :

- Structure of an Linear Programming Problem (LPP)
- Assumptions on an LPP
- Guidelines for formulation of an LPP
- Steps for formulation of an LPP

STRUCTURE

2.1 INTRODUCTION

Linear Programming (LP) is the general technique of optimum allocation of ‘limited’ resources, such as labour, material, machine, capital, energy, etc. to several competing activities, such as product, services, jobs, new equipment, projects, etc. on the basis of a given criterion of optimality. The term 'limited' here is used to describe availability of scarce resources during planning period. The criterion of optimality, generally is either performance, return on investment, utility, time, distance, etc. The optimality means either Minimization or Maximization.

The word linear in LPP is used to describe the relationship of two or more variables in a model. Thus a given change in one variable will always cause, a resulting proportional change in another variable. The word programming here is used to specify a sort of planning that involves the economic allocation of limited resources by adopting a particular strategy amongst various alternative strategies to achieve the desired objective.

Out of several courses of action available, the best or optimal is selected. A course of action is said to be most desirable or optimal if it optimizes (maximizes or minimizes) some measure of criterion of optimality such as profit, cost, rate of return, time, distance, utility, etc.

2.2 STRUCTURE OF AN LP PROBLEM

The general structure of any LP model consists essentially of three components.

2.2.1 The activities (variables) and their relationships

The activity values represent the extent to which each activity is performed. These are represented by x_1, $x_2 \ldots x_n$. For example in a product-mix problem, the activities of interest are the production of several products under consideration. These activities are also known as decision variables because they are under the decision maker's control. These decision variables, usually are inter-related in terms of consumption of limited resources, require simultaneous solutions. All decision variables are continuous (unless otherwise specified), controllable and non-negative. That is, $x_1 \geq 0$, $x_2 \geq 0$, ….. , $x_n \geq 0$.

2.2.2 The objective function

The objective function of each LP problem is a mathematical representation of the objective in terms of a measurable quantity such as profit, cost, revenue, distance etc. It is represented in one of two forms:

Optimize (Maximize or Minimize) $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ where Z is the measure-of-performance variable, which is a function of $x_1x_2, \ldots, x_n$. The $c_1, c_2, \ldots, c_n$ are parameters or uncontrollable variables that give the contribution of a unit of the respective variable $x_1x_2, \ldots, x_n$. to the measure-of-performance Z. The optimal value of the given objective function is obtained by the graphical method or simplex method. Here, $c_1c_2, \ldots, c_n$. are normally constants.

2.2.3 The constraints

There are always certain limitations (or constraints) on the use of limited resources, e.g., labour, machine, raw material, space, money, etc. Such constraints must be expressed as linear equalities or inequalities in terms of decision variables. The solution of an LP model must satisfy these constraints. The condition $x_1 \geq 0, \ldots x_n \geq 0$ is called a non-negativity constraint.

2.3 ASSUMPTIONS OF LINEAR PROGRAMMING

The following four basic assumptions are necessary for all linear programming models :

2.3.1 Parameters

In all LP models it is assumed that all model parameters such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit of decision variable must be known and constant. In some cases these may be either random variables represented by a known distribution (general or statistical) or tend to change; then the given problem can be solved by a stochastic LP model or parametric programming.

2.3.2 Nature of variables

The solution values of decision in variables and resources are assumed to have either continuous values or whole numbers (integers) or a mix of both (integer and fractional). However, if only integer variables are desired, e.g. machines, employees, etc. then the integer programming method may be applied to get the desired values. They are either deterministic or random variables.

2.3.3 Type of LP Model

The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum of the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision variable respectively. For example, the total profit earned by the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed by A and B must be equal to the sum of resources used for A and B individually.

2.3.4 Linearity

All the decision variables, present in the objective function and constraints should be linear in nature.

2.4 GENERAL MATHEMATICAL MODEL OF A LINEAR PROGRAMMING PROBLEM

The general linear programming problem with n decision variables and m constraints can be stated in the following form :

Find the values of decision variables $x_1, x_2, \ldots, x_n$ so as to

Optimize (Max or Min) $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to the linear constraints,

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ (\leq, =, \geq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ (\leq, =, \geq)\ b_2$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \ (\leq, =, \geq)\ b_m$

and $x_1, x_2, \ldots, x_n \geq 0$

The above formulation can also be expressed in a compact form using summation sign :

Optimize (Max. or Min.) $Z = \sum_{j=1}^{n} c_j x_j$ (objective function) (2.4.1)

subject to the linear constraints

$\sum_{j=1}^{n} a_{ij}\, x_i \ (\leq, =, \geq)\ b_i$, $i = 1, 2, \ldots, m$ (constraints) (2.4.2)

and $x_j \geq 0$; $j = 1, 2, \ldots, n$ (non-negativity condition) (2.4.3)

Here, the c_j's are coefficients representing the per unit contribution of decision variable x_j, to the value of objective function. The a_{ij}'s are called the technological coefficients or input-output coefficients. These represent the amount of i^{th} resource consumed per unit of variable x_j. In the given constraints, the a_{ij}'s can be positive, negative or zero. The b_i represent the total availability of the i^{th} resource. The term resource is used in a very general sense to include any numerical value associated with the right hand side of a constraint. It is assumed that $b_i \geq 0$ for all i. However, if any $b_i < 0$, then both sides of constraint 'i' can be multiplied by -1 to make $b_i > 0$ and reverse the inequality of the constraint in the middle.

In the general LP problem, the expression ($\leq$, =, $\geq$) means that in any specific problem each constraint may take only one of the five possible forms :

(i) less than or equal to ($\leq$)

(ii) greater than or equal to ($\geq$)

(iii) equal to (=)

(iv) Strictly greater (>)

(v) Strictly less (<)

2.5 GUIDELINES ON LINEAR PROGRAMMING MODEL FORMULATION

The usefulness of linear programming as a tool for optimal decision making on resource allocation is based on its applicability to many diversified decision problems. The effective use and application require, as a first step, the mathematical formulation of the LP model when the problem is presented in words.

2.5.1 Steps of LP Model Formulation

Step-1: Define Decision Variables

(a) Express each constraint in words. For this first see whether the constraint is of the form, $\geq$ (at least as large as), or of the form $\leq$ (no longer than) or = (exactly equal to), not less than (>), not more than (<).

(b) Then express the objective function in words.

(c) Step 1 (a) and (b) should then allow you to verbally identify the decision variables.

If there are several decision alternative available, then to identify the decision variables you have to ask yourself the question : What decisions must be made in order to optimize the objective function?

Having accomplished step 1 (a) through (c), decide symbolic notation for the decision variables and specify their units of measurement. Such specification of units of measurement would help in interpreting the final solution of the LP problem.

Step-2 : Formulate the Constraints

Formulate all the constraints imposed by the resource availability and express them as linear equality or inequality in terms of the decision variables defined in step 1.

These constraints define the range within which values of decision variables can lie. Wrong formulation can lead to either solutions which are not feasible or excluding some solutions which are really feasible and possibly optimal.

Step-3 : Formulate the Objective Function

Define the objective function. That is, determine whether the objective function is to be maximized or minimized. Then express it as a linear function of decision variables multiplied by their profit or cost contributions.

The following are certain examples of LP model formulations.

2.6 STEPS FOR MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING PROBLEM

Following the above guidelines, the procedure for Mathematical formulation of a linear programming problem consists of the following major steps.

Step-1 : Study the given situation to find the key decisions to be made.

Step-2 : Identify the variables involved and designate them by

$$x_1, x_2, \ldots, x_j.$$

Step-3 : State the feasible alternatives which generally are : $x_j \geq 0$ for all j.

Step-4 : Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables (x_j) input -output coefficients (a_{ij}) and availability coefficients (b_i), i = 1, 2, ..., m and j = 1, 2, .., n.

Step-5 : Identify the objective function and express it as a linear function of the decision variables $x_1, x_2, \ldots, x_n$ and cost coefficents $c_1, c_2, \ldots, c_n$.

2.7 EXAMPLE OF LPP FORMULATION

A company, engaged in producing tinned food, has 300 trained employees on the rolls each of whom can produce one tin of food in a week. Due to the developing taste of the public for this kind of food, the company plans to add the existing labour force by employing 150 people, in a phased manner, over the next five weeks. The newcomers would have to undergo a two-week training programme before being put to work. The training is to be given by employees from amongst the existing ones and it is known that one employee can train three trainees. Assume that there would be no production from the trainers and the trainees during training period as the training is off-the-job. However, the trainees would be remunerated at the rate of Rs. 300 per week, the same rate as for the trainers.

The company has booked the following number of tins to supply during the next five weeks:

Weeks :	1	2	3	4	5
No. of cans :	280	298	305	360	400

Assume that the production in any week would not be more than the number of tins ordered for, so that every delivery of the food would be 'fresh'.

Formulate a LP model to develop a training schedule that minimize the labour cost over the five week period.

LP model formulation: The data of the problem is summarized as given below :

Tins supplied

(i)	Weeks	:	1	2	3	4	5
	No. of cans	:	280	298	305	360	400

(ii) Each trainee has to undergo a two-week training,

(iii) One employee is required to train three trainees.

(iv) Every trained worker producing one Tin/week but no production from trainers and trainees during training.

(iv) Number of employees to be employeed = 150

(v) The production in any week not to exceed the Tins required

(vi) Number of weeks for which newcomers would be employed : 5,4, 3, 2, 1.

From the given information you may observe following facts :

(a) Workers employed at the beginning of the first week would get salary for all the five weeks, those employed at the beginning of the second weeks would get salary for four weeks and so on.

(b) The value of the objective .function would be obtained by multiplying it by 300 because each person would get a salary of Rs. 300 per week.

(c) Inequalities have been used in the constraints because some workers might remain idle in some week(s).

Decision variables : Let

x_1, x_2, x_3, x_4, x_5 = number of trainees appointed in the beginning of weeks 1, 2, 3, 4 and 5 respectively.

The LP model

Minimize (total labour force) $Z = 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5$ subject to the constraints

(i) Capacity constraints

$300 - \frac{x_1}{3} \geq 280$

$300 - \frac{x_1}{3} - \frac{x2}{3} \geq 298$

$300 + x_1 - \frac{x2}{3} - \frac{x3}{3} \geq 305$

$300 + x_1 + x_2 - \frac{x3}{3} - \frac{x4}{3} \geq 360$

$300 + x_1 + x_2 + x_3 - \frac{x4}{3} - \frac{x5}{3} \geq 400$

(ii) New recruitment constraints

$x_1 + x_2 + x_3 + x_4 + x_5 = 150$

and $x_1, x_2, x_3, x_4, x_5 \geq 0$

2.8 SUMMARY

In this lesson the structure and assumptions of a linear programming problem (LPP) are discussed. Further general mathematical model of linear programming problem, guidelines for formulation of linear programming problem and steps for the formulation of LPP are also discussed. The steps are illustrated with an example.

2.9 EXERCISE

1. A company sells two different products A and B, making a profit of Rs. 40 and Rs. 30 per unit on them respectively. They are produced in a common production process and are sold in two different markers. The production process has a total capacity of 30,000 man-hours. It takes three hours to produce a unit of A and one hour to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8,000 units and that of B is 12,000 units.

Subject to these limitations, products can be sold in any combination. Formulate this problem as a linear programming model.

2. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the input and output per production run are given as follows:

Process (units)	Input (units)		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amount available of crude A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run from process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Formulate this problem as a linear programming model.

3. A firm places an order for a particular product at the beginning of each month and the product is received at the end of the month. The firm sells during the month from the stocks and it can sell any quantity.

The prices at which the firm buys and sells vary every month. The following table shows the projected buying and selling prices for the next four months :

Month	Selling price (Rs.) (during the month)	Purchase Price (Rs.) (beginning of the month)
April	-	75
May	90	75
June	60	60
July	75	-

The firm has no stocks on hand as on April 1, and does not wish to have any stocks at the end of July. The firm has a warehouse of limited size, which can hold a maximum of 150 units of the product. The firm has no stocks on hand as on April 1, and does not wish to have any stocks at the end of July. The firm has a warehouse of limited size, which can hold a maximum of 150 units of the product.

The problem is to determine the number of units to buy and sell each-month to maximize the profits from its operations. Formulate this problem as a linear programming problem.

4. A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw material (A and B) of which 4000 and 6000 units respectively are available. The raw material requirements per unit of the three models are as follows:

Raw Material	Requirement per unit of given model		
	I	II	III
A	2	3	5
B	4	2	7

The labour time of each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2,500 units of model I. A market survey indicates that the minimum demand of the three models are : 500, 500 and 375 units respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II and III are Rs. 60, 40 and 100 respectively.

Formulate this problem as an LP model in order to determine the number of units of each product which will maximize profit.

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Lesson – 3

Graphical Method

3.0 Objective :

- Basic definitions
- Graphical method of solving a linear programming problem
- Identification of special cases in graphical method

Structure

3.1 Introduction

An "optimal as well as feasible" solution to an LP problem is obtained by choosing among several values of decision variables x_1, x_2,..., x_n the one set of values that satisfy the given set of constraints simultaneously and also provide the optimal (maximum or minimum) value of the given objective function.

For LP problems that have only two variables it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints on paper to locate a best (optimal) solution. The technique used to identify the optimal solution is called the graphical solution technique for an LP problem with two variables.

In this chapter we shall discuss the graphical solution techniques.

(i.e., Extreme point enumeration approach) to find the optimum solution to an LP problem.

However, before working on the LP problem solution, certain important concepts are defined.

3.2 Important Definitions

Solution : The values of decision variables x_j (j = 1, 2,..., n) which satisfy the constraints of a general LP model is called the solution to that LP model.

Feasible solution : Solution values of decision variables x_j (j = 1, 2,.., n) which satisfy the constraints and non-negativity conditions of a general LP model are said to constitute the feasible solution to that LP model.

Basic solution : For a set of m equations in n variables ($n > m$), a solution that has m variables at positive level ($x_j > 0$) and the remaining (n - m) variables at 'zero' ($x_j = 0$) level is called a basic solution.

The (n - m) variables whose value did not appear in this solution are called non-basic variables and the remaining m variables are called basic variables.

Basic feasible solution : A feasible solution to an LP problem which is also the basic solution is called the basic feasible solution. That is, all basic variables assume non-negative values. Basic feasible solutions are of two types:

(a) *Degenerate :* A basic feasible solution is called degenerate if one or more of the basic variables take zero value. That is, basic variables taking positive values less than m in number.

(b) *Non-degenerate :* A basic feasible solution is called non-degenerate if all m basic variables are nonzero and positive.

Optimum basic feasible solution : A basic feasible solution which optimizes (maximizes or minimizes) the objective function of the given LP model is called an optimum basic feasible solution.

Unbounded solution : A solution which can be increase or decrease or the value of objective function of LP problem indefinitely, is called an unbounded solution.

3.3 Graphical Solution Method of LP Problems

While obtaining the optimal solution to the LP problem by the graphical method, the statements of the following theorems of linear programming are used.

(i) The collection of all feasible solutions to an LP problem constitutes a convex -set whose extreme points correspond to the basic feasible solutions.

(ii) If the convex set of the feasible solutions of the system $\mathbf{Ax = b, x \geq 0}$, is a convex polyhedron, then at least one of the extreme points gives an optimal solution.

(in) If the optimal solution occurs at more than one extreme point, then the value of the objective function will be the same for all convex combinations of these extreme points.

Remarks

1. A convex set is a polyhedron and by 'convex' we mean that if any two points of the polygon are selected arbitrarily, then a straight line segment joining these two points lies completely within the polygon.

2. The extreme points of the convex polyhedron give the basic solution to the LP problem.

3.4 Graphical Solution Method – Steps

The major steps in the solution of a linear programming problem by graphical method are summarised as follows :

Step-1 : Identify the problem - the decision variables, the objective and the restrictions.

Step-2 : Set up the mathematical formulation of the problem.

Step-3 : Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step-4 : The feasible region obtained in step 3 may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step-5 : Find out the value of the objective function at each corner (solution) point determined in step 4.

Step-6 : Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

3.5 Examples for Bounded Solutions

3.5.1 Example : Use the graphical method to solve the following LP Problem

Maximize $Z = 15x_1 + 10x_2$

subject to the constraints

$4x_1 + 6x_2 \leq 360$

$3x_1 + 0x_2 \leq 180$

and $0x_1 + 5x_2 \leq 200$

$x_1, x_2 \geq 0$

Solution: ***Step-1 : State the problem in mathematical form.*** The given LP problem is already in mathematical form.

Step-2 : Plot the constraints on a graph paper and find the feasible region.

We shall plot x_1 on the horizontal axis and x_2 on the vertical axis. Each constraint will be plotted on the graph by treating it as a linear equation and then appropriate inequality conditions will be used to mark the area of feasible solutions.

Consider the first constraint $4x_1 + 6x_2 \leq 360$. Treat it as the equation, $4x_1 + 6x_2 = 360$.

Re-write the constraint as

$$\frac{x_1}{360/4} + \frac{x_2}{360/4} = 1 \; or \; \frac{x_1}{90} + \frac{x_2}{60} = 1$$

This equation indicates that, when it is plotted, the graph cuts an x_1 inter cut of 90 and x_2 inter cut of 60. These two points are then connected by a straight line as shown in Fig. 3.5.1. But the inequality and non-negativity condition can only be satisfied by the shaded area (feasible region) as shown in Fig. 3.5.1.

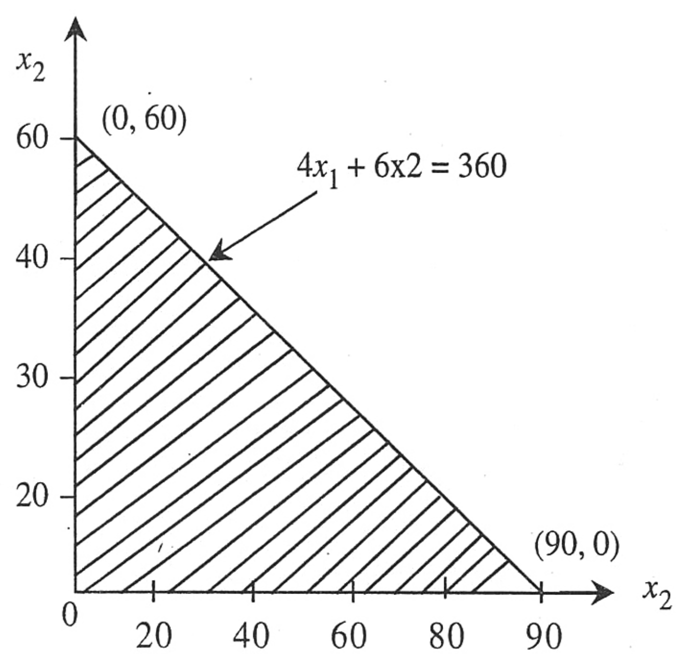


Fig. 3.5.1 Graphical Solution of LP Problem

Similarly the constraints $3\ x_1 \leq 180$ and $5x_2 \leq 200$ are also plotted on the graph and indicated by the shaded area as shown in Fig. 3.5.2.

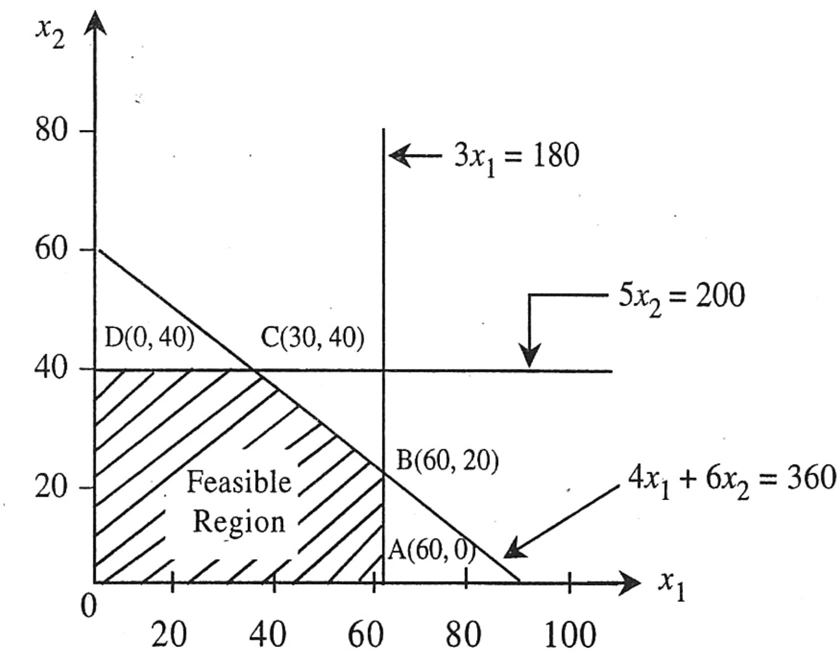


Fig. 3.5.2 Graphical Solution of LP Problem

Since all constraints have been graphed, the area which is bounded by all the constraints lines, including all the boundary points is called the feasible region or solution space. The feasible region is shown in Fig, 3.5.2 by the shaded area OABCD.

Step 3 : Determine coordinates of extreme points

Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates. The coordinates of extreme points of the feasible region are :

0 = (0, 0), A = (60, 0), B = (60, 20), C = (30, 40), D = (0,40).

Step 4: Evaluate the value of objective at extreme points.

Evaluate objective function value at each extreme point of the feasible region is shown below :

Extreme point	Coordinates (x_1, x_2)	Objective function value $Z = 15x_1 + 10x_2$
O	(0, 0)	15(0)+ 10(0) = 0
A	(60, 0)	15(60) + 10(0) = 900
B	(60, 20)	15(60) + 10(20)= 1100
C	(30, 40)	15(30) + 10(40) = 850
D	(0, 40)	15(0) + 10(40) = 400

Step 5: Determine the optimal value of the objective function

Since we desire Z to be a maximum, therefore from step 4, we conclude that maximum value of Z - 1,100 is achieved at the point B(60, 20). Hence the optimal solution to the given LP problem is:

$x_1 = 60$, $x_2 = 20$ and Max Z = 1,100

Remark : To determine which side of a constraint equation is in the feasible region, examine whether the origin (0, 0) satisfies the constraints. If it does, then all points on and below the constraint equation towards the origin are feasible points. If it does not satisfy, then all points on and above the constraint equation away from the origin are feasible points.

3.5.2 Example : Use the graphical method to solve the following LP problem.

Minimize $Z = 3x_1 + 2x_2$

subject to the constraints

$5x_1 + x_2 \geq 10$

$x_1 + x_2 \geq 6$

$x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$

Solution : Rewriting each constraints as follows :

$$\frac{x_1}{2}+\frac{x_2}{10}\geq 1$$

$$\frac{x_1}{6}+\frac{x_2}{6}\geq 1$$

$$\frac{x_1}{12}+\frac{x_2}{3}\geq 1$$

Graph each constraint by first treating it as a linear equation in the. same way as discussed in earlier examples. Use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.5.3.

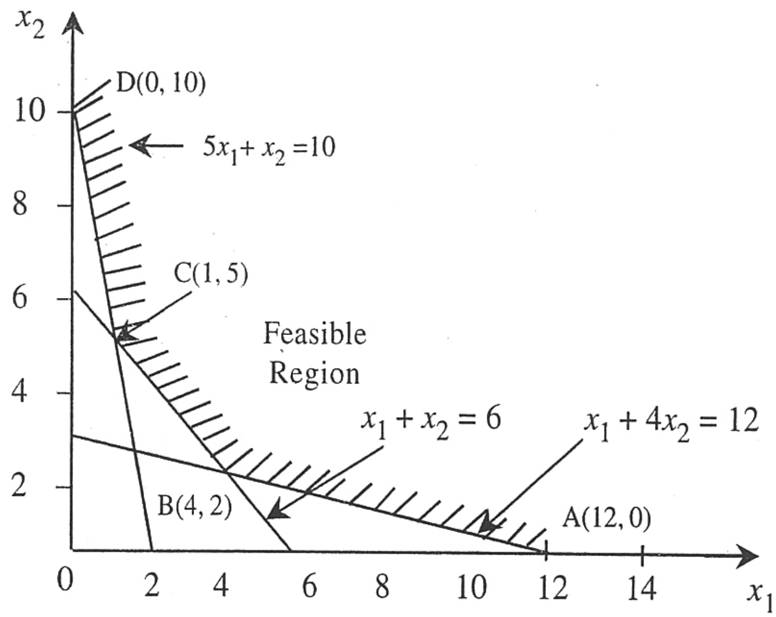


Fig. 3.5.3 Graphical Solution of LP Problem

The coordinates of the extreme point of the feasible region (bounded from below) are, A = (12, 0) B = (4, 2), C = (1, 5) and D = (0, 10). The value of objective function at each of these extreme points is as follows :

Extreme point	Coordinates (x_1, x_2)	Objective function value $Z = 3x_1 + 2x_2$
A	(12,0)	3(12) + 2(0) = 36
B	(4,2)	3(4) + 2(2)= 16
C	(1,5)	3(1) + 2(5)= 13
D	(0, 10)	3(0) + 2(10) = 20

The minimum value of the objective function Z = 13 occurs at the extreme point C(l, 5). Hence the optimal solution to the given L.P, Problem is:

$x_1 = 1$, $x_2 = 5$ and Min. Z = 13.

3.6 Exceptional Cases in Linear Programming

3.6.1 Alternative (or Multiple) Optimal Solution

So far we have seen that the optimal solution of any linear programming problem occurs at an extreme point of the feasible region and the solution is unique and finite, i.e., no other solution yields the same value of the objective function. However, in certain cases a given LP problem may have more than one optimal solution yielding the same objective function value.

There are two conditions that should be satisfied in order that an alternative optimal solution exists :

a) (i) The given objective function is parallel to a constraint that forms the boundary (or edge) of the feasible solutions region. In other words, the slope of the objective function is same as that of the constraint forming the boundary of the feasible solutions region, and

(ii) The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraint should be an active constraint.

b) The objective function assumes the same value at two different extreme points. In this case also, all the points on the line joining these two points will give the same value to the objective function.

Remark. : The constraint is said to be active or binding or tight, if at optimality, the left-hand side of a constraint equals the right-hand side. In other words, an equality constraint is always active. An inequality constraint may or may not be active.
Geometrically, an active constraint is one that passes through one of the extreme points of the feasible solution space.

Example 3.6.1 : Use the graphical method to solve the following LP problem.

$$\text{Maximize } Z = 10x_1 + 6x_2$$

subject to the constraints

$$5x_1 + 3x_2 \leq 30$$

$$x_1 + 2x_2 \leq 18$$

and $x_1, x_2 \geq 0$

Solution : The problem is depicted graphically in Fig. 3.6.1 . The feasible region is shaded and is bound by 0, A, B and C.

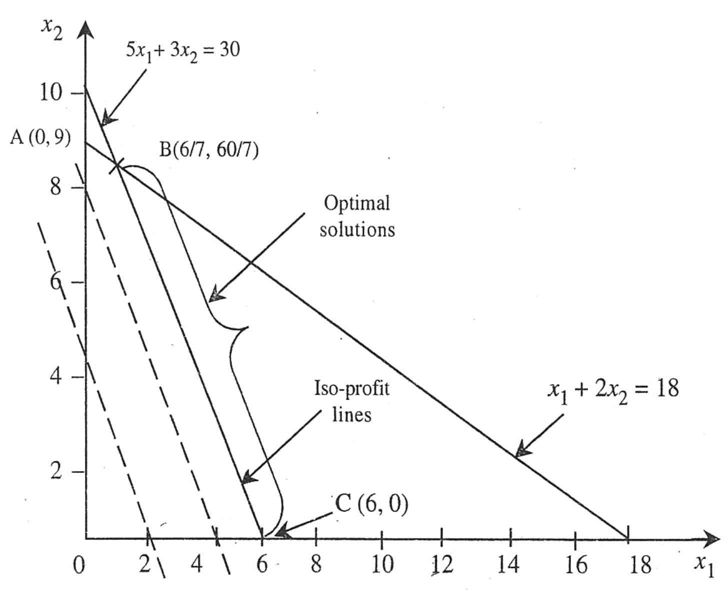


Fig. 3.6.1. Graphical Solution – Multiple Optima

We observe that the objective function (iso-profit line) is. parallel to the line BC (or the first constraint), which forms the boundary of the feasible region. Thus as we move the iso-profit line away from the origin, it coincides with the portion BC of the constraint line which forms the boundary of the feasible region. This implies that any point including extreme points B and C on the same line between B and C is an optimal solution. Therefore, in fact an infinite number of values of x_1, x_2 give the same value of objective function.

We may disregard all other solutions obtained on the line segment BC and consider only those obtained at extreme points B and C to establish that the solution to a linear programming problem shall always lie at one of the extreme point of the feasible region.

The evaluation of four extreme points is shown below :

Extreme point	**Coordinates** (x_1, x_2)	**Objective function value** $Z = 300x_1 + 400x_2$
O	$x_1 = 0, x_2 = 0$	0
A	$x_1 = 0, x_2 = 9$	54
B	$x_1 = 6/7, x_2 = 60/7$	60 } Maximum
C	$x_1 = 6, x_2 = 0$	60 } Maximum

Example 3.6.2 : Using graphical method solve the following LP problem.

Minimize $Z = 2x_1 + 3x_2$

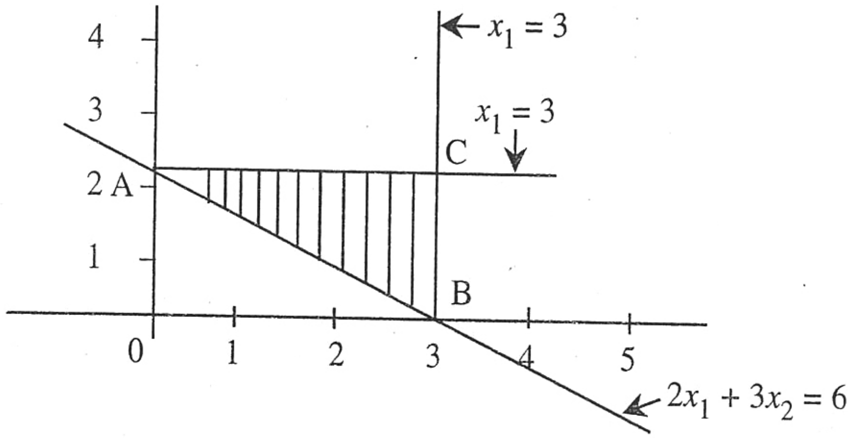
subject to the constraints

$$2x_1 + 3x_2 \geq 6$$

$$x_1 \leq 3$$

and $$x_2 \geq 2$$

$$x_1, x_2, x_3 \geq 0$$



Solution : This problem can be shown on the graph in the following way. Feasible regions is the shaded portion of the graph and its verticies are A(0, 2), B(3, 0) and C(3, 2). The objective function values at the verticies are 6, 6 and 12 respectively. Observe that the minimum value is occuring at the two verticies A and B. This is a situation where multiple solutions exist for the given LP. The two optimum solutions are

i) $x_1 = 3, x_2 = 0$ and Min $z = 6$ and

ii) $x_1 = 0, x_2 = 2$ and Min $z = 6$

Remark : Though all the points on the line joining the two verticies A and B give optimum value for the objective function, as per theorem (iii) Optimum solution exists only at the verticies of the feasible regions.

3.6.2 An Unbounded Solution

When the value of decision variables in linear programming is permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded. Here, the objective function value can also be increased infinitely. However, an unbounded feasible region may also yield some definite value of the objective function some times.

Example 3.6.2 : Use the graphical method to solve the following LP problem:

Maximize $Z = 3x_1 + 4x_2$

subject to the constraints

$$x_1 - x_2 \leq -1$$

$$-x_1 + x_2 \leq 0$$

and $\quad x_1, x_2 \geq 0$

Solution : Graph each constraint by first treating it as a linear equation in the same way as discussed earlier, Then use the inequality condition of each constraint to mark the feasible region as shown in Fig. 3.6.2

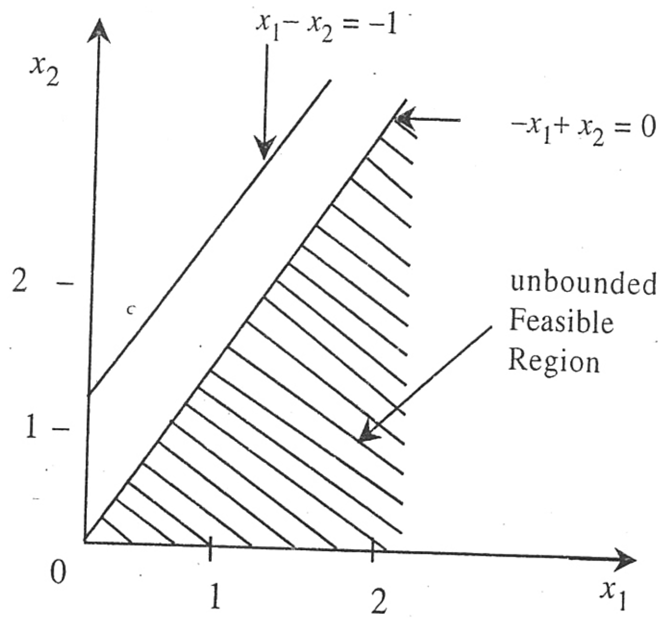


Fig. 3.6.2 Unbounded Solution

It may be noted from Fig. 3.6.2 that there exist infinite number of points in the convex region for which the value of the objective function increases as we move from O = (0, 0) to the right. That is, both the variables x_1 and x_2 can be made arbitrarily large and the value of objective function Z is also increased. Thus, the problem has an unbounded solution.

3.6.3 An Infeasible Solution

If it is not possible to find a feasible solution that satisfies all the constraints, then LP problem is said to have an infeasible solution or alternatively, inconsistence.

Infeasibility depends solely on the constraints and has nothing to do with the objective function.

Example 3.6.3 : (Problem with inconsistence system of constraints) Use graphical method to solve the following LP problem :

Maximize $Z = 6x_1 - 4x_2$

subject to the constraints

$2x_1 + 4x_2 \leq 4$

$4x_1 + 8x_2 \leq 16$

and $x_1, x_2 \geq 0$

Solution: The problem is shown graphically in Fig. 3.6.3. The two inequalities that form the constraint set are inconsistent. Thus, there is no set of points that satisfies all the constraints. Hence, there is no feasible solution to this problem.

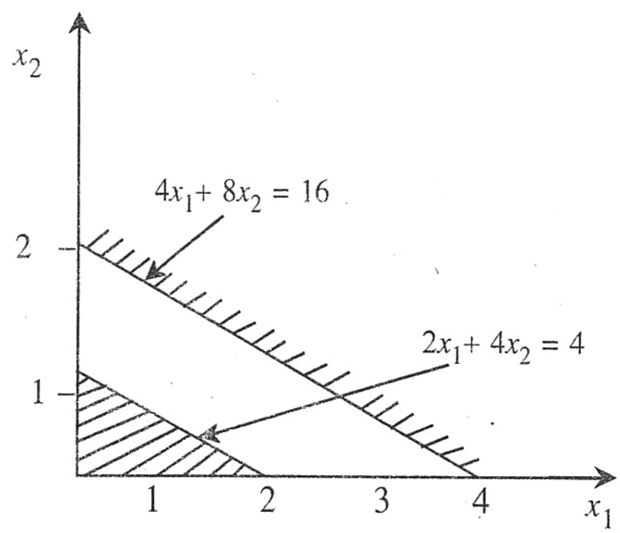


Fig. 3.6.3 An Infeasible Solution

3.7. Summary

In this, lesson some important definitions concerned to Linear Programming Problem are presented. Further, the graphical method of solving a linear programming problem is explained with illustrations. Some typical cases of LPP are also discussed.

3.8 Exercise

Solve the following linear programming problems graphically and state what your solution indicates.

1. Max. $Z = 2x + 5y$

   subject to $x \leq 4$; $y \leq 3$

   $x + 2y \leq 8$

   and $x, y \geq 0$

2. Max. $Z = 5x_1 + 3x_2$

   subject to $3x_1 + 5x_2 \leq 15$

   $5x_1 + 2x_2 \leq 10$

   and $x_1, x2 \geq 0$

3. Max. $Z = 5x_1 + 7x_2$

   subject to $x_1 + x_2 \leq 4$

   $3x_1 + 8x_2 \leq 24$

   $10 x_1 + 7x_2 \leq 35$

   and $x_1, x_2 \geq 0$

4. Min. $Z = 4x_1 - 2x_2$

   subject to $x_1 + x_2 \leq 14$

   $3x_1 + 2x_2 \geq 36$

   $2x_1 + x_2 \leq 24$

   and $x_1, x_2 \geq 0$

5. Min. $Z = 3x_1 + 5x_2$

   subject to $-3x_1 + 4x_2 \leq 12$

   $2x_1 - x_2 \geq -2$

   $2x_1 + 3x_2 \geq 12$

   $x_1 \leq 4$; $x_2 \geq 2$

   and $x_1, x_2 \geq 0$

6. Max. $Z = 2x_1 - x_2$

subject to $x_1 + x_2 \leq 5$

$x_1 + 2x_2 \leq 8$

and $x_1, x_2 \geq 0$

3.9 References

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Lesson – 4

Simplex Method

4.0 Objective :

- Standard form of Linear Programming (LP)
- Slack Variables Problems
- Simplex Algorithm
- Demonstration of Algorithm

Structure

4.1 Introduction

4.2 Standard From of Linear Programming Problem

4.3 Simplex Algorithm (Maximization case)

4.4 Examples

4.5 Summary

4.6 Exercise

4.7 References

4.1 Introduction

The simplex method is an efficient iterative method and is widely used in solving linear programming problems containing several variables and constraints. It was formulated by G.B. Dantzig in 1947.

This method helps in generating the extreme points of the feasible solution space in a systematic manner in the search of an optimal solution of an LP problem.

The simplex method assures an improvement in the solution as we move from one iteration (extreme point) to another in a finite number of steps since the number of extreme points (corner or vertices) of feasible solution space are finite, and also signals when an unbounded solution is reached.

4.2 Standard Form of LP Problem

The use of the simplex method to solve an LP problem requires that the problem be converted into its standard form. The standard form of the LP problem should have the following characteristics :

(i) All the constraints should be expressed as equations by adding slack or surplus and / or artificial variables.

(ii) The right hand side of each constraint should be made non-negative; if it is not, this should be done by multiplying both sides of the constraint by -1.

(iii) The objective function should be of the maximization or minimization type.

For your ready reference the standard form of the LP problem expressed as:

Optimize (Max or Min) $Z = c_1x_1 + c_2x_2 + - + c_nx_n + 0\ s_1 + 0\ s_2 + \ldots + 0\ s_m$ subject to the linear constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 = b_2$$

. . .

. . .

. . .

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m = b_m$$

and $x_1, x_2, \dots x_n, s_1 s_2, \dots. S_m \geq 0.$

In matrix notations the standard form is expressed as :

Optimize (Max or Min) $\mathbf{Z = cx + 0s}$

subject to linear constraints

$$\mathbf{Ax : s = b}$$

and $\mathbf{x, s \geq 0}$

where $c = (c_1, c_2, \dots, c_n)$ is the row vector;

$x = (x_1, x_2, \dots. x_n)^T$, $b = (b_1, b_2, \dots, b_m)^T$ and $s = (s_1, s_2 \dots, s_m)$

are column vectors, and

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is the m x n matrix of coefficients of variables $x_1, x_2, \dots, x_n$ in the constraints.

Remarks : Three types of additional variables, namely

(i) slack variables (s)

(ii) surplus variables (-s), and

(iii) artificial variables (A)

are added in the given LP problem to convert it into standard form for two reasons :

(a) To convert an inequality to an equality to have a standard form of an LP model, and

(b) To get an initial feasible solution represented by the columns of the identity matrix.

The summary of the extra variables needed to add in the given LP problem to convert it into standard form is given in Table 4.2.1

Table 4.2.1

	Types of constraint	Extra variables to be added	Coefficient of extra variables in the objective function		Presence of variables in the initial solution mix
			MaxZ	MinZ	
(i)	Less than or equal to (≤)	Add only slack variable	0	0	Yes
(ii)	Greater than or equal to (≥)	• Subtract surplus variable and add	0	0	No
		• artificial variable	-M	+M	Yes
(iii)	Equal to (=)	Add only artificial variable	-M	+M	Yes

Optimization Proceedure:

To identify the leaving variable and entering variable into the basis at any iteration the following two proceedures are followed.

a) θ **- rule :** To locate the leaving variable we identify a min θ among all θ s and a θ is defined as [b_j / a_{ij}; $a_{ij} > 0$]. The selection of this minimum θ should be done in such a way that only one of the original variables will get a zero value and all other variables are non negative. So,

$$\theta = \underset{a_{ij} < 0}{Min} \left[\frac{b_j}{a_{ij}} \right]$$

b) **Rule of steepest ascent :** Once 'θ' is identified, the corresponding variable will leave the basis. Now, the entering variable is expected to improve the value of objective function. So we white

$$Z_{New} = Z_{Previous} + \theta \, (c_j - z_j).$$

For a maximization problem we expect

$$Z_N > Z_p.$$

So, for a maximization problem we have

$$Z_N < Z_P$$

Hence, we select the max (c_j - zj) or ($Z_j - C_j$) such that (z_j, $-c_j$) < 0 and this j for which ($z_j - c_j$) is most negative will correspond to the entering variable into the basis.

4.3 Simplex Algorithm (Maximization Case)

The steps of the simplex algorithm to obtain an optimal solution (if it exists) to a standard linear programming problem are as follows :

Step-1 : Formulation of the mathematical model

(i) Formulate the mathematical model of the given linear programming problem.

(ii) If the objective function is of minimization, then convert it into one of maximization by using the following relationship.

Minimize Z = - Maximize Z*

where Z* = - Z

(iii) Check whether all the b_i(i = 1, 2, ..., m) values are positive. If any one of them is negative, multiply the corresponding constraint by -1 in order to make that $b_i > 0$.

(iv) Express the mathematical model of the given LP problem in the standard form by adding additional variables to the left side of each constraint and assign a zero cost coefficient to these in the objective function.

Step-2 : Set up the initial solution

Write down the coefficients of all the variables in the LP model in the tabular form, as shown in Table 4.3.1, to get an initial basic feasible solution **[$x_B = B^{-1}b$].**

Table 4.3.1 Initial Simplex Table

		$c_j \rightarrow$		c_1	c_2	$\ldots c_n$	0	$0 \ldots 0$
Coefficient of basic variables (C_B)	Basic variables	Value of basic variables $b(=x_B)$		x_1	x_2	$\ldots x_n$	s_1	$s_2 \ldots s_n$
c_{B1}	s_1	$x_{B1} = b_1$		a_{11}	a_{12}	$\ldots a_{1n}$	1	$0 \ldots 0$
c_{B2}	s_2	$x_{B2} = b_2$		a_{21}	a_{22}	$\ldots a_{2n}$	0	$1 \ldots 0$
.	.	.	.	.	.	.	.	. .
.	.	.	.	.	.	.	.	. .
.	.	.	.	.	.	.	.	. .
c_{Bm}	s_m	$x_{Bm} = b_m$		a_{m1}	a_{m2}	$\ldots a_{mn}$	0	$0 \ldots 1$
$Z = \sum c_{Bi} x_{Bi}$		$z_j = \sum c_{Bi} x_j$		0	0	$\ldots 0$	0	$0 \ldots 0$
		$z_j = c_j$		$z_1 - c_1$	$z_2 - c_2$	$\ldots z_n - c_n$		$0 \ldots 0$

After having set up the initial simplex table, locate the identity matrix and column variables involved in it. This matrix contains all zeros except a diagonal of positive 1's. This identity matrix is always a square matrix and its size is determined by the number of constraints.

The identity matrix so obtained is also called a basis matrix [because basic feasible solution is represented by $\boldsymbol{B} = \boldsymbol{I}$]. Assign the values of the constants (b_i's) to the column variables in the identity matrix [because $\mathbf{x_B} = \mathbf{B^{-1}\ b = Ib = b}$].

The variables corresponding to the columns of the identity matrix are called basic variables and the remaining ones non-basic variables. In general, if an LP model has n variables and m(< n) constraints, then m variables would be basic and n - m variables non-basic.

The value z_j represent the amount by which the value of objective function Z would be decreased (increased) if one unit of given variable is added to the new solution. Each of the values in the $c_j - z_j$ row signifies the net amount of increase (decrease) in the objective function that would occur when one unit of variable represented by the column head is introduced into the solution. That is:

z_j (outgoing total profit) - c_j (incoming unit profit) = $z_j - c_j$ (net effect)

Step-3 : Test for optimality

Calculate the $z_j - c_j$ value for all non-basic variables. To obtain the value of z_j multiply each element under "variables column head' (columns of the coefficient matrix, a_j) with the corresponding elements under "coefficient of basic variables column" (C_B column). Examine the values of $z_j - c_j$. There may arise three cases.

(i) If all $z_j - c_j \geq 0$, then the basic feasible solution is optimal.

(ii) If at least, one column of the coefficients matrix (i.e. a_k) for which $z_k - c_k < 0$ and all elements are negative (i.e. $a_{ik} < 0$), then there exists an unbounded solution is optimal.

(iii) If at least one $z_j - c_j < 0$ and each of these has at least one positive element (i.e. a_{ij}) for some row, then it indicates that an improvement in the value of objective function Z is possible.

Step-4: Select the variable to enter the basis

If case (iii) of step 3 holds, then select a variable that has the most negative $z_j - c_j$ value to enter into the new solution. That is,

$$Z_k - c_k = \text{Min} \{ (z_j - c_j) \, ; \, z_j - c_j < 0 \}$$

The column to be entered is called the key or pivot column.

Step-S : Test for feasibility (variable to leave the basis)

After identifying the variable to become basic variable, the variable to be removed from the basic variables set is determined. For this, bi values are divided by the corresponding values in the pivot column and we select the row for which this ratio, = [(constant column) / (key column)] is non-negative and minimum. This ratio is called the replacement ratio. That is,

$$\frac{x_{Br}}{a_{rj}} = \underset{j}{Min}\left\{\frac{x_{Bi}}{a_{rj}}; a_{rj} > 0\right\}$$

Column a_r is removed and replaced by column vector a_k.

The row selected in this manner is called the key or pivot row -and represents the variable which will leave the solution.

The element lies at the intersectioin of key row and key column of the simplex table is called key or pivot element.

Step-6: Finding the new solution

(i) If the key element is 1, then the row remains the same in the new simplex table.

(ii) If the key element is other than 1, then divide each element in the key row (including b_i) by the key element, to find the new values for that row.

(iii) The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero.

In other words, for each row other than the key row, we use the formula:

New row = Old row element $\pm$ (Intersectural element x Corresponding element in the replacement row)

Mathematical criterion

$$\widehat{y}_{ij} = y_{ij} - \left\{\frac{y_{rj}}{y_{rk}}\right\} y_{ij};\ I = 1, 2, \ldots, m$$

and $\widehat{y}_{rj} = \dfrac{y_{rj}}{y_{rk}}$

The new entries in c_B and x_B columns are updated in the new simplex table of the current solution.

Step-7 : Repeat the procedure

Go to step 3 and repeat the procedure until all entries in the z_j - c_j row are either positive or zero.

4.4 Examples

Example-4.4.1 : Use the simplex method to solve the following LP problem.

Maximize $Z = 3x_1 + 5\,x_2 + 4\,x_3$

subject to the constraints

$$2\,x_1 + 3\,x_2 \leq 8$$

$$2\,x_2 + 5\,x_3 \leq 10$$

$$3x_1 + 2\,x_2 + 4\,x_3 \leq 15$$

and $x_1, x_2, x_3 \geq 0$

Solution :

Step-1 : Introducing non-negative slack variables s_1, s_2 and s_3 to convert inequality constraints to equality. Then the LP problem becomes

Maximize $Z = 3x_1 + 5\,x_2 + 4\,x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$2x_1 + 3x_2 + s_1 = 8$$

$$2x_2 + 5x_3 + s_3 = 10$$

$$3x_1 + 2x_2 + 4x_3 + s_3 = 15$$

and $x_2\,,\, x_2, x_3,\; s_2\,,\, s_2, s_3 \geq 0$

Step-2 : Since all $b_i > 0$, i = 1, 2, 3, we can choose our initial basic feasible solution as :

$x_1 = x_2 = x_3 = 0$

$s_1 = 8$, $s_2 = 10$, $s_3 = 15$ and Max Z = 0

Step-3 : To see whether the current solution given the Table, 4.4.1 below is optimal or not, calculate $z_j - c_j = \{\; c_B\, B^{-1}\, a_j - c_j = c_B\, y_j - c_j \}$ for non-basic variables x_1, x_2 and x_3 as follows :

$$z_1 - c_1 = c_B y_1 - c_1 = (0, 0, 0)\begin{bmatrix}2\\0\\3\end{bmatrix} - 3 = -3$$

Table 4.4.1 Initial Solution

c_B	Variables in basic B	Solution values $b(=x_B)$	$c_j \rightarrow$: c_1 x_1	c_2 x_2	c_n x_3	0 s_1	0 s_2	0 s_3	Min ratio x_B/x_2
0	s_1	8	2	(3)	0	1	0	0	8/3 →
0	s_2	10	0	2	5	0	1	0	10/2
0	s_3	15	3	2	4	0	0	1	15/2
Z = 0		z_j	0	0	0	0	0	0	
		$z_j - c_j$	-3	-5	-4	0	0	0	
			↑						

$$z_2 - c_2 = c_B y_2 - c_2 = (0, 0, 0)\begin{bmatrix}3\\2\\2\end{bmatrix} - 5 == -5, \text{ and}$$

$$z_3 - c_3 = c_B y_3 - c_3 = (0, 0, 0)\begin{bmatrix}0\\5\\4\end{bmatrix} - 4 = -4.$$

Since all $z_j - c_j \leq 0$, j = 1, 2, 3, the current solution is not optimal. Thus variable x_2 is chosen to enter into the basis as it corresponds to the largest net contribution in the x_2-column where all elements are positive.

Step-4 : The variable leaving the basis is determined by dividing the values in the x_B(constant) column by their corresponding elements in the key column as shown in Table 4.4.1.

Since the minimum ratio is in row 1, the variable solution s_1 should leave the solution.

Step-5 : *(Iteration 1)*. Since the key element enclosed in circle in Table 4.4.1 is not 1, divide all elements of the key row by 3 to obtain new values of the elements in this row. The new values of the elements in the remaining rows for the new table can be obtained by performing the following elementary row operations on all rows so that all elements except the key element 1 in the key column are zero.

R_1 (new) $\rightarrow$ R_1 (old)/3 (key element); R_2 (new) $\rightarrow$ R_2 (old) - $2R_1$ (new)

R_3 (new) $\rightarrow$ R_3 (old) - $2R_1$ (new).

The new solution is shown in Table 4.4.2.

Table 4.4.2

		$c_j \rightarrow$	3	5	4	0	0	0	
c_B	Variables in basic B	Solution values b($=x_B$)	x_1	x_2	x_3	s_1	s_2	s_3	Min ratio x_B/x_3
5	x_1	8/3	2/3	1	0	1/3	0	0	-
0	x_2	14/3	-4/3	0	(5)	-2/3	1	0	(14/3)/5$\rightarrow$
0	x_3	29/3	5/3	0	4	-2/3	0	1	(29/3)/4
Z = 40/3		z_j	10/3	5	0	5/3	0	0	
		$z_j - c_j$	+1/3	0	-4	+5/3	0	0	
					$\uparrow$				

The improved basic feasible solution can be read from Table 4.4.2 as : $x_2 = 8/2$, $s_2 = 14/3$, $s_3 = 29/3$ and $x_1 = x_2 = s_1 = 0$. The improved value of the objective function is Z - 40/3.

As shown in Table 4.4,2, $z_3 - c_3 < 0$, the current solution is not optimal.

Step-6: *(Iteration 2)* Repeat steps 3 to 5. Applying following row operations to enter variable x_3 into the basis and to drive out s_2 from the basis.

R_2 (new) = R_2 (old)/5 (key element); R_3 (new) $\rightarrow$ R_3 (old) - $4R_2$ (new)

The new solution is shown in Table 4.4.3

Table 4.4.3

		$c_j \rightarrow$	3	5	4	0	0	0	
c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	Min ratio x_B/x_1
5	x_2	8/3	2/3	1	0	1/3	0	0	(8/3)/(2/3)
4	x_3	14/15	-4/15	0	1	-2/15	1/5	0	-
0	s_3	89/15	(41/15)	0	0	-2/15	-4/5	1	(89/15)/(41/15) $\rightarrow$
Z = 256/15		z_j	34/15	5	0	17/15	4/5	0	
		$z_j - c_j$	-11/15	0	-4	+17/15	+4/5	0	
			↑						

Iteration 3 : The variable x_1 enters the basis and x_3 leaves the basis. For this we shall apply the following row operations.

R_3 (new) $\rightarrow$ R_3 (old) x (15/41); R_2 (new) $\rightarrow$ R_2 (old) + (15/41) R_3 (new)

R_1 (new) $\rightarrow$ R_1 (old) - $(2/3)R_3$ (new).

The new solution is shown in Table 4.4.4.

Table 4.4.3

		$c_j \rightarrow$	3	5	4	0	0	0
c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
5	x_2	50/41	0	1	0	15/41	8/41	-10/41
4	x_3	62/41	0	0	1	-6/41	5/41	4/41
0	x_1	89/41	1	0	0	-2/41	-12/41	15/41
Z = 765/41		z_j	3	5	4	45/41	24/41	11/41
		z_j - c_j	0	0	0	+45/41	+24/41	+11/41

In Table 4.4.4, all $z_j - c_j \geq 0$ for non-basic variables. Therefore the optimal solution is reached with, $x_1 = 89/41$, $x_2 = 50/41$, $x_3 = 62/41$ and the optimal value of Z = 765/41.

4.5 Summary

In this lesson the simplex method to solve a linear programming problem for maximization is discussed at length along with its algorithm. The same is explained with a numerical example.

4.6 Exercise

Solve the following LP problems using the simplex method.

1. Max. $Z = 3x_1 + 2x_2$

   subject to $x_1 + x_2 \leq 4$

   $x_1 - x_2 \leq 2$

   and $x_1, x_2 \geq 0$

2. Max. $Z = 5x_1 + 3x_2$

subject to $x_1 + x_2 \leq 2$

$5x_1 + 2x_2 \leq 10$

$3x_1 + 8x_2 \leq 12$

and $x_1, x_2 \geq 0$

3. Max. $Z = 3x_1 + 2x_2 + 5x_3$

subject to $x_1 + 2x_2 + x_3 \leq 430$

$3x_1 + 2x_3 \leq 460$

$x_1 + 4x_3 \leq 420$

and $x_1, x_2, x_3 \geq 0$

4. Max. $Z = x_1 - 3x_2 + 2x_3$

subject to $3x_1 - x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

and $x_1, x_2 \geq 0$

4.7 References

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Lesson – 5

Simplex Method with Artificial Variables

5.0 Objective :

- Artificial Variables
- Surplus Variables
- Solution to linear Programming Problem Minimization
- Penalty method or Big-M method

Structure

5.1 Introduction

In certain situations, discussed below, it is difficult to obtain an initial basic feasible solution. They arise

When the constraints are of the '≥' form

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i,\ x_j \geq 0$$

(a) if some right hand side constants are negative [i.e. $b_i < 0$]. Then in this case after adding the non-negative slack variables s_i, i = 1, 2, ..., m, the initial solution so obtained will be $s_i = - b_i$ for some i. It is not the feasible solution because it violates the non-negativity conditions of slack variables (i.e. $s_i \geq 0$).

(b) when the constraints are of the '≥' form

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i,\ x_j \geq 0$$

In this case to convert the inequalities into equation form, adding surplus (negative slack) variables,

$$\sum_{j=1}^{n} a_{ij}x_j - s_i = b_i,\ x_j \geq 0,\ s_1 \geq 0$$

Letting $x_j = 0$, j = 1 , 2, ..., n, we get an initial solution $- s_i = b_i$ or $s_i = - b_i$. It is also not a feasible solution as it violates the non-negativity conditions of surplus variables (i.e. $s_i \geq 0$). In this case, we add artificial variables, A_i, i = 1, 2, ..., m to get an initial basic feasible solution. The resulting system of equations.

$$\sum_{j=1}^{n} a_{ij}x_j - s_i + A_i = b_i$$

$$x_j,\ s_i,\ A_i \geq 0,\ i = 1,2,\ldots,m$$

has m equations and (n + m + m) variables (i.e. n decision variables, m artificial variables and m surplus variables). An initial basic feasible solution of the new system can be obtained by equating (n + 2m - m) = (n + m) variables equal to zero.

Thus the new solution to the given LP problem is: $A_i = b_i$ i = 1, 2,…, m, which does not constitute a solution to the original system of equations because the two systems of equation are not equivalent. Thus to get back to the original problem, artificial variables must be driven to zero in the optimal solution. There are two methods for eliminating these variables from the solution.

1. Big-M Method or Method of Penalties
2. Two Phase Method

In this lesson only Big-M method and Two Phase method are explained with examples.

Remark : Artificial variables are only a tool to get the simplex method started by generating an initial basic feasible solution. These variables will eventually be equated to zero in the solution in order to attain feasibility in the original problem. (2) These variables are added to those constraints with equality (=) and greater than or equal to (≥) sign.

5.2 The Big – M Method

The Big-M Method is one of the methods of removing artificial variables from the basis. In this method, we assign coefficients to artificial variables, undesirable from the objective function point of view. If objective function Z is to be minimized, then a very large positive price (called penalty) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called penalty) is assigned to each of these variables. The penalty will be designated by - M for a maximization problem and +M for a minimization problem, where M > 0.

5.3 Big – M Method Algorithm

The Big-M Method for solving an LP problem for maximizing the objective function can be summarized in the following steps.

Step-1: Express the LP problem in the standard form by adding surplus variables and artificial variables. Assign a zero coefficient to surplus variables and a very large number - M to artificial variable in the objective function.

Step-2 : The initial basic feasible solution is obtained by assigning zero value to original variables.

Step-3 : Calculate the values of $z_j - c_j$ in last row of the simplex table and examine these values.

(i) If all $z_j - c_j \geq 0$ then the current basic feasible solution is optimal.

(ii) If for a column, k, $z_k - c_k$ is most negative and all entries in this column are negative, then the problem has an unbounded optimal solution.

(iii) If one or more $z_j - c_j < 0$ then select the variable to enter into the basis with the smallest neative $z_j - c_j$ value. That is,

$$z_k - c_k = \text{Min } \{z_j - c_j : z_j - c_j < 0\}$$

The column to be entered is called key or pivot column.

Step-4 : Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

Note : 1) If it is a minimization problem step 3 has to be performed for positive values of $z_j - c_j$ in the same way. A minimization problem can be solved as it is by changing the c_j - z_j sign or one can convent it into Maximization problem and solve it. The conversion is

$$\text{Max } Z = -(\text{Min-Z}).$$

2) For minimization problem an artificial variable will have its cost as +M, where 'M' is a huge positive number.

5.4 Examples

Example-5.4.1 : The ABC Printing Company is facing a tight financial squeeze and is attempting to cut costs wherever possible. At present it has only one printing contract and, luckily, the book is selling well in both the 'hardcover and paperback form. The printing cost for hardcover books is Rs. 600 per 100 while that for paperbacks is only Rs. 500 per 100. Although the company is attempting to economize, it does not wish to lay off any employee. Therefore, it feels obliged to run its two printing presses at least 80 and 60 hours per week, respectively. Press I can produce 100 hardcover books in

2 hours or 100 paperback books in 1 hour. Press II can produce 100 hardcover books in 1 hour or 100 paperback books in 2 hours.

Determine how many books of each type should be printed in order to minimize costs.

Solution : Let x_1, x_2 be the number of batches containing 100 hard and paperback books respectively. The LP problem can be formulated as follows:

Minimize $Z = 600\, x_1 + 500\, x_2$

subject to the constraints

$$2x_1 + x_2 \geq 80;$$

$$x_1 + 2x_2 \geq 60;$$

and $x_1, x_2 \geq 0$

Standard form

By introducing surplus variables s_1, s_2 and artificial variables A_1, A_2 in the inequalities of the constraints, the standard form of the LP problem becomes

Minimize $Z = 600\, x_1 + 500\, x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

$$2x_1 + x_2 - s_1 + A_1 = 80;$$

$$x_1 + 2\, x_2 - s_2 + A_2 = 60$$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$.

Solution by simplex method

The initial basic feasible solution is obtained by setting $x_1 = x_2 = s_1 = s_2 = 0$. Then we shall have

$A_1 = 80$, $A_2 = 60$ and $Z = 80M + 60M = 140M$

This initial basic feasible solution is shown in Table 5.4.1.

Table 5.4.1 Initial Solution

c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min ratio x_B/x_1
		$c_j \rightarrow$	600	500	0	0	M	M	
M	A_1	80	(2)	1	-1	0	1	0	80/2 →
M	A_2	60	1	2	0	-1	0	1	60/1
Z = 140 M		z_j	3M	3M	-M	-M	M	M	
		$z_j - c_j$	3M-600	3M-500	-M	-M	0	0	
			↑						

Entering variable x_1 to replace basic bariable A_1 into the basis. For this, apply row operations:

R_1 (new) = R_1 (old)/2 ; R_2 (new) $\rightarrow$ R_1 (old) – R_1 (new) to get the new solution as shown in Table 5.4.2

Table 5.4.2

c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min ratio x_B/x_2
		$c_j \rightarrow$	600	500	0	0	M	M	
600	x_1	40	1	1/2	-1/2	0	-1/2	0	80/(1/2)
M	A_2	20	0	(3/2)	1/2	-1	-1/2	1	20/(3/2)→
Z=20M+2400		z_j	600	+300	-300	-M	+300	M	
Z=40/3				+3M/2	+M/2		-M/2		
		$z_j - c_j$	0	-200	-300	-M	+300	0	
				+3M/2	+M/2		-3M/2		
				↑					

Entering variable x_2 to replace basic variable A_2 into the basis. For this, apply row operations :.

$$R_2\text{ (new)} \rightarrow R_2\text{ (old) x } \frac{2}{3};\quad R_1\text{ (new)} \rightarrow R_1\text{ (old) - } \frac{1}{2} R_2\text{ (new)}$$

to get the new solution as shown in Table 5.4.3.

Since all the numbers in the z_j- c_j row are either zero or positive as well as well as both artificial variables have been reduced to zero, an optimum solution has been arrived at with x_1 = 100/3 batches of hard books, x_2 = 40/3 batches of paper back books and total minimum cost, Z = Rs. 80,000/3.

Table 5.4.3

		$c_j \rightarrow$	600	500	0	0	M	M
c_B	Variables in basic B	Solution values b(=x_B)	x_1	x_2	s_1	s_2	A_1	A_2
600	x_1	100/3	1	0	-2/3	1/3	2/3	-1/3
500	x_2	40/3	0	1	1/3	-2/3	-1/3	2/3
Z = 80000/3		z_j	600	500	-700/3	-400/3	700/3	400/3
		z_j - c_j	0	0	-700/3	-400/3	-M+700/3	-M+400/3

Example- 5.4.2 : Use the penalty (Big-M) method to solve the following LP problem.

Minimize $Z = 5x_1 + 3x_2$

subject to the constraints

$2x_1+4x_2 \leq 12$

$2x_1 + 2x_2 = 10$

$5x_1 + 2x_2 \geq 10$; and x_1 , $x_2 \geq 0$

Solution : Introduce slack variable s_1, surplus variable s_2 and artificial variables A_1 and A_2 in the constraints of the given LP problem. The standard form of the LP problem is stated as follows :

Minimize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

$2x_1 + 4x_2 + s_1 = 12$

$2x_1 + 2x_2 + A_1 = 10$

$5x_1 + 2x_2 - s_2 + A_2 = 10$; and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

An initial basic feasible solution is obtained by letting $x_1 = x_2 = s_1 = s_2 = 0$. Therefore, the initial basic feasible solution is : $s_1 = 12$, $A_1 = 10$, $A_2 = 10$ and Min $Z = 10M + 10M = 20M$. Here it may be noted that the columns which correspond to current basic variables and form the basis (identity) matrix are, s_1 (slack variable), A_1 and A_2 (both artificial variables). The initial basic feasible solution is given in Table 5.4.4.

Since the value $z_1 - c_1 = 7M - 5$ is the smallest value, therefore x_1 becomes the entering variable. To decide which basic variable should leave the basis, the minimum ratio is calculated as shown in Table 5.4.4

Table 5.4.4

		$c_j \rightarrow$	5	3	0	0	M	M	
c_B	Variables in basis B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2	Min ratio x_B/x_1
0	s_1	12	2	4	1	0	0	0	12/2
M	A_1	10	2	2	0	0	1	0	10/2
M	A_2	10	(5)	2	0	-1	0	1	10/5 $\rightarrow$
Z = 20M		z_j	7M	4M	0	-M	M	M	
		$z_j - c_j$	-5+7M	-3+4M	0	-M	0	0	
			$\uparrow$						

Iteration-1 : Introduce variable x_1 into the basis and remove A_2 from the basis by applying the following row operations. The row solution is shown in Table 5.4.5.

$R_3(\text{new}) \rightarrow R_3(\text{old})/5$; $R_2(\text{new}) \rightarrow R_2(\text{old}) - 2R_3(\text{new})$;

$R_1(\text{new}) \rightarrow R_1(\text{old}) - 2R_3(\text{new})$.

Table 5.4.5

		$c_j \rightarrow$	5	3	0	0	M	
c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	Min ratio x_B/x_2
0	s_1	8	0	16/5	1	2/5	0	8/(16/5) $\rightarrow$
M	A_1	6	0	(6/5)	0	2/5	1	6/(6/5)
5	x_1	2	1	2/5	0	-1/5	0	2/(2/5)
Z = 10+6M		z_j	5	(6M/5)+2	0	(2M/5)-1	M	
		$z_j - c_j$	0	(+6M/5)-1	0	(2M/5)-1	0	
				$\uparrow$				

Iteration-2 : Introduce variable x_2 into the basis and remove s_1 from the basis by applying the following elementary row operations. The new solution is shown in Table 5.4.6.

$$R_1\text{ (new)} \rightarrow R_1\text{ (old) x } \frac{5}{10};\quad R_2\text{ (new)} \rightarrow R_2\text{ (old)} - \frac{6}{5} R_1\text{ (new)};$$

$$R_3\text{ (new)} \rightarrow R_3\text{ (old) x} + \frac{2}{5}\ R_1\text{(new)}.$$

Table 5.4.6

c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	Min ratio x_B/s_1
		$c_j \rightarrow$	5	3	0	0	M	
3	x_2	5/2	0	1	5/16	1/8	0	(5/2)/(1/8)
M	A_1	3	0	0	-3/8	(1/4)	1	3/(1/4)→
5	x_1	1	1	0	-1/8	-1/4	0	
Z = 5+3M		z_j	5	3	-3M/8+5/16	M/4-7/8	M	
		$z_j - c_j$	0	0	-3M/8+5/16	M/4-7/8		
						↑		

Iteration-3 : Introduce s_2 into the basis and remove A_1 from the basis by applying the following row operations :

$$R_2\text{ (new)} \rightarrow R_2\text{ (old) } 4;\ \ R_1\text{ (new)} \rightarrow R_1\text{ (old) } - \frac{1}{8} R_2\text{ (new)};$$

$$R_3\text{ (new)} \rightarrow R_3\text{ (old) x } \frac{1}{4}\ R_2\text{(new)}.$$

The new solution is shown in Table 5.4.7.

Table 5.4.7

c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1
		$c_j \rightarrow$	5	3	0	0	M
3	x_2	1	0	1	1/2	0	-1/2
0	s_2	12	0	0	-3/2	1	4
5	x_1	4	1	0	-1/2	0	1
Z = 23		z_j	5	3	-1	0	7/2
		$z_j - c_j$	0	0	-1	0	-M+7/2

In Table 5.4.7, it is observed that all $z_j - c_j \leq 0$. Thus an optimal solution has arrived at with value of variables as : $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and Min z = 23.

5.5 Two Phase Method

In the first phase of this method the sum of the artificial variables is minimized subject to the given constraints to get a basic feasible solution of the LP problem. The second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase. Since the solution of the LP problem is completed in two phases, this is called the two phase method.

5.5.1 Advantages of the method

1. No assumptions on the original system of constraints are made, i.e. the system may be redundant, inconsistent or not solvable in non-negative numbers. . .
2. It is easy to obtain an initial basic feasible solution for phase I.
3. The basic feasible solution (if it exists) obtained at the end of phase I is used to start phase II.

5.5.2 Two phase method Algorithm

Phase I

Step 1: (a) If all the constraints in the given LP problem are ($\leq$) type, then phase II can be directly used to solve the problem. Otherwise, a sufficient number of artificial variables are added to get a basis matrix (identity matrix).

(b) If the given LP problem is of minimization, then convert it to the maximization type by the usual method.

Step 2: Solve the following auxiliary LP problem by assigning a coefficient of-1 to each artificial variable and zero to all other variables in the objective function and with the basic feasible solution $x_1 = x_2 = \ldots = x_n = 0$ and $A_i = b_i$; i = 1, 2, ..., m.

$$\text{Maximize } Z^* = \sum_{i=1}^{m} (-1)A_i; \quad Z^* = -Z$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j + A_i = b_i, \qquad i = 1,2, ..., m$$

and $x_j, A_i \geq 0$

Apply the simplex algorithm to solve this LP problem. The following three cases may arise at optimality.

(i) Max $Z^* = 0$ and at least one artificial variable is present in the basis with position value. Then no feasible solution exists for the original LP problem.

(ii) Max $Z^* = 0$ and no artificial variable is present in the basis. Then the basis consists of only decision variables (x'_j s) and hence we may move to phase II to obtain an optimal basic feasible solution to the original LP problem.

(iii) Max $Z^* = 0$ and at least one artificial variable is present in the basis at zero value. Then a feasible solution to the above LP problem is also a feasible solution to the original LP problem. Now in order to arrive at the basic feasible solution we may proceed directly to phase II or else eliminate the artificial basic variable and then proceed to phase II.

Once an artificial variable has left the basis it has served its purpose and can therefore be removed from the simplex table. An artificial variable is never considered for re-entry into the basis.

Remark: The LP problem defined above is also called auxiliary problem. The value of the objective function in this problem is bounded from above by zero because the objective function represents the sum of artificial variables with negative unit coefficients. Thus, the solution to this problem can be obtained in a finite number of steps.

Phase II

Step 3: Assign actual coefficients to the variables in the objective function and zero to the artificial variables which appear at zero value in the basic at the end of phase I, that is, the last simplex table of phase I can be used as the initial simplex table for phase II. Then apply the usual simplex algorithm to the modified simplex table to get the optimal solution to the original problem. Artificial variables which do not appear in the basis may be removed.

5.6 Example

Solve the following LP problem by using the two-phase simplex method.

Minimize $Z = x_1 + x_2$

subject to the constraints

$$2x_1 + x_2 \geq 4$$

$$x_1 + 7x \geq 7$$

and $x_1, x_2 \geq 0$

Solution: After adding surplus variables s_1 and s_2 and artificial variables A_1 and A_2, the problem becomes:

Maximize $Z^* = -x_1 - x_2$

subject to the constraints

$2x_1 + 4x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

where $Z^* = -Z$

Phase I

This phase starts by considering the following auxiliary LP problem.

Maximize $Z^* = -A_1 - A_2$

subject to the constraints

$2x_1 + x_2 - s_1 + A_1 = 4$

$x_1 + 7x_2 - s_2 + A_2 = 7$

and $x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

The initial table is presented in Table 5.6.1

Table 5.6.1

		$c_j \rightarrow$	0	0	0	0	-1	-1
c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2
-1	A_1	4	2	1	-1	0	1	0
-1	A_2	7	1	(7)	0	-1	0	1 $\rightarrow$
Z = -11		z_j	-3	-8	1	1	-1	-1
		$z_j - c_j$	-3	-8	+1	+1	0	0
				$\uparrow$				

Iteration 1: Applying following row operations to get a new solution by entering variable x_2 in the basis and removing variable A_2 from the basis as shown in Table 5.6.2.

$R_2 \text{ (new)} \rightarrow R_2 \text{ (old)}/7; \quad R_1 \text{ (new)} \rightarrow R_1 \text{ (old)} - R_2 \text{ (new)}$

Table 5.6.2

c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2^*	
		$c_j \rightarrow$	0	0	0	0	-1	-1	
-1	A_1	3	13/7	0	-1	(1/7)	1	-1/7	→
0	x_2	1	1/7	1	0	-1/7	0	1/7	
$Z^* = -3$		z_j	-13/7	0	1	-1/7	-1	1/7	
		$z_j - c_j$	-13/7	0	+1	-1/7	0	+8/7	
						↑			

*This column may be removed forever at this stage.

Iteration 2: Apply the following row operations to get a new solution by entering variables s_2 in the basis and removing variable A_1 from the basis as shown in Table 5.6.3. Here it may be noted that if variable x_1 is chosen to enter into the basis, then it will lead to an infeasible solution

$$R_1(\text{new}) \rightarrow R_1\ (\text{old}) \times 7; \quad R_2\ (\text{new}) \rightarrow R_2\ (\text{old}) + \frac{1}{7} R_1(\text{new})$$

Table 5.6.3

c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	A_1	A_2
		$c_j \rightarrow$	0	0	0	0	-1	-1
0	s_2	21	13	0	-7	1	7	-1
0	x_2	4	2	1	-1	0	1	0
$Z^* = 0$		z_j	0	0	0	0	0	0
		$z_j - c_j$	0	0	0	0	+1	+1

Since all $z_j - c_j \geq 0$ correspond to non-basic variables, the optimal solution: $x_1 = 0$, $x_2 = 4$, $s_1, = 0$, $s_2 = 2l$, $A_1 = 0$, $A_2 = 0$ with $Z^* = 0$ is reached. However, this solution may or may not be the basic feasible solution to the original LP problem. Thus, we have to move to phase II to get an optimal solution to our original LP problem.

Phase II: The modified simplex table obtained from Table 5.6.3 is represented in Table 5.6.4.

Table 5.6.4.

c_B	Variables in basic B	$c_j \rightarrow$ Solution values $b(=x_B)$	-1 x_1	-1 x_2	0 s_1	0 s_2	Min ratio x_n/x_1
0	s_2	21	(13)	0	-7	1	21/13 $\rightarrow$
-1	x_2	4	2	1	-1	0	4/2
$Z^* = -4$		z_j	-2	-1	1	0	
		$z_j - c_j$	-1	0	+1	0	
			$\uparrow$				

Iteration I: Introduce variable x_1 into the basis and remove variable s_2 from the basis by applying the following row operations.

$$R_1(\text{new}) \rightarrow R_1(\text{old})/13; \qquad R_2\ (\text{new}) \rightarrow R_2\ (\text{old}) - 2R_1\ (\text{new})$$

The improved solution so obtained is given in Table 5.6.5. Since in Table 5.6.5. $z_j - c_j \geq 0$ for all non-basic variables, the current solution is optimal. Thus, the optimal basic feasible solution to the given LP problem is:

$$x_1 = 21/13, \qquad x_2 = 10/13$$

and $$\text{Max } Z^* = -31/13 \quad \text{or} \quad \text{Min } Z = 31/l3$$

Table 5.6.5.

c_B	Variables in basic B	Solution values $b(=x_B)$	$c_j \rightarrow$ -1 x_1	-1 x_2	0 s_1	0 s_2
-1	x_1	21/13	1	0	-7/13	1/13
-1	x_2	10/13	0	1	1/13	-2/13
$Z^* = -31/13$		z_j	-1	-1	6/13	1/13
		$z_j - c_j$	0	0	+6/13	+1/13

5.7 Summary

In this lesson the Linear Programming problem solution by Penalty method or Big-M and Two Phase method are discussed in detail. Also, the method is explained with an example.

5.8 Exercise

Solve the following LP problems :

1. Max. $Z = 3x_1 - x_2$

   subject to $2x_1 + x_2 \geq 2$

   $x_1 + 3x_2 \leq 2$

   $x_2 \leq 4$

   and $x_1, x_2 \geq 0$

2. Min. $Z = 3x_1 - x_2$

   subject to $2x_1 + x_2 \geq 2$

   $x_1 + 3x_2 \leq 2$

   $x_2 \leq 4$

   and $x_1, x_2 \geq 0$

3. Max. $Z = 5x_1 + 8x_2$

subject to $3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

and $x_1, x_2 \geq 0$

4. Max. $Z = 3x_1 + 2x_2$

subject to $2x_1 + x_2 \geq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1, x_2 \geq 0$

5. Min. $Z = 2x_1 + x_2$

subject to $3x_1 + x_2 = 3;$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1, x_2 \geq 0$

6. Min. $Z = 3x_1 + 2x_2 + 3x_3$

subject to $2x_1 + x_2 + x_3 \leq 2;$

$3x_1 + 4x_2 + 2x_3 \geq 8$

and $x_1, x_2, x_3 \geq 0$

7. Min. $Z = 5x_1 + 2x_2 + 10x_3$

subject to $x_1 - x_3 \leq 10;$

$x_2 + x_3 \geq 10$

and $x_1, x_2, x_3 \geq 0$

8. Min. $Z = 5x_1 + 6x_2$

subject to $2x_1 + 5x_2 \leq 1500;$

$3x_1 + x_2 \geq 1200$

and $x_1, x_2 \geq 0$

5.9 References

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Lesson – 6

Dual Simplex Method

6.0 Objective :

- When Dual – Simplex is applicable
- Dual – Simplex algorithm

Structure

6.1 Introduction

The simplex method is an algorithm that always deals with a basic feasible solution and the algorithm is terminated as soon as an optimal solution is achieved. That is, the procedure should be stopped when all $z_j - c_j \geq 0$ for maximization problem and $z_j - c_j \leq 0$ for minimization problem. However, if one or more solution values (i.e. x_{Bi}) are negative and optimality condition on $z_j - c_j$ (both for maximization and minimizatiom) is satisfied, then current optimal solution may not be feasible because $z_j - c_j = c_B B^{-1} a_j - c_j$ completely independent of the vector b. In such cases, it is possible to find a starting basic, but not feasible solution that is dual feasible, i.e. all $z_j - c_j \geq 0$ for a maximization problem. In all such cases a variant of the simplex method called ***dual-simplex method*** would be used. In the dual simplex method we always attempt to retain optimality (i.e. $z_j - c_j \geq 0$) while bringing the primal back to feasibility (i.e. $x_{Bi} \geq 0$ for all i).

6.2 Dual – Simplex Algorithm

The steps of a dual - simplex algorithm may be summarized as follows :

Step-1 : Determine an initial solution

Convert the given LP problem into the standard form by adding slack, surplus and artificial variables and obtain an initial basic feasible solution. Display this solution in the initial dual-simplex table.

Step-2 : Test optimality

If. all solution values are positive (i.e. $x_{Bi} \geq 0$, for all i) then there is no need of applying a dual-simplex method because improved solution can be obtained by simplex method itself. Otherwise go to step 3.

Step-3 : Test inconsistency

If there exists a row, say r, for which solution value is negative (i.e. $x_{Br} < 0$) and all elements in row r and column j are positive (i.e. $y_{rj} \geq 0$ for all j), then current solution is infeasible; hence go to step 4.

Step-4 : Obtain improved solution

(i) Select a basic variable associated with the row (called key row) having the largest negative solution value, i.e.

$$x_{Br} = \text{Min } \{x_{Bi};\ x_{Bi} < 0\}$$

(ii) Determine the minimum ratios only for those columns having a negative element in row r. Then select a non-basic variable for entering into the basis associated with the column for which

$$\frac{z_k - c_k}{y_{rk}} = \underset{j}{Min}\left\{\frac{z_j - c_j}{y_{rj}};\ y_{rj} < 0\right\}$$

The element (i.e. y_{rk}) laying at the intersection of key row and key column is called key element. The improved solution can then be obtained by making y_{rk} as 1 and all other element of the key column zero. Here, it may be noted that key element is always positive.

Step-5 : Revise the solution

Repeat steps 2 to 4 until either an optimal solution is reached or there exists no feasible solution.

A flow chart of solution procedure of dual – simplex method is shown in Fig. 6.2.1.

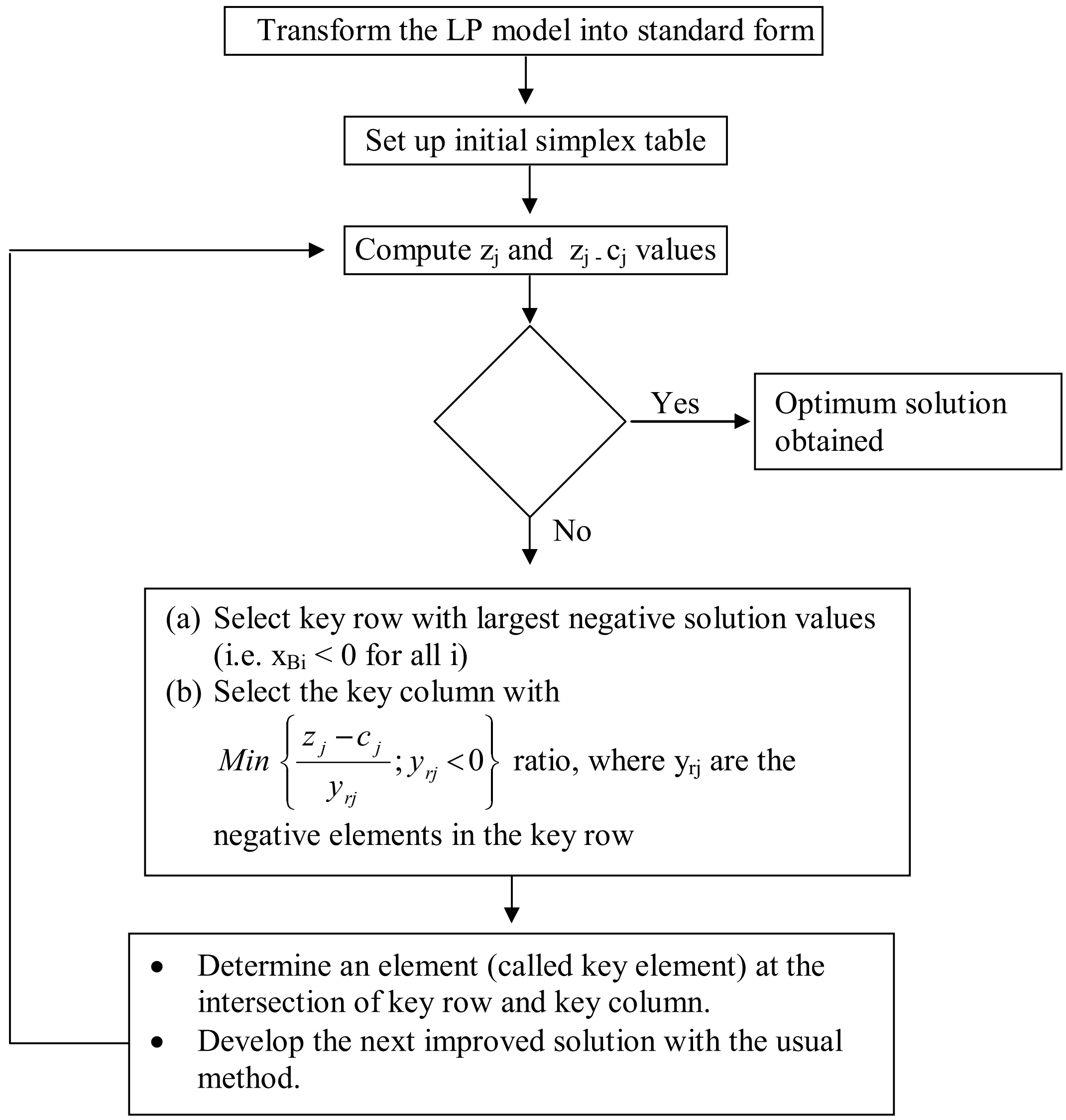

Fig. 6.2.1. Flow Chart of Dual – Simplex Method (Maximization Problem)

6.3 Worked out Example

6.3.1 Example : Use the dual simplex method to solve the LP problem :

Maximize $Z = -3x_1 - 2x_2$

subject to the constraints

$$x_1 + x_2 \geq 1$$

$$x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \geq 10$$

$$x_2 \leq 3$$

and $x_1, x_2 \geq 0$

Solution : In order to apply the dual simplex method, make all the constraints of the type $\leq$ by multiplying by -1 and then add slack variables in the constraints of the given LP problem. Thus the problem becomes

Maximize $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to the constraints

$$-x_1 - x_2 + s_1 = -1$$

$$x_1 + x_2 + s_2 = 7$$

$$-x_1 - 2x_2 + s_3 = -10$$

$$x_2 + s_4 = 3$$

and $x_1, x_2, s_1, s_2, s_3, s_4 \geq 0$

An initial basic solution (infeasible) is obtained by setting xl = x2 = 0 as shown in Table 6.3.1. This gives the solution values as :

$s_1 = -1$, $s_2 = 7$, $s_3 = -10$, $s_4 = 3$ and Max $Z = 0$.

Table 6.3.1 Initial Solution

		$c_j \rightarrow$	-3	-2	0	0	0	0
c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	-1	-1	-1	1	0	0	0
0	s_2	7	1	1	0	1	0	0
0	s_3	-10	-1	(-2)	0	0	1	0
0	s_4	3	0	1	0	0	0	1
Z = 0		z_j	0	0	0	0	0	0
		$z_j - c_j$	+3	+2	0	0	0	0
	$\delta = (z_j - c_j) / y_{rj};\ y_{rj} < 0$		+3	+1	-	-	-	-
				↑				

Since all $z_j - c_j \geq 0$ and all solution values, (i.e. x_{Bi}) are not non-negative, an optimal but infeasible solution has been obtained. Now in order to obtain a feasible solution, we select a basic variable to leave the basis and a non-basic variable to enter into the basis as follows:

Variable to leave the basis $= \text{Min}\ \{x_{Bi} : x_{Bi} < 0\}$

$= \text{Min}\ \{-1,-10\} = -10(=x_{B3})$

That is, the basic variable s_3 leaves the basis.

Variable to enter the basis $= \text{Min}\ \{\delta\ ;\ y_{rj} < 0\}$

$= \text{Min}\ \{(-3/-1), (-2/-2)\}$

$= \text{Min}\{3, 1\} = 1$

That is, variable x_2 enters the basis.

Iteration 1: The new solution is obtained after introducing x_2 into the basis and dropping s_3 from the basis as shown in Table 6.3.2.

Table 6.3.2 Initial Solution

		$c_j \rightarrow$	-3	-2	0	0	0	0
c_B	Variables in basic B	Solution values $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	s_4
0	s_1	4	-1/2	0	1	0	-1/2	0
0	s_2	2	1/2	0	0	1	1/2	0
-2	x_2	5	1/2	1	0	0	-1/2	0
0	s_4	-2	(-1/2)	0	0	0	1/2	1 $\rightarrow$
Z = 10		z_j	0	-2	0	0	1	0
		$z_j - c_j$	+3	0	0	0	+1	0
	$\delta = (z_j - c_j) / y_{rj};\ y_{rj} < 0$		4	-	-	-	-	-
			$\uparrow$					

Table 6.3.2 shows that the solution is still infeasible (because $s_4 = -2$) but optimal. Thus we proceed to iteration 2.

Iteration 2 : Variable x_1 enters the basis and s_4 leaves the basis. The key row is the s_4 row, since it is the only row with negative solution (i.e. x_B) value. The key column can be determined as usual as in the previous iteration. Since y_{41} is the only negative element in the key row, the x_1-colurnn is the key column. The new solution is shown in Table 6.3.3

Table 6.3.3

c_B	Variables in basic B	$c_j \rightarrow$ Solution values $b(=x_B)$	-3 x_1	-2 x_2	0 s_1	0 s_2	0 s_3	0 s_4
0	s_1	6	0	0	1	0	-1	-1
0	s_2	0	0	0	0	1	1	1
-2	x_2	3	0	1	0	0	0	1
-3	x_1	4	1	0	0	0	-1	-2
Z = -18		z_j	-3	2	0	0	3	4
		$z_j - c_j$	0	0	0	0	+3	+4

As shown in Table 6.3.3 all $z_j - c_j \geq 0$ and all solution values are also positive (i.e. $x_{Bi} \geq 0$) thus the current solution is the optimal solution. Hence the optimal basic feasible solution to the given LP problem is :

$$x_1 = 4,\ x_2 = 3 \text{ and Max } Z = -18.$$

Example 6.3.2 : Solve the following problem by dual simplex method

Min. $z = 2x_1 + x_2$,

subject to $3x_1 + x_2 \geq 3$,

$4x_1 + 3x_2 \geq 6$,

$x_1 + 2x_2 \geq 3$,

and $x_1, x_2 \geq 0$.

Solution : Step 1: First, we rewrite the problem in the following form :

Max. $z' = -2x_1 - x_2, z' = -z$,

subject to $-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 - 2x_2 \leq -3,$

$x_1, x_2 \geq 0.$

Step 2: Now, adding the slack variables x_3, x_4, x_5 in each constraint, we get

$-3x_1 - x_2 + x_3 = -3$

$-4x_1 - 3x_2 + x_4 = -6$

$-x_1 - 2x_2 + x_5 = -3$

Consequently, in matrix form, the constraint equations can be written as :

$$\begin{bmatrix} -3 & -1 & 1 & 0 & 0 \\ -4 & -3 & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ -3 \end{bmatrix}$$

Step 3: We now construct the starting simplex table

Table 6.3.4

		$c_j \rightarrow$	-2	-1	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	X_4	X_5	
x_3	0	-3	-3	-1	1	0	0	
x_4	0	-6	-4	-3	0	1	0	$\rightarrow$
x_5	0	-3	-1	-2	0	0	1	
Z = 0	$z_j - c_j$		2	1	0	0	0	
	δ		-1/2	-1/3	-	-	-	
				$\uparrow$				

Step 4: To find the leaving vector

Since, $x_{Br} = \min [x_{Bi}, x_{Bi} < 0] = \min [x_{B1}, x_{B2}, x_{B3}]$

$= \min [-3, -6, -3] = -6 = x_{B2}.$

Hence r = 2. So we must remove X_4 from the basis matrix.

Step 5: To find the entering vector for predetermined value of r = 2.

$$\text{Min } \delta = \text{Min}\left[\frac{z_j - c_j}{x_{2k}}; x_{2k} < 0\right]$$

$$= \text{Min}\left[\frac{2}{-4}, \frac{1}{-3}\right] = \frac{-1}{3}$$

Hence k = 2. So we must enter the vector X_2.

Step 6: In Table 6.3.4 the key elements is - 3. We get the following transformed table in the usual manner.

Table 8.3.5

Basic Variables	C_B	X_B	X_1	X_2	X_3	X_4	X_5	
		$c_j \rightarrow$	-2	-1	0	0	0	
x_3	0	-1	(-5/3)	0	1	-1/3	0	→
x_2	-1	2	4/3	1	0	-1/3	0	
x_5	0	1	5/3	0	0	-2/3	1	
Z = -0	$z_j - c_j$		-2/3	0	0	-1/3	0	
	δ		+2/5	-	-	-1	-	
			↑					

Step7: To find the leaving vector

Since x_{Br} = min. [x_{Bi}, $x_{Bi} < 0$] = min. [x_{B1}] = (x_{B2} and x_{B3} both are positive, hence we have ignored). Therefore, r - l. So we must remove the vector X_3.

Step 8. To find the entering vector for predetermined value of r (=1).

Since

$$\text{Min } \delta = \text{Min} \left[\frac{z_j - c_j}{x_{1k}} ; x_{1k} < 0 \right]$$

$$= \text{Min} \left[\frac{2/3}{-5/3}, \frac{1/3}{-1/3} \right] = -\frac{2}{5}$$

Therefore k=1. So we must enter the vector X_1.

Step 9. To find the transformed table.

In Table 6.3.5, the key element is -5/3. Now, we get the transformed Table 8.3.6.

Since the basic solution $x_1 = 3/5$, $x_2 = 6/5$, $x_3 = x_4 = x_5 = 0$ is feasible at this stage, so this is the required optimal solution giving, z =12/5.

Table 6.3.6

		$c_j \rightarrow$	-2	-1	0	0	0
Basic Variables	C_B	X_B	X_1	X_2	X_3	X_4	X_5
x_1	-2	3/5	1	0	-3/5	1/5	0
x_2	-1	6/5	0	1	4/5	-3/5	0
x_5	0	0	0	0	1	-1	1
Max Z = -12/5 Min z = 12/5		$z_j - c_j$	0	0	2/5	1/5	0

Observe in table 6.3.6. all $z_j - c_j \geq 0$ and all solution values are also non-negative (i.e., $X_{Bi} \geq 0$). Thus, this solution is the optimal solution. Hence, the optimal basic feasible solution to the given LP problem is $x_1 = 3/5$, x_2 - 6/5 and Min. z = 12/5.

Dual Solution : From the final table of Dual Simplex method also one can read the optimal solution of dual problem of the given LPP (Primal Problem). Dual of this example is given by

Maximize $Z' = 3w_1 + 6w_2 + 3w_3$

subject to the constraints

$3w_1 + 4w_2 + w_3 \leq 2$

$w_1 + 3w_2 + 2w_3 \leq 1$

$w_1,\ w_2, w_3 \geq 0.$

Its solution from the final Dual Simplex table is nothing but the values of $z_j - c_j$ corresponding to the slack variables x_3, x_4 and x_5. So its optimal solution is

w_j - 2/5, w_2 = 1/5 and $w_3 = 0$

Max. z = l2/5.

6.4 Summary

In this lesson it is clearly explained when Dual-Simplex method is used to solve a linear programming problem by Dual-simplex algorithm. The algorithm is demonstrated with an example also.

6.5 Exercise

Use dual simplex method to solve the following LP problems

1. Min. $Z = x_1 + x_2$

subject to $2\,x_1 + x_2 \geq 2$

$-\,x_1 - x_2 \geq 1$

and $x_1, x_2 \geq 0$

2. Min. $Z = 3x_1 - x_2$

subject to $x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

and $x_1, x_2 \geq 0$

3. Max. $Z = 2x_1 - 2x_2 \,_{-} 4x_3$

subject to $2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 \leq 3$

$x_1 + 4x_2 + 6x_3 \leq 5$

and $x_1, x_2, x_3 \geq 0$

4. Max. $Z = x_1 + 2x_2 + 3x_3$

subject to $x_1 - x_2 + x_3 \geq 4$

$x_1 + x_2 + 2x_3 \leq 8$

$x_2 - x_3 \geq 2$

and $x_1, x_2, x_3 \geq 0$

subject to $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$

$3x_1 - x_2 + 7x_3 - 2x_4 \leq 2$

$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$

and $x_1, x_2, x_3, x_4 \geq 0$

6.6 References

1. Operation Research - R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.
2. Operations Research - Kanthi Swarup and others, Sultan Chand & Sons.
3. Operations Research - S.D. Sarma, Kedar Nath Ram Nath and Co. Publishers, Meerut.
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Lesson – 7

Inventory Models

7.0 Objective :

- Costs associated with Inventory models
- Economic order quantity (EOQ)
- Assumptions in Inventory Models
- Different inventory models without shortages

Structure

7.1 Introduction

7.2 Costs associates with Inventories

7.3 Factors affecting inventory control

7.4 Economic Order Quantity

7.5 Assumptions

7.6 EOQ with equal production run length

7.7 EOQ with unequal production run length

7.8 Examples

7.9 EOQ with finite replacement

7.10 Example

7.11 Summary

7.12 Exercise

7.13 References

7.1 Introduction

Inventory consists of usable but idle resources. The resource may be of any type-men, materials, machines, etc. For example, if a company purchases a machine or appoints an expert in anticipation of the requirement of their services in future, these resources work as inventory. Generally, inventory of men or machines is not carried and managements hire machines or consult experts whenever required, as an economical alternative. When the resource involved is material or goods, it is referred as stock or simply as 'Inventory'. The term is generally used to indicate raw materials in process, finished product, packaging, spares and others-stocked in order to meet an expected demand or distribution in the future. Though inventory of materials is an idle resource-it is not meant for immediate use-it is almost essential to maintain some inventories for the smooth functioning of an enterprise. Traditionally, inventory is viewed as a necessary evil-too little of it causes costly interruptions and too much of it results in idle capital. Inventory control aims at maintaining the balance between these two extremes.

7.2 Costs Associated with Inventories

Various costs associated with inventory control are often classified as follows :

Set-up cost. This is the cost associated with the setting up of machinery before starting production. Set-up cost is generally assumed. to be independent of the quantity ordered for or produced.

Ordering cost. This is a cost associated with ordering of raw material for production purposes. Advertisements, consumption of stationery and postage, telephone charges telegrams, rent for space used by the purchasing department, travelling expenditures incurred, etc., constitute the ordering cost.

Purchase (or production cost). The cost of purchasing (or producing) a unit of an item is known as purchase (or production) cost. The purchase price will become important when quantity discounts are allowed for purchases above a certain quantity or when economies of scale suggest that the per unit production cost can be reduced by a larger production run.

Carrying (or holding) cost. The carrying cost is associated with carrying (or holding) inventory. This cost generally includes the costs such as rent for space used for storage, interest on the money locked-ups insurance of stored equipment, production, taxes, depreciation of equipment and furniture used, etc.

Shortage (or stock out) cost. The penalty cost for running out of stock (i.e., when an item cannot be supplied on the customer's demand) is known as shortage cost. This cost includes the loss of potential profit through sales of items and loss of goodwill, in terms of permanent loss of customers and its associated lost profit in future sales.

Salvage cost (or selling price). When the demand for certain commodity is affected by the quantity stocked, decision problem is based on a profit maximization criterion that includes the revenue from selling. Salvage value may be combined with the cost of storage and hence is generally neglected.

Revenue cost. When it is assumed that both the price and the demand of the product are not under control of the organisation, the revenue from the sales is independent of the company's inventory policy and may be neglected except for the situation when the organisation cannot meet the demand and the sale is lost. Therefore, the revenue cost may or may not be included in the study of inventory policy.

7.3 Factors Affecting Inventory Control

Besides the costs that determine the profitability, other factors which play an important role in the study of inventory control are the following :

Demand. The number of units required per period is called demand. The demand pattern of a commodity may be either deterministic or probabilistic.

Lead time. The time gap between placing of an order and its actual arrival in the inventory is known as lead time.

Order cycle. The time period between placement of two successive orders is referred to as an order cycle.

Time horizon. The time period over which the inventory level will be controlled is called the time horizon. This horizon may be finite or infinite depending upon the nature of the demand for the commodity.

Re-order level. The level between maximum and minimum stock, at which purchasing (or manufacturing) activities must start for replenishment, is known as re-order level.

Stock replenishment. Although an inventory problem may operate with lead time, the actual replacement of stock may occur instantaneously or uniformly. Instantaneous replenishment occurs in case the stock is purchased from outside sources whereas the uniform replenishment may occur when the product is manufactured by the company.

7.4 Economic Order Quantity (EOQ)

By the 'order quantity' we mean the quantity produced or procured during one production cycle. When the size of order increases, the ordering costs (cost of purchasing, inspection, etc.) will decrease whereas the inventory carrying costs (cost of storage, insurance, etc.) will increase. Thus in the production process there are two opposite costs, one encourages the increase in the order size and the other discourages. Economic Order Quantity (EOQ) is that size of order which minimizes total annual costs of carrying inventory and cost of ordering.

The two opposite costs can be shown graphically by plotting them against the order size as shown in Fig. 7.1 below :

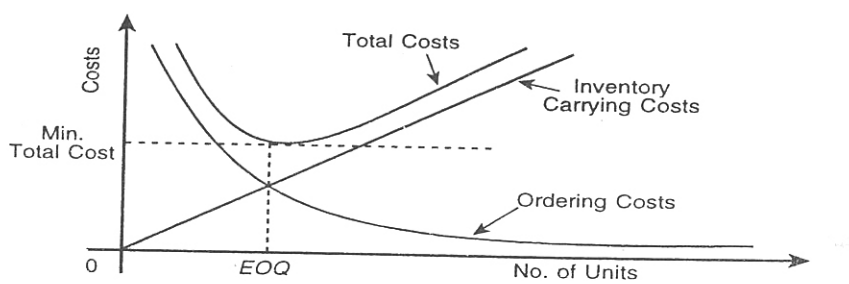

Fig. 7.1. Graph of EOQ

It is evident from above that the minimum total cost occurs at the point where the ordering costs and inventory carrying costs are equal.

7.5 Assumptions

Any inventory problem will assume the following assumptions :

(i) Demand is known and uniform.

(ii) Let D denote the total number of units purchased / produced or supplied per time period and Q denote the lot size in each production run.

(iii) Shortages are not permitted, i.e., as soon as the level of the inventory reaches zero, the inventory is replenished.

(iv) Production or supply of commodity is instantaneous (Abundant Availability).

(v) Lead time is zero.

(vi) Set-up cost per production run on procurement cost is C_s. (or A).

(vii) Holding cost is C_1 per unit in inventory for a unit, i.e., C_1 = IC, where C is the unit cost, I is called inventory carrying charge expressed as a % of the value of the average inventory.

7.6. EOQ Problem with Several production runs with equal length

This fundamental situation can be shown on an inventory-time diagram, with Q on the vertical axis and time on the horizontal axis. The total time period (one year) is divided into n parts :

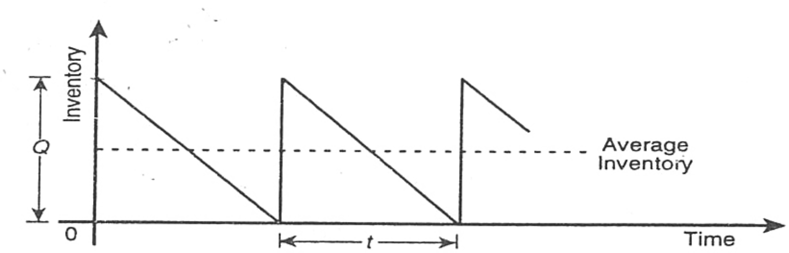

Fig. 7.2. EOQ Problem with uniform demand

Here it is assumed that after each time t, the quantity Q is produced / purchased or supplied throughout the entire time period, say one year. Now, if n denotes the total number of runs of the quantity produced or purchased during the year, then clearly we have

1 = nt and D = nQ.

It may be clear that the average amount of inventory at hand on any day is then $\frac{1}{2}$ Q, as shown by dotted line in Fig. 7.2. Total inventory over the time period t days is clearly the area of the first triangle (= $\frac{1}{2}$ Qt). Thus the average inventory at any time on any given day in the t period is $\frac{1}{2}$ Qt / t = $\frac{1}{2}$ Q.

Now, since each of the triangles in Fig. 7.2 over a year period looks the same, $\frac{1}{2}$ Q remains the average amount of inventory in each interval of length t during the entire period. Annual inventory holding cost is therefore given by

$$f(Q) = \frac{1}{2} QC_1.$$

Annual costs associated with runs of size Q are given by

$$g\ (G) = nCs = \frac{D}{Q}\ Cs$$

Since the minimum total cost occurs at the point where ordering cost and the total inventory carrying cost are equal, we must have

$$f(Q) = g\ (Q)$$

This implies, that

$$\frac{1}{2} QC_1 = \frac{D}{Q} C_s$$

Hence, the optimum value of Q is

$$Q^o = \sqrt{2DC_s / C_1}$$

This is known as the economic (optimum) lot size formula due to R.H. Wilson.

The above EOQ formula can also be expressed in terms of the economic order value terms as follows :

$$Q^o = \sqrt{\frac{2AD}{IC}}$$

Characteristics of Case 1.

1. Optimum number of orders placed per year

$$n^o = D/Q^o = \sqrt{DC_1 / 2C_s}$$

2. Optimum length of time between orders

$$t^o = T/n^o = T\sqrt{2C_s / DC_1} \quad \text{or} \quad \sqrt{2C_s / DC_1}$$

when T (total time horizon) is one year.

3. Minimum total annual inventory cost

$$TC^o = \frac{1}{2} Q^o C_1 + DCs/Q^o = \sqrt{2DC_1C_s}$$

Remark. If the carrying cost is given as a percentage of average value of inventory held, then total annual carrying cost may be expressed as $C_1 = C \times 1$ or $P \times 1$. The total annual inventory cost then becomes

$$TC = \frac{1}{2} Q \times C \times 1 + DCs/Q$$

The optimum order quantity Q^o will, then, be

$$Q^o = \sqrt{2DC_s / CI}$$

Corollary 1. In the EOQ problem discussed above, if the set-up cost is $C_s + bQ$ instead of being fixed (where b is set-up cost per unit item, produced) then there is no change in the optimum order quantity produced due to change in the set-up cost.

Proof. In this case, the annual cost is given by

$$TC = \frac{1}{2}QC_1 + \frac{D}{Q}(Cs + bQ).$$

For the optimum value of Q, we see that

$$\frac{d}{dQ}\text{ (TC)} = 0 \Rightarrow Q = \sqrt{\frac{2DCs}{C_1}} \text{ and } \frac{d^2}{dQ^2}\text{ (TC)} > 0 \text{ for } Q > 0.$$

Hence $$Q^o = \sqrt{\frac{2DCs}{C_1}}$$

This shows that there is no change in Q^o in spite of change in the set-up cost.

Corollary 2. In the above EOQ problem lead time has been assumed to be zero. But in most of the business situations, there exists a positive lead time, say L, from the time the order is placed until it is actually supplied.

Proof. If the inventory consumption rate is 'K' units per day and L is the lead time in days, the total inventory requirements during the lead time will be 'LK'. Therefore, as soon as the inventory level becomes 'LK' an order Q is placed. This is called the re-order point p = LK. This is equivalent to continuously observing the level of inventory, until the re-order point is obtained. Because of this reason, EOQ problem is sometimes called the continuous review problem. Fig. 7.3 shows the re-order points :

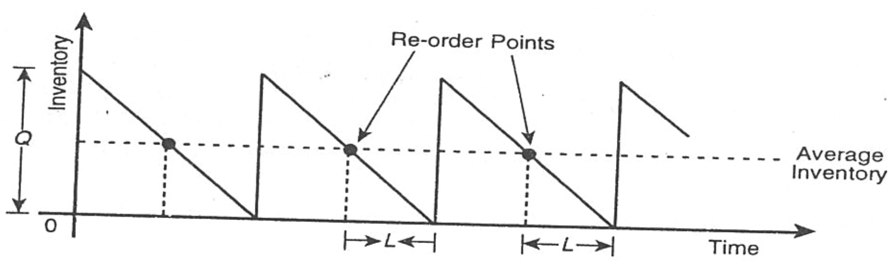

Fig. 7.3

Figure given above assumes that the lead time is less than the cycle length, which is not necessarily the case in general. To account for this situation, we define the effective lead time as

$$Le = L\text{-}mt^o,$$

where m is the largest integer not exceeding L/t^o. This result is justified because after m cycles of t^o each, the inventory situation acts as if the interval between placing an order and receiving another is L_e.

7.7 EOQ Problem with Several Production Runs of Unequal Length

In this problem all the assumptions are same as in Case I except that the demand is uniform and the production runs differ in units.

Let $t_1, t_2, t_3, \ldots, t_n$ denote the times of successive production runs, such that

$$t_1 + t_2 + \ldots + t_n = 1 \text{ year.}$$

Thus, the fundamental situation can be represented graphically as shown in Fig. 7.4.

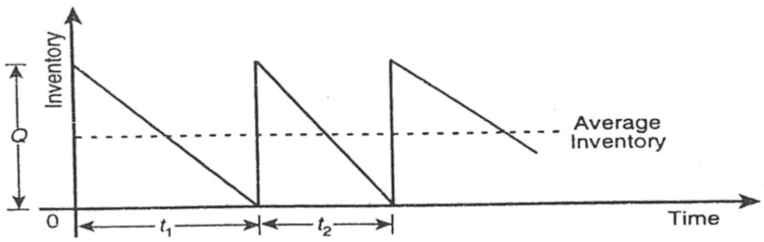

Fig. 7.4

Obviously, the annual inventory holding cost is given by

$$f(Q) = \left(\frac{1}{2}Qt_1\right)C_1 + \left(\frac{1}{2}Qt_2\right)C_1 + \ldots + \left(\frac{1}{2}Qt_n\right)C_1$$

$$= \frac{1}{2}Q\,(t_1 + t_2 + \ldots + t_n)\,C_1 = \frac{1}{2}QC_1$$

and the set-up costs associated with runs of size Q are given by

$$g(Q) = \frac{D}{Q}Cs, \quad \text{since} \quad nQ = D$$

$\therefore$ Total annual cost is

$$TC = f(Q) + g(Q) = \frac{1}{2}\,QC_1 + \frac{D}{Q}Cs.$$

This cost is the same as was obtained in Case 1 and hence the optimum quantities are

$$Q^o = \sqrt{2DC_s/C_1} \qquad \text{and} \qquad TC^o = \sqrt{2DC_1C_s}$$

Remark. If the total time period is T instead of one year, then the optimum order quantity becomes $Q^o = \sqrt{2C_s D / C_1 T}$ and the minimum cost becomes

$TC^o = \sqrt{2 C_1 C_s D / T}$. Thus, the uniform rate of demand is replaced by average rate of demand, i.e., D is replaced by D/T.

7.8 Examples

Example 7.8.1 An oil engine manufacturer purchases lubricants at the rate of Rs. 42 per piece from a vendor. The requirement of these lubricants is 1,800 per year. What should be the order quantity per order, if the cost per placement of an order is Rs. 16 and inventory carrying charge per rupee per year is only 20 paise.

Solution. We are given

D = 1,800 lubricants per year, Cs = Rs. 16 per order

C_1 = Rs. 42 x Re. 0.20 = Rs. 8.40

$Q° = \sqrt{2 x 1800 x 16 / 8.40}$ = 82.8 or 83 lubricants.

Example 7.8.2 A manufacturing company purchases 9,000 parts of a machine for its annual requirements, ordering one month usage at a time. Each part costs Rs. 20. The ordering cost per order is Rs. 15 and the carrying charges are 15% of the average inventory per year. You have been assigned to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it save the company per year?

Solution. We are given

D = 9,000 parts per year, Cs = Rs. 15 per order

C_1 = 15% of the average inventory per year

= Rs. 20 x 15/100 = Rs. 3 each part per year

$Q^o = \sqrt{\frac{2 x 15 x 9{,}000}{3}}$ = 300 units and $t^o = \frac{300}{9,000} = \frac{1}{30}$ year

Minimum average yearly cost = $\sqrt{2 x 3 x 15 x 9,000}$ = Rs. 900. If the company follows the policy of ordering every month, then the annual ordering cost is Rs. 12 x 15 or Rs. 180, and lot size of inventory each month = 9,000/12 = 750 (= Q).

Average inventory at any time = $\frac{1}{2}$Q = 750/2 = 375.

∴ Storage cost at any time = 375 C_1 = 375 x 3 = Rs. 1,125.

Total annual cost = Rs. 1,125 + Rs. 180 = Rs. 1,305.

Hence, the company should purchase 300 parts at time intervals of 1/30 year instead of ordering 750 parts each month. The net saving of the company will be

Rs. 1,305 - Rs. 900 = Rs. 405 per year.

Example 7.8.3. Neon lights in an industrial park are replaced at the rate of 100 units per day. The physical plant orders the neon lights periodically. It costs Rs. 100 to initiate a purchase order. A neon light kept in storage is estimated to cost about Re. 02 per day. The lead time between placing and receiving an order is 12 days. Determine the optimum inventory policy for ordering the neon lights.

Solution. We are given

D = 100 units per day, Cs = Rs. 100 per order,

C_1 = Re. 02 per unit per day, L = 12 days.

∴ $Q^o = \sqrt{\frac{2 x 100 x 100}{.02}}$ = 1,000 neon lights

$t^o = \frac{Q^o}{D} =$ 10 days

Now, since the lead time, L (=12 days) exceeds the cycle length (=10 days), we compute L_e.

Thus, $L_e = L - mt^o =$ 12 – 1 x 10 = 2 days,

where m = (largest integer < L/t^o) = (largest integer ≤ 12/10 = 1.

The re-order point, therefore, occurs when the inventory level drops to

$$L_e \times D = 2 \times 100 = 200 \text{ neon lights.}$$

Hence, the inventory policy for ordering neon lights is : Order 100 units whenever the inventory level drops to 200 units.

7.9 EOQ Problem with Finite Replenishment (Production)

In this problem all the assumptions are same as in case 1, except that of instantaneous replenishment. Assume that each production run of length t consists of two parts, say t_1 and t_2, such that

(i) the inventory is building up at a constant rate of (k - r) units, per unit of time during t_1, $k > r$;

(ii) there is no replenishment (or production) during time t_2 and the inventory is decreasing at the rate of r per unit of time.

The graphical representation of the situation is shown in Fig. 7.5.

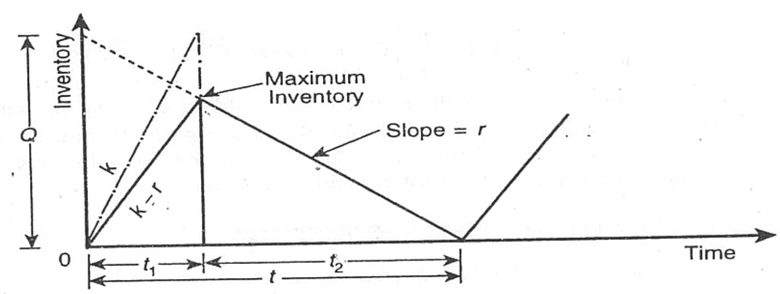

Fig. 7.5

Here, the total (order) quantity Q is produced over a period, t_1, which is defined by the production rate k. Since the inventory does not pile up in one shot but rather continuously over a time period and is also consumed simultaneously, the average inventory level would be determined not only by the lot size Q, but also be affected by the production rate k and depletion (demand) rate r.

To determine the average inventory, we proceed as follows :

Since t_1 is the time required to produce Q at a rate k, we shall have

$$Q = kt_1 \quad \text{or} \quad t_1 = Q/k.$$

During production period t_1, inventory is increasing at the rate of k and simultaneously decreasing at the rate of r. Thus inventory accumulates at the rate of (k-r) units. Therefore, the maximum inventory level shall be equal to t_1 (k - r).

$$\therefore \quad \text{Average inventory} = \frac{1}{2}\, t_1\,(k - r) = \frac{1}{2} Q\,(1 - r/k), \qquad \text{since } t_1 = Q/k$$

Now, with C_1 as the holding cost per unit per year, the total annual holding cost is given by

$$f(Q) = \text{Average inventory} \times C_1 = \frac{1}{2} QC_1\,(1 - r/k)$$

Annual ordering cost is given by

$$g(Q) = C_s \times D/Q,$$

since D is the total demand in a year.

Since the minimum total cost occurs at the point where annual ordering cost and annual holding cost are equal, we must have

$$f(Q) = g(Q)$$

This implies that

$$\frac{1}{2} QC_1 \left(1 - \frac{r}{k}\right) = C_s \frac{D}{Q}$$

Hence, the optimum value of Q is

$$Q^o = \sqrt{\frac{2DC_s}{C_1(1 - r/k)}} = \sqrt{\frac{2DC_s}{C_1}\left(\frac{k}{k - r}\right)}$$

Characteristics of Case 3

1. Optimum number of production runs per year

$$n^o = D/Q^o = \sqrt{\frac{DC_1}{2C_s}(1 - r/k)}$$

2. Optimum length of each lot size production run

$t_1^o = Q^o / k = \sqrt{\dfrac{2DC_s}{C_1 k (k - r)}}$

3. Total minimum production inventory cost

$$TC^o = \frac{C_s D}{Q^o} + \frac{1}{2}\, Q^o \left(1 - \frac{r}{k}\right) C_1 = \sqrt{2DC_s C_1 (1 - r/k)}$$

Note. If k $\to$ r, then C $\to$ 0. This shows that there will be no holding cost and no set-up cost.

If k $\to \infty$, i.e., when the production rate becomes infinite, the above problem reduces to the one considered in case 1. The inventory holding cost per unit of time is reduced from the cost discussed in case 1 in the ratio (1 - r/k) : 1 for minimum cost, although the set-up cost remains the same.

7.10 Example

A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is Rs. 2 and the set-up cost of a production run is Rs. 1,800. How frequently should production run be made?

Solution. We are given

D = 10,000 x 300 = 30,00,000 bearings (Assuming 300 working days in a year)

C_1 = Rs. 2 per bearing per year, C_s = Rs. 1,800 per production run

r = 10,000 bearings per day, k = 25,000 bearings per day

$$\therefore \quad Q^o = \sqrt{\frac{2 \, x \, 30{,}00{,}000 \; x \; 1{,}800}{2}} \; \sqrt{\frac{25{,}000}{25{,}000 - 10{,}000}} = 94{,}868 \text{ bearings}$$

$$t^o = \frac{94{,}868}{10{,}000} = 9.49 \text{ days}$$

$$\text{Length of production cycle} = \frac{94{,}868}{25{,}000} = 4 \text{ days (approx.)}$$

Thus the production cycle starts at an interval of 9.49 days and production continues for 4

7.11 Summary:

The present lesson deals with the Inventory to be maintained, for smooth production process, Costs associated with inventory models, Factors affecting Inventory control, EOQ, along with its assumptions. The theory is supported by suitable examples.

7. 12 Exercises

1. A shipbuilding firm uses rivets at a constant rate of 20,000 numbers per year. Ordering costs are Rs. 30 per year. The rivets cost Rs. 1.50 per number. The holding cost of rivets is estimated to be 12.5% of unit cost per year. Determine the EOQ.

2. A manufacturer has to supply his customer with 600 units of his product per year Shortages are not allowed and the storage cost amounts to Re. 0.60 per unit per year. The set-up cost per run is Rs. 80.00. Find the optimum run-size and the minimum average yearly cost.

3. A shopkeeper has a uniform demand of an item at the rate of 600 items per year. He buys from a supplier at a cost of Rs. 8 per item and the cost of ordering is Rs. 12 each time. If the stock holding costs are 20% per year of stock value, how frequently should he replenish his stocks and what is the optimal order quantity?

4. A certain item costs Rs. 235 per tonne. The monthly requirement is 5 tonnes and each time the stock is replenished there is a set-up cost of Rs. 1,000. The cost of carrying inventory has been estimated at 10% of the value of the stock per year. What is the Optimal Order Quantity?

5. A manufacturer has to supply his customer with 24,000 units of his product per year. This demand is fixed and known. Since the unit is used by the customer in an assembly line operation and the customer has no storage space for the units, the manufacturer must ship a day's supply each day. If the manufacturer fails to supply the required units, he will lose the amount and probably his business. Hence, the cost of a shortage is assumed to be infinite and, consequently, none will be tolerated. The inventory holding cost amounts to 0.10 per unit per month, and the set-up cost per production run is Rs. 350. Find the optimum lot size and the length of optimum production run.

6. An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 items per day. If the set-up cost is Rs. 100.00 and holding cost is Re. .01 per unit of item per day, find the economic lot size for one run, assuming that the shortages are not permitted. Also find the time of cycle and minimum total cost for one run.

7. An item is produced at the rate of 128 units per day. The annual demand is 6,400 units. The set-up cost for each production run is Rs. 24 and inventory carrying cost is Rs. 3 pr unit per year. There are 250 working days for production each year. Develop an inventory policy for this item.

8. A contractor has to supply 20,000 units per day. He can produce 30,000 units per day. The cost of holding a unit in stock is Rs. 3 per year and the set-up cost per run is Rs. 50. How frequently, and of what size, the production runs be made?

9. Amit manufactures 50,000 bottles of tomato ketchup in an year. The factory cost per bottle is Rs. 5, the set-up cost per production run is estimated to be Rs. 90, and the carrying costs on finished goods inventory amount to 20% of the cost per annum. The production rate is 600 bottles per day, and sales amount to 150 bottles per day. What is the optimum production lot size and the number of production runs?

If the factory costs increase to Rs. 7.50 per bottle, what will be the optimum production lot size ?

7.13 References

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Lesson – 8

Inventory Models with Shortages

8.0 Objective :

- Meaning of Shortage
- Models with Shortages
- Computation of EOQ

Structure

8.1 Introduction

In a business concern, if shortages occur then these can be classified into the following two categories :

(a) as soon as the desired units of a certain commodity arrive in inventory, the back orders arc satisfied,

(b) shortages are lost sales.

In the first category demand of the customer is met in the beginning of new production run, whereas in the second category the customer moves to some other firm to fulfil his requirements. This case deals with those problems of shortages where back orders are entertained.

8.2 EOQ Problem with Instantaneous Production and Variable Order Cycle Time

The problem that we now discuss is same as was discussed in Lesson 7 section 7.6 with the difference that the shortages are now permitted. Let C_2 be the shortage cost per unit of time per unit quantity. This inventory situation can also be illustrated graphically as shown in Fig. 8.2.1.

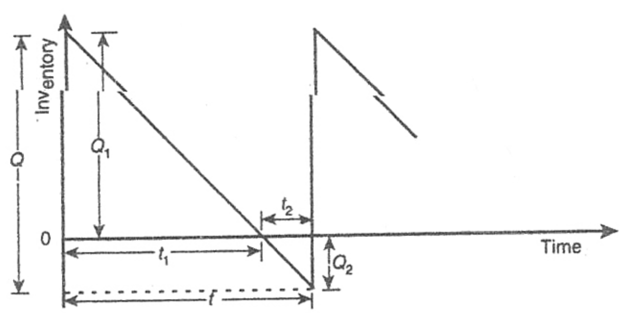

Fig 8.2.1.

Here the total time period is one year and is divided into equal parts, say of interval t. Further, this time interval t is divided into two parts t_1 and t_2, such that $t = t_1 + t_2$.

During the interval t_1 the items are drawn from the inventory as needed and during t_2, orders for the item are being accumulated but not filled. Then at the end of the interval t an amount Q is produced (or delivered). The amount Q has been divided into Q_1 and Q_2

such that $Q = Q_1 + Q_2$, where Q_1 denotes the amount which goes into inventory, and Q_2 denotes the amount which is immediately taken to satisfy past orders or unfilled demand.

The problem now is concerned with the areas of triangles above the time axis (representing items in inventory) and below the same axis (representing items in shortage).

Now, Total inventory over the time period t - $\frac{1}{2} Q_1 t_1$

Average inventory at any time $= \left(\frac{1}{2} Q_1 t_1\right) / t$

Annual inventory holding cost $= C_1 \left(\frac{1}{2} Q_1 t_1\right) / t$

Similarly,

Total amount of shortage over time period t $= \frac{1}{2} Q_2 t_2$

Annual shortage costs= $C_1 \left(\frac{1}{2} Q_2 t_2\right) / t$

Annual costs associated with runs of size Q = nCs = $\frac{D}{Q} C_s$,

since D/Q runs are produced in each year.

$\therefore$ Total annual cost is given by

$$TC = \left[C_1\left(\frac{1}{2} Q_1 t_1\right) + C_2\left(\frac{1}{2} Q_2 t_2\right)\right] / t + \frac{D}{Q} C_s,$$

Now using the relationship for similar triangles, we have

$$\frac{t_1}{t} = \frac{Q_1}{Q} \text{ and } \frac{t_2}{t} = \frac{Q_2}{Q}$$

i.e., $t_1 = \frac{Q_1}{Q}$ t and $t_2 = \frac{Q_2}{Q}$ t.

Making use of these values, we get

$$TC = \frac{1}{2} C_1 \left(\frac{Q_1^{\,2}}{Q}\right) + C_2 \left[\frac{(Q-Q_1)^2}{Q}\right] + C_s\left(\frac{D}{Q}\right),$$

since $Q_2 = Q - Q_1$.

For determining the optimum values of Q_1 and Q so as to optimize TC, we have

$$\frac{\partial (TC)}{\partial Q_1} = 0 \Rightarrow Q_1 = C_2Q/(C_1 + C_2),$$

$$\frac{\partial (TC)}{\partial Q} = 0 \Rightarrow Q = \sqrt{\frac{2C_sD + C_1Q_1^{\,2}}{C_2} + Q_1^{\,2}}$$

and $\frac{\partial^2 (TC)}{\partial Q_1^{\,2}} > 0$, $\frac{\partial^2 (TC)}{\partial Q^2} > 0$ for these values of Q_1 and Q_2.

Thus the optimum quantities are given by (on simplification)

$$Q^o = \sqrt{\frac{2C_sD}{C_1}}\sqrt{\frac{(C_1+C_2)}{C_2}}$$

$$Q_1^{\,o} = \left(\frac{C_2}{C_1+C_2}\right) Q^o = \sqrt{\frac{C_2}{C_1+C_2}}\sqrt{\frac{2C_sD}{C_1}} \quad \text{(Optimum Stock Level)}$$

8.2.1. Characteristics fo this model

1. Time between receipt of orders (when to order)

$$t^o = Q^o/D = \sqrt{\frac{2C_s}{DC_1}}\sqrt{\frac{C_1+C_2}{C_2}}$$

2. Total optimum inventory cost

$$TC^o = \sqrt{2DC_sC_1}\sqrt{\frac{C_2}{C_1+C_2}}$$

3. Maximum inventory level

$$Q^o - Q_2^{\,o} = Q^o\left(1 - \frac{C_1}{C_1+C_2}\right)$$

$$= \sqrt{\frac{2C_sD}{C_1}}\sqrt{\frac{C_2}{C_1+C_2}}$$

Remarks :

1. If $C_1 > 0$ and $C_2 = \infty$, shortages are prohibited. In this case $Q_1^o = Q^o = \sqrt{2C_sD/C_1}$ and each batch Q^o is used entirely for inventory.

2. If $C_1 = \infty$ and $C_2 > 0$, inventories are prohibited. In this case $Q_1^o = 0$, $Q^o = \sqrt{2C_sD/C_2}$ and each batch is used only to fill back orders.

3. If shortage costs are negligible, then $C_1 > 0$ and $C_2 \to 0$. In this case $Q_1^o \to 0$ and $Q^o \to \infty$.

4. If inventory casts are negligible, then $C_1 \to 0$ and $C_2 > 0$. In, this case $Q^o \to \infty$ and $Q_1^o \to \infty$ i.e. $Q_1^o \to Q^o$. Thus as inventory costs become very small, increasingly large batches should be produced and used entirely as inventory for future demands.

5. When inventories and shortages are equally costly, i.e., when

$$C_1 = C_2, \frac{C_2}{C_1+C_2} = \frac{1}{2}.$$

Thus in this case

$$Q^o = \sqrt{2}\sqrt{\frac{2C_sD}{C_1}} = (1.414)\sqrt{\frac{2C_sD}{C_1}}$$

This shows that the lot size is .414 times as large as earlier when no shortages were allowed.

8.3. EOQ Problem with Instantaneous Production and Fixed Order Cycle

Let t be fixed, i.e., inventory is to be replenished after every time period t. Also let us assume that items are being supplied or produced at the rate of r units per unit of time during this fixed time period.

Here, total inventory over the time period $t_1 = \frac{1}{2} Q_1 t_1$

and total amount of shortages over time period $t_2 = \frac{1}{2} Q_2 t_2$

$\therefore$ Total production cost is given by

$$TC = \frac{1}{2} Q_1 t_1 C_1 + \frac{1}{2} Q_2 t_2 C_2$$

Using now the relationship of similar triangles, viz.,

$$\frac{t_1}{t} = \frac{Q_1}{Q} \text{ and } \frac{t_2}{t} = \frac{Q_2}{Q}$$

the cost equation reduces to

$$TC = \frac{1}{2Q}.C_1 Q_1^2 t + \frac{1}{2Q}.C_2 Q_2^2 t = \frac{1}{2r} C_1 Q_1^2 + \frac{1}{2r} C_2 (rt - Q_1)^2.$$

since $Q_2 = Q - Q_1$ and $Q = rt$.

The optimum value Q_1^o is obtained as follows :

$$\frac{\partial (TC)}{\partial Q_1} > 0 \Rightarrow C_1 Q_1 + C_2(Q_1 - rt) = 0 \text{ or } Q_1 = rtC_2/(C_1 + C_2).$$

Now $\frac{\partial^2 (TC)}{\partial Q_1^2} > 0$ for all values of Q_1, therefore,

$$Q_1^o = rt\left(\frac{C_2}{C_1 + C_2}\right) \qquad \textbf{(Optimum Inventory Level)}$$

Note : In this problem, set-up cost is not considered, because of it being fixed as the time of one production run is fixed. Substituting the value of Q_1^o in the total production cost equation, the optimum inventory cost is

$$TC^o = \left(\frac{C_1 C_2}{C_1 + C_2}\right).rt$$

8.4. EOQ Problem with Finite Replenishment (Production)

In this problem all the assumptions are same in case 1 except that the rate of replenishment of inventory is finite, say k units per unit of time.

Assume that each production run of length t consists of two parts t_1 and t_2 which are further sub-divided into two parts say t_{11} and t_{12}, t_{21} and t_{22}, where :

(i) inventory is building up at a constant rate of (k - r) units per unit of time during time t_{11},

(ii) no replenishment during time t_{12} and inventory is decreasing at the rate r per unit of time,

(iii) shortage is building up at a constant rate of r per unit of time during time t_{21},

(iv) shortages are being filled immediately at the rate of (k - r) units per unit of time during time t_{22}.

The graphical representation of the situation is as follows :

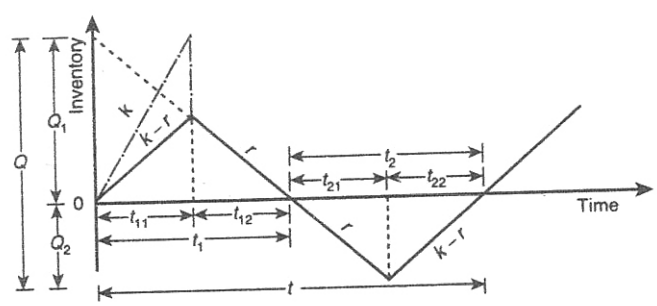

Fig. 8.4.1.

From Fig. 8.4.1 we see that at the end of t_{11}, the level of inventory is Q_1 and at the end of period t_{12} inventory becomes nil. Now shortages start and suppose that the shortages build up of quantity Q_3 up to time t_{21} and let then these shortages be filled up during time t_{22}.

Then obviously,

$$Q_1 = t_{11}(k - r),\ Q_1 = t_{12}r,$$

$$Q_2 = t_{21}r \text{ and } Q_2 = t_{22}\,(k - r)$$

Now if Q is the lot size, then

$$Q_1 = Q - Q_2 - rt_{11} - rt_{22}.$$

Eliminating t_{11} and t_{22} from this, we have

$$Q_1 = Q - Q_2 - r\left(\frac{Q_1}{k-r} + \frac{Q_2}{k-r}\right) \text{ or } Q_1 + Q_2 = (k-r)\,Q/k$$

Production cycle is

$$T = t_{11} + t_{12} + t_{21} + t_{22} = \frac{Q_1}{k-r} + \frac{Q_1}{r} + \frac{Q_2}{r} + \frac{Q_2}{k-r}$$

$$= k\,(Q + Q_2)/r(k-r)$$

Substituting the value of $Q_1 + Q_2$, we get $t = Q/r$.

The average inventory and amount of shortage during production cycle time t are : Average inventory = $\frac{1}{2}$ $Q_1\,(t_{11}, + t_{12})/t$, and Average shortage $\frac{1}{2}$ $Q_2\,(t_{21}\ t_{22}+)/t$.

∴ The total inventory cost is

$$TC = \frac{1}{2}\,Q_1C_1\,\frac{(t_{11}+t_{12})}{t} + \frac{1}{2}\,Q_2C_2\,\frac{(t_{21}+t_{22})}{t} + rC_s/Q$$

$$= \frac{1}{2Q}\,x\,\frac{k}{k-r}\left[C_1\left(\frac{k-r}{k}Q - Q^2\right)^2 + C_2{Q_2}^2\right] + \frac{r}{Q}Cs$$

since $Q_1 + Q_2 = \frac{r(k-r)}{k}t = \frac{r(k-r)}{k}\,x\,\frac{Q}{r} = \frac{k-r}{k}Q.$

Now, $\frac{\partial}{\partial Q_2}TC = 0 \Rightarrow Q_2 = \frac{C_1 Q}{C_1 + C_2}\,(1\text{-}r/k)$

and $\frac{\partial}{\partial Q}TC = 0 \Rightarrow Q = \sqrt{\frac{2Cs\,(C_1 + C_2)}{C_1C_2}}\sqrt{\frac{kr}{k-r}}$

Since $\frac{\partial 2TC}{\partial Q^2} > 0$, and $\frac{\partial^2 (TC)}{\partial {Q_2}^2} > 0$ for all values of Q and Q_2, the optimum values of Q and Q_2 so as to minimize TC are given by

$$Q^o = \sqrt{\frac{2Cs\,(C_1 + C_2)}{C_1C_2}}\sqrt{\frac{kr}{k-r}} \text{ and } {Q_2}^o = \sqrt{\frac{2CsC_1 r}{(C_1 + C_2)C_2}\frac{k-r}{k}}$$

8.4.1 Characteristics of this model

1. Production cycle time is

$$t^o = Q^o/r = \sqrt{\frac{2Cs\ (C_1 + C_2)}{C_1 C_2 r(1 - r/k)}}$$

2. Maximum inventory level is

$$Q_1^{\ o} = \frac{k-r}{k}\ Q^o - Q_2^{\ o} = \sqrt{\frac{2C_2 C_s r}{C_1(C_1 + C_2)}\left(1 - \frac{r}{k}\right)}$$

3. Total minimum production inventory cost is

$$TC^o = \frac{1}{2Q^o} x \frac{k}{k-r}(C_1 Q_1^{\ o2} + C_2 Q_2^{\ o2}) + \frac{r}{Q^o} Cs$$

$$= \sqrt{\frac{2C_1 C_2 C_s r}{C_1 + C_2}(1 - r/k)}$$

Remarks.

1. If $k = \infty$, the problem is in complete agreement with the problem discussed in case 1.
2. If $C_3 = \infty$, the problem reduces to that of case 3 of Lesson 7, Section 7.6.
3. If $k = \infty$ and $C_2 = \infty$, the problem reduces to that discussed in case 1 of Lesson 7 , Section 7.6

8.5 Worked out Examples

Example 8.5.1. A dealer supplies you the following information with regard to a product dealt in by him : Annual demand ; 10,000 units; ordering cost : Rs. 10 per order; price : Rs. 20 per unit. Inventory carrying cost : 20% of the value of inventory per year.

The dealer is considering the possibility of allowing some back-order (stock-out) to occur. He has estimated that the annual cost of back-ordering will be 25% of the value of inventory.

(i) What should be the optimum number of units of the product he should buy in one lot?

(ii) What quantity of the product should be allowed to be back-ordered, if any?

(iii) What would be the maximum quantity of inventory at any lime of the year?

(iv) Would you recommend to allow back-ordering? If so, what would be the annual cost saving by adopting the policy of back-ordering.

Solution. In the usual notations, we are given :

D = 10,000 units, C_s = Rs. 10 per order, C_1 = 20% of Rs. 20 = Rs, 4 per unit per year and C_2 = 25% of Rs. 20 = Rs. 5 per unit per year.

(i) (a) When stock-outs are not permitted :

$$Q^o = \sqrt{\frac{2DC_s}{C_1}} = \sqrt{\frac{2x10.000x10}{4}} = 223.6 \text{ units}$$

(b) When back-ordering is permitted ;

$$Q^o = \sqrt{\frac{2DC_s}{C_1} x \frac{C_1+C_2}{C_2}} = \sqrt{\frac{2x10.000x10}{4} x \frac{4+5}{5}} = 300 \text{ units}$$

(ii) Optimum quantity of the product to be back-ordered is given by :

$$Q_2^o = 300 \text{ x } \frac{4}{4+5} 133 \text{ units (approx.)}$$

(iii) Maximum inventory level =300- 133= 167 units.

(iv) Minimum total variable inventory cost in the cases (a) and (b) are ;

$$\text{TC}(223.6) = \sqrt{2x10.000x10x4} = \text{Rs. } 894.43$$

$$\text{TC }(300) = \sqrt{2x10.000x10x4x\frac{5}{4+5}} = \text{Rs. } 666.67$$

Since TC (223.6) > TC (666.67), the dealer should accept the proposal for back-ordering as this will result in a saving of (894.43-666.67) = Rs. 227.76 per year.

Example 8.5.2. A contractor undertakes to supply Diesel engines to a truck manufacturer at the rate of 25 per day. There is a clause in the contract penalizing him Rs. 10 per engine per day late for missing the scheduled delivery date. He finds that the cost of holding a complete engine in stock is Rs. 16 per month. His production process is such that each month he starts a batch of engines through the shops, and all these engines are available for delivery any time after the end of the month. What should his inventory level be at the beginning of each month?

Solution. We are given

C_1 = Rs. 16.00 per engine per month, C_2 = Rs. 10.00 per engine per day

r = 25 engines per day and t = one month (= 30 days)

The optimal value of Q_1 is,

$$Q_2^o = rt\frac{C_2}{C_1+C_2} = 25x30\frac{10}{10+16/30} = \frac{10x25x30x30}{316} = 712 \text{ engines}$$

Example 8.5.3. The demand for an item in a company is 18,000 units per year, and the company can produce the items at a rate of 3,000 per month. The cost of one set-up is Rs. 500.00 and the holding cost of I unit per month is 15 poise. The shortage cost of one unit is Rs. 20.00 per month. Determine (i) Optimum production batch quantity and the number of strategies, (ii) Optimum cycle time and production time, (iii) Maximum inventory level in the cycle, and (iv) Total associated cost per year if the cost of the item is Rs. 20 per unit.

Solution. Here,

C_1 = Re. 0.15 per month, C_2 = Rs. 20.00, C_s = Rs. 500.00, k = 3,000 units per month,

r = 18,000 units per year or 1,500 units per month.

∴ (i) Optimum production batch quantity is given by

$$\therefore \qquad Q^o = \sqrt{\frac{2Cs\ (C_1+C_2)}{C_1C_2}}\ \sqrt{\frac{kr}{k-r}}$$

$$= \sqrt{\frac{2x500(0.15+20)}{0.15x20}}\ \sqrt{\frac{1{,}500x3{,}000}{3000-1{,}500}}$$

= 4,489 units approx.

Number of shortages is given by

$$Q_2^o = \frac{C_1}{C_1 + C_2} Q^o \left(1 - \frac{r}{k}\right) = \frac{0.15}{0.15 + 20} x\, 4.889 \left[1 - \frac{1{,}500}{3{,}000}\right]$$

= 18 units approx.

(ii) Optimum production time $= \frac{Q^o}{k} = \frac{4{,}489}{3{,}000} = 15$ months.

Optimum cycle time between set-ups = $\frac{Q^o}{r} = \frac{4{,}489}{1{,}500} = 3$ months.

(iii) Maximum inventory level is

$$Q_1^o = \frac{k - r}{k} Q^o - Q_2^o = \frac{1{,}500}{3{,}000} \text{ x } 4{,}489 - 17 = 2.227 \text{ units approx.}$$

(iv) Total associated cost is given by

$$TC^o = \sqrt{\frac{2C_1 C_2 C_s^{\,r}}{C_1 + C_2} \frac{k - r}{k}} = \sqrt{\frac{0.15\, x\, 20\, x\, 50\, x\, 1{,}500}{0.15 + 20}}$$

= Rs. 106.

8.6 Summary

In this lesson Inventory models (Model - I, II & III) with shortages are discussed and computation of EOQ is also dealt with. Worked out examples support the theory for the better understanding of the concept.

8.7. Exercise

1. The demand for a purchased item is 1,000 units/month, and shortages are allowed. If the unit cost is Rs. 1.50 per unit, the cost of making one purchase is Rs. 600, the holding cost for one unit is Rs. 2 per year, and the cost of one shortage is Rs. 10 per year, determine :

   (i) The optimum purchase quantity,

   (ii) The number of orders per year.

   (iii) The optimum total yearly cost.

2. A commodity is to be supplied at a constant rate of 200 units per day. Supplies of any amounts can be had at any required time, but each ordering costs Rs. 50.00, cost of holding the commodity in inventory is Rs. 2.00 per unit per day while the delay in the supply of the item induces a penalty of Rs. 10.00 per unit per delay of 1 day. (i) Find the optimal policy (q, t), where t is the re-order cycle period and q is the inventory level after re-order. (ii) What would be the best policy, if (he penalty cost becomes ∞?

3. The demand for product is 25 units per month and the items are withdrawn uniformly. The set-up cost each time a production is nm is Rs. 15. The inventory holding cost is Re. 0.30 per item per month :

   (i) Determine how often to make production run, if shortages are not allowed.

   (ii) Determine how often to make production run, if shortages cost Rs. 1.50 per item per month.

4. A manufacturer has to supply his customer with 24,000 units of his product every year. This demand is fixed and known. Since the unit is used by the customer in an assembly operation and the customer has no storage space for units the manufacturer must supply a day's requirement each day. If the manufacturer fails to supply the required units, the shortage cost is Rs. 2 per unit per month. The inventory carrying cost is Re. 1 per unit per month, and the set-up cost per run is Rs. 3,500. Determine the optimum run size (Q), the optimum level of inventory (S) at the beginning of any period, the optimum scheduling period, and the minimum total expected relevant yearly cost (TC).

5. A TV manufacturing company produces 2,000 TV sets in a year for which it needs an equal number of tubes of a certain type. Each tube costs Rs. 10 and the cost to hold a tube in stock for a year is Rs. 2.40. Besides, the cost of placing an order is Rs. 150, which is not related to its size. The shortage cost is known to be Rs. 1.60 per unit per annum. Determine EOQ, Optimum level of shortage, maximum inventory level, and total inventory cost.

6. A dealer supplies you the following information with regard to an item of inventory:

Annual demand	:	5,000 units
Ordering costs	:	Rs. 250 per order
Inventory-holding costs	:	30% of the value of inventory per year
Inventory stock-out costs	:	Rs. 10 per unit per year
Price	:	Rs. 100 per unit

Find out : (i) Economic ordering quality, (ii) What quantity should be allowed to be back-ordered? (iii) What will be the maximum inventory at any particular time of the year? And (iv) Cost savings, if any, through back ordering.

8.8 References

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Lesson – 9

Replacement Models

9.0 Objective :

- Replacement Problem
- Different environments of Replacement
- Group Replacement
- Associated Cost considerations

Structure

9.1 Introduction

The study of replacement is concerned with situations that arise when some items such as machines, men, electric-light bulbs, etc., need replacement due to their deteriorating efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. For example, a machine becomes more and more expensive to maintain after a number of years, a railway time-table gradually becomes more and more out of date, an electric-light bulb fails all of a sudden, pipeline is blocked, or an employee loses his job, and so like. In all such situations, there is a need to formulate a most economic replacement policy for replacing faulty units or to take some remedial special action to restore the efficiency of deteriorating units.

Following are the situations when the replacement of certain items needs to be done :

(i) An old item has failed and does not work at all, or the old item is expected to fail shortly.

(ii) The old item has deteriorated and works badly or requires expensive maintenance,

(iii) A better design of equipment has been developed.

Replacement problems can be broadly classified into the following two categories :

(a) When the equipment / assets deteriorate gradually with time and the value of money

  (i) Does not change with time,

  (ii) changes with time.

(b) When the items/units fail completely all of a sudden.

9.2. Replacement of Equipment / Asset that Deteriorates Gradually

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

At this point, a replacement is justified.

9.2.1. Replacement Policy when Value of Money does not change with time

The aim here is to determine the optimum replacement age of an equipment/item whose running/maintenance cost increases with time and the value of money remains static during that period. Let

C	:	capital cost of equipment,
S	:	scrap value of equipment,
n	:	number of years that equipment would be in use,
f(t)	:	maintenance cost functions, and
A (n)	:	Average total annual cost.

Case 1. When t is a continuous variable. If the equipment is used for 'n' years, then the total cost incurred during this period is given by

TC = Capital cost — Scrap value + Maintenance cost

$$= C - S + \int_n^0 f(t)\,dt.$$

Average annual total cost, therefore is

$$A(n) = \frac{1}{n}TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt.$$

For minimum cost, we must have $\frac{d}{dn}$ [A (n)] = 0

or
$$\frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_0^n f(t)\,dt + \frac{1}{n}f(n) = 0$$

or
$$f(n) = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt \equiv A(n).$$

Clearly,

$$\frac{d^2}{dn^2}[A(n)] > 0 \text{ at } f(n) = A(n)$$

This suggests that the equipment should be replaced when maintenance cost equals the average annual total cost.

Case 2. When t is a discrete variable. Here, the period of time is considered as fixed and n, t take the values 1, 2, 3, ... Then

$$A(n) = \frac{C-S}{n} + \frac{1}{n}\sum_{t=1}^{n} f(t)$$

Now, A (n) will be a minimum for that value of n, for which

$$A(n+1) \geq A(n) \quad \text{and} \quad A(n-1) \geq A(n)$$

or

$$A(n+1) - A(n) \geq 0 \quad \text{and} \quad A(n) - A(n-1) \leq 0$$

For this, we write

$$A(n+1) = \frac{C-S}{n} + \frac{1}{n+1}\sum_{t=1}^{n+1} f(t)$$

$$= \frac{1}{n+1}\left[C - S + \sum_{t=1}^{n} f(t) \right] + \frac{1}{n+1} f(n+1)$$

$$= \frac{1}{n+1}[n\,A(n) + f(n+1)]$$

$$\therefore \quad A(n+1) - A(n) = \frac{1}{n+1}[f(n+1) - A(n)]$$

Thus $A(n+1) - A(n) \geq 0 \Rightarrow f(n+1) \geq A(n)$.

Similarly, it can be shown that

$A(n) - A(n-1) \leq 0 \Rightarrow f(n) \leq A(n-1)$

This suggests the optimal replacement policy :

Replace the equipment at the end of n years, if the maintenance cost in the (n +l)th year is more than the average total cost in the nth year and the nth year's maintenance cost is less than the previous year's average total cost.

9.3 Worked out Examples :

Example 9.3.1. A firm, is considering replacement of a machine, whose cost price is Rs. 12,200 and the scrap value. Rs. 200. The running (maintenance and operating) costs in rupees are found from experience to be as follows :

Year	:	1	2	3	4	5	6	7	8
Running cost	:	200	500	800	1,200	1,800	2,500	3,200	4,000

When should the machine be replaced?

Solution. We are given the running cost, f(n), the scrap value S = Rs. 200 and the cost of the machine, C=Rs. 12,200. In order to determine the optimal time n when the machine should be replaced, we calculate an average total cost per year during the life of the machine as shown in table given below :

Year of Service n (1)	Running cost (Rs.) f(n) (2)	Cumulative running cost (Rs.) $\sum f(n)$ (3)	Depreciation cost (Rs.) C – S (4)	Total cost (Rs.) TC (3) + (4) (5)	Average cost (Rs.) A(n) (5) / (1) (6)
1	200	200	12,000	12,200	12,200
2	500	700	12,000	12,700	6,350
3	800	1,500	12,000	13,500	4,500
4	1,200	2,700	12,000	14,700	3,675
5	1,800	4,500	12,000	16,500	3,300
6	2,500	7,000	12,000	19,000	**3,167**
7	3,200	10,200	12,000	22,200	3,171
8	4,000	14,200	12,000	26,200	3,275

From the table it is noted that the average total cost per year, A (n) is minimum in the 6th year (Rs. 3,167). Also the average cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence the machine should be replaced after every 6 years.

Example 9.3.2

(a) Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?

(b) Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one-year old. Should you replace it with B, if so when?

Solution. (a) Let the machine have no resale value when replaced- Then, for machine A, the average total annual cost ATC (n) is computed as follows :

Year(n)	f(n)	∑ f(n)	C - S	TC	A (n)
1	200	200	9,000	9,200	9,200
2	2,200	2,400	9,000	11,400	5,700
3	4,200	6,600	9,000	15,600	**5,200**
4	6,200	12,800	9,000	21,800	5,450
5	8.200	21,000	9,000	30,000	6,000

This table shows that the best age for the replacement of machine A is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(b) For machine B, the average cost per year can similarly be computed as given in the following table :

Year(n)	f(n)	∑ f(n)	C - S	TC	A (n)

1	400	400	10,000	10,400	10,400
2	1,200	1,600	10,000	11,600	5,800
3	2,000	3,600	10,000	13,600	4,533
4	2,800	6,400	10,000	16,400	4,100
5	3,600	10,000	10,000	20,000	**4,000**
6	4,400	14,400	10,000	24,400	4,066

Since the minimum average cost for machine B is lower than that for machine A, machine should be replaced by machine A.

To decide the time of replacement, we should compare the minimum average cost for B (Rs, 4,000) with yearly cost of maintaining and using the machine A. Since there is no salvage value of the machine, we shall consider only the maintenance cost. We would keep the machine A so long as the yearly maintenance cost is lower than Rs. 4,000 and replace when it exceeds Rs. 4,000.

On the one-year old machine A, Rs. 2,200 would be required to be spent in the next year; while Rs. 4,200 would be needed in year following. Thus, we should keep machine A for one year and replace it thereafter.

Example 9.3.3. The data collected in running a machine, the cost of which is Rs. 60,000, are given below

Year		1	2	3	4	5
Resale value (Rs.)	:	42,000	30.000	20,400	14,400	9,650
Cost of spares (Rs.)	:	4,000	4,270	4,880	5,700	6,800
Cost of labour (Rs.)	:	14,000	16,000	18,000	21,000	25,000

Determine the optimum period for replacement of the machine.

Solution. The operating or maintenance cost of machine in successive years is as follows:

Year	:	1	2	3	4	5
Operating cost (Rs.)	:	18,000	20,270	22,880	26,700	31,800

(The cost of spares and labour together determine operating or running or maintenance cost.) The average total annual cost is computed below :

Year of Service n	Operating cost (Rs.) f(n)	Cum. operating cost (Rs.) $\sum f(n)$	Resale value (Rs.) S	Depreciation cost (Rs.) C – S	Total cost (Rs.) TC	Average cost (Rs.) A(n)
1	18,000	18,000	42,000	18,000	36,000	36,000.00
2	20,270	38,270	30,000	30,000	68,270	34,135.00
3	22,880	61,150	20,400	39,600	1,00,750	33,583.30
4	26,700	87,850	14,400	45,600	1,33,450	**33,362.50**
5	31,800	1,19,650	9,650	50,350	1,70,000	34,000.00

The calculations in the above table show that the average cost is lowest during the fourth year. Hence the machine should be replaced after every fourth year.

9.4. Replacement Policy when Value of Money changes with time

When the time value of money is taken into consideration, we shall assume that (i) the equipment in question has no salvage value, and (ii) the maintenance costs are incurred in the beginning of the different time periods.

Since it is assumed that the maintenance cost increases with time and each cost is to be paid just in the start of the period, let the money carry a rate of interest r per year. Thus a rupee invested now will be worth $(1 + r)$ after a year, $(1 + r)^2$ after two years, and so on. In this way a rupee invested today will be worth $(1 + r)^n$, n years hence, or, in other words, if we have to make a payment of one rupee in n years time, it is equivalent to making a payment of $(1 + r)^{-n}$ rupees today. The quantity $(1 + r)^{-n}$ is called the present worth factor (Pwf) of one rupee spent in n years time from now onwards. The expression $(1 + r)^n$ is known as the payment compound amount factor (Caf). of one rupee spent in n years time.

Let the initial cost of the equipment be C and let R_n be the operating cost in year n. Let v be the rate of interest in such a way that $v = (1 + r)^{-1}$ is the discount rate (present worth factor). Then the present value of all future discounted costs V_n associated with a policy of replacing the equipment at the end of each n years is given by

$$V_n = \{(C + R_o) + vR_1 + v^2R_2 + \dots v^{n-1}R_{n-1}\} + \{(C + R_o)v_n + v^{n+1}R_1 + v^{n+2}R_2 + \dots v^{2n-1}R_{n-1}\} + \dots$$

$$= \left[C + \sum_{k=0}^{n-1} v^k R_k \right] x \sum_{k=0}^{\infty} (v^n)^k = \left[C + \sum_{k=0}^{n-1} v^k R_k \right] (1 - v^n)^{-1}$$

Now, V_n will be a minimum for that value of n, for which

$$V_{n+1} - V_n > 0 \text{ and } V_{n-1} - V_n > 0$$

For this, we write

$$V_{n+1} - V_n = \left[C + \sum_{k=0}^{n} v^k R_k \right] (1-v^{n+1})^{-1} - V_n$$

$$= v^n [R_n - (1 - v) V_n] / (1 - v^{n+1})$$

and similarly

$$V_n - V_{n-1} = V^{n-1} [R_{n-1} - (1 - v) V_n] / (1 - v^{m-1})$$

Since v is the depreciation value of money, it will always be less than 1 and therefore 1- v will always be positive. This implies that $v^n/(1 - v^{n+1})$ will always be positive.

Hence, $V_{n+1} - V_n > 0 \Rightarrow R_n > (1 - v) V_n$ and $V_n - V_{n-1} < 0 \Rightarrow R_{n-1} < (1 - v) V_n$

Thus, $R_{n-1} < (1 - v) V_n < R_n$

Or $$R_{n-1} < \frac{C + R_0 + vR_1 + v^2 R_2 + \dots + v^{n-1} R_{n-1}}{1 + v + v^2 + \dots + v^{n-1}} < R_n,$$

Since $(1 - v^n)(1 - v)^{-1} = \sum_{k=0}^{n-1} v^k$.

The expression which lies between R_{n-1} and R_n is called the "weighted average cost" of all the previous n years with weights 1, v, v^2, ..., v^{n-1} respectively.

Hence, the optimal replacement policy of the equipment after n periods is :

(a) Do not replace the equipments if the next period's operating cost is less than the weighted average of previous costs.

(b) Replace the equipments if the next period's operating cost is greater than the weighted average of previous costs.

Remark. Procedure for determining the weighted average of costs (annualized cost) may be summarized in the following steps :

Step 1. Find the present value of the maintenance cost for each of the years, i.e., $\sum Rv^{n-1}$ (n = 1, 2, ...); where $v = (1 + r)^{-1}$

Step 2. Calculate cost plus the accumulated present values obtained in step 1. i.e., $C + \sum Rv^{n-1}$.

Step 3. Find the cumulative present value factor up to each of the years 1, 2, 3, ... i.e., $\sum v^{n-1}$.

Step 4. Determine the annualized cost W (n), by dividing the entries obtained in step 2 by the corresponding entries obtained in step 3. i.e., $[C + \sum v^{n-1}] / \sum v^{n-1}$.

Corollary. When the time value of money is not taken into consideration, the rate of interest becomes zero and hence v approaches unity. Therefore, as $v \rightarrow 1$, we get

$$R_{n-1} < \frac{C + R_0 + R_1 + ... + R_{n-1}}{1 + 1 + ... + n\, times} < R_n$$

or $$R_{n-1} < W(n) < R_n$$

Note. It may be noted that the above result is in complete agreement with the result that was obtained in 18 : 2.1.

9.5 : Selection of the Best Equipment Amongst Two

Following is the procedure for determining a policy for the selection of an economically best item amongst the available equipments :

Step 1. Considering the case of two equipments, say A and B, we first find the best replacement age for both the equipments by making use of

$$R_{n-1} < (1 - v) V_n < R_n.$$

Let the optimum replacement age for A and B comes out to be n_l and n_2 respectively.

Step 2. Next, compute the fixed annual payment (or weighted average cost) for each equipment by using the formula

$$W(n) = \frac{C + R_0 + vR_1 + v^2 R_2 + \ldots + v^{n-1} R_{n-1}}{1 + v + \ldots + v^{n-1}}$$

and substitute n = n_l for equipment A and n = n_2 for equipment B in it.

Step 3. (i) If $w(n_1) < W(n_2)$, choose equipment A.

(ii) If $W(n_1) > W(n_2)$, choose equipment B.

(iii) If $W(n_1) = W(n_2)$, both equipments are equally good.

9.6. Worked out Examples

Example 9.6.1 Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given below :

Year	:	1	2	3	4	5	6
Machine A	:	1,000	200	400	1,000	200	400
Machine B	:	1,700	100	200	300	400	500

Determine which machine should be purchased.

Solution. Since the money carries the rate of interest, the present worth of the money to be spent over in a period of one year is

$$v = \frac{100}{100 + 10} = \frac{10}{11} = 0.9091$$

$\therefore$ The total discounted cost (present worth) of A for 3 years is

$1000 + 200 \text{ x } (0.9091) + 400 \text{ x } (0.9091)^2 = \text{Rs. } 1512$ approx.

Again, the total discounted cost of B for six years is

$1{,}700 + 100 \text{ x } (0.9091) + 200 \text{ x } (0.9091)^2 + 300 \text{ x } (0.9091)^3 + 400 \text{ x } (0.9091)^4$

$+ 500 \text{ x } (0.9091)^5 = \text{Rs. } 2{,}765.$

Average yearly cost of machine A = Rs. 1,512/3 = Rs. 504.

Average yearly cost of machine B = Rs. 2,765/6 = Rs. 461.

This shows that the apparent advantage is with machine B. But, the comparison is unfair since the periods for which the costs are considered are different. So, if we consider 6 years period for machine A also, then the total discounted cost of A will be

$$1,000 + 200 \times (0.9091) + 400 \times (0.9091)^2 + 1,000 \times (0.9091)^3 + 200 \times (0.9091)^4 + 400 \times (0.9091)^5.$$

After simplification this comes out to be Rs. 2,647 which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be purchased.

Example 9.6.2. The cost of a new machine is Rs. 5,000. The maintenance cost of nth year is given by $C_n = 500\,(n-1)$; n = 1, 2, … Suppose that the discount rate per year is 0.5. After how many years it will be economical to replace the machine by a new one ?

Solution. Since the discount rate of money per year is 0.05, the present worth of the money to be spent over a period of one year is

$$v + (1 + 0.05)^{-1} = 0.9523.$$

The optimum replacement time is determined in the following table :

Year (n) (1)	R_{n-1} (2)	v^{n-1} (3)	$R_{n-1}v^{n-1}$ (4)	$C+\sum_k R_{k-1}v^{k-1}$ (5)	$\sum_k v^{k-1}$ (6)	W(n) (7)
1	0	1.0000	0	5,000	1.0000	5,000
2	500	0.9523	476	6,476	1.9523	2,805
3	1,000	0.9070	907	6,383	2.8593	2,232
4	1,500	0.8638	1,296	7,679	3.7231	2,063
5	2,000	0.8227	1,645	9,324	4.5458	**2,051***
6	2,500	0.7835	1,959	11,283	5.3293	2,117

Since, W (n) is minimum for n = 5 and R_4 = 1,500 < W (5) as well as W (5) > R_6 = 2,500 ; it is economical to replace the machine by a new one at the end of five years.

9. 7. Replacement of Equipment that Fails Suddenly

It is difficult to predict that a particular equipment will fail at a particular time. This difficulty can be overcome by determining the probability distribution of failures. Here it is assumed that the failures occur only at the end of the period, say t. Thus the objective becomes to find the value of t which minimizes the total cost involved for the replacement.

We shall consider the following two types of replacement policies :

Individual Replacement Policy. Under this policy, an item is replaced immediately after its failure.

9.7.1. Group Replacement Policy. Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with a provision that if any item fails before the optimal time, it may be individually replaced.

Mortality Tables : These are used to derive the probability distribution of the life span of an equipment. Let

M (t) = number of survivors at any time t,

M (t-1) = number of survivors at any time t-1, and

N = initial number of equipments

Then the probability of failure during time period t is given by

$$P(t) = [M(t-1) - M(t)]/N$$

The probability that an equipment survived till age (t -1), will fail during the interval (t - 1) to t can be defined as the conditional probability of failure. It is given by

$$P_c(t) = [M(t-1) - M(t)] / M(t-1)$$

The probability of survival till age t is given by

$$P_s(t) = M(t) / N.$$

Theorem 9.7.1. (Group Replacement). Let all the items in a system be replaced after a time interval 't' with provisions that individual replacements can be made if and when any item fails during this time period. Then

(a) Group replacement must be made at the end of t^{th} period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of t periods.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacements at the end of period t - 1 is less than the average cost per unit period through the end of period t.

Proof. Let,

N = total number of items in the system,

C_2 = cost of replacing an individual item,

C_1 = cost of replacing an item in group,

$C(t)$ = total cost of group replacement after time period t,

$f(t)$ = number of failures during time period t.

Then, clearly

$$C(t) = NC_1 + C_2 \sum_{x=0}^{t-1} f(x)$$

The average cost of group replacement per unit period of time during a period t, is thus given by

$$A(t) = \frac{C(t)}{t} = \left[NC_1 + C_2 \sum_{x=0}^{t-1} f(x) \right] / t.$$

We shall determine the optimum t so as to minimize C (t) / t.

Note that whenever $\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$ and $\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$, it is better to replace all the items after time period t.

Now, $\frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t) / t;$

And $\frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t) / t$

$\therefore \quad tC_2 f(t - 1) < C(t) < t.\ C_2 f(t).$

or $\quad t\, f(t-1) - \sum_{x=0}^{t-1} f(x) < \frac{NC_1}{C_2} < t\, f(t) - \sum_{x=0}^{t-1} f(x).$

9.8. Worked out Examples

Example 9.8.1. The following failure rates have been observed for a certain type of transistors in a digital computer :

End of the week	:	1	2	3	4	5	6	7	8
Probability of failure to date	:	.05	.13	.25	.43	.68	.88	.96	1.00

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is made to replace all these transistors simultaneously at fixed intervals, and to replace the individual transistors as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? At what group replacement price per transistor would a policy of strictly individual replacement become preferable to the adopted policy?

Solution. Suppose there are 1,000 transistors in use. Let p_i be the probability that a transistor, which was new when placed in position for use, fails during the ith week of its life. Thus, we have

$p_1 \equiv 0.05,$ $\quad p_2 \equiv 0.13 - 0.5 = 0.08,$

$p_3 \equiv 0.25 - 0.13 = 0.12,$ $\quad p_4 \equiv 0.43 - 0.25 = 0,18,$

$p_5 \equiv 0.68 - 0.43 = 0.25,$ $\quad p_6 \equiv 0.88 - 0.68 = 0.20,$

$p_7 \equiv 0.96 - 0.88 = 0.08,$ $\quad p_8 \equiv 1.00 - 0.96 = 0.04.$

Let N_i denote the number of replacements made at the end of the i^{th} week. Then, we have

N_0 = number of transistors in the beginning = 1,000

$N_1 = N_0p_1 = 1{,}000 \times 0.05 = 50$

$N_2 = N_0p_2 + N_1p_1 = 1{,}000 \times 0.08 + 50 \times 0.05 = 82$

$N_3 = N_0p_3 + N_1p_2 + N_2p_1 = 1{,}000 \times 0.12 + 50 \times 0.08 + 82 \times 0.05 = 128$

$N_4 = N_0p_4 + N_1p_3 + N_2p_2 + N_3p_1 = 199$

$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1 = 289$

$N_6 = N_0p_6 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1 = 272$

$N_7 = N_0p_7 + N_1p_6 + N_2p_5 + N_3p_4 + N_4p_3 + N_5p_2 + N_6p_1 = 194$

$N_8 = N_0p_8 + N_1p_7 + N_2p_6 + N_3p_5 + N_4p_4 + N_5p_3 + N_6p_2 + N_7p_1 = 195$

From the above calculations, we observe that the expected number of transistors failing each week increases till 5th week and then starts decreasing and later again increasing from 8th week.

Thus, N_i will oscillate till the system acquires a steady state. The expected life of each transistor is

$$1 \times 0.5 + 2 \times .08 + 3 \times .12 + 4 \times .18 + 5 \times .25 + 6 \times .2 + 7 \times .08 + 8 \times .04$$

$$= 4.62 \text{ weeks.}$$

Average number of failures per week

$$= 1{,}000/4.62 = 216 \text{ approximately.}$$

Therefore, the cost of individual replacement

$$= 216 \times 1.25 = \text{Rs. } 270.00 \text{ per week.}$$

Now, since the replacement of all the 1,000 transistors simultaneously cost 30 paise per transistors and the replacement of an individual transistor on failure cost Rs. 1.25, the average cost for different group replacement policies is given as under :

End of Week	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
1	50	50 x 1.25 + 1,000 x 0.30 = 363	363
2	132	132 x 1.25 + 1,000 x 0.30 = 465	232.50
3	260	260 x 1.25 + 1,000 x 0.30 = 625	208.30
4	459	459 x 1 .25 + 1,000 x 0.30 = 874	218.50

Since the average cost is lowest against week 3, the optimum interval between group replacements is 3 weeks. Further, since the average cost is less than Rs. 270 (for individual replacement), the policy of group replacement is better.

Example 9.8.2. At time zero all items in a system are new. Each item has a probability p of failing immediately before the end of the first month of life, and a probability q = 1 - p of failing immediately before the end of the second month (i.e., all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures f(x) at the end of month x is given by

$$f(x) = \frac{N}{1+q}\,[1-(-q)^{x+1}]$$

where N is the number of items in the system.

If the cost per item of individual replacement is C_1, and the cost per item of group replacement is C_2, find the condition under which

(a) A group replacement policy at the end of each month is the most profitable.

(b) No group replacement policy is better than a policy of pure individual replacement.

Solution. Let

N = number of items in the system in the beginning

N_1 = number of items expected to fail at the end of 1st month

= $N_o p = N(1 - q)$, since $p = 1 - q$.

N_2 = number of items expected to fail at the end of 2nd month

= $N_o q + N_1 p = Nq + N(1-q)^2 = N(1 - q + q^2)$,

N_3 = number of items expected to fail at the end of 3rd month

= $N_1 q + N_2 p = N(1 - q)q + N(1 - q + q^2)(1 - q) = N(1 - q + q^2 - q^3)$,

and so on. In general,

$$N_k = N[1 - q + q^2 - q^3 + ... + (-q)^k].$$

$$\therefore N_{k+1} = N_{k-1} q + N_k p$$

$$= N[1 - q + q^2 + ... + (-q)^{k-1}] q + N[1 - q + q^2 + ... + (-q)^k](1 - q)$$

$$= N[1 - q + q^2 + ... + (-q)^{k+1}]$$

Hence by mathematical induction, the expected number of failures at the end of month x will be given by

$$f(x) = N[1 - q + q^2 + \dots + (-q)^x] = N[1 - (-q)^{x+1}] / (1 + q).$$

The value of $f(x)$ at the end of month x will vary for different values of $(-q)^{x+1}$ and it will reach the steady-state as $x \to \infty$.

Hence, in the steady state case, the expected number of failures will be

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} N[1 - (-q)^{x+1}] / (1 + q)$$

$$= N/(1 + q)\text{ ; since } q < 1 \text{ and } (-q)^{x+1} \to 0 \text{ as } x \to \infty.$$

= Total number of items in the system / Mean age,

Now, since C_1 is the cost of replacement per item individually and C_2 is the cost of an item in group, therefore

(i) If we have a group replacement at the end of each month, then the cost of replacement is NC_2.

(ii) If we have a group replacement policy at the end of every other month, then the cost is $NC_2 + N_pC_1$.

The average cost per month, therefore, is $(NC_2 + N_pC_1)/2$, and

(iii) Average life of an item

$$= 1 \times p + 2 \times q = 1 \times (1 - q) + 2q = 1 + q.$$

Therefore, the average number of failures is $N/(l + q)$ and hence the cost of individual replacement is $NC_1 / (l + q)$.

(a) A group replacement at the end of first month will be better than individual replacement, if total cost of group replacement is less than the average monthly cost of individual replacement.

Thus, $N(1 - q)\, C_1 + NC_2 < NC_1/ (1 + q)$, i.e. $C_2 < c_1q^2 / (1 + q)$.

For a group replacement at the end of every second month, the total cost of replacement will be

$$(N_1 + N_2)\, C_1 + NC_2 = N\,(2 - 2q + q^2)\, C_1 + NC_2.$$

$\therefore$ Average monthly cost of group replacement at the end of second month is

$$[N\,(2 - 2q + q^2)\, C_1 + NC_2] / 2$$

in this case, the group replacement policy will be better than the individual replacement policy, if

Average monthly cost of group replacement < Average monthly cost of individual replacement

or $[N\,(1 - q + q^2/2\, C_1 + NC_2 / 2] < NC_1 / (1 + q)$,

or $C_2 < q^2\,(1 - q)\, C_1/(1 + q)$.

(b) For the individual replacement policy to be better than any of the group replacement policies discussed above, we must have

$$C_2 > C_1q^2 / (q + l) \quad \text{and} \quad C_2 > C_1q^2\,(1 - q) / (q + 1)$$

or $C_1 < C_2 (1 + q) / q^2$ and $C_1 < C_2 (1 + q) / [q^2 (1 - q)]$

But $q < 1$, therefore $(1 + q)/q^2 < (1 + q)/q^2(1 - q)$

Hence, $C_1 < (1 + q) C_2/q^2$.

9.9. Summary :

In this lesson various replacement models are discussed. In particular the equipment that deteriorates gradually and equipment that fails suddenly. Further, when equipment deteriorates suddenly, the two cases when value of money changes with time and does not change with time also discussed. For all these cases theoretical discussion is supported with examples.

9.10. Exercise :

1. The cost of a machine is Rs. 6,100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	100	250	400	600	900	1,200	1,600	2,000

When should the machine be replaced?

2. A firm is considering replacement of a machine whose cost price is Rs. 17,500 and the scrap value is Ps. 500. The maintenance costs (in Rs.) are found from experience to be as follows :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	200	300	3,500	1,200	1,800	2,400	3,300	4,500

When should the machine be replaced?

3. A firm is using a machine whose purchase price is Rs. 13,000. The installation charges amount to Rs. 3,600 and the machine has a scrap value of only Rs.1,600, because the firm has a monopoly of this type of work.

The maintenance cost in various years is given in the following table :

Year	:	1	2	3	4	5	6	7

Cost (Rs.)	:	250	750	1,000	1,500	2,100	2,900	4,000

The firm wants to determine after how many years should the machine be replaced on economic considerations, assuming that the machine replacement can be done only at the year ends.

4. Following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs. 5,000.

Year	:	1	2	3	4	5	6	7	8
Running cost (Rs.)	:	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,000
Resale value (Rs.)	:	3.500	2,500	1,700	1,200	800	500	500	500

At what year is the replacement due?

5. A truck owner finds from his past records that the maintenance costs per year, of a truck whose purchase price is Rs. 8,000 are as given below :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	1,000	1,300	1,700	2,200	2,900	3,800	4,800	6,000
Resale price (Rs.)	:	4,000	2,000	1,200	600	500	400	400	400

Determine at which time it is profitable to replace the truck.

6. Let v = 0.9 and initial price is Rs. 5,000. Running cost varies as follows :

Year	:	1	2	3	4	5	6	7
Running cost (in Rs.)	:	400	500	700	1,000	1,300	1,700	2,100

What would be the optimum replacement interval ?

7. The yearly cost of 2 machines A and B when the money value is neglected is as follows :

Year	:	1	2	3	4	5
Machine A	:	1,800	1,200	1,400	1,600	1,000
Machine B	:	2,800	200	1,400	1,100	600

Find their cost patterns if money value is 10% per year and hence find which machine is most economical.

8. An individual is planning to purchase a car. A new car will cost Rs. 1,20,000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation costs during the first year are Rs. 20,000 and they increase by 15% every year. The minimum resale value of car can be Rs. 40,000.

(i) When should the car be replaced to minimise average annual cost (ignore interest)?

(ii) If interest of 12% is assumed, when should the car be replaced ?

9. In a machine shop, a particular cutting tool costs Rs. 6 to replace. If a tool breaks on the job, the production disruption and associate costs amount to Rs. 30. The past life of a tool is given as follows :

Job No.	:	1	2	3	4	5	6	7
Proportion of broken tools on job	:	.01	.03	.09	.13	.25	.55	.95

After how many jobs should the shop replace a tool before it breaks down?

10. The following failure rates have been observed for a certain type of light bulbs :

Week	:	1	2	3	4	5
Per cent failing by end of week	:	10	25	50	80	100

There are 1,000 bulbs in use, and it costs Rs. 2 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously it would cost 50 paise per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what interval should all the bulbs be replaced?

11. The following failure rates have been observed for a certain type of light bulb :

End of week	:	1	2	3	4	5	6	7
Probability of failure to date	:	0.07	0.15	0.25	0.45	0.75	0.90	1 .00

There are 1,000 bulbs in use. The cost of replacing an individual failed bulb is Rs. 1.25. If all the bulbs are replaced as a group, it costs Rs. 0.60 per bulb. Determine among individual and group replacement policies which one is better.

12. A large computer installation contains 2,000 components of identical nature which are subject to failure as per probability distribution given below :

Week end	:	1	2	3	4	5
Percentage failure to date	:	10	25	50	80	100

Components which fail have to be replaced for efficient functioning of the system. If they are replaced as and when failure occur, the cost of replacement per unit is Rs. 3. Alternatively if all components are replaced in one lot at periodical intervals and individually replaced only as such failures occur between group replacement, the cost of component replaced is Re. 1.

(a) Assess which policy of replacement would be economical.

(b) If group replacement is economical at current costs, then assess at what cost of individual replacement would group replacement be uneconomical.

(c) How high can the cost per unit in group replacement be to make a preference for individual replacement policy?

13. Let p (t) be the probability that a machine in a group of 30 machines would break down in period t. The cost of repairing a broken machine is Rs. 200. Preventive maintenance is performed by servicing all the 30 machines at the end of T units of time. Preventive maintenance cost is Rs. 15 per machine. Find optimum T which will minimize the expected total cost per period of servicing, given that

$$p(t) = \begin{cases} 0.03 & \text{for } t = 1 \\ p(t-1) + 0.01 & \text{for } t = 2, 3, \ldots 10 \\ 0.13 & \text{for } t = 11, 12, 13, \ldots \end{cases}$$

14. India Electric Company operates three large banking ovens used to manufacture electric motors and generators. Each oven contains 2,400 heating resistors. It costs Rs. 80 to replace an individual resistor when it burns out. If all the resistors are replaced (on Sunday night), it costs Rs. 72,000. When a resistor fails, it must be replaced immediately. When group replacement is followed, all resistors are replaced at one time regardless of age. Statistical data provided by the research department of India Electric Company provide the following probability distribution for resistor failure :

Week	:	1	2	3	4	5	6
Probability of failure	:	.02	.08	.15	.30	.25	.20

Is individual or group replacement more desirable?

9.11. References:

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.

2. Operations Research – Kanthi Swarup and others, Sultan Chand & Sons.

3. Operations Research – S.D. Sarma, Kedar Nath Ram Nath and Co. Publishers, Meerut.

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Lesson – 9

Replacement Models

9.0 Objective :

- Replacement Problem

Structure

9.1 Introduction

The study of replacement is concerned with situations that arise when some items such as machines, men, electric-light bulbs, etc., need replacement due to their deteriorating efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. For example, a machine becomes more and more expensive to maintain after a number of years, a railway time-table gradually becomes more and more out of date, an electric-light bulb fails all of a sudden, pipeline is blocked, or an employee loses his job, and so like. In all such situations, there is a need to formulate a most economic replacement policy for replacing faulty units or to take some remedial special action to restore the efficiency of deteriorating units.

Following are the situations when the replacement of certain items needs to be done :

(i) An old item has failed and does not work at all, or the old item is expected to fail shortly.

(ii) The old item has deteriorated and works badly or requires expensive maintenance,
(iii) A better design of equipment has been developed.

Replacement problems can be broadly classified into the following two categories :

(a) When the equipment / assets deteriorate gradually with time and the value of money

 (i) Does not change with time,

 (ii) changes with time.

(b) When the items/units fail completely all of a sudden.

9.2. Replacement of Equipment / Asset that Deteriorates Gradually

Generally, the cost of maintenance and repair of certain items (equipments) increases with time and a stage may come when these costs become so high that it is more economical to replace the item by a new one.

At this point, a replacement is justified.

9.2.1. Replacement Policy when Value of Money does not change with time

The aim here is to determine the optimum replacement age of an equipment/item whose running/maintenance cost increases with time and the value of money remains static during that period. Let

C	:	capital cost of equipment,
S	:	scrap value of equipment,
n	:	number of years that equipment would be in use,
f(t)	:	maintenance cost functions, and
A (n)	:	Average total annual cost.

Case 1. When t is a continuous variable. If the equipment is used for 'n' years, then the total cost incurred during this period is given by

TC = Capital cost — Scrap value + Maintenance cost

$$= C - S + \int_{n}^{0} f(t)\, dt.$$

Average annual total cost, therefore is

$$A(n) = \frac{1}{n} TC = \frac{C-S}{n} + \frac{1}{n}\int_{0}^{n} f(t)\, dt.$$

For minimum cost, we must have $\frac{d}{dn}$ [A (n)] = 0

or
$$\frac{-(C-S)}{n^2} - \frac{1}{n^2}\int_{0}^{n} f(t)\, dt + \frac{1}{n} f(n) = 0$$

or
$$f(n) = \frac{C-S}{n} + \frac{1}{n}\int_{0}^{n} f(t)\, dt \equiv A(n).$$

Clearly,

$$\frac{d^2}{dn^2} [A(n)] > 0 \text{ at } f(n) = A(n)$$

This suggests that the equipment should be replaced when maintenance cost equals the average annual total cost.

Case 2. When t is a discrete variable. Here, the period of time is considered as fixed and n, t take the values 1, 2, 3, ... Then

$$A(n) = \frac{C-S}{n} + \frac{1}{n}\sum_{t=1}^{n} f(t)$$

Now, A (n) will be a minimum for that value of n, for which

$$A(n+1) \geq A(n) \quad \text{and} \quad A(n-1) \geq A(n)$$

or $$A(n+1) - A(n) \geq 0 \quad \text{and} \quad A(n) - A(n-1) \leq 0$$

For this, we write

$$A(n+1) = \frac{C-S}{n} + \frac{1}{n+1}\sum_{t=1}^{n+1} f(t)$$

$$= \frac{1}{n+1}\left[C - S + \sum_{t=1}^{n} f(t) \right] + \frac{1}{n+1} f(n+1)$$

$$= \frac{1}{n+1}[n A(n) + f(n+1)]$$

$$\therefore \quad A(n+1) - A(n) = \frac{1}{n+1}[f(n+1) - A(n)]$$

Thus $A(n+1) - A(n) \geq 0 \Rightarrow f(n+1) \geq A(n)$.

Similarly, it can be shown that

$A(n) - A(n-1) \leq 0 \Rightarrow f(n) \leq A(n-1)$

This suggests the optimal replacement policy :

Replace the equipment at the end of n years, if the maintenance cost in the (n +l)th year is more than the average total cost in the nth year and the nth year's maintenance cost is less than the previous year's average total cost.

Worked out Examples :

Example 9.2.1. A firm, is considering replacement of a machine, whose cost price is Rs. 12,200 and the scrap value. Rs. 200. The running (maintenance and operating) costs in rupees are found from experience to be as follows :

Year	:	1	2	3	4	5	6	7	8
Running cost	:	200	500	800	1,200	1,800	2,500	3,200	4,000

When should the machine be replaced?

Solution. We are given the running cost, f(n), the scrap value S = Rs. 200 and the cost of the machine, C=Rs. 12,200. In order to determine the optimal time n when the machine should be replaced, we calculate an average total cost per year during the life of the machine as shown in table given below :

Year of Service n (1)	Running cost (Rs.) f(n) (2)	Cumulative running cost (Rs.) $\sum f(n)$ (3)	Depreciation cost (Rs.) C – S (4)	Total cost (Rs.) TC (3) + (4) (5)	Average cost (Rs.) A(n) (5) / (1) (6)
1	200	200	12,000	12,200	12,200
2	500	700	12,000	12,700	6,350
3	800	1,500	12,000	13,500	4,500
4	1,200	2,700	12,000	14,700	3,675
5	1,800	4,500	12,000	16,500	3,300
6	2,500	7,000	12,000	19,000	3,167
7	3,200	10,200	12,000	22,200	3,171
8	4,000	14,200	12,000	26,200	3,275

From the table it is noted that the average total cost per year, A (n) is minimum in the 6th year (Rs. 3,167). Also the average cost in 7th year (Rs. 3,171) is more than the cost in the 6th year. Hence the machine should be replaced after every 6 years.

Example 9.2.2

(a) Machine A costs Rs. 9,000. Annual operating costs are Rs. 200 for the first year, and then increase by Rs. 2,000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy is followed, what will be the average yearly cost of owning and operating the machine?

(b) Machine B costs Rs. 10,000. Annual operating costs are Rs. 400 for the first year, and then increase by Rs. 800 every year. You now have a machine of type A which is one-year old. Should you replace it with B, if so when?

Solution. (a) Let the machine have no resale value when replaced- Then, for machine A, the average total annual cost ATC (n) is computed as follows :

Year(n)	f(n)	$\sum$ f(n)	C - S	TC	A (n)
1	200	200	9,000	9,200	9,200
2	2,200	2,400	9,000	11,400	5,700
3	4,200	6,600	9,000	15,600	5,200
4	6,200	12,800	9,000	21,800	5,450
5	8.200	21,000	9,000	30,000	6,000

This table shows that the best age for the replacement of machine A is 3rd year. The average yearly cost of owning and operating for this period is Rs. 5,200.

(b) For machine B, the average cost per year can similarly be computed as given in the following table :

Year(n)	f(n)	∑ f(n)	C - S	TC	A (n)
1	400	400	10,000	10,400	10,400
2	1,200	1,600	10,000	11,600	5,800
3	2,000	3,600	10,000	13,600	4,533
4	2,800	6,400	10,000	16,400	4,100
5	3,600	10,000	10,000	20,000	4,000
6	4,400	14,400	10,000	24,400	4,066

Since the minimum average cost for machine B is lower than that for machine A, machine should be replaced by machine A.

To decide the time of replacement, we should compare the minimum average cost for B (Rs, 4,000) with yearly cost of maintaining and using the machine A. Since there is no salvage value of the machine, we shall consider only the maintenance cost. We would keep the machine A so long as the yearly maintenance cost is lower than Rs. 4,000 and replace when it exceeds Rs. 4,000.

On the one-year old machine A, Rs. 2,200 would be required to be spent in the next year; while Rs. 4,200 would be needed in year following. Thus, we should keep machine A for one year and replace it thereafter.

9.2.3. The data collected in running a machine, the cost of which is Rs. 60,000, are given below

Year		1	2	3	4	5
Resale value (Rs.)	:	42,000	30.000	20,400	14,400	9,650
Cost of spares (Rs.)	:	4,000	4,270	4,880	5,700	6,800
Cost of labour (Rs.)	:	14,000	16,000	18,000	21,000	25,000

Determine the optimum period for replacement of the machine.

Solution. The operating or maintenance cost of machine in successive years is as follows:

Year	:	1	2	3	4	5
Operating cost (Rs.)	:	18,000	20,270	22,880	26,700	31,800

(The cost of spares and labour together determine operating or running or maintenance cost.) The average total annual cost is computed below :

Year of Service n	Operating cost (Rs.) f(n)	Cum. operating cost (Rs.) $\sum f(n)$	Resale value (Rs.) S	Depreciation cost (Rs.) C – S	Total cost (Rs.) TC	Average cost (Rs.) A(n)
1	18,000	18,000	42,000	18,000	36,000	36,000.00
2	20,270	38,270	30,000	30,000	68,270	34,135.00
3	22,880	61,150	20,400	39,600	1,00,750	33,583.30
4	26,700	87,850	14,400	45,600	1,33,450	33,362.50
5	31,800	1,19,650	9,650	50,350	1,70,000	34,000.00

The calculations in the above table show that the average cost is lowest during the fourth year. Hence the machine should be replaced after every fourth year.

PROBLEMS

1. The cost of a machine is Rs. 6,100 and its scrap value is Rs. 100. The maintenance costs found from experience are as follows :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	100	250	400	600	900	1,200	1,600	2,000

When should the machine be replaced?

2. A firm is considering replacement of a machine whose cost price is Rs. 17,500 and the scrap value is Ps. 500. The maintenance costs (in Rs.) are found from experience to be as follows :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	200	300	3,500	1,200	1,800	2,400	3,300	4,500

When should the machine be replaced?

3. A firm is using a machine whose purchase price is Rs. 13,000. The installation charges amount to Rs. 3,600 and the machine has a scrap value of only Rs.1,600, because the firm has a monopoly of this type of work.

The maintenance cost in various years is given in the following table :

Year	:	1	2	3	4	5	6	7
Cost (Rs.)	:	250	750	1,000	1,500	2,100	2,900	4,000

The firm wants to determine after how many years should the machine be replaced on economic considerations, assuming that the machine replacement can be done only at the year ends.

4. Following table gives the running costs per year and resale price of a certain equipment whose purchase price is Rs. 5,000.

Year	:	1	2	3	4	5	6	7	8
Running cost (Rs.)	:	1,500	1,600	1,800	2,100	2,500	2,900	3,400	4,000
Resale value (Rs.)	:	3.500	2,500	1,700	1,200	800	500	500	500

At what year is the replacement due?

5. A truck owner finds from his past records that the maintenance costs per year, of a truck whose purchase price is Rs. 8,000 are as given below :

Year	:	1	2	3	4	5	6	7	8
Maintenance cost (Rs.)	:	1,000	1,300	1,700	2,200	2,900	3,800	4,800	6,000
Resale price (Rs.)	:	4,000	2,000	1,200	600	500	400	400	400

Determine at which time it is profitable to replace the truck.

6. A fleet owner finds from his past records that the costs per year of running a vehicle whose purchase price is Rs. 50,000 are as under :

Year	:	1	2	3	4	5	6	7
Running cost (Rs.)	:	5,000	6,000	7,000	9,000	11,500	16,000	18,000
Resale value (Rs.)	:	30,000	15,000	7,500	9,750	2,000	2,000	2,000

Thereafter, running cost increases by Rs. 2,000, but resale value remains constant at Rs. 2,000.

At what age is the replacement due?

7. Fleet cars have increased their costs as they continue in service due to increased direct operating cost (gas and oil) and increased maintenance (repairs, tyres, batteries, etc.). The initial cost is Rs. 3,500, and the trade-in value drops as time passes until it reaches a constant value of Rs. 500.

Given the cost of operating, maintaining and the trade-in value, determine the proper length of service before cars should be replaced :

Years of Service:	:	1	2	3	4	5
Year end trade-in value (Rs.)	:	1,900	1,050	600	500	500
Annual operating cost (Rs.)	:	1,500	1,800	2,100	2,400	2.700
Annual maintaining cost (Rs.)	:	300	400	600	800	1,000

8. The data on the operating costs per year and resale price of equipment A whose purchase price is Rs. 10,000 are given below :

Year	:	1	2	3	4	5	6	7
Operating cost (Rs.)	:	1,500	1,900	2,300	2,900	3,600	4,500	5,500
Resale value (Rs.)	:	5,000	2,500	1,250	600	400	400	400

(i) What is the optimum period for replacement?

(ii) When equipment A is 2 years old, equipment B, which is a new model for the same usage is available. The optimum period for replacement is 4 years with an average cost of Rs. 3,600. Should we change Equipment A with that of B? If so when?

9. A new tempo costs Rs. 80,000 and may be sold at the end of any year at the following prices :

Year (end)	:	1	2	3	4	5	6
Selling price (in Rs.) (at present value)	:	50,000	33,000	20,000	11,000	6,000	1,000

The corresponding annual operating costs are :

Year (end)	:	1	2	3	4	5	6
Cost/year (in Rs.) (at present value)	:	10,000	12,000	15,000	20,000	30,000	50,000

It is not only possible to sell the tempo after use but also to buy a second hand tempo.

It may be cheaper to do so than to replace by a new tempo.

Age of tempo	:	0	1	2	3	4	5
Purchase price (in Rs.) (at present value)	:	80,000	58,000	40,000	26,000	16,000	10.000

What is the age to buy and to sell tempo so as to minimize average annual cost?

10. (a) A transport manager finds from his past records that the costs per year of running a truck whose purchase price is Rs. 6,000 are as given below :

Year	:	1	2	3	4	5	6	7	8
Running cost (Rs.)	:	1,000	1,200	1,400	1,800	2,300	2,800	3,400	4,000
Resale value (Rs.)	:	3,000	1,500	750	375	200	200	200	200

Determine at what age is replacement due?

(b) Let the owner of a fleet have three trucks, two of which are two years old and the third one year old. The cost price, running cost and resale vale of these trucks are same as given in (a). Now he is considering a new type of truck with 50% more capacity than one of the old ones at a unit price of Rs. 8,000. He estimates that the running costs and resale price for the truck will be as follows :

Year	:	1	2	3	4	5	6	7	8
Running costs (Rs.)	:	1,200	1,500	1,800	2,400	3,100	4,000	5,000	6,100
Resale price (Rs.)	:	4,000	2.000	1,000	500	300	300	300	300

Assuming that the loss of flexibility due to fewer trucks is of no importance, and that he will continue to have sufficient work for three of the old trucks, what should his policy be?

11. Machine A costs Rs. 3,600. Annual operating costs are Rs. 40 for the first year and then increase by Rs. 360 every year. Assuming that machine A has no resale value, determine the best replacement age.

Another machine B, which is similar to machine A, costs Rs. 4,000. Annual running costs are Rs. 200 for the first year and then increase by Rs. 200 every year. It has resale value of Rs. 1,500, Rs. 1,000 and Rs. 500 if replaced at the end of first, second and third years respectively. It has no resale value during fourth year and onwards.

Which machine would you prefer to purchase? Future costs are not to be discounted.

12. Machine A costs Rs. 45,000 and the operating costs are estimated at Rs. 1,000 for the first year, increasing by Rs. 10,000 per year in the second and subsequent years. Machine B costs Rs. 50,000 and operating costs are Rs. 2,000 for the first year, increasing by Rs. 4,000 in the second and subsequent years. If we now have a machine of type A, should we replace it with B? If so when? Assume that both machines have no resale value and future costs are not discounted.

9.2.2. Replacement Policy when Value of Money changes with time

When the time value of money is taken into consideration, we shall assume that (i) the equipment in question has no salvage value, and (ii) the maintenance costs are incurred in the beginning of the different time periods.

Since it is assumed that the maintenance cost increases with time and each cost is to be paid just in the start of the period, let the money carry a rate of interest r per year. Thus a rupee invested now will be worth (1 + r) after a year, $(1 + r)^2$ after two years, and so on. In this way a rupee invested today will be worth $(1 + r)^n$, n years hence, or, in other words, if we have to make a payment of one rupee in n years time, it is equivalent to making a payment of $(1 + r)^{-n}$ rupees today. The quantity $(1 + r)^{-n}$ is called the present worth factor (Pwf) of one rupee spent in n years time from now onwards. The expression $(1 + r)^n$ is known as the payment compound amount factor (Caf). of one rupee spent in n years time.

Let the initial cost of the equipment be C and let R_n be the operating cost in year n. Let v be the rate of interest in such a way that $v = (1 + r)^{-1}$ is the discount rate (present worth factor). Then the present value of all future discounted costs V_n associated with a policy of replacing the equipment at the end of each n years is given by

$$V_n = \{(C + R_o) + vR_1 + v^2R_2 + \dots v^{n-1}R_{n-1}\} + \{(C + R_o)v_n + v^{n+1}R_1 + v^{n+2}R_2 + \dots v^{2n-1}R_{n-1}\} + \dots$$

$$= \left[C + \sum_{k=0}^{n-1} v^k R_k \right] \text{x} \sum_{k=0}^{\infty} (v^n)^k = \left[C + \sum_{k=0}^{n-1} v^k R_k \right] (1 - v^n)^{-1}$$

Now, V_n will be a minimum for that value of n, for which

$$V_{n+1} - V_n > 0 \text{ and } V_{n-1} - V_n > 0$$

For this, we write

$$V_{n+1} - V_n = \left[C + \sum_{k=0}^{n} v^k R_k \right] (1-v^{n+1})^{-1} - V_n$$

$$= v^n [R_n - (1 - v) V_n] / (1 - v^{n+1})$$

and similarly

$$V_n - V_{n-1} = V^{n-1} \; [R_{n-1} - (1 - v)\, V_n] \,/\, (1 - v^{m-1})$$

Since v is the depreciation value of money, it will always be less than 1 and therefore 1- v will always be positive. This implies that $v^n/(1 - v^{n+1})$ will always be positive.

Hence, $V_{n+1} - V_n > 0 \;\Rightarrow\; R_n > (1 - v)\, V_n$ and $V_n - V_{n-1} < 0 \;\Rightarrow\; R_{n-1} < (1 - v)\, V_n$

Thus, $$R_{n-1} < (1 - v)\, V_n < R_n$$

Or $$R_{n-1} < \frac{C + R_0 + vR_1 + v^2 R_2 + \ldots + v^{n-1} R_{n-1}}{1 + v + v^2 + \ldots + v^{n-1}} < R_n,$$

Since $(1 - v^n)\,(1 - v)^{-1} = \sum_{k=0}^{n-1} v^k$.

The expression which lies between R_{n-1} and R_n is called the "weighted average cost" of all the previous n years with weights 1, v, v^2, ..., v^{n-1} respectively.

Hence, the optimal replacement policy of the equipment after n periods is :

(a) Do not replace the equipments if the next period's operating cost is less than the weighted average of previous costs.

(b) Replace the equipments if the next period's operating cost is greater than the weighted average of previous costs.

Remark. Procedure for determining the weighted average of costs (annualized cost) may be summarized in the following steps :

Step 1. Find the present value of the maintenance cost for each of the years, i.e., $\sum Rv^{n-1}$ (n = 1, 2, ...); where $v = (1 + r)^{-1}$

Step 2. Calculate cost plus the accumulated present values obtained in step 1. i.e., $C + \sum Rv^{n-1}$.

Step 3. Find the cumulative present value factor up to each of the years 1, 2, 3, ... i.e., $\sum v^{n-1}$.

Step 4. Determine the annualized cost W (n), by dividing the entries obtained in step 2 by the corresponding entries obtained in step 3. i.e., $[C + \sum v^{n-1}] / \sum v^{n-1}$.

Corollary. When the time value of money is not taken into consideration, the rate of interest becomes zero and hence v approaches unity. Therefore, as $v \to 1$, we get

$$R_{n-1} < \frac{C+R_0+R_1+...+R_{n-1}}{1+1+...+n\,times} < R_n$$

or $$R_{n-1} < W(n) < R_n$$

Note. It may be noted that the above result is in complete agreement with the result that was obtained in 18 : 2.1.

9.2.3 : Selection of the Best Equipment Amongst Two

Following is the procedure for determining a policy for the selection of an economically best item amongst the available equipments :

Step 1. Considering the case of two equipments, say A and B, we first find the best replacement age for both the equipments by making use of

$$R_{n-1} < (1-v)\,V_n < R_n.$$

Let the optimum replacement age for A and B comes out to be n_1 and n_2 respectively.

Step 2. Next, compute the fixed annual payment (or weighted average cost) for each equipment by using the formula

$$W(n) = \frac{C+R_0+vR_1+v^2R_2+...+v^{n-1}R_{n-1}}{1+v+...+v^{n-1}}$$

and substitute $n = n_1$ for equipment A and $n = n_2$ for equipment B in it.

Step 3. (i) If $w(n_1) < W(n_2)$, choose equipment A.

(ii) If $W(n_1) > W(n_2)$, choose equipment B.

(iii) If $W(n_1) = W(n_2)$, both equipments are equally good.

Worked out Examples

Example 9.2.3 Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every six years. The yearly costs of both the machines are given below :

Year	:	1	2	3	4	5	6
Machine A	:	1,000	200	400	1,000	200	400
Machine B	:	1,700	100	200	300	400	500

Determine which machine should be purchased.

Solution. Since the money carries the rate of interest, the present worth of the money to be spent over in a period of one year is

$$v = \frac{100}{100+10} = \frac{10}{11} = 0.9091$$

∴ The total discounted cost (present worth) of A for 3 years is

$1000 + 200 \times (0.9091) + 400 \times (0.9091)^2$ = Rs. 1512 approx.

Again, the total discounted cost of B for six years is

$1{,}700 + 100 \times (0.9091) + 200 \times (0.9091)^2 + 300 \times (0.9091)^3 + 400 \times (0.9091)^4$

$+ 500 \times (0.9091)^5$ = Rs. 2,765.

Average yearly cost of machine A = Rs. 1,512/3 = Rs. 504.

Average yearly cost of machine B = Rs. 2,765/6 = Rs. 461.

This shows that the apparent advantage is with machine B. But, the comparison is unfair since the periods for which the costs are considered are different. So, if we consider 6 years period for machine A also, then the total discounted cost of A will be

$1{,}000 + 200 \times (0.9091) + 400 \times (0.9091)^2 + 1{,}000 \times (0.9091)^3 + 200 \times (0.9091)^4$

$+ 400 \times (0.9091)^5$.

After simplification this comes out to be Rs. 2,647 which is Rs. 118 less costlier than machine B over the same period.

Hence machine A should be purchased.

Example 9.2.4. The cost of a new machine is Rs. 5,000. The maintenance cost of nth year is given by $C_n = 500\,(n-1)$; n = 1, 2, … Suppose that the discount rate per year is 0.5. After how many years it will be economical to replace the machine by a new one ?

Solution. Since the discount rate of money per year is 0.05, the present worth of the money to be spent over a period of one year is

$$v + (1 + 0.05)^{-1} = 0.9523.$$

The optimum replacement time is determined in the following table :

Year (n) (1)	R_{n-1} (2)	v^{n-1} (3)	$R_{n-1}v^{n-1}$ (4)	$C+\sum_k R_{k-1}v^{k-1}$ (5)	$\sum_k v^{k-1}$ (6)	W(n) (7)
1	0	1.0000	0	5,000	1.0000	5,000
2	500	0.9523	476	6,476	1.9523	2,805
3	1,000	0.9070	907	6,383	2.8593	2,232
4	1,500	0.8638	1,296	7,679	3.7231	2,063
5	2,000	0.8227	1,645	9,324	4.5458	**2,051***
6	2,500	0.7835	1,959	11,283	5.3293	2,117

Since, W (n) is minimum for n = 5 and R_4 = 1,500 < W (5) as well as W (5) > R_6 = 2,500 ; it is economical to replace the machine by a new one at the end of five years.

9.2.5 Exercise

1. Let v = 0.9 and initial price is Rs. 5,000. Running cost varies as follows :

Year :	1	2	3	4	5	6	7
Running cost (in Rs.) :	400	500	700	1,000	1,300	1,700	2,100

What would be the optimum replacement interval ?

2. The yearly cost of 2 machines A and B when the money value is neglected is as follows :

Year :	1	2	3	4	5
Machine A :	1,800	1,200	1,400	1,600	1,000
Machine B :	2,800	200	1,400	1,100	600

Find their cost patterns if money value is 10% per year and hence find which machine is most economical.

3. An individual is planning to purchase a car. A new car will cost Rs. 1,20,000. The resale value of the car at the end of the year is 85% of the previous year value. Maintenance and operation costs during the first year are Rs. 20,000 and they increase by 15% every year. The minimum resale value of car can be Rs. 40,000.

(i) When should the car be replaced to minimise average annual cost (ignore interest)?

(ii) If interest of 12% is assumed, when should the car be replaced ?

9. 3. Replacement of Equipment that Fails Suddenly

It is difficult to predict that a particular equipment will fail at a particular time. This difficulty can be overcome by determining the probability distribution of failures. Here it is assumed that the failures occur only at the end of the period, say t. Thus the objective becomes to find the value of t which minimizes the total cost involved for the replacement.

We shall consider the following two types of replacement policies :

Individual Replacement Policy. Under this policy, an item is replaced immediately after its failure.

9.3.1. Group Replacement Policy. Under this policy, we take decisions as to when all the items must be replaced, irrespective of the fact that items have failed or have not failed, with a provision that if any item fails before the optimal time, it may be individually replaced.

Mortality Tables. These are used to derive the probability distribution of the life span of an equipment. Let

M (t) = number of survivors at any time t,

M (t-1) = number of survivors at any time t-1, and

N = initial number of equipments

Then the probability of failure during time period t is given by

$$P(t) = [M(t-1) - M(t)]/N$$

The probability that an equipment survived till age (t -1), will fail during the interval (t - 1) to t can be defined as the conditional probability of failure. It is given by

$$P_c(t) = [M(t-1) - M(t)] / M(t-1)$$

The probability of survival till age t is given by

$$P_s(t) = M(t) / N.$$

Theorem 9.3.1. (Group Replacement). Let all the items in a system be replaced after a time interval 't' with provisions that individual replacements can be made if and when any item fails during this time period. Then

(a) Group replacement must be made at the end of t^{th} period if the cost of individual replacement for the period is greater than the average cost per unit time period through the end of t periods.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacements at the end of period t - 1 is less than the average cost per unit period through the end of period t.

Proof. Let,

N	=	total number of items in the system,
C_2	=	cost of replacing an individual item,
C_1	=	cost of replacing an item in group,
C(t)	=	total cost of group replacement after time period t,
f(t)	=	number of failures during time period t.

Then, clearly

$$C(t) = NC_1 + C_2 \sum_{x=0}^{t-1} f(x)$$

The average cost of group replacement per unit period of time during a period t, is thus given by

$$A(t) = \frac{C(t)}{t} = \left[NC_1 + C_2 \sum_{x=0}^{t-1} f(x) \right] / t.$$

We shall determine the optimum t so as to minimize C (t) / t.

Note that whenever $\frac{C(t-1)}{t-1} > \frac{C(t)}{t}$ and $\frac{C(t+1)}{t+1} > \frac{C(t)}{t}$, it is better to replace all the items after time period t.

Now, $\frac{C(t+1)}{t+1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t) > C(t)/t$;

And $\frac{C(t-1)}{t-1} - \frac{C(t)}{t} > 0 \Rightarrow C_2 f(t-1) < C(t)/t$

$\therefore \quad tC_2 f(t-1) < C(t) < t.\ C_2 f(t).$

or $\quad t\, f(t-1) - \sum_{x=0}^{t-1} f(x) < \frac{NC_1}{C_2} < t\, f(t) - \sum_{x=0}^{t-1} f(x).$

Worked out Examples

Example 9.3.1. The following failure rates have been observed for a certain type of transistors in a digital computer :

End of the week	:	1	2	3	4	5	6	7	8
Probability of failure to date	:	.05	.13	.25	.43	.68	.88	.96	1.00

The cost of replacing an individual failed transistor is Rs. 1.25. The decision is made to replace all these transistors simultaneously at fixed intervals, and to replace the individual transistors as they fail in service. If the cost of group replacement is 30 paise per transistor, what is the best interval between group replacements? At what group replacement price per transistor would a policy of strictly individual replacement become preferable to the adopted policy?

Solution. Suppose there are 1,000 transistors in use. Let p_i be the probability that a transistor, which was new when placed in position for use, fails during the ith week of its life. Thus, we have

p_1	≡	0.05,	p_2	≡	0.13 - 0.5 = 0.08,
p_3	≡	0.25 - 0.13 = 0.12,	p_4	≡	0.43 - 0.25 = 0,18,
p_5	≡	0.68 - 0.43 = 0.25,	p_6	≡	0.88 - 0.68 = 0.20,
p_7	≡	0.96 - 0.88 = 0.08,	p_8	≡	1.00 - 0.96 = 0.04.

Let N_i denote the number of replacements made at the end of the ith week. Then, we have

N_0 = number of transistors in the beginning = 1,000

$N_1 = N_0p_1 = 1,000 \times 0.05 = 50$

$N_2 = N_0p_2 + N_1p_1 = 1,000 \times 0.08 + 50 \times 0.05 = 82$

$N_3 = N_0p_3 + N_1p_2 + N_2p_1 = 1,000 \times 0.12 + 50 \times 0.08 + 82 \times 0.05 = 128$

$N_4 = N_0p_4 + N_1p_3 + N_2p_2 + N_3p_1 = 199$

$N_5 = N_0p_5 + N_1p_4 + N_2p_3 + N_3p_2 + N_4p_1 = 289$

$N_6 = N_0p_6 + N_1p_5 + N_2p_4 + N_3p_3 + N_4p_2 + N_5p_1 = 272$

$N_7 = N_0p_7 + N_1p_6 + N_2p_5 + N_3p_4 + N_4p_3 + N_5p_2 + N_6p_1 = 194$

$N_8 = N_0p_8 + N_1p_7 + N_2p_6 + N_3p_5 + N_4p_4 + N_5p_3 + N_6p_2 + N_7p_1 = 195$

From the above calculations, we observe that the expected number of transistors failing each week increases till 5th week and then starts decreasing and later again increasing from 8th week.

Thus, N_i will oscillate till the system acquires a steady state. The expected life of each transistor is

$$1 \times 0.5 + 2 \times .08 + 3 \times .12 + 4 \times .18 + 5 \times .25 + 6 \times .2 + 7 \times .08 + 8 \times .04$$

$$= 4.62 \text{ weeks.}$$

Average number of failures per week

$$= 1,000/4.62 = 216 \text{ approximately.}$$

Therefore, the cost of individual replacement

$$= 216 \times 1.25 = \text{Rs. } 270.00 \text{ per week.}$$

Now, since the replacement of all the 1,000 transistors simultaneously cost 30 paise per transistors and the replacement of an individual transistor on failure cost Rs. 1.25, the average cost for different group replacement policies is given as under :

End of Week	Individual replacement	Total cost (Rs.) Individual + Group	Average cost (Rs.)
1	50	50 x 1.25 + 1,000 x 0.30 = 363	363
2	132	132 x 1.25 + 1,000 x 0.30 = 465	232.50
3	260	260 x 1.25 + 1,000 x 0.30 = 625	208.30
4	459	459 x 1 .25 + 1,000 x 0.30 = 874	218.50

Since the average cost is lowest against week 3, the optimum interval between group replacements is 3 weeks. Further, since the average cost is less than Rs. 270 (for individual replacement), the policy of group replacement is better.

Example 9.3.2. At time zero all items in a system are new. Each item has a probability p of failing immediately before the end of the first month of life, and a probability q = 1 - p of failing immediately before the end of the second month (i.e., all items fail by the end of the second month). If all items are replaced as they fail, show that the expected number of failures f(x) at the end of month x is given by

$$f(x) = \frac{N}{1+q}\,[1-(-q)^{x+1}]$$

where N is the number of items in the system.

If the cost per item of individual replacement is C_1, and the cost per item of group replacement is C_2, find the condition under which

(a) A group replacement policy at the end of each month is the most profitable.

(b) No group replacement policy is better than a policy of pure individual replacement.

Solution. Let

N = number of items in the system in the beginning

N_1 = number of items expected to fail at the end of 1st month

= $N_o p = N(1 - q)$, since $p = 1 – q$.

N_2 = number of items expected to fail at the end of 2nd month

= $N_o q + N_1 p = Nq + N(1-q)^2 = N(1 - q + q^2)$,

N_3 = number of items expected to fail at the end of 3rd month

= $N_1 q + N_2 p = N(1 - q)q + N(1 - q + q^2)(1 - q) = N(1 - q + q^2 - q^3)$,

and so on. In general,

$$N_k = N[1 - q + q^2 – q^3 + ... + (-q)^k].$$

$$\therefore\ N_{k+1} = N_{k-1} q + N_k p$$

$$= N[1 - q + q^2 + ... + (-q)^{k-1}] q + N[1 - q + q^2 + ... + (-q)^k](1 - q)$$

$$= N[1 - q + q^2 + ... + (-q)^{k+1}]$$

Hence by mathematical induction, the expected number of failures at the end of month x will be given by

$$f(x) = N[1 – q + q^2 + \dots + (-q)^x] = N[1 - (-q)^{x+1}] / (1 + q).$$

The value of f(x) at the end of month x will vary for different values of $(-q)^{x+1}$ and it will reach the steady-state as $x \to \infty$.

Hence, in the steady state case, the expected number of failures will be

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} N[1 - (-q)^{x+1}] / (1 + q)$$

$$= N/(1 + q)\ ;\ \text{since } q < 1 \text{ and } (-q)^{x+1} \to 0 \text{ as } x \to \infty.$$

= Total number of items in the system / Mean age,

Now, since C_1 is the cost of replacement per item individually and C_2 is the cost of an item in group, therefore

(i) If we have a group replacement at the end of each month, then the cost of replacement is NC_2.

(ii) If we have a group replacement policy at the end of every other month, then the cost is $NC_2 + N_pC_1$.

The average cost per month, therefore, is $(NC_2 + N_pC_1)/2$, and

(iii) Average life of an item

$$= 1 \text{ x } p + 2 \text{ x } q = 1 \text{ x } (1 - q) + 2q = 1 + q.$$

Therefore, the average number of failures is $N/(l + q)$ and hence the cost of individual replacement is $NC_1 / (l + q)$.

(a) A group replacement at the end of first month will be better than individual replacement, if total cost of group replacement is less than the average monthly cost of individual replacement.

Thus, $N(1 - q)\, C_1 + NC_2 < NC_1/ (1 + q)$, i.e. $C_2 < c_1q^2 / (1 + q)$.

For a group replacement at the end of every second month, the total cost of replacement will be

$$(N_1 + N_2)\, C_1 + NC_2 = N\,(2 - 2q + q^2)\, C_1 + NC_2.$$

∴ Average monthly cost of group replacement at the end of second month is

$$[N\,(2 - 2q + q^2)\, C_1 + NC_2] / 2$$

in this case, the group replacement policy will be better than the individual replacement policy, if

Average monthly cost of group replacement < Average monthly cost of individual replacement

or $[N\,(1 - q + q^2/2\, C_1 + NC_2 / 2] < NC_1 / (1 + q)$,

or $C_2 < q^2\,(1 - q)\, C_1/(1 + q)$.

(b) For the individual replacement policy to be better than any of the group replacement policies discussed above, we must have

$$C_2 > C_1q^2 / (q + l) \quad \text{and} \quad C_2 > C_1q^2\,(1 - q) / (q + 1)$$

or $\quad C_1 < C_2\,(1 + q) / q^2 \quad$ and $\quad C_1 < C_2\,(1 + q) / [q^2\,(1 - q)]$

But $q < 1$, therefore $(1 + q)/q^2 < (1 + q)/q^2(1 - q)$

Hence, $C_1 < (1 + q)\ C_2/q^2$.

9.4 Summary :

In this lesson various replacement models are discussed. In particular the equipment that deteriorates gradually and equipment that fails suddenly. Further, when equipment deterio rates suddenly, the two cases when value of money changes with time and does not change with time also discussed. For all these cases theoretical discussion is supported with examples.

9.5 Exercise :

1. In a machine shop, a particular cutting tool costs Rs. 6 to replace. If a tool breaks on the job, the production disruption and associate costs amount to Rs. 30. The past life of a tool is given as follows :

Job No.	:	1	2	3	4	5	6	7
Proportion of broken tools on job	:	.01	.03	.09	.13	.25	.55	.95

After how many jobs should the shop replace a tool before it breaks down?

2. The following failure rates have been observed for a certain type of light bulbs :

Week	:	1	2	3	4	5
Per cent failing by end of week	:	10	25	50	80	100

There are 1,000 bulbs in use, and it costs Rs. 2 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously it would cost 50 paise per bulb. It is proposed to replace all bulbs at fixed intervals, whether or not they have burnt out, and to continue replacing burnt out bulbs as they fail. At what interval should all the bulbs be replaced?

3.. The following failure rates have been observed for a certain type of light bulb :

End of week	:	1	2	3	4	5	6	7

Probability of

failure to date	:	0.07	0.15	0.25	0.45	0.75	0.90	1 .00

There are 1,000 bulbs in use. The cost of replacing an individual failed bulb is Rs. 1.25. If all the bulbs are replaced as a group, it costs Rs. 0.60 per bulb. Determine among individual and group replacement policies which one is better.

4. A large computer installation contains 2,000 components of identical nature which are subject to failure as per probability distribution given below :

Week end	:	1	2	3	4	5
Percentage failure to date	:	10	25	50	80	100

Components which fail have to be replaced for efficient functioning of the system. If they are replaced as and when failure occur, the cost of replacement per unit is Rs. 3. Alternatively if all components are replaced in one lot at periodical intervals and individually replaced only as such failures occur between group replacement, the cost of component replaced is Re. 1.

(a) Assess which policy of replacement would be economical.

(b) If group replacement is economical at current costs, then assess at what cost of individual replacement would group replacement be uneconomical.

(c) How high can the cost per unit in group replacement be to make a preference for individual replacement policy?

5. Let p (t) be the probability that a machine in a group of 30 machines would break down in period t. The cost of repairing a broken machine is Rs. 200. Preventive maintenance is performed by servicing all the 30 machines at the end of T units of time. Preventive maintenance cost is Rs. 15 per machine. Find optimum T which will minimize the expected total cost per period of servicing, given that

$$p(t) = \begin{cases} 0.03 & \text{for } t = 1 \\ p(t-1) + 0.01 & \text{for } t = 2, 3, \ldots 10 \\ 0.13 & \text{for } t = 11, 12, 13, \ldots \end{cases}$$

6. India Electric Company operates three large banking ovens used to manufacture electric motors and generators. Each oven contains 2,400 heating resistors. It costs Rs. 80 to replace an individual resistor when it burns out. If all the resistors are replaced (on Sunday night), it costs Rs. 72,000. When a resistor fails, it must be replaced immediately. When group replacement is followed, all resistors are replaced at one time regardless of age. Statistical data provided by the research department of India Electric Company provide the following probability distribution for resistor failure :

Week	:	1	2	3	4	5	6
Probability of failure	:	.02	.08	.15	.30	.25	.20

Is individual or group replacement more desirable?

9.6 References:

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Lesson – 10

Game Theory

10.0 Objective :

- Definition of a Game
- Characteristics of a Game
- Different types of Games
- Saddle Point
- Minimax and Maximin Principles

Structure

10.1 Introduction

10.2 Characteristics of game theory

10.3 Basic definitions

10.4 Minimax (Max-min) Criterion

10.5 Saddle point, optimal strategy and value of the game

10.6 Solution of games with saddle point

10.7 Summary

10.8 Exercise

10.9 References

10.1 Introduction

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an election have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing campaigns by competing business firms, etc.

Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must win and somebody must lose.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the **'minimax (maximin)** criterion' of **J. Von Neumann** (called the father of game theory). This criterion implies the assumption of rationality from which it is argued that each player will act so as **to maximize his minimum gain or minimize his** maximum loss. The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. Therefore, game theory is generally interpreted as an "as " theory, that is, as if rational decision maker (player) behaved in some well defined (but arbitrarily selected) way, such as maximizing the minimum gain.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

10.2 Characteristics of Game Theory

There can be various types of games. They can be classified on the basis of the following characteristics.

(i) **Chance of strategy** : If in a game, activities are determined by skill, it is said to be a ***game of strategy***; if they are determined by chance, it is a game of chance. In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

(ii) **Number of persons :** A game is called an n-person game if the number of persons playing is n. The person means an individual or a group aiming at a particular objective.

(iii) **Number of activities :** These may be finite or infinite.

(iv) **Number of alternatives (choices) available to each person** in a particular activity may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be infinite.

(v) **Information to the players about the past activities of other players** is completely available, partly available, or not available at all.

(vi) **Payoff:** A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i , $1 \leq i \leq n$, in an n-person game. If $\sum_{i=1}^{n} v_i = 0$, then the game is said to be a zero-sum game.

10.3 Basic Definitions

1. **Competitive Game.** A competitive situation is called a ***competitive game*** if it has the following four properties:

(i) There are finite number (n) of competitors (called players) such that $n \geq 2$. In case $n = 2$, it is called a ***two-person game*** and in case $n > 2$, it is referred to as an **n-person game**.

(ii) Each player has a list of finite number of possible activities (the list may not be same for each player).

(iii) A play is said to occur when each player chooses one of his activities. The choices are assumed to be made simultaneously, i.e. no player knows the choice of the other until he has decided on his own.

(iv) Every combination of activities determines an outcome (which may be points, money or any thing else whatsoever) which results in a gain of payments (+ve, -ve or zero) to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

2. Zero-sum and Non-zero-sum Games. Competitive games are classified according to the number of players involved, i.e. as a two person game, three person game, etc. Another important distinction is between zero-sum games and nonzero-sum games. If the players make payments only to each other, i. e. the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be zero-sum.

Mathematically, suppose an n-person game is played by n players $P_1, P_2, \dots, P_n$ whose respective pay-offs at the end of a play of the game are $v_1, v_2, \dots v_n$ then, the game will be called zero-sum if $\sum_{i=1}^{n} v_i = 0$ at each play of the game.

A game which is not zero-sum is called a nonzero-sum game. Most of the competitive games are zero-sum games. An example of a nonzero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. **Strategy.** A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play.

   This strategy may be of two kinds :

   (i) Pute Strategy. : If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. Therefore, the pure strategy is a decision rule always to select a particular course of action.

   A pure strategy is usually represented by a number with which the course of action is associated.

   (ii) Mixed Strategy. [Agra 92; Kerala (Stat.) 83]: If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain.

   Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with m ($\geq$ 2) possible courses of action, is denoted by the set S of m non-negative real numbers whose sum is unity, representing probabilities with which each course of action is chosen. If xi (i= 1, 2, 3,..., m) is the probability of choosing the course i, then

$$S = (x_1, x_2, x_3, \ldots, x_m)$$

subject to the conditions $x_1 + x_2 + x_3 + \ldots + x_m = 1$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \ldots, x_m \geq 0.$

Note. It should be noted that if some $x_i = 1$, (i = 1, 2,..., m) and all others are zero, the player is said to use a pure strategy. Thus, the pure strategy is a particular case of mixed strategy.

4. **Two-Person, Zero-Sum (Rectangular) Games.** A game with only two players (say, Player A and Player B) is called a 'two-person, zero-sum game' if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero.

   Two-person, zero-sum games, are also called rectangular games as these are usually represented by a payoff matrix in rectangular form.

5. **Payoff Matrix.** Suppose the player A has m activities and the player B has n activities. Then a payoff matrix can be formed by adopting the following rules :

   (i) Row designations for each matrix are activities available to player A.

   (ii) Column designations for each matrix are activities available to player B.

   (iii) Cell entry 'v_{ij}' is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.

   (iv) With a 'zero-sum, two person game', the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry 'v_{ij}' in the player A's payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

Table 10.3.1 The player A's pay off matrix

Player B

		1	2	…	j	…	n
	1	v_{11}	v_{12}	…	v_{1j}	…	v_{1n}
	2	v_{21}	v_{22}	…	v_{2j}	…	v_{2n}
Player A	:	:	:		:		:
	i	v_{i1}	v_{i2}	…	v_{ij}	…	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	…	v_{mj}	…	v_{mn}

Note. Further, there is no need to write the B's payoff matrix as it is just the -ve of A's payoff matrix in a zero-sum two-person game. Thus, if 'v_{ij}' is the gain to A, then '$-v_{ij}$' will be the gain to B.

Table 10.3.2

		H	T
A	H	+1	–1
	T	–1	+1

Example 10.3.1: In order to make the above concepts a clear, consider the coin matching game involving two players only. Each player selects either a head H or a tail T. If the outcomes match (H, H or T, T), A wins Re 1 from B, otherwise B wins Re 1 from A . This game is a two-person zero-sum game, since the winning of one player is taken as losses for the other. Each has his choices between two pure strategies (H or T). This yields the following (2 x 2) payoff matrix to player A.

10.4 Minimax (Maximin) Criterion

The 'minimax criterion of optimality' states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an **optimal strategy**.

Example 10.4.1. Consider (two-person, zero-sum) game matrix which represents payoff to the player A. Find the optimal strategy, if any.

Table 10.4.1

		B: I	B: II	B: III	Row minimum
A	I	–3	–2	–3	–3
	II	2	0 (Saddle Point)	2	0 (Maximin Value ($\underline{v}$))
	III	5	–2	–4	–4
Column maximum		5	0	6	
			Minimax Value ($\overline{v}$)		

Solution. The player A wishes to obtain the largest possible 'v_{ij}' by choosing one of his activities (I, II,III), while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). The player A is called the ***maximizing player*** and B the ***minimizing player***.

If the player A chooses the 1st activity, then it could happen that the player B also chooses his 1st activity. In this case the player B can guarantee a gain of at least - 3 to player A , i.e.

$$\min \{-3, -2, 6\} = \textcircled{-3}$$

Similarly, for other choices of the player A , i.e. II and III activities, B can force the player A to get only 0 and - 4, respectively, by his proper choices from (I, II, III), i. e.

$$\min \{2,0,2\} = \textcircled{0} \quad \text{and} \quad \min \{5,-2,-4\} = \textcircled{-4}$$

The minimum value in each row guaranteed by the player A is indicated by 'row minimum' in (Table 10-4.1). The best choice for the player A is to maximize his least gains- 3,0, - 4 and opt II strategy which assures at most the gain 0, i.e.

$$\max \{ \textcircled{-3}, \ 0, \ \textcircled{-4} \} = \boxed{0}$$

In general, the player A should try to maximize his least gains or to find out "$\max_i \ \max_j v_{ij}$"

Player B, on the other hand, can argue similarly to keep A's gain the minimum. He realizes that if he plays his 1st pure strategy, he can loose no more than 5 = max {-3, 2, 5} regardless of A 's selections. Similar arguments can be applied for remaining strategies II and III. Corresponding results are indicated in Table 19-4 by 'column maximum'. The player B will then select the strategy that minimizes his maximum losses. This is given by the strategy II and his corresponding loss is given by

$$\min\{ \boxed{5}, \boxed{0}, \boxed{6} \} = \boxed{0}$$

The player A's selection is called the maximin strategy and his corresponding gain is called the maximin value or lower value ($\underline{v}$) of the game. The player B' s selection is called the minimax value or upper value ($\bar{v}$) of the game. The selections made by player A and B are based on the so called minimax (or maximin) criterion. It is seen from the governing conditions that the minimax (upper) value $\bar{v}$ is greater than or equal to the maximin (lower) value $\underline{v}$. In the case where equality holds i.e.,

$\max_i \min_j v_{ij} = \min_j \max_i v_{ij}$ or $\underline{v} = \bar{v}$, the corresponding pure strategies are called the 'optimal' strategies and the game is said to have a saddle point. It may not always happen as shown in the following example.

Example 10.4.2. Consider the following game :

	B: i \ j	1	2	3
	1	3	−4	8
A	2	−8	5	−6
	3	6	−7	6

As discussed in **Example 10.4.1** $\max_i \min_j v_{ij} = 4$ $\min_j \max_i v_{ij} = 5$.

Also, $\max_i \min_j v_{ij} < \min_j \max_i v_{ij}$

10.5 Saddle Point, Optimal Strategies and Value of the Game

Definitions :

Saddle Point. ***A saddle point*** of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.

Mathematically, if a pay off matrix $\{v_{ij}\}$ is such that $\max_i [\min_j \{v_{ij}\}] = \min_j [\max_i \{v_{ij}\}] = v_{rs}$ (say), then the matrix is said to have a saddle point (r, s).

Optimal Strategies. If the payoff matrix $\{v_{ij}\}$ has the saddle point (r, s), then the players (A and B) are said to have rth and sth optimal strategies, respectively.

3. Value of Game. The payoff (v_{rs}) at the saddle point (r, s) is called the value of game and it is obviously equal to the maximin ($\underline{v}$) and minimax value ($\bar{v}$) of the game.

A game is said to be a **fair game** if $\bar{v} = \underline{v} = 0$. A game is said to be **strictly determinable** if $\bar{v} = v = \underline{v}$.

Note. A saddle point of a payoff matrix is, sometimes, called the equilibrium point of the payoff matrix.

In **Example 10.4.1,** $\underline{v} = \overline{v} = 0$. This implies that the game has a saddle point given by the entry (2,2) of payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy. In this example, it is also seen that no player can improve his position by other strategy.

In general, a matrix need not have a saddle point as defined above. Thus, these definitions of optimal strategy and value of the game are not adequate to cover all cases so need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such generalization.

Rules for Determining a Saddle Point:

1. Select the minimum element of each row of the pay off matrix and mark them by ‘O’.
2. Select the greatest element of each column of the payoff matrix and mark them by '□'.
3. If there appears an element in the payoff matrix marked by 'O' and '□' both, the position of that element is a saddle point of the pay off matrix.

10.6 Solution of games with Saddle Points

To obtain a solution of a rectangular game, it is feasible to find out :

(i) the best strategy for player A (ii) the best strategy for player B, and (iii) the value of the game (v_{rs},).

It is already seen that the best strategies for players A and B will be those which correspond to the row and column, respectively, through the saddle point. The value of the game to the player A is the element at the saddle point, and the value to the player B will be its negative.

10.6.1 Worked out Examples

Example 10.6.1. Player A can choose his strategies from $\{A_1, A_2, A_3\}$ only, while B can choose from the set (B_1, B_2) only. The rules of the game state that the payments should be made in accordance with the selection of strategies :

Strategy Pair Selected	***Payments to be Made***	***Strategy Pair Selected***	***Payments to be Made***
(A_1, B_1)	*Payer A Pays Re. 1 to player B*	(A_2, B_2)	*Player B pays Rs 4 to player A.*
(A_1, B_2)	*Player B pays Rs. 6 to player A*	(A_3, B_1)	*Player A pays Rs 2 to player B*
(A_2, B_1)	*P layer B pays Rs 2 to player A*	(A_3, B_2)	*Player A pays Rs. 6 to player B*

What strategies should A and B play in order to get the optimum benefit of the play ?

Solution. With the help of above rules the following payoff matrix is constructed :

		Player B	
		B_1	B_2
	A_1	−1	6
Player A	A_2	2	4
	A_3	−2	−6

The payoffs marked 'O' represent the minimum payoff in each row and those marked '☐' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Thus, the optimum solution to the game is given by :

(i) the optimum strategy for player A is A_2 ;

(ii) the optimum strategy for player B is B_1 ; and

(iii) the value of the game is Rs. 2 for player A and Rs. (- 2) for player B.

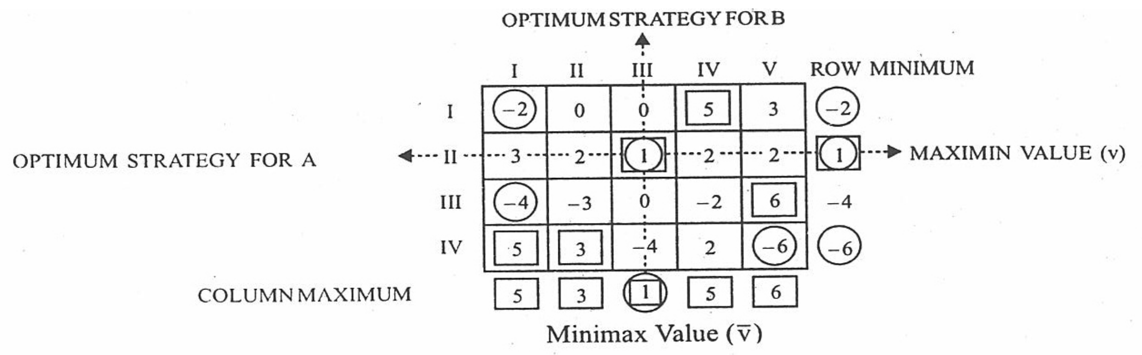
Also, since $v \neq 0$, the game is not fair, although it is strictly determinable.

Example 10.6.2. The payoff matrix of a game is given. Find the solution of the game to the player A and B.

$$
\begin{array}{cc}
 & B \\
 & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\
A \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}
\end{array}
$$

Solution. First find out the saddle point by encircling each row minima and putting squares around each column maxima.

The saddle point thus obtained is shown by having a circle and square both



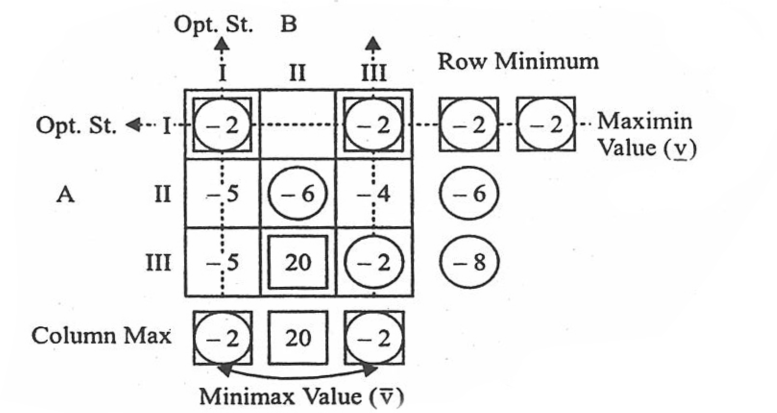
Hence, the solution to this game is given by, (i) the best strategy for player A is 2nd; (ii) the best strategy for player B is 3rd; and (iii) the value of the game is 1 to player A and - 1 to player B.

Example 10.6.3. Solve the game whose payoff matrix is given by

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\
\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{bmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{bmatrix}
\end{array}
$$

Solution.

This game has two saddle points in positions (1, 1) and (1, 3). Thus, the solution to this game is given by,



(i) the best strategy for the player A is I, (ii) the best strategy for the player B is either I or III, i.e. the player B can use either of the two strategies (I, III), and (iii) the value of the game is - 2 for player A and + 2 for player B.

10.7 Summary

In this lesson definitions of a competitive game and other basic terms are given. Also the characteristic of a game and solution of games with saddle point are explained with illustrations.

10.8 Exercise

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games :

(i) Player B

$$\text{Player A} \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}$$

(ii) Player B

$$\text{Player A} \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix}$$

2. Consider the game G with the following payoff matrix:

Player B

$$\text{Player A} \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix}$$

(i) Show that G is strictly determinable whatever μ may be.

(ii) Determine the value of G.

3. Find out whether there is any saddle point in the following problem :

Player B

$$\text{Player A} \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$$

4. For the game with payoff matrix:

Player B

$$\text{Player A} \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

determine the best strategies for players A and Band also the values of the game for them. Is this game (i) fair (ii) strictly determinable ?

5. Find the saddle point (or point) and hence solve the following games :

a.

Player B

		B_1	B_2	B_3
	A_1	15	2	3
Player A	A_2	6	5	7
	A_3	−7	4	0

b.

Player B

		B_1	B_2	B_3	B_4
	A_1	1	7	3	4
Player A	A_2	5	6	4	5
	A_3	7	2	0	3

c.

B

		I	II	III	IV
	I	−5	2	1	20
A	II	5	5	4	6
	III	4	−2	0	−5

d.

$$A \quad \overset{B}{\begin{array}{c} \\ I \\ II \\ III \\ IV \end{array}\begin{array}{ccccc} I & II & III & IV & V \\ \left[\begin{array}{ccccc} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 4 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{array}\right] \end{array}}$$

6. For the following payoff matrix for firm A, determine the optimal strategies for both the firms and the value of the game (using maximin – minimax principle) :

$$\text{Firm A} \quad \overset{\text{Firm B}}{\begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix}}$$

7. Solve the games whose payoff matrices are given below :

(a) **Player B**

$$\textbf{Player A} \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} -3 & -1 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{bmatrix} \end{array}$$

(b) **Player B**

$$\textbf{Player A} \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}$$

(c) **Player B**

$$\textbf{Player A} \begin{bmatrix} -5 & 5 & 0 & 7 \\ 2 & 6 & 1 & 8 \\ -4 & 0 & 1 & -3 \end{bmatrix}$$

10.9 References :

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Lesson – 10

Game Theory

10.0 Objective :

- Definition of a Game
- Characteristics of a Game
- Different types of Games
- Saddle Point
- Minimax and Maximin Principles

Structure

10.1 Introduction

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an election have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing campaigns by competing business firms, etc.

Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must win and somebody must lose.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the **'minimax (maximin)** criterion' of **J. Von Neumann** (called the father of game theory). This criterion implies the assumption of rationality from which it is argued that each player will act so as **to maximize his minimum gain or minimize his** maximum loss. The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. Therefore, game theory is generally interpreted as an "as " theory, that is, as if rational decision maker (player) behaved in some well defined (but arbitrarily selected) way, such as maximizing the minimum gain.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

10.2 Characteristics of Game Theory

There can be various types of games. They can be classified on the basis of the following characteristics.

(i) **Chance of strategy** : If in a game, activities are determined by skill, it is said to be a ***game of strategy***; if they are determined by chance, it is a game of chance. In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

(ii) **Number of persons :** A game is called an n-person game if the number of persons playing is n. The person means an individual or a group aiming at a particular objective.

(iii) **Number of activities :** These may be finite or infinite.

(iv) **Number of alternatives (choices) available to each person** in a particular activity may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be infinite.

(v) **Information to the players about the past activities of other players** is completely available, partly available, or not available at all.

(vi) **Payoff:** A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i , $1 \leq i \leq n$, in an n-person game. If $\sum_{i=1}^{n} v_i = 0$, then the game is said to be a zero-sum game.

10.3 Basic Definitions

1. **Competitive Game.** A competitive situation is called a ***competitive game*** if it has the following four properties:

(i) There are finite number (n) of competitors (called players) such that $n \geq 2$. In case $n = 2$, it is called a ***two-person game*** and in case $n > 2$, it is referred to as an **n-person game**.

(ii) Each player has a list of finite number of possible activities (the list may not be same for each player).

(iii) A play is said to occur when each player chooses one of his activities. The choices are assumed to be made simultaneously, i.e. no player knows the choice of the other until he has decided on his own.

(iv) Every combination of activities determines an outcome (which may be points, money or any thing else whatsoever) which results in a gain of payments (+ve, -ve or zero) to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

2. Zero-sum and Non-zero-sum Games. Competitive games are classified according to the number of players involved, i.e. as a two person game, three person game, etc. Another important distinction is between zero-sum games and nonzero-sum games. If the players make payments only to each other, i. e. the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be zero-sum.

Mathematically, suppose an n-person game is played by n players P_1, P_2 , ... , P_n whose respective pay-offs at the end of a play of the game are v_1 , v_2 , ... v_n then, the game will be called zero-sum if $\sum_{i=1}^{n} v_i = 0$ at each play of the game.

A game which is not zero-sum is called a nonzero-sum game. Most of the competitive games are zero-sum games. An example of a nonzero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. **Strategy.** A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play.

This strategy may be of two kinds :

(i) **Pute Strategy.** : If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. Therefore, the pure strategy is a decision rule always to select a particular course of action.

A pure strategy is usually represented by a number with which the course of action is associated.

(ii) **Mixed Strategy. [Agra 92; Kerala (Stat.) 83]:** If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain.

Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with m ($\geq$ 2) possible courses of action, is denoted by the set S of m non-negative real numbers whose sum is unity, representing probabilities with which each course of action is chosen. If xi (i= 1, 2, 3,..., m) is the probability of choosing the course i, then

$$S = (x_1, x_2, x_3, \dots, x_m)$$

subject to the conditions $x_1 + x_2 + x_3 + \dots + x_m = 1$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \dots, x_m \geq 0.$

Note. It should be noted that if some $x_i = 1$, (i = 1, 2,..., m) and all others are zero, the player is said to use a pure strategy. Thus, the pure strategy is a particular case of mixed strategy.

4. **Two-Person, Zero-Sum (Rectangular) Games.** A game with only two players (say, Player A and Player B) is called a 'two-person, zero-sum game' if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero.

Two-person, zero-sum games, are also called rectangular games as these are usually represented by a payoff matrix in rectangular form.

5. **Payoff Matrix.** Suppose the player A has m activities and the player B has n activities. Then a payoff matrix can be formed by adopting the following rules :

(i) Row designations for each matrix are activities available to player A.

(ii) Column designations for each matrix are activities available to player B.

(iii) Cell entry 'v_{ij}' is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.

(iv) With a 'zero-sum, two person game', the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry 'v_{ij}' in the player A's payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

Table 10.3.1 The player A's pay off matrix

Player B

		1	2	…	j	…	n
	1	v_{11}	v_{12}	…	v_{1j}	…	v_{1n}
	2	v_{21}	v_{22}	…	v_{2j}	…	v_{2n}
Player A	:	:	:		:		:
	i	v_{i1}	v_{i2}	…	v_{ij}	…	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	…	v_{mj}	…	v_{mn}

Note. Further, there is no need to write the B's payoff matrix as it is just the -ve of A's payoff matrix in a zero-sum two-person game. Thus, if 'v_{ij}' is the gain to A, then '$-v_{ij}$' will be the gain to B.

Table 10.3.2

		H	T
A	H	+1	–1
	T	–1	+1

Example 10.3.1: In order to make the above concepts a clear, consider the coin matching game involving two players only. Each player selects either a head H or a tail T. If the outcomes match (H, H or T, T), A wins Re 1 from B, otherwise B wins Re 1 from A . This game is a two-person zero-sum game, since the winning of one player is taken as losses for the other. Each has his choices between two pure strategies (H or T). This yields the following (2 x 2) payoff matrix to player A.

10.4 Minimax (Maximin) Criterion

The 'minimax criterion of optimality' states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an **optimal strategy**.

Example 10.4.1. Consider (two-person, zero-sum) game matrix which represents payoff to the player A. Find the optimal strategy, if any.

Table 10.4.1

		B: I	B: II	B: III	Row minimum
A	I	–3	–2	–3	–3
	II	2	0 (Saddle Point)	2	0 (Maximin Value ($\underline{v}$))
	III	5	–2	–4	–4
Column maximum		5	0 (Minimax Value ($\bar{v}$))	6	

Solution. The player A wishes to obtain the largest possible 'v_{ij}' by choosing one of his activities (I, II,III), while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). The player A is called the ***maximizing player*** and B the ***minimizing player***.

If the player A chooses the 1st activity, then it could happen that the player B also chooses his 1st activity. In this case the player B can guarantee a gain of at least - 3 to player A , i.e.

$$\min\{-3, -2, 6\} = \textcircled{-3}$$

Similarly, for other choices of the player A , i.e. II and III activities, B can force the player A to get only 0 and - 4, respectively, by his proper choices from (I, II, III), i. e.

$$\min\{2,0,2\} = \textcircled{0} \text{ and } \min\{5,-2,-4\} = \textcircled{-4}$$

The minimum value in each row guaranteed by the player A is indicated by 'row minimum' in (Table 10-4.1). The best choice for the player A is to maximize his least gains- 3,0, - 4 and opt II strategy which assures at most the gain 0, i.e.

$$\max\{\textcircled{-3}, 0, \textcircled{-4}\} = \boxed{0}$$

In general, the player A should try to maximize his least gains or to find out "$\max_i \max_j v_{ij}$"

Player B, on the other hand, can argue similarly to keep A's gain the minimum. He realizes that if he plays his 1st pure strategy, he can loose no more than 5 = max {-3, 2, 5} regardless of A 's selections. Similar arguments can be applied for remaining strategies II and III. Corresponding results are indicated in Table 19-4 by 'column maximum'. The player B will then select the strategy that minimizes his maximum losses. This is given by the strategy II and his corresponding loss is given by

$$\min\{\boxed{5}, \boxed{0}, \boxed{6}\} = \boxed{0}$$

The player A's selection is called the maximin strategy and his corresponding gain is called the maximin value or lower value ($\underline{v}$) of the game. The player B' s selection is called the minimax value or upper value ($\bar{v}$) of the game. The selections made by player A and B are based on the so called minimax (or maximin) criterion. It is seen from the governing conditions that the minimax (upper) value $\bar{v}$ is greater than or equal to the maximin (lower) value $\underline{v}$. In the case where equality holds i.e.,

$\max_i \min_j v_{ij} = \min_j \max_i v_{ij}$ or $\underline{v} = \bar{v}$, the corresponding pure strategies are called the 'optimal' strategies and the game is said to have a saddle point. It may not always happen as shown in the following example.

Example 10.4.2. Consider the following game :

		B: j = 1	2	3
A	i = 1	3	−4	8
	2	−8	5	−6
	3	6	−7	6

As discussed in **Example 10.4.1** $\max_i \min_j v_{ij} = 4$ $\min_j \max_i v_{ij} = 5$.

Also, $\max_i \min_j v_{ij} < \min_j \max_i v_{ij}$

10.5 Saddle Point, Optimal Strategies and Value of the Game

Definitions :

Saddle Point. ***A saddle point*** of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.

Mathematically, if a pay off matrix $\{v_{ij}\}$ is such that $\max_i [\min_j \{v_{ij}\}] = \min_j [\max_i \{v_{ij}\}]$ $= v_{rs}$ (say), then the matrix is said to have a saddle point (r, s).

Optimal Strategies. If the payoff matrix $\{v_{ij}\}$ has the saddle point (r, s), then the players (A and B) are said to have rth and sth optimal strategies, respectively.

3. Value of Game. The payoff (v_{rs}) at the saddle point (r, s) is called the value of game and it is obviously equal to the maximin ($\underline{v}$) and minimax value ($\bar{v}$) of the game.

A game is said to be a **fair game** if $\bar{v} = \underline{v} = 0$. A game is said to be **strictly determinable** if $\bar{v} = v = \underline{v}$.

Note. A saddle point of a payoff matrix is, sometimes, called the equilibrium point of the payoff matrix.

In **Example 1,** $\underline{v} = \overline{v} = 0$. This implies that the game has a saddle point given by the entry (2,2) of payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy. In this example, it is also seen that no player can improve his position by other strategy.

In general, a matrix need not have a saddle point as defined above. Thus, these definitions of optimal strategy and value of the game are not adequate to cover all cases so need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such generalization.

Rules for Determining a Saddle Point:

1. Select the minimum element of each row of the pay off matrix and mark them by ‘O’.
2. Select the greatest element of each column of the payoff matrix and mark them by '□'.
3. If there appears an element in the payoff matrix marked by 'O' and '□' both, the position of that element is a saddle point of the pay off matrix.

Examination Problems

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games :

(i) Player B

$$\text{Player A} \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}$$

(ii) Player B

$$\text{Player A} \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix}$$

2. Consider the game G with the following payoff matrix:

Player B

$$\text{Player A} \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix}$$

(i) Show that G is strictly determinable whatever μ may be.

(ii) Determine the value of G.

3. Find out whether there is any saddle point in the following problem :

Player B

$$\text{Player A} \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$$

4. For the game with payoff matrix:

Player B

$$\text{Player A} \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

determine the best strategies for players A and Band also the values of the game for them. Is this game (i) fair (ii) strictly determinable ?

10.6 Solution of games with Saddle Points

To obtain a solution of a rectangular game, it is feasible to find out :

(i) the best strategy for player A (ii) the best strategy for player B, and (iii) the value of the game (v_{rs},).

It is already seen that the best strategies for players A and B will be those which correspond to the row and column, respectively, through the saddle point. The value of the game to the player A is the element at the saddle point, and the value to the player B will be its negative.

10.6.1 Worked out Examples

Example 10.6.1. Player A can choose his strategies from $\{A_1, A_2, A_3\}$ only, while B can choose from the set (B_1, B_2) only. The rules of the game state that the payments should be made in accordance with the selection of strategies :

Strategy Pair Selected	***Payments to be Made***	***Strategy Pair Selected***	***Payments to be Made***
(A_1, B_1)	*Payer A Pays Re. 1 to player B*	(A_2, B_2)	*Player B pays Rs 4 to player A.*
(A_1, B_2)	*Player B pays Rs. 6 to player A*	(A_3, B_1)	*Player A pays Rs 2 to player B*
(A_2, B_1)	*P layer B pays Rs 2 to player A*	(A_3, B_2)	*Player A pays Rs. 6 to player B*

What strategies should A and B play in order to get the optimum benefit of the play ?

Solution. With the help of above rules the following payoff matrix is constructed :

		Player *B*	
		B_1	B_2
	A_1	−1	6
Player *A*	A_2	2	4
	A_3	−2	−6

The payoffs marked 'O' represent the minimum payoff in each row and those marked '□' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Thus, the optimum solution to the game is given by :

(i) the optimum strategy for player A is A_2 ;

(ii) the optimum strategy for player B is B_1 ; and

(iii) the value of the game is Rs. 2 for player A and Rs. (- 2) for player B.

Also, since $v \neq 0$, the game is not fair, although it is strictly determinable.

Example 10.6.2. The payoff matrix of a game is given. Find the solution of the game to the player A and B.

$$A\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}\overset{B}{\begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ \end{array}}\begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix}$$

Solution. First find out the saddle point by encircling each row minima and putting squares around each column maxima.

The saddle point thus obtained is shown by having a circle and square both

OPTIMUM STRATEGY FOR B ↑ (column III)

	I	II	III	IV	V	ROW MINIMUM
I	(−2)	0	0	[5]	3	(−2)
II	3	2	[(1)]	2	2	[(1)] → MAXIMIN VALUE (v)
III	(−4)	−3	0	−2	[6]	−4
IV	[5]	[3]	−4	2	(−6)	(−6)
COLUMN MAXIMUM	[5]	[3]	[(1)]	[5]	[6]	

← OPTIMUM STRATEGY FOR A (row II)

Minimax Value ($\bar{v}$)

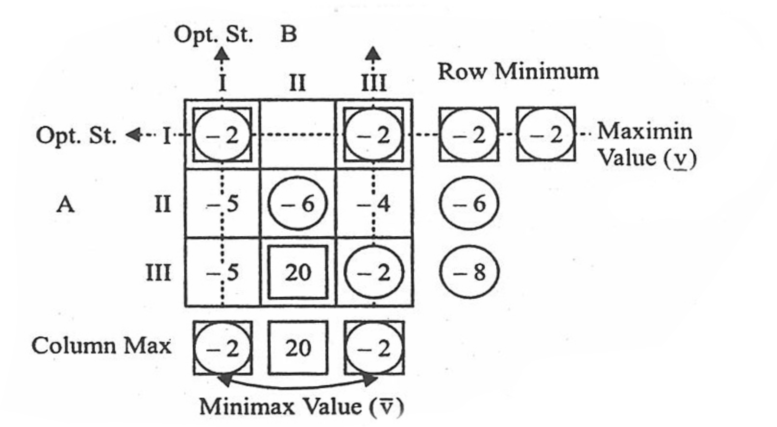
Hence, the solution to this game is given by, (i) the best strategy for player A is 2nd; (ii) the best strategy for player B is 3rd; and (iii) the value of the game is 1 to player A and - 1 to player B.

Example 10.6.3. Solve the game whose payoff matrix is given by

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{bmatrix} \end{array}$$

Solution.

This game has two saddle points in positions (1, 1) and (1, 3). Thus, the solution to this game is given by,



(i) the best strategy for the player A is I, (ii) the best strategy for the player B is either I or III, i.e. the player B can use either of the two strategies (I, III), and (iii) the value of the game is - 2 for player A and + 2 for player B.

10.7 Summary

In this lesson definitions of a competitive game and other basic terms are given. Also the characteristic of a game and solution of games with saddle point are explained with illustrations.

10.8 Exercise

Find the saddle point (or point) and hence solve the following games :

1.

Player B

B_1 B_2 B_3

$$\text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}$$

2. **Player B**

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix} \end{array}$$

3. **B**

$$\begin{array}{cc} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \mathbf{A} \begin{array}{c} I \\ II \\ III \end{array} & \begin{bmatrix} -5 & 2 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{bmatrix} \end{array}$$

4. **B**

$$\begin{array}{cc} & \begin{array}{ccccc} I & II & III & IV & V \end{array} \\ \text{A} \begin{array}{c} I \\ II \\ III \\ IV \end{array} & \begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 4 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \end{array}$$

5. For the following payoff matrix for firm A, determine the optimal strategies for both the firms and the value of the game (using maximin – minimax principle) :

Firm B

$$\text{Firm A} \begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix}$$

6. Solve the games whose payoff matrices are given below :

(a) **Player B**

B_1 B_2 B_3

$$\textbf{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \begin{bmatrix} -3 & -1 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{bmatrix}$$

(b) **Player B**

$$\textbf{Player A}\quad \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} -5 & 5 & 0 & 7 \\ 2 & 6 & 1 & 8 \\ -4 & 0 & 1 & -3 \end{bmatrix}$$

10.9 References :

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Lesson – 12

Graphical and Linear Programming Methods

12.0 Objective :

- 2 x n and m x 2 games
- Graphical method
- 3 x 3 games
- Linear Programming Method

Structure

12.1 Introduction

The optimal strategies for a (2 x n) or (m x 2) matrix game can be located easily by a simple graphical method. This method enables us to reduce the 2 x n or m x 2 matrix game to 2 x 2 game that could be easily solved by the earlier methods.

If the graphical method is used for a particular problem, then the same reasoning can be used to solve any game with mixed strategies that has only two undominated pure strategies for one of the players.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. It is clear that if one player has only two strategies, the other will also use two strategies. Hence, graphical method can be used to find two strategies of the player. The method can be applied to 3 x n or m x 3 games also by carefully drawing three dimensional diagram.

12.2 Graphical Method for 2 x n Games

Consider the (2 x n) game, assuming that the game does not have a saddle, point.

Since the player A has two strategies, it follows that $x_2 = 1 - x_1$, $x_1 \geq 0$, $x_2 \geq 0$. Thus, for each of the pure strategies available to the player B, the expected payoff for the player A, would be as follows :

Table 12.1

			B				
			y_1	y_2	y_3	…	y_n
			B_1	B_2	B_3	…	B_n
A	x_1	A_1	v_{11}	v_{12}	v_{13}	…	v_{1n}
	$1-x_1$	A_2	v_{21}	v_{22}	v_{23}	…	v_{2n}

Table 12.2

B's Pure Strategies	A's Expected Payoff $E_i(x_1)$
B_1	$v_{11}x_1 + v_{21}(1-x_1) = (v_{11}-v_{21})x_1 + v_{21}$
B_2	$v_{12}x_1 + v_{22}(1-x_1) = (v_{12}-v_{22})x_1 + v_{22}$
:	: :
B_n	$v_{1n}x_1 + v_{2n}(1-x_1) = (v_{1n}-v_{2n})x_1 + v_{2n}$

This shows that the player A's expected payoff varies linearly with x_1.

According to the maximin criterion for mixed strategy games; the player A should select the value of x_1 so as to maximize his minimum expected payoff. This may be done by plotting the following straight lines ;

$$E_1(x_1) = (v_{11}-v_{21})\, x_1 + v_{21}$$

$$E_2(x_1) = (v_{12}-v_{22})\, x_1 + v_{22}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$E_n(x_1) = (v_{1n}-v_{2n})\, x_1 + v_{2n}$$

as functions of x_1. The lowest boundary of these lines will give the minimum expected payoff as function of x_1. The highest point on this lowest boundary would then give the maximin expected payoff and the optimum value of x_1 (= x_1^*).

Now determine only two strategies for player B corresponding to those two lines which pass through the maximin point P. This way, it is possible to reduce the game to 2 x 2 which can be easily solved either by using formulae given in arithmetic method.

12.2.1 Outlines of Graphical Method :

To determine maximin value $\underline{v}$, we take different values of x_1 on the horizontal line and values of $E(x_1)$ on the vertical axis. Since $0 \le x_1 \le 1$, the straight line $E_j(x_1)$ must pass through the points $\{0, E_j(0)\}$ and $\{1, E_j(1)\}$, where $E_j(0) = v_{2j}$ and $E_j(1) = v_{1j}$. Thus the lines $E_j(x_1) = (v_{1j} - v_{2j})\, x_1 + v_{2j}$ for j = 1,2, ..., n can be drawn as follows :

Step 1. Construct two vertical axes, axis 1 at the point x_1=0 and axis 2 at the point x_1=1.

Step 2. Represent the payoffs v_{2j}, j = 1, 2,..., n on axis 1 and payoff v_{1j}, j = 1, 2, ..., n on axis 2.

Step 3. Join the point representing v_{1j} on Axis 2 to the point representing v_{2j} on axis 1. The resulting straight line is the expected payoff line $E_j(x_1)$, j = 1,2, ... ,n.

Step 4. Mark the lowest boundary of/the lines $E_j(x_1)$ so plotted, by thick line segments. The highest point on this lowest boundary gives the maximin point P and identifies the two critical moves of player B.

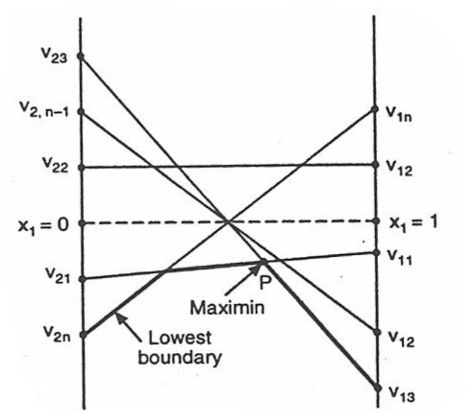

Fig 12.1 Graphical solution of 2 x n games.

If there are more than two lines passing through the maximin point P, there are ties for the optimum mixed strategies for player B. Thus any two such lines with opposite sign slopes will define an alternative optimum for B.

12.3 Graphical Solution of m x 2 Games

The (m x 2) games are also treated in the like manner except that the minimax point P is the lowest point on the uppermost boundary instead of highest point on the lowest boundary.

From this discussion, it is concluded that any (2 x n) or (m x 2) game is basically equivalent to a (2 x 2) game.

Now each point of the discussion is explained by solving numerical examples for (2 x n) and (m x 2) games.

12.4 Examples

Example 12.4.1 Solve the following (2 x 3) game graphically.

Table 12.3

			y_1 I	y_2 II	y_3 III
A	x_1	I	1	3	11
	$1-x_1$	II	8	5	2

Solution. This game does not have a saddle point. Thus the player A's expected payoff corresponding to the player B's pure strategies are given (Table 12.4).

Three expected payoff lines are :

$$E(x_1) = -7x_1 + 8,\ E(x_1) = -2x_1 + 5 \text{ and } E(x_1) = 9x_1 + 2$$

and can be plotted on a graph as follows [see Fig. 12-2]

Table 12.4

B's Pure Strategies	A's Expected Payoff $E(x_1)$
I	$E(x_1) = 1x_1 + 8(1 - x_1) = -7x_1 + 8$
II	$E(x_1) = 3x_1 + 5.(1 - x_1) = -2x_1 + 5$
III	$E(x_1) = 11x_1 + 2(1 - x_1) = 9x_1 + 2$

First, draw two parallel lines one unit apart and mark a scale on each. These two lines will represent two strategies available to the player A. Then draw lines to represent each of player B's strategies.

For example, to represent the player B' s 1st strategy, join mark 1 on scale I to mark 8 on scale II; to represent the player B's second strategy, join mark 3 on scale I to mark 5 on scale II, and so on. Since the expected payoff $E(x_1)$ is the function of x_1 alone, these three expected payoff lines can be drawn by taking x_1 as x-axis and $E(x_1)$ as y- axis.

Points A, P, B, C on the lowest boundary (shown by a thick line in Fig. 12.2) represent the lowest possible expected gain to the player A for any value of x_1 between 0 and 1. According to the maximin criterion, the player A chooses the best of these worst outcomes.

Clearly, the highest point P on the lowest boundary will give the largest expected gain PN to A. So best strategies for the player B are those which pass through the point P. Thus, the game is reduced to 2 x 2 (Table 12.5).

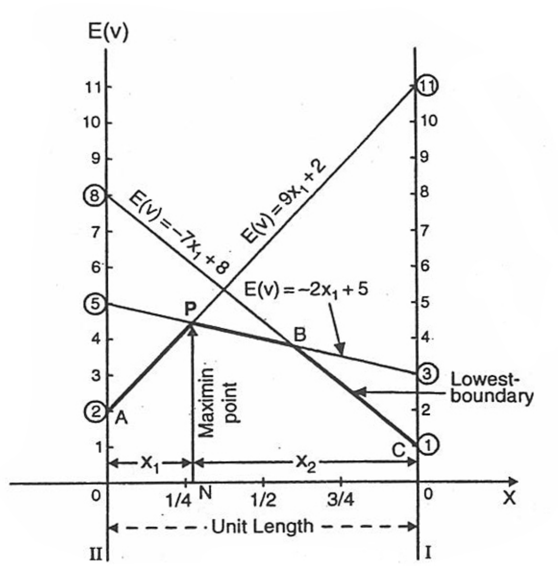

Fig. 12.2 Graphical representation for solving (2 x n) game

Now, by solving the simultaneous equations

$3x_1 + 5x_2 = v$, $11x_1 + 2x_2 = v$, $x_1 + x_2 = 1$ (For player A)

$3y_2 + 11y_3 = v$, $5y_2 + 2y_3 = v$, $y_2 + y_3 = 1$ (For player B)

the solution of the game is obtained as follows :

(i) The player A chooses the optimal mixed strategy $(x_1, x_2) = (3/11, 8/11)$,

(ii) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3) = (0, 2/11, 9/11)$,

(iii) The value of the game to the player A is $v = 49/11$.

Table 12.5

		B II	B III
A	I	3	11
	II	5	2

Example 12.4.2 Solve the game graphically whose payoff matrix for the player A is given in Table 12-6 :

Table 12.6

		B I	B II
	I	2	4
A	II	2	3
	III	3	2
	IV	-2	6

Solution. The game does not have a saddle point. Let y_1 and y_2 $(= 1-y_1)$ be mixed strategies of the player B.

The four straight lines thus obtained are :

$E(y_1) = -2y_1 + 4$, $E(y_1) = -y_1 + 3$,

$E(y_1) = y_1 + 2$, $E(y_1) = -8y_1 + 6$,

and these are plotted in Fig. 12-3. In this case, the minimax point is determined as the lowest point P on the uppermost boundary. Lines intersecting at the minimax point P correspond to the player A's pure strategies I and III. This indicates $x_2 = x_4 = 0$. Thus, the reduced game is given in Table 12.8.

Table 12.7

A's Pure Strategies	B's Expected Payoff E (y_1)
I	$E(y_1) = 2y_1 + 4(1 - y_1)$
II	$E(y_1) = 2y_1 + 3(1 - y_1)$
III	$E(y_1) = 3y_1 + 2(1 - y_1)$
IV	$E(y_1) = - 2y_1 + 6(1 - y_1)$

Table 12.8

		B I	B II
A	I	2	4
	III	3	2

Now, solve this (2 x 2) game by solving the simultaneous equations :

$2x_1 + 3\,x_3 = v,\ 4x_1 + 2x_3 = v,\ x_1 + x_3 = 1$ (For A)

$2y_1 + 4y_2 = v,\ 3y_1 + 2y_2 = v,\ y_1 + y_2 = 1$ (For B) to get the solution :

(i) The player A chooses the optimal mixed strategy, $(x_1, x_2, x_3, x_4) = (1/3, 0, 2/3, 0)$.

(ii) The player B chooses the optimal mixed strategy, $(y_1, y_2) = (2/3, 1/3)$

(iii) The value of the game to the player A is $v = 8/3$.

Remark. If there are more than two lines passing through the maximin (minimax) point P, this would imply that there are many ties for optimal mixed strategies for the player B. Thus, any two lines having opposite signs for their slopes will define an alternative optimum solutions.

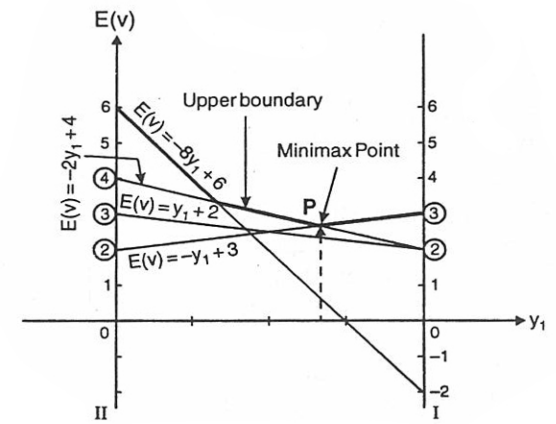

Fig. 12.3. Graphical representation for (m x 2) game.

Example 12.4.3 Solve the following (2 x 4) game.

		B			
		I	II	III	IV
A	I	2	2	3	–1
	II	4	3	2	6

Solution. This game does not have saddle point. Thus, the player A's expected payoffs corresponding to the player B's pure strategies are given below :

Table 12.9

B's Pure Strategies	A's Expected Payoff E (x_1)
I	$E(x_1) = -2x_1 + 4$
II	$E(x_1) = -x_1 + 3$
III	$E(x_1) = x_1 + 2$
IV	$E(x_1) = -7x_1 + 6$

These four straight lines are then plotted in Fig. 12-4.

It follows from Fig. 12.4 that maximin occurs at $x_1 = 1/2$. This is the point of intersection of any two of the lines joining (2) to (3); (3) to (2); (6) to (- 1). As mentioned in the above remark, any two lines having opposite signs for their slopes will define an alternative optimum solution. The combination of lines $E(x_1) = - x_1 + 3$ and $E(x_1) = - 7 x_1 + 6$ must be excluded as being non-optimal. So the game can be reduced to (2 x 2) in the following manner.

It is also important to note that the average of above two payoff matrices (Table 12.10and 12.11) will also be the additional possibility of reducing the game to (2 x 2). Thus, the additional possibility of (2 x 2) game will also yield a new optimal solution which mixes three strategies II, III and IV. Then (2 x 2) game is solved by solving the governing simultaneous equations.

Table 12.10

1st possibility

		B: II	B: III
A	I	2	3
	II	3	2

Table 12.11

2nd possibility

		B: III	B: IV
A	I	3	-1
	II	2	6

Table 12.12

Additional possibility

		B	
A	I	$\frac{2+3}{2} = 5/2$	$\frac{3-1}{2} = 1$
	II	$\frac{3+2}{2} = 5/2$	$\frac{2+6}{2} = 4$

The first possibility of the. solution of (2 x 4). game with reduced (2 x 2) matrix is :

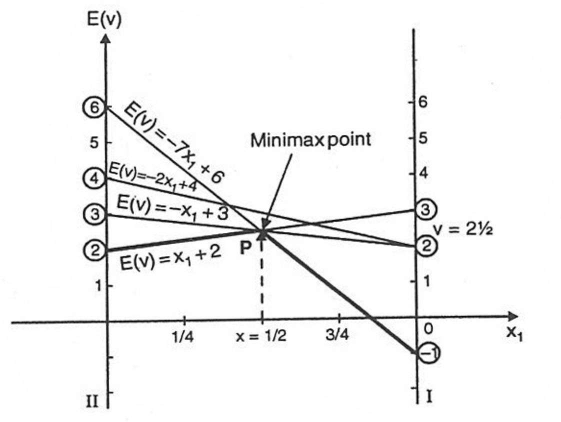

Fig 12.4

(i) The player A chooses the optimal mixed strategy (x_1, x_2) = (l/2, 1/2).

(ii) The player B chooses the optimal mixed strategy (y_1, y_2, y_3, y_4) = (0, 1/2, 1/2, 0).

(iii) The value of the game to the player A is 2 1/2.

Similarly, solution of the game with reduced matrix in 2nd possibility is :

(i) The player A chooses the optimal mixed strategy (1/2, 1/2).

(ii) The player B chooses the optimal mixed strategy (0, 0, 7/8, 1/8).

(iii) The value of the game to the player A is 2 1/2.

Solution of the game with reduced matrix in additional possibility can be obtained easily, because it has a saddle point 5/2. So, the value of the game to the player A is 5/2. It has been observed that the formulae will yield an incorrect solution in this case.

Example 12.4.4 : Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs. 400 per colour set and Rs. 300 per black & white set. Firm B can, on the other hand, make either 300 colour sets, or 150 colour and 150 black & white sets, or 300 black &. while sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set.

Write the pay-off matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game.

Solution. For firm A, the strategies are :

A_1 : make 150 colour sets,

A_2 : make 150 black & white sets.

For firm B, the strategies are :

B_1 : make 300 colour sets,

B_2 : make 150 colour and 150 black & white sets,

B_3 : make 300 black and white sets.

For the combination A_1B_1, the profit to firm A would be: $\frac{150}{150+300}$ x 150 x 400

= Rs. 20,000

wherein (150/150 + 300) represents share of market for A, 150 is the total market for colour television sets and 400 is the profit per set. In a similar way, other profit figures may be obtained as shown in the following pay-off matrix :

B's strategy

$$\text{A's strategy}\quad \begin{matrix} & B_1 & B_2 & B_3 \\ A_1 & 20{,}000 & 30{,}000 & 60{,}000 \\ A_2 & 45{,}000 & 45{,}000 & 30{,}000 \end{matrix}$$

Since no saddle point exists, we shall determine optimum mixed strategy. The data are plotted on graph as shown in the adjoining Fig. 12-5 :

Lines joining the pay-offs on axis I with the pay-offs on axis II represents each of B's strategies. Since firm A wishes to maximize his minimum expected pay-off, we consider the highest point P on the lower envelope of A's expected pay-off equation. This point P represents the maximin expected value of the game for firm A. The lines B_1 and B_3

passing through P, define the relevant moves B_1 and B_3 that alone from B needs to adopt. The solution to the original 2 x 3 game, therefore, reduces to that of the simple game with 2 x 2 pay-off matrix as follows :

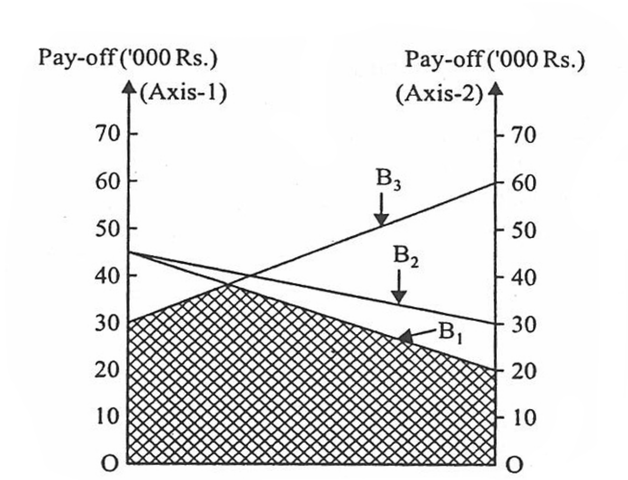

Fig. 12.5 : Graphic solution to the Game

B's strategy

$$\text{A's strategy} \quad \begin{matrix} & B_1 & B_2 \\ A_1 & 20{,}000 & 60{,}000 \\ A_2 & 45{,}000 & 30{,}000 \end{matrix}$$

Correspondingly,

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{30{,}000 - 45{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} = \frac{3}{11}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{30{,}000 - 60{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} = \frac{6}{11}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{21}) - (a_{12} + a_{21})} = \frac{20{,}000x30{,}000 - 60{,}000x45{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} \cong 38{,}182$$

12.5 Linear Programming Method

The two-person zero-sum games can also be solved by linear programming. The major advantage of using linear programming technique is to solve mixed-strategy games of larger dimension payoff matrix.

To illustrate the transformation of a game problem to a linear programming problem, consider a payoff matrix of size m x n. Let a_{ij} be the element in the ith row and jth column of game payoff matrix, and letting pi be the probabilities of m strategies (i = 1, 2, ..., m) for player A. Then the value of the game (expected gains) for player A, for each of player B's strategies will be

$$V_j = \sum_{i=1}^{m} p_i a_{ij}, \quad j = 1,2,\ldots,n$$

The aim of player A is to select a set of strategies with probability pi,(i = 1, 2, ...,m) on any play of game such that he can maximize his minimum expected gains.

Now to obtain values of probability p*i*, the value of the game to player A for all strategies by player B must be at least equal to V. Thus to maximize the minimum expected gains, it is necessary that

$$a_{11}p_1 + a_{21} p_2 + \ldots + a_{m1} p_m \geq V$$
$$a_{12}p_1 + a_{22} p_2 + \ldots + a_{m2} p_m \geq V$$
$$\vdots$$
$$a_{1n}p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m \geq V$$

where $p_1 + p_2 + \ldots + p_m = 1; p_i \geq 0$

Dividing both sides of the *m* inequalities and equation by *V*. The division is valid as long as V > 0. In case, V < 0, the direction of inequality constraints must be reversed. But if V = 0; division would be meaning less. In this case a constant can be added to all entries of the matrix ensuring that the value of the game (V) for the revised matrix becomes more than zero. After optimal solution is obtained, the true value of the game is obtained by substracting the same constant value. Let $pi/V = x_i$, (≥ 0). Then we have

$$a_{11} \frac{P_1}{V} + a_{21} \frac{P_2}{V} + \ldots + a_{m1} \frac{P_m}{V} \geq 1$$

$$a_{12} \frac{P_1}{V} + a_{22} \frac{P_2}{V} + \ldots + a_{m2} \frac{P_m}{V} \geq 1$$

$$\vdots$$

$$a_{1n}\frac{P_1}{V} + a_{2n}\frac{P_2}{V} + \ldots + a_{mn}\frac{P_m}{V} \geq 1$$

$$\frac{P_1}{V} + \frac{P_2}{V} + \ldots + \frac{P_m}{V} = 1$$

Since the objective of player A is to maximize the value of the game, V, which is equivalent to minimizing 1/V. The resulting linear programming problem can be stated as

$$\text{Minimize } Z_p = \frac{1}{V} = x_1 + x_2 + \ldots + x_m$$

subject to the constraints

$$a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m \geq 1$$

$$a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m \geq 1$$

:

:

$$a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{mn}x_m \geq 1$$

$$x_1, x_2, \ldots x_m \geq 0$$

and

Similarly, player B has a similar problem with the inequalities of the constraints reversed, i.e. minimizes the expected loss. Since, minimizing of V is equivalent to maximizing l/V, therefore, the resulting linear programming problem can be stated as:

$$\text{Minimize } Z_q = \frac{1}{V} = y_1 + y_2 + \ldots + y_n$$

Subject to the constraints

$$a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \leq 1$$

$$a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \leq 1$$

:

:

$$a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \leq 1$$

$$y_1, y_2, \ldots y_n \geq 0$$

where, $y_j = \frac{q_j}{V} \geq 0$; $j = 1, 2, \ldots, n$

It may be noted that the LP problem for player B is the dual of LP problem for player A. Therefore, solution of the dual problem can be obtained from the primal simplex table. Since for both the players Zp = Zq, therefore the expected gain to player A in the game will be exactly equal to expected loss to player B.

Remark: Linear programming technique requires all variables to be non-negative and therefore to obtain a non-negative value V of the game, the data to the problem, i.e. a_{ij} in the payoff table should all be non-negative. If there are some negative elements in the payoff table, a constant to every element in the payoff table must be added so as to make the smallest element zero; the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

12.6 Workout Problem

Example 12.6.1: For the following payoff matrix, transform the zero-sum game into an equivalent linear programming problem and solve it by using simplex method.

		Player B		
		B_1	B_2	B_3
Player A	A_1	1	−1	3
	A_2	3	5	−3
	A_3	6	2	−2

Solution: The first step is to find out the saddle point (if any) .in the payoff matrix as shown below:

		Player B			
		B_1	B_2	B_3	Row minima
Player A	A_1	1	−1	3	−1 ← Maximin
	A_2	3	5	−3	−3
	A_3	6	2	−2	−2
Column maxima		6	5	3 ← Minimax	

The given game payoff matrix does not have a saddle point. Since the maximin value is -1, therefore it is possible that the value of game (V) may be negative or zero because -1 < V < 1. Thus a constant which is atleast equal to the negative of maximin value, i.e. more than - 1 is added to all the elements of the payoff matrix. Thus adding a constant number, 4 to all the elements of the payoff matrix, the payoff matrix

becomes:

Player B

		B_1	B_2	B_3	Probability
	A_1	5	3	7	p_1
Player A	A_2	7	9	1	p_2
	A_3	10	6	2	p_3
Probability		q_1	q_2	q_3	

Let p_i (i = 1,2,3) and q_j (j = 1, 2, 3) be the probabilities of selecting strategies A_i (i = 1, 2, 3) and B_j (j =1,2, 3) by players A and B respectively.

The expected gain for player A will be as follows:

$5p_1 + 7p_2 + 10p_3 \geq V$ (if B uses strategy B_1)

$3p_1 + 9p_2 + 6p_3 \geq V$ (if B uses strategy B_2)

$7p_1 + p_2 + 2p_3 \geq V$ (if B uses strategy B_3)

$P_1 + P_2 + P_3 \geq 1$

$P_1, P_2, P_3 > 0$

and

Dividing each inequality and equality by V, we get,

$$5\frac{P_1}{V} + 7\frac{P_2}{V} + 10\frac{P_3}{V} \geq 1$$

$$3\frac{P_1}{V} + 9\frac{P_2}{V} + 6\frac{P_3}{V} \geq 1$$

$$7\frac{P_1}{V} + \frac{P_2}{V} + 2\frac{P_3}{V} \geq 1$$

$$\frac{P_1}{V} + \frac{P_2}{V} + \frac{P_3}{V} = \frac{1}{V}$$

In order to simplify, we define new variables:

$$x_1 = \frac{P_1}{V}, \ x_2 = \frac{P_2}{V} \text{ and } x_3 = \frac{P_3}{V}$$

The problem for player A, therefore becomes,

Minimize $Z_p = 1/V = x_1 + x_2 + x_3$

subject to the constraints

$5x_1 + 7x_2 + 10x_3 \geq 1$

$3x_1 + 9x_2 + 6x_3 \geq 1$

$7x_1 + x_2 + 2x_3 \geq 1$

and $x_1, x_2, x_3 \geq 0$

Player B's objective is to minimize his expected losses which can be reduced to minimizing the value of the game V. Hence, the problem of player B can be expressed as follows:

Maximize $Z_q = 1/V = y_1 + y_2 + y_3$

subject to the constraints

$5y_1 + 3y_2 + 7y_3 \leq 1$

$7y_1 + 9y_2 + y_3 \leq 1$

$10y_1 + 6y_2 + 2y_3 \leq 1$

and $y_1, y_2, y_3 \geq 0$

where $y_1 = \frac{q_1}{V}, \ y_2 = \frac{q_2}{V}$ and $y_3 = \frac{q_3}{V}$

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal.

To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities. The problem becomes

$$\text{Maximize } Z_q = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

$$5\,y_1 + 3\,y_2 + 7\,y_3 + s_1 = 1$$

$$7\,y_1 + 9y_2 + y_3 + s_2 = 1$$

$$10y_1 + 6y_2 + 2y_3 + s_3 = 1$$

and $y_1, y_2, y_3, s_1, s_2, s_3, \geq 0$

The initial solution shown in Table 12.13.

Table 12.13 Initial Solution

		$c_j \rightarrow$	1	1	1	0	0	0	
c_B	*Variables in basis* B	*Solution values* $y_B (= b)$	y_1	y_2	y_3	s_1	s_2	s_3	*Min. ratio* y_B / y_1
0	s_1	1	5	3	7	1	0	0	1/5
0	s_2	1	7	9	1	0	1	0	1/7
0	s_3	1	(10)	6	2	0	0	1	1/10 →
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	1 ↑	1	1	0	0	0	

Proceeding with usual simplex method, the optimal solution is shown in Table 12.14.

Table 12.14 Optimal Solution

		$c_j \rightarrow$	1	1	1	0	0	0
c_B	Variables in basis B	Solution values $y_B (= b)$	y_1	y_2	y_3	s_1	s_2	s_3
1	y_3	1/10	2/5	0	1	3/20	−1/10	0
1	y_2	1/10	11/15	1	0	−1/60	7/60	0
0	s_1	1/5	24/5	0	0	−1/5	−3/5	1
Z = 1/5		z_j	17/15	0	0	2/15	1/15	0
		$c_j - z_j$	−2/15	0	0	−2/15	−1/15	0

The optimal solution (mixed strategies) for B is: y_1 =0; y_2 = 1/10 and the expected value of the game is: Z = 1/V- constant (= 4) = 5-4 = 1

These solution values are now converted back into the original variables: l/V = l/5, then V= 5

$$y_1 = \frac{q_1}{V}, \text{ then } q_1 = y_1 \text{ x } V = 0$$

$$y_2 = \frac{q_2}{V}, \text{ then } q_2 = y_2 \text{ x } V = \frac{1}{10} \text{ x } 5 = \frac{1}{2}$$

$$y_3 = \frac{q_3}{V}, \text{ then } q_3 = y_3 \text{ x } V = \frac{1}{10} \text{ x } 5 = \frac{1}{2}$$

The optimal strategies for player A are obtained from the z_j - c_j row of the Table 12.14.

$$x_1 = \frac{2}{15}, x_2 = \frac{1}{15} \text{ and } x_3 = 0$$

Then $p_1 = x_1 \text{ x } V = \frac{2}{15} \text{ x } 5 = \frac{2}{3}$

$$P_2 = x_2 \text{ x } V = \frac{1}{15} \text{ x } 5 = \frac{1}{3}$$

$$P_3 = x_3 \text{ x } V = 0$$

Hence the probabilities of using strategies for both the players are :

Player A = (2/5, 1/3, 0) ; Player B = (0, ½, ½) and value of the game is, V = 1.

Example 12.6.2. Solve (3 x 3) game by the simplex method of linear programming whose payoff matrix, is given below.

Player B

		1	2	3
	1	3	-1	-3
Player A	2	-3	3	-1
	3	-4	-3	3

Solution. First apply minimax (maximin) criterion to find the minimax ($\bar{v}$) and maximin ($\underline{v}$) value of the game. Thus, the following matrix is obtained (Table 12.15).

Table 12.15

A \ B	1	2	3	Row Minimum.
1	3	−1	−3	−3
2	−3	3	−1	−3
3	−4	−3	3	−4
Column Maximum	3	3	3	

Maximin Value ($\underline{v}$)

Minimax Value ($\bar{v}$)

Since, maximin value is - 3, it is possible that the value of the game (v) may be negative or zero because -3 < v < 3.

Thus, a constant c is added to all elements of the matrix which is at least equal to the - ve of the maximin value, i.e. $c \geq 3$. Let c = 5. The matrix is shown in Table 12.16 Now, following the reasoning, the player B's linear programming problem is :

Maximize $y_0 = Y_1 + Y_2 + Y_3$ subject to the constraints :

$8Y_1, + 4Y_2 + 2Y_3 \leq 1$, $2Y_1 + 8Y_2 + 4Y_3 \leq 1$, $1\ Y_1 + 2Y_2 + 8Y_3 \leq 1$, $Y_1 \geq 0$, $Y_2 \geq 0$,

$Y_3 \geq 0$

Table 12.16

B

	1	8	4	2
A	2	2	8	4
	3	1	2	8

Introducing slack variables, the constraint equations become:

$$\left.\begin{aligned} 8Y_1 + 4Y_2 + 2Y_3 + Y_4 \quad\quad &= 1 \\ 2Y_1 + 8Y_2 + 4Y_3 \quad + Y_5 &= 1 \\ 1Y_1 + 2Y_2 + 8Y_3 \quad + Y_6 &= 1 \end{aligned}\right\}$$

$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \geq 0.$

Now the following simplex table is formed.

Table 12.17. Simplex Table

B	C_B	Y_B	α_1	α_2	α_3	α_4 (β_1)	α_5 (β_2)	α_6 (β_3)	Min. Ratio (Y_B/α_k)
		$c_j \rightarrow$	1	1	1	0	0	0	
α_4	0	1	[8] ←	4	2	1	0	0	1/8 ←
α_5	0	1	2	8	4	0	1	0	1/2
α_6	0	1	1	2	8	0	0	1	1/1
	$y_0 = C_B Y_B = 0$		(− 1)* ↑	− 1	− 1	0 ↓	0	0	$\leftarrow \Delta_j = C_B\alpha_j - c_j$
α_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
α_5	0	3/4	0	7	7/2	− 1/4	1	0	3/14
α_6	0	7/8	0	3/2	← [31/4]	1/8	0	1	7/62 ←
	$y_0 = 1/8$		0	− 1/2	(− 3/4)* ↑	1/8	0	0 ↓	$\leftarrow \Delta_j$
α_1	1	3/31	1	14/31	0	4/31	0	− 1/31	3/14
α_5	0	11/31	0	← [196/31]	0	− 6/31	1	− 14/31	11/196 ←
α_3	1	7/62	0	6/31	1	− 1/62	0	4/31	7/12
	$y_0 = 13/62$		0	(− 11/31)* ↑	0	7/62	0 ↓	3/31	$\leftarrow \Delta_j$
α_1	1	1/14	1	0	0	1/7	1/14	0	
α_2	1	11/196	0	1	0	− 3/98	31/196	− 1/14	
α_3	1	5/49	0	0	1	− 1/98	− 3/98	1/7	
	$y_0 = 45/196$		0	0	0	5/49	11/196	1/14	← all $\Delta_j \geq 0$

Thus, the solution for B's original problem is obtained as :

$$y_1^* = \frac{Y_1}{y_0} = \frac{1/14}{45/196} = \frac{14}{15}, \qquad y_2^* = \frac{Y_2}{y_0} = \frac{11/196}{45/196} = \frac{11}{45}$$

$y_3^* = \frac{Y_3}{y_0} = \frac{5/49}{45/196} = \frac{20}{45}$, $\quad v^* = \frac{1}{y_0} - C = \frac{196}{45} - 5 = -\frac{29}{45}$

The optimal strategies for the player A are obtained from the final table of the above problem. This is given by duality rules :

$$X_0 = y_0 = \frac{45}{196},\ X_1 = \Delta_4 = \frac{5}{49},\qquad X_2 = \Delta_5 = \frac{11}{196},\qquad X_3 = \Delta_6 = \frac{1}{14}$$

Hence $\quad X_1^* = \frac{X_1}{x_0} = \frac{20}{45},\ X_2^* = \frac{X_2}{x_0} = \frac{11}{45},\ X_3^* = \frac{X_3}{x_0} = \frac{14}{45},\ v^* = \frac{29}{45}.$

12.7 Summary : In this lesson graphical method to solve 2 x n and m x 2 games and linear programming method to solve 3 x 3 games are discussed in detail along with illustrations.

12.8 Exercise :

1. Obtain the optimal strategies for both players and the value of the game for two-person zero-sum game whose pay off matrix is given as follows, by graphical method.

(a) **Player B**

		B_1	B_2	B_3
Player A	A_1	1	3	11
	A_2	8	5	2

(b) **Player B**

		B_1	B_2	B_3	B_4	B_5	B_6
Player A	A_1	1	3	−1	4	2	−5
	A_2	−3	5	6	1	2	0

2. Obtain the optimal strategies for both players and the value of the game for two-person zero-sum game whose pay off matrix is given as follows, by graphical method.

(a) **Player B**

$$\textbf{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix} \overset{\begin{matrix} B_1 & \quad & B_2 \end{matrix}}{\begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}}$$

(b) **Player B**

$$\textbf{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{matrix} \overset{\begin{matrix} B_1 & \quad & B_2 \end{matrix}}{\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}}$$

3. For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method.

(a) **Player B**

$$\textbf{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \overset{\begin{matrix} B_1 & B_2 & B_3 \end{matrix}}{\begin{bmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{bmatrix}}$$

(b) **Store XYZ**

$$\textbf{Store ABC}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \overset{\begin{matrix} B_1 & \quad B_2 & \quad B_3 \end{matrix}}{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix}}$$

12.9 References :

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Lesson – 12

Graphical and Linear Programming Methods

12.0 Objective :

- 2 x n and m x 2 games
- Graphical method
- 3 x 3 games
- Linear Programming Method

Structure

12.1 Introduction

The optimal strategies for a (2 x n) or (m x 2) matrix game can be located easily by a simple graphical method. This method enables us to reduce the 2 x n or m x 2 matrix game to 2 x 2 game that could be easily solved by the earlier methods.

If the graphical method is used for a particular problem, then the same reasoning can be used to solve any game with mixed strategies that has only two undominated pure strategies for one of the players.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. It is clear that if one player has only two strategies, the other will also use two strategies. Hence, graphical method can be used to find two strategies of the player. The method can be applied to 3 x n or m x 3 games also by carefully drawing three dimensional diagram.

12.2 Graphical Method for 2 x n Games

Consider the (2 x n) game, assuming that the game does not have a saddle, point.

Since the player A has two strategies, it follows that $x_2 = 1 - x_1$, $x_1 \geq 0$, $x_2 \geq 0$. Thus, for each of the pure strategies available to the player B, the expected payoff for the player A, would be as follows :

Table 12.1

			B				
			y_1	y_2	y_3	…	y_n
			B_1	B_2	B_3	…	B_n
A	x_1	A_1	v_{11}	v_{12}	v_{13}	…	v_{1n}
	$1-x_1$	A_2	v_{21}	v_{22}	v_{23}	…	v_{2n}

Table 12.2

B's Pure Strategies	A's Expected Payoff $E_i(x_1)$
B_1	$v_{11}x_1 + v_{21}(1-x_1) = (v_{11}-v_{21})x_1 + v_{21}$
B_2	$v_{12}x_1 + v_{22}(1-x_1) = (v_{12}-v_{22})x_1 + v_{22}$
:	: :
B_n	$v_{1n}x_1 + v_{2n}(1-x_1) = (v_{1n}-v_{2n})x_1 + v_{2n}$

This shows that the player A's expected payoff varies linearly with x_1.

According to the maximin criterion for mixed strategy games; the player A should select the value of x_1 so as to maximize his minimum expected payoff. This may be done by plotting the following straight lines ;

$$E_1(x_1) = (v_{11}-v_{21})x_1 + v_{21}$$

$$E_2(x_1) = (v_{12}-v_{22})x_1 + v_{22}$$

: : :

$$E_n(x_1) = (v_{1n}-v_{2n})x_1 + v_{2n}$$

as functions of x_1. The lowest boundary of these lines will give the minimum expected payoff as function of x_1. The highest point on this lowest boundary would then give the maximin expected payoff and the optimum value of x_1 ($= x_1^*$).

Now determine only two strategies for player B corresponding to those two lines which pass through the maximin point P. This way, it is possible to reduce the game to 2 x 2 which can be easily solved either by using formulae given in arithmetic method.

Outlines of Graphical Method :

To determine maximin value $\underline{v}$, we take different values of x_1 on the horizontal line and values of $E(x_1)$ on the vertical axis. Since $0 \leq x_1 \leq 1$, the straight line $E_j(x_1)$ must pass through the points $\{0, E_j(0)\}$ and $\{1, E_j(1)\}$, where $E_j(0) = v_{2j}$ and $E_j(1) = v_{1j}$. Thus the lines $E_j(x_1) = (v_{1j} - v_{2j})x_1 + v_{2j}$ for j = 1,2, ..., n can be drawn as follows :

Step 1. Construct two vertical axes, axis 1 at the point x_1=0 and axis 2 at the point x_1=1.

Step 2. Represent the payoffs v_{2j}, j = 1, 2,..., n on axis 1 and payoff v_{1j}, j = 1, 2, ..., n on axis 2.

Step 3. Join the point representing v_{1j} on Axis 2 to the point representing v_{2j} on axis 1. The resulting straight line is the expected payoff line $E_j(x_1)$, j = 1,2, ... ,n.

Step 4. Mark the lowest boundary of/the lines $E_j(x_1)$ so plotted, by thick line segments. The highest point on this lowest boundary gives the maximin point P and identifies the two critical moves of player B.

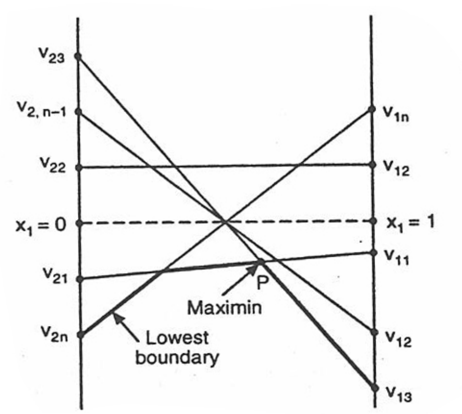

Fig 12.1 Graphical solution of 2 x n games.

If there are more than two lines passing through the maximin point P, there are ties for the optimum mixed strategies for player B. Thus any two such lines with opposite sign slopes will define an alternative optimum for B.

12.3 Graphical Solution of m x 2 Games

The (m x 2) games are also treated in the like manner except that the minimax point P is the lowest point on the uppermost boundary instead of highest point on the lowest boundary.

From this discussion, it is concluded that any (2 x n) or (m x 2) game is basically equivalent to a (2 x 2) game.

Now each point of the discussion is explained by solving numerical examples for (2 x n) and (m x 2) games.

12.4 Examples

Example 12.4.1 Solve the following (2 x 3) game graphically.

Table 12.3

			y_1 I	y_2 II	y_3 III
A	x_1	I	1	3	11
	$1-x_1$	II	8	5	2

Solution. This game does not have a saddle point. Thus the player A's expected payoff corresponding to the player B's pure strategies are given (Table 12.4).

Three expected payoff lines are :

$$E(x_1) = -7x_1 + 8,\ E(x_1) = -2x_1 + 5 \text{ and } E(x_1) = 9x_1 + 2$$

and can be plotted on a graph as follows [see Fig. 12-2]

Table 12.4

B's Pure Strategies	A's Expected Payoff E (x_1)
I	$E(x_1) = 1x_1 + 8(1 - x_1) = -7x_1 + 8$
II	$E(x_1) = 3x_1 + 5.(1 - x_1) = -2x_1 + 5$
III	$E(x_1) = 11x_1 + 2(1 - x_1) = 9x_1 + 2$

First, draw two parallel lines one unit apart and mark a scale on each. These two lines will represent two strategies available to the player A. Then draw lines to represent each of player B's strategies.

For example, to represent the player B' s 1st strategy, join mark 1 on scale I to mark 8 on scale II; to represent the player B's second strategy, join mark 3 on scale I to mark 5 on scale II, and so on. Since the expected payoff $E(x_1)$ is the function of x_1 alone, these three expected payoff lines can be drawn by taking x_1 as x-axis and $E(x_1)$ as y- axis.

Points A, P, B, C on the lowest boundary (shown by a thick line in Fig. 12.2) represent the lowest possible expected gain to the player A for any value of x_1 between 0 and 1. According to the maximin criterion, the player A chooses the best of these worst outcomes.

Clearly, the highest point P on the lowest boundary will give the largest expected gain PN to A. So best strategies for the player B are those which pass through the point P. Thus, the game is reduced to 2 x 2 (Table 12.5).

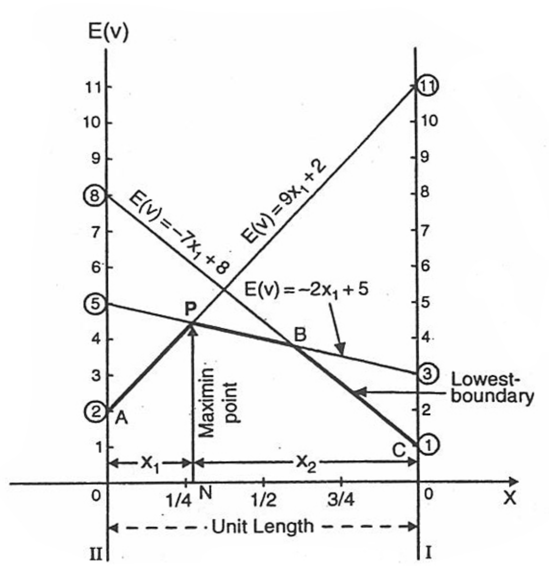

Fig. 12.2 Graphical representation for solving (2 x n) game

Now, by solving the simultaneous equations

$3x_1 + 5x_2 = v$, $11x_1 + 2x_2 = v$, $x_1 + x_2 = 1$ (For player A)

$3y_2 + 11y_3 = v$, $5y_2 + 2y_3 = v$, $y_2 + y_3 = 1$ (For player B)

the solution of the game is obtained as follows :

(i) The player A chooses the optimal mixed strategy $(x_1, x_2) = (3/11, 8/11)$,

(ii) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3) = (0, 2/11, 9/11)$,

(iii) The value of the game to the player A is $v = 49/11$.

Table 12.5

		B: II	B: III
A	I	3	11
	II	5	2

Example 12.4.2 Solve the game graphically whose payoff matrix for the player A is given in Table 12-6 :

Table 12.6

		B: I	B: II
A	I	2	4
	II	2	3
	III	3	2
	IV	-2	6

Solution. The game does not have a saddle point. Let y_1 and y_2 $(= 1-y_1)$ be mixed strategies of the player B.

The four straight lines thus obtained are :

$E(y_1) = -2y_1 + 4$, $E(y_1) = -y_1 + 3$,

$E(y_1) = y_1 + 2$, $E(y_1) = -8y_1 + 6$,

and these are plotted in Fig. 12-3. In this case, the minimax point is determined as the lowest point P on the uppermost boundary. Lines intersecting at the minimax point P correspond to the player A's pure strategies I and III. This indicates $x_2 = x_4 = 0$. Thus, the reduced game is given in Table 12.8.

Table 12.7

A's Pure Strategies	B's Expected Payoff E (y_1)
I	$E(y_1) = 2y_1 + 4(1 - y_1)$
II	$E(y_1) = 2y_1 + 3(1 - y_1)$
III	$E(y_1) = 3y_1 + 2(1 - y_1)$
IV	$E(y_1) = - 2y_1 + 6(1 - y_1)$

Table 12.8

		B I	B II
A	I	2	4
	III	3	2

Now, solve this (2 x 2) game by solving the simultaneous equations :

$2x_1 + 3\,x_3 = v$, $4x_1 + 2x_3 = v$, $x_1 + x_3 = 1$ (For A)

$2y_1 + 4y_2 = v$, $3y_1 + 2y_2 = v$, $y_1 + y_2 = 1$ (For B) to get the solution :

(i) The player A chooses the optimal mixed strategy, $(x_1, x_2, x_3, x_4) = (1/3, 0, 2/3, 0)$.

(ii) The player B chooses the optimal mixed strategy, $(y_1, y_2) = (2/3, 1/3)$

(iii) The value of the game to the player A is $v = 8/3$.

Remark. If there are more than two lines passing through the maximin (minimax) point P, this would imply that there are many ties for optimal mixed strategies for the player B. Thus, any two lines having opposite signs for their slopes will define an alternative optimum solutions.

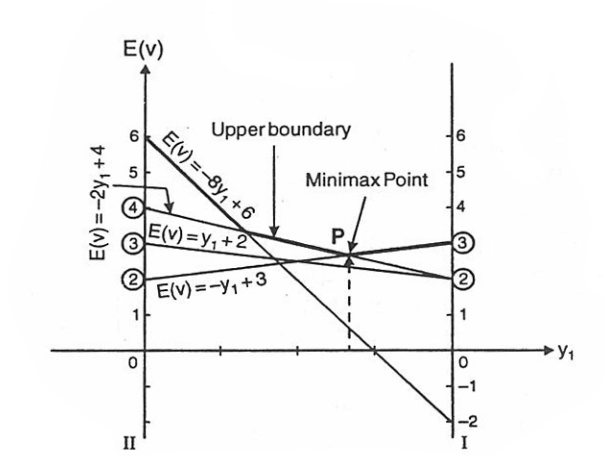

Fig. 12.3. Graphical representation for (m x 2) game.

Example 12.4.3 Solve the following (2 x 4) game.

		B			
		I	II	III	IV
A	I	2	2	3	–1
	II	4	3	2	6

Solution. This game does not have saddle point. Thus, the player A's expected payoffs corresponding to the player B's pure strategies are given below :

Table 12.9

B's Pure Strategies	A's Expected Payoff E (x_1)
I	$E(x_1) = -2x_1 + 4$
II	$E(x_1) = -x_1 + 3$
III	$E(x_1) = x_1 + 2$
IV	$E(x_1) = -7x_1 + 6$

These four straight lines are then plotted in Fig. 12-4.

It follows from Fig. 12.4 that maximin occurs at $x_1 = 1/2$. This is the point of intersection of any two of the lines joining (2) to (3); (3) to (2); (6) to (- 1). As mentioned in the above remark, any two lines having opposite signs for their slopes will define an alternative optimum solution. The combination of lines $E(x_1) = - x_1 + 3$ and $E(x_1) = - 7 x_1 + 6$ must be excluded as being non-optimal. So the game can be reduced to (2 x 2) in the following manner.

It is also important to note that the average of above two payoff matrices (Table 12.10and 12.11) will also be the additional possibility of reducing the game to (2 x 2). Thus, the additional possibility of (2 x 2) game will also yield a new optimal solution which mixes three strategies II, III and IV. Then (2 x 2) game is solved by solving the governing simultaneous equations.

Table 12.10

1st possibility

		B: II	B: III
A	I	2	3
	II	3	2

Table 12.11

2nd possibility

		B: III	B: IV
A	I	3	-1
	II	2	6

Table 12.12

Additional possibility

		B	
A	I	$\frac{2+3}{2} = 5/2$	$\frac{3-1}{2} = 1$
	II	$\frac{3+2}{2} = 5/2$	$\frac{2+6}{2} = 4$

The first possibility of the. solution of (2 x 4). game with reduced (2 x 2) matrix is :

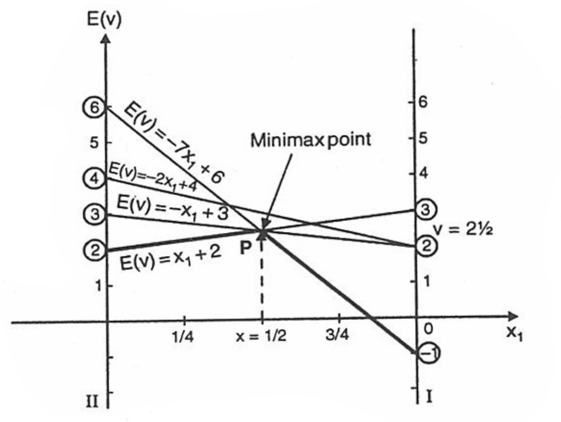

Fig 12.4

(i) The player A chooses the optimal mixed strategy $(x_1, x_2) = (1/2, 1/2)$.

(ii) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3, y_4) = (0, 1/2, 1/2, 0)$.

(iii) The value of the game to the player A is 2 1/2.

Similarly, solution of the game with reduced matrix in 2nd possibility is :

(i) The player A chooses the optimal mixed strategy (1/2, 1/2).

(ii) The player B chooses the optimal mixed strategy (0, 0, 7/8, 1/8).

(iii) The value of the game to the player A is 2 1/2.

Solution of the game with reduced matrix in additional possibility can be obtained easily, because it has a saddle point 5/2. So, the value of the game to the player A is 5/2. It has been observed that the formulae will yield an incorrect solution in this case.

Example 12.4.4 : Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs. 400 per colour set and Rs. 300 per black & white set. Firm B can, on the other hand, make either 300 colour sets, or 150 colour and 150 black & white sets, or 300 black &. while sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set.

Write the pay-off matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game.

Solution. For firm A, the strategies are :

A_1 : make 150 colour sets,

A_2 : make 150 black & white sets.

For firm B, the strategies are :

B_1 : make 300 colour sets,

B_2 : make 150 colour and 150 black & white sets,

B_3 : make 300 black and white sets.

For the combination A_1B_1, the profit to firm A would be: $\frac{150}{150+300}$ x 150 x 400

= Rs. 20,000

wherein (150/150 + 300) represents share of market for A, 150 is the total market for colour television sets and 400 is the profit per set. In a similar way, other profit figures may be obtained as shown in the following pay-off matrix :

B's strategy

$$\text{A's strategy} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 20{,}000 & 30{,}000 & 60{,}000 \\ 45{,}000 & 45{,}000 & 30{,}000 \end{bmatrix} \end{array}$$

Since no saddle point exists, we shall determine optimum mixed strategy. The data are plotted on graph as shown in the adjoining Fig. 12-5 :

Lines joining the pay-offs on axis I with the pay-offs on axis II represents each of B's strategies. Since firm A wishes to maximize his minimum expected pay-off, we consider the highest point P on the lower envelope of A's expected pay-off equation. This point P represents the maximin expected value of the game for firm A. The lines B_1 and B_3

passing through P, define the relevant moves B_1 and B_3 that alone from B needs to adopt. The solution to the original 2 x 3 game, therefore, reduces to that of the simple game with 2 x 2 pay-off matrix as follows :

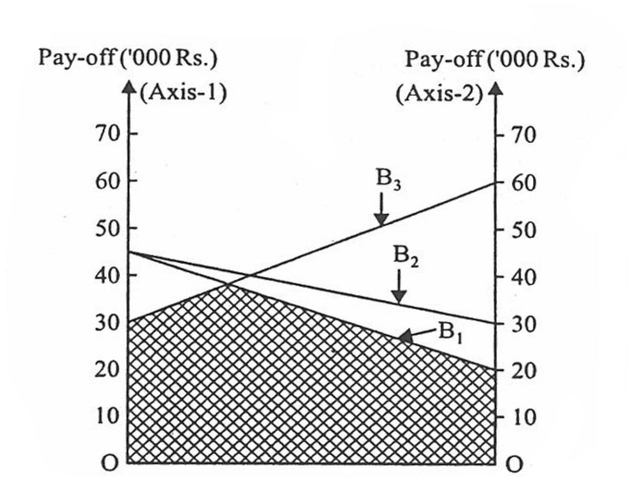

Fig. 12.5 : Graphic solution to the Game

B's strategy

$$\text{A's strategy}\quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 20{,}000 & 60{,}000 \\ 45{,}000 & 30{,}000 \end{bmatrix} \end{array}$$

Correspondingly,

$$P_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{30{,}000 - 45{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} = \frac{3}{11}$$

$$q_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{30{,}000 - 60{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} = \frac{6}{11}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{21}) - (a_{12} + a_{21})} = \frac{20{,}000x30{,}000 - 60{,}000x45{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} \cong 38{,}182$$

12.5 Linear Programming Method

The two-person zero-sum games can also be solved by linear programming. The major advantage of using linear programming technique is to solve mixed-strategy games of larger dimension payoff matrix.

To illustrate the transformation of a game problem to a linear programming problem, consider a payoff matrix of size m x n. Let a_{ij} be the element in the ith row and jth column of game payoff matrix, and letting pi be the probabilities of m strategies (i = 1, 2, ..., m) for player A. Then the value of the game (expected gains) for player A, for each of player B's strategies will be

$$V_j = \sum_{i=1}^{m} p_i a_{ij}, \quad j = 1,2,\ldots,n$$

The aim of player A is to select a set of strategies with probability pi,(i = 1, 2, ...,m) on any play of game such that he can maximize his minimum expected gains.

Now to obtain values of probability p*i*, the value of the game to player A for all strategies by player B must be at least equal to V. Thus to maximize the minimum expected gains, it is necessary that

$$a_{11}p_1 + a_{21} p_2 + \ldots + a_{m1} p_m \geq V$$
$$a_{12}p_1 + a_{22} p_2 + \ldots + a_{m2} p_m \geq V$$
$$\vdots$$
$$a_{1n}p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m \geq V$$

where $p_1 + p_2 + \ldots + p_m = 1;\ p_i \geq 0$

Dividing both sides of the *m* inequalities and equation by *V*. The division is valid as long as V > 0. In case, V < 0, the direction of inequality constraints must be reversed. But if V = 0; division would be meaning less. In this case a constant can be added to all entries of the matrix ensuring that the value of the game (V) for the revised matrix becomes more than zero. After optimal solution is obtained, the true value of the game is obtained by substracting the same constant value. Let $pi/V = x_i$, ($\geq$ 0). Then we have

$$a_{11} \frac{P_1}{V} + a_{21} \frac{P_2}{V} + \ldots + a_{m1} \frac{P_m}{V} \geq 1$$

$$a_{12} \frac{P_1}{V} + a_{22} \frac{P_2}{V} + \ldots + a_{m2} \frac{P_m}{V} \geq 1$$

$$\vdots$$

$$a_{1n}\frac{P_1}{V} + a_{2n}\frac{P_2}{V} + \ldots + a_{mn}\frac{P_m}{V} \geq 1$$

$$\frac{P_1}{V} + \frac{P_2}{V} + \ldots + \frac{P_m}{V} = 1$$

Since the objective of player A is to maximize the value of the game, V, which is equivalent to minimizing 1/V. The resulting linear programming problem can be stated as

$$\text{Minimize } Z_p = \frac{1}{V} = x_1 + x_2 + \ldots + x_m$$

subject to the constraints

$$a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m \geq 1$$

$$a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m \geq 1$$

:

:

$$a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{mn}x_m \geq 1$$

$$x_1, x_2, \ldots x_m \geq 0$$

and

Similarly, player B has a similar problem with the inequalities of the constraints reversed, i.e. minimizes the expected loss. Since, minimizing of V is equivalent to maximizing l/V, therefore, the resulting linear programming problem can be stated as:

$$\text{Minimize } Z_q = \frac{1}{V} = y_1 + y_2 + \ldots + y_n$$

Subject to the constraints

$$a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \leq 1$$

$$a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \leq 1$$

:

:

$$a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \leq 1$$

$$y_1, y_2, \ldots y_n \geq 0$$

where, $y_j = \frac{q_j}{V} \geq 0$; j = 1, 2, …, n

It may be noted that the LP problem for player B is the dual of LP problem for player A. Therefore, solution of the dual problem can be obtained from the primal simplex table. Since for both the players Zp = Zq, therefore the expected gain to player A in the game will be exactly equal to expected loss to player B.

Remark: Linear programming technique requires all variables to be non-negative and therefore to obtain a non-negative value V of the game, the data to the problem, i.e. a_{ij} in the payoff table should all be non-negative. If there are some negative elements in the payoff table, a constant to every element in the payoff table must be added so as to make the smallest element zero; the solution to this new game will give an optimal mixed strategy for the original game. The value of the original game then equals the value of the new game minus the constant.

12.6 Workout Problem

Example 12.6.1: For the following payoff matrix, transform the zero-sum game into an equivalent linear programming problem and solve it by using simplex method.

Player B

$$\text{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix} \overset{\begin{matrix} B_1 & B_2 & B_3 \end{matrix}}{\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}}$$

Solution: The first step is to find out the saddle point (if any) .in the payoff matrix as shown below:

Player B

		B_1	B_2	B_3	Row minima	
Player A	A_1	1	−1	3	[−1]	← Maximin
	A_2	3	5	−3	−3	
	A_3	6	2	−2	−2	
Column maxima		6	5	(3)	← Minimax	

The given game payoff matrix does not have a saddle point. Since the maximin value is -1, therefore it is possible that the value of game (V) may be negative or zero because -1 < V < 1. Thus a constant which is atleast equal to the negative of maximin value, i.e. more than - 1 is added to all the elements of the payoff matrix. Thus adding a constant number, 4 to all the elements of the payoff matrix, the payoff matrix

becomes:

Player B

		B_1	B_2	B_3	Probability
Player A	A_1	5	3	7	p_1
	A_2	7	9	1	p_2
	A_3	10	6	2	p_3
	Probability	q_1	q_2	q_3	

Let p_i (i = 1,2,3) and q_j (j = 1, 2, 3) be the probabilities of selecting strategies

A_i (i = 1, 2, 3) and B_j (j =1,2, 3) by players A and B respectively.

The expected gain for player A will be as follows:

$5p_1 + 7p_2 + 10p_3 \geq V$ (if B uses strategy B_1)

$3p_1 + 9p_2 + 6p_3 \geq V$ (if B uses strategy B_2)

$7p_1 + p_2 + 2p_3 \geq V$ (if B uses strategy B_3)

$P_1 + P_2 + P_3 \geq 1$

$P_1, P_2, P_3 > 0$

and

Dividing each inequality and equality by V, we get,

$$5\frac{P_1}{V} + 7\frac{P_2}{V} + 10\frac{P_3}{V} \geq 1$$

$$3\frac{P_1}{V} + 9\frac{P_2}{V} + 6\frac{P_3}{V} \geq 1$$

$$7\frac{P_1}{V} + \frac{P_2}{V} + 2\frac{P_3}{V} \geq 1$$

$$\frac{P_1}{V} + \frac{P_2}{V} + \frac{P_3}{V} = \frac{1}{V}$$

In order to simplify, we define new variables:

$$x_1 = \frac{P_1}{V}, \; x_2 = \frac{P_2}{V} \text{ and } x_3 = \frac{P_3}{V}$$

The problem for player A, therefore becomes,

Minimize $Z_p = 1/V = x_1 + x_2 + x_3$

subject to the constraints

$5x_1 + 7x_2 + 10x_3 \geq 1$

$3x_1 + 9x_2 + 6x_3 \geq 1$

$7x_1 + x_2 + 2x_3 \geq 1$

and $x_1, x_2, x_3 \geq 0$

Player B's objective is to minimize his expected losses which can be reduced to minimizing the value of the game V. Hence, the problem of player B can be expressed as follows:

Maximize $Z_q = 1/V = y_1 + y_2 + y_3$

subject to the constraints

$5y_1 + 3y_2 + 7y_3 \leq 1$

$7y_1 + 9y_2 + y_3 \leq 1$

$10y_1 + 6y_2 + 2y_3 \leq 1$

and $y_1, y_2, y_3 \geq 0$

where $y_1 = \frac{q_1}{V}$, $y_2 = \frac{q_2}{V}$ and $y_3 = \frac{q_3}{V}$

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal.

To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities. The problem becomes

$$\text{Maximize } Z_q = y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

$$5\,y_1 + 3\,y_2 + 7\,y_3 + s_1 = 1$$

$$7\,y_1 + 9y_2 + y_3 + s_2 = 1$$

$$10y_1 + 6y_2 + 2y_3 + s_3 = 1$$

and $y_1, y_2, y_3, s_1, s_2, s_3, \geq 0$

The initial solution shown in Table 12.13.

Table 12.13 Initial Solution

		$c_j \rightarrow$	1	1	1	0	0	0	
c_B	*Variables in basis* B	*Solution values* $y_B (= b)$	y_1	y_2	y_3	s_1	s_2	s_3	*Min. ratio* y_B / y_1
0	s_1	1	5	3	7	1	0	0	1/5
0	s_2	1	7	9	1	0	1	0	1/7
0	s_3	1	(10)	6	2	0	0	1	1/10 $\rightarrow$
$Z = 0$		z_j	0	0	0	0	0	0	
		$c_j - z_j$	1 $\uparrow$	1	1	0	0	0	

Proceeding with usual simplex method, the optimal solution is shown in Table 12.14.

Table 12.14 Optimal Solution

		$c_j \rightarrow$	1	1	1	0	0	0
c_B	Variables in basis B	Solution values y_B (= b)	y_1	y_2	y_3	s_1	s_2	s_3
1	y_3	1/10	2/5	0	1	3/20	−1/10	0
1	y_2	1/10	11/15	1	0	−1/60	7/60	0
0	s_1	1/5	24/5	0	0	−1/5	−3/5	1
Z = 1/5		z_j	17/15	0	0	2/15	1/15	0
		$c_j - z_j$	−2/15	0	0	−2/15	−1/15	0

The optimal solution (mixed strategies) for B is: y_1 =0; y_2 = 1/10 and the expected value of the game is: Z = 1/V- constant (= 4) = 5-4 = 1

These solution values are now converted back into the original variables: l/V = l/5, then V= 5

$$y_1 = \frac{q_1}{V}, \text{ then } q_1 = y_1 \text{ x } V = 0$$

$$y_2 = \frac{q_2}{V}, \text{ then } q_2 = y_2 \text{ x } V = \frac{1}{10} \text{ x } 5 = \frac{1}{2}$$

$$y_3 = \frac{q_3}{V}, \text{ then } q_3 = y_3 \text{ x } V = \frac{1}{10} \text{ x } 5 = \frac{1}{2}$$

The optimal strategies for player A are obtained from the z_j - c_j row of the Table 12.14.

$$x_1 = \frac{2}{15}, \ x_2 = \frac{1}{15} \text{ and } x_3 = 0$$

Then $p_1 = x_1 \text{ x } V = \frac{2}{15} \text{ x } 5 = \frac{2}{3}$

$$P_2 = x_2 \text{ x } V = \frac{1}{15} \text{ x } 5 = \frac{1}{3}$$

$$P_3 = x_3 \text{ x } V = 0$$

Hence the probabilities of using strategies for both the players are :

Player A = (2/5, 1/3, 0) ; Player B = (0, ½, ½) and value of the game is, V = 1.

Example 12.6.2. Solve (3 x 3) game by the simplex method of linear programming whose payoff matrix, is given below.

Player B

		1	2	3
	1	3	-1	-3
Player A	2	-3	3	-1
	3	-4	-3	3

Solution. First apply minimax (maximin) criterion to find the minimax ($\overline{v}$) and maximin ($\underline{v}$) value of the game. Thus, the following matrix is obtained (Table 12.15).

Table 12.15

		B 1	2	3	Row Minimum.
	1	3	−1	−3	−3
A	2	−3	3	−1	−3
	3	−4	−3	3	−4
Column Maximum		3	3	3	

Maximin Value ($\underline{v}$)

Minimax Value ($\overline{v}$)

Since, maximin value is - 3, it is possible that the value of the game (v) may be negative or zero because -3 < v < 3.

Thus, a constant c is added to all elements of the matrix which is at least equal to the - ve of the maximin value, i.e. $c \geq 3$. Let c = 5. The matrix is shown in Table 12.16 Now, following the reasoning , the player B's linear programming problem is :

Maximize $y_0 = Y_1 + Y_2 + Y_3$ subject to the constraints :

$8Y_1, + 4Y_2 + 2Y_3 \leq 1$, $2Y_1 + 8Y_2 + 4Y_3 \leq 1$, $1\ Y_1 + 2Y_2 + 8Y_3 \leq 1$, $Y_1 \geq 0$, $Y_2 \geq 0$,

$Y_3 \geq 0$

Table 12.16

		B		
	1	8	4	2
A	2	2	8	4
	3	1	2	8

Introducing slack variables, the constraint equations become:

$$\left.\begin{aligned} 8Y_1+4Y_2+2Y_3+Y_4 &= 1 \\ 2Y_1+8Y_2+4Y_3+Y_5 &= 1 \\ 1Y_1+2Y_2+8Y_3+Y_6 &= 1 \end{aligned}\right\}$$

$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \geq 0.$

Now the following simplex table is formed.

Table 12.17. Simplex Table

	$c_j \rightarrow$		1	1	1	0	0	0	
B	C_B	Y_B	α_1 ↑	α_2	α_3	α_4 (β_1)	α_5 (β_2)	α_6 (β_3)	Min. Ratio (Y_B/α_k)
α_4	0	1	[8]	4	2	1	0	0	1/8 ←
α_5	0	1	2	8	4	0	1	0	1/2
α_6	0	1	1	2	8	0	0	1	1/1
	$y_0 = C_B Y_B = 0$		(− 1)* ↑	− 1	− 1	0 ↓	0	0	← $\Delta_j = C_B\alpha_j - c_j$
α_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
α_5	0	3/4	0	7	7/2	− 1/4	1	0	3/14
α_6	0	7/8	0	3/2	[31/4]	1/8	0	1	7/62 ←
	$y_0 = 1/8$		0	− 1/2	(− 3/4)* ↑	1/8	0	0 ↓	← Δ_j
α_1	1	3/31	1	14/31	0	4/31	0	− 1/31	3/14
α_5	0	11/31	0	[196/31]	0	− 6/31	1	− 14/31	11/196 ←
α_3	1	7/62	0	6/31	1	− 1/62	0	4/31	7/12
	$y_0 = 13/62$		0	(− 11/31)* ↑	0	7/62	0 ↓	3/31	← Δ_j
α_1	1	1/14	1	0	0	1/7	1/14	0	
α_2	1	11/196	0	1	0	− 3/98	31/196	− 1/14	
α_3	1	5/49	0	0	1	− 1/98	− 3/98	1/7	
	$y_0 = 45/196$		0	0	0	5/49	11/196	1/14	← all $\Delta_j \geq 0$

Thus, the solution for B's original problem is obtained as :

$$y_1^* = \frac{Y_1}{y_0} = \frac{1/14}{45/196} = \frac{14}{15}, \qquad y_2^* = \frac{Y_2}{y_0} = \frac{11/196}{45/196} = \frac{11}{45}$$

$y_3{*} = \frac{Y_3}{y_0} = \frac{5/49}{45/196} = \frac{20}{45}$, $v{*} = \frac{1}{y_0}$ - C = $\frac{196}{45}$-5=-$\frac{29}{45}$

The optimal strategies for the player A are obtained from the final table of the above problem. This is given by duality rules :

$$X_0 = y_0 = \frac{45}{196},\ X_1 = \Delta_4 = \frac{5}{49},\quad X_2 = \Delta_5 = \frac{11}{196},\quad X_3 = \Delta_6 = \frac{1}{14}$$

Hence $X_1^* = \frac{X_1}{x_0} = \frac{20}{45}$, $X_2^* = \frac{X_2}{x_0} = \frac{11}{45}$, $X_3^* = \frac{X_3}{x_0} = \frac{14}{45}$, $v^* = \frac{29}{45}$.

12.7 Summary : In this lesson graphical method to solve 2 x n and m x 2 games and linear programming method to solve 3 x 3 games in detail along with illustrations.

12.8 Exercise :

1. Obtain the optimal strategies for both players and the value of the game for two-person zero-sum game whose pay off matrix is given as follows, by graphical method.

(a) **Player B**

$$\textbf{Player A}\quad \begin{matrix} \\ A_1 \\ A_2 \end{matrix} \begin{matrix} B_1 & B_2 & B_3 \\ \begin{bmatrix} 1 & 3 & 11 \\ 8 & 5 & 2 \end{bmatrix} \end{matrix}$$

(b) **Player B**

$$\textbf{Player A}\quad \begin{matrix} \\ A_1 \\ A_2 \end{matrix} \begin{matrix} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ \begin{bmatrix} 1 & 3 & -1 & 4 & 2 & -5 \\ -3 & 5 & 6 & 1 & 2 & 0 \end{bmatrix} \end{matrix}$$

2. Obtain the optimal strategies for both players and the value of the game for two-person zero-sum game whose pay off matrix is given as follows, by graphical method.

(a) **Player B**

$$\text{Player A}\quad \begin{matrix} & B_1 & B_2 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{matrix}\begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}$$

(b) **Player B**

$$\text{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{matrix}\overset{\begin{matrix} B_1 & \quad B_2 \end{matrix}}{\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix}}$$

3. For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method.

(a) **Player B**

$$\text{Player A}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix}\overset{\begin{matrix} B_1 & B_2 & B_3 \end{matrix}}{\begin{bmatrix} 9 & 1 & 4 \\ 0 & 6 & 3 \\ 5 & 2 & 8 \end{bmatrix}}$$

(b) **Store XYZ**

$$\text{Store ABC}\quad \begin{matrix} A_1 \\ A_2 \\ A_3 \end{matrix}\overset{\begin{matrix} B_1 & B_2 & B_3 \end{matrix}}{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix}}$$

12.9 References :

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.

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3.Operations Research – S.D. Sarma, Kedar Nath Ram Nath and Co. Publishers, Meerut.

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Lesson – 13

Dominance Property

13.0 Objectives :

- Dominance Property
- Inferior row
- Inferior column
- Generalized dominance property

Structure

13.1 Introduction

For easiness of solutions, it is always convenient to deal with smaller payoff matrices. Fortunately, the size of the payoff matrix can be considerably reduced by using the so called principle of dominance. Before stating this principle, let us define a few important terms.

13.1.2 Inferior and Superior Strategies. Consider two n-tuples **a** = (a_1 ,a_2, ... , a_n) and b = (b_1, b_2, ... ,b_n). If a_i- $\geq$ b_i for all i = l,2, ..., n, then for player A the strategy corresponding to **b** is said to be **inferior** to the strategy corresponding to **a**; and equivalently, the strategy corresponding to **a** is said to **be superior** to the strategy corresponding to **b**.

For player B, the above situation will be reversed, because player A's gain-matrix is player B's loss-matrix.

13.2 Dominance. An n - tuple **a** = (a_1, a_2, ..., a_n) is said to **dominate** the n - tuple **b** = (b_1, b_2, ..., b_n) if $a_i \geq b_i$ for al\ i = 1, 2, ...,n. The superior strategies are said to dominate the inferior ones.

Thus a player would not like to use inferior strategies which are dominated by other's. Now we are able to state the principle of dominance as follows :

13.2.1 Principle of Dominance. If one pure strategy of a player is better or superior than another one (irrespective of the strategy employed by his opponent), then the inferior strategy may be simply ignored by assigning a zero probability while searching for optimal strategies.

Theorem 13.2.1 (Dominance Property). Let A = [v_{ij}] be the payoff matrix of an m x n rectangular game. If the ith row of A is dominated by the rth row of A, then the deletion of ith row of A. does not change the set of optimal strategies for the row player (player A).

Further, if the jth column of A. dominates the kth column of A, then the deletion of jth column of A does not change the set of optimal strategies for the column player (player B).

Proof. Given that

$v_{ij} \leq v_{rj}$, for all j = 1,2,..., n and $v_{ij} \neq v_{rj}$ for at least one j …(1)

Let y* = (y_1* , y_2* , ... , y_n*) be an optimal strategy for the column player B.It follows from (1) that

$$\sum_{j=1}^{n} v_{ij}\, y_j^* < \sum_{j=1}^{n} v_{rj}\, y_j^* \text{ or } E(e_i, y^*) < E(e_r, y^*)$$

$$\therefore \quad v \geq E(e_r, y^*) > E(e_i, y^*) \qquad \ldots(2)$$

Now let $x^* = (x_1^*, x_2^*, \ldots, x_m^*)$ be an optimal strategy for the row player. If possible, let us suppose $x_i^* > 0$, then from (2) $x_i^* v > x_i^* E(e_i, y^*)$

Also, we have $$v = E(x^*, y^*) = \sum_{i=1}^{m} x_i^* E(e_i, y^*)$$

$$= x_r^* E(e_r, y^*) + \sum_{i \neq r}^{m} x_i^* E(e_i, y^*)$$

$$< x_r^* v + \sum_{i \neq r}^{m} x_i^* v = v \sum_{i=1}^{m} x_i^* = v \qquad \left(\because \sum_{i=1}^{m} x_i^* = 1\right)$$

which is a contradiction, and hence $x_i^* = 0$.

Second part can also be proved similarly.

13.3 Generalized Dominance Property

The dominance property is not only based on the superiority of pure strategies , but on the superiority of some convex linear combination of two or more pure strategies also. A given strategy can also be said to be dominated if it is inferior to some convex linear combination of two or more strategies. This concept generalizes the above dominance principle in the following theorem.

Theorem. 13.3.1 (Generlized Dominance). Let $A = [v_{ij}]$ be the pay-off matrix of an m x n rectangular game. If the ith row of A. is strictly dominated by a convex combination of the other rows of A, then the deletion of the ith row of A does not effect the set of optimal strategies for the row player (the player A).

Further, if the jth column of A strictly dominates a convex combination of the other columns, then the deletion of the jth column of A does not effect the optimal strategies for the column player (the player B).

Proof. Let $A = [v_{ij}]$ be the payoff matrix considering the first part, we are given that there exist scalars (probabilities) $x_1, x_2, \ldots, x_m$ ($0 \le x_i \le 1$, $x_r = 0$, $\sum x_i = 1$) such that

$$\sum_{i=1, i\neq r}^{m} x_i v_{ij} \ge v_{rj}, \text{ for } j = 1, 2, \ldots, n$$

or $$\sum_{i=1}^{m} x_i v_{ij} \ge v_{rj}, \text{ for } j = 1, 2, \ldots, n \quad (\because x_r = 0) \qquad \ldots(1)$$

where strict inequality holds for at least one j.

Let $y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ be an optimal strategy for player B. Then it follows from (1) that

$$\sum_{j=1}^{n} v_{rj}\, y_j^* < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\, x_i y_j^*,$$

or $$E(e_r, y^*) < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\, x_i y_j^* \le v \qquad \ldots(2)$$

Let $x^* = (x_1^*, x_2^*, \ldots, x_m^*)$ be an optimal strategy for player A.

If possible, let us suppose that $x_r^* \neq 0$. From (2), we know that $E(e_r, y^*) < v$.

Then since $x_r^* \neq 0$, we must have $x_r^* E(e_r, y^*) < x_i^* v$.

Thus $$E(x^*, y^*) = \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\, x_i^* y_j^* = x_r^* \, E(e_r, y^*) + \sum_{i\neq r} x_i^* \, E(e_i, y^*)$$

This implies $$v < x_r^* v + v \sum_{i\neq r} x_i^* = v \sum_{i=1}^{m} x_i^* = v$$ which is a contradiction.

Hence we must have $x_r^* = 0$. This completes the proof for the first part.

Similarly, we can prove the second part.

Remarks:

1. If $v_{rj} = \sum_{1}^{m} x_i \, v_{ij}$ for all $j = 1, 2, \ldots, n$; the result follows trivially, for then any probability assigned to the rth row can be easily distributed over the other rows, and rth row itself is ignored.

2. It should also be noted here that the dominating column is deleted whereas the row dominated by a convex combination of other rows is deleted.

13.4 Summary of Dominance Rules

The "dominance property " can be summerized in the following rules :

Rule l. If each element in one row, say rth of the payoff matrix $[v_{ij}]$, is less than or equal to the corresponding element in the other row, say sth, then the player A will never choose the rth strategy. In other words, if for all j = 1, 2,..., n, and $v_{rj} \leq v_{sj}$, then the probability x_r of choosing rth strategy will be zero. The value of the game and the non-zero choice of probabilities remain unaltered even if rth row is deleted from the payoff matrix. Such rth row is said to be dominated by the sth row.

Rule 2. Following the similar arguments, if each element in one column, say C_r, is greater than or equal to the corresponding element in the other column, say C_s, then the player B will never use the stategy corresponding to column C_r. In this case, the column C_s dominates the column C_r.

Rule 3. Dominance need not be based on the superiority of pure strategies only. A given strategy can be dominated if it is inferior to an average of two or more other pure strategies. In general, if some convex linear combination of some rows dominates the ith row, then the ith row will be deleted. If the ith row dominates the convex linear combination of some other rows, then one of the rows involving in the combination may be deleted. Similar arguments follow for columns also.

Rule 4. If (x_1, x_2) be the optimal strategy for the player A for the reduced game and (w_1, x_2) be the optimal strategy for the original game, then w_1 is the ith place extension of x_1.

Rule 5. If (y_1, y_2) be the optimal strategy for the player B for the reduced game and (y_1, w_2) be the optimal strategy for the original game, then w_2 is the jth place extension of y_2.

Rule 6. If the dominance holds strictly, then values of optimal strategies do coincide, and when the dominance does not hold strictly, then optimal strategies may not coincide.

Note. Using dominance properties, try to reduce the size of payoff matrix.

13.5 Demonstration of Dominance Properties by Examples

13.5.1. To illustrate first and second properties, consider the example of (3 x 3) game

It is clear that this game has no saddle point. However, consider 1st and IIIrd columns from player B's point of view. It is seen that each payoff (element) in IIIrd column is greater than the corresponding element in 1st column regardless of the player A's strategy. Evidently, the choice of IIIrd strategy by the player B will always result in the greater loss as compared to that of selecting the 1st strategy. The column III is inferior to I as never to be used. Hence, deleting the IIIrd column which is dominated by I, the reduced-size payoff matrix (Table 13.2) is obtained.

Again, if the reduced matrix (Table 13.2) is looked from player A's point of view, it is seen that the player A will never use the II strategy which is dominated by III. Hence, the size of matrix can be reduced further by deleting the II row (Table 13.2). This reduced matrix can be further reduced by deleting II row as shown in Table 13.3. The solution of the reduced 2 x 2 matrix game without saddle point can be easily obtained by solving the following simultaneous equations in usual notations :

$$-4x_1 + 2x_3 = v,\ 6x_1 - 3x_3 = v,\ x_3 + x_1 = 1 \text{ (For A)}$$

and

$$-4y_1 + 6y_2 = v,\ 2y_1 - 3y_2 = v,\ y_1 + y_2 = 1 \text{ (For B)}$$

It is advisable to verify the solution :

Table 13.1

		B: I	II	III
	I	-4	6	3
A	II	-3	-3	4
	III	2	-3	4

Table 13.2

B

		I	II
	I	-4	6
A	II	-3	-3
	III	2	-3

Table 13.3

B

		I	II
A	I	-4	6
	III	2	-3

(i) The player A chooses mixed strategy $(x_1, x_2, x_3) = (1/3, 0, 2/3)$

(ii) The player B chooses mixed strategy $(y_1, y_2, y_3) = (3/5, 2/5, 0)$

(iii) The value of the game is zero, i.e., the game is fair.

13.5.2. To illustrate the third property, consider the following game matrix

None of the pure strategies of the player A is inferior to any of his other pure strategies. However, the average of the player A' s first and second pure strategies gives

$$\left\{\frac{5-1}{2}, \frac{0+8}{2}, \frac{2+6}{2}\right\} \quad \text{or} \quad (2, 4, 4)$$

Obviously, this is superior to the player A's third pure strategy. So the third strategy may be deleted from the matrix. The reduced matrix is shown in Table 13.5.

Table 13.4

B

		1	2	3
	1	5	0	2
A	2	-1	8	6
	3	1	2	3

Table 13.5

		1	2	3
	1	5	0	2
A	2	-1	8	6

Example 13.5.3 Solve the game whose payoff matrix to the player A is given in the table:

		B I	B II	B III
	I	1	7	2
A	II	6	2	7
	III	5	2	6

Solution. Since the row III is inferior to the row II, row III can be deleted from the payoff matrix. Thus the reduced matrix (Table 13.6) is obtained.

Table 13.6

		B I	B II	B III
	I	1	7	2
A	II	6	2	7

Again, column III is dominated by column I, therefore column III can also be deleted from the above matrix. The reduced matrix is given in (Table 13.7).

Table 13.7

B

(y_1)	(y_2)

	(x_1) I	1	7
A	(x_2) II	6	2

This 2 x 2 game without saddle point can be solved either by putting $v_{11} = 1$, $v_{12} = 7$, $v_{21} = 6$, $v_{22} = 2$ in the formulae of Sec. 19.13, or by solving the simultaneous equations :

$1x_1 + 6x_2 = v$, $7x_1 + 2x_2 = v$, $x_1 + x_2 = 1$ (For player A)

$1y_1 + 7y_2 = v$, $6y_1 + 2y_2 = v$, $y_1 + y_2 = 1$ (For player B)

Thus the following solution is obtained :

(i) The player A chooses optimal strategy $(x_1, x_2, x_3) = (2/5 , 3/5 , 0)$.

(ii) The player B chooses optimal strategy $(y_1, y_2, y_3) = (1/2 , 1/2 , 0)$.

(iii) The value of the game to the player A is $v = 4$.

Example 13.5.4. Use the relation of dominance to solve the rectangular game whose payoff matrix to A is given in **Table 13.8**

Table 13.8

		B					
		I	II	III	IV	V	VI
	I	0	0	0	0	0	0
	II	4	2	0	2	1	1
A	III	4	3	1	3	2	2
	IV	4	3	7	-5	1	2
	V	4	3	4	-1	2	2
	VI	4	3	3	-2	2	2

Solution. In the payoff matrix from player, A's point of view, rows I and II are dominated by the row III. Hence the player A will never use strategies I and II in comparison to the strategy III. Thus, deleting I and II rows we obtain the reduced matrix.

Table 13.9

B

		I	II	III	IV	V	VI
	III	4	3	1	3	2	2
	IV	4	3	7	-5	1	2
A	V	4	3	4	-1	2	2
	VI	4	3	3	-2	2	2

Again, from the player B's point of view, columns I, II and VI are dominated by the column V. Therefore, the player B will never use strategies I, II and VI in comparison to the strategy V. Now, delete columns I, II and VI from the matrix to obtain the new matrix.

Table 13.10

B

		III	IV	V
	III	1	3	2
A	IV	7	-5	1
	V	4	-1	2
	VI	3	-2	2

Again the row VI is dominated by the row V from the player A's point of view. Hence, deleting VIth row, obtain the next reduced matrix.

Table 13.11

B

		III	IV	V
	III	1	3	2
A	IV	7	-5	1

V	4	-1	2

None of the pure strategies of the player B is inferior to any of his other strategies. However, the average of player B's III. and IV pure strategies gives,

$$\left\{\frac{1+3}{2}, \frac{7-5}{2}, \frac{4-1}{2}\right\} \quad \text{or } (2, 1, 3/2)$$

which is obviously superior to the player B's Vth pure strategy, because Vth strategy will result much more losses to B. Thus deleting the Vth strategy from the matrix, the revised matrix is obtained:

Table 13.12

		B: III	B: IV
	III	1	3
A	IV	7	-5
	V	4	-1

Also, the average of the player A's III and IV pure strategies give

$$\left\{\frac{1+7}{2}, \frac{3-5}{2}\right\} \quad \text{or } (4, -1).$$

This is obviously the same as the player A's Vth strategy.

In this case, the Vth strategy may be deleted from the matrix. Finally, (2 x 2) reduced matrix (Table 13.13) is obtained.

Table 13.13

	B: (y_3) III	B: (y_4) IV
A (x_3)III	1	3
(x_4)IV	7	-5

Now, for (2 x 2) game, having no saddle point, solve the following simultaneous equations:

$1.x_3 + 7x_4 = v$, $3x_3 - 5x_4 = v$, $x_3 + x_4 = 1$ (For A)

$1.y_3 + 3y_4 = v$, $7y_3 - 5y_4 = v$, $y_3 + y_4 = 1$ (For B)

The solution is :

(i) The player A chooses the optimal strategy (0, 0, 6/7, 1 /7, 0, 0).

(ii) The player B chooses the optimal strategy (0, 0, 4/7, 3/7, 0, 0).

(iii) The value of the game to player A is 13/7 .

Example 13.5.5. Two competitors A and B are competing for the same product. Their different strategies are given in the following payoff matrix :

Table 13.14

Company B

		I	II	III	IV
	I	3	2	4	0
Company A	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

use dominance principle to find the optimal strategies.

Solution. First, we can find that this game does not have a saddle point. Now try to reduce the size of the given payoff matrix by using the principle of dominance.

From the player A's point of view, Ist row is dominated by the IIIrd row. So delete Ist row from the matrix.

Table 13.15

B

	II	III	IV
II	4	2	4

A	III	2	4	0
	IV	4	0	8

Again, from the player B's point of view, 1st column is dominated by the IIIrd column. Hence, Ist column may also be deleted from the matrix. Thus, the reduced payoff matrix [Table 13.15] is obtained.

Table 13.16

		B: III	B: IV
A	III	2	4
	III	4	0
	IV	0	8

In order to check the further reduction of this reduced matrix, the average of the player B's III and IV pure strategies give

$$\left\{\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right\} \quad \text{or} \quad (3,2,4)$$

which is obviously superior to the player B's II pure strategy. Under this condition, the player B will not use II strategy. Hence, II column may be deleted from the matrix. Thus, new matrix (Table 13.16), is obtained.

Table 13.17

	B (y3) III	(y4) IV
A (x3)III	4	0
(x4)IV	0	8

Again, in the new matrix, the average of the player A's III and IV pure strategies give

$$\left\{\frac{4+0}{2}, \frac{0+8}{2}\right\} \quad \text{or } (2, 4)$$

which is obviously the same as the player A's II strategy. Therefore, the player A will gain the same amount even if the II strategy is never used by him. Hence deleting the player A's II strategy from the matrix to obtain the reduced (2 x 2) matrix (Table 13-17).

Since this (2 x 2) payoff matrix has no saddle point, solve the simultaneous equations:

$4.x_3 + 0x_4 = v$, $0x_3 + 8x_3 = v$, $x_3 + x_4 = 1$ (For player A)

$4.y_3 + 0y_4 = v$, $0y_3 + 8y_4 = v$, $y_3 + y_4 = 1$ (For player B)

to get the solution :

(i) Optimal strategy for the player A = (x_1, x_2, x_3, x_4) - (0, 0, 2/3 , 1/3).

(ii) Optimal strategy for the player B = (y_l, y_2, y_3, y_4) = (0, 0, 2/3, 1/3).

(iii) The value of the game to the player A is v = 8/3.

Example 13.5.6. Solve the following game using dominance principle :

Player B

		I	II	III	IV	V
Player A	*I*	3	5	4	9	6
	II	5	6	3	7	8
	III	8	7	9	8	7
	IV	4	2	8	5	3

Solution. In the given payoff matrix, IVth column dominates the 1st column and also Vth column dominates the IInd column. So Ist and IInd columns can be deleted without affecting the optimal strategies of B. Thus we get the reduced payoff matrix (a).

Again, we observe that IIIrd row of the reduced matrix dominates all the other rows. Thus the payoff matrix (b) is obtained.

Again, the IInd column of (b) is dominated by both the 1st and IIIrd columns. Thus the reduced payoff matrix (c) is obtained.

B

		I	II	III
	I	3	5	4
A	II	5	6	3
	III	8	7	9
	IV	4	2	8

(a)

	I	II	III
III	8	7	9

(b)

	II
III	7

(c)

Thus the solution to the game is :

(i) best strategy for player A is III; (ii) best strategy for player B is II; and

(iii) value of the game for player A is 7, and for player B is -7.

Note. If we apply principle of dominance to the payoff matrix having a saddle point, then we get a single element reduced matrix only. So the students are advised to use the principle of dominance for solving the games without saddle point until unless otherwise stated.

13.6 Limitations of Game Theory

Game theory which was initially received in literature with great enthusiasm as holding promise, has been found to have a lot of limitations. The major limitations are summarised below :

1. The assumption that the players have the knowledge about their own pay-offs and payoffs of others is rather unrealistic. He can only make a guess of his own and his rivals' strategies.
2. As the number of players increase in the game, the analysis of the gaming strategies become increaseingly complex and difficult. In practice, there are many firms in an oligopoly situation and game theory cannot be very helpful in such situations.
3. The assumptions of maximin and minimax show that the players are risk-averse and have complete knowledge of the strategies. These do not seem practical.
4. Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion, Then the mixed strategies are not very useful.

However, inspite of its limiations, game theory provides insight into the operations of oligopoly markets.

13.7 Summary : In this lesson Game theory over view is presented. In particular the dominance property and its applications in different types of situations is illustrated by numerical examples.

13.8 Exercise

Explain the principle of dominance and hence solve the following games :

1. **Player B**

$$\textbf{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}$$

2. **Player B**

$$\textbf{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & 2 & 0 & -5 \end{bmatrix} \end{array}$$

3. **Player B**

$$\textbf{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

4. **Player B**

$$\textbf{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 2 & 3 & 1/2 \\ 3/2 & 2 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \end{array}$$

5. **Player B**

$$\textbf{Player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 1 & 8 & 4 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

6. Use dominance principle to reduce the following games to 2 x 2 games and hence solve them.

(i)

$$\begin{bmatrix} 2 & 0 & 3 \\ 3 & -1 & 1 \\ 5 & 2 & -1 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 1 & -1 & 0 \\ -6 & 3 & -2 \\ 8 & -5 & 2 \end{bmatrix}$$

(iii)

$$\begin{bmatrix} 8 & 5 & 8 \\ 8 & 6 & 5 \\ 7 & 4 & 5 \\ 6 & 5 & 6 \end{bmatrix}$$

(iv)

$$\begin{matrix} I \\ II \\ III \end{matrix} \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

7.

$$\begin{matrix} & B_1 & B_2 & B_3 & B_4 & B_5 \\ A_1 & 4 & 4 & 2 & -4 & -6 \\ A_2 & 8 & 6 & 8 & -4 & 0 \\ A_3 & 10 & 2 & 4 & 10 & 12 \end{matrix}$$

13.9 References:

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.

2. Operations Research – Kanthi Swarup and others, Sultan Chand & Sons.

3. Operations Research – S.D. Sarma, Kedar Nath Ram Nath and Co. Publishers, Meerut.

4. Operations Research and Principles – Philips D.D. Ravindran. A and Solberg. J., John Wiky.

Lesson – 14

Queueing Theory
(Waiting Line Models)

14.0 Objective :

- Queue or waiting line
- Different types of queues
- Chief components of a queueing system

Structure

14.1 Introduction

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the ***waiting line***, in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait, (i.e. remains idle), until additional people arrive. Here the arriving people are called the ***customers*** and the person issuing the tickets is called a ***server***.

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the customers and the typist represents the server. A third example is illustrated by a machine breakdown situation. A broken machine represents a customer calling for the service of a repairman. These examples show that the term customer may be interpreted in various number of ways. It is also noticed that a service may be performed either by moving the server to the customer or the customer to the server.

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aeroplanes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations. Further, it is also observed that arriving units may form one line and be serviced through only one station (as in a doctor's clinic), may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations (e.g. at check out counters of supermarket).

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single queue as shown in Fig. 14.1 or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

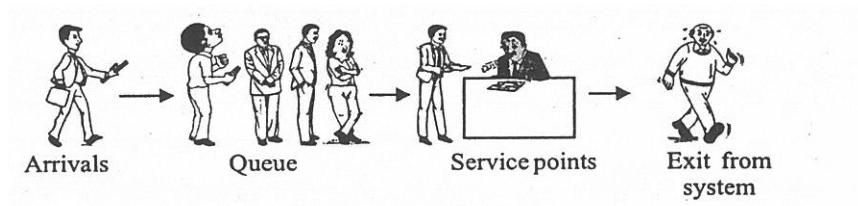


Fig 14.1(a). Queueing system with single queue and single service station.

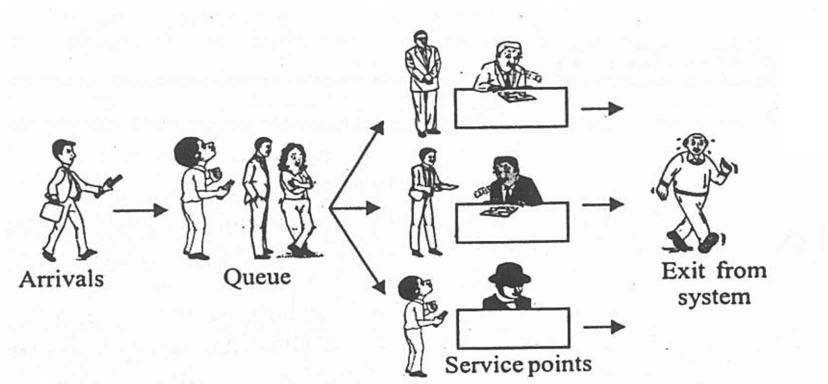


Fig 14.1(b). Queueing system with single queue and several service station.

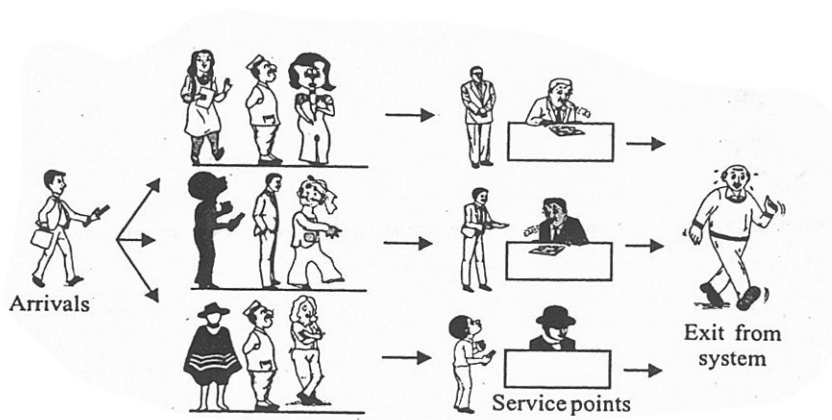


Fig 14.1(c). Queueing system with several queue and several service

Queuqing theory is concerned with the statistical description of the behaviour of queues with finding, e.g., the probability distribution of the number in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer, or the distribution of a server's busy periods can be found. In operational research problems involving queues, investigators must measure the existing system to make an objective assessment of its characteristics and must determine how changes may be made to the system, what effects of various kinds of changes in the system's characteristics would be," and whether, in the light of the costs incurred in the systems, changes should be made to it, A model of the queueing system under study must be constructed in this kind of analysis and the results of queueing theory are required to obtain the characteristics of the model and to assess the effects of changes, such as the addition of an extra server or a reduction in mean service time.

Perhaps the most important general fact emerging from the theory is that the degree of congestion in a queueing system (measured by mean wait in the queue or mean queue length) is very much dependent on the amount of irregularity in the system. Thus congestion depends not just on mean rates at which customers arrive and are served and may be reduced without altering mean rates by regularizing arrivals or service times, or both where this can be achieved.

14.2 Characteristics of Queueing System

A queueing system can be completely described by

(a) the input (or arrival pattern), (b) the service mechanism (or service pattern),

(c) the 'queue discipline' and (d) customer's behaviour.

14.2.1 The input (or arrival pattern). The input describes the way in which the customers arrive and join the system. Generally, the customers arrive in a more or less random fashion which is not worth making the prediction. Thus, the arrival pattern can best be described in terms of probabilities and consequently the probability distribution for inter-arrival times (the time between two successive arrivals) or the distribution of number of customers arriving in unit time must be defined.

The present chapter is only dealt with those queueing systems in which the customers arrive in 'Poisson' or 'completely random' fashion. Other types of arrival pattern may also be observed in practice that have been studied in queueing theory. Two such patterns are observed, where (i) arrivals are of regular intervals; (ii) there is general distribution (perhaps normal) of time between successive arrivals.

14.2.2. The service mechanism (or service pattern). It is specified when it is known how many customers can be served at a time, what the statistical distribution of service time is, and when service is available. It is true in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers (e.g. machines waiting repair) each with a different service time distribution. Service time may be constant or a random variable. Distributions of service time which are important in practice are ***'negative exponential distribution'*** and the related ***'Erlang (Gamma) distribution'.*** Queues with the negative exponential service time distribution are studied in the following sections.

In the present chapter, only those queueing systems are discussed in which the service time follows the 'Exponential and Erlang (Gamma)' probability distributions (see sec. 14.7-1 to 14.7-8).

14.2.3. The queue discipline. The queue discipline is the rule determining the formation of the queue, the manner of the customer's behaviour while waiting, and the manner in which they are chosen for service. The simplest discipline is "first come, first served", according to which the customers are served in the order of their arrival. For example, such type of queue discipline is observed at a ration shop, at cinema ticket windows, at railway stations, etc. If the order is reversed, we have the ***"last come, first served"*** discipline, as in the case of a big godown the items which come last are taken out first. An extremely difficult queue discipline to handle might be ***"service in random order"*** or ***"might is right"***.

Properties of a queueing system which are concerned with waiting times, in general, depend on queue discipline. For example, the variance of waiting time will be much greater with the queue discipline ***'first come, last served'*** than with ***'first come, first served'***, although mean waiting time will remain unaffected.

The following notations are used for describing the nature of service discipline.

FIFO $\rightarrow$ First In, First Out or FCFS $\rightarrow$ First Come, First Served

LIFO $\rightarrow$ Last In, First Out or FILO $\rightarrow$ First In, Last Out.

SIRO $\rightarrow$ Service in Random Order

This chapter shall be concerned only with the customers which are served in the order in which they arrive at the service facility, that is, ***'first come, first served'*** discipline.

14.2.4. Customer's behaviour. The customers generally behave in four ways :

(i) **Balking.** A customer may leave the queue because the queue is too long and he has no time to wait, or there is not sufficient waiting space.

(ii) **Reneging.** This occurs when a waiting customer leaves the queue due to impatience.

(iii) **Priorities.** In certain applications some customers are served before others regardless of their order of arrival. These customers have priority over others.

(iv) **Jockeying.** Customers may jockey from one waiting line to another. It may be seen that this occurs in the supermarket.

14.2.5 Size of a Population : The collection of potential customers may be very large or of a moderate size. In a railway booking counter the total number of potential passengers is so large that although theoretically finite it can be regarded as infinity for all practical purposes. The assumption of infinite population is very convenient for analysing a queuing model. However, this assumption is not valid where the customer group is represented by few machines in workshop that require operator facility from time to time. If the population size is finite then the analysis of queueing model becomes more involved.

14.2.6 Maximum Length of a Queue : Sometimes only a finite number of customers are allowed to stay in the system although the total number of customers in the population may or may not be finite. For example, a doctor may have appointments with k patients in a day. If the number of patients asking for appointment exceeds k, they are not allowed to join the queue. Thus, although the size of the population is infinite, the maximum number permissible in the system is k.

14.3 Queueing Problem

In a specified queueing system, the problem is to determine the following :

(a) **Probability distribution of queue length.** When the nature of probability distributions of the arrival and service patterns is given, the probability distribution of queue length can be obtained. Further, we can also estimate the probability that there is no queue.

(b) **Probability distribution of waiting time of customers.** We can find the time spent by a customer in the queue before the commencement of his service which is called his ***waiting time.*** The total time spent by him in the system is the waiting time plus service time.

(c) **The busy period distribution.** We can estimate the probability distribution of busy periods. If we suppose that the server is free initially and customer arrives, he will be served immediately. During his service time, some more customers will arrive and will be served in their turn. This process will continue in this way until no customer is left unserved and the server becomes free again. Whenever this happens, we say that a **busy period** has just ended. On the other hand, during **idle periods** no customer is present in the system. A busy period and the idle period following it together constitute a busy cycle. The study of the busy period is of great interest in cases where technical features of the server and his capacity for continuous operations must be taken into account.

14.4 Transient and Steady States

Queueing theory analysis involves the study of a system's behaviour over time. ***A system is said to be in "transient state" when its operating characteristics (behaviour) are dependent on time.*** This usually occurs at the early stages of the operation of the system where its behaviour is still dependent on the initial conditions. However, since we are mostly interested in the "long run" behaviour of the sytem, mainly the attention has been paid toward "steady state" results.

A steady state condition is said to prevail when the behaviour of the system becomes independent of time. Let $P_n(t)$ denote the probability that there are n units in the system at time t. In fact, the change of $P_n(t)$ with respect to t is described by the derivative $[dP_n(t)/dt]$ or $P_n'(t)$. Then the queueing system is said to become 'stable' eventually, in the sense that the probability $P_n(t)$ is independent of time, that is, remains the same as time passes ($t \to \infty$). Mathematically, in steady state

$$\lim_{t\to\infty} P_n(t) = P_n(\text{independent of t}) \Rightarrow \lim_{t\to\infty} \frac{dP_n(t)}{dt} = \frac{dP_n}{dt} \Rightarrow \lim_{t\to\infty} P_n'(t) = 0.$$

In some situations, if the arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. In fact, in this case the queue length will increase with time and theoretically it could build up to infinity. Such case is called the ***"explosive state"***.

In this chapter, only the steady state analysis will be considered. We shall not treat the 'transient' and 'explosive' states.

14.5 A List of Symbols

Unless otherwise stated, the following symbols and terminology will be used henceforth in connection with the queueing models. The reader is reminded that a queueing system is defined to include the queue and the service stations both, (see Fig. 15.3).

n = number of units in the system

$P_n(t)$ = transient state probability that exactly n calling units are in the queueing system at time t

E_n = the state in which there are n calling units in the system

P_n = steady state probability of having n units in the system

λ_n = mean arrival rate (expected number of arrivals per unit time) of customers (when n units are present in the system)

μ_n = mean service rate (expected number of customers served per unit time when there are n units in the system)

λ = mean arrival rate when λ_n is constant for all n

μ = mean service rate when μ_n is constant for all $n \geq 1$

s = number of parallel service stations

p = $\lambda / \mu s$ = traffic intensity (or utilization factor) for servers facility, that is, the expected fraction of time the servers are busy

$\phi_T(n)$ = probability of n services in time T, given that servicing is going on throughout T Line length (or queue size)

= number of customers in the queueing system

Queue length

= line length (or queue size)-(number of units being served)

ψ /(w) = probability density function (p.d.f.) of waiting time in the system

L_s = expected line length, i.e., expected number of customers in the system

L_q = expected queue length, i.e., expected number of customers in the queue

W_s = expected waiting time per customer in the system

W_q = expected waiting time per customer in the queue

(W|W>0) = expected waiting time of a customer who has to wait

(L|L>0) = expected length of non-empty queues, i.e., expected number of customers in the queue when there is a queue

P(W>0) = probability of a customer having to wait for service

$\begin{bmatrix} n \\ r \end{bmatrix}$ = the binomial coefficient ^nC_r.

$$= \frac{n!}{r!(n-r)!} = \frac{n(n-1)...(n-r+1)}{r!}$$ for r and n non-negative integers ($r \leq n$).

14.6 Traffic Intensity (or Utilization Factor)

An important measure of a simple queue $(M \mid M \mid 1)$ is its traffic intensity, where

$$\text{Traffic intensity } (\rho) = \frac{mean arrival rate}{mean service rate} = \frac{\lambda}{\mu}$$

i.e.

$$\rho = \frac{1/\mu}{1/\lambda} = \frac{Mean service time}{Mean inter-arrival time}$$

The unit of traffic intensity is Erlang.

Here it should be noted carefully that a necessary condition for a system to have settled down to steady state is that $\rho < 1$ or $\frac{\lambda}{\mu} < 1$ or $\lambda < \mu$, i.e., arrival rate < service rate.

If this is not so, i.e., $\rho > 1$, the arrival rate will be greater than the service rate and consequently, the number of units in the queue tends to increase indefinitely as the time passes on, provided the rate of service is not affected by the length of queue.

14.7. Summary :

In this lesson the concept of a queue or waiting line and characteristics of a queueing system are explained in detail also the study and transient states of queueing system are discussed. At the end some symbols are explained which will be used in the next lesson.

14.8. References

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.

2. Operations Research – Kanthi Swarup and others, Sultan Chand & Sons.

3.Operations Research – S.D. Sarma, Kedar Nath Ram Nath and Co. Publishers, Meerut.

4.Operations Research and Principles – Philips D.D. Ravindran. A and Solberg. J., John Wiky.

Lesson – 15

Probability Distributions in Queueing Theory

15.0 Objective :

- Probability distribution of arrival process
- Probability distribution of interarrival time
- Probability distribution of departures
- Some of their properties

Structure

15.1 Probability Distributions in Queueing Systems

The arrival pattern of customers at a queueing system varies between one system and another, but one pattern of common occurrence in practice, which turns out to be relatively easy to deal with mathematically, is that of *'completely random arrivals'*. This phrase means something quite specific, and we discuss what does it mean before dealing in the subsequent sections with a variety of queueing systems. In particular, we show that, if arrivals are *'completely random'*, the number of arrivals in unit time has a *Poisson distribution*, and the intervals between successive arrivals are distributed *negative exponentially*.

15.2 Distribution of Arrivals 'The Poisson Process' (Pure Birth Process)

In many situations the objective of an analysis consists of merely observing the number of customers that enter the system. The model in which only arrivals are counted and no departures take place are called *pure birth models*. The term *'birth'* refers to the arrival of a new calling unit in the system, and the 'death' refers to the departure of a served unit. As such pure birth models are not of much importance so far as their applicability to real life situation is concerned, but these are very important in the understanding of completely random arrival problems.

Theorem 15.2.1 (Arrival Distribution Theorem). If the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time-interval follows a Poisson distribution.

Proof : In order to derive the arrival distribution in queues, we make the following three assumptions (sometimes called the *axioms*).

1. Assume that there are n units in the system at time t, and the probability that exactly one arrival (birth) will occur during small time interval Δt be given by $\lambda \Delta t$ + O(Δt), where λ is the arrival rate independent of t and O(Δt) includes the terms of higher order of Δt.

2. Further assume that the time Δt is so small that the probability of more than one arrival in time Δt is $O(\Delta t)^2$, i.e., almost zero.

3. The number of arrivals in non-overlapping intervals are statistically independent, i.e., the process has independent increments.

We now wish to determine the probability of *n* arrivals in a time interval of length *t*, denoted by $P_n(t)$. Clearly, *n* will be an integer greater than or equal to zero. To do so, we shall first develop the differential-difference equations governing the process in two different situations.

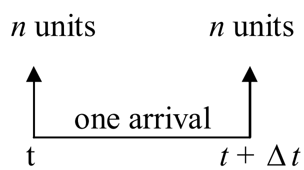


Fig 15.2.1

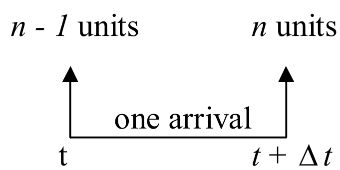


Fig 15.2.2

Case I. When n > 0. For n > 0, there may be two mutually exclusive ways of having *n* units at time $t + \Delta t$.

(i) There are *n* units in the system at time *t* and no arrival takes place during time interval Δt. Hence, there will be *n* units at time $t + \Delta t$ also. This situation is better explained in Fig. 15.2.1

Therefore, the probability of these two combined events will be

= Prob. of *n* units at time *t* x Prob. of no arrival during $\Delta t = P_n(t) .(1-\lambda \Delta t)$ ... (15.1)

[since prob. of exactly one arrival in $\Delta t = \lambda \Delta t$, prob. of no arrival becomes = $1 - \lambda \Delta t$]

(ii) *Alternately*, there are (n-1) units in the system at time *t*, and one arrival takes place during Δt. Hence there will remain *n* units in the system at time $t + \Delta t$. This situation is better explained *in Fig.1.5*.2.2.

Therefore, the probability of these two combined events will be

= Prob. of (n - 1) units at time t x Prob. of one arrival in time $\Delta t = P_{n-1}(t). \lambda \Delta t$

... (15.2)

Note: Since the probability of more than one arrival in Δt is assumed to be negligible, other alternatives do not exist.

Now, adding above two probabilities [given by (15.1) and (15.2)], we get the probability of n arrivals at time $t + \Delta t$, i.e.

$$P_n(t+\Delta t) = P_n(t)(1 - \lambda \Delta t) + P_{n-1}(t)\,\lambda \Delta t \qquad \text{... (15.3)}$$

Case 2. ***When n* = 0.**

$$P_0(t+\Delta t) = \text{Prob. [no unit at time t] x Prob. [no arrival in time } \Delta t]$$

$$\therefore \qquad P_0(t+\Delta t) = P_0(t)(1 - \lambda \Delta t) \qquad \text{... (15.4)}$$

Rewriting the equations (15.3) and (15.4) after transposing the terms $P_n(t)$ and $P_0(t)$ to left hand sides, respectively, we get

$$P_n(t+\Delta t) - P_n(t) = P_n(t)(-\lambda \Delta t) + P_{n-1}(t)\,\lambda \Delta t, \quad n > 0 \qquad \text{... (15.3)'}$$

$$P_0(t+\Delta t) - P_0(t) = P_0(t)(-\lambda \Delta t) \qquad n = 0 \qquad \text{... (15.4)'}$$

Dividing both sides by Δt and then taking limit as $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t) \qquad \text{... (15.5)}$$

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) \qquad \text{... (15.6)}$$

Since by definition of first derivative, $\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \frac{dP_n(t)}{dt} = P_n'(t)$

the equations (15.6) and (15.5) respectively can be written as

$$P_0'(t) = -\lambda P_0(t), \qquad n = 0$$

... (15.7)

$$P_{n}'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \quad n > 0$$

... (15.8)

This is known as the system of differential-difference equations.

To solve the equations (15.7) and (15.8) by iterative method :

Equation (15.7) can be written as

$$\frac{P_0'(t)}{P_0(t)} = -\lambda \text{ or } \frac{d}{dt}[\log P_0(t)] = -\lambda$$

... (15.9)

Integrating both sides w.r.t. '*t*',

$$\log P_0(t) = -\lambda t + A$$

... (15.10)

The constant of integration can be determined by using the boundary conditions :

$$P_n(0) = \begin{cases} 1 & for \quad n = 0 \\ 0 & for \quad n > 0 \end{cases}$$

Substituting $t = 0$, $P_0(0) = 1$ in (15.10), find, A = 0. Thus, (15.10) gives

$$\log P_0(t) = -\lambda t \text{ or } P_0(t) = e^{-\lambda t}$$

... (15.11)

Putting n = 1 in (15.8), $P_1'(t) = -\lambda P_1(t) + \lambda P_0(t)$

or $P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$

... (15.12)

Since this is the linear differential equation of first order, it can be easily solved by multiplying both sides of this equation by the integrating factor, I.F. = $e^{\int \lambda dt} = e^{\lambda t}$.

Thus, eqn. (15.12)becomes

$$e^{\lambda t}\,[P_1'(t) + \lambda\, P_1(t)] = \lambda \;\text{ or }\; \frac{d}{dt}\,[e^{\lambda t}\, P_1(t)] = \lambda$$

Now integrating both sides w.r.t. 't'

... (15.13)

$$e^{\lambda t}\, P_1(t) = \lambda\, t + B,$$

where B is the constant of integration.

In order to determine the constant B, put $t = 0$ in (15.13), and get

$$P_1(0) = 0 + B \text{ or } B = 0 \qquad [\because\; P_1(0) = 0]$$

... (15.14)

Substituting B = 0 in (15.13), $\quad P_1(t) = \dfrac{\lambda t e^{-\lambda t}}{1!}$

Similarly, putting $n = 2$ in (15.8) and using the result (15.14), we get the equation

$$P_2'(t) + \lambda\, P_2(t) = \lambda\; \frac{(\lambda t)e^{-\lambda t}}{1!} \;\text{ or }\; \frac{d}{dt}\,[e^{\lambda t}\, P_2(t)] = \frac{\lambda(\lambda t)}{1!}$$

Integrating w.r.t. 't' $\quad e^{\lambda t}\, P_2(t) = \dfrac{(\lambda t)^2}{2!} + C,$

Put $t = 0$, $P_2(0) = 0$ to obtain C = 0. Hence

$$P_2\,(t) = \frac{(\lambda t)^2\, e^{-\lambda t}}{2!}, \text{ for n = 2}$$

... (15.15)

Similarly, obtain $\quad P_3\,(t) = \dfrac{(\lambda t)^3\, e^{-\lambda t}}{3!}$, for n = 3

Likewise, in general, $\quad P_m\,(t) = \dfrac{(\lambda t)^m\, e^{-\lambda t}}{m!}$, for n = m

... (15.16)

If, anyhow, it can be proved that the result (15.16) is also true for n = m + 1, then by induction hypothesis result (15. 16) will be true for general value of n.

To do so, put n = m + 1 in (15.8) and get

$$P'_{m+1}(t) + \lambda P_{m+1}(t) = \lambda \frac{(\lambda t)^m e^{-\lambda t}}{m!} \qquad \text{[using the results (15.16)]}$$

or $$\frac{d}{dt}[e^{\lambda t} P_{m+1}(t)] = \frac{(\lambda t)^m (\lambda)}{m!}$$

Integrating both sides, $$e^{\lambda t} P_{m+1}(t)] = \frac{(\lambda t)^{m+1}}{(m+1)m!} + D,$$

Again, putting $t = 0$, $P_{m+1}(0) = 0$, we get D = 0. Therefore,

$$\therefore \qquad P_{m+1}(t) = \frac{(\lambda t)^{m+1} e^{-\lambda t}}{(m+1)!}$$

Hence, in general, $$P_n(t) = \lambda \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

... (15.17)

which is a ***Poisson distribution formula***. This completes the proof of the theorem.

Note: After carefully understanding the above procedure, the students can much reduce the number of steps by solving the differential equation of the standard from : y' + P(x) y = Q(x), using the formula

$$y.e^{\int Pdx} = \int Q(x)(e^{\int Pdx})dx + C,$$

where $e^{\int Pdx}$ is the integrating factor (I.F.)

15.3 Properties of Poisson Process of Arrivals

It has already been derived that – if n be the number of arrivals during time interval t, then the law of probability in Poisson process is given by

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, \ldots, \infty$$

... (15.18)

where λ t is the parameter.

(1) Since mean $E(n) = \lambda t$, and var. $(n) = \lambda t$,

... (15.19)

the average (expected) number of arrivals in unit time will be

$$E(n)/t = \lambda = \text{mean arrival rate (or input rate)}.$$

(2) If we consider the time interval $(t, t + \Delta t)$, where Δt is sufficiently small, then

$$P_0(\Delta t) = \text{Prob}\ [\textit{no arrival in time}\ \Delta t]$$

Putting n = 0 and t = Δt in (15. 18)

$$P_0(\Delta t) = \frac{e^{-\lambda\Delta t}}{0!} = e^{-\lambda\Delta t} = 1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2!} - \ldots = 1 - \lambda\Delta t + O(\Delta t)$$

where the term $O(\Delta t)$ indicates a quantity that is negligible compared to Δt, More precisely, $O(\Delta t)$ represents any function of Δt such that

$$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0$$

[For example, $(\Delta t)^2$ can be replaced by $O(\Delta t)$ because $\lim_{\Delta t \to 0} \frac{(\Delta t)^2}{\Delta t} = 0$. This notation will be very useful for summarizing the negligible terms which do not enter in the final result]

$\therefore$ $P_0(\Delta t) = 1 - \lambda\Delta t$ ... (15.20)

which means that the probability of no arrival in Δt is $1 - \lambda\Delta t$. In the similar fashion, $P_1(\Delta t)$ can be written as

$$P_1(\Delta t) = \frac{(\lambda\Delta t)e^{-\lambda\Delta t}}{1!} \qquad [\text{putting } n = 1,\ t = \Delta t \text{ in (15. 18)}]$$

$$= \lambda\Delta t \left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2!} - \ldots\right] = \lambda\Delta t + O(\Delta t)$$

Neglecting the term $O(\Delta t)$, $P_1(\Delta t) = \lambda\Delta t$, ... (15.21) which means that the probability of one arrival in time Δt is $\lambda\Delta t$.

Similarly, $$P_2(\Delta t) = \frac{(\lambda\Delta t)^2 e^{-\lambda\Delta t}}{2!} = (\lambda\Delta t)^2 \left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2!} - \ldots\right] = O(\Delta t).$$

Again neglecting the term $O(\Delta t)$, we have $P_2(\Delta t) = 0$, ... (15.22)

ans so on. Thus, it is concluded from the property of Poisson process that the probability of more than one arrival in time Δt is negligibly small, provided the terms of second and higher order of Δt are considered to be negligibly small. Symbolically,

$$P_n(\Delta t) = \text{negligibly small for all } n > 1. \qquad \ldots (15.23)$$

15.4 Distribution of Inter-Arrival Times (Exponential Process)

Let T be the time between two consecutive arrivals (called the inter-arrival time), and a(T) denotes the probability density function of T. Then the following important theorem can be proved.

Theorem 15.6.1. If n, the number of arrivals in time t, follows the Poisson distribution,

$$P_n(t) = (\lambda t)^n e^{-\lambda t}/n!, \qquad \ldots (15.24)$$

then T (the inter-arrival time) obeys the negative exponential law

$$a(T) = \lambda e^{-\lambda T} \qquad \ldots (15.25)$$

and vice-versa.

Proof. Suppose that t_0 = instant of an arrival initially.

Since there is no arrival in the intervals $(t_0, t_0 + T)$ and $(t_0 + T, t_0 + T + \Delta T)$, therefore $(t_0 + T + \Delta T)$ will be the instant of subsequent arrival.

Therefore, putting $t = T + \Delta T$ and $n = 0$ in (15.24),

$$P_0(T + \Delta T) = \frac{[\lambda(T + \Delta T)]^0 . e^{-\lambda(T+\Delta T)}}{0!} = e^{-\lambda(T+\Delta T)}$$

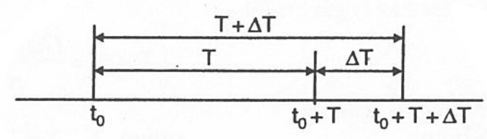


Fig. 15.3

$= e^{-\lambda T} . e^{-\lambda \Delta T} = e^{-\lambda T} [1 - \lambda\Delta T + O(\Delta T)]$

Since $P_0(T) = e^{-\lambda T}$ from (15.24),

$$P_0(T + \Delta T) = P_0(T) [1 - \lambda\Delta T + O(\Delta T)]$$

or $\quad P_0(T + \Delta T) - P_0(T) [1 - \lambda\Delta T + O(\Delta T)]$

Dividing both sides by ΔT,

$$\frac{P_0(T+\Delta T) - P_0(T)}{\Delta T} = -\lambda P_0(T) + \frac{O(\Delta T)}{\Delta T} P_0(T)$$

Now taking limit on both sides as $\Delta T \to 0$,

$$\lim_{\Delta T \to 0} \frac{P_0(T+\Delta T) - P_0(T)}{\Delta T} = \lim_{\Delta T \to 0} \left[-\lambda P_0(T) + \frac{O(\Delta T)}{\Delta T} P_0(T) \right]$$

or $$\frac{dP_0(T)}{dT} = -\lambda P_0(T). \left[\text{since } \lim_{\Delta T \to 0} \frac{O(\Delta T)}{\Delta T} = 0 \right] \qquad \ldots (15.26)$$

But, L.H.S. of (15.26) is denoting the probability density function of T, say a(T). Therefore,

$$*a(T) = ** \lambda P_0(T). \qquad \ldots(15.27)$$

But, from equation (15.24), $P_0(T) = e^{-\lambda T}$. Putting this value of $P_0(T)$ in (15.27),

$$a(T) = \lambda e^{-\lambda T} \qquad \ldots(15.28)$$

which is the exponential law of probability for T with mean $1/\lambda$, and variance $1/\lambda^2$, i.e.,

$$E(T) = 1/\lambda, \text{ Var. } (T) = 1/\lambda^2.$$

In a similar fashion, the converse of this theorem can be proved.

Q. 1. Give the axioms characterizing a Poisson process. If the number of arrivals in some time interval follows a Poisson distribution, show that the distribution of the time interval between two consecutive arrivals is exponential.

2. Show that if the inter-arrival times are negative exponentially distributed, the number of arrivals in a time period is a Poisson process and conversely.

3. If the intervals between successive arrivals are i.i.d. random variables which follow the negative exponential distribution with mean $1/\lambda$, then show that the arrivals form a Poisson Process with mean λt.

4. Show that inter-arrival times are distributed exponentially, if arrival is a Poisson process. Prove the converse also.

5. State the three axioms underlying the exponential process. Under exponential assumptions can two events occur during a very small interval.

15.5 Markovian Property of Inter-arrival Times

Statement. The Markovian property of inter-arrival times states that at any instant the time until the next arrival occurs is independent of the time that has elapsed since the occurrence of the last arrival. That is to say,

$$\text{Prob, } [T \geq t_1 \mid T \geq t_0] = \text{Prob. } [0 \leq T \leq t_1 - t_0]$$

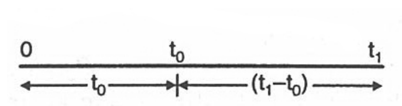


Fig. 15.7

Proof. Consider

$$\text{Prob. } [T \geq t_1 \mid T \geq t_0] = \frac{\text{Prob.}[(T \geq t_1) \text{ and } (T \geq t_0)]}{\text{Prob.}[T \geq t_0]} \text{ (formula of conditional probability)}$$

… (15.29)

Since the inter-arrival times are exponentially distributed, the right hand side of equation (15.29) can be written as

$$\frac{\int_{t_0}^{t_1} \lambda e^{-\lambda t} dt}{\int_{t_0}^{\infty} \lambda e^{-\lambda t} dt} = \frac{e^{-\lambda t_1} - e^{-\lambda t_0}}{-e^{-\lambda t_0}}$$

$\therefore$ Prob. $[T \geq t_1 \mid T \geq t_0] = 1 - e^{-\lambda(t_1 - t_0)}$

… (15.30)

* According to probability distributions d/dx[F(x)] = f(x), where F(x) is the 'distribution function' and f(x) is the 'probability density function'. Hence by the similar argument, we may write $d/dT[P_0(T)] = a(T)$, where $P_0(T)$ is the probability distribution function for no arrival in time T, and a(T) is denoting the corresponding probability density function of T.

** Since 'probability density function' is always non-negative, so neglect the negative sign from right side of equation (15.26).

But,

$$\text{Prob. } [0 \leq T \leq t_1 - t_0] = \int_0^{t_1 - t_0} \lambda e^{-\lambda t} dt = 1 - e^{-\lambda(t_1 - t_0)}$$

…(15.31)

Thus, by virtue of equations (15.30) and (15.31), it can be concluded that

$$\text{Prob. } [T \geq t_1 \mid T \geq t_0] = \text{Prob. } [0 \leq T \leq t_1 - t_0].$$

This proves the Markovian property of inter-arrival times.

15.6 Distribution of Departures (or Pure Death Process)

In this process assume that there are N customers in the system at time t = 0. Also, assume that no arrivals (births) can occur in the system. Departures occur at a rate μ per unit time, i.e., output rate is μ. We wish to derive the distribution of departures from the system on the basis of the following three axioms :

(1) Prob. [one departure during Δt] = $\mu\ \Delta t + O(\Delta t)^2 = \mu\ \Delta t$ [$\because$ $O(\Delta t)^2$ is negligible]

(2) Prob. [more than one departure during Δt] = $O(\Delta t)^2 = 0$.

(3) The number of departures in non-overlapping intervals are statistically independent and identically distributed random variable, i.e., the process N(t) has independent increments.

First obtain the differential difference equation in three mutually exclusive ways :

Case I. When 0 < n < N. Proceeding exactly as in the Pure Birth Process,

$$P_n(t + \Delta t) = P_n(t)\,[1 - \mu\ \Delta t] + P_{n+1}(t)\ \mu\ \Delta t \qquad \ldots(15.32)$$

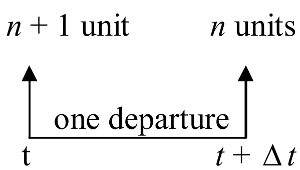


Fig 15.8

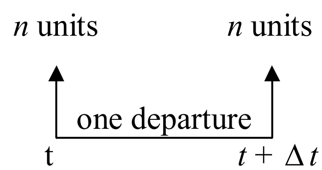


Fig 15.9

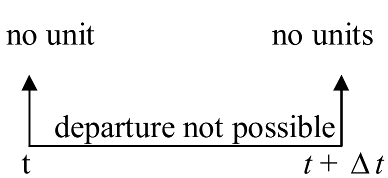
Case II. When n = N. Since there are exactly N units in the system, $P_{n+1}(t) = 0$,

$$\therefore \qquad P_N(t + \Delta t) = P_N(t)\,[1 - \mu\ \Delta t] \qquad \ldots(15.33)$$

Case III. When n = 0.

$$P_0(t + \Delta t) = P_0(t) + P_1(t)\ \mu\ \Delta t \qquad \ldots(15.34)$$

Since there is no unit in the system at time t, the question of any departure during Δt does not arise. Therefore, probability of no departure is unity in this case.



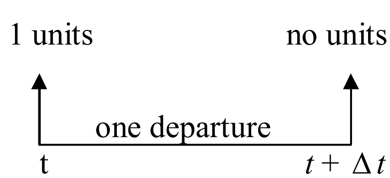


Fig 15.10 **Fig 15.11**

Now, re-arranging the terms and dividing by Δt, and also taking the limit $\Delta t \to 0$ the equations (15.33), (15.32) and (15.34), respectively, become

$$P_N'(t) = -\mu P_N(t), \quad n = N \quad \ldots(15.35)$$

$$P_n'(t) = -\mu P_n(t) + \mu P_{n+1}(t) \quad 0 < n < N \quad \ldots(15.36)$$

$$P_0'(t) = -\mu P_1(t), \quad n = 0 \quad \ldots(15.37)$$

To solve the system of equations (15.35), (15.36) and (15.37) :

Iterative method can be used to solve the system of three equations.

Step l. From equation (15.35) obtain

$$\frac{P_N'(t)}{P_N(t)} = -\mu \text{ or } \frac{d}{dt}\log P_N(t) = -\mu$$

To find s(t) for the Poisson departure case, it has been observed that the probability of no service during time 0 to t is equivalent to the probability of having no departure during the same period.

Thus, Prob. [service time $T \geq t$] = Prob. [no departure during t] = $P_N(t)$

where there are N units in the system and no arrival is allowed after N. Therefore,

$P_N(t) = e^{-\mu t}$

$S(t) = \text{Prob.}(T \leq t) = 1 - \text{Prob.}[T \geq t]$ or $S(t) = 1 - e^{-\mu t}$

Differentiating both sides, w.r.t. 't', we get

$$\frac{d}{dt}S(t) = s(t) = \begin{cases} \mu e^{-\mu t}, t \geq 0 \\ 0, t < 0 \end{cases}$$

Thus, it is concluded that the service time distribution is 'Exponential' with mean $1/\mu$ and variance $1/\mu^2$.

Thus, mean service time= $1/\mu$.

15.7 Summary :

In this lesson the probability distributions of arrival process, inter arrival time and departures are derived. Some properties of these processes are also discussed.

15.8 References :

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.

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Lesson – 16

Queueing Models - I

16.0 Objective :

- Types of waiting line models
- Poisson Queueing Models
- M / M / 1 Queueing Model and its Characteristics
- M / M / S Queueing Model and its Characteristics

Structure

16.1 Introduction

Generally queueing model may be completely specified in the following symbolic form :

$$(a / b / c) : (d / e)$$

The first and second symbols denote the type of distributions of inter-arrival times and of inter-service times, respectively. Third symbol specifies the number of servers, whereas fourth symbol stands for the capacity of the system and the last symbol denotes the queue discipline.

If we specify the following letters as :

M ≡ Poisson arrival or departure distribution,

E_k ≡ Erlangian or Gamma inter-arrival for service time distribution,

GI ≡ General input distribution,

G ≡ General service time distribution,

then $(M / E_k / l) : (\infty/FIFO)$ defines a queueing system in which arrivals follow Poisson distribution, service times are Erlangian, single server, infinite capacity and "first in, first out" queue discipline.

16.2 Transient and Steady States

A queueing system is said to be a transient state when its operating characteristic (like input, output, mean queue length, etc.) are dependent upon time.

If the characteristic of the queueing system becomes independent of time, then an at steady-state condition is said to prevail.

If $P_n(t)$ denotes the probability that there are n customers in the system at time t, then in the steady-state case, we have

$$\lim_{t\to\infty} P_n(t) = P_n \text{ (independent of t)}$$

Due to practical viewpoint of the steady-state behaviour of the systems, the present chapter is amply focused on studying queueing systems under the existence of steady-state conditions. However, the differential-difference equations which can be used for deriving transient solutions will be presented.

16.3 Poisson Queueing Systems

Queues that follow the Poisson arrivals (exponential inter-arrival time) and Poisson services (exponential service time) are called Poisson queues. In this section, we shall study a number of Poisson queues with different characteristics.

16.3.1 Model I {(M / M /1) : (∞/FIFO)}. This model deals with a queueing system having single service channel. Poisson input, Exponential service and there is no limit on the system capacity while the customers are served on a "first in, first out" basis.

The solution procedure of this queueing model can be summarized in the following three steps :

Step 1. Construction of Differential-Difference Equations. Let $P_n(t)$ be the probability that there are *n* customers in the system at time t. The probability that the system has *n* customers at time $(t + \Delta t)$ can be expressed as the sum of the joint probabilities of the four mutually exclusive and collectively exhaustive events as follows :

$$\begin{aligned} P_n(t + \Delta t) = {} & P_n(t).P[\text{no arrival in } \Delta t].P[\text{no service completion in } \Delta t] \\ & + P_n(t) . P[\text{one arrival in } \Delta t] . P[\text{one service completed in } \Delta t] \\ & + P_{n+1}(t) . P[\text{no arrival in } \Delta t] . P[\text{one service completed in } \Delta t] \\ & + P_{n-1}(t) . P[\text{one arrival in } \Delta t] . P[\text{no service completion in } \Delta t] \end{aligned}$$

This is re-written as :

$$P_n(t + \Delta t) = P_n(t)[1 + \lambda \Delta t + o(\Delta t)] [1 - \mu \Delta t + o(\Delta t)] + P_n(t) [\lambda \Delta t] [\mu \Delta t]$$

$$+ P_{n+1}(t) [1 - \lambda \Delta t + o(\Delta t)] [\mu \Delta t + o(\Delta t)] + P_{n-1}(t) [\lambda \Delta t + o(\Delta t)] [1 - \mu \Delta t + o(\Delta t)]$$

or $P_n(t + \Delta t) - P_n(t) = -(\lambda + \mu) \Delta t\, P_n(t) + \mu \Delta t\, P_{n+1}(t) + \lambda \Delta t\, P_{n-1}(t) + o(\Delta t)$

Since Δt is very small, terms involving $(\Delta t)^2$ can be neglected. Dividing the above equation by Δt on both sides and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_n(t) = -(\lambda + \mu) P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t) \; ; \; n \geq 1.$$

Similarly, if there is no customer in the system at time $(t + \Delta t)$, there will be no service completion during Δt. Thus for n = 0 and $t \geq 0$, we have only two probabilities instead of four. The resulting equation is

$$P_o(t+\Delta t) = P_o(t)\{1-\lambda\Delta t + {}_o(\Delta t)\} + P_1(t)\{\mu\Delta t + o(\Delta t)\}\{1-\lambda\Delta t + o(\Delta t)\}$$

or $$P_o(t+\Delta t) - P_o(t) = -\lambda\Delta t\, P_o(t) + \mu\Delta t\, P_1(t) + o(\Delta t).$$

Dividing both sides of this equation by Δt and then taking limit as $\Delta t \to 0$, we get

$$\frac{d}{dt} P_o(t) = -\lambda P_o(t) + \mu P_1(t);\ n = 0.$$

Step 2. Deriving the Steady-State Difference Equations. In the steady-state, $P_n(t)$ is independent of time t and $\lambda < \mu$ when $t \to \infty$. Thus $P_n(t) \to P_n$ and

$$\frac{d}{dt} P_n(t) \to 0 \text{ as } t \to \infty.$$

Consequently the differential-difference equations obtained in Step 1 reduce to

$$0 = -(\lambda+\mu)P_n + \mu P_{n+1} + \lambda P_{n-1};\ n \geq 1$$

and $$0 = -\lambda P_n + \mu P_1;\ n = 0.$$

These constitute the steady-state difference equations.

Step 3. Solution of the Steady-State Difference Equations. For the solution of the above difference equations there exist three methods, namely, the iterative method, use of generating functions and the use of linear operators. Out of these three the first one is the most straightforward and therefore the solution of the above equations will be obtained here by using the iterative method.

Using iteratively, the difference-equations yield

$$P_1 = \frac{\lambda}{\mu} P_o.\ P_2 = \frac{\lambda+\mu}{\mu} P_1 - \frac{\lambda}{\mu} P_o = \left(\frac{\lambda}{\mu}\right)^2 P_o$$

$$P_3 = \frac{\lambda+\mu}{\mu} P_2 - \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^3 P_o, \text{ and in general } P_n = \left(\frac{\lambda}{\mu}\right)^n P_o.$$

Now, $P_{n+1} = \frac{\lambda + \mu}{\mu} P_n - \frac{\lambda}{\mu} P_{n-1}, \; n \geq 1$

Substituting the values of P_n and P_{n-1}, the equation yields

$$P_{n+1} = \frac{\lambda + \mu}{\mu} \left(\frac{\lambda}{\mu}\right)^n P_o - \frac{\lambda}{\mu} \left(\frac{\lambda}{\mu}\right)^{n-1} P_o = \left(\frac{\lambda}{\mu}\right)^{n+1} P_{o.}$$

Thus by the principle of mathematical induction, the general formulae for P_n, is valid for $n \geq 0$.

To obtain the value of P_o, we make use of the boundary condition $\sum_{n=0}^{\infty} P_n = 1$.

$$\therefore \qquad 1 = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_o = P_o \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n; \quad \text{since } P_n = \left(\frac{\lambda}{\mu}\right)^n P_o$$

$$= P_o \frac{1}{1 - \lambda/\mu}, \text{ since } (\lambda/\mu) < 1.$$

This gives $\quad P_o = 1 - \lambda/\mu$.

Hence, the steady-state solution is

$$P_n = (\lambda/\mu)^n (1 - \lambda/\mu) = p^n (1 + p); \quad p = \lambda/\mu < 1, \quad \text{and} \quad n \geq 0.$$

This expression gives us the probability distribution of queue length.

16.3.2 Characteristics of Model I

(i) Probability of queue size being greater than or equal to n, the number of customers is given by

$$P(\geq n) = \sum_{k=n}^{\infty} P_k = \sum_{k=n}^{\infty} (1-p)\, p^k = (1-p)\, p^n \sum_{k=n}^{\infty} p^{k-n} = (1-p)\, p^n \sum_{k-n=0}^{\infty} p^{k-n}$$

$$= \frac{(1-p)p^n}{1-p} = p^n.$$

(ii) Average number of customers in the system is given by

$$E(n) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\,(1 - p)\,p^n = (1 - p) \sum_{n=0}^{\infty} np^n = p(1 - p) \sum_{n=1}^{\infty} np^{n-1}$$

$$= p(1 - p) \sum_{n=0}^{\infty} \frac{d}{dp} p^n = p\,(1 - p) \frac{d}{dp} \sum_{n=0}^{\infty} p^n, \text{ since } p < 1$$

$$= p(1 - p) \frac{1}{(1-p)^2} = \frac{p}{1-p} = \frac{\lambda}{\mu - \lambda}.$$

(iii) Average queue length is given by

$$E(m) = \sum_{m=0}^{\infty} mP_n,$$

where m = n- 1 being the number of customers in the queue, excluding the customer which is in service.

$$\therefore\ E(m) = \sum_{n=1}^{\infty} (n-1)\,P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=0}^{\infty} nP_n - [\sum_{n=0}^{\infty} P_n - P_o]$$

$$= \frac{p}{1-p} - [1 - (1 - p)] = \frac{p}{1-p} - p$$

$$= p^2/(1 - p) = \lambda^2 / \mu\ (\mu - \lambda).$$

(iv) Average length of non-empty queue is given by

$$E(m\,/m > 0) = \frac{E(m)}{P(m > 0)} = \frac{\lambda^2}{\mu(\mu - \lambda)} \cdot \frac{1}{(\lambda / \mu)^2} = \frac{\mu}{\mu - \lambda},$$

since $$P\,(m > 0) = P\,(n > 1) = \sum_{n=0}^{\infty} P_n - P_o - P_1 = \left(\frac{\lambda}{\mu}\right)^2$$

(v) The fluctuation (variance) of queue length is given by

$$V\,(n) = \sum_{n=0}^{\infty} [n - E(n)]^2\ P_n = \sum_{n=0}^{\infty} n^2P_n - [E\,(n)]^2.$$

Using some algebraic transformations and the value of P_n, the result reduces to

$$V(n) = (1-p)\ \frac{p+p^2}{(1-p)^3} - \left[\frac{p}{1-p}\right]^2 = \mu \frac{p}{(1-p)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

16.4 Waiting Time Distribution for Model I. The waiting time of a customer in the system is, for the most part, a continuous random variable except that there is a non-zero probability that the delay will be zero, that is a customer entering service immediately upon arrival. Therefore, if we denote the time spent in the queue by w and if $\psi_w(t)$ denotes its cumulative probability distribution then from the complete randomness of the Poisson distribution, we have

$\psi_w(0) = P(w = 0) = P$ (No customer in the system upon arrival)

$= P_o = (1 - p).$

It is now required to find $\psi_w(t)$ for $t > 0$.

Let there be n customers in the system upon arrival, then in order for a customer to go into service at a time between 0 and t, all the n customers must have been served by time t. Let $s_1, s_2, ..., s_n$ denote service times of n customers respectively. Then

$$w = \sum_{i=1}^{n} s_i, \ (n \geq 1) \quad \text{and} \quad w = 0 \quad (n = 0).$$

The distribution function of waiting time, w, for a customer who has to wait is given by

$$P(w \leq t) = P\ [\sum_{i=1}^{n} si, \leq t\ ];\quad n \geq 1 \quad \text{and} \quad t > 0.$$

Since the service time for each customer is independent and identically distributed, therefore its probability density function is given by $\mu e^{-\mu t}$ $(t > 0)$, where μ is the mean service rate. Thus

$\psi_n(t) = \sum_{n=1}^{\infty}$ P_n x P (n - 1 customers are served at time t) x P (1 customer is served in time Δt)

$$= \sum_{n=1}^{\infty} \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n \frac{(\mu t)^{n-1}e^{-\mu t}}{(n-1)!}.\ \mu\Delta t$$

The expression for ψ_w (t), therefore, can be written as

$$\psi_w\ (t) = P\ (w \le 1) = \sum_{n=1}^{\infty} P_n \int_0^t \psi_n\ (t)\ dt$$

$$= \sum_{n=1}^{\infty} (1 - p)\ p^n \int_0^t \frac{(\mu t)^{n-1}}{(n-1)!} e^{-\mu t}\ .\mu\ dt = (1 - p)\ p \int_0^t \mu e^{-\mu t} \sum_{n=1}^{\infty} \frac{(\mu t p)^{n-1}}{(n-1)!}\ .dt$$

$$= (1 - p)\ p \int_0^t \mu e^{-\mu t\ (1-p)}\ dt.$$

Hence, the waiting time of a customer who, has to wait is given by

$$\psi\ (w) = \frac{d}{dt}\ [\psi_w(t)]\ = p\ (1 - p).\ \mu e^{-\mu t\ (1-p)}\ = \lambda\left(1 - \frac{\lambda}{\mu}\right) e^{-(\mu-\lambda)t}$$

16.4.1 Characteristic of Waiting Time Distribution

(i) Average waiting time of a customer (in the queue) is given by

$$E(w)\ = \int_0^{\infty} t.\ \psi\ (w)\ dt = \int_0^{\infty} t.\ \rho\mu\ (1 - p)\ e^{-\mu t(1-p)}\ dt$$

$$= p \int_0^{\infty} \frac{xe^{-x}}{\mu(1-p)}\ dx, \text{ for } \mu\ (1 - p)\ t = x$$

$$= \frac{p}{\mu(1-p)} = \frac{\lambda}{\mu(\mu-\lambda)}.$$

(ii) Average waiting time of an arrival who has to wait is given by

$$E\ (w \mid w > 0) = \frac{E(w)}{P(w > 0)} = \left\{\frac{\lambda}{\mu(\mu-\lambda)}\right\} / \left(\frac{\lambda}{\mu}\right) = \frac{1}{\mu - \lambda}.$$

[Here P (w > 0) = 1 - P (w = 0) = 1 - (1 - p) = p.]

(iii) For the busy period distribution, let the random variable v denote the total time that a customer has to spend in the system including service. Then the probability density of its cumulative density function is given by

$$\psi\ (w \mid w > 0) = \frac{\psi(w)}{P(w>0)} = \left[\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}\right] / \left(\frac{\lambda}{\mu}\right)$$

$$= (\mu-\lambda)e^{-(\mu-\lambda)t} \qquad t \geq 0$$

(iv) Average waiting time that a customer spends in the system including service is given by

$$E(v) = \int_0^\infty t.\ \psi\ (w \mid w > 0)\ dt = \int_0^\infty t.\ (\mu-\lambda)e^{-(\mu-\lambda)t}\ dt$$

$$= \frac{1}{\mu-\lambda} \int_0^\infty x^{e-x}\ dx, \text{ for } (\mu-\lambda)\ t = x$$

$$= \frac{1}{\mu-\lambda}.$$

16.4.2. Relations between Average Queue Length and Average Waiting Time - Little's Formula.

We have derived above the following important characteristics of M/M/l queueing system :

$$E(n) = \frac{\lambda}{\mu-\lambda}, \quad E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)}, \quad E(w) = \frac{\lambda}{\mu(\mu-\lambda)} \text{ and } E(v) = \frac{1}{\mu-\lambda}.$$

Using these expressions, we observe that

$$E(n) = \lambda E(v), \quad E(m) = \lambda E(w) \quad \text{and} \quad E(v) = E(w) + 1/\mu.$$

16.5 Examples on Model - I

Example 16.5.1. A T.V. repairman finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs sets in the order in which they came in, and if the arrival of sets is approximately Poisson with an average rate of 10 per 8-hour day, what is repairman 's expected idle time each day? How many jobs are ahead of the average set just brought in?

Solution. We are given,

λ = 10 sets per day, and μ = 16 sets per day.

$\therefore \quad p = \lambda / \mu = 10/16 = 0.625$

The probability for the repairman to be idle is

$$P_o = 1 - p = 1 - 0.625 = 0.375$$

(i) Expected idle time per day = 8 x 0.375 = 3 hours.

(ii) Expected (or average) number of T.V. sets in the system

$$E(n) = \frac{p}{1-p} = \frac{0.625}{1-0.625} = \frac{5}{3} = 2 \text{ (approx.) T.V. sets.}$$

Example 16.5.2. In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average 36 minutes. Calculate the following :

(i) the mean queue size (line length), and

(ii) the probability that the queue size exceeds 10.

If the input of trains increases to an average 33 per day, what will be the change in (i) and (ii) ?

Solution. Here, we have

$$\lambda = \frac{30}{60x24} = \frac{1}{48} \quad \text{and} \quad \mu = \frac{1}{36} \text{ trains per minute.}$$

$\therefore \quad p = \lambda / \mu = 36/48 = 0.75$

(i) $\quad E\,(m) = \frac{p}{1-p} = \frac{0.75}{1-0.75} = 3$ trains.

(ii) $\quad p\,(\geq 10) = p^{10} = (0.75)^{10} = 0.06.$

When the input increases to 33 trains per day, we have

$$\lambda = \frac{33}{60x24} = \frac{11}{480} \text{ and } \mu = \frac{1}{36} \text{ trains per minute.}$$

$\therefore \quad p = \lambda / \mu = \frac{11}{480} \text{ x } 36 = 0.83$

Then, we get

(i) $E(n) = \frac{p}{1-p} = \frac{0.83}{1-0.83} = 4.9$ or 5 trains (approx.)

(ii) $P(\geq 10) = p^{10} = (0.83)^{10} = 0.2$ (approx.)

16.5.3. A road transport company has one reservation clerk on duty at a time. He handles information of bus schedules and makes reservations. Customers arrive at a rate of 8 per hour and the clerk can service 12 customers on an average per hour. After stating your assumptions, answer the following :

(i) What is the average number of customers waiting for the service of the clerk?

(ii) What is the average time a customer has to wait before getting service?

(iii) The management is contemplating to install a computer system to handle the information and reservations. This is expected to reduce the service time from 5 to 3 minutes. The additional cost of having the new system works out to Rs. 50 per day. If the cost of goodwill of having to wait is estimated to be 12 paise per minute spent waiting before being served. Should the company install the. computer system? Assume 8 hours working day.

Solution. We are given

λ = 8 customers per hour and μ = 12 customers per hour. Average number of customers waiting for the service of the clerk (in the system) :

$$E(n) = \frac{\lambda}{\mu-\lambda} = \frac{8}{12-8} = 2 \text{ customers.}$$

(i) The average number of customers waiting for the service of the clerk (in the queue) :

$$E(m) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{8x8}{12(12-8)} \text{ or } 1.33 \text{ customers.}$$

(ii) The average waiting time of a customer (in the system) before getting service :

$$E\ (w) = \frac{1}{\mu - \lambda} = \frac{1}{12-8} \text{ hour or 15 minutes.}$$

The average waiting' time of a customer (in the queue) before getting service :

$$E\ (n) = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{8}{12(12-8)} = \frac{1}{6} \text{hour or 10 minutes.}$$

(iii) We now calculate the difference between the goodwill cost of customers with one system and the goodwill cost of customers with an additional computer system. This difference will be compared with the additional cost (of Rs. 50 per day) of installing another computer system.

An arrival waits for E (v) hours before being served and there are λ arrivals per hour. Thus expected waiting time for all customers in an 8-hour day with one system = 8λ.

$E\ (v) = 8 \text{ x } 8 \text{ x } \frac{1}{6}$ hrs. or $\frac{64}{6}$ x 60 minutes, i.e., 640 minutes.

The goodwill cost per day with one system = 640 x Re. 0.12 = Rs. 76.80.

The expected waiting time of a customer before getting service when there is an additional computer system is :

$$E(v^*) = \frac{8}{20(20-8)} = \frac{8}{20x12} \text{ or } \frac{1}{30} \text{ hr.}$$

Thus expected waiting time of customers in an 8-hour day with an additional computer system is 8λ x E(v*)

$$= 8 \text{ x } 8 \text{ x } \frac{1}{30} \text{ hr.} = 128 \text{ minutes.}$$

The total goodwill cost with an additional computer system = 128 x Re. 0.12 = Rs. 15.36.

Hence reduction in goodwill cost with the installation of a computer system

= Rs. 76.80- Rs. 15.36 = Rs. 61.44.

Whereas the additional cost of a computer system is Rs. 50 per day, Rs. 61.44 is the reduction in goodwill cost when additional computer system is installed, hence there will be net saving of Rs. 11.44 per day. It is, therefore, worthwhile to install a computer.

16.6. Model II {(M/M/S) : (∞/FIFO)}. This model is a special case of Model IV in the sence that here we consider C parallel service channels. The arrival rate is λ and the service rate per service channel is μ .

The effect of using C parallel service channels is a proportionate increase in the service rate of the facility to nμ if n $\leq$ C and Cμ if n > C. Thus, in terms of the generalized model (Model IV), λ_n and μ_n are defined as

$$\lambda_n = \lambda \ , \quad n > 0$$

and μ_n = nμ if $1 \leq n \leq C$ and Cμ , if n $\geq$ C.

Utilizing the above values of λ_n and μ_n, the steady-state probabilities of Model IV becomes

$$P_n = \begin{cases} \dfrac{\lambda^n p_o}{n\mu(n-1)\mu ...(1)\mu}; 1 \leq n \leq C, \\ \dfrac{\lambda^n p_o}{\underbrace{(c\mu)(c\mu)...(c\mu)}_{n\text{ - C terms}}(c\mu)(c-1)\mu(c-2)\mu...(1)\mu}; n > C \end{cases}$$

$$= \frac{\lambda^n P_o}{n!\mu^n} \text{ if } 1 \leq n \leq C \text{ and } \frac{\lambda^n P_o}{C^{n-c} C! \mu^n} \text{ if } n > C,$$

$$= \frac{1}{n!} p^n p_o \text{ if } 1 \leq n \leq C \text{ and } \frac{1}{C^{n-c} C!} p^n p_o \text{ if } n > C,$$

To find the value of P_o, we use the boundary condition $\sum_{n=0}^{\infty} P_n = 1$. Thus, we have

$$\sum_{n=0}^{C-1} P_n + \sum_{n=C}^{\infty} P_n = 1$$

or
$$\left[\sum_{n=0}^{C-1} \frac{1}{n!} p^n + \sum_{n=C}^{\infty} \frac{1}{C^{n-C} C!} p^n\right] P_o = 1$$

$$\Rightarrow \qquad P_o = \left[\sum_{n=0}^{C-1} \frac{1}{n!} p^n + p^C \sum_{n=C}^{\infty} \frac{1}{C!}\left(\frac{p}{C}\right)^{n-C}\right]^{-1}$$

$$= \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!}\left(\frac{\lambda}{\mu}\right)^C . \frac{C\mu}{C\mu - \lambda}\right]^{-1}$$

Remark. The result obtained above is valid only if $\frac{\lambda}{C\mu} < 1$; that is, the mean arrival rate must be less than the mean maximum potential service rate of the system. If C = 1, then the value of P_o is in complete agreement with the value of P_o for Model I.

16.6.1 Characteristics of Model II

(i) P (n ≥ C) = Probability that an arrival has to wait

$$= \sum_{n=C}^{\infty} P_n = \sum_{n=C}^{\infty} \frac{1}{C! C^{n-C}} (\lambda / \mu) \; P_o = \frac{(\lambda/\mu)^C C\mu}{C!(C\mu - \lambda)} P_o$$

(ii) Probability that an arrival enters the service without wait

$$= 1 - P\,(n \geq C) \text{ or } 1 - \frac{C(\lambda/\mu)^C}{C!(C - \lambda/\mu)} P_o$$

(iii) Average queue length is given by

$$E\,(m) = \sum_{n=C}^{\infty} (n - C)\, P_n = \sum_{x=0}^{\infty} x\, Px + c, \quad \text{for} \quad x = n - C$$

$$= \sum_{x=0}^{\infty} x.\ \frac{1}{C! C^x} (\lambda/\mu)^{C+x} P_o$$

or
$$E(m) = \frac{1}{C!}(\lambda/\mu)^C \sum_{x=0}^{\infty} x.\ (\lambda / C\mu)^x P_o$$

$$= \frac{1}{C!}(\lambda/\mu)^C P_o \sum_{x=0}^{\infty} \left(\frac{d}{dy} y^x\right).y, \text{ where } y = \frac{\lambda}{C\mu}$$

$$= \frac{1}{C!}(\lambda/\mu)^C \; P_o \, y \, \frac{d}{dy}\left(\frac{1}{1-y}\right)$$

$$= \frac{\lambda\mu(\lambda/\mu)^C P_o}{(C-1)!(C\mu-\lambda)^2}$$

(iv) Average number of customers in the system is given by

$$E(n) = E(m) + \frac{\lambda}{\mu} = \frac{\lambda\mu(\lambda/\mu)^C P_o}{(C-1)!(C\mu-\lambda)^2} + \frac{\lambda}{\mu}$$

(v) Average waiting time of an arrival is given by

$$E(w) = \frac{1}{\lambda} \; E(m) = \frac{\mu(\lambda/\mu)^C P_o}{(C-1)!(C\mu-\lambda)^2}.$$

(vi) Average waiting time an arrival spends in the system is given by

$$E(v) = E(w) + \frac{1}{\mu} = \frac{\mu(\lambda/\mu)^C P_o}{(C-1)!(C\mu-\lambda)^2} + \frac{1}{\mu} \text{ or } E(v) = E(n)/\lambda$$

(vii) Average number of idle servers is equal to

C - Average number of customers served.

16.7. Examples on Model - II

Example 16.7.1 A supermarket has two girls serving at the counters. The customers arrive in a Poisson fashion at the rate of 12 per hour. The service time for each customer is exponential with mean 6 minutes. Find (i) the probability that an arriving customer has to wait for service, (ii) the average number of customers in the system, and (iii) the average time spent by a customer in the super market.

Solution. We are given

λ = 12 customers per hour. μ = 10 per hour, and C = 2 girls.

$$\therefore \qquad P_o = \left[\sum_{n=0}^{2-1} \frac{1}{n!} \left(\frac{12}{10} \right)^n + \frac{1}{2!} \left(\frac{12}{10} \right)^2 . \frac{2x10}{20-12} \right]^{-1} = \frac{1}{4} \text{ (or 0.25).}$$

(i) Probability of having to wait for service

$$P\ (w > 0) = \frac{1}{C!} (\lambda / \mu)^C \frac{C\mu}{(C\mu - \lambda)} P_o$$

$$= \frac{1}{2} (12/10)^2 \frac{20}{20-12} x \frac{1}{4} = 0.45$$

(ii) Average queue length is

$$E(m) = \frac{\lambda\mu(\lambda/\mu)^C P_o}{(C-1)!(C\mu-\lambda)^2} = \frac{12x10x(1.2)^2 x0.25}{(2-1)!(20-12)^2} = \frac{27}{40}$$

Average number of customers in the system

$$E(n) = E(m) + \frac{\lambda}{\mu} = \frac{27}{40} + \frac{12}{10} = 1.87 \text{ (or 2 customers) approx.}$$

(iii) Average time spent by customer in supermarket

$E(v) = E(n)/\lambda$ = 1.87/12 = 0.156 hours or 9.3 minutes.

Example 16.7.2 A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles deposits only. It has been found that the service time distribution for both deposits and withdrawals is exponential with mean service time 3 minutes per customer. Depositors are found to arrive in Poisson fashion throughout the day with mean arrival rate of 16 per hour. Withdrawers also arrive in Poisson fashion with mean arrival rate of 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits? What could be the effect if this could be accomplished by increasing the mean service time to 3.5 minutes?

Solution. Initially we have two independent queueing systems for withdrawers and depositors with input as Poisson distribution and service as exponential distribution.

For withdrawers : λ = 14/hour; μ = 3/minute or 20/hour

Average waiting time. in the queue

$$E(w) = \frac{14}{20(20-14)} = \frac{14}{20x6} = \frac{7}{60} \text{ hour or 7 minutes.}$$

For depositors : λ = 16/hour ; μ = 3/minute or 20/hour.

Average waiting time in the queue

$$E(w) = \frac{16}{20(20-16)} = \frac{16}{20x4} = \frac{1}{5} \text{ hour or 12 minutes.}$$

If each teller could handle both withdrawals and deposits, we have a common queue with two servers. The queueing system is thus with 2 service channels with λ = 14 + 16 = 30/hour and μ = 20/hour.

$$\therefore \quad P_o = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{30}{20}\right)^n + \frac{1}{2!}\left(\frac{30}{20}\right)^2 \frac{2x20}{(2x20-30)}\right]^{-1} = 1/7$$

Average waiting time of arrivals in the queue

$$E(w) = \left(\frac{30}{20}\right)^2 \text{ x } \frac{20}{(40-30)^2} \text{ x } \frac{1}{7} = \frac{9}{140} \text{ hour or 3.86 minutes.}$$

When the service time is increased to 3.5 minutes,

λ = 30/hour and μ = 120/7 or 17.14/hour.

$$\therefore \quad P_o = \left[\sum_{n=0}^{2-1} \frac{1}{n!}\left(\frac{21}{12}\right)^n + \frac{1}{2!}\left(\frac{21}{12}\right)^2 \frac{2x17.14}{(2x17.14-30)}\right]^{-1} = 1/15$$

Average waiting time of arrivals in the queue

$$E(w) = \left(\frac{21}{12}\right)^2 \text{ x } \frac{17.14}{(34.28-30)^2} \text{ x } \frac{1}{15} = \frac{343}{30} \text{ hour or 11.43 minutes.}$$

16.8. Summary

In this lesson the types of waiting line models, M/M/1 Queuing models and M/M/S Queuing models along with its characteristics are discussed. Suitable examples are given to understand the concept much better.

16.9. Exercise

1. A foreign bank is considering opening a drive-in window for customer service. Management estimates that customers will arrive for service at the rate of 12 per hour. The teller whom it is considering to staff the window can serve customers at the rate of one every three minutes. Assuming Poisson arrivals and Exponential service, find:

   (i) Utilisation of teller,

   (ii) Average number in the system,

   (iii) Average waiting time in the line, and

   (iv) Average waiting time in the system.

2. At a one-man barber shop, customers arrive according to Poisson distribution with a mean arrival rate of 5 per hour and his hair cutting time was exponentially distributed with an average haircut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following :

   (i) Average number of customers in the shop and the average number of customers waiting for a hair-cut.

   (ii) The percentage of time an arrival can walk right in without having to wait.

   (iii) The percentage of customers who have to wait prior to getting into the barber's chair.

3. Customers arrive at a sales counter manned by a single person according to a Poisson process with a mean rate of 20 per hour. The time required to serve a customer has an exponential distribution with a mean of 100 seconds. Find the average waiting time of a customer.

4. Patients arrive at a clinic at an average of 6 patients per hour. Some patients require only the repeat prescription, some come for minor check-up while some others require thorough inspection for diagnosis. This takes the doctor six minutes per

patient on an average. It can be assumed that arrivals follow a Poisson distribution and the doctor's inspection time follows an exponential distribution. Determine :

(i) the percentage of times a patient can walk right inside the doctor's cabin, without having a wait,

(ii) the average number of patients in the clinic,

(iii) the average number of patients waiting for their turn, and

(iv) the average time a patient spends in the clinic.

5. A supermarket has a single cashier. During the peak hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Calculate :

   (i) the probability that the cashier is idle,

   (ii) the average number of customers in the queueing system,

   (iii) the average time a customer spends in the system,

   (iv) the average number of customers in the queue, and

   (v) the average time a customer spends in the queue waiting for service.

6. A petrol pump station has two pumps. The service times follows the exponential distribution with a mean of 4 minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait, for service. What proportion of time the pumps remain idle?

7. Given an average arrival rate of 20 per hour, is it better for a customer to get service at a single channel with mean service rate of 22 customers or at one of two channels in parallel, with mean service rate of 11 customers for each of the two channels? Assume that both queues are of M/M/S type.

8. A telephone exchange has two long-distance operators. The telephone company finds that during the peak load, long-distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length of 5 minutes.

(a) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

(b) If the subscribers wait and are serviced in turn, what is the expected waiting time? Establish the formulae used.

9. A telephone company is planning to install telephone booths in a new airport. It has established the policy that a person should not have to wait more than 10 per cent of the times he tries to use a phone. The demand for use is estimated to be Poisson with an average of 30 per hour. The average phone call has an exponential distribution with a mean time of 5 minutes. How many phone booths should be installed ?

10. A library wants to improve its service facilities in terms of the waiting time of its, borrowers. The library has two counters at present and borrowers arrive according to Poisson distribution with arrival rate 1 every 6 minutes and service time follows exponential distribution with a mean of 10 minutes. The library has relaxed its membership rules and a substantial increase in the number of borrowers is expected. Find the number of additional counters to be provided if the arrival rate is expected to be twice the present value and the average waiting time of the borrower must be limited to half the present value.

16.10 References

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Lesson – 17

Queueing Models - II

17.0 Objective :

- Non – Poisson Queues
- M / E_k / 1 Queueing Model
- M / G / 1 Queueing Model

Structure

17.1 Non – Poisson Queueing Model

The queues in which arrivals and/or departures may not follow the Poisson axioms are called Non-Poisson queues. The development of these queueing systems becomes more difficult, mainly because the Poisson axioms no longer hold good. However, following techniques are usually adopted for the development of non-Poisson queues:

(a) Phase Technique. This technique is used when an arrival demands phases of service, say k in number.

(b) Imbedded Markov Chain Technique. The technique by which non-Markovian queues are reduced to Markovian is termed as Imbedded Markov Chain Technique.

(c) Supplementary Variable Technique. When one or more random variables are added to convert a non-Markovian process into a Markovian one, the technique involved is called Supplementary Variable Technique. This technique is used for the queueing models : GI | G | C, M | G | 1, GI | M | S , GI | E_k | 1, D | E_k | 1. But we have not presented these models here initially.

However we shall introduce the queueing system M | Ek | 1 and the steady state results of M | G | 1.

17.2 Model III {(M/E_k/1) : (∞/FIFO)}. This model consist of a single service channel queueing system in which there are n phases in the system (waiting or in service). It has been assumed that a new arrival creates k-phases of service and departure of one customer reduces k-phases of service. Let

n - number of customers in the system,

$\lambda_n = \lambda$, constant arrival rate per unit time,

$\mu_n = k\mu$, k phases of service per unit time.

When P_n denotes the steady-state probability of n phases in the system, the transition-rate diagram of the model under consideration is :

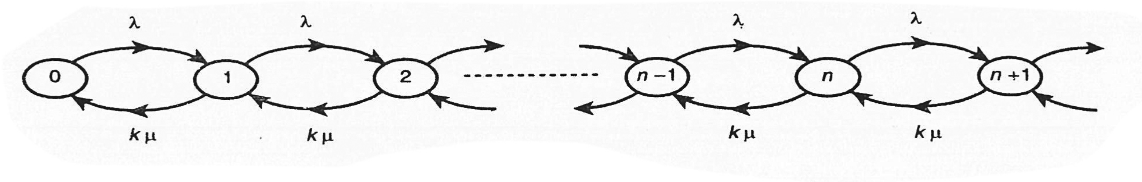

Fig. 17.4

The balance equations, therefore, are

$$\lambda P_{n-k} + k\mu P_{n+1} = \lambda P_n + k\mu P_n, \qquad n \geq 1$$

and $$k\mu P_1 = \lambda P_0, \qquad n = 0$$

Letting $(\lambda/k\mu) = p$, these equations are

$$(1 + p)P_n = p\,P_{n-k} + P_{n+1}, \qquad n \geq 1$$

and $$P_1 = p\,P_0\,, \qquad n = 0.$$

The solution of these difference equations is beyond the scope of this book. However, we discuss the characteristics of this model using these difference equations.

17.2.1. Characteristics of Model I

(i) Average number of phases in the system $E(n_p)$ is obtained as follows :

Multiplying by n^2 on both sides and then taking summation, the above difference equation gives

$$(1+p)\sum_{n=1}^{\infty} n^2P_n = p\sum_{n=k}^{\infty} n^2P_{n-k} + \sum_{n=1}^{\infty} n^2P_{n+1} = p\sum_{x=0}^{\infty}(x+k)^2P_x + \sum_{y=2}^{\infty}$$

$(y - l)^2Py$, (where $n - k = x$ and $n + 1 = y$)

$$= p\sum_{x=0}^{\infty}(x^2 + 2xk + k^2)P_x + \sum_{y=1}^{\infty}(y - 2y + 1)P_y,$$

$$\text{since } \sum_{y=2}^{\infty}(y-1)^2P_y = \sum_{y=1}^{\infty}(y-1)^2Py$$

$$= (1+p)\sum_{n=1}^{\infty} n^2P_n + p\sum_{n=0}^{\infty}(2nk + k^2)P_n + \sum_{n=1}^{\infty}(-2n+1)P_n$$

or $0 = p\left[\ 2k \sum_{n=0}^{\infty} nP_n + k^2 \sum_{n=0}^{\infty} P_n\ \right] + \sum_{n=1}^{\infty} P_n - 2 \sum_{n=1}^{\infty} nP_n$

or $2(1 - kp) \sum_{n=0}^{\infty} nP_n = pk^2 - P_0 + 1,$

since $\sum_{n=0}^{\infty} P = 1$ and $\sum_{n=1}^{\infty} nP_n = \sum_{n=0}^{\infty} nP_n.$

$$\therefore\ E(n_p) = \frac{pk^2 + 1 - 1(1 - kp)}{2(1 - kp)} = \frac{k(k+1)p}{2(1 - kp)}$$

i.e. $E(n_p) = \frac{k(k+1)}{2} \cdot \frac{\lambda / k\mu}{1 - k\lambda / k\mu} = \frac{k+1}{2} \frac{\lambda}{\mu - \lambda}.$

(ii) Average waiting time of the phases in the system is given by

$$E(w_p) = \frac{E(n_p)}{\mu} = \frac{k+1}{2\mu} \frac{\lambda}{\mu - \lambda}.$$

(iii) Average waiting time of an arrival is given by

$$E(w) = \frac{E(w_p)}{k} = \frac{k+1}{2k} \frac{\lambda}{\mu(\mu - \lambda)}.$$

(iv) Average time an arrival spends in the system is given by

$$E(v) = E(w) + \frac{1}{\mu} = \frac{k+1}{2k} \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

(v) Average number of units in the system is given by

$$E(n) = \lambda\ E(v) = \frac{k+1}{2k} \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

(vi) Average queue length is given by

$$E(m) = E(n) - \frac{\lambda}{\mu} = \frac{k+1}{2k} \frac{\lambda^2}{\mu(\mu - \lambda)}.$$

17.3 Examples on Model - III

Example 17.3.1. A hospital clinic has a doctor examining every patient brought in for a general check-up. The doctor spends 4 minutes on each phase of the check-up although the distribution of time spent on each phase is approximately exponential. If each patient goes through four phases in the check-up and if the arrivals of the patients to the doctor's office are approximately Poisson at the average rate of three per hour, what is the average time spent by a patient waiting in the doctor's office? What is the average time spent in the check-up? What is the most probable time spent in the check-up?

Solution. We are given

k = 4 and Mean arrival rate = 3 patients per hour, i.e., λ = 3 per hour.

Service time per phase = $1/4\,\mu$ = 4 minutes

$$\therefore \quad \mu = \frac{1}{4x4} = \frac{1}{16} \text{ patients per minute}$$

and $$E(w) = \frac{4+1}{4x4}\,\frac{3}{\frac{15}{4}\left(\frac{15}{4}-3\right)} = 40 \text{ minutes.}$$

Average time spent in the examination $1/\mu$ = 16 minutes.

Most probable time spent in the examination

$$= \frac{k-1}{k\mu} = \frac{4-1}{4x\frac{1}{16}} = \frac{3}{1/4} = 12 \text{ minutes}$$

Example 17.3.2. An airline maintenance base has facilities for overhauling only one airplane engine at a time. Hence, to return the airplanes into use at the earliest, the policy is to stagger the overhauling of the 4 engines of each airplane. In other words only one engine is overhauled each time in airplane comes into the base. Under this policy, airplanes have arrivals according to a Poisson process at a mean rate of 1/day. The time required for an engine overhaul has an exponential distribution with a mean of 1/2 day.

A proposal has been made to change the policy so as to overhaul all four engines consecutively each time an aeroplane comes into the shop. It is pointed out that although this will quadruple the expected service time, each plane would need to come into the shop only one-fourth time as often. Compare the two alternatives on a meaningful basis.

Solution. The two alternatives will be compared on the basis of the waiting time cost of the airplanes requiring overhauling.

First alternative : {M / M / 1) : (∞/FIFO} queueing system.

Given that λ = 1 airplane per day ; μ = 2 airplanes per day.

Therefore, average number of airplanes in the system are

$$E(n) = \frac{\lambda}{\mu - \lambda} = \frac{1}{2-1} = 1.$$

Second alternative : {M/E_k/1) : (∞/FIFO} queueing system.

Given that λ = 1/4 airplane per day ; k = 4.

Since service time per airplane is 4 x (1/2) = 2 days, therefore mean service rate, μ = 1/2 airplane per day. Thus

Average number of airplanes in the system are

$$E(n) = \frac{k+1}{2k} \cdot \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu} = \frac{4+1}{2.(4)} \cdot \frac{(1/4)^2}{\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)} + \frac{1/4}{\frac{1}{2}} = \frac{13}{16} \text{ or } 0.81.$$

Since E_n(= 0.81) in the second alternative is less than its value in the first alternative, therefore the waiting cost for requiring overhauling in the second alternative will be less. Hence, the proposal should be accepted.

Example 17.3.3. At a certain airport it takes exactly 5 minutes to land an aeroplane, once it is given the signal to land. Although Incoming planes have scheduled arrival times the wide variability in arrival times produces an effect which makes the incoming planes appear to arrive in a Poisson fashion at an average rate of 6 per hour. This produces occasional stockups at the airport which can be dangerous and costly. Under these circumstances, how much time will a pilot expect to spend circuling the field waiting to land?

Solution. From the data of the problem, we have

λ = 6 per hour or 1/10 per minute ; μ = 1/5 per minute and k = ∞, as service time is constant. Hence, the average time which a pilot expects to spend circling the field waiting to land is given by

$$E(w) = \lim_{k\to\infty} \frac{k+1}{2k} \cdot \frac{\lambda}{\mu(\mu-\lambda)} = \lim_{k\to\infty} \frac{1}{2}\left(1+\frac{1}{k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right)$$

$$= \frac{1}{2}\left(1+\frac{1}{\infty}\right)\frac{1/10}{\left(\frac{1}{5}\right)\left(\frac{1}{5}-\frac{1}{10}\right)}$$

= 5/2 or 2.5 minutes.

17.4 Model IV {(M/G/1) : (∞/GD)}.

This model consists of a single server, Poisson arrivals and general service time distribution. In this case, the probability that a customer leaves the system in $(t, t+\Delta t)$ is a quantity dependent on when the service of that particular customer begins, i.e., they would involve not merely t, but also another time variable, say t' measured from the beginning of the current service.

Let q_0 denote the queue length when the service of customer n terminates, and q_1 be the queue length when the service of customer (n+1) terminates. Further, let k (k = 1, 0, 2, ...) denote the number of customers who arrive during the service of customer (n + 1). Then we can have the following relations between q_o, q_1 and k :

$$q_1 = \begin{cases} k; & \text{if } q_0 = 0, \text{ i.e., the queue length is zero after serving nth customer} \\ (q_0 - l) + k; & \text{if } q_o > 0, \text{ i.e., the queue has } q_o \text{ units after serving nth customer.} \end{cases}$$

These two relations together can be written as $q1 = q_0 - \delta + k$, where $\delta = 0$, if $q_o = 0$ and 1 if $q_0 > 0$.

The expected value of the above relation is obtained as, $E(q_1) = E(q_o) - E(\delta) + E(k)$

Since $E(q_1) = E(q_0)$ in the steady state, it follows that $E(\delta) = E(k)$.

Further, $q_1^2 = (q_0 - \delta + k)^2 = (q_o^2 + \delta^2 + 2k\,q_o - 2k\delta - 2\delta - 2\delta\,q_o)$.

But by definition, $\delta^2 = \delta$ and $\delta q_o = q_o$

$\therefore \qquad q_1^2 = q_o^2 + k^2 + 2kq_o + \delta - 2k\delta - 2\delta q_o$

or $\qquad E(q_1^2) = E(q_o^2) + E(k^2) + 2E(kq_o) + E(\delta) - 2E(k\delta) - 2E(q_o).$

Again, since $E(q_1^2) = E(q_o^2)$ in me steady-state, it follows that

$$2E(q_o) - 2E(kq_o) = E(k^2) + E(\delta) - 2E(k\delta)$$

or
$$E(q_o) = \frac{E(k^2) + E(k) - 2E^2(k)}{2[1 - E(k)]}$$

since $E(\delta) = E(k)$ and q_o, k and δ are independent.

If λ is the mean arrival rate, the number of arrivals in a service time of length t has a Poisson distribution with mean λt. The probability of k arrivals in time t, therefore, is given by

$$\frac{(\lambda t)^k e^{-\lambda t}}{k!}, \; k \geq 1,$$

and the probability of k arrivals during the service time of a customer is

$$\int_0^\infty \frac{(\lambda t)^t e^{-\lambda t}}{k!} p(t)\, dt,$$

where p (t) is the probability density function of service time.

The mean number of arrivals in the service time of a customer is given by

$$E(k) = \lambda E(t) = \lambda\ (l/\mu); \quad \text{where } \mu. \text{ is the mean service rate.}$$

The variance of arrivals in the service time of a customer is

$$E(k^2) = \lambda E(t) + \lambda^2 E(t^2)$$

$= \lambda$ [mean service time] $+ \lambda^2$ [variance of service time + square of mean service time]

$$= (\lambda/\mu) + [\lambda^2 (\sigma^2 + 1/\mu^2)] = p + \lambda^2 \sigma^2 + p^2.$$

Substituting the values of E (k) and $E(k^2)$ in the relation for $E(q_o)$ obtained above

$$E(q_o) = \frac{(p + \lambda^2\sigma^2 + p^2) + p - 2p^2}{2(1-p)} = p + \frac{\lambda^2\sigma^2 + p^2}{2(1-p)}$$

Further, if the customer (n + 1) has to wait for time ω before being taken into service, the length of the queue must be q_l during the time $\omega + t$.

$$\therefore \quad E(q_1) = \lambda E(\omega + r) = \lambda E(\omega) + \lambda E(r)$$

i.e, $\lambda E(\omega)$ = Average waiting time = $E(q_1) - (\lambda/\mu)$, where $E(r) = 1/\mu$

or
$$E(\omega) = \frac{1}{\lambda}\left[p + \frac{\lambda^2\sigma^2 + p^2}{2(1-p)} - p\right] = \frac{\lambda^2\sigma^2 + p^2}{2\lambda(1-p)}.$$

since $E(q_1) = E(q_o)$ in the steady state.

We can now summarize the various formulae for (M/G/1) : (∞/GD) as follows :

Average number of customers in the system = $\dfrac{\lambda^2\sigma^2 + p^2}{2(1-p)} + p.$

Average queue length = $\left[\dfrac{\lambda^2\sigma^2 + p^2}{2(1-p)} + p\right] - p = \dfrac{\lambda^2\sigma^2 + p^2}{2(1-p)}.$

Average waiting time of a customer in the queue = $\dfrac{\lambda^2\sigma^2 + p^2}{2\lambda(1-p)}.$

Average waiting time that a customer spends in the system = $\dfrac{\lambda^2\sigma^2 + p^2}{2\lambda(1-p)} + \dfrac{1}{\mu}$

Remark. The formulae for the average queue length and the average number of customers in the system are known as Pollaczek - Khintchine (or P-K) formulae.

17.5. Examples on Model - IV

Example 17.5.1. In a heavy machine shop, the overhead crane is 75 per cent utilised. Time study observations gave the average slinging time as 10.5 minutes with a standard deviation of 8.8 minutes. What is the average calling rate for the services of the crane, and what is the average delay in getting service? If the average-service time is cut to 8.0 minutes, with standard deviation of 6.0 minutes, how much reduction will occur, on average, in the delay of getting served?

Solution. This is a (M / G / 1) : (∞ / FCFS) process.

The average delay in getting service is given by

$$E(w) = \frac{p(1+\mu^2\sigma^2)}{2\mu(1-p)}$$

Initial situation : p = 0.75, $\mu = \frac{60}{10.5} = 5.71$ per hour, $\lambda = p \times \mu = 0.75 \times 5.71 =$ 4.29 per hour

$$E(w) = \frac{0.75}{2(1-0.75)} \left[1+(5.71)^2\left(\frac{8.8}{60}\right)^2\right] \text{x } \frac{60}{5.71}$$

$$= \frac{0.75}{0.5} \text{ x } 1.70 \text{ x } \frac{60}{5.71} = 26.8 \text{ minutes.}$$

If service time is cut to 8 minutes, then

$$\mu = \frac{60}{8} = 7.5 \text{ per hour, and} \quad p = \frac{4.29}{7.5} = 0.571$$

or utilisation of the crane reduced to 57.1 per cent.

Then $$E(w) = \frac{0.571}{2(1-0.571)} \left[1+(7.5)^2 x\left(\frac{6.0}{60}\right)^2\right] \text{x } \frac{60}{7.5}$$

$$= \frac{0.571}{2x0.429} \text{ x } 1.562 \text{ x } 8 = 8.3 \text{ minutes.}$$

a reduction of 1 8.5 minutes or approximately 70 per cent.

17.6 Summary

In the above lesson the Non Poisson Queues and two Queuing Models M / E_k / 1 Queuing Model and M / G / 1 Queuing Model are discussed elaborately also few worked out examples are given in support to the theory.

17.7 Exercise

1. A barber with a one-man takes exactly 25 minutes to complete one haircut. If customers arrive in a Poisson fashion at an average rate of one every 40 minutes, how long on the average must a customer wait for service? Also find the average time a customer spends in the barber shop.

2. A tailoring shop with one man takes exactly one day to stitch a suit. Customer's arrival follow a Poisson pattern with mean rate of arrival of one in every two day. How long, on an average, a customer is expected to wait in such a situation?

3. In a car manufacturing plant, a loading crane takes exactly 10 minutes to load a car into a wagon and again come back to position to load another car. If the arrival of cars is a Poisson stream at an average of one every 20 minutes, calculate the average waiting time of a car.

4. A barber runs his own saloon. It takes him exactly 25 minutes to complete one haircut. Customers arrive in a Poisson fashion at an average rate of one every 35 minutes.

   (a) For what percent of time would the barber be idle?

   (b) What is the average time of a customer spent in the shop?

5. The repair of a Lathe requires four steps to be completed one after another in a certain order. The time taken to perform each step follows exponential distribution with a mean of 5 minutes and is independent of other steps. Machine breakdown follows Poisson process with mean rate of 2 breakdowns per hour. Answer the following :

   (i) What is the expected idle time of the machine, assuming there is only one repairman available in the workshop?

   (ii) What is the average waiting time of a break down machine in the queue?

   (iii) What is the expected number of broken down machines in the queue?

6 . Consider a queueing system where arrivals are according to a Poisson distribution with mean 5. Find expected waiting time in the system if the service time distribution is (i) Uniform from t= 5 min to t = 15 min, (ii) Normal with mean 3 min. and variance 4 min^2.

17.8 References

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Lesson – 18

Project Management

18.0 Objective :

- Networks basic components
- Rules for networks construction
- Numbering events
- Drawing network diagrams

Structure

18.1 Introduction

Network Scheduling is a technique used for planning and scheduling large projects in the fields of construction, maintenance, fabrication, purchasing, computer system installation, research and development designs, etc. The technique is a method of minimizing trouble spots, such as, production bottlenecks, delays and interruptions, by determining critical factors and coordinating various parts of overall job.

There are two basic planning and control techniques that utilize a network to complete a pre-determined project or schedule. These are : *Program Evaluation and Review Technique (PERT)*; and the *Critical Path Method (CPM)*. Several variations of these have also been developed, one such important variation being the *Review Analysis of Multiple Projects (RAMP)* which is useful for guiding the 'activities' of several projects at one time.

18.2 Basic Components

A *network* is a graphic representation of a project's operations and is composed of activities and events that must be completed to reach the end objective of a project, showing the planning sequence of their accomplishments, their dependence and inter-relationships. The basic components of a network are :

18.2.1. Activity. An *activity* is a task, or item of work to be done, that consumes time, effort, money or other resources. It lies between two events, called the 'preceding' and 'succeeding' ones. An activity is represented by an arrow with its head indicating the sequence in which the events are to occur.

18.2.2. Event. An *event* represent the start (beginning) or completion (end) of some activity and as such it consumes no time. It has no time duration and does not consume any resources. It is also known as a *node*. An event is not complete until all the activities flowing into it are completed. An event is generally represented on the network by a circle, rectangle, hexagon or some other geometric shape.

Activities are identified by the numbers of their starting (tail or initial) event and ending (head, or terminal) event. An arrow (i, j) extended between two events; the tail event *i* represents the start of the activity and the head event j, represents the completion of the activity as shown below :

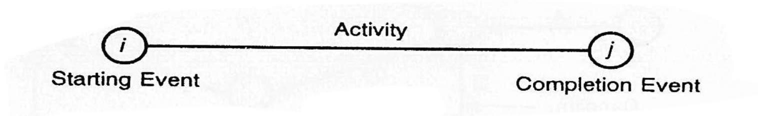

Fig. 18.1

The activities can be further classified into the following three categories :

1. *Predecessor activity*. An activity which must be completed before one or more other activities start is known as predecessor activity.

2. *Successor activity*. An activity which started immediately after one or more of other activities are completed is known as successor activity.

3. *Dummy activity*. An activity which does not consume either any resource and time is known as dummy activity. A dummy activity is depicted by dotted line in the network diagram.

Remark. A dummy activity in the network is added only to represent the given precedence relationships among activities of the project and is needed when (a) two or more parallel activities in a project have same head and tail events, or (b) two or more activities have some (but not all) of their immediate predecessor activities in common.

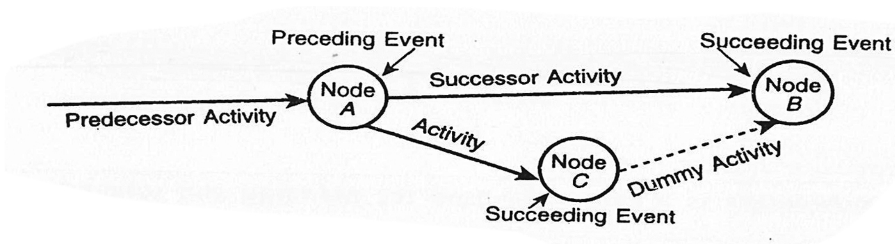

Fig . 18.2

18.3. Logical Sequencing

All the projects consist of certain activities that can begin only after certain others are completed. In fact, the entire project may be considered as a series of activities which may begin only after another activity or activities are completed. In a network schedule, these types of relationships are called *constraints* and are represented by inequalities. For example, A < B will indicate that the activity A must be completed before the start of the activity B.

For example, a project of laying a pipeline consists of the activities : trenching, laying pipe, and welding of pipe.

In logical sequencing, following two types of errors are most common while drawing a network diagram :

1. *Looping*. If an activity were represented as going back in time, a closed loop would occur as shown in Fig. 18.3.

A closed loop would produce an endless cycle in computer programmes without a built-in routine for detection or identification of the cycle. This situation can be avoided by checking the precedence relationship of the activities and by numbering them in a logical order. Thus one property of a correctly constructed network diagram is that it is "non-cyclic".

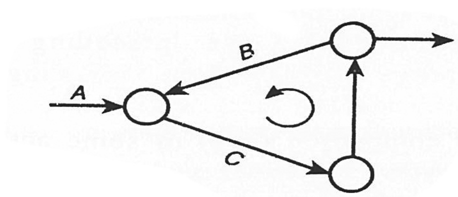

Fig. 18.3 Closed loop

2. *Dangling*. No activity should end without being joined to the end event. If it is not so, a dummy activity is introduced in order to maintain the continuity of the system. Such end-events other than the end of the project as a whole are called dangling events.

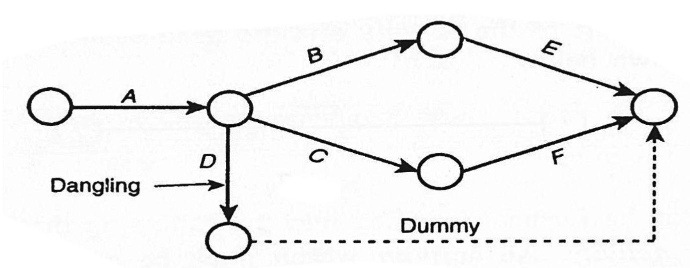

Fig. 18.4 Dangling

In the above network, activity *D* leads to dangling. A dummy activity is therefore introduced to avoid this dangling.

18.4. Rules of Network Construction

For the construction of a network, generally, the following rules are followed :

1. Each activity is represented by one and only one arrow.
2. Each activity must be identified by its *starting* and *end* node which implies that
   (i) two activities should not be identified by the same completion events, and
   (ii) activities must be represented either by their symbols or by the corresponding ordered pair of starting-completion events.
3. Nodes are numbered to identify an activity uniquely. Tail node (starting point) should be lower than the head node (end point) of an activity.
4. Between any pair of nodes, there should be one and only one activity; however more than one activity may emanate from and terminate to a node.
5. Arrows should be kept straight and not curved or bent.
6. The logical sequence (or inter-relationship) between activities must follow the following rules :
   (i) An event cannot occur until all the incoming activities into it have been completed.
   (ii) An activity cannot start unless all the preceding activities on which it depends, have been completed.
   (iii) Dummy activities should only be introduced if absolutely necessary.

18.5 Numbering the Events

After the network is drawn in a logical sequence, every event is assigned a number. The number sequence must be such so as to reflect the flow of the network. In event numbering, the following rules should be observed :

(a) Event numbers should be unique.

(b) Event numbering should be carried out on a sequential basis from left to right.

(c) The initial event which has all outgoing arrows with no incoming arrow is numbered 0 or 1.

(d) The head of an arrow should always bear a number higher than the one assigned at the tail of the arrow.

(e) Gaps should be left in the sequence of event numbering to accommodate subsequent inclusion of activities, if necessary.

Remark. The above procedure of assigning the numbers to various events of a network is known as Fulkerson's Rule.

18.6. Example

Example 18.6.1. A television is manufactured in six steps, labelled A through F. Because of its size and complexity, the television is produced one at a time. The production control manager thinks that network scheduling techniques might be useful in planning future production. He recorded the following information :

A is the first step and precedes B and C,

C precedes D and E,

B follows D and precedes E,

D is successor of F.

(a) Draw an activity-on-node diagram for the production manager.

(b) On checking with the records, the production manager corrects his last note to read, "D is a predecessor of F". Draw a new diagram for the revised network incorporating this new change.

(c) After pondering over the network and rechecking from the records, it was found that B was really a predecessor of D rather than vice versa. Draw a revised network incorporating this new change.

(d) Draw an arrow-diagram representation of the network in part

(c) How many dummy activities did you use in your network?

Solution. (a)

(i) A is the first step which follows B and C.

(ii) C precedes D and E.

(iii) F follows E and D and is the successor of F.

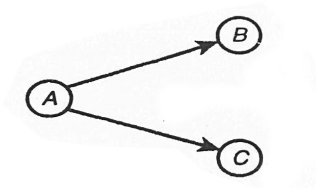

Fig. 18.5

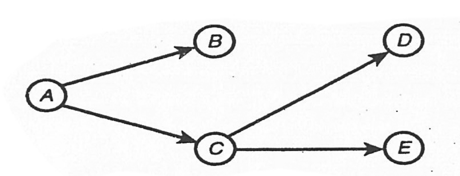

Fig 18.6

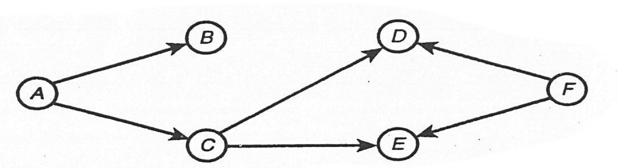

Fig. 18.7

(iv) Now since B follows D and precedes E, the complete network drawing is shown in Fig. 18.8 below :

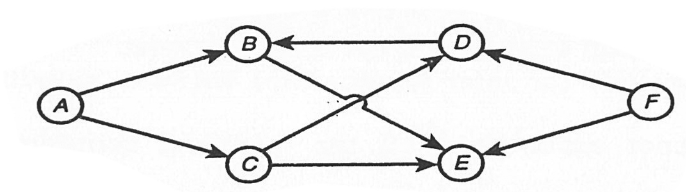

Fig. 18.8

Evidently, this network contains a cycle as shown in Fig. 18.9 below

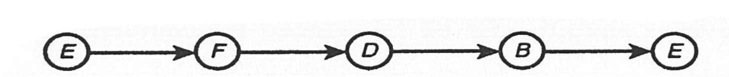

Fig. 18.9

(b) Revised network when D is a predecessor of F is as follows :

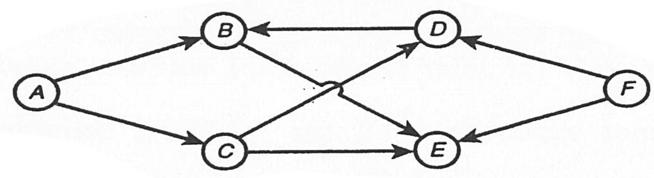

Fig. 18.10

(c) Revised network when D is a predecessor of F and B is a predecessor of D is given below :

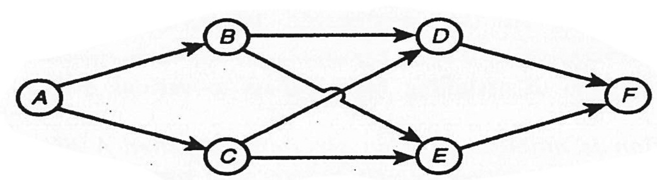

Fig. 18.11

(d) The arrow diagram of the network presented in part (c) can be represent as

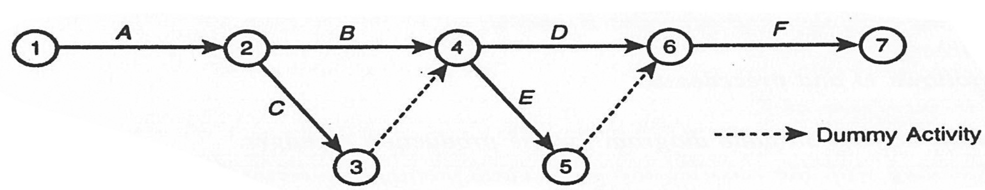

Fig. 18.12

The events of the diagram are numbered such that their ascending order indicates the direction of progress in the project.

Example 18.6.2. Construct the network diagram comprising activities B, C,..., Q and N such that the following constraints are satisfied :

B<E, F; C<G,L; E, G < H ; L, H < I; L<M; H < N ; H<J; I, J<P; P < Q.

The notation X < Y means that the activity X must be finished before Y can begin.

Solution. The resulting network is shown in Fig. 18.13. The dummy activities D_1, D_2 and D_3 are used to establish the correct precedence relationships. D_4 is used to identify the activities *I* and *J* with unique end nodes. The nodes of the project are numbered such that their ascending order indicates the direction of progress in the project :

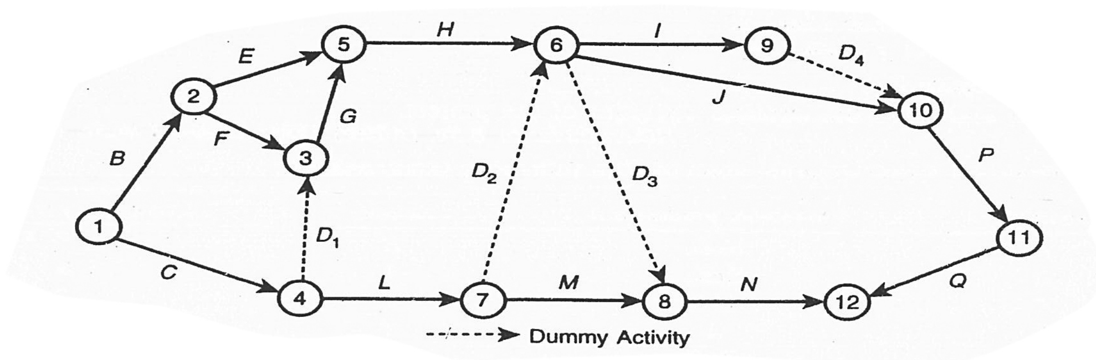

Fig. 18.13

Example 18.6.3. In a boiler overhauling project following activities are to be performed :

A. Inspection of boiler by boiler engineer and preparation of list of parts to be replaced/repaired.

B. Collecting quotations for the parts to be purchased.

C. Placing the orders and purchasing.

D. Dismantling of the defective parts from the boiler.

E. Preparation of necessary instructions for repairs.

F. Repair of parts in the workshop.

G. Cleaning of the various mountings and fittings.

H. Installation of the repaired parts.

I. Installation of the purchased parts.

J. Inspection.

K. Trial run.

Assuming that the work is assigned to the boiler engineer who has one boiler mechanic and one boiler attendant at his disposal, draw a network showing the precedence relationships.

Solution. Looking at the list of activities, we note that activity A (inspection of boiler) is to be followed by dismantling of defective parts (D) and only after that it can be decided which parts can be repaired and which will have to be replaced. Now the repairing and purchasing can go side by side; But the instructions for repairs may be prepared after sending the letters for quotations. Note that it becomes a partial constraint, also started after activity D. Now we assume that repairing will take less time than purchasing. But the installation of repaired parts can be started only when the cleaning is completed. This results in the use of a dummy activity. After the installation of repaired parts, installation of purchased parts can be taken up. This will be followed by inspection and trial run.

The network showing the precedence relationships is given below :

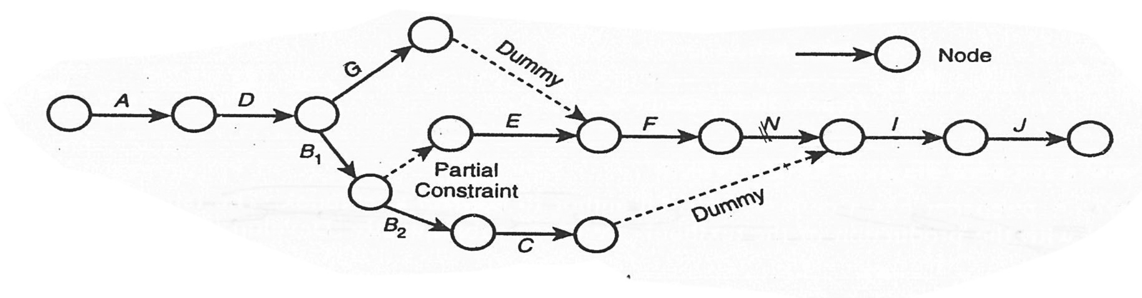

Fig. 18.14

The dummy activities are used to identify the activities C, H and E,G with unique end nodes.

18.7 Summary

The basic components in a network diagram are defined in the present lesson. The rules for Network construction and Fulkerson's Rule are also explained. Further this lesson also enables a student know how to draw a network diagram. Examples and the Exercises given here will help to understand the concept much better.

18.8 Exercise

1. Construct the arrow diagram comprising activities A, B, ... and L such that the following relationships are satisfied :

(i) A, B and C, the first activities of the project, can start simultaneously,

(ii) A and B precede D,

(iii) B precedes E, F and H,

(iv) F and C precede G,

(v) E and H precede I and J,

(vi) C, D, F and J precede K,

(vii) K precedes L,

(viii) I, G and L are the terminal activities of the project.

2. Draw an arrow diagram showing the following relationships :

Activity	Immediate predecessor	Activity	Immediate predecessor
A	None	H	D, E, F
B	None	I	D
C	None	J	G
D	A	K	G
E	B, C	L	H, J
F	A	M	K
G	C	N	I, L

3. Following are the activities which are to be performed for a building site preparation. Determine the precedence relationship and draw the network :

1. Clear the site.
2. Survey and layout.
3. Rough grade.
4. Excavate for sewer.
5. Excavate for electrical manholes.
6. Install sewer and bask fill.
7. Install electrical manholes.

8. Construct the boundary wall.

4. Listed in the table are the activities and sequencing necessary for a maintenance job on the heat exchangers in a refinery :

Activity	Description	Predecessor activity
A	Dismantle pipe connections	--
B	Dismantle header, closure, and floating front	A
C	Remove tube bundle	B
D	Clean bolts	B
E	Clean header and floating head front	B
F	Clean tube bundle	C
G	Clean shell	C
H	Replace tube bundle	F, G
I	Prepare shell pressure test	D, E, H
J	Prepare tube pressure test and reassemble	I

Draw a network diagram for the project.

18.9 References

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.

2. Operations Research – Kanthi Swarup and others, Sultan Chand & Sons.

3. Operations Research – S.D. Sarma, Kedar Nath Ram Nath and Co. Publishers, Meerut.

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Lesson – 19

Critical Path Method
(CPM)

19.0 Objective :

- Critical Path Technique
- Forward Pass Computations
- Backward Pass Computations
- Critical path
- Floats

Structure

19.1 Introduction – Critical Path Analysis

The objective of critical path analysis is to estimate the total project duration and to assign starting and finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project. To achieve this objective, we carry out the special computations that produce the following information :

(a) Total duration needed for the completion of the project.

(b) Categorization of the activities of the project as being *critical* or *non-critical*.

An activity in a network diagram is said to be *critical*, if the delay in its start will further delay the project completion time. A *non-critical* activity allows some scheduling slack, so that the start time of the activity may be advanced or delayed within limits without affecting the completion date of the entire project. The following terms shall be used in critical path calculations :

E_i = Earliest occurrence time of event i

L_j = Latest occurrence time of event j

t_{ij} - Duration of activity (i, J)

The critical path calculations are done in the following two ways : (a) Forward Pass Calculations, and (b) Backward Pass Calculations.

19.2 Forward Pass Calculations. We start from the initial node 1 (event 1) with starting time of the project as zero. Proceed through the network visiting nodes in an increasing order of node number and end at the final node of the network. At each node, we calculate earliest start and finish times for each activity by considering F_i as the earliest occurrence of node i. The method may be summarized as follows :

Step 1. Set $E_1 = 0$; i = 1 (initial node)

Step 2. Set the earliest start time for each activity that begins at node i as

$ES_{ij} = Ei$; for all activities (i, j) that start at node i.

Step 3. Compute the earliest finish time of each activity that begins at node i by adding the earliest start time of the activity to the duration of the activity. Thus

$$EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$$

Step 4. Move on to next node, say node j (j > i) and compute the earliest occurrence for node j, using

$$E_j = \max_i \{EF_{ij}\} - \max_i \{E_i + t_{ij}\}$$

for all immediate predecessor activities.

Step 5. If j = n (final node), then the earliest finish time for the project is given by

$$E_n = \max \{EF_{ij}\} = \max \{E_{n-1} + t_{ij}\} .$$

19.3. Backward Pass Calculations. We start from the final (last) node (event) n of the network, proceed through the network visiting nodes in the decreasing order of node numbers and end at the initial node 1. At each node, we calculate the least finish and start times for each activity by considering L_j as the latest occurrence of node j. The method may be summarized below :

Step 1. $L_n = E_n$; for $j = n$.

Step 2. Set the latest finish time of each activity that ends at node j as

$$LF_{ij} = L_j.$$

Step 3. Compute the latest occurrence times of all activities ending at j by subtracting the duration of each activity from the latest finish time of the activity. Thus

$$LS_{jj} = LF_{ij} - t_{ij} = L_j - t_{ij} .$$

Step 4. Proceed backward to the node in the sequence, that decrease j by 1. Also compute the latest occurrence time of node i (i < j) using

$$L_i = \min_j \{LS_{ij}\} = \min_j \{L_j - t_{ij}\}$$

Step 5. If j = 1 (initial node), then

$$L_1 = \min \{LS_{ij}\} = \min \{L_2 - t_{ij}\}$$

Based on the above calculations, an activity (i, j) will be critical if it satisfies the following conditions :

(i) $E_i = L_i$ and $E_j = L_j$

(ii) $E_j - E_i = L_j - L_i = t_{ij}$.

An activity that does not satisfy the above conditions is termed as non-critical.

19.4. Critical Path. The critical activities of a network that constitute an uninterrupted path which spans the entire network from start to finish is known as *critical path.*

19.5 Float or Slack

The float of an activity is the amount of time by which it is possible to delay its completion time without affecting the total project completion time.

19.5.1. Event float. The *float* (also called 'slack') of an event is the difference between its latest time (L_i) and its earliest time (E_i). That is

$$\text{Event float} = L_i - E_i$$

It is a measure of how much later than expected a particular event could occur without delaying the completion of the entire project.

19.5.2. Activity float. As mentioned earlier, it is the float (or slack) in the activity time estimates. There are mainly three types of activity floats as discussed below :

(i) Total float. The *total float* of an activity represents the amount of time by which an activity can be delayed without delay in the project completion date. In other words, it refers to the amount of the free time associated with an activity which can be used before, during or after the performance of this activity. Total float is the positive difference between the earliest finish time and the latest finish time, or the positive difference between the earliest start time and the latest start time of an activity depending upon which way it is defined.

(ii) Free float. *Free float* is that portion of the total float within which an activity can be manipulated without affecting the float of subsequent activities. It is computed for an activity by subtracting the head event slack from its total float. The head event slack is the difference between the latest and earliest event timings of an activity.

(iii) Independent float. It is that portion of the total float within which an activity can be delayed for start without affecting floats of the preceding activities. It is computed by subtracting the tail event slack from the free float of the activity. If the result is negative, it is taken as zero.

Illustration. Consider the following activity of certain network :

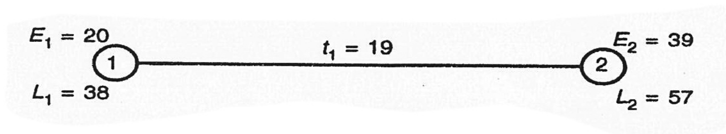

Fig. 19.1

Total Float = $L_2 - (E_1 + t_{12}) = 57 - (20 + 19) = 18$

Free Float = $E_2 - (E_1 + t_{12}) = 39 - (20 + 19) = 0$

Independent Float = $E_2 - (L_1 + t_{12}) = 39 - (38 + 19) = - 18$

Remark :

1. The basic difference between slack and float times is that slack is used for events only, whereas float is applied for activities.
2. The difference between total float and free float is known as interference float.
3. An activity is critical if its total float is zero, otherwise it is non-critical.

19.6. Examples

Example 19.6.1. A project consists of a series of tasks labelled A, B, ..., H, I with the following relationships (W < X,Y means X and Y cannot start until W is completed; X, Y < W means W cannot start until both X and Y are completed). With this notation construct the network diagram. having the following constraints :

$$A<D,E; \quad B,D<F, \quad C<G; \quad B,G<H; \quad F,G<I.$$

Find also the minimum time of completion of the project, when the time (in days) of completion of each task is as follows :

Task:	A	B	C	D	E	F	G	H	I
Time :	23	8	20	16	24	18	19	4	10

Solution. Using the given constraints, the resulting network is shown in Fig. 19.2. The dummy activities D_1 and D_2 are introduced to establish the correct precedence relationships. The events of the projects are numbered in such a way that their ascending order indicates the direction of progress in the project.:

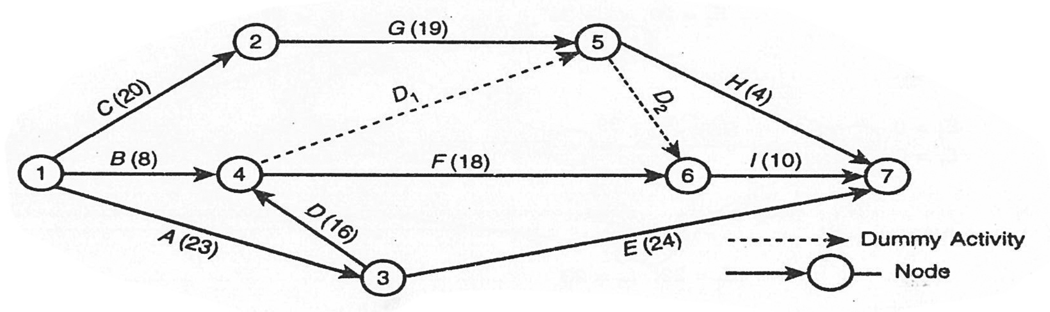

Fig. 19.2

To determine the minimum time of completion of the project (critical path), we compute E_j and E_i for each of the task (i, j) of the project. The critical path calculations as applied to Fig. 19.2 are as follows :

Forward calculations

Node 1: Set $E_1 = 0$

Node 2 : $E_2 = E_1 + t_{12} = 0 + 20 = 20$

Node 3 : $E_3 = E_1 + t_{13} = 0 + 23 = 23$

Node 4: $E_4 = \max\limits_{i=1,3} \{E_i + t_{i4}\} = \max \{0 + 8, 23+16\} = 39$

Node 5 : $E_5 = \max\limits_{i=2,4} \{E_i + t_{i5}\} = \max \{20+19, 39 + 0\} = 39$

Node 6: $E_6 = \max\limits_{i=4,5} \{E_i + t_{i6}\} = \max \{39+ 18, 39 + 0\} = 57$

Node 7: $E_6 = \max\limits_{i-3,5,6} \{E_i + t_{i6}\} = \{23 + 24, 39 + 4, 57+10\} = 67$

Backward calculations

Node 7: Set $L_7 = E7 = 67$

Node 6: $L_6 = \min_{j=7} \{L_j - t_{6j}\} = L_7 - t_{67} = 57$

Node 5: $L_5 = \min_{j-6,7} \{L_j - t_{5j}\} = \min \{57-0, 67-4\} = 57$

Node 4: $L_4 = \min_{j=5,6} \{L_j - t_{4j}\} = \min \{57 - 0, 57-18\} = 39$

Node 3: $L_3 = \min_{j=4,7} \{L_j - t_{3j}\} = \min \{39 - 1\ 6, 67 - 24\} = 23$

Node 2: $L_2 = L_5 - t_{25} - 57 - 1\ 9 = 38$

Node 1: $L_1 = \min_{j=2,3,4} \{L_j - t_{ij}\} = \min \{38 - 20, 23 - 23, 39 - 8\} = 0$

To evaluate the critical nodes, all these calculations are -put in the flowing Table :

Task	Normal	Earliest time		Latest time		Float		.
i, j	time (days)	Start (E_i)	Finish (E_j)	Start (L_i)	Finish (L_j)	Total	Free	Independent
(1, 2)	20	0	20	18	38	18	0	0
(1, 3)	23	0	23	0	23	0	0	0
(1, 4)	8	0	8	31	39	31	31	31
(2, 5)	19	20	39	38	57	18	0	-18
(3, 4)	16	23	39	23	39	0	0	0
(3, 7)	24	23	47	43	67	20	20	20
(4, 5)	0	39	39	57	57	18	0	0
(4, 6)	18	39	57	39	57	0	0	0
(5, 6)	0	39	39	57	57	18	18	0
(5, 7)	4	39	43	63	67	24	24	6
(6, 7)	10	57	67	57	67	0	0	0

The above table shows that the critical nodes are for the tasks (1, 3), (3, 4), (4, 6) and (6, 7).

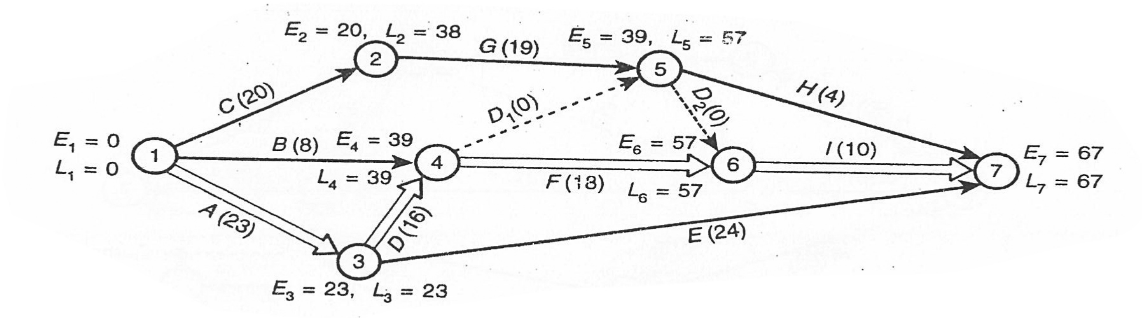

Fig. 19.3

It is apparent from Fig. 19.3 that the critical path comprises the tasks (1,3), (3,4), (4,6) and (6, 7). This path represents the shortest time to complete the entire project.

Example 19.6.2. A small project consists of seven activities for which the. relevant data are given below :

Activity	Preceding Activities	Activity Duration (Days)
A	--	4
B	--	7
C	--	6
D	A, B	5
E	A, B	7
F	C, D, E	6
G	C, D, E	5

(i) Draw the network and find the project completion time.

(ii) Calculate total float for each of the activities.

(iii) Draw the time scaled diagram.

Solution. (i) Using the precedence relationship among the activities, the resulting network is shown in Fig. 19.4 below

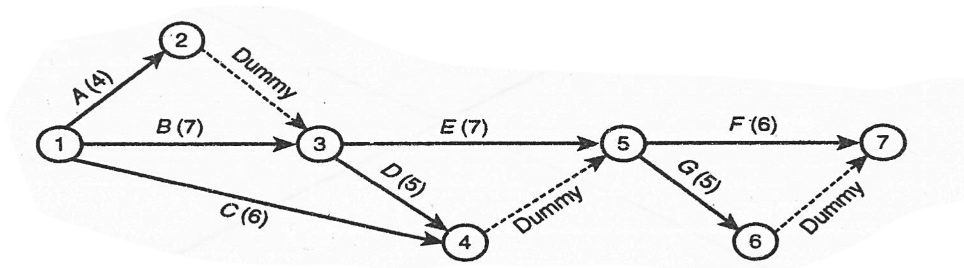

Fig. 19.4

(ii) To determine the project completion time, we compute the earliest time, E_i, and late finish time, L_j, for each activity of the project. For it we proceed as follows :

Forward calculations

Set $E_1 = 0$

$E_2 = E_1 + t_{12} = 0 + 4 = 4$

$E_3 = \max(E_1 + t_{13}, E_2 + t_{23})$

$= \max(0 + 7, 4 + 0) = 7$

$E_4 = \max(E_1 + t_{14}, E_3 + t_{34})$

$= \max(0 + 6, 7 + 5) = 12$

$E_5 = \max(E_3 + t_{35}, E_4 + t_{45})$

$= \max(7 + 7, \ 12 + 0) = 14$

$E_6 = E_5 + t_{56} = 14 + 5 = 19$

$E7 = \max(E_5 + t_{57}, E_6 + t_{67})$

$= \max(14 + 6, 19 + 0) = 20$

Backward calculations

Set $L_7 = E_7 = 20$

$L_6 = L_7 - t_{67} = 20 - 0 = 20$

$L_5 = \min(L_6 - t_{56}, L_7 - t_{57})$

$= \min(20–5, 20–6) = 14$

$L_4 = L_5 - t_{45} = 14 - 0 = 14$

$L_3 = \min(L_4 - t_{34}, L_5 - t_{35})$

$= \min(14-5, 14-7) = 7$

$L_2 = L_3 - t_{23} = 7 - 0 = 7$

$L_1 = \min(L_2 - t_{12}, L_3 - t_{13}, L_4 - t_{14})$

$= \min(7-4, 7-7, 14-6) = 0$

To evaluate the critical nodes and total float, the above calculations are displayed in the following table :

Activity (i, j)	Normal time (days)	Earliest time E_i	Earliest time $E_i + t_{ij}$	Latest time $L_j - t_{ij}$	Latest time L_j	Total float
1-2	4	0	4	3	7	3
1-3	7	0	7	0	7	0
1-4	6	0	6	8	14	8
3-4	5	7	12	9	14	2
3-5	7	7	14	7	14	0
5-7	6	14	20	14	20	0
5-6	5	14	19	20	20	1

From the above table, we observe that the critical nodes (events) are : (1,3), (3,5) and (5,7). Thus, we have

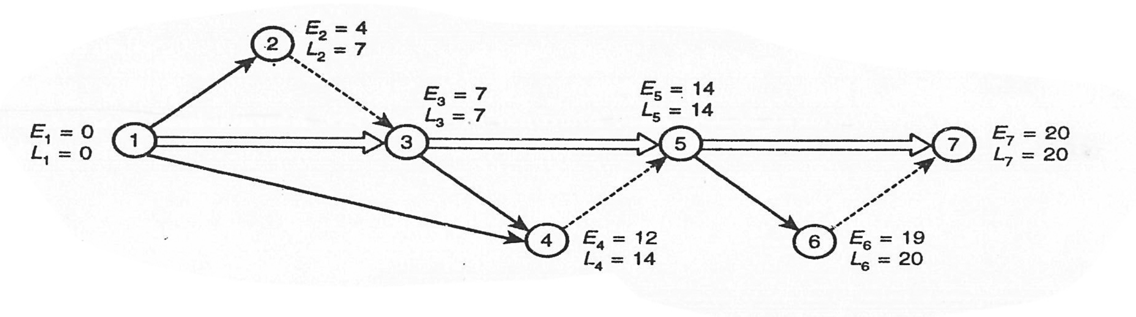

Fig. 19.5

CPM is $1 \rightarrow 3 \rightarrow 5 \rightarrow 7$ and Project completion time is 20 days.

19.7 Summary

The above lesson explains the Critical path method to find the critical activities of a network. It also covers Forward and Backward pass calculations to calculate the Critical path. Activity Floats such as Total Float, Free Float and Independent Float are also discussed. Worked out examples will bring a good clarity of the concept.

19.8 Exercise

1. The following are the details of estimated times of activities of a certain project.

Activity	Immediate predecessors	Estimated time (weeks)
A	-	2
B	A	3
C	A	4
D	B, C	6
E	-	2
F	E	8

(a) Find the critical path and the expected time of the project.

(b) Calculate the earliest start time and earliest finish time for each activity,

(c) Calculate the slack for each activity.

2. The following table gives the activities in a construction project and time duration :

Activity	Preceding activity	Normal time (days)
1-2	-	20
1-3	-	25
2-3	1-2	10
2-4	1-2	12
3-4	1-3, 2-3	5

4-5	2-4, 3-4	10

(a) Draw the activity network of the project.

(b) Find the total float and free-float for each activity.

(c) Determine the critical path and the project duration.

3. Tasks A, B, C, ..., H, I constitute a project. The notation $X < Y$ means that the task X must be finished before Y can begin. With this notation

$$A < D, \quad A < E, \quad B < F, \quad D < F, \quad C < G, \quad C < H, \quad F < I, \quad G < I.$$

Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project, when the time (in days) of completion of each task is as follows :

Task :	A	B	C	D	E	F	G	H	1
Time :	8	10	8	10	16	17	18	14	9

4. Given the following information :

Activity:	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7	
Duration :	2	8	10	6	3	3	7	5	2	8	(in days)

(i) Draw the arrow diagram.

(ii) Identify critical path and find the total project duration.

(iii) Determine total, free and independent floats.

5. A small maintenance project consists of the following 10 jobs whose precedence relationships are identified by their node numbers :

Job	Node Numbers	Estimated duration (days)	Job	Node Numbers	Estimated duration (days)
a	(1, 2)	2	f	(4, 6)	6

b	(2, 3)	3	g	(4, 7)	2
c	(2, 4)	5	h	(5, 8)	8
d	(3, 5)	4	i	(6, 8)	7
e	(3, 6)	1	j	(7, 8)	4

(a) Draw an arrow diagram representing the project.

(b) Calculate early and late start and finish times for each job.

(c) What jobs are critical ?

(d) How much slack does job (3, 5) have ?

6. Draw the network for the following project and compute the earliest and latest times for each event and also find critical path :

Activity	Immediate predecessor	Time (days)
1-2	--	5
1-3	--	4
2-4	1-2	6
3-4	1-3	2
4-5	2-4	1
4-6	2-4 & 3-4	7
5-7	4-5	8
6-7	4-6	4
7-8	6-7 & 5-7	3

7. Draw the network for the data given below and compute :

(i) Critical path, (ii) Early start and Late start times for each activity, and (iii) Total slack for each activity :

Activity :	A	B	C	D	E	F	G	H	I
Predecessor :	-	-	-	A	B	C	D, E	B	H, F
Estimated time (weeks) :	3	5	4	2	3	9	8	7	9

19.9 References

1. Operation Research – R. Pannearselvam, Prentice Hall of India (P) Ltd. Edition, 2006.
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Lesson – 20

Project Evaluation and Review Technique (PERT)

20.0 Objective :

- Project Evaluation and Review Technique
- Comparison of CPM and PERT

Structure

20.1 PERT - Introduction

The network methods discussed so far may be termed as deterministic, since estimated activity times are assumed to be the expected values. But no recognition is given to the fact that expected activity time is the mean of a distribution of possible values which could occur.

Under the conditions of uncertainty, the estimated time for each activity for PERT network is represented by a probability distribution. This probability distribution of activity time is based upon three different time estimates made for each activity. These are as follows :

t_0 = the *optimistic time*, is the shortest possible time to complete the activity if all goes well.

t_p = the *pessimistic time*, is the longest time that an activity could take if every thing goes wrong.

t_m = the *most likely time*, is the estimate of the normal time an activity would take. If only one time were available, this would be it. Otherwise it is the mode of the probability distribution.

The range specified by the optimistic time (t_0) and pessimistic time (t_p) estimates supposedly must enclose every possible estimate of the duration of the activity. The most likely time (t_m) estimate may not coincide with the midpoint $(t_0 + t_p)/2$ and may occur to its left or to its right as shown in Fig. 20.1 :

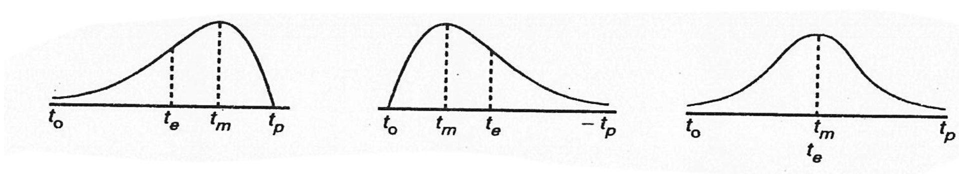

Fig. 20.1

Keeping in view of the above-mentioned properties, it may be justified to assume that the duration of each activity may follow Beta (β) distribution with its unimodal point occurring at t_m and its end points t_0 and t_p

In Beta distribution the mid-point $(t_0 + t_p)/2$ is assumed to weigh half as much as the most likely point (t_m). Thus the expected value of the activity duration can be approximated as the arithmetic mean of $(t_0 + t_p)/2$ and 2 t_m. Thus, we have

$$t_3 = 1/3\ [2t_m + (t_o + t_p)/2] = (t_0 + 4t_m + t_p)/6$$

Since almost (99%) all values of random variables fall within ± 3 standard deviation from the mean or fall within the range approximately 6 standard deviation in length, therefore the interval (t_0, t_p) is assumed to enclose about 6 standard deviation of a symmetric distribution. Thus, if σ denotes the standard deviation, then

$$6\ \sigma \cong t_p - t_0 \quad \text{or} \quad \sigma = (t_p - t_0)/6.$$

The variance, therefore, is : $\sigma^2 = \{(t_p - t_0)/6\}^2$.

Remark. In PERT analysis, a Beta distribution is assumed because it is unimodal, has non-negative end points, and is approximately symmetric.

20.2 Probability of Meeting the Schedule Time

With PERT, it is possible to determine the probability of completing a contract on schedule. The scheduled dates are expressed as a number of time units from the present time. Initially they may be the latest times, T_L, for each event, but after a project is started we shall know how far it has progressed at any given date, and the scheduled time will be the latest time if the project is to be completed on its original schedule.

The probability distribution of times for completing an event can be approximated by the normal distribution due to the central limit theorem. Thus the probability of completing the project by scheduled time (T_s) is given by

$$\text{Prob}\left(Z < \frac{T_s - T_e}{\sigma_e}\right)$$

The standard normal variate (SNV) is given by,

$$Z = \frac{T_s - T_e}{\sigma_e}$$

where, T_e = expected completion time of the project and

σ_e = number of standard deviations the scheduled time lies from the expected (mean) time, i.e., the standard deviation of the scheduled time.

Using the cumulative normal distribution tables, the corresponding value of the standard normal variate is read off. This will give the required probability of completing the project on scheduled time.

20.3 Examples

Example 20.3.1. A project consists of eight activities with the following relevant information :

Activity	Immediate predecessor	Estimated duration (days)		
		Optimistic	Most likely	Pessimistic
A	-	1	1	7
B	-	1	4	7
C	-	2	2	8
D	A	1	1	1
E	B	2	5	14
F	C	2	5	8
G	D, E	3	6	15
H	F, G	1	2	3

(i) Draw the PERT network and find out the expected project completion time.

(ii) What duration will have 95% confidence for project completion?

(iii) If the average duration for activity F increases to 14 days, what will be its effect on the expected project completion time which will have 95% confidence?

(For standard normal Z = 1.645, area under the standard normal curve from 0 to Z is 0.45)

Solution. The expected time and variance of each activity is computed in table below :

Activity	t_o	t_m	t_p	$t_e=(t_0 + 4t_m + t_p)/6$	$[(t_p - t_o)/6]^2$
A	1	1	7	2	1
B	1	4	7	4	I
C	2	2	8	3	1
D	1	1	1	1	0

E	2	5	14	6	4
F	2	5	8	5	1
G	3	6	15	7	4
H	1	2	3	2	$(2/6)^2$

(i) Using the precedence relationship among the activities, the resulting network is shown in Fig. 20.3.1 :

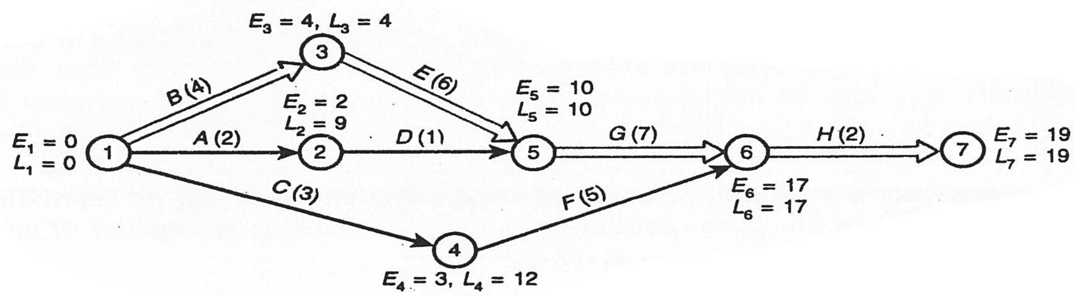

Fig. 20.3.1

From the above network diagram, we observe :

CPM : $1 \to 3 \to 5 \to 6 \to 7$, i.e., $B \to E \to G \to H$.

Expected duration of the project is 19 days.

The variance of the project length is : $\sigma^2 = 1 + 4 + 4 + 0.108$

Thus $\sigma^2 = 1 + 4 + 4 + 0.108 = 9.108$ or $\sigma_e = \sqrt{9.108} = 3.02$.

(ii) Since $P(z \leq 1.645) = 0.5 + 0.45$, i.e., 0.95;

$$\frac{Ts - T_e}{\sigma_e} = 1.645.$$

This implies that

$$T_s = 19 + 3.02 \times 1.645 = 24 \text{ days}.$$

Hence, 24 days of project completion time will have 95%. confidence of completion in the scheduled time.

(iii) When the average duration of activity F increases to 14, the path $C \rightarrow F \rightarrow H$ also becomes critical. The standard deviation of new critical path is : $\sigma_e = \sqrt{9.108 + 2} =$ 3.36.

$\therefore$ P(z< 1.645) = 0.95 gives us

$$\frac{T_s - 19}{3.36} = 1.645, \quad \text{i.e.,} \quad T_s = 19 + 3.36 \times 1.645 = 24.52 \text{ days.}$$

Hence, the project completion duration of 24.52 days will have 95% confidence.

Example 20.3.2. A small project is composed of seven activities whose time estimates are listed in the table as follows :

Activity		Estimated duration (weeks)		
i	j	Optimistic	Most likely	Pessimistic
1	2	1	1	7
1	3	1	4	7
1	4	2	2	8
2	5	1	1	1
3	5	2	5	14
4	6	2	5	8
5	6	3	6	15

(a) Draw the project network.

{b) Find the expected duration and variance of each activity. What is the expected project length?

(c) Calculate the variance and standard deviation of project length. What is the probability that the project will be completed :

(i) at least 4 weeks earlier than expected?

(ii) no more than 4 weeks later than expected?

(d) If the project due date is 19 weeks, what is the probability of meeting the due date ?

Given :	z	0.50	0.67	1.00	1.33	2.00
	P	0.3085	0.2514	0.1587	0.0918	0.0228

Solution. (a) The problem is same as in example problem 20.3.1, except that activity 6-7 has been deleted. As such the network diagram remains the same as given in Fig. 20.3.1. after deleting activity 6-7.

Thus, we have

(b) CPM : $1 \rightarrow 3 \rightarrow 5 \rightarrow 6$, Duration of project = 17 days.

Variance of project length is given by

$\sigma_e^2 = 1 + 4 + 4 = 9$ or $\sigma_e = 3$.

(c) The standard normal variate is :.

$$z = \frac{\text{Duedate} - \text{Expecteddateofcompletion}}{\sqrt{\text{Variance}}}$$

∴ We compute

(i) $z = \frac{13-17}{3} = -\frac{4}{3} = -1.33$

$P(z \leq -1.33) = 0.5 - \Phi(1.33) = 0.5 - .4082$ (from Normal Tables)

$= .0918 = 9.18\%$

(ii) $z = \frac{21-17}{3} = \frac{4}{3} = 1.33$

∴ The probability of meeting the due date (4 weeks later than expected) is

$P(z \leq 1.33) = 0.5 + \Phi(1.33) = 0.5 + 0.4082 = 0.9082$

$= 0.9082 = 90.82\%$.

(d) When the due date is 19 weeks,

$z = \frac{19-17}{3} = \frac{2}{3} = 0.67$

∴ Probability of meeting the due date is

$P(z \leq .67) = .5 + \Phi\ (.67) = .7486 = 74.86\%$.

Thus the probability of not meeting the due date is 1 -0.7486, i.e., 0.2514 or 25.14%.

20.4 Pert and CPM - Comparison

Both PERT and CPM are managerial techniques for planning and control of large complex projects. Both are techniques to network analysis wherein a network is prepared to analyse interrelationships between different activities of a project. However, there are several differences between the two techniques :

1. CPM is used for *repetitive* jobs like planning the construction of a house. On the other hand, PERT is .used for *non-repetitive jobs* like planning the assembly of the space platform.

2. PERT is a *probabilistic model* with uncertainty in activity duration. Multiple time estimates are made to calculate the probability of completing the project within scheduled time. On the contrary, CPM is a deterministic model with well-known activity (single) times based upon past experience. It, therefore, does not deal with uncertainty in project duration.

3. PERT is said to be *event-oriented* as the results of analysis are expressed in terms of events or distinct points in time indicative of progress. *CPM* is, on the other hand, activity-oriented as the results of calculations are considered in terms of activities or operations of the project.

4. PERT is applied mainly for planning and scheduling research programmes. On the other hand, CPM is employed in construction and business problems.

5. PERT incorporates statistical analysis and thereby enables the determination of probabilities concerning the time by which each activity and the entire project would be completed. On the other hand, CPM does not incorporate statistical analysis in determining time estimates because time is precise and known.

6. PERT serves a useful control device as it assists the management in controlling a project by calling attention through constant review to such delays in activities which might lead to a delay in the project completion date. But it is difficult to use

CPM as a controlling device for the simple reason that one must repeat the entire evaluation of the project each time the changes are introduced into the network.

20.5 Summary

In this lesson construction of a network diagram by PERT is discussed which is drawn under the conditions of uncertainty in the duration of the activities where the estimated time for each activity is calculate separately. Also at the end Comparison of PERT with CPM is made to bring clarity between the two methods.

20.6 Exercise

1. A small project consists of seven activities, the details of which are given below :

Activity	Duration (in days)			Immediate predecessor
	Most likely	Optimistic	Pessimistic	
A	3	1	7	-
B	6	2	14	A
C	3	3	3	A
D	10	4	22	B, C
E	7	3	15	B
F	5	2	14	D, E
G	4	4	4	D

(i) Draw the network, number of nodes, find the critical path, the expected project completion time and the next most critical path.

(ii) What project duration will have 95% confidence of completion?

2. Consider the following project :

Activity	Time estimates in weeks			Predecessors
	t_o	t_m	t_p	
A	3	6	9	None

B	2	5	8	None
C	2	4	6	A
D	2	3	10	B
E	1	3	11	B
F	4	6	8	C, D
G	1	5	15	E

Find the path and a standard deviation. What is the probability that the project will be completed by 18 weeks?

3. A project is composed of eleven activities, the time estimates for which are given below :

Activity	a (days)	b (days)	m (days)
1 - 2	7	17	9
1 - 3	10	60	20
1 – 4	5	15	10
2 - 5	50	110	65
2 - 6	30	50	40
3 - 6	50	90	55
3 - 7	1	9	5
4 - 7	40	68	48
5 - 8	5	15	10
6 - 8	20	52	27
7 - 8	30	50	40

(a) Draw the network diagram for the project.

(b) Calculate slacks for each node.

(c) Determine the critical path.

(d) What is the probability of completing the project in 125 days?

20.7 References

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