SOLID GEOMETRY AND REAL ANALYSIS

(DSMAT21/DBMAT21)

(BSC, BA MATHS - II)

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Euclid of Alexandria:

Born: about 325 BC - Died: about 265 BC in Alexandria, Egypt.

Euclid (also referred to as Euclid of Alexandria) (c. 325-c. 265 BC) a Greek mathematician, who lived in Alexandria, Hellenistic Egypt, almost certainly during the reign of Ptolemy I (323 BC - 283 BC), is often considered to be the "father of geometry". His most popular work, ***Elements***, is thought to be one of the most successful textbooks in the history of mathematics. Within it, the properties of geometrical objects are deduced from a small set of axioms, thereby founding the axiomatic method of mathematics.

Euclid also wrote works on perspective, conic sections, spherical geometry and possibly quadric surfaces. Neither the year nor place of his birth have been established, nor the circumstances of his death.

Although many of the results in ***Elements*** originated with earlier mathematicians, one of Euclid's accomplishments was to present them in a single, logically coherent framework. In addition to providing some missing proofs, Euclid's text also includes sections on number theory and three - dimensional geometry. In particular, Euclid's proof of the infinitude of prime numbers is in Book IX, Proposition 20.

The geometrical system described in ***Elements*** was long known simply as "the" geometry. Today, however, it is often referred to as Euclidean geometry to distinguish it from other so-called non-Euclidean geometries which were discovered in the 19th century. These new geometries grew out of more than two millennia of investigation into Euclid's fifth postulate, one of the most-studied axioms in all of mathematics. Most of these investigations involved attempts to prove the relatively complex and presumably non-intuitive fifth postulate using the other four (a feat which, if successful, would have shown the postulate to be in fact a theorem).

Lesson - 0

PRELIMINARIES

Three Dimensional Geometry:

Lesson zero is divided in to two parts, the first part contains some basic information required for three dimentional coordinate geometry and the second part which goes with analysis contains some fundamental concepts required for analysis.

1. Axioms and Definitions:

Enclidean geometry begins with a set, called the space. The elements of this set are called points. Some special subsets of the space are lines, planes and so on. If P, Q are points, L, M are lines and π is a plane. We say that

(i) P lies on L or L passes through P if $P \in L$.

(ii) L is in π or π contains L if $L \subset E$

(iii) P, Q are collinear if there is a line passing through P and Q.

(iv) Lines L, M are coplanar if there is a plane either containing L and M or parallel to L and M.

Euclidean geometry is based on the following axioms.

Axiom 1: One and only one line passes through two distinct points.

Axiom 2: One and only one plane passes through three non collinear points.

Axiom 3: If a plane contains two distinct points P, Q then

the line $\overleftrightarrow{PQ}$ determined by P, Q is contained in the plane.

Axiom 4: The intersection of two planes is either empty or else a line.

Axiom 5: Every line has at least two points and every plane has at least three non collinear points.

Axiom 6 : The space contains at least four noncoplanar points.

Axiom 7: Associated with every pair of points in a set S there is a real number called the distance between P and Q. and is denoted by $d(P,Q)$ and also by PQ.

The distance function defined on $S \times S$ satisfies:

(1) $d(P,Q) \in \mathbb{R}\ \forall\ P,\ Q$ in S and $d(P, Q) \geq 0$

(2) $d(P, Q) = 0$ if and only if $P = Q$.

(3) $d(P, Q) = d(Q, P)$

Definition 8: If L is line and $f : L \to \mathbb{R}$ is a one - one map then $d(P, Q) = |f(P) - f(Q)|$ defined a distance funtion on L and is called a coordinate system on L.

$x \in f(P)$ is called the coordinate of P.w.r.t.f.

Axiom 9: Every line has a coordinate system. This axiom is called the Ruler Postulate.

Some Consequences:

Every line has infinitely many coordinate systems. For any points $P \neq Q$ on a line L there corresponds a coordinate system such that the coordinate of P is zero and of Q is positive.

Definition 10: Two lines in a plane are parallel if either they are disjoint or equal. If L_1, L_2 are parallel we write $L_1 \parallel L_2$.

Axiom 11: A line has one and only one parallel line through a point.

Based on these axioms and definitions and other definitions such as betweenness, line segment, congruence of segments convexity, ray, angle, angle measurement pastulate and so on geometry is developed in a rigourous way. We assume validity of these concepts without attempting to prove here.

2. Basic Results:

We now present a few results without proof that are necessary for solid analytical geometry.

1. Two distinct lines do not intersect in more than a point.

2. If L is a line and π is a plane then either $L \subseteq \pi$ or $L \cap \pi$ has atmost one point.

3. A line and a point not on it are contained in a unique plane.

4. Two intersecting lines are contained in a unique plane.

   Definition: Suppose a line L intersects a plane E in a point P. If L is perpendicular to every line in E passing thorugh P we say that L is perpendicular to E at P and call P the foot of the perpendicular L. L is also called the normal to E at P.

5. If a line L contains the mid point of the segment PQ and any point on L is equidistant from P and Q then $L \perp PQ$.

6. If a line L is perpendicular to two intersecting lines at the point of intersection then L is perpendicular to the plane containing the lines.

7. If L is a line and P is a point on L then there is one and only one plane perpendicular to L at P.

8. Any two perpendiculars of a plane are (coplanar and) parallel.

9. A plane π has one and only one perpendicular through a point P. If M is the foot of the perpendicular PM is called the perpendicular distance from P to π.

10. Any two planes perpendicular to a line are parallel.

In the sequel we may use some other results that are not stated above, as well.

Definitions: L_1 and L_2 are said to be skew lines if they are not parallel and not intersecting.

11. If L_1 and L_2 are skew lines then there is one and only one plane containing L_1 and parallel to L_2 and vice versa.

12. The feet of the perpendiculars from the points on a line L to a plane σ form a line.

13. There is one and only one line perpendicular to each one of a pair of skew lines. The length of the segment on this line joining the feet of the perpendiculars is the shortest distance between the skew lines.

14. There exists a unique plane containing two distinct points (i.e., a line) and perpendicular to a given plane.

15. There exists a unique plane containing a point and perpendiuclar to two given planes.

3. Vectors:

Just as some complicated problems in two dimensional coordinate geometry are better handled and solved by vector methods, these methods help getting better insight in three dimensional problems and as equally useful in solving problems in three dimenssional coordinate geometry . As such we will have a brief account of vectors in the three dimensional Euclidean space.

An "Ordered triad" of real numbers is called a three dimensional real vector or simply a vector. If $P=(x,y,z)$ is a three dimensional real vector we call x,y,z as the first (or x), the second (or y) and the third (or z) component respectively. A vector is usually denoted by a pair of letters with a bar (or arrow) on them. The vector $(0,0,0)$ is called the null (zero) vector and is denoted by $\bar{0}$. Two vector are said to be equal iff the cooresponding components are equal. Addition, subtraction, multiplication by a scalar, dot product, length and cross product of vectors are defined in the usual way (parallel to the two dimensional case). A unit vector is a vector of length one.

The angles between the non zero vectors $\bar{P}$ and $\bar{Q}$ is defined by

$$\cos\theta = \pm \frac{\bar{P}\cdot\bar{Q}}{\sqrt{\bar{P}\cdot\bar{P}}\;\sqrt{\bar{Q}\cdot\bar{Q}}}$$

$\overline{P}$ and $\overline{Q}$ are perpendicular if $\overline{P}\cdot\overline{Q}=0$ and are parallel or collinear if $\left|\overline{P}\cdot\overline{Q}\right|=\sqrt{P\cdot P}\,\sqrt{Q\cdot Q}$

The scalar triple product of $\overline{P}, \overline{Q}, \overline{R}$ denoted by

$\left[\overline{P}\,\overline{Q}\,\overline{R}\right]$ and is defined to be

$$\left[\overline{P}\,\overline{Q}\,\overline{R}\right]=\overline{P}\cdot\left(\overline{Q}\times\overline{R}\right)$$

If $\overline{P}=(x_1, y_1, z_1), \overline{Q}=(x_2, y_2, z_2)$ and $\overline{R}=(x_3, y_3, z_3)$

$$\left[\overline{P}\ \overline{Q}\ \overline{R}\right]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

We have mentioned above some frequently used concepts only. We will use other results as well when needed.

4. Coordinate Systems:

We fix a point 'O' in the three dimensional space and call O, the origin of the system. We then fix two mutually perpendicular lines X^1OX and Y^1OY through O. These lines determine a plane XY. We then consider the perpendicular Z^1OZ at O to the plane XY. The planes XOY, YOZ and ZOX are called the XY plane, YZ plane and ZX plane respectively. We fix points I, J, K on $\overrightarrow{OX}$ $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ respectively such that $OI = OJ = OK$. The origin O and these points I, J, K determine coordinate systems on the lines X^1OX, Y^1OY and Z^1OZ respectively such that $OI = OJ = OK = 1$. The coordinate of a point on X^1OX is called the x coordinate, of a point on Y^1OY is called the y coordinate and of a point on Z^1OZ is called the Z coordinate of the corresponding point. We call $\overline{XOX^1}$ the x axis, $\overline{YOY^1}$ the y axis and $\overline{ZOZ^1}$ the Z axis respectively. The planes XY, YZ, ZX are called the coordinate planes and the axes x, y, z are called the coordinate axes. The frame work OXYZ is called the frame of reference or the coordinate system.

If P is any point in the space and L, M, N are the feet of the perpendiculars from P to the coordinate axes x, y, z respectively and if the x coordinate of L is x, y coordinate of M is y and z coordinate of N is z we call x, y, z as the x, y, z a coordinates of P respectively and write $P=(x, y, z)$. Thus with reference to the coordinate system OXYZ any point P in the space can be uniquely associate with an ordered triad (x, y, z) of numbers.

Conversely given any ordered triad of numbers (x, y, z) we fix L, M, N on the coordinate axes such that $OL = x, OM = y$ and $ON = Z$. The planes perpendicular to the x axis at L, perpendicular to the y axis at M and to the z axis at N meet at a unique point P and the coordinates of P are respectively x, y, z so that $P = (x, y, z)$.

Thus corresponding to every ordered triad (x, y, z) of real numbers there is a unique point P in the space such that $P = (x, y, z)$.

Suppose a point P has coordinates x, y, z with respect to a frame work $OX\ YZ$ in the space, The planes through P parallel to the coordinate planes form a rectangular parallelopiped. PM^1LN^1 is parallel to YZ plane, hence perpendicualr to x axis. L is the foot of the perpendicular from P to the x axis thus

$$OL = |x|$$

Similarly $OM = |y|$ and $ON = |z|$

The coordinates of N^1 are $(x, y, 0)$.

Since $\overline{PN^1} \perp \overline{XOY}, \overline{PN^1} \perp \overline{N^1L} \Rightarrow PL^2 = PN^{1^2} + N^1L^2 = y^2 + z^2 \Rightarrow PL = \sqrt{y^2 + z^2}$

$\Rightarrow$ The distance from P to the x axis is $\sqrt{y^2 + z^2}$

Similarly the distance from P to the Y axis is $\sqrt{z^2 + x^2}$ and the distance from P to the Z axies is $\sqrt{x^2 + y^2}$

Also $OP^2 = OL^2 + PL^2 = x^2 + y^2 + z^2$ so $OP = \sqrt{x^2 + y^2 + z^2}$

Thus the distance from the origin to P is

$$OP = \sqrt{x^2 + y^2 + z^2}$$

4. Position Vector:

We have seen that there is a one one correspondence between points in the space and ordered triads of numbers with respect to a fixed frame work. We also defined any ordered triad a position vector. Thus with respect to a given coordinate system every point in the space is associated with a unique position vector. We call this position vector, the postiiton vector of the point with respect to the fixed coordinate system. Thus the position vectors of

O is (0, 0, 0)

I is (1, 0, 0)

J is (0, 1, 0)

K is (0, 0, 1)

If P = (x, y, z) then $P = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$$= xI + yJ + zK$$

So that $\overline{OP} = xI + y\bar{J} + zK$

5. Translation of Axes:

If OA, $O_1 A_1$ are parallel lines, $OA = O_1A_1$ and OO_1A_1A is a parallelogram then we say that OA, O_1A_1 are in the same direction. Let $OXYZ, O_1X_1Y_1Z_1$ be coordinate systems. We say that $O_1X_1Y_1Z_1$ is a translation of $OXYZ$ if the coordinate axis OX is parallel to O_1X_1 OY is parallel to O_1Y_1 and OZ is parallel to O_1Z_1 and OX, O_1X_1 are in the same direction, OY and O_1Y_1 are in the same direction and OZ and O_1Z_1 are in the same direction. If the coordinates of a point P with respect to the system OXYZ are x, y, z and with respect to $O_1X_1Y_1Z_1$ are X, Y, Z and if the coordinates of O_1 w.r.t. $OXYZ$ are (λ, μ, υ) then

$X = x - \lambda,\ Y = y - \mu$ and $Z = z - \upsilon$

i.e., $x = X + \lambda,\ Y = y + \mu$ and $Z = Z + \upsilon$

6. Rotation of Axes:

If the axes of a coordinate system OXYZ are changed without shifting the origin then the transformation of axes is said to be by rotation. If the new axes $\overline{OX^1}, \overline{OY^1}, \overline{OZ^1}$ are muttually perpendicular and have direction consines $\ell_1, m_1, n_1 ; \ell_2, m_2, n_2$ and ℓ_3, m_3, n_3 respectively with respect to the system $OXYZ$ then the relations between the coordinates (x, y, z) of a point w.r.t. the OXYZ system and (X, Y, Z) of the same point w.r.t. $OX^1Y^1Z^1$ are given by

$x = X\ell_1 + Y\ell_2 + Z\ell_3$ and $X = \ell_1 x + m_1 y + n_1 z$

$y = Xm_1 + Ym_2 + Zm_3$ $\quad Y = \ell_2 x + m_2 y + n_2 z$

$z = Xn_1 + Yn_2 + Zn_3$ $\quad Z = \ell_3 x + m_3 y + n_3 z$

The matrix $A = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix}$ is called the transformation matrix.

	x	y	z
X	ℓ_1	m_1	n_1
Y	ℓ_2	m_2	n_2
Z	ℓ_3	m_3	n_3

We state the following theorems without proof:

Theorem:

If $f(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is transformed to

$$g(X,Y,Z) = a^1X^2 + b^1Y^2 + c^1Z^2 + 2f^1YZ + 2g^1ZX + 2h^1XY$$

by transformation of axes v to $OX^1Y^1Z^1$ then

(1) $a + b + c = a^1 + b^1 + c^1$

(2) $ab + bc + ca - f^2 - g^2 - h^2 = a^1b^1 + b^1c^1 + c^1a^1 - f^{1^2} - g^{1^2} - h^{1^2}$

and (3) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a^1 & h^1 & b^1 \\ h^1 & b^1 & f^1 \\ g^1 & f^1 & c^1 \end{vmatrix}$

Theorem:

If $f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$ is transformed to $g(X,Y,Z) = a^1X^2 + b^1Y^2 + c^1Z^2 + 2f^1YZ + 2g^1ZX + 2h^1XY + 2h^1X + 2v^1Y + 2w^1Z + d^1$.

Under a transformation of axes $OXYZ$ to $OX^1Y^1Z^1$ by rotation then

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = \begin{vmatrix} a^1 & h^1 & g^1 & u^1 \\ h^1 & b^1 & f^1 & v^1 \\ g^1 & f^1 & c^1 & w^1 \\ y^1 & v^1 & w^1 & d^1 \end{vmatrix}$$

7 Direction Ratios and Direction Cosines:

The direction cosines of a line L are defined to be the cosines of the angles made by the line with the coordinate axes.

Let L be a line and L^1 be the line parallel to L passing through the origin. The coordinates x, y, z of any point $P \neq 0$ on L^1 are called the direction ratios of L. Since the coordinates of $P(x, y, z)$ and $P^1(x^1, y^1, z^1)$ on L^1 where $P \neq 0 \neq P^1$ satisfy $\frac{x}{x^1} = \frac{y}{y^1} = \frac{z}{z^1}$. Any pair of direction ratios $(x, y, z), (x^1, y^1, z^1)$ of L satisfies $x = \lambda x^1,\ y = \lambda y^1,\ z = \lambda z^1$ for some $\lambda \neq 0$ in $\mathbb{R}$.

Therefore the line L^1 has only two pairs $(\ell, m, n), (-\ell, -m, -n)$ on it so that $\ell^2 + m^2 + n^2 = 1$. These directionratios are called the direction cosines of L. Geometrically this is interpreted as follows. If L makes the angles α, β, γ with the x axis, y axis and z axis respectively measured from the axes to the line in the anticlock wise direction, then $\cos\alpha,\ \cos\beta,\ \cos\gamma$ are the direction cosines of L. It can be proved that $\ell^2 + m^2 + n^2 = 1$.

8 Algebraic Surfaces:

Let $F : \mathbb{R}^3 \to R$ be a function. For every point $P(x, y, z)$ there corresponds a real number $F(x, y, z)$. Let S be the set of all points in $\mathbb{R}^3$ satisfying the equation $F(x, y, z) = 0$, $S = \{P/P = (x, y, z)\ ;\ F(x, y, z) = 0\}$. S is called a surface and the equation $F(x, y, z) = 0$ represents S. Also we say that $F(x, y, z) = 0$ is an equation of S. If $F(x, y, z)$ is a non - zero polynomial in x, y, z then F is a finite sum of the terms of the form $ax^\alpha y^\beta z^\gamma$ where $a \neq 0, \alpha \geq 0, \beta \geq 0$ and $\gamma \geq 0$. The largest sum $\alpha + \beta + \gamma$ is called the degree of $F(x, y, z)$. The surface represented by $F(x, y, z) = 0$ is called an algebraic surface and the degree of $F(x, y, z)$ is called the degreee of the surface. A first degree equation in x, y, z is said to be a linear equation and a second degree surface is called a quadric.

Lesson Writer
Prof.I. RAMABHADRA SARMA

Rene Descartes (1595 - 1650):

"I am thinking therefore I exist.":

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Rene Descartes (March 31, 1596 - February 11, 1650) also known as Cartesius, was a noted French philosopher, mathematician and scientist. Dubbed the "Founder of Modern Philosophy" and the "Father of Modern Mathematics", he ranks as one of the most important and influential thinkers of modern times. Descartes was one of the key thinkers of the Scientific Revolution in the Western World. He is also honoured by having the Cartesian coordinate system used in plane geometry and algebra named after him.

As the inventor of the Cartesian coordinate system, Descartes founded analytic geometry, that bridge between algebra and geometry crucial to the invention of the calculus and analysis. His most famous statement is *Cogito ergo sum* (French: Je pense, donc je suis or in English: I think, therefore I am), found in §7 of Principles of Philosophy (Latin) and part IV of Discourse on Method (French).

Mathematical legacy:

Descartes's theory provided the basis for the calculus of Newton and Leibniz, by applying infinitesimal calculus to the tangent problem, thus permitting the evolution of that branch of modern mathematics. Descartes' rule of signs is also a commonly used method in modern mathematics to determine possible quantities of positive and negative zeros of a function.

Descartes also made contributions in the field of optics;

Lesson - 1

PLANES

1.1 Objective of the lesson:

In this lesson the student is introduced to various aspects of the plane such as, the equation of a plane, angles between planes, distance from a plane, distance between planes and systems of planes.

1.2 Structure:

This lesson contains the following components:

1.3 Introduction

1.4 Equations of a plane

1.5 Angles between two planes

1.6 Distance of a point from a plane

1.7 Systems of planes

1.8 Projections on a plane and volume of tetrahedron

1.9 Answers to S.A.Q.

1.10 Summary

1.11 Technical Terms

1.12 Model Examination questions

1.13 Exercises

1.14 Model Practical Problem with Solution

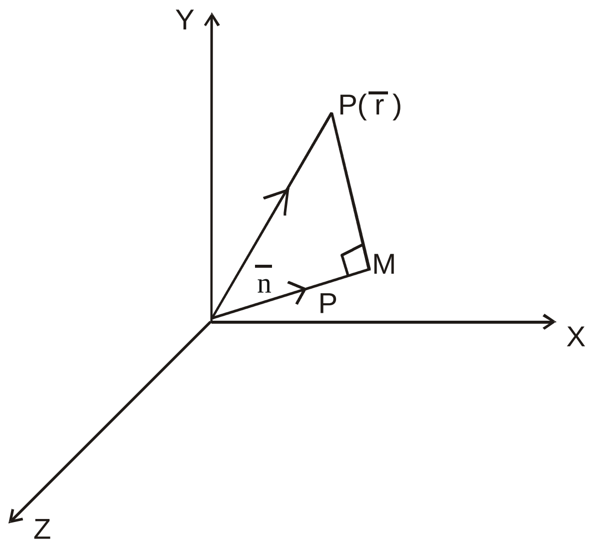
1.3 Introduction:

As mentioned in the preliminaries (2.7), a plane is uniquely determined as the perpendicular to a line at a point (foot of the perpendicular) on it. Using this result we obtain the equation of a plane in the normal form from which we derive several types of equations to the plane. We also observe that the equation of a plane is a first degree (linear) equation in x,y,z.

1.4 Equations of a Plane:

1.4.1 Theorem: The equation of a plane is of first degree. More precisely a plane is the locus of points (x, y, z) satisfying a first degree equation.

Proof: We first derive the vector form of the equation of a plane. If a point lies outside a plane, then there is a unique line through the point that is perpendicular to every line in the plane.



Let π be the given plane, M the foot of the perpendicular from the origin to π and $p = OM\left(p \geq 0\right)$.

Let $\bar{n}$ be the unit vector along this perpendicular $\overline{OM}$

If $P \in \pi \cdot \ \bar{n} \cdot \overline{PM} = 0$

Since $\overline{OM} = \overline{OP} + \overline{PM}, \ \overline{PM} = \overline{OM} - \overline{OP}$

So $\bar{n} \cdot \left(\overline{OM} - \overline{OP}\right) = 0$

$$\Rightarrow \bar{n} \cdot \overline{OM} = \bar{n} \cdot \overline{OP}$$

Let $\overline{OP} = \bar{r}$ then $\bar{n} \cdot \bar{r} = OM = p$

Thus the position vector $\bar{r}$ of a point $P \in \pi$ satisfies the equation $\bar{r} \cdot \bar{n} = p$.

Conversely suppose $\bar{r} \cdot \bar{n} = p \cdot \quad \bar{r} = \overline{OP}$ is the position. Vector of a point P in the space. We show that $P \in \pi$

Then $\bar{n} \cdot \overline{PM} = \bar{n} \cdot \left(\overline{PO} + \overline{OM}\right)$ where M is as above.

$$= -\bar{n} \cdot \bar{r} + \bar{n} \cdot \overline{OM}$$

$$= -\bar{n} \cdot \bar{r} + p = 0$$

$\Rightarrow \overline{PM}$ is perpendicular to π

$\Rightarrow PM \perp OM$

$\Rightarrow P \in \pi$ (by 2.7 in lesson zero)

Hence $P \in \pi$ iff $\bar{r} \cdot \bar{n} = p$

Hence the vector form of the equation of the plane π is

$$\bar{r} \cdot \bar{n} = p$$

Using this vector equation we show that the equation of a plane is of first degree.

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $\bar{n} = \ell\bar{i} + m\bar{j} + n\bar{k}$, then the equation $\bar{r} \cdot \bar{n} = p$ becomes $\ell x + my + nz = p$.

This is the cartesian form of the equations of the plane.

Since ℓ, m, n are the direction ratios of $\bar{n}$ atleast one of ℓ, m, n is non-zero.

Thus the equation of the plane is a first degree equation.

1.4.2 Definition:

Normal form of the equation of a plane: The equation $\ell x + my + nz = p$ where p is the length of perpendicular from the origin to the plane and ℓ, m, n are the d.cs. of the normal to the plane is called the normal form of the equation of the plane.

1.4.3 Corollary: Every equation of first degree in x, y, z represents a plane.

Proof: Consider the equation $ax + by + cz + d = 0$.................(1)

where one of a, b, c is non-zero

We have $a^2 + b^2 + c^2 > 0$. We can take $d \le 0$ i.e. $-d \ge 0$

$$ax + by + cz + d = 0$$

$$\Leftrightarrow ax + by + cz = -d$$

$$\Leftrightarrow \frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

$$\Leftrightarrow (x, y, z) \cdot \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

$\Leftrightarrow \bar{r}\cdot\bar{n}=p$ ....(2) where $\bar{r}=(x,y,z)$

$$\bar{n}=\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right) \text{ and } p=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

Thus (2) holds $\Leftrightarrow$ (1) holds.

Since equation (2) is the equation of a plane, (1) represents the equation of a plane

1.4.4 Condition for two linear equations to represent the same plane:

Theorem: The equations $a_1x+b_1y+c_1z+d_1=0$.....................(1)

$a_2x+b_2y+c_2z+d_2=0$.....................(2)

represent the same plane iff $a_1:b_1:c_1:d_1=a_2:b_2:c_2:d_2$ ....(3)

Proof: The normal form of (1) is $\bar{r}\cdot\bar{n}_1=-d_1$ where $\overline{n_1}=(a_1, b_1, c_1)$ and of (2) is $\bar{r}\cdot\bar{n}_2=-d_2$ where $\overline{n_2}=(a_2, b_2, c_2)$.

Since $\bar{n}_1$ and $\bar{n}_2$ are normals to the planes (1) and (2), If the equations represent the same plane, $\overline{n_1}=k\overline{n_2}$ for some k, so that $a_1=ka_2$, $b_1=kb_2$ and $c_1=kc_2$........(4)

Moreover for any x,y,z in the plane

$$0=d_2+a_2x+b_2y+c_2z=d_1+a_1x+b_1y+c_1z$$

$$=d_1+k(a_2x+b_2y+c_2z)$$

$$=d_1-kd_2$$

$$\Rightarrow d_1=kd_2$$

Thus $a_1:a_2=b_1:b_2=c_1:c_2=d_1:d_2=k:1$

Conversely if $a_1:a_2=b_1:b_2=c_1:c_2=d_1:d_2=k:1$, then

$a_1=ka_2,\ b_1=kb_2,\ c_1=kc_2,\ d_1=kd_2$

so $a_1x+b_1y+c_1z+d_1=0 \Leftrightarrow a_2x+b_2y+c_2z+d_2=0$

i.e. the two equations represent the same plane.

1.4.5 Note:

1. $\bar{r}\cdot\bar{m} = q$ and $\bar{m} \neq \bar{0}$. The equation $\bar{r}\cdot\bar{m} = q$ represents a plane which is at a distance $\frac{|q|}{|\bar{m}|}$ from the origin in the direction of $\bar{m}$.

2. The equation of a plane through the origin is a first degree homogeneous equation of the form $\ell x + my + nz = 0$.

3. Direction ratios of a normal to the plane $ax + by + cz + d = 0$ are $a, b, c.$

4. The distance of the origin from the plane $ax + by + cz + d = 0$ is $\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

5. The normal form of the plane $ax + by + cz + d = 0$ is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

Intercepts:

1.4.6 Definition: If a plane cuts the x-axis at A (a, 0, 0), y- axis at B(0, b, 0) and z - axis at C(0,0, c) then a, b, c are called x intercept, y intercept and z intercept of the plane respectively.

Intercept form of a plane:

1.4.7 Theorem: The equation of the plane making intercepts a, b, c on the x, y, z coordinate axes respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Proof:

Let $\ell x + my + nz = d$ be the equation of the plane π making intercepts a, b, c with the axes.

Then as (a, 0, 0), (0, b, 0) and (0, 0, c) are on the plane,

$$\ell a = mb = nc = d$$

Since $a \neq 0 \neq b \neq 0 \neq c \neq 0$,

$$\ell = \frac{d}{a},\ m = \frac{d}{b},\ n = \frac{d}{c}$$

The equation representing the given plane becomes

$$\frac{dx}{a}+\frac{dy}{b}+\frac{dz}{c}=d$$

Since one of ℓ, m, n is non-zero, $d \neq 0$.

This equation is equivalent to

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

1.4.8 (a) Planes parallel to the axes:

If the plane $ax+by+cz+d=0$ is parallel to the x - axis, then the normal to the plane is perpendicular to the x-axis, hence

$$a(1)+b(0)+c(0)=0$$

$\Rightarrow a=0$ (since d.cs. of x - axis are 1, 0, 0)

Therefore the equation of the plane is

$$by+cz+d=0$$

Similarly equations of the planes parallel to y-axis and z - axis can be written as

$ax+cz+d=0$ and $ax+by+d=0$ respectively.

(b) Planes parallel to coordinate planes:

Theorem: The equation of xy- plane is $z=0$, yz plane is x = 0 and zx - plane is y = 0.

Proof: Clearly the origin lies in xy - plane and z - zxis is a normal to xy - plane

d.cs. of z- axis are 0, 0, 1.

So, the equation of xy - plane is x = 0 and the equation of zx plane is y = 0.

1.4.9 Corollary: The equation of any plane parallel to yz - plane is of the form $ax+d=0$, parallel to zx - plane is $by+d=0$ and parallel to xy - plane is $cz+d=0$.

Equation of the plane through the non collinear points:

1.4.10 Theorem: The vector equation of the plane passing through three non collinear points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$ and $C(x_3,y_3,z_3)$ is $\left[\left(\bar{r}-\bar{a}\right)\ \left(\bar{b}-\bar{a}\right)\ \left(\bar{c}-\bar{a}\right)\right]=0$

where $\bar{a}, \bar{b}, \bar{c}$ are the position vectors of A, B, C respectively.

The cartesian equation of this plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Proof: Let π be the plane containing A, B, C and $P(x,y,z)$ be any point with position vector $\bar{r}$. Then P lies in π if and only if AB, AC, AP lie in π

i.e. AB, AC, AP are coplanar

The condition for coplanarity of these lines is that

$$\left[\overline{AP} \quad \overline{AB} \ \overline{AC}\right] = 0$$

i.e. $\left[\bar{r}-\bar{a}, \bar{r}-\bar{b}, \bar{r}-\bar{c}\right] = 0.........(1)$

Thus the vector equation of π is given by (1)

Since $\bar{r} - \bar{a} = (x-x_1,\ y-y_1,\ z-z_1)$

$$\bar{b} - \bar{a} = (x_2-x_1,\ y_2-y_1, z_2-z_1)$$

$$\bar{c} - \bar{a} = (x_3-x_1,\ y_3-y_1,\ z_3-z_1)$$

$$\left[\bar{r}-\bar{a} \quad \bar{b}-\bar{a} \quad \bar{c}-\bar{a}\right] = \Delta = \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}$$

Thus the cartesian equation of π is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

1.4.11 Corollary: If $A = (x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C = (x_3, y_3, z_3)$ are non collinear then the equation of the plane π containing A, B, C is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Proof: The above determinant reduces to Δ in 1.4.10 when we subtract the second row from every row and expand in terms of the last column.

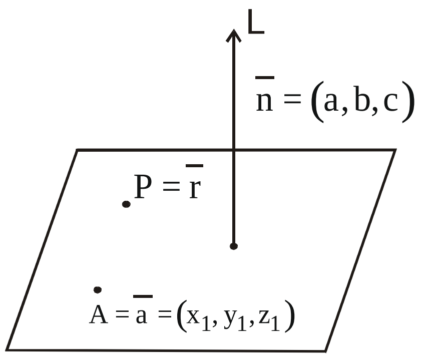
Plane through a point and perpendicular to a line:

1.4.12 Theorem: The vector equation to the plane π through a point $A=(x,y,z)$ with position vector $\bar{a}$ and perpendicular to line (L) with d.r.s a, b, c is $(\bar{r}-\bar{a})\cdot\bar{n}=0$ where $\bar{n}$ is the unit vector along L.

The cartesian equation of this plane is given by

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

Proof:



Let $P=\bar{r}=(x,y,z)$ be any point

Then $\overline{AP}=\bar{r}-\bar{a}=(x-x_1,\ y-y_1,\ z-z_1)$

$P\in\pi$ iff AP lies in $\pi \Leftrightarrow \overline{AP}\perp\bar{n} \Leftrightarrow (\bar{r}-\bar{a})\cdot\bar{n}=0$

Then $P\in\pi$ iff $(\bar{r}-\bar{a})\cdot\bar{n}=0$

$(\bar{r}-\bar{a})\cdot\bar{n}=0$ is the vector equation of π.

Since $(\bar{r}-\bar{a})\cdot\bar{n}=(x-x_1,\ y-y_1,\ z-z_1)\cdot(a,b,c)$

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ is the cartesian form of π.

Parametric form of the equation of a plane through three non collinear points:

1.4.13 Theorem: The equation to the plane passing through three non-collinear points $A(\bar{a}), B(\bar{b}), C(\bar{c})$ is $\bar{r}=(1-s-t)\bar{a}+s\bar{b}+t\bar{c}$ where s, t are scalars.

Proof: A point P with position vector $\bar{r}$ lies in the plane of A, B, C iff $\overline{AP}, \overline{AB}, \overline{AC}$ are coplanar.

iff $\bar{r}-\bar{a}$, $\bar{b}-\bar{a}$, $\bar{c}-\bar{a}$ are coplanar

$\Leftrightarrow$ each one of $\overline{AP}, \overline{AB}, \overline{AC}$ is a linear combination of the other two vectors.

$\Leftrightarrow \bar{r}-\bar{a}=s(\bar{b}-\bar{a})+t(\bar{c}-\bar{a})$ where s, t are scalars

$\Leftrightarrow \bar{r}=(1-s-t)\bar{a}+s\bar{b}+t\bar{c}$

Examples:

1.4.14 Find the intercepts of the plane $2x-3y+4z=12$

Solution: Method 1:

Given equation of the plane is $2x-3y+4z=12$

Intercept form of the plane is $\frac{2x}{12}-\frac{3y}{12}+\frac{4z}{12}=1$

i.e. $\frac{x}{6}-\frac{y}{4}+\frac{z}{3}=1$

Hence x - intercept is 6, y - intercept is - 4, z - intercept is 3.

Method 2:

If the plane meets the axes in $(x_0,0,0),(0,y_0,0)$ and $(0,0,z_0)$ then $2x_0=12 \Rightarrow x_0=6$ and similarly $y_0=-4, z_0=3$.

Therefore x - intercept is 6, y - intercept is -4, z - intercept is 3.

1.4.15 What are the d.rs. of a normal to the plane $2x-2y+z=5$. Express equation of the plane in its normal form.

Solution:

By definition d.rs. of normal to the plane are coefficients of x, y, z in the given equation.

So the d.rs. of normal to the plane $2x-2y+z-5=0$ are 2, -2, 1.

Normal form of the plane $ax+by+cz+d=0$ is

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

The normal form of the given plane is $\frac{2}{3}x-\frac{2}{3}y+\frac{1}{3}z=\frac{5}{3}$

1.4.16 Find the equation of the plane through the points (2, 2, -1), (3, 4, 2), (7, 0, 6).

Solution: Let π be the plane through the given points

$A=\bar{a}=(2,2,-1),\ B=\bar{b}=(3,4,2),\ C=\bar{c}=(7,0,6)$

Let $P=\bar{r}=(x,y,z)$ be any point on the plane π.

Hence $\overline{AP}, \overline{AB}, \overline{AC}$ are coplanar.

$\Rightarrow\left[\overline{AP}, \overline{AB}, \overline{AC}\right]=0$

$\overline{AP}=(x-2,\ y-2,\ z+1),\ \overline{AB}=(1,2,3),\ \overline{AC}=(5,-2,7)$

$$\left[\overline{AP}, \overline{AB}, \overline{AC}\right]=\begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}$$

$$=20(x-2)+8(y-2)-12(z+1)=20x+8y-12z-68$$

Hence the equation of π is $20x+8y-12z-68=0$

i.e. $5x+2y-3z-17=0$

1.4.17 The foot of the perpendicular from the origin to a plane is $(2,-3,4)$. Find the equation of the plane.

Solution: The point of intersection of the plane and the perpendicular line from the given point to the plane is called the foot of the perpendicular of the given point.

Since the foot of the perpendicular from the origin 0 to a plane is $A=(2,-3,4)$.

Then $\overline{OA}$ is normal to the required plane, A is a point on the plane and the d.rs. of OA are 2, -3, 4.

Hence the equation of the plane is

$$2(x-2)-3(y+3)+4(z-4)=0$$

i.e. $2x-3y+4z-29=0$

1.4.18 A plane meets the coordinate axes in A, B, C. If the centroid of ΔABC is (a,b,c) show that the equation to the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.

Solution: Let $A=(x_1,0,0)$, $B=(0,y_1,0)$, $C=(0,0,z_1)$ be the points on x,y,z axes respectively.

Centroid of ΔABC is $\left(\frac{x_1}{3},\frac{y_1}{3},\frac{z_1}{3}\right)$

But given centroid is (a,b,c)

$$a=\frac{x_1}{3},\ b=\frac{y_1}{3},\ c=\frac{z_1}{3} \Rightarrow x_1=3a,\ y_1=3b,\ z_1=3c$$

Hence $A=(3a,0,0)$ $B=(0,3b,0)$ $C=(0,0,3c)$

Equation to the plane ABC is $\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$

1.4.19 **S.A.Q:** Show that the line joining the points $(6,-4,4)$, $(0,0,-4)$ intersects the line joining the points $(-1,-2,-3)$, $(1,2,-5)$.

1.4.20 Obtain the equation of the plane containing $(0,4,3)$ and the line through the points $(-1,-5,-3)$, $(-2,-2,1)$. Hence show that $(0,4,3)$, $(-1,-5,-3)$, $(-2,-2,1)$ and $(1,1,-1)$ are coplanar.

1.5 Angles between two planes:

For any pair of planes π_1,π_2 two angles are formed between their normals drawn from an arbitrary point P in the space. These angles are supplementary and are independent of the choice of P.

1.5.1 **Definition:** An angle between two planes is an angle between their corresponding normals.

1.5.1A **Angular bisector of two planes:** Angular bisector of two planes is the plane through the line of intersection of the planes that makes equal angles with the planes.

1.5.2 Theorem: If θ is an angle between the planes π_1, π_2 represented by $\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$.

Then $\cos\theta = \pm\dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Proof: Since a_1, b_1, c_1 are the direction ratios of the normal to the plane π_1, the normal vector to π_1 is $\overline{m}_1 = (a_1, b_1, c_1)$

Similarly the normal vector to π_2 is $\overline{m}_2 = (a_2, b_2, c_2)$.

If θ is an angle between $\overline{m}_1, \overline{m}_2$ then

$$\cos\theta = \pm\frac{\overline{m}_1 \cdot \overline{m}_2}{|\overline{m}_1|\ |\overline{m}_2|} = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

1.5.3 Corollaries:

(i) The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel iff $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$ and are perpendicular iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

(ii) The equation of any plane parallel to the plane $ax + by + cz + d = 0$ is of the form $ax + by + cz + k = 0 \quad (k \in \mathbb{R})$.

(iii) The equation of the plane passing through $\overline{r}_0 = (x_0, y_0, z_0)$ and parallel to $ax + by + cz + d = 0$ is

$(\overline{r} - \overline{r}_0) \cdot \overline{n} = 0$ where $\overline{n} = (a, b, c)$

i.e. $(a - a_0)x + (b - b_0)y + (c - c_0)z = 0$

(iv) The equation of the plane passing through $\overline{r}_0 = (x_0, y_0, z_0)$ and perpendicular to the line whose direction ratios are a, b, c is $(\overline{r} - \overline{r}_0) \cdot \overline{n} = 0$ where $\overline{n} = (a, b, c)$

i.e. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Proof: (i) and (ii) follow from 1.4.2

(iii) and (iv) are equivalent.

Thus it is enough if we prove (iv)

Proof: Let $A=(x_0,y_0,z_0)$ be the given point in the plane π with position vector $\bar{r}_0$ and $P=(x,y,z)$ be any point in π with position vector $\bar{r}$.

$$\overline{AP}=\bar{r}-\bar{r}_0=(x-x_0,\ y-y_0,\ z-z_0)$$

Let L be a line perpendicular to the plane π with d.rs. a,b,c

Let $\bar{n}=(a,b,c)$ be the vector along L.

$\Leftrightarrow$ L is perpendicular to every line in π

$\Leftrightarrow$ L is perpendicular to AP

$\Leftrightarrow$ $\bar{n}$ is perpendicular to $\overline{AP}$

$\Leftrightarrow$ $(\bar{r}-\bar{r}_0)\cdot\bar{n}=0$

$\Leftrightarrow$ $(x-x_0,\ y-y_0,\ z-z_0)\cdot(a,b,c)=0$

$\Leftrightarrow$ $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ is the equation of π through A and perpendicular to $\bar{n}$.

1.5.4 Examples:

Find the equation of the plane through the point $(4,0,1)$ and parallel to the plane $4x+3y-12z+8=0$.

Solution: The equation of the plane parallel to $4x+3y-12z+8=0$ is of the form $4x+3y-12z+k=0$

This passes through the point $(4,0,1)$

$4(4)+3(0)-12(1)+k=0$

$\Rightarrow k=-4$

$\Rightarrow$ equation of the plane is $4x+3y-12z-4=0$

1.5.5 Find the equation of the plane π through the points $(2,2,1)$, $(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$.

Solution: Let A = (2,2,1) and B=(9,3,6) and a, b, c be d.rs. of normal to the plane.

d.rs. of AB are 7, 1, 5

Since AB is perpendicular to the normal of the plane.

$$7a + b + 5c = 0 \cdots\cdots\cdots\cdots\cdots (1)$$

The plane π is perpendicular to the plane $2x + 6y + 6z = 9$.

$\Leftrightarrow$ Then corresponding normals are perpendicular to each other.

i.e. $2a + 6b + 6c = 0 \cdots\cdots\cdots\cdots (2)$

Solving (1) and (2)

$$\begin{array}{cccc} & a & b & c \\ 1 & 5 & 7 & 1 \\ 6 & 6 & 2 & 6 \end{array}$$

$$\frac{a}{6-30} = \frac{b}{10-42} = \frac{c}{42-2}$$

$$\Rightarrow \frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$$

$$\Rightarrow \frac{a}{-3} = \frac{b}{-4} = \frac{c}{5}$$

Hence -3, -4, -5 are d.rs. of π.

The equation of π is therefore

$$-3(x-2) - 4(y-2) + 5(z-1) = 0$$

$$3x + 4y - 5z - 9 = 0.$$

1.5.6 Show that the equation of the plane passing through the points $A(1,-2,4)$, $B(3,-4,5)$ and perpendicular to x y plane is $x + y + 1 = 0$.

Solution: Let $A = (1,-2,4)$, $B = (3,-4,5)$

d.rs. of AB are 2, -2, 1

Let $ax + by + cz + d = 0$ be the equation of the required plane.

Then the normal L to the plane has d.rs. a, b, c.

Since AB lies in the plane, AB and L are perpendicular.

$\Rightarrow\ 2a-2b+c=0\cdots\cdots\cdots\cdots(1)$

Since the plane is perpendicular to xy - plane

L is perpendicular to Z - axis

Since d.cs. of Z axis are 0, 0, 1.

$$a(0)+b(0)+c=0$$

$$\Rightarrow c=0$$

$$\Rightarrow 2a-2b=0$$

$$\Rightarrow a=b \Rightarrow \frac{a}{1}=\frac{b}{1}=\frac{c}{0}$$

Hence d.rs. of L are 1, 1, 0

Equation of the required plane is

$$1(x-1)+1(y+2)+0(z-4)=0$$

$$x+y+1=0$$

1.5.7 Find the equation of the plane through the point (-1, 3, 2) and perpendicular to the planes $x+2y+2z=5$ and $3x+3y+2z=8$.

Solution: Let $A=(-1,3,2)$

and $\pi \equiv ax+by+cz+d=0$ be the required plane

Then d.rs. of normal to π are a,b,c

Given planes are $x+2y+2z=5\cdots\cdots\cdots\cdots(1)$

and $3x+3y+2z=8\cdots\cdots\cdots\cdots(2)$

π is perpendicular to the plane (1) $\Rightarrow\ a+2b+2c=0\cdots\cdots\cdots\cdots(3)$

Plane π is perpendicular to the plane (2) $\Rightarrow 3a+3b+2c=8\cdots\cdots\cdots\cdots(4)$

From (3) and (4)

$$\begin{array}{cccccccc} & a & & b & & c & & \\ 2 & & 2 & & 1 & & 2 \\ 3 & & 2 & & 3 & & 3 \end{array}$$

$$\frac{a}{4-6} = \frac{b}{6-2} = \frac{c}{3-6}$$

i.e. $\frac{a}{-2} = \frac{b}{4} = \frac{c}{-3}$

Hence d.rs. of the normal to the plane π are $-2, 4, -3$ the required equation is

$$-2(x+1) + 4(y-3) - 3(z-2) = 0 \Rightarrow 2x - 4y + 3z + 8 = 0$$

1.5.8 Find the equation of the plane through the point $(2, -3, 1)$ and perpendicular to the line through the points $(3, 4, -1)$ and $(2, -1, 5)$.

Solution: Let $A = (2, -3, 1)$, $P = (3, 4, -1)$ and $Q = (2, -1, 5)$

d.rs. of PQ are 1, 5, -6.

Let π be the plane with given conditions.

The plane π is perpendicular to PQ.

Then the normal to π is parallel to PQ.

Therefore d.rs. of the normal to the plane π are 1, 5, -6.

Hence the equation of the plane π is

$$1(x-2) + 5(y+3) - 6(z-1) = 0$$

i.e. $x + 5y - 6z + 19 = 0$

1.5.9 Show that the points $(-6, 3, 2), (-13, 17, -1), (3, -2, 4)$ and $(5, 7, 3)$ are coplanar.

Solution: Let $A = (-6, 3, 2)$, $B = (-13, 17, -1)$, $C = (3, -2, 4)$ and $D = (5, 7, 3)$.

Let $P = (x, y, z)$ be any point on the plane containing A, B and C.

Equation of the plane through A, B, C is

$$\begin{vmatrix} x+6 & y-3 & z-2 \\ -13+6 & 17-3 & -1-2 \\ 3+6 & -2-3 & 4-2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x+6 & y-3 & z-2 \\ -7 & 14 & -3 \\ 9 & -5 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 13(x+6) - 13(y-3) + (-91)(z-2) = 0 \Rightarrow x+6-y+3-7z+14=0$$

$$x-y-7z+23=0 .... (1)$$

consider $x-y-7z+23 = 5-7-21+23=0$

The point $D(5,7,3)$ lies on the plane (1)

Hence the given points A, B, C, D are coplanar.

1.5.10 Find the angles between the planes $x+2y+3z=5,\ 3x+3y+z=9$.

Solution: Let θ be the angle between the planes

$$x+2y+3z=5 \cdots\cdots\cdots\cdots (1)$$

$$3x+3y+z=9 \cdots\cdots\cdots\cdots\cdots (2)$$

$$\cos\theta = \pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \pm\frac{(1)(3)+(2)(3)+(3)(1)}{\sqrt{1+4+9}\ \sqrt{9+9+1}} = \pm\frac{3+6+3}{\sqrt{14}\ \sqrt{19}}$$

$$\cos\theta = \pm\frac{12}{\sqrt{14}\ \sqrt{19}}$$

$$\theta = \cos^{-1}\left(\frac{12}{\sqrt{14}\ \sqrt{19}}\right) \text{ or } \pi - \cos^{-1}\left(\frac{12}{\sqrt{14}\ \sqrt{19}}\right)$$

1.5.11 Find the angle between the planes $2x-y+z=0,\ x+y+2z=7$.

Solution: Let θ be the angle between the planes

$$2x-y+z=0 \cdots\cdots\cdots\cdots (1) \qquad x+y+2z=7 \cdots\cdots\cdots (2)$$

$$\cos\theta = \pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} = \pm\frac{(2)(1)-(1)(1)+(1)(2)}{\sqrt{4+1+1}\sqrt{1+1+4}}$$

$$= \pm\frac{3}{\sqrt{6}\ \sqrt{6}} = \pm\frac{3}{6} = \pm\frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\pm\frac{1}{2}\right) = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

1.5.12 S.A.Q.: Prove that the equation of the plane through the points $(1,-2,4)$ and $(3,-4,5)$ and parallel to x - axis is $y+2z=6$.

1.5.13 S.A.Q.: Find the equation of the plane bisecting the line segment joining $(2,0,6)$ and $(-6,2,4)$ and perpendicular to the line segment.

1.5.14 S.A.Q.: Find the angles between the planes $2x-y+4z+11=0,\ \ 3x-2y-3z+27=0$.

1.6 Distance of a point from a plane:

1.6.1 Definition: 1. Two points A, B are on the same side of a plane π if the segment AB doesn't intersects π.

2. Two points A, B are on the opposite sides of a plane π if the segment AB intersects π.

1.6.2 Theorem: $A=(x_1,y_1,z_1),\ B=(x_2,y_2,z_2)$ lie on the same side or opposite side of a plane $\pi = ax+by+cz+d=0$ iff $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ have same sign or opposite signs.

Proof: If a point $A(x_1,y_1,z_1)$ doesn't lie in $\pi \equiv ax+by+cz+d=0$ then $ax_1+by_1+cz_1+d>0$ or $ax_1+by_1+cz_1+d<0$.

Any point P on the segment AB other than A and B is represented by

$$P=\left(\lambda x_1+(1-\lambda)x_2,\quad \lambda y_1+(1-\lambda)y_2,\ \lambda z_1+(1-\lambda)z_2\right)$$

where $0<\lambda<1$.

This point lies on π iff

$$0=a\left(\lambda x_1+(1-\lambda)x_2\right)+b\left(\lambda y_1+(1-\lambda)y_2\right)+c\left(\lambda z_1+(1-\lambda)z_2\right)+d$$

$$=\lambda\left(ax_1+by_1+cz_1+d\right)+(1-\lambda)\ \left(ax_2+by_2+cz_2+d\right)$$

$$\Rightarrow \frac{\lambda}{1-\lambda}=\frac{-\left(ax_2+by_2+cz_2+d\right)}{ax_1+by_1+cz_1+d}>0 \qquad (\because\ 0<\lambda<1)$$

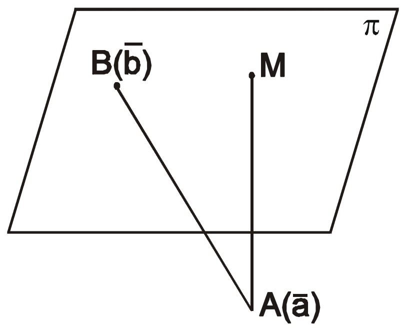
The segment AB meets π if and only if

$ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ have opposite signs. Consequently AB does not meet π if and only $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ have same signs. This completes the proof of the theorem.

Definitions: If π is a plane and P is a point it can be proved that there is one and only one line L through P such that if M is the point of intersection of π and L then PM is perpendicular to every line in π passing through M. The line L is called the perpendicular to π through P, M the foot of the perpendicular and the length PM is called the perpendicular distance (simply the distance) from π to P. This distance is also called the distance from P to π.

1.6.3 Theorem: The distance of $A(x_1, y_1, z_1)$ from the plane π represented by the linear equation $ax+by+cz+d=0$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

Proof:



Let $A=\bar{a}=(x_1, y_1, z_1)$.

The vector equation of π is $\bar{r}\cdot\bar{m}=q$

where $\bar{r}=(x,y,z)$, $\bar{m}=(a,b,c)$ and $q=-d$

Clearly $|\bar{m}|=\sqrt{a^2+b^2+c^2}$

Let AM be the perpendicular from A to π and M be the foot of the perpendicular.

Let $B=\bar{b}$ on π so that $\bar{b}\cdot\bar{m}=q$ where $\overline{AM}=\bar{m}$

AM = Projection of $\overline{AB}$ along $\overline{AM}$

$$= \frac{\left|\overline{AB} \cdot \overline{AM}\right|}{\left|\overline{AM}\right|} = \frac{\left|\left(\overline{b} - \overline{a}\right) \cdot \overline{m}\right|}{\left|\overline{m}\right|}$$

$$= \frac{\left|\overline{b} \cdot \overline{m} - \overline{a} \cdot \overline{m}\right|}{\left|\overline{m}\right|} = \frac{\left|\overline{a} \cdot \overline{m} - \overline{b} \cdot \overline{m}\right|}{\left|\overline{m}\right|} = \frac{\left|\overline{a} \cdot \overline{m} - q\right|}{\left|\overline{m}\right|}$$

$$= \frac{\left|\left(x_1, y_1, z_1\right) \cdot \left(a, b, c\right) + d\right|}{\sqrt{a^2 + b^2 + c^2}} = \frac{\left|ax_1 + by_1 + cz_1 + d\right|}{\sqrt{a^2 + b^2 + c^2}}$$

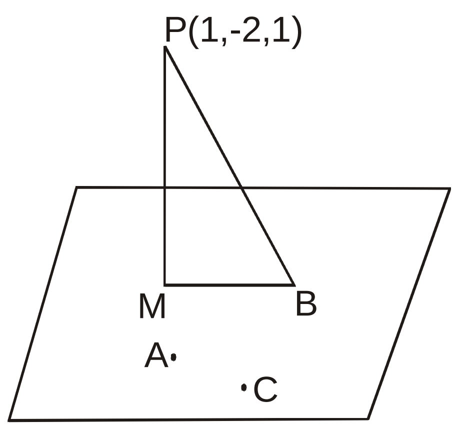
AM is the distance of the point $\left(x_1, y_1, z_1\right)$ from the plane π.

Note: The distance of origin from the plane $ax + by + cz + d = 0$ is $\dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

Examples:

1.6.4 Show that the distance of the point $P(1, -2, 1)$ from the plane $\overleftrightarrow{ABC}$ where $A = (2, 4, 1)$, $B = (-1, 0, 1)$, $C = (-1, 4, 2)$ is $\dfrac{14}{13}$ without finding the equation to the plane $\overleftrightarrow{ABC}$.

Solution: $\overline{AB} \times \overline{AC}$ is a vector perpendicular to the plane $\overleftrightarrow{ABC}$.



$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ -3 & -4 & 0\\ -3 & 0 & 1\end{vmatrix}=-4\bar{i}+3\bar{j}-12\bar{k}$$

Let $\bar{n}$ be the unit vector along $\overline{AB}\times\overline{AC}$

$$\bar{n}=\frac{(-4,3,-12)}{\sqrt{16+9+144}}=\frac{(-4,3,-12)}{13}$$

Also $\overline{PB}=(-2,2,0)$

Perpendicular distance of P from $\overleftrightarrow{ABC}$ = PM = Projection of $\overline{PB}$ along $\overline{PM}$

$$=\left|\bar{n}\cdot\overline{PB}\right|=\frac{\left|(-4,3,-12)\cdot(-2,2,0)\right|}{13}=\frac{8+6+0}{13}=\frac{14}{13}$$

1.6.5. Two systems S_1, S_2 of rectangular axes have the same origin. If a plane intersects the x,y,z axes in S_1 at distances a_1,b_1,c_1 and those in S_2 at a_2,b_2,c_2 respectively from the origin prove that $a_1^{-2}+b_1^{-2}+c_1^{-2}=a_2^{-2}+b_2^{-2}+c_2^{-2}$.

Solution: Let the systems be OX_i, OY_i, OZ_i for $i=1,2$.

Let the equation of the plane with respect to one set of rectangular axes OX_1, OY_1, OZ_1 be

$$\frac{x}{a_1}+\frac{y}{b_1}+\frac{z}{c_1}=1\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The equation of the given plane with respect to the system S_2 a set of rectangular axes OX_2, OY_2, OZ_2 is

$$\frac{x}{a_2}+\frac{y}{b_2}+\frac{z}{c_2}=1\ldots\ldots\ldots\ldots\ldots(2)$$

Since both the systems have the same origin O, the length of perpendicular from the origin to the plane in both the cases is the same.

Therefore $\dfrac{1}{\sqrt{\dfrac{1}{a_1^2}+\dfrac{1}{b_1^2}+\dfrac{1}{c_1^2}}}=\dfrac{1}{\sqrt{\dfrac{1}{a_2^2}+\dfrac{1}{b_2^2}+\dfrac{1}{c_2^2}}} \Rightarrow \dfrac{1}{a_1^2}+\dfrac{1}{b_1^2}+\dfrac{1}{c_1^2}=\dfrac{1}{a_2^2}+\dfrac{1}{b_2^2}+\dfrac{1}{c_2^2}$

$\Rightarrow a_1^{-2}+b_1^{-2}+c_1^{-2} = a_2^{-2}+b_2^{-2}+c_2^{-2}$

1.6.6. A variable plane is at a constant distance $3p$ from the origin and meets the axes OX, OY, OZ in A, B, C respectively. Show that the locus of the centroid of the $\Delta\, ABC$ satisfies $x^{-2}+y^{-2}+z^{-2}=P^{-2}$.

Solution: If $A=(a,0,0),\ B=(0,b,0),\ C=(0,0,c)$

then the equation of the variable plane is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1\cdots\cdots\cdots\cdots(1)$

Let $G=(x_1,y_1,z_1)$ be the centroid of ΔABC.

Then $x_1=\dfrac{a}{3},\ y_1=\dfrac{b}{3},\ z_1=\dfrac{c}{3} \Rightarrow a=3x_1,\ b=3y_1,\ c=3z_1$

Substituting in (1) $\dfrac{x}{3x_1}+\dfrac{y}{3y_1}+\dfrac{z}{3z_1}=1\cdots\cdots\cdots\cdots(2)$

Distance of the origin from the plane (2) is

$$\frac{1}{\sqrt{\dfrac{1}{9x_1^2}+\dfrac{1}{9y_1^2}+\dfrac{1}{9z_1^2}}}=3p \Rightarrow \frac{1}{9x_1^2}+\frac{1}{9y_1^2}+\frac{1}{9z_1^2}=\frac{1}{9p^2} \Rightarrow x_1^{-2}+y_1^{-2}+z_1^{-2}=p^{-2}.$$

$G=(x_1,y_1,z_1)$ satisfies $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

Conversely suppose $G=(x_1,y_1,z_1)$ satisfies $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

There are many triangles having G as centroid.

But there is a unique triangle with centroid G and vertices on the coordinate axes.

Let the triangle be ABC where A, B, C are on X, Y, Z axes respectively.

Then $A=(3x_1,0,0),\ B=(0,3y_1,0),\ C=(0,0,3z_1)$

Equation to the plane containing ΔABC is

$$\frac{x}{3x_1}+\frac{y}{3y_1}+\frac{z}{3z_1}=1 \Rightarrow \frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=3 \cdots\cdots\cdots\cdots (3)$$

Distance of origin from the plane (3) is

$$\frac{|-3|}{\sqrt{\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}}}=\frac{3}{\sqrt{x_1^{-2}+y_1^{-2}+z_1^{-2}}}=\frac{3}{\sqrt{p^{-2}}}=\frac{3}{p^{-1}}=3P$$

Hence the problem.

1.6.7. P is a point such that the sum of the squares of its distances from the planes $x+y+z=0,\ x+y-2z=0,\ x-y=0$ is 5. Show that the locus of P satisfies $x^2+y^2+z^2=5$.

Solution: The distance from an arbitrary point $P=(x_1, y_1, z_1)$ to

(i) the plane $x+y+z=0$ is $\frac{|x_1+y_1+z_1|}{\sqrt{3}}$

(ii) the plane $x+y-2z=0$ is $\frac{|x_1+y_1-2z_1|}{\sqrt{6}}$

(iii) the plane $x-y=0$ is $\frac{|x_1-y_1|}{\sqrt{2}}$

Now $\frac{(x_1+y_1+z_1)^2}{3}+\frac{(x_1+y_1-2z_1)^2}{6}+\frac{(x_1-y_1)^2}{2}=5$

$\Leftrightarrow 2(x_1+y_1+z_1)^2+(x_1+y_1-2z_1)^2+3(x_1-y_1)^2=30$

$\Leftrightarrow 6x_1^2+6y_1^2+6z_1^2=30 \ \Leftrightarrow x_1^2+y_1^2+z_1^2=5$

Hence $P(x_1, y_1, z_1)$ satisfies $x^2+y^2+z^2=5$

1.6.8. S.A.Q.: Prove that the distance from P to the plane π is the shortest distance.

1.6.9. S.A.Q.: Find the locus of the point whose distance from the origin is three times it's distance from the plane $2x-y+2z=3$.

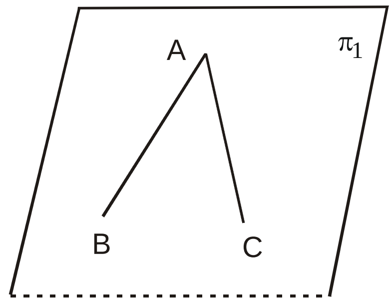
1.6.10. S.A.Q: A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

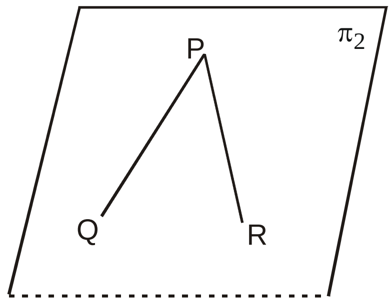
Distance Between Parallel Planes:

We know that the two planes are parallel when the d.rs. of their normals are in proportion and they do not have common points. If π_1 and π_2 are parallel planes, any line L is perpendicular to π_1 iff L is perpendicular to π_2.

1.6.11. Theorem: If two intersecting lines in one plane are respectively parallel to two intersecting lines in another plane then the two planes are parallel.

Proof: Let the two intersecting lines be AB, AC in a plane π_1 which are parallel to PQ, PR respectively in a plane π_2.





$AB \| PQ \Rightarrow AB \| \pi_2$

Suppose the plane π_1 through AB intersects π_2 in a line ℓ.

Now AB and ℓ are in π_1.

Since $AB \| \pi_2$ and $\ell \subset \pi_2, AB$ doesn't intersect ℓ

Thus AB and ℓ lie in the plane π_1 and they don't intersect.

Hence AB and ℓ are parallel.

Similarly AC and ℓ are parallel.

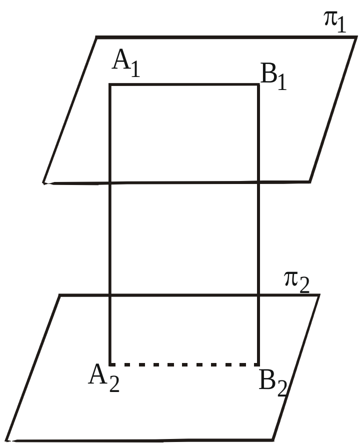
Which is contradiction

Hence π_1 and π_2 don't intersect.

π_1 and π_2 are parallel planes.

1.6.12. **Definition:** If π_1 and π_2 are parallel planes, then the perpendicular distance of any point in π_1 from π_2 (or viceversa) is called the distance between π_1 and π_2.

1.6.13. **Remark:** If π_1, π_2 are parallel planes and $A_1, B_1 \in \pi_1$ and $A_2, B_2 \in \pi_2$, $A_1 A_2$ is perpendicular to π_1 and B_1B_2 is perpendicular to π_2, then $A_1B_1B_2A_2$ is a rectangle so that $A_1A_2 = B_1B_2 = d$ say. Then A_1A_2 is independent of the choice of A_1 and is called the distance from π_1 to π_2. It is also true that d is the distance from π_2 to π_1.



1.6.14. **Theorem:** The distance between the parallel planes $ax + by + cz + d_1 = 0$, $ax + by + cz + d_2 = 0$ is $\dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

Proof: Let the given planes be $\pi_1 \equiv ax + by + cz + d_1 = 0$ and $\pi_2 \equiv ax + by + cz + d_2 = 0$.

Let $P(x_1, y_1, z_1)$ be a point in the plane $ax + by + cz + d_2 = 0$

$\Rightarrow ax_1 + by_1 + cz_1 + d_2 = 0 \Rightarrow -d_2 = ax_1 + by_1 + cz_1$

Distance between the parallel planes = perpendicular distance from P to π_1

$$= \frac{|ax_1 + by_1 + cz_1 + d_1|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

1.6.15. **Example:** Prove that the distance between the parallel planes $2x - 2y + z + 3 = 0 \cdots\cdots(1)$ and $4x - 4y + 2z + 5 = 0 \cdots\cdots\cdots(2)$ is $\frac{1}{6}$.

Solution: (1) and $4x - 4y + 2z + 6 = 0$ represent the parallel planes.

From theorem 1.5.10 the distance between (1) and (2)

$$= \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|6-5|}{\sqrt{4^2 + 4^2 + 2^2}} = \frac{1}{\sqrt{36}} = \frac{1}{6}$$

1.7 Systems of Planes:

The general equation of a plane is $ax + by + cz + d = 0$. Since one of a, b, c is not zero we may assume without loss of generality that $a \neq 0$. Therefore $\frac{d}{a}, \frac{b}{a}, \frac{c}{a}$ uniquely determine the plane. Three ratios are usually determined by three conditions on the plane.

For example:

(i) Three non collinear points in the plane.

(ii) Two distinct points in the plane and a plane perpendicular to the plane.

(iii) A point in the plane and two planes perpendicular to the plane.

With less than three conditions the ratios $\frac{d}{a}, \frac{b}{a}, \frac{c}{a}$ may be determined in several ways yielding systems of planes. Finding the equation of the plane under the condition (i) is discussed in 1.4.10. Conditions (ii) and (iii) yield the ratios uniquely (Ref 2.15 of preliminaries).

We know that there is one and only are plane passing through three non collinear points. For example $x + y + z - 1 = 0$ represents the plane passing through $(1,0,0), (0,1,0)$ and $(0,0,1)$. There are infinitely many planes that contain $(1,0,0)$ and $(0,1,0)$. In fact any equation of the type $x + y + kz - 1 = 0 \cdots\cdots(1)$ represents a plane that contains these points. Also if $ax + by + cz + d = 0$ contains these two points then $a + d = 0$ and $b + d = 0$ so that $a = b = -d$. This gives the general form of the equation of a plane that contains $(1,0,0)$ and $(0,1,0)$ namely $x + y + cz - 1 = 0 \cdots\cdots\cdots\cdots(2)$

We thus come across with equations representing a family of planes rather than a single plane. Such equations contain "arbitrary" coefficients (as k and c in the above (1) and (2)) called parameters.

1.7.1. Definition: The equation of a plane satisfying two conditions will involve one arbitrary constant which can be chosen in an infinite number of ways, thus giving rise to an infinite collection of planes, called a system of planes. Infinite number of ways, thus giving rise to an infinite collection.

The arbitrary constant which is different for different members of the system is called a parameter.

Similarly the equation of a plane satisfying one condition will involve two parameters.

The following are the equations of a few systems of planes involving one or two parameters.

1. The equation $ax + by + cz + k = 0$ represents the system of planes parallel to a given plane $ax + by + cz + d = 0,$ k being the parameter.

2. Given a, b, c not all zero, the equation $ax + by + cz + d = 0$ represents the system of planes perpendicular to given line with direction ratios a, b, c; d being the parameter.

3. The equation $(ax + by + cz + d) + k(a_1x + b_1y + c_1z + d_1) = 0 \cdots\cdots (1)$ represents the system of planes through the line of intersection of the planes

$$ax + by + cz + d = 0 \cdots\cdots\cdots\cdots (2)$$

$$a_1x + b_1y + c_1z + d_1 = 0 \cdots\cdots\cdots\cdots (3)$$

k being the parameter.

for the equation (1), being of the first degree in x, y, z represents a plane.

1.7.2. Theorem: $\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$

$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ represent two intersecting planes.

(i) If $(\lambda_1, \lambda_2) \neq 0$ then $S \equiv \lambda_1\pi_1 + \lambda_2\pi_2 = 0$ represents a plane passing through the line L of intersection of π_1 and π_2.

(ii) Any plane through the line L of intersection of π_1 and π_2 is given by $S \equiv \lambda_1\pi_1 + \lambda_2\pi_2 = 0$ for $(\lambda_1, \lambda_2) \neq 0$.

Proof: (i) Since π_1, π_2 are not parallel, $(a_1, b_1, c_1) \neq 0$, $(a_2, b_2, c_2) \neq 0$ and $(\lambda_1, \lambda_2) \neq 0$.

$S \equiv \lambda_1\pi_1 + \lambda_2\pi_2 = 0$ is a first degree equation in x, y, z.

Hence S represents a plane.

Let P be any point on the line L.

$P \in \pi_1$ and $P \in \pi_2 \Rightarrow P \in S$.

S contains all the common points of π_1 and π_2.

Since π_1, π_2 are intersecting planes, all the common points lie on L

$$L \subset S$$

The plane S contains L.

$$\lambda_2 = 0 \Rightarrow S = \pi_1, \quad \lambda_1 = 0 \Rightarrow S = \pi_2.$$

If $(\lambda_1, \lambda_2) \neq 0$, S represents a plane passing through the line of intersection of π_1 and π_2 and different from π_1 and π_2.

(ii) Let $S \equiv \ell x + my + nz + t = 0$ be a plane containing the line

Let $\overline{n_1} = (a_1, b_1, c_1)$, $\overline{n_2} = (a_2, b_2, c_2)$

$L \perp \overline{n_1}$, $L \perp \overline{n_2} \Rightarrow L \parallel \overline{n_1} \times \overline{n_2}$

$\Rightarrow$ d.rs. of L are $b_1c_2 - b_2c_1$, $c_1a_2 - c_2a_1$, $a_1b_2 - a_2b_1$

Since at least one of them is not zero, without loss of generality we may suppose $a_1b_2 - a_2b_1 \neq 0$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \Rightarrow \exists\, \lambda_1, \lambda_2 \text{ such that } \lambda_1 a_1 + \lambda_2 a_2 = \ell \text{ and } \lambda_1 b_1 + \lambda_2 b_2 = m$$

Then $\ell x + my + nz + t = (\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + nz + t$

$$= \lambda_1 (a_1x + b_1y + c_1z + d_1) + \lambda_2 (a_2x + b_2y + c_2z + d_2) + (n - \lambda_1c_1 - \lambda_2c_2)z + (t - \lambda_1d_1 - \lambda_2d_2)$$

Let $\alpha = n - \lambda_1c_1 - \lambda_2c_2$; $\beta = t - \lambda_1d_1 - \lambda_2d_2$

$$\ell x + my + nz + t \equiv \lambda_1\pi_1 + \lambda_2\pi_2 + \alpha z + \beta \cdots\cdots\cdots\cdots\cdots (1)$$

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be any two points on the line L. Since the planes π_1, π_2 and S contain the line L, the points P, Q on L belong to the three planes π_1, π_2 and S.

from (1) $\quad \ell x_1 + my_1 + nz_1 + t \equiv \lambda_1\pi_1 + \lambda_2\pi_2 + \alpha z_1 + \beta$

$$\ell x_2 + my_2 + nz_2 + t \equiv \lambda_1\pi_1 + \lambda_2\pi_2 + \alpha z_2 + \beta$$

$\Rightarrow \ 0 = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \alpha z_1 + \beta$ and $0 = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 + \alpha z_2 + \beta$

$\Rightarrow \ \alpha z_1 + \beta = 0; \quad \alpha z_2 + \beta = 0 \ \Rightarrow \alpha(z_1 - z_2) = 0$

Since L is not parallel to the XY - plane, $z_1 \neq z_2$

Therefore $\alpha = 0$ and so $\beta = 0$

Then $\ell x + my + nz + t \equiv \lambda_1\pi_1 + \lambda_2\pi_2$

$$\ell x + my + nz + t = 0$$

$$\Leftrightarrow \lambda_1(a_1x + b_1y + c_1z + d_1) + \lambda_2(a_2x + b_2y + c_2z + d_2) = 0$$

Thus the equation of every plane containing the line L is of the form

$$\lambda_1\pi_1 + \lambda_2\pi_2 = 0$$

1.7.3. Theorem: If two intersecting planes π_1, π_2 are represented by

$S_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$, $S_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ respectively where $d_1d_2 > 0$ then the equations of the planes bisecting the angles between two given planes π_1, π_2 are

(i) $$\frac{S_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{S_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) $$\frac{S_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{S_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Proof: If $P(x_1, y_1, z_1)$ is a point on any one of the angle bisector then the distance from P to the planes

$S_1 \equiv a_1x + b_1y + c_1z + d_1 = 0,\ S_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ are equal.

$$\frac{|a_1x_1 + b_1y_1 + c_1z_1 + d_1|}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{|a_2x_1 + b_2y_1 + c_2z_1 + d_2|}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Leftrightarrow \frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x_1 + b_2y_1 + c_2z_1 + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$\Leftrightarrow$ the equation of the locus of P is $\dfrac{S_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \dfrac{S_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Examples:

1.7.4. Find the equation of the plane through the line of intersection of the planes $x - 3y + 2z + 3 = 0$ and $3x - y - 2z - 5 = 0$ and through the origin.

Solution: Let the equation of the plane through the line of intersection of the planes $x - 3y + 2z + 3 = 0$ and $3x - y - 2z - 5 = 0$ be

$(x - 3y + 2z + 3) + \lambda(3x - y - 2z - 5) = 0 \cdots\cdots\cdots (1)$ for any λ.

(1) Passes through origin $(0,0,0)$ Therefore $3 - 5\lambda = 0 \Rightarrow \lambda = \dfrac{3}{5}$

Therefore the equation to the required plane is $(x - 3y + 2z + 3) + \dfrac{3}{5}(3x - y - 2z - 5) = 0$

$$14x - 18y + 4z = 0 \quad 7x - 9y + 2z = 0$$

1.7.5. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y - z = -4$ and parallel to x - axis.

Solution: Let the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y - z = -4$ be

$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0 \cdots\cdots\cdots\cdots (1)$ for any λ.

(1) is parallel to x - axis $\Rightarrow$ normal of (1) is perpendicular to x - axis

Let a, 0, 0 be d.rs. of x- axis.

$$(1+2\lambda)a+(1+3\lambda)0+(1-\lambda)0=0 \Rightarrow (1+2\lambda)a=0 \Rightarrow 1+2\lambda=0 \qquad \because a\neq 0$$

$$\Rightarrow \lambda=-\frac{1}{2}$$

Equation to the required plane is $(x+y+z-1)-\frac{1}{2}(2x+3y-z+4)=0$

$$\Rightarrow -y+3z-6=0 \Rightarrow y-3z+6=0$$

1.7.6. Find the equation of the plane through the intersection of the planes $x+y+z=1,\ 2x+3y+4z=5$ and perpendicular to the plane $x-y+z=0$.

Solution: Let the equation of the plane through the intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ be

$$(x+y+z-1)+\lambda(2x+3y+4z-5)=0 \cdots\cdots\cdots\cdots\cdots(1)$$ for any λ.

Plane (1) is perpendicular to the plane $x-y+z=0 \cdots\cdots\cdots\cdots(2)$

d.rs. of normal to the plane (1) are $1+2\lambda,\ 1+3\lambda,\ 1+4\lambda$

d.rs. of normal to the plane (2) are $1, -1, 1$.

$$1(1+2\lambda)-(1+3\lambda)+(1+4\lambda)=0 \quad 3\lambda+1=0 \Rightarrow \lambda=-\frac{1}{3}$$

Equation of the plane is

$$(x+y+z-1)-\frac{1}{3}(2x+3y+4z-5)=0 \Rightarrow x-z+2=0$$

1.7.7. Find the equations of the planes bisecting the angles between the planes $x+2y+2z=19,\ 4x-3y+12z+3=0$.

Solution: Equations to the bisecting planes between the planes (1) and (2) are

$$\frac{x+2y+2z-19}{\sqrt{1+4+4}}=\pm\frac{4x-3y+12z+3}{\sqrt{16+9+144}}$$

$$\frac{x+2y+2z-19}{3}=\pm\frac{4x-3y+12z+3}{13}$$

$$13(x+2y+2z-19)=\pm 3(4x-3y+12z+3)$$

$$13x + 26y + 26z - 247 = \pm(12x - 9y + 36z + 9)$$

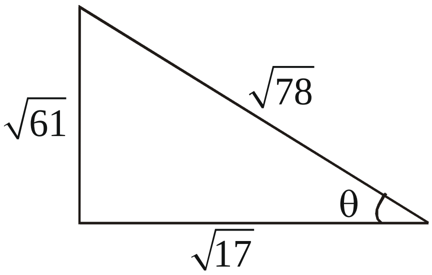
$$x + 35y - 10z - 256 = 0 \cdots\cdots\cdots\cdots\cdots (3)$$

$$25x + 17y + 62z - 238 = 0 \cdots\cdots\cdots\cdots (4)$$

Let θ be the acute angle between (1) and (3)

$$\cos\theta = \frac{1 + 70 - 20}{\sqrt{1+4+4}\ \sqrt{1+1225+100}} = \frac{51}{3\sqrt{1326}} = \frac{17}{\sqrt{1326}} = \frac{17}{\sqrt{17}\ \sqrt{78}} = \frac{\sqrt{17}}{\sqrt{78}}$$

$\tan\theta = \dfrac{\sqrt{61}}{\sqrt{17}} > 1$ Therefore $\theta > \dfrac{\pi}{4}$



Hence, 2θ, the angle between the planes (1) and (2) is greater than $\dfrac{\pi}{2}$.

Therefore equation (3) represents to the plane bisecting the obtuse angle between (1) and (2).

Equation (4) represents the plane bisecting the acute angle between (1) and (2).

1.7.8. A variable plane passes through a fixed point (a,b,c). It meets the axes of reference in A, B and C. Show that the locus of the point of intersection of the planes through (i) A and parallel to YOZ plane, (ii) through B and parallel to ZOX plane, (iii) through C and parallel to XOY plane is $ax^{-1} + by^{-1} + cz^{-1} = 1$.

Solution: Let the variable plane meeting the coordinate axies in A, B, C be

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1 \cdots\cdots\cdots\cdots (1)$$

Therefore $A = (x_1, 0, 0),\ B = (0, y_1, 0),\ C = (0, 0, z_1)$

Also (1) passes through the fixed point (a, b, c)

$$\Rightarrow \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1 \cdots\cdots\cdots\cdots (2)$$

But equation of the plane through A, B, C and parallel to the coordinate planes, YOZ plane, ZOX plane and XOY plane respectively are

$$x = x_1,\ y = y_1,\ z = z_1.$$

Clearly they intersect at $P = (x_1, y_1, z_1)$

from (2) P satisfies $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ i.e. $ax^{-1} + by^{-1} + cz^{-1} = 1$

conversely (2) holds for $P = (x_1, y_1, z_1)$

Then P is the intersection of the planes.

$$x = x_1,\ y = y_1,\ z = z_1.$$

1.7.9. S.A.Q.: Find the equation of the plane through the intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 3z + 1 = 0$ and perpendicular to the plane $x + y + z + 9 = 0$.

1.7.10 S.A.Q.: Find the equation of the plane bisecting the acute angle between the planes $3x - 2y - 6z + 2 = 0$, $-2x + y - 2z - 2 = 0$.

Pair of planes:

Consider the equations

$$a_1x + b_1y + c_1z + d_1 = 0 \cdots\cdots\cdots\cdots (1),\ a_2x + b_2y + c_2z + d_2 = 0 \cdots\cdots\cdots (2).$$

These equations represent two planes. If a point P lies either in plane (1) or in plane (2), then P satisfies the equation.

$$(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0 \cdots\cdots\cdots (3)$$

Conversely if P is a point satisfying the equation (3) then P satisfies either (1) or (2) and hence P lies in (1) or (2). Thus equation (3) represents the pair of the planes (1) and (2). On multiplication, (3) takes the form.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

It is a second degree equation in x, y and z.

We state the following theorem without proof.

1.7.11. Theorem: The necessary and sufficient condition for

$H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0$ to represent a pair of planes passing through the origin is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0,\ f^2 \geq bc,\ g^2 \geq ac,\ h^2 \geq ab$ and $d = 0$.

We give the following theorem without proof.

1.7.12. Theorem: If θ is an angle between the pair of planes.

$H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ then

$$\cos\theta = \frac{a+b+c}{\sqrt{(a+b+c)^2 + 4\left(f^2 + g^2 + h^2 - ab - bc - ca\right)}}$$

1.7.13. Corollary:

(1) The two planes are perpendicular $\Leftrightarrow \theta = \frac{\pi}{2}$

$\Leftrightarrow \cos\theta = 0, \quad \Leftrightarrow a + b + c = 0$

$\Leftrightarrow$ coeff $\cdot$ of x^2 + coeff $\cdot$ of y^2 + coeff $\cdot$ of $z^2 = 0$

2. The planes are identical (Coincident) $\Leftrightarrow \theta = 0^0$

$\Leftrightarrow f^2 = bc,\ g^2 = ac,\ h^2 = ab$.

3. If θ is an angle between the pair of planes

$H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$ then

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a+b+c}, \text{ if } a+b+c \neq 0.$$

4. Direction ratios of the line of intersection of the planes represented by $H \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ are $\sqrt{f^2 - bc}$, $\sqrt{g^2 - ac}$, $\sqrt{h^2 - ab}$.

Proof: Let $H \equiv 0$ represent the planes $\ell_1 x + m_1 y + n_1 z = 0,\ \ell_2 x + m_2 y + n_2 z = 0$.

$$(\ell_1 x + m_1 y + n_1 z)\ (\ell_2 x + m_2 y + n_2 z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$\Rightarrow \ell_1\ell_2 = a,\ m_1m_2 = b,\ n_1n_2 = c,\ \ell_1m_2 + \ell_2m_1 = 2h,\ \ell_1n_2 + \ell_2n_1 = 2g,\ m_1n_2 + m_2n_1 = 2f$$

Let (ℓ, m, n) be the d.cs. of the line of intersection of $H \equiv 0$.

Then $\ell\ell_1 + mm_1 + nn_1 = 0,\ \ell\ell_2 + mm_2 + nn_2 = 0$

$$\frac{\ell}{m_1n_2 - m_2n_1} = \frac{m}{n_1\ell_2 - n_2\ell_1} = \frac{n}{\ell_1m_2 - \ell_2m_1}$$

$$\Rightarrow \frac{\ell}{\sqrt{4f^2 - 4bc}} = \frac{m}{\sqrt{4g^2 - 4ac}} = \frac{n}{\sqrt{4h^2 - 4ab}}$$

$$\Rightarrow \frac{\ell}{\sqrt{f^2 - bc}} = \frac{m}{\sqrt{g^2 - ac}} = \frac{n}{\sqrt{h^2 - ab}}$$

d.rs. of the line of intersection of the planes are $\sqrt{f^2 - bc},\ \sqrt{g^2 - ac},\ \sqrt{h^2 - ab}$

Examples:

1.7.14. Prove that the equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents a pair of planes, and find the acute angle between them.

Solution: The given equation is $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0 \cdots\cdots\cdots(1)$

Comparing the equation with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \cdots\cdots\cdots(2)$

$$a = 2,\ b = -6,\ c = -12,\ f = 9,\ g = 1,\ h = \frac{1}{2}$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & \frac{1}{2} & 1 \\ \frac{1}{2} & -6 & 9 \\ 1 & 9 & -12 \end{vmatrix} = 2(72-81) - \frac{1}{2}(-6-9) + 1\left(\frac{9}{2} + 6\right) = -18 + \frac{15}{2} + \frac{21}{2} = 0$$

$f^2 = 81, bc = 72 \Rightarrow f^2 > bc$

$g^2 = 1, \; ac = -24 \Rightarrow g^2 > ac$

$h^2 = \frac{1}{4}, \; ab = -12 \Rightarrow h^2 > ab$

Given equation represents a pair of planes through the origin.

Let θ be the acute angle between the planes.

$$\cos\theta = \left| \frac{a+b+c}{\sqrt{(a+b+c)^2 + 4\left(f^2+g^2+h^2-ab-bc-ca\right)}} \right|$$

$$= \left| \frac{2-6-12}{\sqrt{\left\{(-16)^2 + 4\left(81+1+\frac{1}{4}+12-72+24\right)\right\}}} \right| = \frac{16}{21}$$

$$\theta = \cos^{-1}\left(\frac{16}{21}\right)$$

1.7.15. Show that the equation $x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6=0$ represents a pair of parallel planes and find the distance between them.

Solution: $x^2+4y^2+9z^2-12yz-6zx+4xy = (x+2y-3z)^2$

Therefore if $x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6$

$$\equiv (x+2y-3z+k)\,(x+2y-3z+\ell)$$

$k+\ell = 5, \; k\ell = 6 \Rightarrow k + \frac{6}{k} = 5 \Rightarrow k^2 - 5k + 6 = 0 \Rightarrow k = 3, \; \ell = 2$

Given equation represents the pair of planes $x+2y-3z+3=0, \; x+2y-3z+2=0$ which are parallel.

The distance between the parallel planes $= \frac{|3-2|}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}$

1.7.16. If $a^2+b^2+c^2-2bc-2ca-2ab>0$ then show that the equation $\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$ represents a pair of planes.

Solution: Given equation is $\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$

$$\Rightarrow a(z-x)(x-y)+b(y-z)(x-y)+c(y-z)(z-x)=0$$

$$\Rightarrow a\left(xz-x^2-yz+xy\right)+b\left(xy-zx-y^2+yz\right)+c\left(yz-z^2-xy+zx\right)=0$$

$$\Rightarrow ax^2+by^2+cz^2-(b+c-a)yz-(c+a-b)zx-(a+b-c)xy=0$$

From (1) the conditions for the equation

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ to represent a pair of parallel planes are (1), (2), (3) and (4).

Here $A=a,\ B=b,\ C=c,\ F=-\frac{1}{2}(b+c-a),\ G=-\frac{1}{2}(c+a-b),\ H=-\frac{1}{2}(a+b-c)$

(1) Clearly $\begin{vmatrix} a & -\frac{1}{2}(a+b-c) & -\frac{1}{2}(c+a-b) \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix}$

$$\text{by } R_1\to R_1+R_2+R_3=\begin{vmatrix} 0 & 0 & 0 \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix}=0$$

(2) Also $F^2-BC=\frac{1}{4}(b+c-a)^2-bc=\frac{1}{4}\left(b^2+c^2+a^2+2bc-2ca-2ab\right)-bc$

$$=\frac{1}{4}\left(a^2+b^2+c^2-2ab-2bc-2ca\right)>0$$

$$\Rightarrow F^2 > BC$$

Similarly (3) $G^2 > AC$; (4) $H^2 > AB$.

Hence given equation represents a pair of planes.

d.rs. of the line of intersection are $\pm\sqrt{F^2 - BC},\ \pm\sqrt{G^2 - AC},\ \pm\sqrt{H^2 - AB}$

$$\equiv \pm 1, \pm 1, \pm 1$$

1.8 Projections on a plane and volume of Tetrahedron:

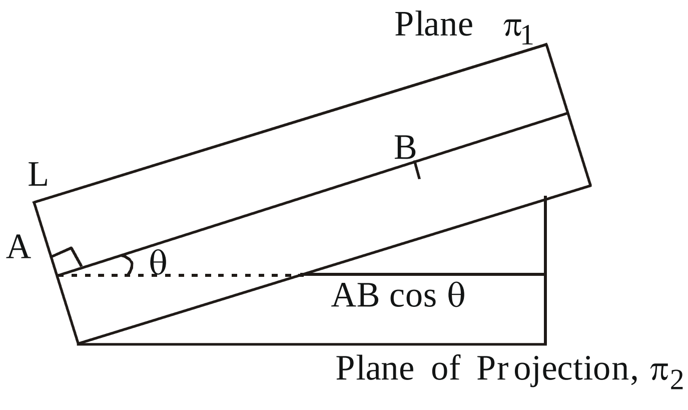
Orthogonal Projection on a Plane: Corresponding to the notion of projection on a line, we also have that of projection on a plane.

1.8.1. Definition: Orthogonal projection on a plane.

1. The foot of the perpendicular from a point to a given plane is called the orthogonal projection of the point on the plane.
2. The projection of a curve on a plane is the locus of the projections on the plane of points on the curve.
3. The projection on a given plane of the region enclosed by a curve in a plane is the region enclosed by the projection of the curve on the plane.

The following simple results in solid Geometry are assumed without proof:-

1. If a line segment AB in a plane π_1 perpendicular to the line L of intersection of this plane π_1 with the plane of projection π_2, then the length of its projection on π_2 is $AB\,|\cos\theta|$; θ being an angle between the two planes.



2. The area of the projection of a region with area A, enclosed by a curve in a plane is $A\left|\cos\theta\right|$; θ being an angle between the plane of the curve containing the given region and the plane of projection.

1.8.2. Theorem: If A_x, A_y, A_z are the areas of the projections on the coordinate planes $z=0, y=0$ and $x=0$ of a region in a plane π with area A respectively then $A^2 = A_x^2 + A_y^2 + A_z^2$.

Proof: Let ℓ, m, n be the direction cosienes of the plane π. Then by (2)

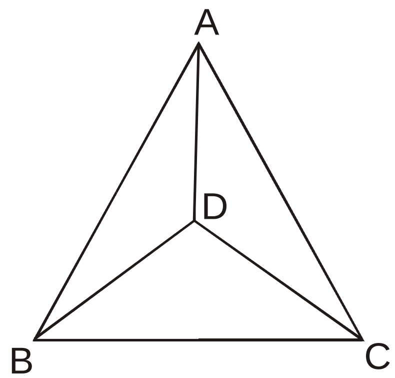
$$A_x = A\left|\ell\right|$$

$A_y = A\left|m\right|$ and $A_z = A\left|n\right|$

Hence $A_x^2 + A_y^2 + A_z^2 = A^2\left(\ell^2 + m^2 + n^2\right) = A^2$

1.8.3. Volume of Tetrahedron:

We are familiar with the formula for the volume V of a tetrahedron as $V = \frac{1}{6}\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$ where $\bar{a}, \bar{b}, \bar{c}$ are co-terminus vectors. We present a description of a tetrahedron in terms of planes.



Given three non collinear points A, B, C, there is a unique plane π containing A, B, C. If D is any point not in π then a plane π_1 is uniquely determined that contains D and the line AB. Similarly let π_2 be the plane containing D and BC and π_3 be the plane containing D and AC. Then the triangles DAB, DBC, DCA and ABC form a non coplanar geometric figure in the 3D space called the TETRAHEDRON. D, A, B, C are called the vertices of the tetrahedron.

By definition, the volume V of the tetrahedron ABCD is $V = \frac{1}{3}$ height X area of the base triangle.

So $V = \frac{1}{3}\left|\left[\overline{AB}\,,\ \overline{AC},\ \overline{AD}\right]\right|$

If $A = (x_1, y_1, z_1),\ B = (x_2, y_2, z_2),\ C = (x_3, y_3, z_3)$ and $D = (x_4, y_4, z_4)$

$\overline{AB} = \overline{OB} - \overline{OA} = (x_2 - x_1\ ,\ y_2 - y_1,\ z_2 - z_1)$ and similarly

$\overline{AC} = \overline{OC} - \overline{OA} = (x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1)$ and

$\overline{AD} = \overline{OD} - \overline{OA} = (x_4 - x_1,\ y_4 - y_1,\ z_4 - z_1)$ so that

$V = \frac{1}{6}$ absolute value of $\left[\overline{AB},\ \overline{AC},\ \overline{AD}\right]$

$$= \frac{1}{6} \text{ absolute value of } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix}$$

$$= \frac{1}{6} \text{ absolute value of } \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

1.8.4. Find the areas of the projection of the region bounded by the triangle whose vertices are the points $P\,(1,2,3),\ Q\,(-2,1,-4),\ R\,(3,4,-2)$ on the co-ordinate planes.

Solution: Let A be the area of the triangle P Q R.

Let A_x, A_y, A_z be the areas of the projection of the area of $\Delta\ PQR$ on $YZ,\ ZX,\ XY$ planes respectively.

$\overline{PQ} = (3,1,7) \quad \overline{PR} = (-2,-2,5)$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 1 & 7 \\ -2 & -2 & 5 \end{vmatrix} = 19\bar{i} - 29\bar{j} - 4\bar{k}$$

Area of $\Delta\, PQR$, $A = \frac{1}{2}\left|\overline{PQ} \times \overline{PR}\right| = \frac{1}{2}\sqrt{19^2 + 29^2 + 4^2} = \frac{1}{2}\sqrt{1218}$

d.rs. of the normal to the plane containing the triangle with area A are 19, -29, -4.

d.cs. ℓ, m, n are $\frac{19}{\sqrt{1218}}, \frac{-29}{\sqrt{1218}}, \frac{-4}{\sqrt{1218}}$

$$A_x = \ell A = \left|\frac{19}{\sqrt{1218}} \times \frac{\sqrt{1218}}{2}\right| = \frac{19}{2}$$

$$A_y = mA = \left|\frac{-29}{\sqrt{1218}} \times \frac{\sqrt{1218}}{2}\right| = \left|\frac{-29}{2}\right|$$

$$A_z = nA = \left|\frac{-4}{\sqrt{1218}} \times \frac{\sqrt{1218}}{2}\right| = |-2|$$

1.9 Answers to S.A.Q.:

1.4.19. S.A.Q.: Let $A = (6, -4, 4)$ $\quad B = (0, 0, -4)$

$C = (-1, -2, -3)$ $\quad D = (1, 2, -5)$

$\overline{AB} = (-6, +4, -8)$ and $\overline{CD} = (2, 4, -2)$

Since $\overline{AB} \cdot \overline{CD} = -12 + 16 + 16 = 20 \neq 0$

and $\overline{AB} \neq \lambda\, \overline{CD}$ for any λ

$\overline{AB}$ is neither perpendicular nor parallel to $\overline{CD}$.

If we prove that A, B, C, D are coplanar, then that AB intersects CD.

Equation to the plane $\overleftrightarrow{ABC}$ is $\begin{vmatrix} x-6 & y+4 & z-4 \\ -6 & 4 & -8 \\ -7 & 2 & -7 \end{vmatrix} = 0$

$$\Rightarrow -12(x-6) + 14(y+4) + 16(z-4) = 0$$

$$\Rightarrow 6(x-6) - 7(y+4) - 8(z-4) = 0$$

$\Rightarrow 6x - 7y - 8z - 32 = 0 \cdots\cdots\cdots(1)$

Substituting $D(1,2,-5)$ in L.H.S. of (1) we get

$$6 - 14 + 40 - 32 = 0$$

$$\therefore\ D \in \overleftrightarrow{ABC}$$

AB and CD intersect.

1.4.20. S.A.Q.: Let π be the required plane and a, b, c be d.rs. of normal to the plane.

Let $A = (0,4,3)$, $B = (-1,-5,-3)$, $C = (-2,-2,1)$

d.rs. of AB are 1, 9, 6 and d.rs. of AC are 2, 6, 2

Since π contains A, B, C the lines AB, AC lie in the plane π

$$a + 9b + 6c = 0 \cdots\cdots\cdots(1)$$

$$2a + 6b + 2c = 0 \cdots\cdots(2)$$

Solving (1) and (2)

$$\left.\begin{array}{ccccccc} & a & & b & & c & \\ 9 & \times & 6 & \times & 1 & \times & 9 \\ 6 & & 2 & & 2 & & 6 \end{array}\right\}$$

$$\frac{a}{18-36} = \frac{b}{12-2} = \frac{c}{6-18}$$

i.e. $\dfrac{a}{9} = \dfrac{b}{-5} = \dfrac{c}{6}$

So the equation to the plane π is $9(x-0) - 5(y-4) + 6(z-3) = 0$

i.e. $9x - 5y + 6z + 2 = 0$

Clearly $(1,1,-1)$ lies in this plane.

Points are coplanar.

1.5.12. S.A.Q.: Let the points on the plane be $A = (1,-2,4)$, $B = (3,-4,5)$

Let d.rs. of normal to the plane be a, b, c d.rs. of AB are 2, -2, 1

$2a - 2b + c = 0 \cdots\cdots (1)$

Since the d.cs. of the x - axis are 1, 0, 0 and the plane is parallel to the x-axis $\Leftrightarrow$ its normal is perpendicular to x-axis,

$a(1) + b(0) + c(0) = 0$ i.e. a = 0 $\Rightarrow -2b + c = 0$ $\Rightarrow 2b = c \Rightarrow \frac{b}{1} = \frac{c}{2}$

$\Rightarrow$ $0,1,2$ are d.rs. of normal to the required plane

$\Rightarrow$ Equation of the required plane is $0(x-1) + 1(y+2) + 2(z-4) = 0$

i.e. $y + 2 + 2z - 8 = 0$ i.e. $y + 2z - 6 = 0$

1.5.13. S.A.Q.: Let the given points be $A = (2,0,6)$, $B = (-6,2,4)$

Let P be the mid point of AB

$$P = \left(\frac{2-6}{2}, \frac{0+2}{2}, \frac{6+4}{2}\right) = (-2,1,5)$$

d.rs. of AB are 8, -2, 2.

So the equation of the plane through the point P and perpendicular to the line joining A and B is

$$8(x+2) - 2(y-1) + 2(z-5) = 0$$

i.e. $8x - 2y + 2z + 8 = 0$

i.e. $4x - y + z + 4 = 0$

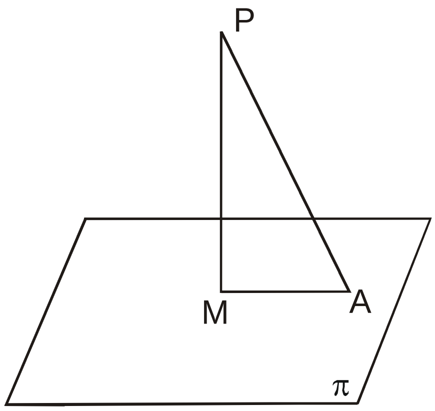
1.5.14. S.A.Q.: If θ is an angle between the given planes

$2x - 3y + 4z + 11 = 0 \cdots\cdots (1)$, $3x - 2y - 3z + 27 = 0 \cdots\cdots\cdots\cdots (2)$

Then $\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{(2)(3) + (-3)(-2) + 4(-3)}{\sqrt{4+9+16}\ \sqrt{9+4+9}}$

$$= \frac{6+6-12}{\sqrt{29}\ \sqrt{22}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

1.6.8. S.A.Q.: Let M be the foot of the perpendicular and A be any point other than M in π. PM is the perpendicular distance to π.



In Δ PMA, AP is hypotenuese.

$\Rightarrow$ PM < PA

$\Rightarrow$ PM < PA $\forall$ A $\in \pi$

Hence the distance from P to π is the shortest distance.

1.6.9. S.A.Q.: Let O be the origin and P be the point (x_1, y_1, z_1) such that OP is equal to 3 times its distance from the plane

$$2x - y + 2z = 3$$

$$\Rightarrow OP^2 = 9 \frac{(2x_1 - y_1 + 2z_1 - 3)^2}{4+1+4}$$

$$\Leftrightarrow x_1^2 + y_1^2 + z_1^2 = 4x_1^2 + 4y_1^2 + 4z_1^2 + 9 - 4x_1y_1 - 4y_1z_1 + 8x_1z_1 - 12x_1 + 6y_1 - 12$$

$$\Leftrightarrow 3x_1^2 + 3z_1^2 - 4x_1y_1 - 4y_1z_1 + 8x_1z_1 - 12x_1 + 6y_1 - 12z_1 + 9 = 0$$

$\Leftrightarrow$ P satisfies

$$3x^2 + 3z^2 - 4xy - 4yz + 8xy - 12x + 6y - 12z + 9 = 0$$

1.6.10. S.A.Q.: Let the variable plane that meets the axes in A, B, C be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \cdots\cdots\cdots (1)$$

where $A=(a,0,0)$, $B=(0,b,0)$, $C=(0,0,c)$ and a, b, c not zero.

Let $P(x_1,y_1,z_1)$ be the centroid of tetrahedron OABC.

$$\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)=(x_1,y_1,z_1) \Rightarrow a=4x_1,\ b=4y_1,\ c=4z_1$$

substituting in (1) $\dfrac{x}{4x_1}+\dfrac{y}{4y_1}+\dfrac{z}{4z_1}=1\cdots\cdots(2)$

$\Rightarrow$ the distance of the origin O from the plane (2) is

$$p=\frac{1}{\sqrt{\frac{1}{16x_1^2}+\frac{1}{16y_1^2}+\frac{1}{16z_1^2}}} \qquad \Rightarrow \frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}=\frac{16}{p^2}$$

Conversely suppose the tetrahedron be OABC where A, B, C are on X, Y, Z axes respectively.

Then $A=(4x_1,0,0)$, $B=(0,4y_1,0)$, $C=(0,0,4z_1)$

Equation of the plane containing $\Delta\ ABC$ is

$$\frac{x}{4x_1}+\frac{y}{4y_1}+\frac{z}{4y_1}=1 \Rightarrow \frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=4\cdots\cdots\cdots\cdots\cdots(3)$$

Distance of origin from the plane (3) is

$$\frac{|-4|}{\sqrt{\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}}}=\frac{4}{\sqrt{x_1^{-2}+y_1^{-2}+z_1^{-2}}}=\frac{4}{\sqrt{p^{-2}}}=4p$$

Hence the locus of P is given by the equation

$$x^{-2}+y^{-2}+z^{-2}=16p^{-2}$$

1.7.9. S.A.Q.: Let the plane through the intersection of the planes

$x+2y+3z+4=0,\quad 4x+3y+3z+1=0$ be $(x+2y+3z+4)+\lambda(4x+3y+3z+1)=0$

$$\Rightarrow (1+4\lambda)x+(2+3\lambda)y+(3+3\lambda)z+(4+\lambda)=0\cdots\cdots\cdots(1)$$

If (1) is perpendicular to $x+y+z+9=0$, then

$$(1+4\lambda)\cdot 1+(2+3\lambda)\cdot 1+(3+3\lambda)\cdot 1=0$$

i.e. $10\lambda=-6 \Rightarrow \lambda=\frac{-3}{5}$

Required plane is $7x+7y+6z-17=0$

1.7.10. S.A.Q.: Given planes are $3x-2y+6z+2=0\cdots\cdots\cdots\cdots(1)$ and

$$-2x-y+2z+2=0\cdots\cdots\cdots(2)$$

Equations to the bisecting planes between the given planes are

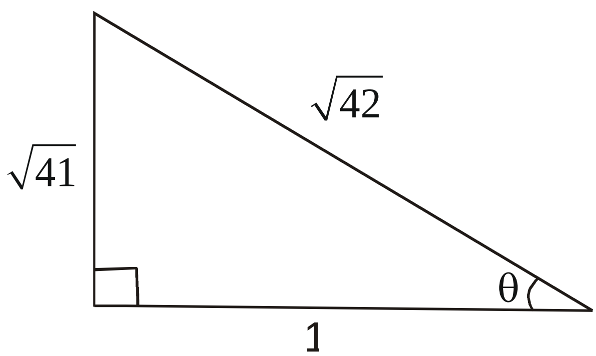
$$\frac{3x-2y+6z+2}{\sqrt{9+4+36}}=\pm\frac{(-2x-y+2z+2)}{\sqrt{4+1+4}}$$

$$\frac{3x-2y+6z+2}{7}=\pm\frac{(-2x-y+2z+2)}{3}$$

$$3(3x-2y+6z+2)=\pm 7(-2x-y+2z+2)$$

$$5x-y-4z+8=0\cdots\cdots\cdots(3)$$

$$23x-13y+32z+20=0\cdots\cdots\cdots\cdots\cdots\cdots(4)$$



Let θ be the acute angle between (2) and (3)

$$\cos\theta=\left|\frac{10+1-8}{\sqrt{9}\ \sqrt{42}}\right|=\frac{1}{\sqrt{42}}$$

$$\tan\theta = \sqrt{41} > 1 \Rightarrow \theta > \frac{\pi}{4}$$

Hence 2θ, the angle between the planes (1) and (2) is greater than $\frac{\pi}{2}$ i.e. obtuse. Equation (3) is the plane bisecting the obtuse angle between (1) and (2). Equation (4) is the plane bisecting the acute angle between (1) and (2).

1.10 Summary:

After completing this lesson the student shall be able to find equations of the plane in various forms, angles between two planes, distance of a point from the plane, equations for systems of planes and volume of a tetrahedron using both vector methods and cartesian methods.

1.11 Technical Terms:

1. Plane
2. Intercept
3. Parametric Equation
4. Projection of a Vector
5. Orthogonal Projection
6. Perpendicular distance
7. Parallel Planes
8. Tetrahedron

1.12 Model Examination Questions:

1. Show that the equation of the plane through the points $(2,2,-1)$, $(3,4,2)$, $(7,0,6)$ is $5x+2y-3z-17=0$.

2. Find the equation of the plane through the points $(2,2,1)$, $(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$.

3. Show that the points $(-6,3,2)$, $(-13,17,-1)$, $(3,-2,4)$ and $(5,7,3)$ are coplouar.

4. Find the angle between the planes $2x-y+z=0$, $x+y+2z=7$.

5. Two systems ofrectangular axes have the same origin. If a plane intersects them at distances a, b, c and a_1, b_1, c_1 respectively from the origin, prove that $a^{-2}+b^{-2}+c^{-2}=a_1^{-2}+b_1^{-2}+c_1^{-2}$.

6. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y-z=-4$ and is parallel to x - axis.

7. Find the equations of the planes bisecting the angles between planes $x+2y+2z=19,\ 4x-3y+12z+3=0$.

8. Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of planes and find the angle between them.

9. Find the areas of the projection of the area of the triangle whose vertices are the points $(1,2,3),\ (-2,1,-4),\ (3,4,-2)$ on the co-ordinate planes.

1.13 Exercises:

1. Find the equation of the plane in which the foot of the perpendicular from origin is $(-3,5,-8)$.

Ans: $3x-5y+8z+98=0$.

2. Find the perpendicular distance from $(2,3,4)$ to the plane $3x-6y+2z+11=0$.

Ans: $\frac{16}{7}$

3. Find the angles between the planes $3x-4y+5z=0$, $2x-y-2z=5$.

Ans: $\frac{\pi}{2}$

4. Find the equation of the plane passing through the point $(2,-1,3)$ and parallel to the plane $3x+4y-7z+5=0$.

Ans: $3x+4y-7z+19=0$.

5. Find the distance between the parallel planes $3x-y+2z+4=0,\ 6x-2y+4z+5=0$.

Ans: $3/\sqrt{56}$ unit.

6. Find the equation ofthe plane passing through the points $(a,0,0),\ (0,b,0),\ (0,0,c)$.

Ans: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

7. Show that the four points $(0,4,3), (-1,-5,-3), (-2,-2,1),\ (1,1,-1)$ are coplanar.

8. Find the equation of the plane through the points $(1,-2,4),(3,-4,5)$ and perpendicular to XY - plane.

Ans: $x+y+1=0$

9. Find the equation of the plane through the point $(4,4,0)$ and perpendicular to each of the planes $x+2y+2z-5=0$ and $3x+3y+2z-8=0$.

Ans: $2x-4y+3z+8=0$

10. Find the locus of the point, the sum of the squares of whose distances from the planes $x+y+z=0,\ x-z=0,\ x-2y+z=0$ is 9.

Ans: $x^2+y^2+z^2=9$

11. Find the equation of the plane through the line of intersection of the planes $2x-7y+4z-3=0$, $3x-5y+4z+11=0$ and the point $(-2,1,3)$.

Ans: $15x-47y+28z-7=0$

12. Find the equation of the plane through the line of intersection of the planes $x+2y+3z+4=0,\quad 4x+3y+3z+1=0$ and perpendicular to the plane $x+y+z+9=0$.

Ans: $7x-y-6z-17=0$

13. Find the equations to the planes through the line of intersection of $2x+y+3z-2=0,\ x-y+z+4=0$ such that each plane is at a distance of 2 unit from the origin.

Ans: $15x-12y+16z+50=0,\ \ x+2y+2z-6=0$

14. Show that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents pair of planes find the angle between them.

Ans: $\cos^{-1}\left(\frac{16}{21}\right)$

15. Show that $x^2+4y^2+9z^2-12yz-6zx+4xy+5x+10y-15z+6=0$ represents a pair of parallel planes and find the distance between them.

1.14 Model Practical Problem with solution:

Problem: Find the equations of the bisecting planes of the angles between the planes $S_1 \equiv 3x-2y-6z+2=0$ and $S_2 \equiv -2x+y-2z-2=0$. Find the obtuse angular bisector.

Definitions:

1. An angle between two planes is an angle between their corresponding normals.

2. An angular bisector of two planes is the plane through the line of intersection of the planes that makes equal angles with the planes.

Results Used:

1. If θ is an angle between the planes π_1, π_2 represented by $\pi_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ then $\text{Cos } \theta = \pm \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$.

2. If two intersecting planes π_1, π_2 are represented by $S_1 \equiv a_1x + b_1y + c_1z + d_1 = 0$, $S_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$ respectively where $d_1\, d_2 > 0$ then the equations of the planes bisecting the angles between π_1, π_2 are

(i) $$\frac{S_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{S_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(ii) $$\frac{S_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{S_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Stepwise division of the problem:

Step 1: To find the equations of the bisecting planes between the given planes S_1, S_2.

Step 2: To find $\text{Cos } \theta$ and $\tan \theta$ where θ is an angle between bisecting plane and one of S_1 and S_2.

Step 3: To determine whether θ is acute or obtuse.

Step 4: To distinguish the acute angular bisector and obtuse angular bisector.

Stepwise Solution:

Step 1: Equations of the given planes are

$$S_1 \equiv 3x - 2y - 6z + 2 = 0 \cdots\cdots (1)$$

$$S_2 \equiv -2x + y - 2z - 2 = 0 \cdots\cdots\cdots (2)$$

Equations of the planes bisecting the angles between the given planes are

$$\frac{S_1}{\sqrt{9+4+36}} = \frac{S_2}{\sqrt{4+1+4}} \quad \text{and} \quad \frac{S_1}{\sqrt{49}} = -\frac{S_2}{\sqrt{9}}$$

$\Rightarrow (3x-2y-6z+2)\,3 = (2x-y+2z+2)7$

and $(3x-2y-6z+2)3 = -(2x-y+2z+2)7$

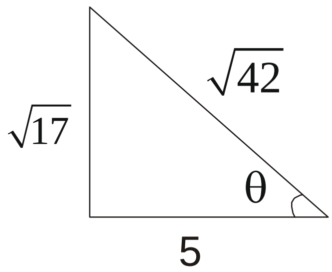
$\Rightarrow\; 5x-y+32z+8=0 \cdots\cdots\cdots\cdots (3)$

$23x-13y-4z+20=0 \cdots\cdots\cdots\cdots (4)$

Step 2: Let $2\,\theta$ be an angle between given planes containing angular bisector (3) and θ be an angle between the planes (2) and (3).

$$\text{Cos }\theta = \frac{10+1+64}{\sqrt{4+1+4}\;\sqrt{25+1+1024}}$$

$$= \frac{75}{3\sqrt{1050}} = \frac{5}{\sqrt{42}}$$



From figure $\tan\theta = \dfrac{\sqrt{17}}{5} < 1$

Step 3: Since $\tan\theta = \dfrac{\sqrt{17}}{5} < 1 \;\Rightarrow\; \tan\theta < \tan\dfrac{\pi}{4} \;\Rightarrow \theta < \dfrac{\pi}{4} \;\Rightarrow\; 2\theta < \dfrac{\pi}{2}$

Hence the angle between the given planes containing angular bisector (3) is an acute angle.

Step 4: Equation of the plane bisecting acute angle between the planes (1) and (2) is

$$5x-y+32z+8=0$$

Equation fo the plane bisecting obtuse angle between the planes (1) and (2) is

$$23x-13y-4z+20=0$$

Lesson Writer

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Lesson - 2

LINES - I

2.1 Objective of the lesson:

In this lesson the student is introduced to various aspects of the line such as, the equation of a line, angles between lines and planes, projection of a point or line in a plane or line and distance of a point from a line.

2.2 Structure:

This lesson contains the following components:

2.3 Introduction

2.4 Equations of a line

2.5 Angles between lines and planes

2.6 Foot of the perpendicular of a point

2.7 Answers to SAQ

2.8 Summary

2.9 Technical Terms

2.10 Model Examination Questions

2.11 Exercises

2.12 Model Practical Problem with solution

2.3 Introduction:

A line is uniquely determined by a point on it and a line parallel to it. Equivalently a point and the angles with the coordinate axes determine a line uniquely. A line is also the intersection of two planes. These facts enable us to derive the equations of a line in various forms. We also obtain expression for the angle between lines and planes using vector methods. Conditions for perpendicularity and parallelism are also obtained in this lesson. Finally coordinates of the foot of the perpendicular of a point and the perpendicular distance are obtained.

2.4 Equations of a line:

Vector Form:

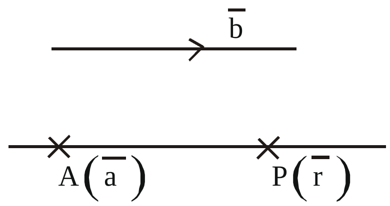
2.4.1. Theorem: The vector equation of a line parallel to a vector $\bar{b} \neq \bar{0}$ and passing through the point $A(\bar{a})$ is

$$\bar{r} = \bar{a} + t\bar{b} \qquad (t \in \mathbb{R})$$

Proof: Let L be the line passing through A and parallel to $\bar{b}$.

For any point $P(\bar{r})$, $\bar{r} = \overline{OP} = \overline{OA} + \overline{AP} = \bar{a} + \overline{AP}$

So $\overline{AP} = \bar{r} - \bar{a}$



Now P lies on L $\Leftrightarrow \overline{AP} \parallel \bar{b}$

$\Leftrightarrow \bar{r} - \bar{a} \parallel \bar{b}$

$\Leftrightarrow \bar{r} - \bar{a} = t\bar{b}$ for some $t \in \mathbb{R}$

$\Leftrightarrow \bar{r} = \bar{a} + t\bar{b}$ for some $t \in \mathbb{R}$

Thus P lies on L iff there exists a scalar t such that $\bar{r} = \bar{a} + t\bar{b}$

So the vector equation $\bar{r} = \bar{a} + t\bar{b}$ with scalar parameter t represents the line L.

2.4.2. Corollary: The parametric form of the vector equation of a line passing through the distinct points $A(\bar{a})$ and $B(\bar{b})$ is $\bar{r} = t\bar{b} + (1-t)\bar{a}$, t being a scalar.

Proof: Let L be the line AB

$$\overline{AB} = \overline{OB} - \overline{OA} = \bar{b} - \bar{a} \neq \bar{0}$$

So L is parallel to $\bar{b} - \bar{a}$.

Since L passes through $\bar{a}$ and is parallel to $\bar{b} - \bar{a}$ the vector equations of L (by 2.3.1) is

$$\bar{r} = \bar{a} + t(\bar{b} - \bar{a})$$

$$\bar{r} = t\bar{b} + (1-t)\bar{a}$$

2.4.3. **Corollary:** Equations of a line passing through $A(x_1, y_1, z_1)$ and having direction ratios ℓ, m, n are

$$x = x_1 + \ell t, \quad y = y_1 + mt, \quad z = z_1 + nt \qquad (t \in \mathbb{R})$$

Proof: Let L be the line through A, $\bar{a} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$ and $\bar{b} = \ell\bar{i} + m\bar{j} + n\bar{k}$.

Then L is the line through $\bar{a}$ and parallel to $\bar{b}$.

So the vector equation of L is $\bar{r} = \bar{a} + t\bar{b} \quad (t \in \mathbb{R})$

(by 2.4.1)......................... (1)

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, (1) can be written as

$$x = x_1 + \ell t, \; y = y_1 + mt, \; z = z_1 + nt \qquad (t \in \mathbb{R})$$

These are the parameteric equations of the line L.

2.4.4. Symmetric form of the equations of a line L passing through $A(x_1, y_1, z_1)$ and having direction cosines ℓ, m, n is

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

By 2.3.3 the cartesian equations of L are given by

$$x = x_1 + \ell t, \; y = y_1 + mt, \; z = z_1 + nt \qquad (t \in \mathbb{R}) \cdots\cdots\cdots\cdots (1)$$

Since ℓ, m, n are the direction cosines of L, $\ell^2 + m^2 + n^2 = 1$ and hence one of ℓ, m, n is non zero.

(1) can be expressed as

$$x - x_1 : \ell = y - y_1 : m = z - z_1 : n$$

equivalently $\dfrac{x - x_1}{\ell} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$

These are called the symmetric form of the equations of the line L.

2.4.5. **Note:** (i) If one of ℓ, m, n say $\ell = 0$, we get $\dfrac{x - x_1}{0}$ in the symmetric equations. This does not mean that $x - x_1$ is divided by 0. $\dfrac{x - x_1}{0} = t$ means that $x = x_1$.

For example consider the equations

$$\frac{x-1}{0}=\frac{y-2}{1/\sqrt{2}}=\frac{z-0}{1/\sqrt{2}}$$

These equations represent the line consisting of the points (x, y, z) where $x=1+0t,\ y=2+\frac{1}{\sqrt{2}}t$ and $z=0+\frac{1}{\sqrt{2}}t$

i.e. $(x, y, z)=\left(1, 2+\frac{1}{\sqrt{2}}t,\ \frac{1}{\sqrt{2}}t\right)$

(ii) If $P(x, y, z)$ is a point on the line L through $A(x_1, y_1, z_1)$ with d.c.s. ℓ, m, n then $\exists\ t \in \mathbb{R} \ni$

$$x=x_1+\ell t,\ y=y_1+mt,\ z=z_1+nt$$

So that $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=t^2\left(\ell^2+m^2+n^2\right)=t^2$

Thus $|t|=\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}=AP$.

2.4.6. Corollary: If a line L passes through $A(x_1, y_1, z_1)$ and has direction ratios ℓ, m, n then the equations of the line L are $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

Proof: If $k=\sqrt{\ell^2+m^2+n^2},\ k \neq 0$ and $\frac{\ell}{k}, \frac{m}{k}, \frac{n}{k}$ are the direction cosines of the line L. Hence the equations of L are

$$\frac{x-x_1}{\left(\frac{\ell}{k}\right)}=\frac{y-y_1}{\left(\frac{m}{k}\right)}=\frac{z-z_1}{\left(\frac{n}{k}\right)}=r. \qquad (r \in \mathbb{R})$$

i.e. $x=x_1+\ell\left(\frac{r}{k}\right),\ y=y_1+m\left(\frac{r}{k}\right),\ z=z_1+n\left(\frac{r}{k}\right),\ \frac{r}{k} \in \mathbb{R}$

Since $\frac{r}{k}$ is arbitrary we replace $\frac{r}{k}$ by the parameter t and get $x=x_1+\ell t,\ y=y_1+mt,\ z=z_1+nt$, so that the symmetric form reduces to

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

2.4.7. Corollary: The parametric equations of the Line L passing through two distinct points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are

$$x=t\ x_2+(1-t)x_1$$

$$y=t\ y_2+(1-t)y_1 \qquad (t\in\mathbb{R})$$

$$z=t\ z_2+(1-t)z_1$$

Proof: If the position vectors of A and B are respectively $\bar{a}$, $\bar{b}$ then $\bar{a}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$ and $\bar{b}=x_2\ \bar{i}+y_2\ \bar{j}+z_2\ \bar{k}$ and $\bar{b}-\bar{a}=(x_2-x_1)\ \bar{i}+(y_2-y_1)\bar{j}+(z_2-z_1)\bar{k}$.

Hence the vector equation of L is

$$\bar{r}=t\bar{a}+(1-t)\ \bar{b},\quad t\in\mathbb{R}$$

If $\bar{r}=x\ \bar{i}+y\ \bar{j}+z\bar{k}$, the vector equation is

$$x\ \bar{i}+y\bar{j}+z\bar{k}=\left(t\ x_1+(1-t)\ x_2\right)\bar{i}+\left(ty_1+(1-t)y_2\right)\bar{j}+\left(tz_1+(1-t)z_2\right)\bar{k}$$

This is equivalent to

$$x=t\ x_1+(1-t)\ x_2$$

$$y=t\ y_1+(1-t)\ y_2$$

$$z=t\ z_1+(1-t)\ z_2$$

2.4.8. Corollary: The equations of the line passing through two distinct points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Proof: If $\bar{a}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$ and $\bar{b}=x_2\bar{i}+y_2\bar{j}+z_2\bar{k}$ then

$$\bar{b}-\bar{a}=(x_2-x_1)\ \bar{i}+(y_2-y_1)\bar{j}+(z_2-z_1)\bar{k}$$

Since if $\bar{r} = x\,\bar{i} + y\bar{j} + z\bar{k}$, then equation of L is

$$\bar{r} = -a + t\left(\bar{b} - \bar{a}\right)$$

i.e. $x = x_1 + t(x_2 - x_1)$

$$y = y_1 + t(y_2 - y_1)$$

$$z = z_1 + t(z_2 - z_1)$$

So that $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

These are the equations of L.

2.4.9. Equations of a line in terms of planes (Unsymmetric form):

Theorem: The direction ratios of the line of intersection of two intersecting planes

$$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$$

and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$

are $b_1c_2 - b_2c_1, \quad c_1a_2 - c_2a_1, \quad a_1b_2 - a_2b_1$

Proof: If $\overline{n_1} = (a_1, b_1, c_1)$ and $\overline{n_2} = (a_2, b_2, c_2)$ then $\bar{n}_1$, is perpendicular to π_1 and $\bar{n}_2$ to π_2 .

Hence L is perpendicular to $\overline{n_1}$ and $\overline{n_2}$.

So L is parallel to $\overline{n_1} \times \overline{n_2}$

$$\overline{n_1} \times \overline{n_2} = (a_1, b_1, c_1) \times (a_2, b_2, c_2)$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= (b_1c_2 - b_2c_1)\,\bar{i} + (c_1a_2 - c_2a_1)\bar{j} + (a_1b_2 - a_2b_1)\bar{k}$$

$$= (b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1)$$

Thus the direction ratios of $\overline{n_1} \times \overline{n_2}$ are

$b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1$

Since L and $\overline{n_1} \times \overline{n_2}$ are parallel these are the direction ratios of L as well.

2.4.10. Lines as intersection of planes: Unsymmetric form:

Since the intersection of two planes is either the empty set or a line, the equations for a line may be described in terms of those of planes.

If a line L is the intersection of the planes

$$\pi_1 \quad : \quad a_1x + b_1y + c_1z + d_1 = 0 \cdots\cdots\cdots\cdots(1)$$

and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0 \cdots\cdots\cdots\cdots(2)$

(x, y, z) lies on L iff (x, y, z) satisfies (1) and (2).

i.e., $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 \cdots\cdots\cdots(3)$

Since L is the locus of (x, y, z) satisfying (3), the equations of L are

$$a_1x + b_1y + c_1z - d_1 = 0 = a_2x + b_2y + c_2z - d_2$$

These are called the unsymmetric form of the equations of L.

2.4.11 Theorem: If L is the intersection of the planes π_1 and π_2

$$\pi_1 \quad : \quad a_1x + b_1y + c_1z + d_1 = 0$$

$$\pi_2 \quad : \quad a_2x + b_2y + c_2z + d_2 = 0$$

then the direction ratios of L are

$b_1c_2 - b_2c_1$, $c_1a_2 - c_2a_1$ and $a_1b_2 - a_2b_1$

Hence the symmetric equations of L are

$$\frac{x-\alpha}{b_1c_2 - b_2c_1} = \frac{y-\beta}{c_1a_2 - c_2a_1} = \frac{z-\gamma}{a_1b_2 - a_2b_1} \cdots\cdots\cdots(1)$$

where (α, β, γ) is any point on L.

Proof: Let L be the line of intersection of the planes π_1 and π_2.

Since L is the line of intersection of the planes π_1 and π_2, from Theorem 2.3.9 the direction ratios of the line L are

$b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ \ a_1b_2 - a_2b_1$

If we fix a point on L say $(\alpha,\ \beta,\ \gamma)$ the equations of L are given by (1). This completes the proof.

Note: Since the line L can't be parallel to all the coordinate planes at a time, line L should intersect one of the three coordinate planes.

Suppose the line L intersects XY - plane at $P(x_1,\ y_1,\ 0)$

$$a_1x + b_1y_1 + d_1 = 0$$

$$a_2x_1 + b_2y_1 + d_2 = 0$$

Solving these two equations for $x_1,\ y_1$

$$x_1 = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1} \quad , \quad y_1 = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}$$

Hence a point on line L is

$$P = (x_1,\ y_1,\ 0) = \left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1} \quad , \quad \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1} \quad , \quad 0\right)$$

The equations of the line L in symmetric form is

$$\frac{x - x_1}{b_1c_2 - b_2c_1} = \frac{y - y_1}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

2.4.12 Symmetric equations into unsymmetric form :

Theorem: The line represented by the equations

$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \cdots\cdots\cdots(1)$ is the intersection of two planes.

Proof: Since ℓ, m, n are the direction ratios of the given line at least one of $\ell,\ m,\ n$ is not zero. We assume that $\ell \neq 0$.

Any point $(x,\ y,\ z)$ on the given line satisfies

$$m(x - x_1) - \ell(y - y_1) = 0 = \ell(z - z_1) - n(x - x_1)$$

i.e., $mx - ly = mx_1 - \ell y_1 \ \cdots\cdots\cdots\cdots(2)$

and $\ell z-nx=\ell z_1-nx_1\cdots\cdots\cdots\cdots(3)$

Since $\ell \neq 0$ the equations (2) and (3) represent planes and any (x,y,z) on L lies on the planes (2) and (3).

Since the intersection of two planes is either the empty set or a line it follows that the line (1) is the intersection of the planes (2) and (3).

the proof is similar if $m\neq 0$ or if $n\neq 0$.

Examples:

2.4.13 Find the point of intersection of the line $L\ \frac{x+1}{1}=\frac{y+3}{3}=\frac{z+2}{-2}$ with the plane $\pi = 3x+4y+5z-5=0$.

Solution: Given equation of the line L is $\frac{x+1}{1}=\frac{y+3}{3}=\frac{z+2}{2}=t\ (\text{say})$

$\Rightarrow (t-1,\ 3t-3, -2t-2)$ is any point on L

Let $P=(t-1,\ 3t-3, -2t-2)$

If P lies on the plane π then

$$3(t-1)+4(3t-3)+5(-2t-2)-5=0$$

$\Rightarrow 5t-30=0 \Rightarrow t=6$

The point of intersection of L and π is

$$P=(5,15,-14).$$

2.4.14 Find the point of intersection of the line L, $x-2y+4z+4=0\cdots\cdots(1)$, $x+y+z-8=0\cdots\cdots(2)$ with the plane $\pi=x-y+2z+1=0$.

Solution: Let $\overline{n_1}$, $\overline{n_2}$ be the unit vectors along the normals of (1) and (2) respectively.

$$\overline{n_1}=(1,-2,4) \qquad \overline{n_2}=(1,1,1)$$

Then d.rs. of the line L are -6, 3, 3.

Putting $z=0$ in (1) and (2) we get

$$x-2y+4=0$$

$x+y-8=0$

Solving these two equations we get $x=4,\ y=4$

The point on the line L is $P=(4,4,0)$

The equations of L in symmetric form is

$$\frac{x-4}{-6}=\frac{y-4}{3}=\frac{z}{3}=t\ (\text{say})$$

Any point on the line $Q=(-6t+4,\ 3t+4,\ 3t)$

If Q lies on π then $-6t+4-3t-4+6t+1=0$

$$\Rightarrow t=\frac{1}{3}$$

The point of intersection of L and π is $Q=(2,5,1)$.

2.4.15 Find the point of intersection of the line L through $A(-2,3,4),\ B(1,2,3)$ with $xz-\text{plane}$.

Solution: D.rs. of the line L through A, B are $3,\ -1,\ -1$.

Equation of the line L is $\frac{x+2}{3}=\frac{y-3}{-1}=\frac{z-4}{-1}=t(\text{say})$

Any point on L is $P=(3t-2,\ -t+3,\ -t+4)$

Equation of $xz-$ plane is $y=0$.

If P lies on $y=0$ then $-t+3=0\Rightarrow\ t=3$

point of intersection of L and $xz-$ plane is $P=(7,0,1)$.

2.4.16 Find the symmetric form the equation of the line

$x+y+z+1=0=4x+y-2z+2$.

Solution: Let $\pi_1\equiv x+y+z+1=0$

$\pi_2\equiv 4x+y-2z+2=0$ and L be the line of intersection of π_1 and π_2.

Let $\overline{n}_1$ and $\overline{n}_2$ be the unit vectors along the normals of π_1 and π_2. respectively.

$\bar{n}_1 = (1,1,1) \qquad \bar{n}_2 = (4,1,-2)$

$\bar{n}_1 \times \bar{n}_2$ is a vector along L.

d.rs. of L are $-2-1,\ 4+2,\ 1-4$ i.e. $-3, 6, -3$

$\Rightarrow$ d.rs. of L are $1, -2, 1$

Putting $z=0$ in π_1 and π_2 we get

$x+y+1=0$

$4x+y+2=0$

Solving these two equations $3x+1=0 \Rightarrow x=-\frac{1}{3}$

$x=-\frac{1}{3}$ and $y=-\frac{2}{3}$

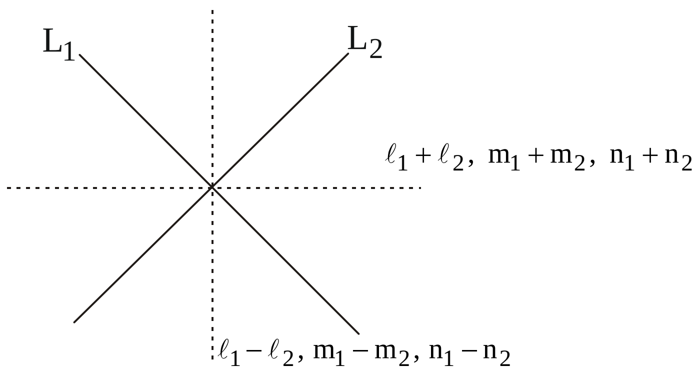
$\Rightarrow$ the point $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ is on L.

Equations of L in symmetric form are

$$\frac{x+\frac{1}{3}}{1} = \frac{y+\frac{2}{3}}{-2} = \frac{z}{1}$$

Note: We use the following result with which we are already familiar.

If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are the d.cs. of two lines L_1 and L_2 respectively then the d.rs. of the lines bisecting the angles between the lines L_1 and L_2 are $\ell_1 \pm \ell_2, m_1 \pm m_2, n_1 \pm n_2$ and they pass through the point of intersection of L_1 and L_2.



2.4.17 Find the equations of the lines bisecting the angles between the line

$$\frac{x-1}{1}=\frac{y+4}{2}=\frac{z-5}{-2}\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

and $$\frac{x-1}{4}=\frac{y+4}{3}=\frac{z-5}{12}\cdots\cdots\cdots\cdots(2)$$

Solution: The d.c.s. of the line (1) are $\frac{1}{3},\frac{2}{3},\frac{-2}{3}$

The d.c.s of the line (2) are $\frac{4}{13},\frac{3}{13},\frac{12}{13}$

Then the d.rs. of the lines bisecting the angles between (1) and (2) are

$$\frac{1}{3}\pm\frac{4}{13},\frac{2}{3}\pm\frac{3}{13},\frac{-2}{3}\pm\frac{12}{13}$$ (see the note)

i.e., $\frac{1}{3}+\frac{4}{13},\frac{2}{3}+\frac{3}{13},\frac{-2}{3}+\frac{12}{13}\ ;\ \frac{1}{3}-\frac{4}{13},\frac{2}{3}-\frac{3}{13},\frac{-2}{3}-\frac{13}{13}$

i.e., $\frac{25}{39},\frac{35}{39},\frac{10}{39}\ ;\ \frac{1}{39},\frac{17}{39},\frac{-62}{39}$

i.e., $25,\ 35,\ 10,\ ;\ 1,\ 17,\ -62$

i.e., $5,\ 7,\ 2\ ;\ 1,\ 17,\ -62$

So the equations of the angular bisectors of (1) and (2) are

$$\frac{x-1}{5}=\frac{y+4}{7}=\frac{z-5}{2};$$

$$\frac{x-1}{1}=\frac{y+4}{17}=\frac{z-5}{-62}$$

2.5 Angles between lines and planes:

We assume the knowledge of the following:

(a) If $\bar{a}$, $\bar{b}$ are any two non zero vectors and θ is an angle between $\bar{a}$ and $\bar{b}$ then

$$\text{Cos}\,\theta=\pm\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\ |\bar{b}|}$$

(b) If ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 are d.rs. of the lines L_1 and L_2 respectively and θ is an angle between L_1 and L_2 then

$$\text{Cos } \theta = \pm \frac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{\ell_2^2 + m_2^2 + n_2^2}}$$

2.5.1 Theorem: An angle between the lines L_1 and L_2 represented by $L_1 : \bar{r} = \overline{a_1} + t\,\overline{b_1}$ and $L_2 : \bar{r} = \overline{a_2} + t\,\overline{b_2}$ is given by $\cos\theta = \pm \dfrac{\overline{b_1} \cdot \overline{b_2}}{|\overline{b_1}|\ |\overline{b_2}|}$

If the symmetric equations for L_1 and L_2 are given by

$$\left.\begin{array}{lcccccc} L_1 & : & \dfrac{x - x_1}{\ell_1} & = & \dfrac{y - y_1}{m_1} & = & \dfrac{z - z_1}{n_1} \\ L_2 & : & \dfrac{x - x_2}{\ell_2} & = & \dfrac{y - y_2}{m_2} & = & \dfrac{z - z_2}{n_2} \end{array}\right\} (A)$$

Then $\cos\theta = \pm \dfrac{(\ell_1\ell_2 + m_1m_2 + n_1n_2)}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{\ell_2^2 + m_2^2 + n_2^2}}$

Proof: Since L_1 is parallel to $\overline{b_1}$ and L_2 is parallel to $\overline{b_2}$, angles between L_1, L_2 and $\overline{b_1}$, $\overline{b_2}$ are the same. But we know that the angles between $\overline{b_1}$ and $\overline{b_2}$ are given by

$$\cos\theta = \pm \frac{\overline{b_1} \cdot \overline{b_2}}{|\overline{b_1}|\ |\overline{b_2}|}$$

From (A) it is clear that

L_1 has direction ratios ℓ_1, m_1, n_1 and

L_2 has direction ratios ℓ_2, m_2, n_2.

Since $\overline{b_1}$ is parallel to L_1, there exists a scalar k_1 such that

$$\overline{b_1} = k_1 \left(\ell_1 \bar{i} + m_1 \bar{j} + n_1 \bar{k}\right)$$

Similarly there exists a scalar k_2 such that

$$\overline{b_2} = k_2\left(\ell_2\bar{i} + m_2\bar{j} + n_2\bar{k}\right)$$

Hence $\cos\theta = \pm\dfrac{\overline{b_1}\cdot\overline{b_2}}{\left|\overline{b_1}\right|\ \left|\overline{b_2}\right|} = \pm\dfrac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{\ell_2^2 + m_2^2 + n_2^2}}$

2.5.2 Conditions for perpendicularity and parallelism:

Corollary: The lines L_1, L_2 with vector equations

$$L_1 : \bar{r} = \overline{a_1} + t\overline{b_1} \quad \text{and} \quad L_2 : \bar{r} = \overline{a_2} = t\overline{b_2}$$

are perpendicular iff $\overline{b_1}\cdot\overline{b_2} = 0$.

If the d.r.s. of L_1, L_2 are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 respectively the condition for perpendicularity is

$$\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$$

Proof: If θ is an angle between the lines L_1, L_2 then,

L_1, L_2 are perpendicular iff $\theta = \dfrac{\pi}{2}$ i.e. $\cos\theta = 0$

Since $\cos\theta = \pm\dfrac{\overline{b_1}\cdot\overline{b_2}}{\left|\overline{b_1}\right|\ \left|\overline{b_2}\right|} = \pm\dfrac{\ell_1\ell_2 + m_1m_2 + n_1n_2}{\sqrt{\ell_1^2 + m_1^2 + n_1^2}\ \sqrt{\ell_2^2 + m_2^2 + n_2^2}}$

L_1, L_2 are perpendicular iff $\overline{b_1}\cdot\overline{b_2} = 0$

iff $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$

2.5.3 Corollary: Under the above hypothesis, L_1, L_2 are parallel

iff $\overline{b_1} = k\overline{b_2}$ for some $k \in \mathbb{R}$

iff $\ell_1 : \ell_2 = m_1 : m_2 = n_1 : n_2$

Proof: L_1, L_2 are parallel iff $\overline{b_1}, \overline{b_2}$ are parallel

iff for some $k \in \mathbb{R}$, $\overline{b_1} = k\overline{b_2}$

Since $\overline{b_1} = \ell_1\bar{i} + m_1\bar{j} + n_1\bar{k}$ and $\overline{b_2} = \ell_2\bar{i} + m_2\bar{j} + n_2\bar{k}$,

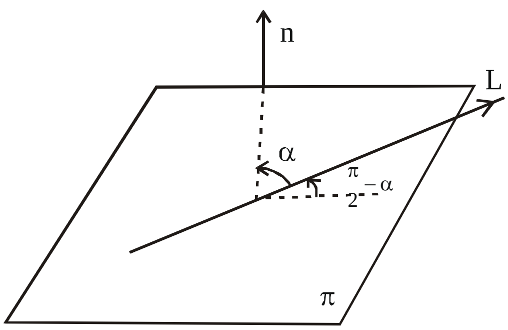
$$\overline{b_1} = k\overline{b_2} \Leftrightarrow \ell_1 = k\ell_2,\ m_1 = km_2,\ n_1 = kn_2$$

$$\Leftrightarrow \ell_1 : \ell_2 = m_1 : m_2 = n_1 : n_2$$

Now we consider angles between a line and a plane

Angles between a line and a plane:

2.5.4 Definition: Let L be a line and π be a plane and $\overline{n}$ be a normal to the plane π. If α is an angle between $\overline{n}$ and L such that $0 \le \alpha \le \frac{\pi}{2}$ then $\left(\frac{\pi}{2} - \alpha\right)$ is known as the angle between L and π.



Note:

1. Since all the normals are parallel, we may choose any $\overline{n}$.
2. If $\alpha = 0$ then L is normal to the plane. The angle between the line and the plane is $\frac{\pi}{2}$.
3. If $\alpha = \frac{\pi}{2}$ then L is parallel to the plane. The angle between the line and the plane is 0.

2.5.6 Corollary: The line $L : \frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is parallel to the plane

$\pi :\ ax + by + cz + d = 0$ iff $a\ell + bm + cn = 0$.

Proof: The direction ratios of the normal to π are (a, b, c).

Since the normal is perpendicular to L, $a\ell + bm + cn = 0$

Conversely if $a\ell + bm + cn = 0$, the line L and the vector $\bar{n} = (a, b, c)$ are perpendicular.

Hence L is parallel to the plane whose normal has direction ratios a, b, c.

Since the normal to π has direction ratios a, b, c, L is parallel to π.

2.5.7 Corollary: The line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ lies in the plane

$$ax + by + cz + d = 0 \Leftrightarrow ax_1 + by_1 + cz_1 + d = 0, \quad a\ell + bm + cn = 0$$

Proof: The line lies in the plane

$\Leftrightarrow$ the point (x_1, y_1, z_1) lies in the plane and the line is parallel to the plane.

$\Leftrightarrow$ $ax_1 + by_1 + cz_1 + d = 0$ and $a\ell + bm + cn = 0$.

2.5.8 Corollary: If a, b, c are chosen such that $a\ell + bm + cn = 0$ then there exists a unique plane containing the line $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and its equation is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \cdots\cdots\cdots (1)$$

Proof: Since $a\ell + bm + cn = 0$, any line with direction ratios ℓ, m, n is perpendicular to the vector (a, b, c).

Hence the normal to any plane containing the given line must have d.rs. (a, b, c).

Thus the equation of any such plane is of the form $ax + by + cz + d = 0$

Since the point (x_1, y_1, z_1) lies in the plane $ax + by + cz + d = 0$, we have $ax_1 + by_1 + cz_1 + d = 0$.

Hence the equation of the plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

2.5.9 Theorem: If θ is an angle between a line L : $\bar{r} = \bar{a} + t\bar{b}$ and a plane $\pi : \bar{r} \cdot \bar{n} = q$ then

$$\text{Sin } \theta = \pm \frac{\bar{b} \cdot \bar{n}}{|\bar{b}|\,|\bar{n}|}.$$

Proof: The line L is parallel to $\bar{b}$ and π is the plane with normal $\bar{n}$.

If α is an angle between $\bar{b}$ and $\bar{n}$ then $\alpha = \frac{\pi}{2} \pm \theta$

Therefore $\text{Cos}\ \alpha = \pm\ \text{Sin}\ \theta$

But $\text{Cos}\ \alpha = \frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\ |\bar{n}|} \Rightarrow \text{Sin}\,\theta = \pm \frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\ |\bar{n}|}$

If $\bar{a} = (x_1, y_1, z_1)$ $\bar{b} = (\ell, m, n)$ and $\bar{n} = (a, b, c)$ then the line is

$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane is $ax+by+cz+d=0$ where $d=-q$.

Now $\text{Sin}\ \theta = \pm \frac{(a\ell+bm+cn)}{\sqrt{a^2+b^2+c^2}\ \sqrt{\ell^2+m^2+n^2}}$

Examples:

2.5.10 Show that the line $L: \frac{x+3}{-2} = \frac{y+4}{3} = \frac{z+4}{5}$ is perpendicular to the plane $\pi \equiv -2x+3y+5z=0$.

Solution: Since d.rs. of normal to the plane π are $-2, 3, 5$ and d.rs. of the line L are $-2, 3, 5$.

Hence L is parallel to normal of π.

L is perpendicular to π.

2.5.11 Find the acute angle between the lines $\frac{x-2}{1} = \frac{y-4}{0} = \frac{z-5}{-1}$ ....(1)

$\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$........(2).

Solution: d.rs of the line (1) are 1, 0, -1

d.rs. of the line (2) are 3, 4, 5

If θ is the acute angle between the lines (1) and (2) then

$$\text{Cos}\ \theta = \frac{|(1)(3)+(4)(0)+(5)(-1)|}{\sqrt{1+1}\ \sqrt{9+16+25}} = \frac{+2}{\sqrt{2}\ \sqrt{50}} = \frac{+1}{5}$$

$$\theta = \text{Cos}^{-1}\left(\frac{1}{5}\right)$$

2.5.12 Find the value of k if the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}\cdots\cdots\cdots(1)$

$\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$ are perpendicular.

Solution: d.rs. of the line (1) are $-3, 2k, 2$

d.rs. of the line (2) are $3k, 1, -5$

The lines (1) and (2) are perpendicular

$$\Leftrightarrow (-3)3k + 2k(1)+2(-5)=0 \Leftrightarrow -9k+2k-10=0 \Leftrightarrow -7k-10=0 \Leftrightarrow k=-\frac{10}{7}$$

2.5.13 Find two points on the line $\frac{x-z}{1}=\frac{y-3}{-2}=\frac{z-5}{2}\cdots\cdots\cdots\cdots(1)$ on either side of $(2,-3,-5)$ and at a distance 3 from it.

Solution: Given point on the line $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z+5}{2}=t$ (say) is $A(2,-3,-5)$.

Any point on the line (1) is $P=(t+2, -2t-3, 2t-5)$

If the distance between A and P is 3 units then

$$AP=3 \Rightarrow AP^2=9 \Rightarrow t^2+4t^2+4t^2=9 \Rightarrow t=\pm 1$$

The two points are $(3,-5,-3)$ and $(1,-1,-7)$

2.5.14 Find the condition that the two lines $x=az+b,\ y=cz+d\cdots\cdots\cdots(1)$

$x=a_1z+b_1$, $y=c_1z+d_1\cdots\cdots\cdots(2)$ are perpendicular.

Given lines are $x-az-b=0=y-cz-d\cdots\cdots\cdots\cdots(1)$

$$x-a_1z-b_1=0=0\ y-c_1z-d_1\cdots\cdots\cdots(2)$$

d.rs. of the line (1) are

$$0(c)-1(-a), (-a)0-(-c)1, 1-0(0)$$

i.e., a, c, 1

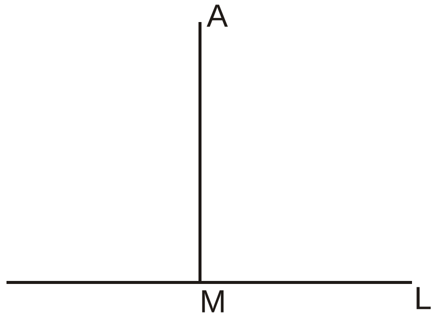
Similarly d.rs. of the line (2) are $a_1, c_1, 1$

The two lines (1) and (2) are perpendicular

$\Leftrightarrow aa_1 + cc_1 + 1(1) = 0$

$\Leftrightarrow aa_1 + cc_1 + 1 = 0$

2.6 Foot of the perpendicular of a point:



We recall that if L be a line and A be any point, the point of intersection M of the perpendicular line through A to L is called the foot of the perpendicular from A to L and A M is called the perpendicular distance from A to L.

2.6.1 Theorem:

(i) The foot of the perpendicular from $A(\alpha, \beta, \gamma)$ to the line $L : \frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is $Q = (x_1 + \ell t_1,\ y_1 + mt_1, z_1 + nt_1)$.

$$\text{Where } t_1 = \frac{\ell(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{\ell^2 + m^2 + n^2}$$

(ii) The equation of the perpendicular line AQ is

$$\frac{x - \alpha}{x_1 - \alpha - \ell\, t_1} = \frac{y - \beta}{y_1 - \beta - mt_1} = \frac{z - \gamma}{z_1 - \gamma - nt_1}$$

(iii) and the length AQ is

$$AQ = \sqrt{(x_1 - \alpha - \ell\, t_1)^2 + (y_1 - \beta - mt_1)^2 + (z_1 - \gamma - nt_1)^2}$$

Proof: If a line through A has d.rs. ℓ_1, m_1, n_1 and is perpendicular to L then its equation is

$$\frac{x-\alpha}{\ell_1}=\frac{y-\beta}{m_1}=\frac{z-\gamma}{n_1}$$

If the point of intersection of these lines is Q, Q lies on L, so then for some t_1,

$$Q=\left(x_1+\ell t_1,\ y_1+mt_1,\ z_1+nt_1\right)$$

So the line AQ has d.rs.

$$\alpha-x_1-\ell t_1\ ,\ \beta-y_1-mt_1,\ \gamma-z_1-nt_1$$

and these are proportional to ℓ_1, m_1, n_1.

If condition for perpendicularity becomes

$$\ell\left(\alpha-x_1-\ell t_1\right)+m\left(\beta-y_1-mt_1\right)+n\left(\gamma-z_1-nt_1\right)=0$$

$$\Rightarrow\ \ell\left(\alpha-x_1\right)+m\left(\beta-y_1\right)+n\left(\gamma-z_1\right)-\left(\ell^2+m^2+n^2\right)t_1=0$$

$$\Rightarrow t_1=\frac{\ell\left(\alpha-x_1\right)+m\left(\beta-y_1\right)+n\left(\gamma-z_1\right)}{\ell^2+m^2+n^2}$$

If $x_2=x_1+\ell t_1,\ y_2=y_1+mt_1,\ z_2=z_1+nt_1$ then the direction ratios of the perpendicular AQ to L are

$$x_1-x_2=x_1-\alpha-\ell t_1$$

$$y_1-y_2=y_1-\beta-mt_1$$

and $\quad z_1-z_2=z_1-\gamma-nt_1$

Hence the equations of the line AQ are

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

Also $\quad AQ=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$

$$=\sqrt{\left(x_1-\alpha-\ell t_1\right)^2+\left(y_1-\beta-mt_1\right)^2+\left(z_1-\gamma-nt_1\right)^2}$$

2.6.2 Theorem: Let $P(x_1, y_1, z_1)$ be a point and $\pi : ax + by + cz + d = 0$ be a plane, M the foot of the perpendicular to π through P.

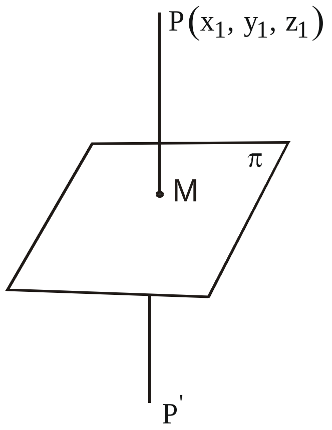
Then

(i) The equations of the perpendicular line from P to π are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

(ii) The coordinates of M are $(x_1 + at,\ \ y_1 + bt,\ \ z_1 + ct)$

where $t = \dfrac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$



(iii) $PM = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

Proof: If the perpendicular line through P to the plane meets at M then M is the foot of perpendicular from P in the plane π.

Let $P' \neq P$ be the point on $\overline{PM}$ such that $PM = MP'$

and $P' = (x_2,\ y_2,\ z_2)$ and $M = (\alpha, \beta, \gamma)$.

Since $PM = MP'$ then M is the mid point of the line PP'.

$$\Rightarrow (\alpha, \beta, \gamma) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$\Rightarrow x_2 = 2\alpha - x_1,\ y_2 = 2\beta - y_1,\ z_2 = 2\gamma - z_1$

Since PM is the normal to the plane π, direction ratios of the normal PM are a, b, c.

(i) Equations to the line PM are

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \cdots\cdots\cdots\cdots(1)$$

$$\Rightarrow \quad x = x_1 + at,\ y = y_1 + bt,\ z = z_1 + ct,\ (t \in \mathbb{R})$$

(ii) Since M lies on PM there is a real no. t such that

$$M = (x_1 + at,\ y_1 + bt,\ z_1 + ct)$$

M lies in the plane $\pi \Rightarrow a(x_1 + at) + b(y_1 + bt) + c(c_1 + ct) + d = 0$

$$\Rightarrow t = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

and the coordinates of M are

$$(x_1 + at,\ y_1 + bt,\ z_1 + ct) \cdots\cdots\cdots\cdots(2)$$

(iii) $PM^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 = (-at)^2 + (-bt)^2 + (-ct)^2 = (a^2 + b^2 + c^2)t^2$

$$= (a^2 + b^2 + c^2)\frac{(ax_1 + by_1 + cz_1 + d)^2}{(a^2 + b^2 + c^2)^2}$$

$$\Rightarrow PM = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

The perpendicular distance from P to the plane is

$$PM = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \cdots\cdots\cdots\cdots(3)$$

2.6.3 Theorem: Equations of the projection of a line

$L : \dfrac{x - x_1}{\ell} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ in the plane

$\pi \equiv ax + by + cz + d = 0$ when

(i) L is parallel to π are $\dfrac{x-\alpha}{\ell} = \dfrac{y-\beta}{m} = \dfrac{z-r}{n}$

where $\alpha = x_1 + at_0,\quad \beta = y_1 + bt_0,\quad \gamma = z_1 + ct_0$

and $t_0 = \dfrac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(ii) L is not parallel to π are $\dfrac{x-\alpha}{\alpha_1 - \alpha} = \dfrac{y-\beta}{\beta_1 - \beta} = \dfrac{z-\gamma}{\gamma_1 - \gamma}$

where $\alpha_1 = x_1 + at, \beta_1 = y_1 + bt,\quad \gamma_1 = z_1 + ct$

and $t = \dfrac{-(ax_1 + by_1 + cz_1 + d)}{a\ell + bm + cn}$

Proof: Let $A = (x_1, y_1, z_1)$

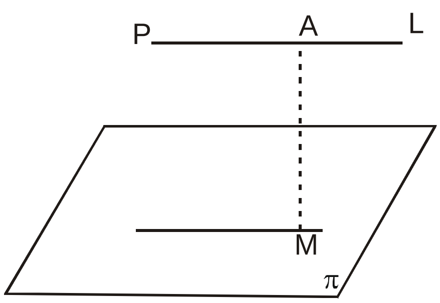
Any point on the line L is $P = (x_1 + \ell t\ ,\ \ y_1 + mt,\ z_1 + nt),\quad t \in \mathbb{R}$

Let M be the foot of the perpendicular of A on π and $M = (\alpha, \beta, \gamma)$.

By theorem 2.4.11; M is given by $M = (\alpha, \beta, \gamma) = (x_1 + at_0,\ y_1 + bt_0, z_1 + ct_0)$

where $t_0 = \dfrac{-(ax_1 + by_1 + cz_1 + d_1)}{a^2 + b^2 + c^2}$

(i) If L is parallel to π

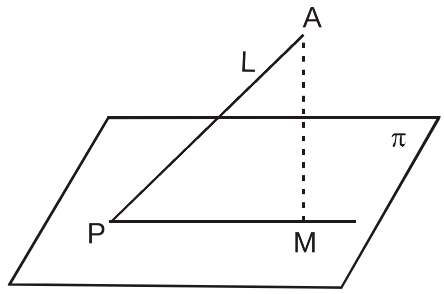


d.rs. of the projection of L on π are ℓ, m, n and M lies on this line.

Hence equation of the projection of the line L on π is

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

(ii) If L is not parallel to π



and P is the point of intersection of L and π, then P lies on π.

If $P=\left(x_1+\ell t,\ y_1+mt, z_1+nt\right)$

$$a\left(x_1+\ell t\right)+b\left(y_1+mt\right)+c\left(z_1+nt\right)+d=0$$

$$\Rightarrow\ t=\frac{-\left(ax_1+by_1+cz_1+d\right)}{a\ell+bm+cn}$$

Let $P=\left(\alpha_1, \beta_1, \gamma_1\right)$

Hence the projection of L on π is the line passing through M and P.

Therefore the equations of MP are

$$\frac{x-\alpha}{\alpha_1-\alpha}=\frac{y-\beta}{\beta_1-\beta}=\frac{z-\gamma}{\gamma_1-\gamma}$$

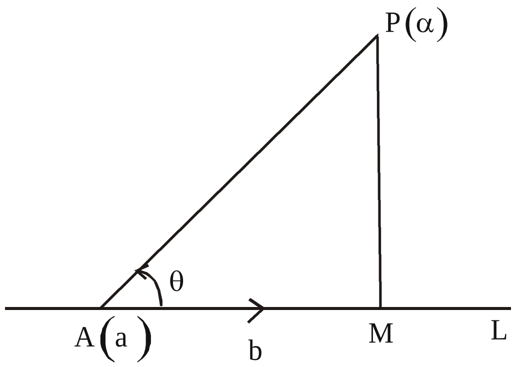
Length of the perpendicular From a point to a line:

We derive a formula for the perpendicular distance from a point to the given line.

2.6.4 Theorem: In vector form the perpendicular distance from a point with position vector $\bar{\alpha}$ to the line $\bar{r}=\bar{a}+t\,\bar{b}$ is $\dfrac{\left|\left(\bar{\alpha}-\bar{a}\right)\times\bar{b}\right|}{\left|\bar{b}\right|}$

Proof: Let L be the given line. L passes through $A\left(\bar{a}\right)$ and parallel to the vector $\bar{b}$.

Let M be the foot of perpendicular of $P(\bar{\alpha})$ on L.



Let $\underline{|PAM} = \theta$

L is parallel to $\bar{b}$ then the angle between $\overline{AP}$ and $\bar{b}$ is θ.

$$\left|\overline{AP} \times \bar{b}\right| = \left|\overline{AP}\right| \left|\bar{b}\right| \text{ Sin } \theta = (\text{AP Sin } \theta)\left|\bar{b}\right| = \text{PM}\left|\bar{b}\right|$$

$$\Rightarrow \text{PM} = \frac{\left|\overline{AP} \times \bar{b}\right|}{\left|\bar{b}\right|} = \frac{\left|(\bar{\alpha} - \bar{a}) \times \bar{b}\right|}{\left|\bar{b}\right|}$$

2.6.5 Theorem: The perpendicular distance from (x_1, y_1, z_1) to the line

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ is } \sqrt{\frac{\sum\{(y_1-\beta)n - (z_1-\gamma)m\}^2}{\sqrt{\ell^2 + m^2 + n^2}}}$$

Proof: Let $\bar{\alpha} = (x_1, y_1, z_1)$ $\quad \bar{a} = (\alpha, \beta, \gamma),\ b = (\ell,\ m,\ n)$

Then $(\bar{\alpha} - \bar{a}) \times \bar{b} = (x_1 - \alpha,\ y_1 - \beta,\ z_1 - \gamma) \times (\ell, m, n) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_1 - \alpha & y_1 - \beta & z_1 - \gamma \\ \ell & m & n \end{vmatrix}$

$$= \left(\{(y_1-\beta)n - (x_1-\gamma)m\},\ \{(z_1-\gamma)\ell - (x_1-\alpha)n\},\ \{(x_1-\alpha)m - (y_1-\beta)\ell\}\right)$$

$$\left|(\bar{\alpha} - \bar{a}) \times \bar{b}\right| = \sqrt{\sum\{(y_1-\beta)n - (z_1-\gamma)m\}^2}$$

$$|\bar{b}| = \sqrt{\ell^2 + m^2 + n^2}$$

Perpendicular distance from a point to the given line

$$= \frac{|(\bar{\alpha} - \bar{a}) \times \bar{b}|}{|\bar{b}|} = \frac{\sqrt{\sum\{(y_1 - \beta)n - (z_1 - \gamma)m\}^2}}{\sqrt{\ell^2 + m^2 + n^2}}$$

Examples:

2.6.6 Find the distance of the point $P(1,-2,3)$ from the plane $\pi \equiv x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6} \cdots\cdots\cdots\cdots (1)$.

Solution: D.rs. of the line parallel to the line (1) are 2, 3, -6.

Equation of the line through $P(1,-2,3)$ and having d.rs. 2, 3, -6 is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = t \quad (\text{say})$$

Any point on the line $Q = (2t+1,\ 3t-2,\ -6t+3)$

If Q lies on the plane π then $2t+1-3t+2-6t+3-5=0 \Rightarrow -7t+1=0 \Rightarrow t = \frac{1}{7}$

$$Q = \left(\frac{9}{7},\ \frac{-11}{7},\ \frac{15}{7}\right)$$

The distance $PQ = \sqrt{\left(\frac{9}{7}-1\right)^2 + \left(\frac{11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2} = \frac{49}{49} = 1$

2.6.7 Find the foot of the perpendicular and perpendicular distance from the point $(-1,3,9)$ to the line $\frac{x-13}{5} = \frac{y+8}{-8} = \frac{z-31}{1} \cdots\cdots\cdots\cdots (1)$

Solution: Any point on the line (1) is $P = (5t+13,\ -8t-8,\ t+31)$

If P is the foot of the perpendicular from $A(-1,3,9)$ to the line (1) then P lies on the line (1)

d.rs. of AP are $5t+14,\ -8t-11,\ t+22$

d.rs. of the line (1) are 5, -8, 1

Thus AP is perpendicular to the line (1)

$\Leftrightarrow\ 5(5t+14)-8(-8t-11)+1(t+22)=0 \quad \Leftrightarrow\ 25t+70+64t+88+t+22=0$

$\Leftrightarrow 90t+180=0 \quad \Leftrightarrow\ t=-2$

$\Rightarrow$ The foot of perpendicular from A to the line (1) is

$$P=(-10+13, 16-8, -2+31)=(3, 8, 29)$$

The perependicular distance from A to the line (1)

$AP=\sqrt{(3+1)^2+(8-3)^2+(29-9)^2} = \sqrt{16+25+400}=\sqrt{441}=21$ units.

2.6.8 Find the equations of the line through $A(-1,3,2)$ and perpendicular to the plane $\pi : x+2y+2z=3$.

Solution: d.rs. of the normal to π are $1, 2, 2$

A line through A line through A and $\perp r$ to π is parallel to the normal of π.

d.rs. of the required line are 1, 2, 2

Equations of the required line is

$$\frac{x+1}{1}=\frac{y-3}{2}=\frac{z-2}{2}$$

2.6.9 Find the equations of the line through the point $(-3, 2, 5)$ and parallel to the line $2x+3y-4z=0=3x-4y+z$.

Solution: Let $\pi_1 \equiv\ 2x+3y-4z=0$

$\pi_2 \equiv 3x-4y+z=0$ represent a line L.

Let $\overline{n_1}$ and $\overline{n_2}$ be the unit vectors along the normals of π_1 and π_2 respectively.

$\overline{n_1} \times \overline{n_2}$ is a vector along L

$\overline{n_1}=(2,3,-4) \qquad \overline{n_2}=(3,-4,1)$

d.rs. of L are 3 - 16, -12 - 2, -8 - 9

ie., -13, -14, -17

Equations of L are $\frac{x+3}{13}=\frac{y-2}{14}=\frac{z-5}{17}$

2.6.10 Find the angle between the lines $x+2y-2z=0,\ x-2y+z=7;\ \frac{x-1}{1}=\frac{y+2}{-2}=\frac{z}{2}$

Solution: We recall the formula for the angle between the lines when d.rs. of the lines are given.

Let $\pi_1 \equiv x+2y-2z=0$

$\pi_2 \equiv x-2y+z-7=0$ represent the line L,

Given the line $L_2 : \frac{x-1}{1}=\frac{y+2}{-2}=\frac{z}{2}$

Let $\overline{n_1}$ and $\overline{n_2}$ be the unit vectors along the normals of π_1 and π_2 respectively.

$\overline{n_1}=(1,2,-2) \qquad \overline{n_2}=(1,-2,1)$

$\overline{n_1}\times\overline{n_2}$ is a vector along the line L_1

d.rs. of L_1 are 2 - 4, -2 - 1, -2 - 2

ie., -2, -3, -4 i.e., 2, 3, 4

d.rs. of L_2 are 1, -2, 2

If θ is the angle between the lines L_1 and L_2 then

$$\cos\theta=\frac{2(1)+(3)(-2)+4(2)}{\sqrt{4+9+16}\ \sqrt{1+4+4}}$$

$$=\frac{4}{\sqrt{29}\times 3}=\frac{4}{3\sqrt{29}}$$

$$\theta=\cos^{-1}\left(\frac{4}{3\sqrt{29}}\right).$$

2.6.11.: Find the an angle between the lines in which the planes

$3x-7y-5z=1,\ 5x-13y+3z+2=0$ cut the plane $8x-11y+2z=0$.

Solution: Let $\pi_1 \equiv 3x-7y-5z-1=0$

$\pi_2 \equiv 5x-13y+3z+2=0$

and $\pi_3 \equiv 8x-11y+2z=0$ be the given planes

and π_1 and π_3 represent the line L_1 and π_2 and π_3 represent the line L_2.

Let $\overline{n_1}, \overline{n_2}$ and $\overline{n_3}$ be the unit vectors along the normals of π_1, π_2 and π_3 respectively.

$\overline{n_1} = 3, -7, -5, \; \overline{n_2} = 5, -13, 3 \qquad \pi_3 = 8, -11, 2$

$\overline{n_1} \times \overline{n_3}$ and $\overline{n_2} \times \overline{n_3}$ are the vectros along L_1 and L_2 respectively.

d.rs. of L_1 are $-14-55, \; -40-6, \; -33+56$.

i.e., $-69, \; -46, \; 23$

i.e., $3, 2, -1$

d.rs. of L_2 are $-26+33, \; 24-10, \; -55+104$

i.e., $7, 14, \; 49$ i.e, $1, 2, 7$

If θ is the angle between L_1 and L_2 then

$$\text{Cos}\,\theta = \frac{1(3)+2(2)-7}{\sqrt{9+4+1}\,\sqrt{1+4+49}} = 0$$

$$\Leftrightarrow \; \theta = \frac{\pi}{2}$$

2.6.12 Show that the line L, $\dfrac{x+1}{-1} = \dfrac{y+2}{3} = \dfrac{z+5}{5} \cdots\cdots\cdots\cdots(1)$ lies in the plane

$\pi \equiv \; x+2y-z=0 \cdots\cdots\cdots\cdots(2)$

Solution: Let $A = (-1, -2, -5)$

d.rs. of L are -1, 3, 5.

d.rs. of the normal to π are 1, 2, -1

Since $-1+2(-2)-1(-5)=0$

A lies in the plane

Since $(-1)(-1)+3(2)+5(-1)=0$

Line L lies int he plane π.

2.6.13 Show that the line L, $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z+4}{5}$ is parallel to

$\pi \equiv 3x+4y+z-4=0$

Solution: d.rs. of L are 1, -2, 5

d.rs. of normal to π are 3, 4, 1

Since (1) (3) - 2(4) + 5(1) = 0

L is perpendicular to the normal of π

$\Rightarrow$ L is parallel to π

2.6.14 Show that the line L : $\frac{x+3}{-2}=\frac{y+4}{3}=\frac{z+4}{5}$ is perpendicular to $\pi \equiv -2x+3y+5z=0$.

Solution: d.rs. of L are -2, 3, 5

d.rs. of normal to π are -2, 3, 5

$\Rightarrow$ L is parallel to normal to π

$\Rightarrow$ L is perpendicular to π

2.6.15 Find the angle between the line L: $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $\pi \equiv 3x+y+z-7=0$.

Solution: d.rs. of L are $2,3,6$

d.rs. of the normal to π are 3, 1, 1

If θ is an angle between L and π then

$$\text{Sin } \theta = \pm \frac{a\ell + bm + cn}{\sqrt{a^2+b^2+c^2}\ \sqrt{\ell^2+m^2+n^2}} = \pm \frac{2(3)+3(1)+6(1)}{\sqrt{4+9+36}\ \sqrt{9+1+1}}$$

$$= \pm \frac{15}{7\sqrt{11}}$$

$$\theta = \text{Sin}^{-1}\left(\pm \frac{15}{7\sqrt{11}}\right)$$

2.6.16 Find the equations of the line through the point $(1, 2, 3)$ and parallel to the line $x-y+2z=5,\ 3x+y+z=6$.

Solution: Let $\pi_1 \equiv x-y+2z-5$

$\pi_2 \equiv 3x+y+z-6$

$\pi_1 = \pi_2 = 0$ represents a line

If $\overline{n_1} = (1, -1, 2)$ and $\overline{n_2} = (3,1,1)$ are the unit vectors along the normals of π_1 , π_2 respectively then $\overline{n_1} \times \overline{n_2}$ is the unit vector along the line L.

Also $\overline{n_1} \times \overline{n_2} = (-3, 5, 4)$

$\Rightarrow$ Equations of line through $(1, 2, 3)$ and parallel to L having d.rs. $-3, 5, 4$ are

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

2.6.17 Find the equation of the plane containing the line $x-y+3z+5=0=2x+y-2z+6$ and passing through the point $(3,1,1)$.

Solution: $\pi_1 \equiv x-y+3z+5=0$

$\pi_2 \equiv 2x+y-2z+6=0$ represent the line L.

Plane through the line L is same as the plane through intersection of π_1 and π_2 which is in the form $\pi_3 \equiv \lambda_1\pi_1 + \lambda_2\pi_2 = 0,$ for $(\lambda_1, \lambda_2) \neq (0, 0)$

The plane π_3 passes through the point $(3, 1,1)$

Hence $\lambda_1(3-1+3+5)+\lambda_2(6+1-2+6)=0$

$$10\lambda_1 + \lambda_2(11) = 0$$

$$\frac{\lambda_1}{\lambda_2} = \frac{-11}{10}$$

Equation of the plane π_3 is

$$-11(x-y+3z+5)+10(2x+y-2z+6)=0$$

$$9x+21y-53z+5=0$$

2.6.18 Find the equations of the plane containing the parallel lines

$$\frac{x-4}{1}=\frac{y-3}{-4}=\frac{z-2}{5}\cdots\cdots\cdots(1),\quad \frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}\cdots\cdots(2)$$

Solution: d.rs. of the given parallel lines are 1, -4, 5

Let a, b, c be d.rs. of normal to the required plane π

Since the plane π contains the lines (1) and (2)

$$a(1)+b(-4)+c(5)=0$$

$$\Rightarrow a-4b+5c=0\cdots\cdots\cdots\cdots\cdots(3)$$

Points on the plane are $A(4,3,2)$, $B=(3,-2,0)$

d.rs. of the line AB on the plane π are 1, 5, 2

Then $a+5b+2c=0\cdots\cdots\cdots\cdots(4)$

Solving (3) and (4)

$$\frac{a}{-8-25}=\frac{b}{5-2}=\frac{c}{5+4}$$

i.e., $$\frac{a}{-33}=\frac{b}{3}=\frac{c}{9}$$

i.e., $$\frac{a}{11}=\frac{b}{-1}=\frac{c}{-3}$$

Equation of π is

$$11(x-3)-1(y+2)-3(z)=0$$

$$11x-y-3z-35=0$$

2.6.19 Find the equation of the plane containing the line

$$\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}\cdots\cdots\cdots\cdots(1)$$

and perpendicular to the plane $x+2y+z-12=0$.

solution: Let π be the required plane and a, b, c be d.rs. of the normal to π

Since line (1) lies on π

$$2a - b + 4c = 0 \cdots\cdots(2)$$

Since π is perpendicular to $\pi_1 \equiv x + 2y + z - 12 = 0$

$$a + 2b + c = 0 \cdots\cdots(3)$$

From (2) and (3)

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1} \Rightarrow \frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$$

$A(1, -1, 3)$ lies in π.

Hence equation of π is

$$-9(x-1) + 2(y+1) + 5(z-3) = 0$$

$$9x - y - 5z + 4 = 0$$

2.6.20 Find the perpendicular distance of a point $(1, 6, 3)$ to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Solution: Let $A = (0,1,2)$, $P = (1, 6, 3)$, $\bar{b} = (1, 2, 3)$

$$\overline{AP} = (1, 5, 1)$$

$$\overline{AP} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 5 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 13\,\bar{i} - 2\bar{j} - 3\bar{k}$$

$$= (13, -2, -3)$$

Required perpendicular distance $= \frac{|\overline{AP} \times \bar{b}|}{|\bar{b}|} = \frac{\sqrt{13^2 + 4 + 9}}{\sqrt{1+4+9}}$

$$= \sqrt{\frac{182}{14}} = \sqrt{13}$$

2.6.21 Find the perpendicular distance from the point $3\bar{i} - 2\bar{j} + \bar{k}$ on the line joining the points $\bar{i} - 3\bar{j} + 5\bar{k}$, $2\bar{i} + \bar{j} - 4\bar{k}$.

Solution: Let $\overline{OP} = 3\bar{i} - 2\bar{j} + \bar{k}$, $\overline{OA} = \bar{i} - 3\bar{j} + 5\bar{k}$, $\overline{OB} = 2\bar{i} + \bar{j} - 4\bar{k}$

$\overline{AP} = \overline{OP} - \overline{OA} = 2\bar{i} + \bar{j} - 4\bar{k}$, $\overline{AB} = \overline{OB} - \overline{OA} = \bar{i} + 4\bar{j} - 9\bar{k}$

$$\overline{AP} \times \overline{AB} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 1 & -4 \\ 1 & 4 & -9 \end{vmatrix} = 7\bar{i} + 14\bar{j} + 7\bar{k}$$

$$\left|\overline{AP} \times \overline{AB}\right| = 7\sqrt{6} \quad , \quad \left|\overline{AB}\right| = \sqrt{1+16+81} = \sqrt{98} = 7\sqrt{2}$$

Perpendicular distance from P to $AB = \dfrac{\left|\overline{AP} \times \overline{AB}\right|}{\left|\overline{AB}\right|} = \dfrac{7\sqrt{6}}{7\sqrt{2}} = \sqrt{3}$

2.6.22 Find the perpendicular distance of the origin from the line L.

$$2x + 3y + 4z + 5 = 0 \cdots\cdots\cdots(1), \quad x + 2y + 3z + 4 = 0 \cdots\cdots(2)$$

Solution: Put $z = 0$ in the equations of planes that intersect in L.

$$2x + 3y + 5 = 0 \cdots\cdots\cdots(3)$$

$$x + 2y + 4 = 0 \cdots\cdots\cdots(4)$$

From (3) and (4) $x = 2$, $y = -3$

Hence a point on L is $A = (2, -3, 0)$

From (1) and (2) d.rs. of L are

$$\begin{array}{cccc} & a & b & c \\ 3 & 4 & 2 & 3 \\ 2 & 3 & 1 & 2 \end{array}$$

$$\frac{a}{1} = \frac{b}{-2} = \frac{c}{1}$$

The equations of L are $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z}{1}$

$B=(t+2,\,-2t-3,\,t)$ is any point on L

d.rs. of OB are $t+2,\,-2t-3,\,t$

If OB is perpendicular to L, then

$$1(t+2)-2(-2t-3)+t=0 \Rightarrow t=\frac{-4}{3}$$

$$B=\left(\frac{2}{3},\frac{-1}{3},\frac{-4}{3}\right)$$

Perpendicular distance of the origin from L is

$$OB=\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{16}{9}}=\sqrt{\frac{7}{3}}\text{ units}$$

2.6.23 S.A.Q.: Find the equation of the plane through the points $(1,0,-1)$, $(3,2,2)$ and parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$.

2.6.24 S.A.Q.: Find the equation of the plane through the line $\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and parallel to another line with d.cs. ℓ_2, m_2, n_2.

2.6.25 S.A.Q.: Find the equation of the plane through the line $\frac{x-x_2}{\ell}=\frac{y-y_2}{m}=\frac{z-z_2}{n}$ and through the point (x_1, y_1, z_1).

2.6.26 S.A.Q.: Find the equation of the plane which passes through the line $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ and is parallel to the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$.

2.6.27 S.A.Q.: Find the equation of the plane through the point (α, β, γ) and perpendicular to the straight line $L:\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

2.6.28 S.A.Q.: If ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 are direction cosines, show that the equations to the planes through the lines which bisect the angle between the lines $\frac{x}{\ell_1}=\frac{y}{m_1}=\frac{z}{n_1}$; $\frac{x}{\ell_2}=\frac{y}{m_2}=\frac{z}{n_2}$ and at right angles to the plane containing them are $(\ell_1 \pm \ell_2)\,x+(m_1 \pm m_2)\,y+(n_1 \pm n_2)\,z=0$.

2.6.29 S.A.Q.: A variable plane makes intercepts on the coordinate axes, the sum of whose squares is k^2 (a constant). Show that the locus of the foot of the perpendicular from the origin to the plane is $\left(x^{-2}+y^{-2}+z^{-2}\right)\left(x^2+y^2+z^2\right)^2=k^2$.

2.6.30 S.A.Q.: The plane $\ell x+my+nz=p,\ \ell^2+m^2+n^2=1,\ p>0$ meets the axes in P, Q, R and G is the centroid of $\Delta\,PQR$. If the perpendicular line to the plane at G meets the coordinate planes in A, B, C then prove that $\frac{1}{GA}+\frac{1}{GB}+\frac{1}{GC}=\frac{3}{p}$.

2.6.31 S.A.Q.: Find the equations of the projection of the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ on the plane $x+2y+z=6$.

2.7 Answers to S.A.Q.'s:

2.6.23 Let π be the required plane and a, b, c be d.rs. of the normal to the plane π.

$A=(1,0,-1),\ B=(3,2,2)$ are the points on π.

The line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$ is parallel to π

$\Rightarrow\ a-2b+3c=0\cdots\cdots\cdots(1)$

d.rs. of AB are 2, 2, 3

Since AB lies in π, $2a+2b+3c=0\cdots\cdots(2)$

From (1) and (2)

$$\frac{a}{-6-6}=\frac{b}{6-3}=\frac{c}{2+4}\ \text{ i.e., }\ \frac{a}{-12}=\frac{b}{3}=\frac{c}{6}\ \text{ i.e., }\ \frac{a}{+4}=\frac{b}{-1}=\frac{c}{-2}$$

$\Rightarrow$ Equation of π is $4(x-1)-1(y-0)-2(z+1)=0$

i.e., $4x-y-2z-6=0$

2.6.24 Let π be the required plane and a, b, c be the d.rs. of the normal to π.

π contains the line $\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and passes through the point $A=(x_1, y_1, z_1)$.

$\Rightarrow \quad \ell_1 a+m_1 b+n_1 c=0 \cdots\cdots\cdots\cdots\cdots(1)$

π is parallel to the line L with d.cs. ℓ_2, m_2, n_2

$\Rightarrow$ L is perpendicular to the normal of π

$\Rightarrow \ell_2 a+m_2 b+n_2 c=0 \cdots\cdots(2)$

From (1) and (2)

$$\frac{a}{m_1 n_2-m_2 n_1}=\frac{b}{n_1\ell_2-n_2\ell_1}=\frac{c}{\ell_1 m_2-\ell_2 m_1}$$

$\Rightarrow$ Equation of π is

$$(m_1 n_2-m_2 n_1)(x-x_1)+(n_1\ell_2-n_2\ell_1)(y-y_1)+(\ell_1 m_2-\ell_2 m_1)(z-z_1)=0$$

i.e. $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}=0$

2.6.25 Let π be the required plane and a, b, c be the d.rs. of normal to π.

π contains the line $\frac{x-x_2}{\ell}=\frac{y-y_2}{m}=\frac{z-z_2}{n}$ and passes through the points $A=(x_2, y_2, z_2)$, $B=(x_1, y_1, z_1)$.

$\Rightarrow \ell a+mb+nc=0 \cdots\cdots\cdots\cdots(1)$

Since d.rs. of the line AB are $x_1-x_2,\ y_1-y_2,\ z_1-z_2$

$$(x_1-x_2)a+(y_1-y_2)b+(z_1-z_2)c=0 \cdots\cdots\cdots(2)$$

Equation of π through A is

$$a(x-x_2)+b(y-y_2)+c(z-z_2)=0 \cdots\cdots\cdots(3)$$

Eliminating a, b, c from (1), (2) and (3) equation of the plane π is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ x_1-x_2 & y_1-y_2 & z_1-z_2 \\ \ell & m & n \end{vmatrix} = 0$$

2.6.26 Let $\pi_1 \equiv a_1x+b_1y+c_1z+d_1=0$

$\pi_2 \equiv a_2x+b_2y+c_2z+d_2=0$ together represent a line L_1. Let π be the required plane through L_1 and parallel to the line L_2, $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$.

The plane π through L_1 is the plane through the intersection of π_1 and π_2 which is of the form $\lambda_1\pi_1+\lambda_2\pi_2=0$, for $(\lambda_1, \lambda_2) \neq (0,0)$.

$\Rightarrow$ $\lambda_1(a_1x+b_1y+c_1z+d_1)+\lambda_2(a_2x+b_2y+c_2z+d_2)=0$

$\Rightarrow$ $(a_1\lambda_1+a_2\lambda_2)x+(b_1\lambda_1+b_2\lambda_2)y+(c_1\lambda_1+c_2\lambda_2)z+(d_1\lambda_1+d_2\lambda_2)=0$

This plane is parallel to the line L_2

$\Rightarrow$ $\ell(a_1\lambda_1+a_2\lambda_2)+m(b_1\lambda_1+b_2\lambda_2)+n(c_1\lambda_1+c_2\lambda_2)=0$

$\Rightarrow$ $\frac{\lambda_1}{\lambda_2}=\frac{-(\ell a_2+mb_2+nc_2)}{\ell a_1+mb_1+nc_1}$

Equation of the required plane is

$$(\ell a_2+mb_2+nc_2)(a_1x+b_1y+c_1z+d_1)-(\ell a_1+mb_1+nc_1)(a_2x+b_2y+c_2z+d_2)=0$$

2.6.27 Find the equation of the plane through the point (α, β, γ) and perpendicular to the straight line L : $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

Solution: Let π be the required plane through the point $A(\alpha, \beta, \gamma)$

Given line L is $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

d.rs. of L are ℓ, m, n

π is perpendicular to L

$\Leftrightarrow$ normal of π is parallel to L

$\Leftrightarrow$ d.rs. of normal of π and L are proportional.

d.rs. of normal to π are ℓ, m, n

Equation of the plane π is

$$\ell(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$

2.6.28 Let L_1 represent the line $\frac{x}{\ell_1}=\frac{y}{m_1}=\frac{z}{n_1}$

and L_2 represent the line $\frac{x}{\ell_2}=\frac{y}{m_2}=\frac{z}{n_2}$

where ℓ_1, m_1, n_1 ; ℓ_2, m_2, n_2 and d.cs. of L_1, L_2 respectively.

Then d.rs. of the lines L_3, L_4 bisecting the angles between L_1, L_2 are $\ell_1+\ell_2, m_1+m_2, n_1+n_2$; $\ell_1-\ell_2, m_1-m_2, n_1-n_2$.

We known that L_3, L_4 are perpendicular to each other.

Let π_1 , π_2 be the required planes through L_3, L_4 respectively and are perpendicular to L_4, L_3 respectively.

π_1 , π_2 pass through origin.

Equation of π_1 is $(\ell_1-\ell_2)x+(m_1-m_2)y+(n_1-n_2)z=0$

Equation of π_2 is $(\ell_1+\ell_2)x+(m_1+m_2)y+(n_1+n_2)z=0$

Hence Equations of the required planes are

$$(\ell_1 \pm \ell_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0.$$

2.6.29 Let the equation of the variable plane π that have x intercept a, y intercept b, z intercept c respectively be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \cdots\cdots (1)$$

It is given that $a^2+b^2+c^2=k^2$

d.rs. of the normal to the plane π are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

Equation of the line passing through the origin and perpendicular to the plane is $\frac{x-0}{\frac{1}{a}} = \frac{y-0}{\frac{1}{b}} = \frac{z-0}{\frac{1}{c}} = t\ (\text{say}) \cdots\cdots (2)$

Any point on the line (2) is $Q = \left(\frac{t}{a}, \frac{t}{b}, \frac{t}{c}\right)$

If Q lies in the plane π, then $\frac{t}{a^2} + \frac{t}{b^2} + \frac{t}{c^2} = 1$

$$\Rightarrow t = \frac{1}{a^{-2} + b^{-2} + c^{-2}}$$

Let $P(x_1, y_1, z_1)$ be the foot of the perpendicular of the origin to π. Then $(x_1, y_1, z_1) = \left(\frac{t}{a}, \frac{t}{b}, \frac{t}{c}\right)$

$$x_1^2 + y_1^2 + z_1^2 = t^2\left(a^{-2} + b^{-2} + c^{-2}\right) = \frac{a^{-2} + b^{-2} + c^{-2}}{\left(a^{-2} + b^{-2} + c^{-2}\right)^2} = t$$

$$\frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2} = \frac{a^2 + b^2 + c^2}{t^2} = \frac{k^2}{t^2}$$

$$\left(x_1^2 + y_1^2 + z_1^2\right)^2 \left(x_1^{-2} + y_1^{-2} + z_1^{-2}\right) = t^2 \cdot \frac{k^2}{t^2} = k^2$$

$\Rightarrow$ $P(x_1, y_1, z_1)$ satisfies

$$\left(x^2 + y^2 + z^2\right)^2 \left(x^{-2} + y^{-2} + z^{-2}\right) = k^2$$

Conversely suppose that $P(x_1, y_1, z_1)$ satisfies the equation

$$\left(x^2 + y^2 + z^2\right)\left(x^{-2} + y^{-2} + z^{-2}\right) = k^2$$

$$\Rightarrow \left(x_1^2+y_1^2+z_1^2\right)\left(x_1^{-2}+y_1^{-2}+z_1^{-2}\right)=k^2 \cdots\cdots\cdots(1)$$

If P is the foot of the perpendicular from the origin to the plane, then equation of that plane through P and having $x_1-0,\ y_1-0,\ z_1-0$ as d.rs. of the normal is

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$$

i.e., $x_1x+y_1y+z_1z = x_1^2+y_1^2+z_1^2$

i.e., $$\frac{x}{\left(\frac{x_1^2+y_1^2+z_1^2}{x_1}\right)}+\frac{y}{\left(\frac{x_1^2+y_1^2+z_1^2}{y_1}\right)}+\frac{z}{\left(\frac{x_1^2+y_1^2+z_1^2}{z_1}\right)}=1$$

$x_1\neq 0,\ y_1\neq 0$ and $z_1\neq 0$

The sum of the squares of the intercepts

$$=\left(\frac{x_1^2+y_1^2+z_1^2}{x_1}\right)^2+\left(\frac{x_1^2+y_1^2+z_1^2}{y_1}\right)^2+\left(\frac{x_1^2+y_1^2+z_1^2}{z_1}\right)^2$$

$$=\left(x_1^2+y_1^2+z_1^2\right)^2\left(\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}\right)$$

$$=\left(x_1^2+y_1^2+z_1^2\right)^2\left(x_1^{-2}+y_1^{-2}+z_1^{-2}\right)=k^2$$

2.6.30 Let π be the plane $\ell x+my+nz=p,\ \ \ell^2+m^2+n^2=1,\ p>0$. π meets X - axis, Y - axis and Z - axis in P, Q and R respectively and G be the centroid of $\Delta\,PQR$.

$$P=\left(\frac{p}{\ell},0,0\right),\ Q=\left(0,\frac{p}{m},0\right),\ R=\left(0,0,\frac{p}{n}\right)$$

$$\Rightarrow\ G=\left(\frac{p}{3\ell},\frac{p}{3m},\frac{p}{3n}\right)$$

Let L be the line perpendicular to π at G.

The equation of L is $\dfrac{x-\frac{p}{3\ell}}{\ell}=\dfrac{y-\frac{p}{3m}}{m}=\dfrac{z-\frac{p}{3n}}{n}=t\ (\text{say})$

L meets the YZ - plane i.e. x = 0 in A.

Hence $GA = \left|\frac{0 - p/3\ell}{\ell}\right| = \left|\frac{-p}{3\ell^2}\right|$

$$\Rightarrow \frac{1}{GA} = \frac{3\ell^2}{p}, \quad (p>0)$$

Similarly $\frac{1}{GB} = \frac{3m^2}{p}, \quad \frac{1}{GC} = \frac{3n^2}{p}$

Hence $\frac{1}{GA} + \frac{1}{GB} + \frac{1}{GC} = \frac{3\ell^2 + 3m^2 + 3n^2}{p} = \frac{3}{p}$

2.6.31 The equation of the given plane is $x + 2y + z = 6 \cdots\cdots (1)$

The equation of the given line L is $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} = t \quad (\text{say})$

The equation of any plane through the line L is

$a(x-1) + b(y+1) + c(z-3) = 0 \cdots\cdots (2)$ where a, b, c are d.rs. of normal to the plane.

Since L lies i n (2) $2a - b + 4c = 0 \cdots\cdots\cdots (3)$

If the plane (2) is perpendicular to the plane (1) then

$$a + 2b + c = 0 \cdots\cdots\cdots\cdots (4)$$

Solving (3) and (4)

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1}$$

i.e. $\frac{a}{-9} = \frac{b}{2} = \frac{c}{5}$

From (2) we get

$$-9(x-1) + 2(y+1) + 5(z-3) = 0$$

i.e. $9x - 2y - 5z + 4 = 0 \cdots\cdots\cdots (5)$

Hence (1) and (5) together represent the line of projection of L on (1).

2.8 Summary:

After reading the lesson and working out the examples and exercises the student should get working knowledge on finding the equation of a line in various forms, angles between lines and planes and perpendicular distance from a point to a line.

2.9 Technical Terms:

(i) Symmetric form

(ii) Foot of the perpendicular

(iii) Projection

2.10 Model Examination Questions:

1. Show that the line joining $(2,-3,1),(3,-4,-5)$ intersects the plane $2x+y+z=7$ in the points $(1,-2,7)$.

2. Show that the line $\frac{x-3}{3}=\frac{2-y}{4}=\frac{z+1}{1}$ intersects the line $2x+4y+3z+3=0,\ x+2y+3z=0$ in the point $(9,-6,1)$.

3. Find the distance of the point $(3,-4,5)$ from the plane $2x+5y-6z=16$ measured along a line with d.cs. proportional to $(2,1,-2)$.

4. Find the equation to the plane through (x_1, y_1, z_1) and parallel to the lines $\frac{x}{\ell_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{\ell_2}=\frac{y}{m_2}=\frac{z}{n_2}$.

5. Show that the equations of the perpendicular from the point $(1,6,3)$ to the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ are $x-1=0,\ \frac{y-6}{-3}=\frac{z-3}{2}$ and the foot of the perpendicular is $(1,3,5)$ and the length of the perpendicular is $\sqrt{(13)}$.

6. Find the equation of the line of projection of the line.

$$\frac{x-1}{2}=\frac{y}{-1}=\frac{z+2}{1} \text{ in the plane } 2x+y-3z-4=0.$$

7. Show that the line $\frac{x+1}{-1}=\frac{y+2}{3}=\frac{z+5}{5}$ lies in the plane $x+2y-z=0$.

8. Show that the line $\frac{x+3}{-2}=\frac{y+4}{3}=\frac{z+4}{5}$ is perpendicular to the plane $-2x+3y+5z=0$.

9. Find the equation of the plane containing the parallel lines

$$\frac{x-3}{4}=\frac{y-2}{-5}=\frac{z-4}{-1} \text{ and } \frac{x+2}{-4}=\frac{y}{5}=\frac{z-3}{1}$$

10. Find the perpendicular distance of the point $(2, 4, -1)$ from the line through the points $(-5,-3,6)$ whose d.rs. are 1, 4, 9.

2.11 Exercise:

1. Find the point of intersection of the line $\frac{x-2}{1}=\frac{y-2}{1}=\frac{z+3}{-4}$ and the plane $x+y+z-3=0$.

Ans: (1, 1, 1)

2. Find the point of intersection of the line through $(2, -3, 1)$, $(3, -4, 5)$ with the plane $2x+y+z=7$.

Ans: (3, -4, 5)

3. Find the distance of the point $(1, 3, 4)$ from the plane $2x-y+z=3$ measured parallel to the line $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-1}$.

Ans: 0

4. Find the distance of the point $(-1, -5, -10)$ and the point of intersection of the line $\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=5$.

Ans: 13

5. Find the foot of the perpendicular and perpendicular distance from $(3, -1, 11)$ to the join of the points $(0, 2, 3)$, $(4, 8, 11)$.

Ans: $(2, 5, 7)$, $\sqrt{53}$.

6. Find in symmetric form the equations of the line $3x+2y-z-4=0$, $4x+y-2z+3=0$.

Ans: $\frac{x+2}{-3}=\frac{y-5}{2}=\frac{z}{-5}$

7. Find the equation of the plane through the point $(1,1,1)$ and perpendicular to the line $x-y+z=2,\ 4x+3y-z+1=0$.

Ans: $x-5y-11z+15=0$.

8. Find the foot and equation of the perpendicular from $(2,4,-1)$ to $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$.

Ans: $(-4,1,-3)$, $\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$.

9. Find the angle between the lines $x-2y+z=0=x+y-z-3$; $x+2y+z-5=0=8x+12y+5z$.

Ans: $\text{Cos}^{-1}\left(\frac{8}{\sqrt{406}}\right)$

10. Show that the lines $2x+3y-4z=0=3x-4y+z-7$;

$5x-y-3z+12=0=x-7y+5z-6$ are parallel.

11. Show that the line $\frac{x-2}{1}=\frac{y-2}{2}=\frac{z+1}{3}$ lies in the plane $5x+2y-3z-17=0$.

12. Show that the line $\frac{x-1}{1}=\frac{y-2}{-4}=\frac{z-3}{3}$ is parallel to $6x+3y+2z-6=0$.

13. Show that the line $\frac{x-2}{1}=\frac{y-2}{-2}=\frac{z+3}{1}$ is perpendicular to $x-2y+z+5=0$.

14. Find the equations of the line through the point (x_1, y_1, z_1) and parallel to the line $\ell_1x+m_1y+n_1z=p_1$, $\ell_2x+m_2y+n_2z=p_2$.

Ans: $\frac{x-x_1}{m_1n_2-m_2n_1}=\frac{y-y_1}{n_1\ell_2-n_2\ell_1}=\frac{z-z_1}{\ell_1m_2-\ell_2m_1}$

15. Find the equation of the plane containing the line $4x-3y+5=0=y-2z-5$ through the point $(2,-1,1)$.

Ans: $4x-y-4z=0$.

16. Find the equation of the plane containing the parallel lines $\frac{x-3}{4}=\frac{y-2}{-5}=\frac{z-4}{-1}$; $\frac{x+2}{-4}=\frac{y}{5}=\frac{z-3}{1}$.

Ans: $x+3y-11z+35=0$.

17. Find the equations of the plane through the points $(1,0,-1)$, $(0,-8,1)$ and parallel to the line $6(x+1)=3-3y=2z+4$.

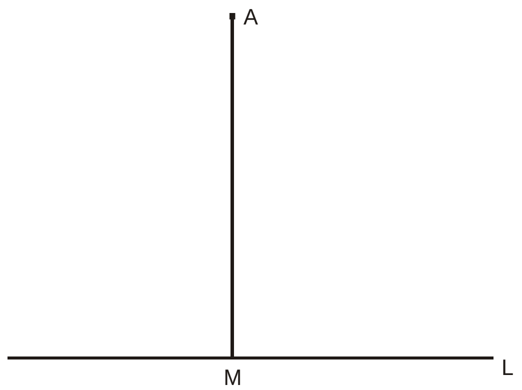
Ans: $4x-y-2z-6=0$.

2.12 Model Practical Problem with Solution:

Problem: Find the foot of the perpendicular and perpendicular distance from the point $(-1, 3, 9)$ to the line $\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-31}{1}$.

Definitions:

1. Foot of the perpendicular of a point:



If L is a line and A is any point, the point of intersection, M of the perpendicular line through A to L is called the foot of the perpendicular from A to L and AM is called the perpendicular distance from A to L.

Results Used:

1. (i) The foot of the perpendicular from $A(\alpha, \beta, \gamma)$ to the line

$$L : \frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \text{ is}$$

$$Q = (x_1 + \ell t_1, y_1 + mt_1, z_1 + nt_1)$$

where $t_1 = \dfrac{\ell(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{\ell^2 + m^2 + n^2}$

(ii) The equation of the perpendicular line AQ is

$$\frac{x-\alpha}{x_1 - \alpha - \ell t_1} = \frac{y-\beta}{y_1 - \beta - m t_1} = \frac{z-\gamma}{z_1 - \gamma - n t_1}$$

(iii) and the length AQ is

$$AQ = \sqrt{(x_1 - \alpha - \ell t_1)^2 + (y_1 - \beta - mt_1)^2 + (z_1 - \gamma - nt_1)^2}$$

Solution:

Given line is $\dfrac{x-13}{5} = \dfrac{y+8}{-8} = \dfrac{z-31}{1} = t \; (\text{say}) \cdots\cdots\cdots\cdots\cdots (1)$

Any point on line (1) is $P = (5t + 13, \; -8t - 8, \; t + 31)$

If P is the foot of the perpendicular from $A(-1, 3, 9)$ to the line (1), then P lies on the line (1).

d.rs. of AP are $5t + 14, \; -8t - 11, \; t + 22$

d.rs. of the line (1) are $5, -8, 1$.

Thus AP is perpendicular to the line (1).

$\Leftrightarrow \; 5(5t + 14) - 8(-8t - 11) + 1(t + 22) = 0$

$\Leftrightarrow \; 25t + 70 + 64t + 88 + t + 22 = 0 \; \Leftrightarrow \; 90t + 180 = 0 \; \Leftrightarrow \; t = -2$

Hence the foot of the perpendicular from A to the line (1) is

$$P = (-10 + 13, 16 - 8, -2 + 31) = (3, 8, 29)$$

The perpendicular distance from A to the line (1) is

$$AP = \sqrt{(3+1)^2 + (8-3)^2 + (29-9)^2} = \sqrt{16 + 25 + 400} = \sqrt{441} = 21 \text{ units.}$$

Lesson Writer

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Lesson - 3

LINES - II

3.1 Objective of the lesson:

In this lesson the student is introduced to find a image of a point and a line with respect to a plane, and derive the conditions for coplanarity of lines. The expressions for the shortest distance and the equations of the line of the shortest distance and the nature of intersections of three planes are studied.

3.2 Structure:

This lesson contains the following components:

3.3 Introduction

3.4 Images

3.5 Coplanar Lines

3.6 Shortest Distance Between Two Skew Lines

3.7 Intersection of Three Planes

3.8 Answers to SAQ

3.9 Summary

3.10 Technical Terms

3.11 Model Examination Questions

3.12 Exercises

3.13 Model Practical Problem with Solution

3.3 Introduction:

The concepts of the foot of the perpendicular from a point enable us to find out a image of a point and a line with respect to a plane. We then study the nature of the lines in space and derive the conditions for coplanarity of lines. Using these facts and conditions we find out the expression for the shortest distance between any two lines. Finally making use of the concepts and conditions from Lesson 1 and Lesson 2 we study the nature of the intersection of the planes which enables us to obtain characterizations for intersections of three planes.

3.4 Images:

We define the image of a point in a line and also in a plane.

3.4.1 Definition: Image of a point in a plane (Line) :

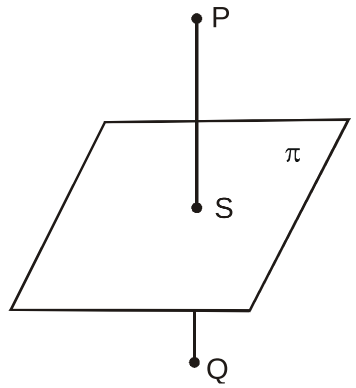
Let W be a line or a plane and P be a point not in W and S the foot of perpendicular of P in W. If Q is the point on the line joining P and S such that $P \neq Q$ and $PS = SQ$, then Q is called the image of P in W.

Note: If P belongs to W then $P = S$ so $Q = S = P$. Hence the image of P is itself. Moreover if the image of P is Q, then the image Q is P.

3.4.2 Theorem: If $Q(x_2, y_2, z_2)$ is the image of the point $P(x_1, y_1, z_1)$ with respect to the plane $\pi \equiv ax + by + cz + d = 0$, then

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

Proof: Since the d.rs. of the line PQ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and PQ is parallel to the normal of the plane π.



$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = t \ (\text{say})$$

$$\Rightarrow x_2 = x_1 + at, \ y_2 = y_1 + bt, \ z_2 = z_1 + ct$$

The mid point of PQ say $S = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

S lies on $\pi \Rightarrow a\left(\frac{x_1 + x_2}{2}\right) + b\left(\frac{y_1 + y_2}{2}\right) + c\left(\frac{z_1 + z_2}{2}\right) + d = 0$

$$\Rightarrow ax_1 + by_1 + cz_1 + ax_2 + by_2 + cz_2 + 2d = 0$$

$$\Rightarrow \; ax_1 + by_1 + cz_1 + a(x_1 + at) + b(y_1 + bt) + c(z_1 + ct) + 2d = 0$$

$$\Rightarrow \; 2(ax_1 + by_1 + cz_1 + d) + t(a^2 + b^2 + c^2) = 0$$

$$\Rightarrow \qquad t = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

The point $(x_1 + at,\ y_1 + bt,\ z_1 + ct)$ where

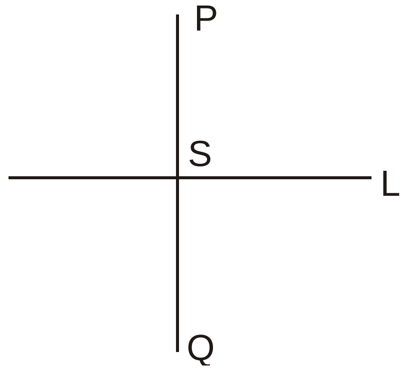
$$t = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

is the image of $P(x_1, y_1, z_1)$ with respect to the plane π.

3.4.3 Theorem: If $Q(x_2, y_2, z_2)$ is the image of the point $P(x_1, y_1, z_1)$ with respect to the line $L: \frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-r}{n}$, then $x_2 = 2(\alpha + \ell t) - x_1,\ y_2 = 2(\beta + mt) - y_1,\ z_2 = 2(r + nt) - z_1$.

Where $t = -\left\{\dfrac{\ell(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{\ell^2 + m^2 + n^2}\right\}$

Proof:



$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = t \quad (\text{say})$$

$\Rightarrow$ any point on the line L is given by

$(\alpha + \ell t, \beta + mt, \gamma + nt)$ where $t \in \mathbb{R}$

If $S = (\alpha + \ell t, \beta + mt, \gamma + nt)$ is mid point of PQ, then

$$\alpha + \ell t = \frac{x_1 + x_2}{2},\ \beta + mt = \frac{y_1 + y_2}{2},\ \gamma + nt = \frac{z_1 + z_2}{2}$$

$$\Rightarrow x_2 = 2(\alpha + \ell t) - x_1,\ y_2 = 2(\beta + mt) - y_1,\ z_2 = 2(\gamma + nt) - z_1$$

d.rs. of PQ are $2(\alpha + \ell t - x_1),\ 2(\beta + mt - y_1),\ 2(\gamma + nt - z_1)$

Since PQ is perpendicular to L,

$$2\ell(\alpha + \ell t - x_1) + 2m(\beta + mt - y_1) + 2n(\gamma + nt - z_1) = 0$$

$$\Rightarrow \ell(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1) + t(\ell^2 + m^2 + n^2) = 0$$

$$\Rightarrow\ t = -\frac{\ell(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{\ell^2 + m^2 + n^2}$$

for this value of t the point

$$(2(\alpha + \ell t) - x_1,\ 2(\beta + mt) - y_1,\ 2(\gamma + nt) - z_1)$$

is the image of P with respect of L.

Image of a line in a plane π:

3.4.4 Definition: The locus of the images of the points on L with respect to the plane π is called the image of the line L in a plane π.

3.4.5 Theorem: the image of the line $L : \frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ with respect to the plane $\pi \equiv ax + by + cz + d = 0$.

(i) is $\frac{x - \alpha}{\ell} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$ when L is parallel to π.

where $\alpha = x_1 + at_0,\ \beta = y_1 + bt_0,\ \gamma = z_1 + ct_0$

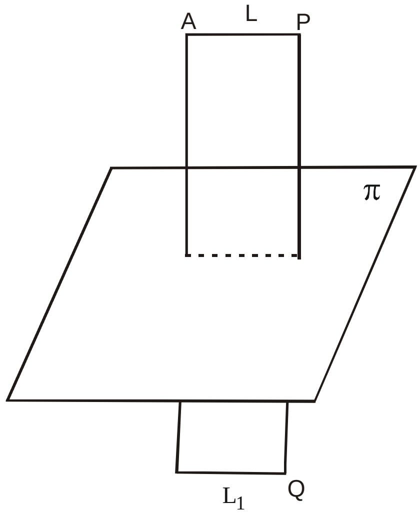
and $t_0 = \dfrac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(ii) and when L is not parallel to π, then the image is $\frac{x-\alpha}{\alpha_1-\alpha}=\frac{y-\beta}{\beta_1-\beta}=\frac{z-\gamma}{\gamma_1-\gamma}$

where $\alpha_1 = x_1+\ell t,\ \beta_1 = y_1+mt,\ \gamma_1 = z_1+nt$

$t=\frac{-(ax_1+by_1+cz_1+d)}{a\ell+bm+cn}$ and α, β, γ are same as in (i)

Proof: Case (i): Let $Q=(\alpha, \beta, \gamma)$ be the image of P with respect to π, where $P=(x_1, y_1, z_1)$



From theorem image of P on L with respect to π,

$Q=(x_1+at_0,\ y_1+bt_0,\ z_1+ct_0)$

where $t_0=\frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

Let L_1 be the image of L with respect to π.

L is parallel to L_1

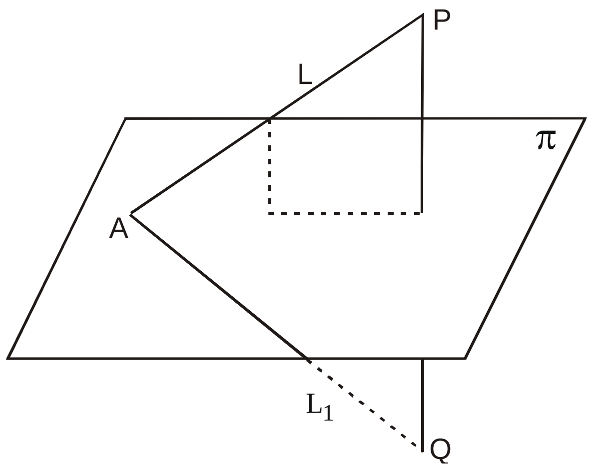
$\Rightarrow$ d.rs. of L_1 are ℓ, m, n.

Equation of the line L_1 is

$$\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Case (ii): Assume that $a\ell + bm + cn \neq 0$ and L intersects π at A.

$$A = (x_1 + \ell t,\ y_1 + mt,\ z_1 + nt) = (\alpha_1, \beta_1, \gamma_1) \text{ (say)}$$



Since A lies in π, then

$$a(x_1 + \ell t_0) + b(y_1 + mt_0) + c(z_1 + nt_0) + d = 0$$

$$\Rightarrow t_1 = \frac{-(ax_1 + by_1 + cz_1 + d)}{a\ell + bm + cn}$$

The image of the point P is Q (α, β, γ) as in case (i)

Hence AQ is the image of AP with respect to the plane π.

If $A = (\alpha_1, \beta_1, \gamma_1)$ and $Q = (\alpha, \beta, \gamma)$, then the equation of the line AQ is

$$\frac{x-\alpha}{\alpha_1 - \alpha} = \frac{y-\beta}{\beta_1 - \beta} = \frac{z-\gamma}{\gamma_1 - \gamma}$$

Examples:

3.4.6 Find the image of the point $A(2,-1,3)$ in the plane $\pi \equiv 3x - 2y - z = 9$.

Solution: D.rs. of the normal to the plane π are 3, -2, -1.

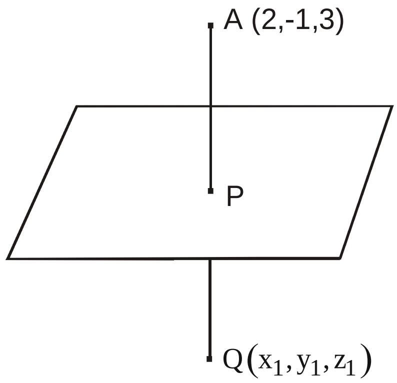
The equations of the line perpendicular to π and passing through A is

$$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{-1}=t\cdots\cdots\cdots(1)$$

Any point on the line (1) is $P=(3t+2,\,-2t-1,\,-t+3)\ (t\in\mathbb{R})$

If P is the foot of the perpendicular from A to π, then P lies on π.

$$3(3t+2)-2(-2t-1)-1(-t+3)=9$$



$$\Leftrightarrow\quad 9t+6+4t+2+t-3-9=0\quad\Leftrightarrow\quad t=\frac{2}{7}$$

$$P=\left(\frac{6}{7}+2,\ \frac{-4}{7}-1,\ \frac{-2}{7}+3\right)\quad=\left(\frac{20}{7},\ \frac{-11}{7},\ \frac{19}{7}\right)$$

Let Q be the image of A in π.

$$Q=\left(\frac{40}{7}-2,\ \frac{-22}{7}+1,\ \frac{38}{7}-3\right)\quad=\left(\frac{26}{7},\ \frac{-15}{7},\ \frac{17}{7}\right)$$

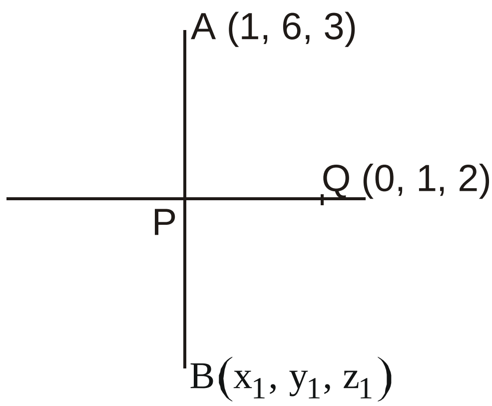
3.4.7 Find the image of the point $A\,(1,6,3)$ in the line

$$L:\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}\cdots\cdots\cdots\cdots(1)$$

Solution:

Any point on the line (1) is of the form

$$P=(t,\ 2t+1,\ 3t+2)\qquad(t\in\mathbb{R})$$



If P is the foot of the perpendicular from A (1, 6, 3) in the line (1), then P lies in (1). AP is perpendicular to L.

Since d.rs. of AP are $t-1,\ 2t-5,\ 3t-1$ and

d.rs. of (1) are 1, 2, 3.

and AP is perpendicular to L

$1(t-1)+2(2t-5)+3(3t-1)=0 \Rightarrow 14t-14=0 \Rightarrow t=1 \Rightarrow P=(1,3,5)$.

Let $B=(x_1, y_1, z_1)$ be the image of A in (1)

$\Rightarrow$ P is mid point of AB.

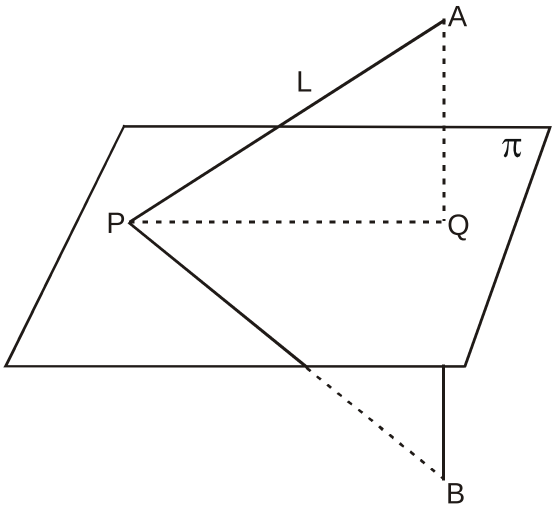
$\Rightarrow B=(2(1)-1,\ 2(3)-6,\ 2(5)-3)=(1,0,7)$

3.4.8 Find the image of the line

$L: \frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ in the plane $\pi \equiv x+y+z-1=0$

Solution: $A=(1,2,3)$ lies on L. Let B be the image of A in π and Q be the mid point of AB.

$B=(1+t_0,\ 2+t_0, 3+t_0)$



where $t_0 = \dfrac{-2(1+2+3-1)}{1+1+1} = \dfrac{-10}{3}$, $B = \left(\dfrac{-7}{3}, \dfrac{-4}{3}, \dfrac{-1}{3}\right)$ $P = (1+2t,\ 2+3t,\ 3+4t)$

where $t = \dfrac{-(1+2+3-1)}{2+3+4} = \dfrac{-5}{9}$, $P = \left(\dfrac{-1}{9}, \dfrac{1}{3}, \dfrac{7}{9}\right)$

PB is the image of PA in π

Also d.rs. of PB are $-\dfrac{1}{9}+\dfrac{7}{3},\ \dfrac{1}{3}+\dfrac{4}{3},\ \dfrac{7}{9}+\dfrac{1}{3}$ i.e. $\dfrac{20}{9}, \dfrac{15}{9}, \dfrac{10}{9}$

These are proportional to 4, 3, 2. $\Rightarrow$ equations of B are $\dfrac{x+\frac{7}{3}}{4} = \dfrac{y+\frac{4}{3}}{3} = \dfrac{z+\frac{1}{3}}{2}$

3.5 Coplanar Lines:

Two lines are either in a plane i.e. coplanar or not in the same plane. If the lines are coplanar they are either intersecting or parallel. If they do not intersect and are not parallel they are not coplanar. Let us recall that three vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar if and only if $[\bar{a}\ \bar{b}\ \bar{c}] = 0$. We use this fact to derive conditions for coplanarity of two lines.

3.5.1 Theorem: Let L_1, L_2 be two lines whose equations are respectively

$$\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ then}$$

L_1, L_2 are coplanar iff $\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

Proof: Let $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$. Then A lies in L_1 and B lies in L_2 and

$$\overline{BA} = (x_1 - x_2, \quad y_1 - y_2, \quad z_1 - z_2)$$

Let $\overline{n_1} = (\ell_1, m_1, n_1)$, $\overline{n_2} = (\ell_2, m_2, n_2)$. Then $\overline{n_1}$ is parallel to L_1 and $\overline{n_2}$ is parallel to L_2.

L_1, L_2 are coplanar $\Leftrightarrow$ $\overline{BA}, \overline{n_1}, \overline{n_2}$ are coplanar $\Leftrightarrow \left[\overline{BA} \quad \overline{n_1} \quad \overline{n_2}\right] = 0$

$$\Leftrightarrow \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

3.5.2 Corollary: The lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + t\bar{d}$ are coplanar

$\Leftrightarrow$ $\bar{a} - \bar{c}, \bar{b}, \bar{d}$ are coplanar $\Leftrightarrow \left[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}\right] = 0 \Leftrightarrow \left[\bar{a}\, \bar{b}\, \bar{d}\right] = \left[\bar{c}\, \bar{b}\, \bar{d}\right]$

3.5.3 Theorem: If L_1 and L_2 are the two lines whose equations are respectively $\dfrac{x-x_1}{\ell_1} = \dfrac{y-y_1}{m_1} = \dfrac{z-z_1}{n_1}$ and $\dfrac{x-x_2}{\ell_2} = \dfrac{y-y_2}{m_2} = \dfrac{z-z_2}{n_2}$, then the equations of the plane containing L_1 and parallel to L_2 are

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$$

Proof: Let $A = (x_1, y_1, z_1)$ $\overline{n_1} = (\ell_1, m_1, n_1)$, $\overline{n_2} = (\ell_2, m_2, n_2)$

Let π be the required plane containing L_1 and parallel to L_2.

Let $P\,(x, y, z)$ be any point in the plane π.

A lies in the plane π and $\overline{n_1}$, $\overline{n_2}$ are parallel to the lines L_1, L_2 respectively.

$\overline{AP}$, $\overline{n_1}$, $\overline{n_2}$ are coplanar $\Leftrightarrow \left[\overline{AP}\ \ \overline{n_1}\ \ \overline{n_2}\right] = 0$

$\Leftrightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$ is the equation of π.

3.5.4 Corollary: If L_1 and L_2 are two coplanar lines whose equations are respectively $\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ then the equation of the plane containing L_1, L_2 is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0$

3.5.5 Theorem: The lines $\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$

and $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ are coplanar if and only if

$$\frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{a_1\ell + b_1m + c_1n} = \frac{a_2x_1 + b_2y_1 + c_2z_1 + d_2}{a_2\ell + b_2m + c_2n}$$

Proof: Equation of any plane passing through the line,

$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ is

$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$

i.e., $(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + d_1 + \lambda d_2 = 0 \cdots\cdots\cdots(1)$

The line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in the plane (1)

$\Leftrightarrow (a_1 + \lambda a_2)\ell + (b_1 + \lambda b_2)m + (c_1 + \lambda c_2)n = 0$

and $(a_1 + \lambda a_2)x_1 + (b_1 + \lambda b_2)y_1 + (c_1 + \lambda c_2)z_1 + d_1 + \lambda d_2 = 0$

$\Leftrightarrow (a_1\ell + b_1m + c_1n) + \lambda(a_2\ell + b_2m + c_2n) = 0$

and $(a_1x_1 + b_1y_1 + c_1z_1 + d_1) + \lambda(a_2x_1 + b_2y_1 + c_2z_1 + d_2) = 0$

Taking the ratio $\lambda : 1$ from both the equations

$$\frac{-(a_1\ell + b_1m + c_1n)}{a_2\ell + b_2m + c_2n} = \frac{-(a_1x_1 + b_1y_1 + c_1z_1 + d_1)}{a_2x_1 + b_2y_1 + c_2z_1 + d_2}$$

$$\frac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{a_1\ell + b_1m + c_1n} = \frac{a_2x_1 + b_2y_1 + c_2z_1 + d_2}{a_2\ell + b_2m + c_2n}$$

3.5.6 Theorem: The lines $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ and

$a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4$ are coplanar iff

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

Proof: Let $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ represent a line L_1 and

$a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4$ represent a line L_2.

Equation of the plane passing through the line L_1 is

$$a_1x + b_1y + c_1z + d_1 + \lambda_1(a_2x + b_2y + c_2z + d_2) = 0$$

$$\Rightarrow (a_1 + \lambda_1a_2)x + (b_1 + \lambda_1b_2)y + (c_1 + \lambda_1c_2)z + d_1 + \lambda_1d_2 = 0 \cdots\cdots\cdots(1)$$

Equation of the plane passing through the line L_2 is

$$a_3x + b_3y + c_3z + d_3 + \lambda_2(a_4x + b_4y + c_4z + d_4) = 0$$

$$\Rightarrow (a_3 + \lambda_2a_4)x + (b_3 + \lambda_2b_4)y + (c_3 + \lambda_2c_4)z + (d_3 + \lambda_2d_4) = 0 \cdots\cdots\cdots\cdots(2)$$

Given lines are coplanar

$$\Leftrightarrow a_1 + \lambda_1a_2 = \lambda(a_3 + \lambda_2a_4), \quad b_1 + \lambda_1b_2 = \lambda(b_3 + \lambda_2b_4)$$

$c_1 + \lambda_1c_2 = \lambda(c_3 + \lambda_2c_4), \quad d_1 + \lambda_1d_2 = \lambda(d_3 + \lambda_2d_4)$ for some $\lambda \neq 0$

$\Leftrightarrow$ The system of equations

$$a_1 + \lambda_1a_2 - \lambda a_3 - \lambda\lambda_2a_4 - 0, \quad b_1 + \lambda_1b_2 - \lambda b_3 - \lambda\lambda_2b_4 = 0$$

$$c_1 + \lambda_1 c_2 - \lambda c_3 - \lambda\lambda_2 c_4 = 0, \quad d_1 + \lambda_1 d_2 - \lambda d_3 - \lambda\lambda_2 d_4 = 0$$

has non zero solution

$$\Leftrightarrow \; \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

3.5.7 Determination of lines satisfuing given conditions:

The general equations of a line are

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} \cdots\cdots\cdots (1), \quad \frac{y - y_1}{m} = \frac{z - z_1}{n} \cdots\cdots\cdots\cdots (2).$$

Since one of ℓ, m in (1) and m, n in (2) is not zero we may assume without loss of generality that $m \neq 0$ in (1) and $n \neq 0$ in (2). The equations (1) and (2) are equivalent to

$$x = \frac{\ell}{m} y + \frac{(mx_1 - \ell y_1)}{m} \; ; \; y = \frac{m}{n} z + \frac{(ny_1 - mz_1)}{n}$$

respectively so that $\frac{\ell}{m}, \frac{m}{n}, \frac{mx_1 - \ell y_1}{m}, \frac{ny_1 - mz_1}{n}$ are

the four arbitary constants or parameters. These four ratios are usually determined by various sets of conditions.

For Example:

(1) Passing through a given point and intersecting two given lines.

(2) Intersecting two given lines and having a given direction.

(3) Intersecting a given line at right angles and passing through a given point.

(4) Intersecting two given lines at right angles.

Examples:

3.5.8 Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Find the point of intersection and the plane containing the lines.

Solution: Let the line L_1 be $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r\ (say)\cdots\cdots\cdots\cdots(1)$

and the line L_2 be $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}=s\ (say)\cdots\cdots\cdots(2)$

Any point on L_1 is $P=(2r+1,\ 3r+2,\ 4r+3)$

Any point on L_2 is $Q=(3s+2,\ 4s+3,\ 5s+4)$

If P = Q then $2r+1=3s+2 \Rightarrow 2r-3s=1\cdots\cdots\cdots\cdots(3)$

$3r+2=4s+3 \Rightarrow 3r-4s=1\cdots\cdots\cdots\cdots\cdots(4)$

$4r+3=5s+4 \Rightarrow 4r-5s=1\cdots\cdots\cdots\cdots(5)$

Solving (3) and (4) we get $r=s=-1$

These values satisfy theequation (5)

Hence the lines L_1 and L_2 are intersecting $\Rightarrow$ L_1 and L_2 are coplanar.

The point of intersection of L_1 and L_2 is $(-1,-1,-1)$

Let π be the plane containing the lines L_1 & L_2 and a, b, c be d.rs. of normal to π.

d.rs. of L_1 are 2,3,4 $\Rightarrow 2a+3b+4c=0\cdots\cdots\cdots\cdots(6)$

d.rs. of L_2 are 3, 4, 5 $\Rightarrow 3a+4b+5c=0\cdots\cdots\cdots\cdots(7)$

From (6) and (7) $\frac{a}{-1}=\frac{b}{2}=\frac{c}{-1}$ i.e. $\frac{a}{1}=\frac{b}{-2}=\frac{c}{1}$

Hence the equation of π is $1(x+1)-2(y+1)+1(z+1)=0$ i.e. $x-2y+z=0$.

3.5.9 Show that the lines $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$; $3x-2y+z+5=0=2x+3y+4z-4$ are coplanar. Find the point of intersection and the plane containing the lines.

Solution: Let the equations $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ represent the line L_1.

Any point on L_1 is $P=(3t-4,\ 5t-6,\ -2t+1)$ where $t\in\mathbb{R}$

Let π_1 be the plane $3x - 2y + z + 5 = 0$ and

π_2 be the plane $2x + 3y + 4z - 4 = 0$ and $\pi_1 \cap \pi_2$ be the line L_2.

If P lies in π_1 then $3(3t - 4) - 2(5t - 6) + (-2t + 1) + 5 = 0$

$\Rightarrow \; 9t - 12 - 10t + 12 - 2t + 6 = 0$

$\Rightarrow \; -3t + 6 = 0 \Rightarrow t = 2$

$\Rightarrow \; P = (2, 4, -3)$. This satisfies the equation of the plane π_2.

The general equation of any plane π through the intersection of π_1 and π_2 is $\pi_1 + \lambda\pi_2 = 0$.

$(3x - 2y + z + 5) + \lambda(2x + 3y + 4z - 4) = 0$

i.e. $(3 + 2\lambda)x + (-2 + 3\lambda)y + (1 + 4\lambda)z + (5 - 4\lambda) = 0$

π contains the line L_1

$\Leftrightarrow \; (3 + 2\lambda)3 + (-2 + 3\lambda)5 + (1 + 4\lambda) - 2 = 0$

$\Leftrightarrow \; 13\lambda - 3 = 0$

$\Leftrightarrow \; \lambda = \dfrac{3}{13}$

Thus the equation of the plane π is

$45x - 17y + 25z + 53 = 0$

3.5.10 Show that the lines $x + y + z - 3 = 0 = 2x + 3y + 4z - 5 \cdots\cdots\cdots\cdots(1)$

$4x - y + 5z - 7 = 0 = 2x - 5y - z - 3 \cdots\cdots\cdots(2)$

are coplanar. Find the plane in which they lie.

Solution:

$$\begin{vmatrix} 1 & 1 & 1 & -3 \\ 2 & 3 & 4 & -5 \\ 4 & -1 & 5 & -7 \\ 2 & -5 & -1 & -3 \end{vmatrix}$$

Apply $c_2 \to c_2 - c_1,\ c_2 \to c_3 - c_1,\ c_4 \to c_4 + 3c_1$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 4 & -5 & 1 & 5 \\ 2 & -7 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ -5 & 1 & 5 \\ -7 & -3 & 3 \end{vmatrix}$$

Apply $c_2 \to c_2 - 2c_1,\ c_3 \to c_3 - c_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -5 & 11 & 10 \\ -7 & 11 & 10 \end{vmatrix} = 0$$

Hence the given lines are coplanar.

Equation of any plane through the line (1) is

$$(x + y + z - 3) + \lambda(2x + 3y + 4z - 5) = 0 \cdots\cdots (3)$$

Equation of any plane through the line (2) is

$$(4x - y + 5z - 7) + \mu(2x - 5y - z - 3) = 0 \cdots\cdots (4)$$

If the equations (3) and (4) represent the same plane, then

$$1 + 2\lambda = k(4 + 2\mu) \cdots\cdots\cdots\cdots (5) \Rightarrow 1 + 2\lambda = 4k + 2\mu k$$

$$1 + 3\lambda = k(-1 - 5\mu) \cdots\cdots\cdots\cdots (6) \Rightarrow 1 + 3\lambda = -k - 5\mu k$$

$$1 + 4\lambda = k(5 - \mu) \cdots\cdots\cdots\cdots (7) \Rightarrow 1 + 4\lambda = 5k - \mu k$$

$$-3 - 5\lambda = k(-7 - 3\mu) \cdots\cdots\cdots\cdots (8) \Rightarrow -3 - 5\lambda = -7k - 3\mu k$$

Eliminating μk from (5), (7) and (5), (8) we get

$$-14k + 10\lambda + 3 = 0 \cdots\cdots\cdots (9), \quad 2k - 4\lambda - 3 = 0 \cdots\cdots\cdots (10)$$

From (9) and (10) $k = -\frac{1}{2}, \lambda = -1$

Substituting these values in (7) we get $\mu = -1$

The values of k, λ, μ satisfy the equation (6).

Hence the equation of the required plane is

$$-x-2y-3z+2=0 \Rightarrow x+2y+3z-2=0$$

3.5.11 S.A.Q.: Find the equations of the line intersecting the lines.

$$\frac{x-5}{1}=\frac{y}{1}=\frac{z-5}{1},\ \frac{x+5}{1}=\frac{y}{1}=\frac{z+5}{2}$$ and parallel to the line $\frac{x-5}{2}=\frac{y-5}{1}=\frac{z-10}{3}$.

3.5.12 S.A.Q.: Find the equations of the line intersecting the lines

$2x+y-1=0=x-2y+3z$; $3x-y+z+2=0=4x+5y-2z-3$ and parallel to the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$.

3.5.13 S.A.Q.: Prove that the lines $\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$, $\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$,

$\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ will lie in one plane if $\left(\frac{\ell}{\alpha}\right)(b-c)+\left(\frac{m}{\beta}\right)(c-a)+\left(\frac{n}{\gamma}\right)(a-b)=0$

3.5.14 S.A.Q.: Find the equations of the line which passes through the point $(2,-1,1)$ and intersects the lines $2x+y-4=0=y+2z$; $x+3z=4,\ 2x+5z=8$.

3.6 Shortest Distance Between Two Skew Lines:

3.6.1 Skew Lines:

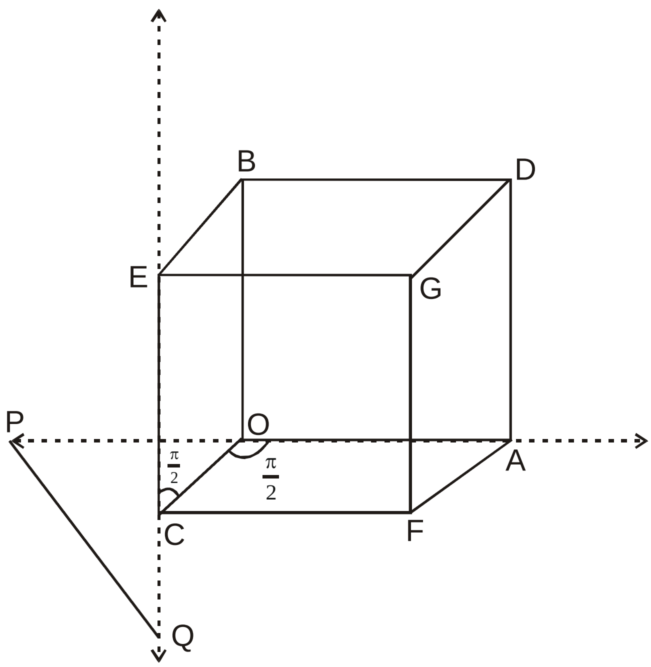
Definition: Any two lines which are non-parallel and non-intersecting (i.e. non-coplanar) are called skew lines.

Let us find out the shortest distance between two lines. The shortest distance between two lines is the perpendicular distance between the given lines. If the two lines are

(i) intersecting, then the shortest distance (perpendicular distance) between them is zero.

(ii) skew lines, then the shortest distance between them lies along the line meeting both of them at right angles.

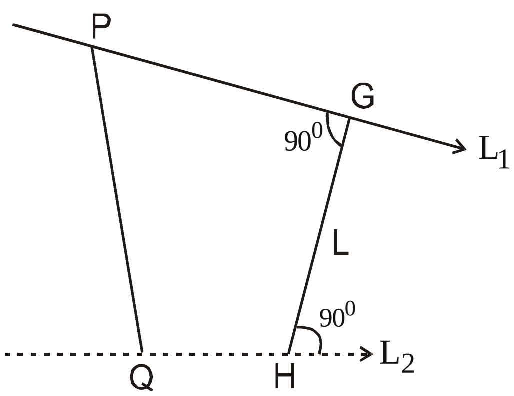
(iii) parallel, then the shortest distance between them is the perpendicular distance from any point on one line to the other line.



From the figure OA, CE are non - coplanar lines. OC is perpendicular to both OA and CE. Consider a point P on OA and Q on CE. If the length of PQ is changed such that PQ = OC then the length PQ is the shortest. Let us prove the following theorem.

3.6.2 Theorem: Given L_1, L_2 are non - coplanar lines. Another line L intersects L_1, L_2 at G, H respectively and L is perpendicular to both L_1, L_2. Then the length of GH is the shortest distance between L_1 and L_2.

Proof: Let P, Q be any two points on L_1, L_2 respectively.



Let the line L meet L_1, L_2 at G, H respectively.

$GH \perp L_1$ and $GH \perp L_2$

If θ is the angle between PQ and GH then the length of GH is the length of projection of $\overline{PQ}$ along $\overline{GH}$.

Hence $$GH = \frac{\left|\overline{PQ} \cdot \overline{GH}\right|}{\left|\overline{GH}\right|} = \frac{\left|\overline{PQ}\right| \left|\overline{GH}\right| \left|\text{Cos}\left(\overline{PQ}, \overline{GH}\right)\right|}{\left|\overline{GH}\right|}$$

$$GH = \left|\overline{PQ}\right| \left|\text{Cos } \theta\right| = PQ \left|\text{Cos } \theta\right| \le PQ \qquad \left(\because \left|\text{Cos } \theta\right| \le 1\right)$$

The shortest distance between L_1 and L_2 is GH.

3.6.3 Theorem: Given two skew lines L_1, L_2 there exist uniquely determined parallel planes π_1, π_2 such that π_1 contains L_1 and π_2 contains L_2.

Proof: Let the equations of L_1 be $\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and the equations of L_2 be $\frac{x-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$.

If such planes π_1, π_2 exist then their normals have the same d.cs.

We may assume that the equations of π_1, π_2 are respectively

$$\pi_1 \equiv ax + by + cz + d_1 = 0$$

$$\pi_2 \equiv ax + by + cz + d_2 = 0$$

The condition that L_1 lies in π_1 is

$a\ell_1 + bm_1 + cn_1 = 0$ and $ax_1 + by_1 + cz_1 + d_1 = 0$

Similarly $a\ell_2 + bm_2 + cn_2 = 0$ and $ax_2 + by_2 + cz_2 + d_2 = 0$

These conditions hold good iff

$$\frac{a}{m_1 n_2 - m_2 n_1} = \frac{b}{n_1 \ell_2 - n_2 \ell_1} = \frac{c}{\ell_1 m_2 - \ell_2 m_1}$$

Let $m_1n_2 - m_2n_1 = \alpha,\ n_1\ell_2 - n_2\ell_1 = \beta,\ \ell_1m_2 - \ell_2m_1 = \gamma$

Since $L_1,\ L_2$ are skew lines $\ell_1 : m_1 : n_1 \neq \ell_2 : m_2 : n_2$

so that at least one of α, β, γ is non zero.

Thus the planes exist and are given by

$$\pi_1 \equiv \alpha x + \beta y + \gamma z + d_1 = 0$$

$$\pi_2 \equiv \alpha x + \beta y + \gamma z + d_2 = 0$$

where $d_1,\ d_2$ are uniquely fixed by fixing the point (x_1, y_1, z_1) on L_1 and the point (x_2, y_2, z_2) on L_2.

3.6.4 Theorem: If the equations of two skew lines are

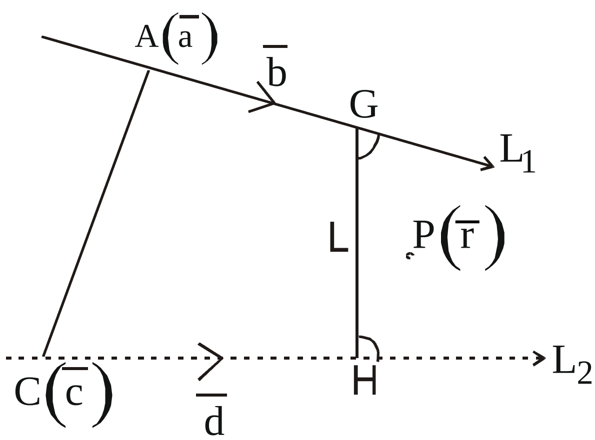
$\bar{r} = \bar{a} + t\bar{b} \cdots\cdots\cdots\cdots (1);\ \bar{r} = \bar{c} + s\bar{d} \cdots\cdots\cdots (2)$. The distance (shortest distance) between them is $\dfrac{\left|(\bar{a} - \bar{c}) \cdot (\bar{b} \times \bar{d})\right|}{\left|\bar{b} \times \bar{d}\right|}$ and the equation of the line which isthe common perpendicular is

$$\left[\bar{r} - \bar{a} \quad \bar{b} \quad \bar{b} \times \bar{d}\right] = 0 = \left[\bar{r} - \bar{c} \quad \bar{d} \quad \bar{b} \times \bar{d}\right]$$

Proof: Let L_1, L_2 be the lines represented by the given equations

$\bar{r} = \bar{a} + t\bar{b}$, $\bar{r} = \bar{c} + s\bar{d}$ respectively.

Let L be the line of shortest distance which is perpendicular to L_1 and L_2 and meet L_1 and L_2 at G and H respectively.



L is perpendicular to both L_1 and L_2.

Therefore $\overline{GH} = \left(\bar{b} \times \bar{d}\right)$ for some scalar u.

Shortest distance between L_1 and L_2 = Length of projection of $\overline{AC}$ along $\overline{GH}$.

$$= \frac{\left|\overline{AC} \cdot \overline{GH}\right|}{\left|\overline{GH}\right|} = \frac{\left|\left(\bar{c} - -a\right) \cdot u\left(\bar{b} \times \bar{d}\right)\right|}{\left|u\left(\bar{b} \times \bar{d}\right)\right|} = \frac{\left|\left(\bar{a} - \bar{c}\right) \cdot \left(\bar{b} \times \bar{d}\right)\right|}{\left|\bar{b} \times \bar{d}\right|} = \frac{\left|\left[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}\right]\right|}{\left|\bar{b} \times \bar{d}\right|}$$

Let π be the plane containing L_1 & GH and π' be the plane containing L_2 & GH. $\pi, \ \pi'$ intersect in GH.

Equation of GH is $\pi = 0 = \pi'$

Let P be any point with position vector $\bar{r}$ on the plane π.

$P \in \pi \Leftrightarrow \overline{AP}, \ \bar{b}, \overline{GH}$ are coplanar.

$$\Leftrightarrow \left[\overline{AP} \quad \bar{b} \quad \overline{GH}\right] = 0 \ \Leftrightarrow \left[\bar{r} - \bar{a} \quad \bar{b} \quad \bar{b} \times \bar{d}\right] = 0$$

Equation of π is $\left[\bar{r} - \bar{a} \quad \bar{b} \quad \bar{b} \times \bar{d}\right] = 0$

Similarly equation of π' is $\left[\bar{r} - \bar{c} \quad \bar{d} \quad \bar{b} \times \bar{d}\right] = 0$

Therefore equation of GH is $\left[\bar{r} - \bar{a} \quad \bar{b} \quad \bar{b} \times \bar{d}\right] = 0 = \left[\bar{r} - \bar{c} \quad \bar{d} \quad \bar{b} \times \bar{d}\right]$

3.6.5 Theorem: Given the lines L_1 is $\dfrac{x - x_1}{\ell_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and L_2 is $\dfrac{x - x_2}{\ell_2} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$. The shortest distance between $L_1, \ L_2$ is the absolute value of

$$\frac{\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\Sigma\left(m_1 n_2 - m_2 n_1\right)^2}} \ldots\ldots\ldots\ldots\ldots(1)$$

and equation of the line of shortest distance

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \text{s i}\ \ell_1 & m_1 & n_1 \\ m_1n_2-m_2n_1 & n_1\ell_2-n_2\ell_1 & \ell_1m_2-\ell_2m_1 \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ \ell_2 & m_2 & n_2 \\ m_1n_2-m_2n_1 & n_1\ell_2-n_2\ell_1 & \ell_1m_2-\ell_2m_1 \end{vmatrix} \cdots(2)$$

Proof: In the Theorem 3.6.4 put $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (\ell_1, m_1, n_1)$

$$\bar{c} = (x_2, y_2, z_2),\ \bar{d} = (\ell_2, m_2, n_2)$$

and $\bar{r} = (x, y, z)$

$$\bar{b} \times \bar{d} = (m_1n_2 - m_2n_1,\quad n_1\ell_2 - n_2\ell_1,\quad \ell_1m_2 - \ell_2m_1)$$

$$|\bar{b} \times \bar{d}| = \sqrt{\Sigma(m_1n_2 - m_2n_1)^2}$$

$$\bar{a} - \bar{c} = (x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2)$$

$$\bar{r} - \bar{a} = (x - x_1,\ y - y_1,\ z - z_1)$$

$$\bar{r} - \bar{b} = (x - x_2,\ y - y_2,\ z - z_2)$$

From theorem 3.6.4 we get the required formulae for shortest distance between L_1 and L_2 is (1) and the equations of the line of shortest distance is (2).

Examples:

3.5.6 If $A=(1,-2,-1)$, $B=(4,0,-3)$, $C=(1,2,-1)$, $D=(2,-4,-5)$, then find the distance between AB and CD.

Solution: Equation of AB is $\bar{r} = \bar{a} + t\bar{b}$ where $\bar{a} = \overline{OA} = (1,-2,-1)$ and

$$\bar{b} = \overline{AB} = \overline{OB} - \overline{OA} = (3,2,-2)$$

Equation of CD is $\bar{r} = \bar{c} + s\bar{d}$ where $\bar{c} = \overline{OC} = (1,2,-1)$ and $\bar{d} = \overline{CD} = (1,-6,-4)$

$$\bar{a} - \bar{c} = \overline{OA} - \overline{OC} = (0,-4,0)$$

$$\left[\bar{a} - \bar{c}\ \ \bar{b}\ \ \bar{d}\right] = \begin{vmatrix} 0 & -4 & 0 \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix} = 4(-12+2) = -40$$

$$\bar{b}\times\bar{d}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 3 & 2 & -2\\ 1 & -6 & -4\end{vmatrix}=(-8-12)\bar{i}-(-12+2)\bar{j}+(-18-2)\bar{k}=-20\bar{i}+10\bar{j}-20\bar{k}$$

$$\left|\bar{b}\times\bar{d}\right|=\sqrt{400+100+400}=\sqrt{900}=30$$

Shortest distance between $\overline{AB}$ and $\overline{CD}$ is

$$\frac{\left|\left[\bar{a}-\bar{c}\ \ \bar{b}\ \ \bar{d}\right]\right|}{\left|\bar{b}\times\bar{d}\right|}=\frac{40}{30}=\frac{4}{3}\ \text{unit}$$

3.6.7 Find the shortest distance and the equations of the line of shortest distance line between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}\cdots\cdots\cdots(1),\ \frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}\cdots\cdots(2)$

Solution: A point on the line (1) is $O=(0,0,0)$

A point on the line (2) is $A=(2,1,-2)$

d.rs. of the line (1) are $2,-3,1\Rightarrow$ vector along (1) is $\bar{a}=(2,-3,1)$

d.rs. of the line (2) are $3,-5,2\Rightarrow$ vector along (2) is $\bar{b}=(3,-5,2)$

$$\overline{OA}=(2,1,-2)$$

Let PQ be the shortest distance between the lines (1) and (2)

so that P is a point on (1) and Q is a point on (2)

Now PQ is perpendicular to both (1) and (2).

$\Rightarrow$ PQ is parallel to $\bar{a}\times\bar{b}$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & -3 & 1\\ 3 & -5 & 2\end{vmatrix}=-\bar{i}-\bar{j}-\bar{k}$$

$$\left|\bar{a}\times\bar{b}\right|=\sqrt{1+1+1}=\sqrt{3}$$

A unit vector along PQ is $\dfrac{-\bar{i}-\bar{j}-\bar{k}}{\sqrt{3}}$

So the shortest distance = PQ = Projection of OA on PQ

$$= \left|\overline{OA}\cdot\left(\text{Unit vector along PQ}\right)\right|$$

$$= \frac{\left|\overline{OA}\cdot\overline{PQ}\right|}{\left|\overline{PQ}\right|} = \frac{\left|(2,1,-2)\cdot(-1,-1,-1)\right|}{\sqrt{3}} = \frac{\left|-2-1+2\right|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The equation of the plane containing the line (1) and the line PQ is

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ -1 & -1 & -1 \end{vmatrix} = 0 \Rightarrow 4x + y - 5z = 0$$

Similarly the equation of the plane containing the line (2) and the line PQ is

$$\begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ -1 & -1 & -1 \end{vmatrix} = 0 \Rightarrow 7x + y - 8z - 31 = 0$$

Equations of the line of shortest distance is $4x + y - 5z = 0 = 7x + y - 8z - 31$

We solve this type of problems in the following method as well

3.6.8 Find the length and equations of the shortest distance line between the lines $\dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-1}{2}$; $\dfrac{x-4}{4} - \dfrac{y-5}{5} = \dfrac{z-2}{3}$

Solution: Given lines are $L_1 : \dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-1}{2} = t(\text{say})$

and $L_2 : \dfrac{x-4}{4} = \dfrac{y-5}{5} = \dfrac{z-2}{3} = s\ (\text{say})$

Any point on L_1 is $(3t+2,\ 4t+3,\ 2t+1)$

Any point on L_2 is $(4s+4,\ 5s+5,\ 3s+2)$

If $G = (3t+2,\ 4t+3,\ 2t+1)$ and $H = (4s+4,\ 5s+5,\ 3s+2)$

and GH is perpendicular to both L_1 and L_2 then

d.rs. of GH are $3t-4s-2,\ 4t-5s-2,\ 2t-3s-1$

$$GH \perp L_1 \Rightarrow 3(3t-4s-2)+4(4t-5s-2)+2(2t-3s-1)=0$$

$$GH \perp L_2 \Rightarrow 4(3t-4s-2)+5(4t-5s-2)+3(2t-3s-1)=0$$

Then $29t-38s-16=0$ and $38t-50s-21=0$

Solving these two equations

$$\frac{t}{(-38)(-21)-(-50)(-16)}=\frac{s}{(-16)(38)-(-21)(29)}=\frac{1}{(29)(-50)-(-38)(38)}$$

i.e. $\dfrac{t}{-2}=\dfrac{s}{1}=\dfrac{1}{-6} \Rightarrow t=\dfrac{1}{3},\ \ s=\dfrac{-1}{6}$

$$\Rightarrow G=\left(3,\frac{13}{3},\frac{5}{3}\right) \qquad H=\left(\frac{10}{3},\frac{25}{6},\frac{3}{2}\right)$$

$$\Rightarrow GH=\sqrt{\left(\frac{10}{3}-3\right)^2+\left(\frac{13}{3}-\frac{25}{6}\right)^2+\left(\frac{5}{3}-\frac{3}{2}\right)^2}=\sqrt{\frac{1}{9}+\frac{1}{36}+\frac{1}{36}}=\frac{1}{\sqrt{6}}$$

$\Rightarrow$ d.rs. of GH are $\dfrac{1}{3},\dfrac{1}{6},\dfrac{1}{6}$ i.e. 2, 1, 1

Equations of the line GH in shortest distance are

$$\frac{x-3}{2}=\frac{y-\frac{13}{3}}{1}=\frac{z-\frac{5}{3}}{1}$$

i.e. $\dfrac{x-3}{2}=\dfrac{3y-13}{3}=\dfrac{3z-5}{3}$

3.6.9 Find the shortest distance between the lines $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{1}$ and $x+y+2z-3=0=2x+3y+3z-4$ and the equations for the line of shortest distance.

Solution: Given lines are $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{1}=t\ (\text{say})\cdots\cdots\cdots(1)$

and $x+y+2z-3=0=2x+3y+3z-4 \cdots\cdots\cdots\cdots(2)$

Equation of any plane through (2) is

$$x+y+2z-3+\lambda(2x+3y+3z-4)=0$$

$$\Rightarrow (1+2\lambda)x+(1+3\lambda)y+(2+3\lambda)z+(-3-4\lambda)=0 \cdots\cdots\cdots\cdots\cdots(3)$$

If the plane (3) is parallel to the line (1) then

$$1(1+2\lambda)+2(1+3\lambda)+1(2+3\lambda)=0$$

$$\Rightarrow 11\lambda+5=0 \Rightarrow \lambda=\frac{-5}{11}$$

Equation of the plane (3) is

$$\frac{1}{11}x-\frac{4}{11}y+\frac{7}{11}z--\frac{13}{11}=0$$

$$\Rightarrow x-4y+7z-13=0 \cdots\cdots(4)$$

Length of the shortest distance between the lines (1) and (2)

= Length of perpendicular from the point $(0,0,0)$ of the line (1) to the plane (4)

$$=\frac{|-13|}{\sqrt{1+9+49}}=\frac{13}{\sqrt{59}} \text{ units}$$

To find the equation of the line of the shortest distance:

Equation of the plane through the line (1) and perpendicular to the plane (4) is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 1 \\ 1 & -4 & 7 \end{vmatrix}=0 \Rightarrow 18x-6y-6z=0$$

$$\Rightarrow 3x-y-z=0 \cdots\cdots(5)$$

If the equation (3) represents a plane perpendicular to (4) then

$$1(1+2\lambda)-4(1+3\lambda)+7(2+3\lambda)=0$$

$$\Rightarrow 11\lambda+11=0 \Rightarrow \lambda=-1$$

Equation of the plane through the line (2) and perpendicular to the plane (4) is

$$-x-2y-z+1=0$$

$$x+2y+z-1=0 \cdots\cdots\cdots(6)$$

Equations of the line of the shortest distance is

$$3x-y-z=0=x+2y+z-1=0$$

3.6.10 Find the shortest distance and the equations of the line of shortest distance between the lines $3x-9y+5z=0=x+y-z$; $6x+8y+3z-10=0=x+2y+z-3$.

Solution: Given lines are $3x-9y+5z=0=x+y-z \cdots\cdots\cdots(1)$

and $6x+8y+3z-10=0=x+2y+z-3 \cdots\cdots\cdots(2)$

Any plane through (1) is

$$(3x-9y+5z)+\lambda(x+y-z)=0$$

$$\Rightarrow (3+\lambda)x+(-9+\lambda)y+(5-\lambda)z=0 \cdots\cdots\cdots\cdots(3)$$

Any plane through (2) is

$$(6x+8y+3z-10)+\mu(x+2y+z-3)=0$$

$$\Rightarrow (6+\mu)x+(8+2\mu)y+(3+\mu)z+(-10-3\mu)=0 \cdots\cdots\cdots(4)$$

If (3) and (4) are parallel then $3+\lambda=k(6+\mu) \Rightarrow \lambda-6k-k\mu=-3 \cdots\cdots\cdots\cdots\cdots(5)$

$$-9+\lambda=(8+2\mu) \Rightarrow \lambda-8k-2k\mu=9 \cdots\cdots\cdots(6)$$

$$5-\lambda=k(3+\mu) \Rightarrow \lambda+3k+k\mu=5 \cdots\cdots\cdots(7)$$

From (5) and (7) $\quad 2\lambda-3k=2$

From (5) & (6) $\quad \lambda-4k=-15$

Solving $5k=32 \Rightarrow k=\dfrac{32}{5} \quad \lambda=\dfrac{53}{5}$

then $\mu=\dfrac{1}{k}(3+\lambda)-6 = \dfrac{5}{32}\left(3+\dfrac{53}{5}\right)-6 = \dfrac{-31}{8}$

Equations to the planes through (1) and (2) and parallel to each other from (3) and (4) are $17x+2y-7z=0 \cdots\cdots\cdots\cdots(8)$, $17x+2y-7z-13=0 \cdots\cdots\cdots(9)$

A point on (8) is $(0,0,0)$

Shortest distance between (1) and (2) = Distance of $(0,0,0)$ from (9)

$$=\left|\frac{-13}{\sqrt{289+4+49}}\right|=\frac{13}{3\sqrt{38}}$$

If (3) is perpendicular to (8) then $17(3+\lambda)+2(-9+\lambda)-7(5-\lambda)=0 \Rightarrow \lambda=\frac{1}{13}$

Similarly $\mu=\frac{-97}{14}$

$\Rightarrow$ Equation of the plane through (1) and perpendicular to (8) is

$$3x-9y+5z+\frac{1}{13}(x+y+z)=0 \Rightarrow 10x-29y+16z=0 \cdots\cdots(10)$$

Also equation to the plane through (2) and perpendicular to (8) is

$$13x+82y+55z-151=0 \cdots\cdots\cdots\cdots(11)$$

Hence the equations of the line of the shortest distance are

$$10x-29y+16z=0=13x+82y+55z-151=0$$

S.A.Q.s:

3.6.11 S.A.Q.: Show that the equation to the plane containing the line $\frac{y}{b}+\frac{z}{c}=1,\ x=0$ and parallel to the line $\frac{x}{a}-\frac{z}{c}=1,\ y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d is the shortest distance prove that $\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

3.6.12 S.A.Q.: Planes through OX, OY include an angle α. Show that their lines of intersection lies on the cone $Z^2\left(x^2+y^2+z^2\right)=x^2y^2\tan^2\alpha$.

3.6.13 S.A.Q.: Show that the shortest distance between the diagonals of a rectangular parallelopiped and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$ where a, b, c are the lengths of the edges.

3.7 Intersection of Three Planes:

We know that any two different planes in space are either parallel or intersect in a line. Let us discuss the intersection of three planes.

Given three distinct planes such that no two of them are parallel. We have the following three possibilities in respect of their intersection.

The three planes may

(i) have only one point in common

(ii) have a line in common so that the three planes are coaxial.

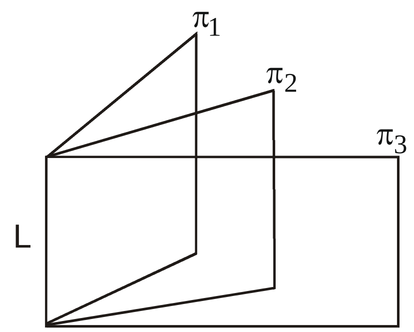
(iii) form a triangular prism, where (i) and (ii) do not hold and the triangular prism is defined as

3.7.1 Definition: The region formed by three planes no two of which are parallel and which are neither concurrent at a point nor pass through a line is called a triangular prism.

3.7.2 Theorem: If three planes intersect in pairs, their three lines of intersection are either coincident, concurrent or parallel.

Proof: Let π_1, π_2, π_3 be three planes, each of which meets the other two.

Case (i): If the plane π_1 intersects the planes π_2 and π_3 along the same line L, then L lies in the planes π_1, π_2 and π_3.



$\Rightarrow$ It is the intersection of π_1, π_2 and π_3.

Hence L is the coincident line of intersection of three planes.

Case (ii): Let the plane π_1 intersect π_2 and π_3 along <u>the two intersecting straight lines</u> L_2 and L_3 respectively.

Let the planes π_2 and π_3 intersect along the line L_1

Let the point of intersection of L_2 and L_3 be P.

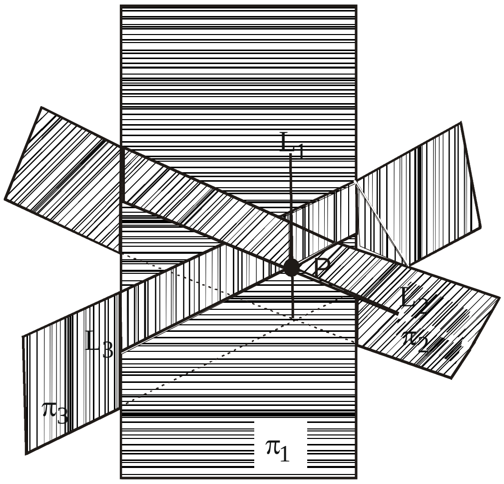
$\Rightarrow$ P lies on L_2 and P lies on L_3

$\Rightarrow$ P lies on π_2 and P lies on π_3.

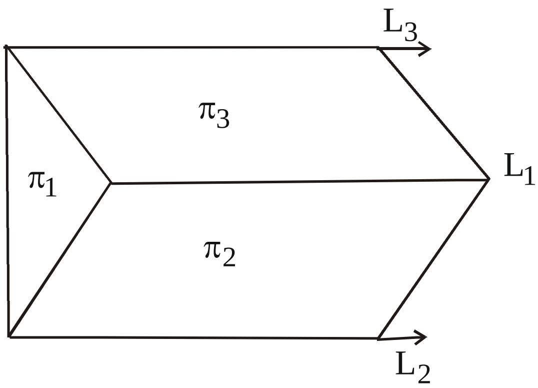
$\Rightarrow$ P lies in the line of intersection of π_2 and π_3

$\Rightarrow$ P lies on L_1

Thus the three lines L_1, L_2 and L_3 are concurrent at P.



Case (iii): Let the plane π_1 intersect π_2 and π_3 along the two parallel straight lines L_2 and L_3 respectively.



Let the planes π_2 and π_3 intersect in a line L_1.

Here L_2 and L_3 are two parallel lines each on two intersecting planes. Then by theorem 1.6.11 in planes we say that L_2 and L_3 are parallel to L_1 the line of intersection of the planes π_2 & π_3 containing them respectively. Hence L_1, L_2 and L_3 are parallel to each other.

For our convenience we make use of the following notations:

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

3.7.3 Theorem: Suppose $\pi_1 \equiv a_1x + b_1y + c_1z + d_1 = 0 \cdots\cdots\cdots (1)$

$$\pi_2 \equiv a_2x + b_2y + c_2z + d_2 = 0 \cdots\cdots (2)$$

$$\pi_3 \equiv a_3x + b_3y + c_3z + d_3 = 0 \cdots\cdots (3)$$

are planes no two of which are parallel. The planes (1), (2) and (3)

(i) intersect in a unique point iff $\Delta \neq 0$

(ii) intersect in a line iff $\Delta = 0,\ \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

(iii) form a triangular prism iff $\Delta = 0$ and $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$ or $\Delta_3 \neq 0$.

Proof: We assume that the planes π_1 and π_2 are not parallel.

Hence the equations $\pi_1 = 0, \pi_2 = 0$ together represent a line.

Equations of the line of intersection of π_1 and π_2 (by taking Z = 0) is

$$\frac{x - (b_1d_2 - b_2d_1)/(a_1b_2 - a_2b_1)}{b_1c_2 - b_2c_1} = \frac{y - (a_2d_1 - a_1d_2)/(a_1b_2 - a_2b_1)}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1} \cdots\cdots\cdots (4)$$

(i) The planes π_1, π_2 and π_3 intersect in a unique point

$\Leftrightarrow$ line (4) is not parallel to π_3

$\Leftrightarrow$ The normal of π_3 is not perpendicular to the line (4)

$\Leftrightarrow$ (a_3, b_3, c_3) is not perpendicular to $(b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$

$\Leftrightarrow$ $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) \neq 0$

$$\Leftrightarrow \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

(ii) The planes $\pi_1,\ \pi_2$ and π_3 intersect in a line

$\Leftrightarrow$ line (4) lies in the plane π_3

$\Leftrightarrow$ line (4) is parallel to the plane π_3 and $P = \left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1},\ \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1},\ 0\right)$

lies in the plane π_3.

$\Leftrightarrow$ The normal of π_3 is perpendicular to the line (4) and P lies in π_3.

$\Leftrightarrow$ (a_3, b_3, c_3) is perpendicular to $(b_1c_2 - b_2c_1, c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1)$

and $a_3\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}\right) + d_3(0) + d_3 = 0$

$\Leftrightarrow a_3\ (b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + d_3(a_1b_2 - a_2b_1) = 0$

and $a_3\ (b_1d_2 - b_2d_1) + d_3(d_1a_2 - d_2a_1) + d_3(a_1b_2 - a_2b_1) = 0$

$\Leftrightarrow$ $\Delta = 0$ and $\Delta_3 = 0$

Similarly when $x = 0$ and $y = 0$ we get $\Delta_1 = 0$ and $\Delta_2 = 0$ respectively.

(iii) The three planes π_1, π_2 and π_3 form a triangular prism

$\Leftrightarrow$ the line (4) is parallel to the plane π_3 without lying in the same.

$\Leftrightarrow$ the normal of the plane π_3 is perpendicular to the line (4) and P doesn't lie in π_3.

$\Leftrightarrow$ (a_3, b_3, c_3) is perpendicular to $(b_1c_2 - b_2c_1,\ \ c_1a_2 - c_2a_1,\ \ a_1b_2 - a_2b_1)$ and

$a_3\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}\right) + c_3(0) + d_3 \neq 0$

$\Leftrightarrow$ $a_3\ (b_2c_1 - b_1c_2) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

and $a_3\ \ (b_1d_2 - b_2d_1) + b_3(a_2d_1 - a_1d_2) + d_3(a_1b_2 - a_2b_1) \neq 0$

$\Leftrightarrow$ $\Delta = 0$ and $\Delta_3 \neq 0$

Similarly in the cases of $x = 0$ and $y = 0$ we get $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$ respectively.

Hence the three planes π_1, π_2 and π_3 form a triangular prism

$\Leftrightarrow \Delta = 0$ and

$\Delta_1 \neq 0$ or $\Delta_2 \neq 0$ or $\Delta_3 \neq 0$

3.7.4 **Note:** It can be proved that if the three planes intersect in a unique point then the point is $\left(\frac{-\Delta_1}{\Delta}, \frac{-\Delta_2}{\Delta}, \frac{-\Delta_3}{\Delta}\right)$.

Examples:

3.7.5 Examine the nature of intersection of the planes $\pi_1 \equiv x - y + z = 0$, $\pi_2 \equiv 2x + 5y + 3z = 0$, $\pi_3 \equiv 3x - 2y - 6z + 1 = 0$.

Solution: Given $a_1 = 1,\ b_1 = -1,\ c_1 = 1$

$a_2 = 2,\ b_2 = 5,\ c_2 = 3$

$a_3 = 3,\ b_3 = -2,\ c_3 = -6$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 5 & 3 \\ 3 & -2 & -6 \end{vmatrix}$

$= 1(-30 + 6) + 1(-12 - 9) + 1(-4 - 15)$

$= -24 - 21 - 19 = -64 \neq 0$

So given planes intersect in a point.

3.7.6 Examine the nature of intersection of the planes $\pi_1 \equiv x - y + z - 4 = 0$, $\pi_2 \equiv 2x - y - z + 4 = 0$, $\pi_3 \equiv x + y - 5z + 14 = 0$.

Solution: Given $a_1 = 1 \quad b_1 = -1 \quad c_1 = +1 \quad d_1 = -4$

$a_2 = 2 \quad b_2 = -1 \quad c_2 = -1 \quad d_2 = 4$

$a_3 = 1 \quad b_3 = 1 \quad c_3 = -5 \quad d_3 = 14$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -5 \end{vmatrix} = 1(5+1) + 1(-10+1) + 1(2+1)$

$= 6 - 9 + 3 = 0$

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -4 \\ 2 & -1 & 4 \\ 1 & 1 & 14 \end{vmatrix} = 1(-14-4)+1(28-4)-4(2+1)$

$$= -18+24-12 = -6 \neq 0$$

So given planes form a prism.

3.7.7 Examine the nature of intersection of the planes $\pi_1 \equiv 2x - y + z - 4 = 0$, $\pi_2 \equiv 5x + 7y + 2z = 0$, $\pi_3 \equiv 3x + 4y - 2z + 3 = 0$.

Solution: Given $a_1 = 2, \quad b_1 = -1 \quad c_1 = 1, \quad d_1 = -4$

$a_2 = 5, \quad b_2 = 7, \quad c_2 = 2, \quad d_2 = 0$

$a_3 = 3, \quad b_3 = 4, \quad c_3 = -2, \quad d_3 = 3$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 5 & 7 & 2 \\ 3 & 4 & -2 \end{vmatrix} = 2(-14-8)+1(-10-6)+1(20-21)$

$$= 44-16-1 = -61 \neq 0$$

$\Rightarrow$ Three planes intersect at a point.

3.7.8 Examine the nature of intersection of the planes $\pi_1 \equiv x + y + z + 6 = 0$, $\pi_2 \equiv x + 2y + 2z + 6 = 0$, $\pi_3 \equiv x + 3y + 3z + 6 = 0$.

Solution: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{vmatrix} = 0$

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 6 \\ 1 & 3 & 6 \end{vmatrix} = 0$$

Three planes have a common line.

3.7.9 For what values of λ do the planes $\pi_1 \equiv x - y + z + 1 = 0$, $\pi_2 \equiv \lambda x + 3y + 2z - 3 = 0$, $\pi_3 \equiv 3x + \lambda y - z - 2 = 0$. (i) intersect at a point, (ii) form a triangular prism.

Solution: Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ \lambda & 3 & 2 \\ 3 & \lambda & -1 \end{vmatrix}$

(i) planes intersect at a point

$\Leftrightarrow \Delta \neq 0 \Leftrightarrow -3-2\lambda+(-\lambda-6)+\lambda^2-9 \neq 0 \Leftrightarrow \lambda^2-3\lambda-18 \neq 0$

for $\lambda = -3$ and 6, $\Delta = 0$

$\therefore$ planes intersect at a point for all real values of λ other than -3 and 6.

(ii) For $\lambda = -3$ and 6, $\Delta = 0$

$\Rightarrow$ planes form a triangular prism.

3.7.10 Prove that the planes $\pi_1 \equiv x - cy - bz = 0$, $\pi_2 \equiv az + cx - y = 0$, $\pi_3 \equiv bx + ay - z = 0$ pass through one line if $a^2+b^2+c^2+2abc=1,$ and show that the line of intersection then has the equations

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$

Solution: Given

$$\begin{array}{llll} a_1 = 1, & b_1 = -c, & c_1 = -b, & d_1 = 0 \\ a_2 = -c, & b_2 = 1, & c_2 = -a, & d_2 = 0 \\ a_3 = -b, & b_3 = -a, & c_3 = 1, & d_3 = 0 \end{array}$$

Given planes have a common line

$\Rightarrow \Delta = 0,\ \Delta_1 = 0$

$$\Rightarrow \begin{vmatrix} 1 & -c & -b \\ -c & 1 & -a \\ -b & -a & 1 \end{vmatrix} = 0 \Rightarrow 1-a^2+c(-c-ab)-b(ac+b) = 0$$

$$\Rightarrow 1-a^2-b^2-c^2-2abc = 0 \Rightarrow a^2+b^2+c^2+2abc = 1$$

Since three planes pass through origin, the common line L, also passes through origin.

Let $\ell, \mathrm{m}, \mathrm{n}$ be the d.rs. of common line L.

$$L \perp \pi_1,\ L \perp \pi_2,\ L \perp \pi_3$$

$$\Rightarrow \ell - mc - bn = 0 \cdots\cdots\cdots\cdots(1),\quad -c\ell + m - an = 0 \cdots\cdots\cdots\cdots(2),$$

$$-b\ell - am + n = 0 \cdots\cdots\cdots\cdots(3)$$

Solving (1) and (2) we get

$$\frac{\ell}{ac+b} = \frac{m}{bc+a} = \frac{n}{1-c^2} \cdots\cdots\cdots\cdots(4)$$

Solving (2) and (3) we get

$$\frac{\ell}{1-a^2} = \frac{m}{ab+c} = \frac{n}{ca+b} \cdots\cdots\cdots(5)$$

Solving (3) & (1) we get

$$\frac{\ell}{ab+c} = \frac{m}{1-b^2} = \frac{n}{bc+a} \cdots\cdots\cdots(6)$$

taking first two terms of (5) & (6) and multiplying we get

$$\frac{\ell^2}{(ab+c)\left(1-a^2\right)} = \frac{m^2}{(ab+c)\left(1-b^2\right)} \Rightarrow \frac{\ell^2}{1-a^2} = \frac{m^2}{1-b^2}$$

Similarly we get $\dfrac{m^2}{1-b^2} = \dfrac{n^2}{1-c^2}$

$$\frac{\ell^2}{1-a^2} = \frac{m^2}{1-b^2} = \frac{n^2}{1-c^2} \Rightarrow \frac{\ell}{\sqrt{1-a^2}} = \frac{m}{\sqrt{1-b^2}} = \frac{n}{\sqrt{1-c^2}}$$

Equation of the line L is

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$

3.7.11 Show that the planes $bx - ay = n,\ cy - bz = \ell,\ az - cx = m$ intersect in a line if $a\ell + bm + cn = 0$.

Solution: Given the planes are $\pi_1 \equiv bx - ay - oz - n = 0$

$$\pi_2 \equiv cy - bz - \ell = 0$$

$$\pi_3 \equiv -cx + oy + az - m = 0$$

If π_1, π_2 & π_3 intersect in a line then

$\Delta = 0, \Delta_1 = 0$

$$\Rightarrow \begin{vmatrix} b & -a & 0 \\ 0 & c & -b \\ -c & 0 & a \end{vmatrix} = 0 \text{ and } \begin{vmatrix} b & -a & -n \\ 0 & c & -\ell \\ -c & 0 & -m \end{vmatrix} = 0$$

$$\Rightarrow \quad b(-mc) + a(-c\ell) - n\left(c^2\right) = 0 \Rightarrow -cbm - ca\ell - c^2n = 0 \Rightarrow a\ell + bm + cn = 0$$

3.8 Answers S.A.Q.:

3.5.11 S.A.Q.: Let the line L_1 be $\dfrac{x-5}{1} = \dfrac{y}{1} = \dfrac{z-5}{1} = r \ (\text{say})$

& the line L_2 be $\dfrac{x+5}{1} = \dfrac{y}{1} = \dfrac{z+5}{2} = s \ (\text{say})$

Any point on L_1 is $P = (r+5, r, r+5)$

Any point on L_2 is $Q = (s-5, s, 2s-5)$

D.rs of PQ are $r - s + 10, \ r - s, \ r - 2s + 10$.

If PQ is parallel to the line $L_3, \ \dfrac{x-5}{2} = \dfrac{y-5}{1} = \dfrac{z-10}{3}$

then $\dfrac{r-s+10}{2} = \dfrac{r-s}{1} = \dfrac{r-2s+10}{3} = \lambda \ (\text{say})$

$\Rightarrow 10 + r - s = 2\lambda \cdots\cdots(1) \quad r - s = \lambda \cdots\cdots\cdot(2) \quad r - 2s + 10 = 3\lambda \cdots\cdots\cdot(3)$

(1) - (2) $\Rightarrow \lambda = 10$, (1) - (3) $\Rightarrow s = -10$, $r = 0$

$\Rightarrow P = (5,0,5), \ Q = (-15,-10,-25)$

PQ is parallel to $L_3 \Rightarrow$ d.rs. of PQ are 2, 1, 3

$\Rightarrow$ Equation of the line through P, Q and parallel to L_3 is

$$\frac{x-5}{2}=\frac{y}{1}=\frac{z-5}{3}$$

3.5.12 S.A.Q.: Let the planes $\pi_1 \equiv 2x+y-1=0$, $\pi_2 \equiv x-2y+3z=0$ together represent a line L_1 and the planes $\pi_3 \equiv 3x-y+z+2=0$, $\pi_4 \equiv 4x+5y-2z-3=0$ together represent a line L_2.

Equation of any plane through L_1 is $\pi_1+\lambda\pi_2=0$

$$(2x+y-1)+\lambda(x-2y+3z)=0$$

$$(2+\lambda)x+(1-2\lambda)y+3\lambda z+(-1)=0\cdots\cdots(1)$$

Equation of any plane through L_2 is $\pi_3+\mu\pi_4=0$

$$(3x-y+z+2)+\mu(4x+5y-2z-3)=0$$

$$(3+4\mu)x+(-1+5\mu)y+(1-2\mu)z+(2-3\mu)=0\cdots\cdots(2)$$

If the equations (1) and (2) repersent the required line, then π is parallel to the line L, $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$.

$\Rightarrow$ L is parallel to the plane (1) & L is parallel to the plane (2)

$\Rightarrow 1(2+\lambda)+2(1-2\lambda)+3(3\lambda)=0$ & $(3+4\mu)1+(-1+5\mu)2+(1-2\mu)3=0$

$\Rightarrow \lambda=-\frac{2}{3}$ and $\mu=-\frac{1}{2}$

Hence the equations of the required line are

$$4x+7y-6z=3,\ 2x-7y+4z=-7.$$

3.5.13 S.A.Q.: Assume $\alpha\beta\gamma \neq 0$

Given the lines $L_1, \frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}$

line $L_2, \frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}$

line $L_3, \frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$

The lines L_1, L_2, L_3 are concurrent at the origin $(0, 0, 0)$.

Vectors along L_1, L_2 and L_3 are $(\alpha, \beta, \gamma), (a\alpha, b\beta, c\gamma)$ and (ℓ, m, n) respectively.

L_1, L_2 and L_3 are coplanar

$$\Rightarrow \left[(\alpha,\beta,\gamma),(a\alpha,b\beta,c\gamma),(\ell,m,n)\right]=0 \quad \Rightarrow \begin{vmatrix} \alpha & \beta & \gamma \\ a\alpha & b\beta & c\gamma \\ \ell & m & n \end{vmatrix} = 0$$

$$\Rightarrow \ell\beta\gamma(c-b)-m\alpha\gamma(c-a)+n\alpha\beta(b-a)=0$$

$$\Rightarrow \frac{\ell}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0$$

3.5.14 S.A.Q.: The line $2x+y-4+\lambda_1(y+2z)=0, \ \ x+3z-4+\lambda_2(2x+5z-8)=0$

intersects the two given lines for all values of λ_1, λ_2

This line will pass through $(2, -1, 1)$, if

$-1+\lambda_1=0$ and $1+\lambda_2=0$

$\Rightarrow \lambda_1=1$ and $\lambda_2=-1$

Here equations of the required line

$x+y+z=2$ and $x+2z=4$

3.6.11 S.A.Q.: Given lines are $\frac{y}{b}+\frac{z}{c}-1=0, \ x=0 \cdots\cdots\cdots\cdots(1)$

and $\frac{x}{a}-\frac{z}{c}-1=0, \ y=0 \cdots\cdots\cdots\cdots(2)$

$$\Rightarrow \frac{x-a}{a}=\frac{z}{c}, \ y=0$$

Equation of any plane through the line (1) is

$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0\cdots\cdots\cdots(3)$$

If this plane is parallel to line (2) then

$$\lambda a+\frac{1}{b}(0)+\frac{1}{c}\cdot c=0 \Rightarrow \lambda a+1=0 \Rightarrow \lambda=\frac{-1}{a}$$

Equation to the plane containing line (1) and parallel to (2) is

$$\frac{y}{b}+\frac{z}{c}-1-\frac{x}{a}=0 \Rightarrow \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0\cdots\cdots\cdots\cdots(4)$$

A point on line (2) is $(a, 0, 0)$.

Since 2d is the shortest distance between (1) and (2).

2d = distance of (a, 0, 0) from the plane (4)

$$\Rightarrow 2d=\left|\frac{\frac{a}{a}-\frac{0}{b}-\frac{0}{c}+1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| \Rightarrow \frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

3.6.12 S.A.Q.: The equations of the line OX are $y=0, z=0$.

The equation of any plane through OX is $y+\lambda_1 z=0\cdots\cdots(1)$

Similarly the equation of any plane through OY is $x+\lambda_2 z=0\cdots\cdots(2)$

Since α is an angle between the planes (1) & (2)

$$\text{Cos }\alpha=\frac{0+0+\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\ \sqrt{1+\lambda_2^2}} \Rightarrow \left(1+\lambda_1^2\right)\left(1+\lambda_2^2\right)=\lambda_1^2\lambda_2^2\ \text{Sec}^2\alpha$$

$$\Rightarrow 1+\lambda_1^2+\lambda_2^2=\lambda_1^2\lambda_2^2\left(\text{Sec}^2\alpha-1\right)=\lambda_1^2\lambda_2^2\,\text{Tan}^2\alpha\cdots\cdots\cdots\cdots(3)$$

from (1) and (2) $\lambda_1=\frac{-y}{z}, \lambda_2=\frac{-x}{z}$

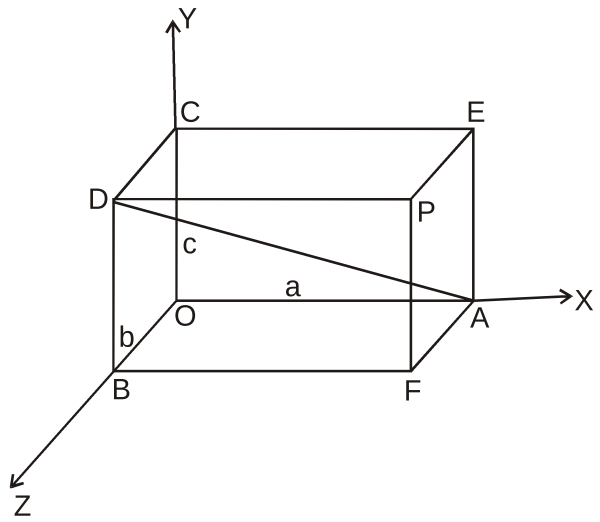
Substituting these values in (3)

$$1+\frac{y^2}{z^2}+\frac{x^2}{z^2}=\frac{y^2x^2}{z^4}\tan^2\alpha$$

$$\Rightarrow z^2\left(x^2+y^2+z^2\right)=x^2y^2 \tan^2\alpha .$$

3.6.13 S.A.Q.: Take the three coterminous edges of the rectangular parallelopiped as the coordinate axes. Let $OA = a, OB = b$ and $OC = c$.

Then the coordinates of different vertices are as follows:



$O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$

$C(0,0,c)$, $D(0,b,c)$, $E(a,0,c)$

$F(a,b,0)$, $P(a,b,c)$

AD is a diagonal of the parallelopiped and OB is the edge not meeting AD.

We shall find the shortest distance between AD and OB.

Equation of AD is $\frac{x-a}{0-a}=\frac{y}{b-0}=\frac{z}{c-0} \Rightarrow \frac{x-a}{-a}=\frac{y}{b}=\frac{z}{c}\cdots\cdots(1)$

The equation of OB is $\frac{x}{0}=\frac{y-b}{1}=\frac{z}{0}\cdots\cdots\cdots(2)$

A vector along the line (1) is $(-a,b,c)$

A vector along the line (2) is $(0,1,0)$

Let the line of shortest distance between the lines (1) and (2) be L.

Since L is perpendicular to both the lines (1) and (2), a vector along L is

$(0,1,0)\times(-a,b,c)$

$$(0,1,0)\times(-a,b,c) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 1 & 0 \\ -a & b & c \end{vmatrix} = c\bar{i} + o\bar{j} + a\bar{k} = (c,0,a)$$

unit vector along $L = \dfrac{(c,0,a)}{\sqrt{c^2+a^2}}$

Now the length of L between AD and OB

= the projection of the join of $(a,0,0)$ and $(0,b,0)$ along L.

$$= \frac{(a-0,\ 0-b,\ 0)\cdot(c,0,a)}{\sqrt{c^2+a^2}} = \frac{ac - b(0) + 0(a)}{\sqrt{c^2+a^2}} = \frac{ac}{\sqrt{c^2+a^2}}$$

Similarly the shortest distance between the other pairs of lines can be obtained as

$$\frac{ab}{\sqrt{a^2+b^2}}, \frac{bc}{\sqrt{b^2+c^2}}$$

3.9 Summary:

After studying this lesson the student should get knowledge on finding a image of a point and a line in a plane, the shortest distance between two lines and the characterizations for intersections of three planes.

3.10 Technical Terms:

(i) Image

(ii) Coplanarity

(iii) Skew lines

(iv) Shortest Distance

(v) Triangular Prism

3.11 Model Examination Questions:

1. Find the image of the point $A(2,-1,3)$ in the plane $\pi \equiv 3x-2y-z=9$.

2. Find the image of the point $A(1,6,3)$ in the line $L:\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

3. Find the image of the line $L:\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ in the plane $\pi \equiv x+y+z+1=0$.

4. Show that the lines $\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}$; $3x+2y+z-2=0$, $x-3y+2z-13=0$ are coplanar. Find the equation of the plane in which they lie.

5. Find the shortest distance and the equations of the line of the shortest distance between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$; $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$.

6. Find the shortest distance and the equations of the line of the shortest distance between the lines $\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2}$; $5x-2y-3z+6=0=x-3y+2z-3$.

7. Find the shortest distance between Z - axis and the line

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2.$$

8. Find the area and the length of the edges of a normal section of the prism $2x+y+z-3=0$, $x-y+2z-4=0$, $x+z-2=0$.

9. Show that the planes $2x-3y-7z=0$, $3x-14y-13z=0$, $8x-31y-33z=0$ pass through one line and find its equations.

10. Examine the nature of intersection of the set of planes

$$x+y+z+3=0,\ 3x+y-2z+2=0,\ 2x+4y+7z-7=0$$

3.12 Exercises:

1. Find the image of the point $(1,3,4)$ in the plane $2x-y+z+3=0$.

Ans: $(-3, 5, 2)$

2. Find the image of the point $(-2, 1, 3)$ in the line $\frac{x-2}{2}=\frac{y-2}{3}=\frac{z-6}{2}$.

Ans: $(2, -3, 5)$

3. Find the image of the line in the plane $3x-3y+10z-26=0$.

Ans: $$\frac{x-4}{9}=\frac{y+1}{-1}=\frac{z-7}{-3}$$

4. Show that the lines $x=\frac{y-2}{2}=\frac{z+3}{3}$; $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ are coplanar. Find the point of intersection and the plane containing the lines.

Ans: $(2,6,3)$, $x-2y+z+7=0$

5. Find the equations of the line passing through the point $(1,0,-1)$ and intersecting the lines $x=2y=2z$ and $3x+4y=1,\ 4x+5z=2$.

Ans: $x-3y+z=0=16x-12y+11z-5$.

6. Find the length and equations of the S.D. line between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4};\quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

Ans: $\frac{1}{\sqrt{6}}$, $11x+2y-7z+6=0=7x+y-5z+7$

7. Find the length and equations of the S.D. between the lines $2x-3y+4z=0=x-y+z$; $x+y+2z-3=0=2x+3y+3z-4$.

Ans: $\frac{13}{\sqrt{66}}$, $3x-y-z=0=x+2y+z-1$

8. Show that the shortest distance between a diagonal and an edge not meeting it in a cube of edge a is a $\sqrt{2}$

9. Find the point on the line through the points $(-6,1,-10)$, $(-3,7,-13)$ which is nearest to the line $3x+2y-15z-8=0,\quad 3x-y-3z+32=0$.

Ans: $$\left(\frac{-3433}{321},\frac{-2113}{321},\frac{-1693}{321}\right)$$

10. Examine the nature of intersection of the set of planes $2x-5y+z=3,$ $x+y+4z=5,\ x+3y+6z=1$.

Ans: The planes form a prism.

11. Show that the planes $2x-3y-7z=0, 3x-14y-13z=0, \quad 8x-31y-33z=0$ pass through one line and find its equations.

Ans: $\frac{x}{59}=\frac{y}{-5}=\frac{3}{19}$

3.13 Model Practical Problem with Solution:

Problem: Find the length and equations of the shortest distance line between the lines $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-1}{2};\ \frac{x-4}{4}=\frac{y-5}{5}=\frac{z-2}{3}$.

Definitions:

1. **Skew lines:** Any two lines which are non - parallel and non - intersecting are called skew lines.

2. **Shortest distance:** The shortest distance between two lines is the perpendicular distance between the given lines.

Results Used:

1. Given L_1, L_2 are non - coplanar lines. Another line L intersects L_1, L_2 at G, H respectively and L is perpendicular to both L_1, L_2. Then the length of GH is the shortest distance between L_1 and L_2.

2. Given the lines L_1 is $\frac{x-x_1}{\ell_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and L_2 is $\frac{x-x_2}{\ell_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$. The shortest distance between L_1, L_2 is the absolute value of

$$\frac{1}{\sqrt{\Sigma\left(m_1n_2-m_2n_1\right)^2}}\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}$$

Stepwise Division of Problem:

Given lines are in symmetric form:

Step (1): To find the points G, H on the given lines respectively such that the line joining of the two points is perpendicular to both the given lines.

Step (2): To find the distance between G and H which is the shortest distance between two lines.

Step (3): To find theequation of the line joining G and H.

Solution:

Step (1): Given lines are $L_1 : \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2} = t$ (say)

$L_2 : \frac{x-4}{4} = \frac{y-5}{5} = \frac{z-2}{3} = s$ (say)

Any point on L_1 is $(3t+2, 4t+3, 2t+1)$ $(t \in \mathbb{R})$

Any point on L_2 is $(4s+4, 5s+5, 3s+2)$ $(s \in \mathbb{R})$

If $G = (3t+2, 4t+3, 2t+1)$ and $H = (4s+4, 5s+5, 3s+2)$

and GH is perpendicular to both L_1 and L_2 then

d.rs. of GH are $3t-4s-2, 4t-5s-2, 2t-3s-1$

$GH \perp L_1 \Rightarrow 3(3t-4s-2)+4(4t-5s-2)+2(2t-3s-1)=0$

$GH \perp L_2 \Rightarrow 4(3t-4s-2)+5(4t-5s-2)+3(2t-3s-1)=0$

Then $29t - 38s - 16 = 0$

$38t - 50s - 21 = 0$

Solving these two equations

$$\frac{t}{-2} = \frac{s}{1} = \frac{1}{-6} \Rightarrow t = \frac{1}{3}, \quad s = -\frac{1}{6}$$

$$G = \left(3, \frac{13}{3}, \frac{5}{3}\right), \; H = \left(\frac{10}{3}, \frac{25}{6}, \frac{3}{2}\right)$$

Step (2): $GH = \sqrt{\left(\frac{10}{3}-3\right)^2 + \left(\frac{13}{3}-\frac{25}{6}\right)^2 + \left(\frac{5}{3}-\frac{3}{2}\right)^2}$

$$= \sqrt{\frac{1}{9}+\frac{1}{36}+\frac{1}{36}} = \frac{1}{\sqrt{6}}$$

Step (3): d.rs. of GH are $\frac{1}{3}, \frac{1}{6}, \frac{1}{6}$ i.e. $2,1,1$.

Equations of the line GH in shortest distance are

$$\frac{x-3}{2} = \frac{y-\frac{13}{3}}{1} = \frac{z-\frac{5}{3}}{1} \quad ; \quad \frac{x-3}{2} = \frac{3y-13}{3} = \frac{3z-5}{3}$$

Lesson Writer

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Lesson - 4

SPHERE 1 - EQUATIONS OF A SPHERE

4.1 Objective of the lesson:

After studying this lesson, the student should be able to

- Find the equation of a sphere with given radius and centre
- Know some characterizations that a second degree equation is x, y, z represents a sphere
- Find the equation of a sphere through four noncoplanar points the equation of a sphere that contains a circle on it.

4.2 Structure:

This lesson contains the following components:

4.3 Introduction

4.4 Equation of Sphere

4.5 Examples

4.6 Plane section of a sphere

4.7 Intersection of two spheres

4.8 Examples

4.9 Summary

4.10 Technical Terms

4.11 Exercise

4.12 Model Examination Questions

4.13 Model Practical Problem with Solution

4.3 Introduction:

The locus of the points in a plane at a given distance from a fixed point in the plane is a circle and is represented by the quadratic equation.

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

It is natural to consider this locus in the three dimensional space also. The locus of a point in t he three dimensional space at a distance from a fixed point in the space is called a sphere. In this lesson we investigate various types of representations for a sphere. The equation of the sphere, as defined above, the equation of the sphere through four non coplanar points, section of a sphere and a plane as well of two spheres are discussed in this lesson.

4.4 Equations of a sphere:

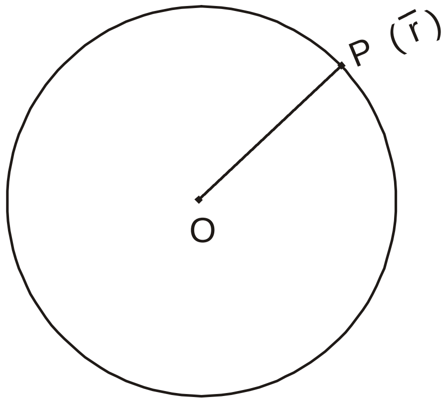
4.4.1 Def: A sphere is the locus of a point which is at a constant distance from a fixed point. The constant distance is called the radius and the fixed point the centre of the sphere.

4.4.2 Def: A sphere of radius zero is called a point sphere.

4.4.3 Theorem: The vector equation of the sphere with centre $A(x_1y_1z_1)$ having position vector $\bar{d}$ and radius a is $\left|\bar{r}-\bar{d}\right|^2 = a^2$ and the cartesian form is

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=a^2$$

Proof: Let $P(x_1y_1z_1)$ be any point with position vector $\bar{r}$ then



$$\overline{OA}=\bar{d}, \overline{OP}=\bar{r} \quad \text{and} \quad \overline{AP}=\overline{OP}-\overline{OA}=\bar{r}-\bar{d}.$$

P lies on the sphere $\Leftrightarrow$ AP = a

$$\Leftrightarrow \left|\overline{AP}\right| = a \quad \Leftrightarrow \left|\bar{r}-\bar{d}\right| = a \quad \Leftrightarrow \left|\bar{r}-\bar{d}\right|^2 = a^2$$

$$\Leftrightarrow |\bar{r}|^2 - 2\,\bar{r}\cdot\bar{d} + |\bar{d}|^2 - a^2 = 0$$

Thus the vector equation of a sphere with centre $A\left(\bar{d}\right)$ and radius a is

$$\boxed{\left|\bar{r}-\bar{d}\right|^2 = a^2}$$

and the cartesian equation is

$$\boxed{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=a^2}$$

4.4.4 Corollary: The vector equation of the sphere with centre at the origin and radius a is $|\bar{r}|^2=a^2$. Its cartesian form is $x^2+y^2+z^2=a^2$.

4.4.5 Note: It can be observed that

(i) The equation of a sphere is a second degree equation in x, y, z.

(ii) The coefficients of $x^2,\ y^2, z^2$ in the equation of a sphere are equal.

(iii) The equation of a sphere is of the form

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

More precisely we have the following.

4.4.6 Theorem: If $u^2+v^2+w^2-d>0$ then the equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

represents a sphere with centre $(-u,-v,-w)$ and radius $\sqrt{u^2+v^2+w^2-d}$.

Proof: Given equation is $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$$\Rightarrow x^2+2ux+u^2+y^2+2uy+v^2+z^2+2wz+w^2=u^2+v^2+w^2-d$$

$$\Rightarrow (x+u)^2+\left(y+v^2\right)+(z+w)^2=\left(\sqrt{u^2+v^2+w^2-d}\right)^2$$

$$\Rightarrow [x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2=\left(\sqrt{u^2+v^2+w^2-d}\right)^2$$

which represents a sphere with centre $(-u,-v,-w)$ and radius

$\sqrt{u^2+v^2+w^2-d}$.

4.4.7 Theorem: The necessary and sufficient conditions that a general second degree equation in x, y, z.

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \cdots\cdots\cdots (1)$$

represents a sphere are

(i) $a = b = c \neq 0$ (ii) $f = g = h = 0$ (iii) $u^2 + v^2 + w^2 - d > ad$

Proof: Necessity:

Let $S = 0$ be a sphere, with radius $r(>0)$ and centre $A = (\xi, \eta, \zeta)$. Shift the origin to A. If (x^1, y^1, z^1) are the new coordinates of the point (x, y, z) then

$$x = x^1 + \xi,\ y = y^1 + \eta,\ z = z^1 + \zeta$$

Now, S becomes

$$S(x^1 + y^1 + z^1) \equiv ax^{1^2} + by^{1^2} + cz^{1^2} + 2fy^1z^1 + 2gz^1x^1 + 2hx^1y^1 + 2x^1(a\xi + h\eta + g\zeta + u)$$

$$+2y^1(h\xi + b\eta + f\zeta + v)$$

$$+2z^1(g\xi + f\eta + e\zeta + w) + S(\xi, \eta, \zeta) = 0 \cdots\cdots\cdots (1)$$

For every θ, the distance between two points $(r \text{ Cos } \theta, r \text{ Sin } \theta, 0)$ and $(0,0,0)$ is r.

The points $(r \text{ Cos } \theta, r \text{ Sin } \theta, 0), \forall\, \theta$ lie one the sphere $S = 0$

Write $\xi_1 = a\xi + h\eta + g\zeta + u$

$\eta_1 = h\xi + b\eta + f\zeta + v$

$\zeta_1 = f\xi + f\eta + e\zeta + w$

$d_1 = S(\xi, \eta, \zeta)$

and $x^1 = r \text{ Cos } \theta,\ y^1 = r \text{ Sin } \theta, z^1 = 0$ in (1) we get

$$r^2\left(a\cos^2\theta + b\sin^2\theta + 24\sin\theta\cos\theta\right) + 2r(\xi_1\cos\theta + \eta_1\sin\theta) + d_1 = 0 \text{ ....(2)}$$

Similarly, substitue $x^1 = r\cos(\theta + \pi),\ y^1 = r\sin(\theta + \pi),\ z^1 = 0$ in (1) we get

$$r^2\left(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta\right) - 2r\left(\xi_1\cos\theta + \eta_1\sin\theta\right) + d_1 = 0 \cdots\cdots(3)$$

From (2) and (3) we get $\xi_1\cos\theta + \eta_1\sin\theta = 0 \cdots\cdots\cdots\cdots(4)$

Then (2) becomes

$$r^2\left(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta\right) + d_1 = 0,\ \forall\,\theta \cdots\cdots\cdots(5)$$

For $\theta = 0$ and $\theta = \frac{\pi}{2}$ equation (5) becomes

$$ar^2 + d_1 = 0,\ br^2 + d_1 = 0 \Rightarrow a = b$$

If $a = 0$ then $d_1 = 0 \Rightarrow S(\xi, \eta, \zeta) = 0 \Rightarrow$ the centre (ξ, η, ζ) lies on the sphere $S = 0$. i.e. radius of the sphere r is zero. Which contradicts our assumption that $r > 0$. So $a \neq 0$.

For $\theta = \frac{\pi}{4}$ and $\theta = \frac{-\pi}{4}$, (5) becomes

$$r^2\left(\frac{a+b}{2} + h\right) + d_1 = 0,\ r^2\left(\frac{a+b}{2} - h\right) + d_1 = 0 \Rightarrow h = 0$$

Thus we prove that the point $(r\cos\theta,\ \sin\theta, 0)$ lies on the sphere whose centre is origin and radius r, then we get $a = b \neq 0$ and $h = 0$.

Similarly by considering the points $(0, r\cos\theta, r\sin\theta)$ and $(r\cos\theta, 0, r\sin\theta)$ on the sphere $S = 0$ we get,

$$b = e \neq 0,\ f = 0 \text{ and } c = a \neq 0,\ g = 0$$

Then, $S = 0$ becomes

$$S \equiv a\left(x^2 + y^2 + z^2\right) + 2ux + 2vy + 2wz + d$$

$$\equiv a\left\{\left[x + \frac{u}{a}\right]^2 + \left(y + \frac{v}{a}\right)^2 + \left(x + \frac{w}{a}\right)^2 - \frac{u^2 + v^2 + w^2 - ad}{a^2}\right\} \cdots\cdots(6)$$

$$\Rightarrow \frac{u^2+v^2+w^2-ad}{a^2} = r^2 > 0$$

$$\therefore\ u^2+v^2+w^2-ad > 0$$

Sufficiency:

If $a = b = e \neq 0,\ f = g = h = 0$ then $S = 0$ becomes

$$S \equiv x^2+y^2+z^2+\frac{2u}{a}x+\frac{2v}{a}y+\frac{2w}{a}z+\frac{d}{a}=0$$

Since $u^2+v^2+w^2 > ad,\ \frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}-\frac{d}{a}=\frac{u^2+v^2+w^2-ad}{a^2}>0$

S = 0 becomes

$$S \equiv \left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}-\frac{d}{a}$$

$$=\frac{u^2+v^2+w^2-ad}{a^2}>0$$

$$\Rightarrow S \equiv \left[x-\left(\frac{-u}{a}\right)\right]^2+\left[y-\left(\frac{-v}{a}\right)\right]^2+\left[z-\left(\frac{-w}{a}\right)\right]^2=\left(\frac{\sqrt{u^2+v^2+w^2-ad}}{a^2}\right)^2$$

Which represents a sphere with centre $\left(\frac{-u}{a},\frac{-v}{a},\frac{-w}{a}\right)$ and radius $=\frac{\sqrt{u^2+v^2+w^2-ad}}{a}$.

4.4.8 Example: Find the centre and raius of the following spheres:

(i) $x^2+y^2+z^2-6x+8y-10z+1=0$

(ii) $x^2+y^2+z^2+2x-2y-6z+5=0$

(iii) $2\left(x^2+y^2+z^2\right)-2x+4y+2z+3=0$

Solution:

(i) centre $=(3,-4,5)$ and radius $=\sqrt{3^2+(-4)^2+5^2-1}=7$

(ii) centre $=(-1, 2, 3)$ and radius $= \sqrt{(-1)^2+2^2+3^2-5}=3$

(iii) Given equation can be written as

$$x^2+y^2+z^2-x+2y+z+\frac{3}{2}=0$$

centre $=\left(\frac{1}{2}, -1, \frac{-1}{2}\right)$, radius $=\sqrt{\left(\frac{1}{2}\right)^2+(-1)^2+\left(\frac{-1}{2}\right)^2-\frac{3}{2}}=0$

4.4.9 Example: A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere.

Solution: Take the centre of the cube as the origin and the planes through the centre parallel to its faces as coordinate planes.

Let each of the edge of the cube be equal to 2a.

Then the equations of the faces of the cube are

$$x=a,\ x=-a,\ y=a,\ y=-a,\ z=a,\ z=-a$$

$P(f, g, h)$ be a point of the locus.

$\Leftrightarrow (f-a)^2+(f+a)^2+(g-a)^2+(h-a)^2+(h+a)^2=k^2$ (k constant)

$\Leftrightarrow 2\left(f^2+g^2+h^2+3a^2\right)=k^2$

The locus of P is $2\left(x^2+y^2+z^2+3a^2\right)=k^2$

which is a sphere.

4.4.10 Example: A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular drawn to it from the origin is the sphere

$$x^2+y^2+z^2-ax-by-cz=0.$$

Solution: Let $A=(a,b,c)$ be a fixed point on the plane π. $M(x_1, y_1, z_1)$ is the foot of the perpendicular from origin to the plane π.

d.rs. of $OM = x_1, y_1, z_1$

d.rs. of $MA = a-x_1, b-y_1, c-z_1$

$$OM \perp MA$$

$$\Leftrightarrow x_1(a-x_1)+y_1(b-y_1)+z_1(c-z_1)=0$$

$$\Leftrightarrow x_1^2+y_1^2+z_1^2-ax_1-by_1-cz_1=0$$

$\Leftrightarrow M(x_1, y_1, z_1)$ lies on the sphere

$$x^2+y^2+z^2-ax-by-cz=0$$

Note: The above example can also be stated as follows:

The equation of the sphere through origin and making intercepts a, b, c on the axes is $x^2+y^2+z^2-ax-by-cz=0$.

4.4.11 Example: Through a point P three mutually perpendicualr straight lines are drawn; one passes through a fixed point C on the z - axis; while the others intersect the x - axis and y - axis respectively, show that the locus of P is a sphere of which C is the centre.

Solution: Let $P(x_1, y_1, z_1)$ be any point on the space.

Let PA, PB, PC be three mutually perpendicular straight lines through P satisfying the hypothesis.

Let the lines PA, PB, PC be meet the X, Y, Z axes at the points $A(a,0,0), B(0,b,0), C(0,0,c)$ respectively.

D.r's of PA are (x_1-a, y_1, z_1)

D.r's of PB are (x_1, y_1-b, z_1)

D.r's of PC are (x_1, y_1, z_1-c)

P lies on the locus $\Leftrightarrow$ PA, PB, PC are three mutually perpendicular line

$$\Leftrightarrow PA \perp PB, PB \perp PC \text{ and } PC \perp PA$$

$$\Leftrightarrow (x_1-a)x_1+y_1(y_1-b)+z_1^2=0,$$

$$x_1^2+y_1(y_1-b)+z_1(z_1-c)=0 \text{ and}$$

$$x_1(x_1-a)+y_1^2+z_1(z_1-c)=0$$

(or) $$x_1^2+y_1^2+z_1^2-ax_1-by_1=0 \cdots\cdots\cdots(1),$$

$$x_1^2 + y_1^2 + z_1^2 - by_1 - cz_1 = 0 \cdots\cdots\cdots (2)$$ and

$$x_1^2 + y_1^2 + z_1^2 - ax_1 - cz_1 = 0 \cdots\cdots\cdots (3)$$

$\Leftrightarrow$ $(2)+(3)-(1) \Rightarrow x_1^2 + y_1^2 + z_1^2 - 2cz_1 = 0$

$\Leftrightarrow$ The point $P(x_1, y_1, z_1)$ lies on the sphere

$x^2 + y^2 + z^2 - 2cz = 0$ with centre $C(0,0,c)$.

4.4.12 Definition: Let A, B be any two points on the sphere. The segment AB is called a <u>diameter</u> of the sphere it contains the centre of the sphere.

4.4.13 Definition: Spheres with the same centre are called <u>concentric</u> spheres.

4.4.14 Example: Find the equation of the sphere of radius 4, concentric with the sphere

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 1 = 0$$

Solution: Centre of the given sphere is $(1,1,1)$.

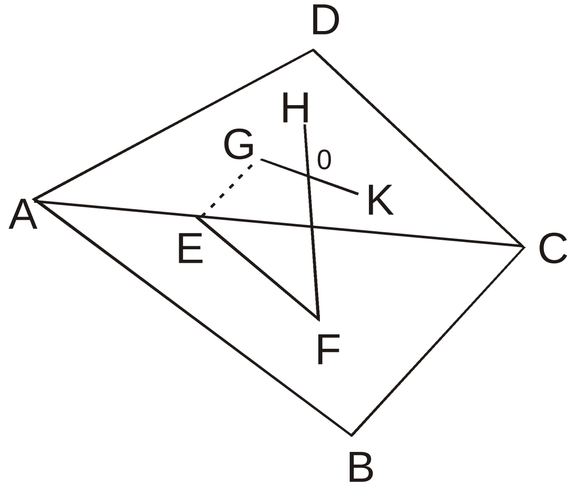
The equation of the required sphere is

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 4^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 2y - 2z - 13 = 0$$

4.4.15 Theorem: One sphere and only one can pass through any four points not in the same plane.

Proof:



Let A, B, C, D be four points not in the same plane; and let F and G be the circum - centres of the triangles ABC, ADC.

Let FH, GK be the perpendiculars to the planes ABC, ADC through F and G respectively.

Then every point in FH is equidistant from A, B, C; and every point in GK is equidistant from A, D, C; hence every point in FH and in GK is equidistant from A nd C.

But the locus of points equidistante from A and C is the plane which bisects AC at right angles.

So FH and GK both lie in this plane, and since they cannot be perpendicular (being perpendiculars to intersecting pianes) they must meet at some point O.

Then 0, the only point common to FH and GK is equidistant from A, B, C and D.

A sphere having its centre at 0 and radius OA, will pass through A, B, C and D; and this is the only sphere that can pass through the four given points.

4.4.16 Thorem: The equation of a sphere passing through four non - coplanar points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+x_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \cdots\cdots\cdots(I)$$

Proof: Let $x^2+y^2+z^2+2ux+2vy+2wz+d=0 \cdots\cdots(1)$

be the general equation of the sphere.

The sphere (1) passes through the given four point A, B, C, D iff

$$\left.\begin{aligned} 2x_1u+2y_1v+2z_1w+1\cdot d &= S_1 \\ 2x_2u+2y_2v+2z_2w+1\cdot d &= S_2 \\ 2x_3u+2y_3v+2z_3w+1\cdot d &= S_3 \\ 2x_4u+2y_4v+2z_4w+1\cdot d &= S_4 \end{aligned}\right\} \cdots\cdots\cdots\cdots(II)$$

where $S_i = -x_i^2 - y_i^2 - z_i^2;\ i = 1, 2, 3, 4.$

This system has a unique solution iff $\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \neq 0$

This is true because A, B, C, D are non coplanar.

If (u, v, w, d) is the unique solution of (II) and

$$u = \frac{\Delta_1}{\Delta}, v = \frac{\Delta_2}{\Delta}, \ w = \frac{\Delta_3}{\Delta}, d = \frac{\Delta_4}{\Delta}.$$

Where Δ_i is obtained by replacing the i^{th} column of Δ by the column vector (S_1, S_2, S_3, S_4).

Replacing u, v, w and d by $\frac{\Delta_1}{\Delta}, \frac{\Delta_2}{\Delta}, \frac{\Delta_3}{\Delta}$ and $\frac{\Delta_4}{\Delta}$ respectively in (1) we get on simplification (I).

4.5 Examples:

4.5.1 Example: Find the equation of the sphere through the four points $(4,-1,2), (0,-2,3), (1,-5,-1), (2,0,1)$

Solution: If the given points lie on the sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

we get on substitution,

$$8u - 2v + 4w + d = -21 \cdots\cdots\cdots\cdots (1)$$

$$-4v + 6w + d = -13 \cdots\cdots\cdots\cdots (2)$$

$$2u - 10v - 2w + d = -27 \cdots\cdots\cdots (3)$$

$$4u + 2w + d = -5 \cdots\cdots\cdots (4)$$

Solving $(1),(2),(3)$ and (4) we get $u = -2, v = 3, w = -1, d = 5$

$\therefore$ The required sphere equation is

$$x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$$

4.5.2 Example: Find the equation of the sphere through the four points $(0,0,0,)$, $(-a,b,c)$, $(a,-b,.c)$, $(a,b,-c)$ and determine its radius.

Solution: If the given points lie on the sphere.

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

we get on substitution,

$$d = 0 \cdots\cdots\cdots\cdots (1)$$

$$-2au + 2bv + 2wc = -\left(a^2 + b^2 + c^2\right) \cdots\cdots\cdots\cdots (2)$$

$$2au - 2bv + 2wc = -\left(a^2 + b^2 + c^2\right) \cdots\cdots\cdots (3)$$

$$2au + 2bv - 2wc = -\left(a^2 + b^2 + c^2\right) \cdots\cdots\cdots (4)$$

By solving (2), (3), (4) we get

$$u = \frac{-\left(a^2 + b^2 + c^2\right)}{2a}, \quad v = \frac{-\left(a^2 + b^2 + c^2\right)}{2b}, \quad w = \frac{-\left(a^2 + b^2 + c^2\right)}{2c}$$

Thus the required sphere equation is

$$x^2 + y^2 + z^2 - \left(a^2 + b^2 + c^2\right)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

radius $= \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{\frac{\left(a^2 + b^2 + c^2\right)}{4a^2} + \frac{\left(a^2 + b^2 + c^2\right)^2}{4b^2} + \frac{\left(a^2 + b^2 + c^2\right)^2}{4c^2} - 0}$

$$= \frac{\left(a^2 + b^2 + c^2\right)}{2}\sqrt{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}$$

4.5.3 Example: Obtain the equation of the sphere circumscribing the tetrahedron whose faces are $x = 0,\ y = 0,\ z = 0,\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \cdots\cdots\cdots (1)$

Solution: Solving the equation in (1) we get the vetices of the tertrahidron. They are

$$(0,0,0),\ (a,0,0),\ (0,b,0),\ (0,0,c)$$

If the equation of the sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

passes through the vertices of tetrahedron then we get

$$d = 0\,,\ 2au = -a^2 \Rightarrow u = {}^{-a}\!/_2$$

similarly $\quad v = {}^{-b}\!/_2\,,\ \ w = {}^{-c}\!/_2$

Thus, the required sphere equation is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

4.5.4 Example: Show that the equation of the sphere passing through the three points $(3,0,2), (-1,1,1,), (2,-5,4)$ and having its centre on the plane $2x + 3y + 4z = 6$ is $x^2 + y^2 + z^2 + 4x - 6z - 1 = 0$.

Solution: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2uy + 2wz + d = 0 \cdots\cdots\cdots\cdots(1)$$

If the sphere (1) passes through $(3,0,2), (-1,1,1), (2,-5,4)$, we get

$$6u + 4w + d = -13 \cdots\cdots(2)$$

$$-2u + 2v + 2w + d = -3 \cdots\cdots\cdots\cdots(3)$$

$$4u - 10v + 8w + d = -45 \cdots\cdots\cdots\cdots(4)$$

If the centre $(-u, -v, -w)$ lies on the plane $2x + 3y + 4z = 6$, we get

$$-2u - 3v - 4w = 6 \cdots\cdots\cdots(5)$$

Solving (2), (3), (4) and (5) we get

$$u = 0, v = 2,\ w = -3,\ d = -1$$

Thus the equation of the required sphere is

$$x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$$

4.5.5 Example: Obtain the equation of the sphere which passes through three points $(1,0,0), (0,1,0), (0,0,1)$ and has its radius, as small as possible.

Solution: Let the required equation of the sphere be

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \cdots\cdots(1)$$

If the sphere (1) passes through $(1,0,0), (0,1,0), (0,0,1)$, we get

$$1 + 2u + d = 0 \Rightarrow u = \frac{-(d+1)}{2},$$

$$1 + 2v + d = 0 \Rightarrow v = \frac{-(d+1)}{2},$$

$$1 + 2w + d = 0 \Rightarrow w = \frac{-(d+1)}{2}.$$

$$\Rightarrow u = v = w = \frac{-(d+1)}{2}$$

Let $r = \sqrt{u^2 + v^2 + w^2 - d}$ be the radius of the sphere.

$$\Rightarrow r^2 = u^2 + v^2 + w^2 - d = \left(\frac{-(d+1)}{2}\right)^2 + \left(\frac{-(d+1)}{2}\right)^2 + \left(\frac{-(d+1)}{2}\right)^2 - d$$

$$= \frac{3}{4}(d+1)^2 - d.$$

Let $f(d) = \frac{3}{4}(d+1)^2 - d.$

If r is least then r^2 is least i.e. f is the least. So $f^1(d) = 0$

$$\Rightarrow \frac{3}{4} 2(d+1) - 1 = 0 \Rightarrow d = \frac{-1}{3}$$

This value of d makes $f^{11}(d)$ positive and hence f is the least and here r is the least when $d = -1/3$.

Thus $u = v = w = \dfrac{-\left(\frac{-1}{3}+1\right)}{2} = \dfrac{-1}{3}$

The equation of the required sphere is

$$x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$$

i.e., $3\left(x^2+y^2+z^2\right)-2(x+y+z)-1=0$

4.5.6 Example: Obtain the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the two points $(0,-2,-4)$, $(2,-1,-1)$.

Solution: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0\cdots\cdots\cdots\cdots\cdots(1)$$

Its centre $(-u,-v,-w)$ lies in the given line then

$$-5v-2w=0\cdots\cdots\cdots(2)$$

$$2u-3v=0\cdots\cdots\cdots\cdots(3)$$

If the sphere passes through the points $(0,-2,-4)$, $(2,-1,-1)$.

Then $-4v-8w+d=-20\cdots\cdots\cdots\cdots\cdots(4)$

$$4u-2v-2w+d=-6\cdots\cdots\cdots\cdots(5)$$

Solving (2), (3), (4) and (5) we get

$$u=-3,\ v=-2,\ w=+5,\ d=12$$

Thus the required sphere equation is

$$x^2+y^2+z^2-6x-4y+10z+12=0$$

4.5.7 Example: A sphere whose centre lies in the positive octant passes through the origin and cuts the planes $x=0$ $y=0$, $z=0$ in circles of radii $a\sqrt{2}$, $b\sqrt{2}$, $c\sqrt{2}$ respectively; show that its equation is

$$x^2+y^2+z^2-2\sqrt{\left(b^2+c^2-a^2\right)}\ x-2\sqrt{\left(c^2+a^2-b^2\right)}\ y-2\sqrt{\left(a^2+b^2-c^2\right)}z=0$$

Solution: Since the sphere passes through origin so its equation may be taken as

$$x^2+y^2+z^2+2ux+2vy+2wz=0\cdots\cdots\cdots(1)$$

Its centre is $(-u,-v,-w)$ which lies in the positive octant and hence each coordinate will be positive, so that each of u, v, w be chosen as negative.

The sphere (1) cuts the plane $z=0$ in the circle.

$x^2+y^2+2ux+2vy=0$ whose radius is $c\sqrt{2}$

$$\Rightarrow \sqrt{u^2+v^2} = c\sqrt{2} \;\Rightarrow\; u^2+v^2=2c^2 \cdots\cdots\cdots(2)$$

Similarly we have $v^2+w^2=2a^2 \cdots\cdots\cdots\cdots(3)$

$$u^2+w^2=2b^2 \cdots\cdots\cdots(4)$$

adding (2), (3) and (4) we get $u^2+v^2+w^2=a^2+b^2+c^2 \cdots\cdots\cdots(5)$

Subtracting each of the equations (2), (3), (4) from (5) we get

$$u^2=b^2+c^2-a^2, \quad v^2=c^2+a^2-b^2, \quad w^2=a^2+b^2-c^2$$

Since u,v,w are negative.

$$u=-\sqrt{b^2+c^2-a^2}, \; v=-\sqrt{c^2+a^2-b^2}, \quad w=-\sqrt{a^2+b^2-c^2}$$

Hence, the equation of the sphere is

$$x^2+y^2+z^2-2\sqrt{\left(b^2+c^2-a^2\right)}x-2\sqrt{\left(c^2+a^2-b^2\right)}\,y-2\sqrt{\left(a^2+b^2-c^2\right)}z=0$$

4.5.8 Example: A variable plane passes through a fixed point (a,b,c) and cuts the axes in A,B,C. Show that the locus of the centre of the sphere $OABC$ is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

Solution: Since such a plane cannot be any of the coordinate planes, we may take the equation of the plane to be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \cdots\cdots\cdots(1)$$

The plane (1) meets the coordinate axes in $A(\alpha,0,0)$, $B(0,\beta,0)$ and $c\,(0,0,\gamma)$.

Equation of the sphere OABC is

$$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0 \cdots\cdots\cdots\cdots(2)$$

Let $P(x_1, y_1, z_1)$ be any point in the space.

$P(x_1, y_1, z_1)$ is the centre of the sphere (2)

$$\Leftrightarrow (x_1, y_1, z_1) = \left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right)$$

$$\Leftrightarrow \alpha = 2x_1,\ \beta = 2y_1,\ \gamma = 2z_1$$

The equation of the plane (1) becomes

$$\frac{x}{2x_1} + \frac{y}{2y_1} + \frac{z}{2z_1} = 1$$

This plane passes through a fixed point (a, b, c) then

$$\frac{a}{2x_1} + \frac{b}{2y_1} + \frac{c}{2z_1} = 1$$

or $$\frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$$

$\Leftrightarrow$ The point $P(x_1, y_1, z_1)$ lies on the plane.

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

Which is the required locus.

4.5.9 Example: A sphere of constant radius r passes through the origin o and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from 0 to the plane ABC.

Solution: Let the plane ABC be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \cdots\cdots\cdots (1)$$

$$\therefore\ A = (a, 0, 0), \quad B = (0, b, 0), \quad C = (0, 0, c)$$

Equation of the sphere through O,A,B,C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \cdots\cdots\cdots (2)$$

Its radius is $r^2 = \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}$

$$\Rightarrow \quad a^2 + b^2 + c^2 = 4r^2 \cdots\cdots\cdots\cdots(3)$$

Equations to the line 0 and perpendicular to the plane (1) are

$$\frac{x-0}{1/a} = \frac{y-0}{1/b} = \frac{z-0}{1/e} = \lambda \text{ (say)} \cdots\cdots\cdots\cdots(4)$$

Any point on the line (4) is $\left(\frac{\lambda}{a}, \frac{\lambda}{b}, \frac{\lambda}{c}\right)$.

$P(x_1, y_1, z_1)$ is the foot of the perpendicular from '0' to the plane $\overleftrightarrow{ABC}$.

$\Leftrightarrow$ P lies on the line (4) and is of the form $\left(\frac{\lambda}{a}, \frac{\lambda}{b}, \frac{\lambda}{c}\right)$

$$\Leftrightarrow x_1 = \frac{\lambda}{a},\ y_1 = \frac{\lambda}{b},\ z_1 = \frac{\lambda}{c}$$

$$\Leftrightarrow \quad a = \frac{\lambda}{x_1},\ b = \frac{\lambda}{y_1},\ \ c = \frac{\lambda}{z_1} \cdots\cdots\cdots\cdots(5)$$

from (3) and (5) $\frac{\lambda^2}{x_1^2} + \frac{\lambda^2}{y_1^2} + \frac{\lambda^2}{z_1^2} = 4r^2$

$$\Rightarrow \lambda^2 \left[\frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2}\right] = 4r^2 \cdots\cdots\cdots\cdots(6)$$

i.e. $\lambda^2 \left[x_1^{-2} + y_1^{-2} + z_1^{-2}\right] = 4r^2$

P lies on plane (1) $\Rightarrow \frac{x_1}{a} + \frac{y_1}{b} + \frac{z_1}{c} = 1$

$$\Rightarrow \frac{x_1^2}{\lambda} + \frac{y_1^2}{\lambda} + \frac{z_1^2}{\lambda} = 1$$

$$\Rightarrow \frac{1}{\lambda}\left(x_1^2+y_1^2+z_1^2\right)=1 \cdots\cdots\cdots\cdots (7)$$

from (6) and (7) $\left(x_1^2+y_1^2+z_1^2\right)\left(x_1^{-2}+y_1^{-2}+z_1^{-2}\right)=4r^2$

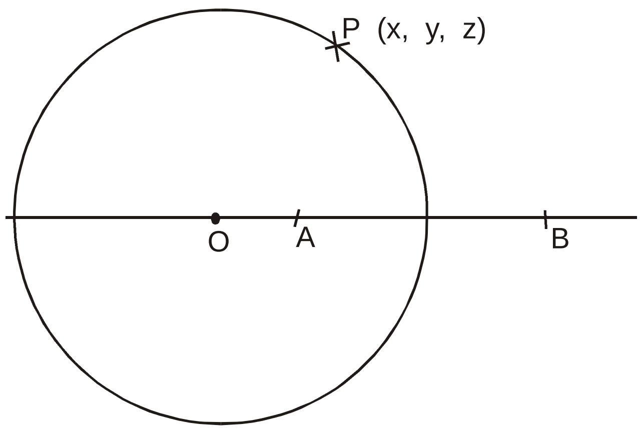
Foot of the perpendicular from 0 to the plane $\overleftrightarrow{ABC}$

lies on

$$\left(x^2+y^2+z^2\right)\left(x^{-2}+y^{-2}+z^{-2}\right)=4r^2$$

4.5.10 Example: If O be the centre of a sphere of radius unity and A, B be two points in a line with O such that $OA \cdot OB = 1$ and if P be a variable point on the sphere, show that $PA \cdot PB = \text{Cons tan t}$.

Solution: Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be given points.



Let $P(x, y, z)$ be any point on the sphere with centre origin O and radius unity. Then

$$x^2+y^2+z^2=1.$$

DR's of $OA = x_1-0,\ y_1-0,\ z_1-0 = x_1,\ y_1, z_1$.

DR's of $OB = x_2-0,\ y_2-0,\ z_2-0 = x_2, y_2, z_2$.

Since O, A, B are collinear, $\frac{x_2}{x_1}=\frac{y_2}{y_1}=\frac{z_2}{z_1}=k \quad (\text{say})$

$$\Rightarrow x_2 = kx_1,\ y_2 = ky_1,\ z_2 = kz_1$$

$$OA^2 = x_1^2 + y_1^2 + z_1^2, \quad OB^2 = x_2^2 + y_2^2 + z_2^2 = k^2\left(x_1^2 + y_1^2 + z_1^2\right) = k^2 OA^2$$

Given that $OA \cdot OB = 1 \Rightarrow OA^2 \cdot OB^2 = 1$

$$\Rightarrow OA^2 \cdot K^2 OA^2 = 1$$

$$\Rightarrow \boxed{OA^2 = \frac{1}{k}} \quad \text{and} \quad \boxed{OB^2 = k}.$$

Now,

$$\frac{PA^2}{PB^2} = \frac{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)}$$

$$= \frac{x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z_1^2 - 2(xx_1 + yy_1 + zz_1)}{x^2 + y^2 + z^2 + x_2^2 + y_2^2 + z_2^2 - 2(xx_2 + yy_2 + zz_2)}$$

$$= \frac{1 + OA^2 - 2(xx_1 + yy_1 + zz_1)}{1 + OB^2 - 2(xkx_1 + yky_1 + zkz_1)} = \frac{1 + \frac{1}{k} - 2(xx_1 + yy_1 + zz_1)}{1 + k - 2k(xx_1 + yy_1 + zz_1)}$$

$$= \frac{1 + \frac{1}{k} - 2(xx_1 + yy_1 + zz_1)}{k\left[\frac{1}{k} + 1 - 2(xx_1 + yy_1 + zz_1)\right]} = \frac{1}{k} \quad \Rightarrow \quad \frac{PA^2}{PB^2} = \frac{1}{k} = \text{ Constant}$$

$$\Rightarrow \frac{PA}{PB} = \sqrt{\frac{1}{k}} = \text{ Constant}$$

$$\Rightarrow PA : PB = \text{ Constant}$$

4.5.11 Example: A sphere of constant radius 2k passes through the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is the sphere

$$x^2 + y^2 + z^2 = 4k^2.$$

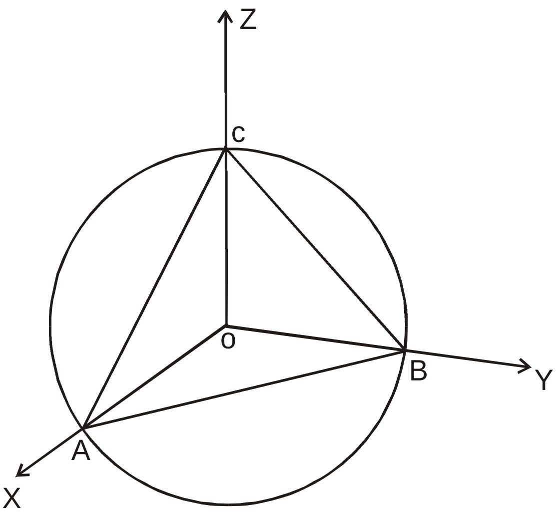
Solution: Let $A(a,0,0),\ B(0,b,0),\ C(0,0,c)$

The centroid of the tetrahedron OABC is $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$.

The sphere through O, A, B, C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

i.e., $x^2 + y^2 + z^2 - (ax + by + cz) = 0$



Now radius $= \sqrt{u^2 + v^2 + w^2 - d}$

$$\Rightarrow \quad 2k = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}$$

$$\Rightarrow \quad 4k^2 = \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}$$

The centroid $\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$ lies on the sphere

$$x^2 + y^2 + z^2 = k^2$$

4.6 Plane Section of a Sphere:

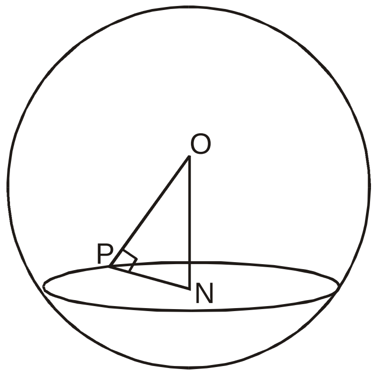
4.6.1 Definition: The inter section of a sphere and a plane is called the plane section of the sphere.

4.6.2 Theorem: Any plane section of a sphere is either circle or the empty set.

Proof: Let σ be a sphere with centre 'O' and radius $r\,(\geq 0)$ and π be a plane.

If $r=0$ then the plane section of σ and π is either the empty or single point

We may assume that $\pi \cap \sigma \neq \phi$



Let $r>0$ and P be a point on the plane section of σ and π.

Draw ON perpendicular to the plane π from the origin.

(i) If $N=P$ then the point N lies on the sphere.

So $ON = OP = r =$ radius of the sphere.

Let Q be any point on the plane section of σ and π other than P.

Then $ON \perp NQ$

$\Rightarrow NQ^2 = OQ^2 - ON^2 = r^2 - r^2 = 0$ which is impossible.

Hence, The common point of σ and π is unique.

i.e. the plane section of σ and π is point circle.

(ii) If $N \neq P$; then ON is perpendiculat to π.

Since NP is any line in π.

$\therefore$ Every line in π is perpendicular to ON.

$$\Rightarrow ON \perp NP \Rightarrow OP^2 = ON^2 + NP^2$$

$$\Rightarrow NP = \sqrt{OP^2 - ON^2} = \sqrt{r^2 - ON^2}$$

Since O and N are fixws points and OP is the radius of the sphere, PN is constant.

Hence locus of P is a circle whose centre is N and radius is $NP = \sqrt{r^2 - ON^2}$.

The plane section of σ and π is a circle.

4.6.3 Definition: Great Circle: The section of a sphere by a plane through the centre of the sphere is called a great circle. Its centre and radius are the same as those of the given sphere.

4.6.4 Definition: Small Circle: If a plane does not pass through the centre of the sphere and intersects the sphere, then the plane section is called a small circle.

4.6.5 Note: A plane and a sphere intersect iff the distance of the centre of the sphere from the plane is less than or equal to the radius of the sphere. Thus the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ will intersect the plane

$\ell x + my + nz + p = 0$ iff

$$\left|\frac{\ell(-u) + m(-v) + n(-w) + P}{\ell^2 + m^2 + n^2}\right| \leq \sqrt{u^2 + v^2 + w^2 - d}$$

4.6.6 Equation of the plane sections of a sphere: The equation of a sphere and plane together represent the equation of the circle which is the intersection of the sphere and the plane.

4.7 Intersection of Two Spheres:

4.7.1 Theorem: The intersection of two distinct spheres is either empty or a circle.

Proof: The set of common points of the spheres $S = 0, S^1 = 0$ is either empty or a plane $S - S^1 = 0 \cdot$

$$\left\{(x, y, z) \middle/ S = 0, S^1 = 0\right\} = \left\{(x, y, z) \middle/ S = 0, S - S^1 = 0\right\} \cdots\cdots\cdots\cdots (1)$$

But, the set of common points of the sphere and the plane π is a circle (by 4.6.2)

Thus from (1) the common points of the spheres $S = 0, S^1 = 0$ form a circle.

4.7.2 Theorem: If a circle 'C' passes three points A, B, C then it lies on any sphere S through them.

Proof: Since the circle passes through the points A, B, C. The points A, B, C are non collinear.

Let π be the plane through A, B, C. If a sphere S passes through A, B, C then by 4.6.2 the plane section of the sphere S and plane π is a circle. Clearly C is the only circle through A, B, C. Hence the sphere S contains the circle.

4.7.3 Theorem: The equation of the sphere drawn on the join of $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$$

Proof: Let σ be the sphere such that AB as diameter.

$P(x,y,z) \in \sigma \Leftrightarrow$ either P = A or P = B or

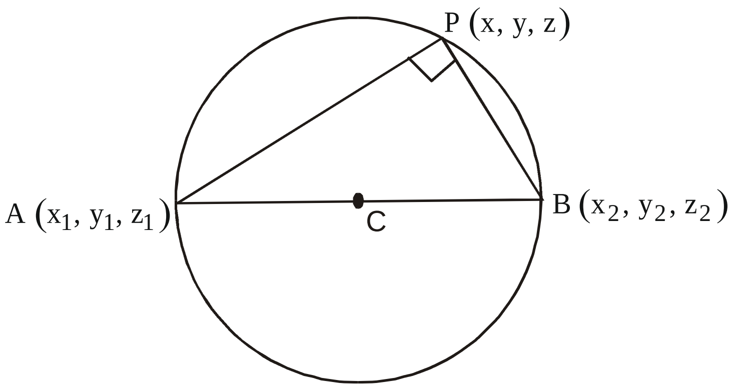
P lies on a circle with AB as diameter.

$\Leftrightarrow \; PA \perp PB$

$\Leftrightarrow \; \overline{PA} \cdot \overline{PB} = 0$

$\Leftrightarrow \; (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

This is the equation of the sphere with AB as diameter.



4.7.4 Theorem: The equation of any sphere through a given circle which is the intersection of the sphere.

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0 \cdots\cdots (1)$$

and the plane

$$\pi \equiv \ell x+my+nz+p=0 \cdots\cdots\cdots\cdots (2)$$

is of the form $S+\lambda L=0$ where λ is any constant.

Proof: Let us consider the equation $S+\lambda\pi=0$

$S+\lambda\pi = x^2+y^2+z^2+2ux+2vy+2wz+d+\lambda(\ell x+my+nz+p)$

$$= x^2+y^2+z^2+(2u+\lambda\ell)x+(2v+\lambda m)y+(2w+\lambda n)z+d+\lambda P$$

The equation $S+\lambda\pi=0,\ \forall\lambda$ is satisfied by the coordinates of the points which satisfy both $S=0$ and $\pi=0$. That is any point on the circle formed by S=0, $\pi=0$ must satisfy the equation $S+\lambda\pi=0$.

But $S+\lambda\pi=0$ is a second degree in x, y, z in which the coefficients of x^2, y^2, z^2 are all equal and there are no terms of xy, yz and zx.

$\therefore$ $S+\lambda\pi=0$ represents a sphere through the circle given by $S=0,\ \pi=0$ together.

Note (1): Suppose two spheres $S=0, S^1=0$ intersect. Then the parametirc equation $S+\lambda S^1=0$ can be shown to represent all spheres passing through the circle $S=0=S^1$.

Note (2): Where $\lambda=-1$, the equation $S+\lambda S^1=0$ becomes $S-S^1=0$ which is a plane and is common to the spheres.

$S=0, S^1=0$ are two spheres intersecting in a circle. Then the equations to the circle are $S=0, S-S^1=0$ as well as $S^1=0, S-S^1=0$.

4.8 Examples:

4.8.1 Example: Find the equation of the sphere through the circle $x^2+y^2+z^2+2x+3y+6=0,\ x-2y+4z-9=0$ and the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$.

Solution: The equation of the sphere passing through the given circle is

$$x^2+y^2+z^2+2x+3y+6+\lambda(x-2y+4z-9)=0\cdots\cdots(1)$$

The center of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$ is $(1, -2, 3)$

Gvien that the sphere (1) passes through $(1,-2,3)$. Then

$$1+4+9+2-6+6+\lambda(1+4+12-9)=0 \Rightarrow \quad 16+8\lambda=0 \Rightarrow \lambda=-2$$

$\therefore$ The required sphere is

$$x^2+y^2+z^2+2x+3y+6+(-2)(x-2y+4z-9)=0$$

i.e. $x^2 + y^2 + z^2 + 2x + 3y + 6 - 2x + 4y - 8z + 18 = 0$

$\Rightarrow x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$

4.8.2 Example: Show that the equation of the sphere having its centre on the plane

$$4x - 5y - z = 3$$

and pass through the circle with equation

$$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0, \quad x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$$

is $x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$.

Solution: The given plane is $4x - 5y - z = 3 \cdots\cdots (1)$

Given spheres

$$S \equiv x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0 \cdots\cdots\cdots\cdots (2)$$

$$S^1 \equiv x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0 \cdots\cdots\cdots (3)$$

the plane section of the spheres $S = 0, S^1 = 0$ is $S - S^1 = 0$

i.e. $-6x - 8y + 10z + 6 = 0$

$$3x + 4y - 5z - 3 = 0 \cdots\cdots\cdots\cdots (4)$$

The equation of the sphere through the given circle is same as the equation of the sphere passing through the circle whose equations (2) and (4).

$\therefore$ Equation of the required sphere equation is $S + \lambda\left(S - S^1\right) = 0$.

i.e., $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 + \lambda(3x + 4y - 5z - 3) = 0$

i.e. $x^2 + y^2 + z^2 + (-2 + 3\lambda)x + (-3 + 4\lambda)y + (4 - 5\lambda)z + 8 - 3\lambda = 0$

Its centre $\left(\left(\dfrac{-2+3\lambda}{2}\right), \dfrac{-(-3+4\lambda)}{2}, \dfrac{-(4-5\lambda)}{2}\right)$

But this centre lies on the plane (1).

$$-4\left(\frac{-2+3\lambda}{2}\right)+5\left(\frac{-3+4\lambda}{2}\right)+\left(\frac{4-5\lambda}{2}\right)=3$$

$$\Rightarrow \quad 8-12\lambda-15+20\lambda+4-5\lambda=6$$

$$\Rightarrow \quad 3\lambda=9 \Rightarrow \lambda=3$$

Required sphere equation is

$$x^2+y^2+z^2-2x-3y+4z+8+3(3x+4y-5z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+7x+9y-11z-1=0.$$

4.8.3 Example: Obtain the equation of the sphere having the circle

$x^2+y^2+z^2+10y-4z-8=0,\ x+y+z=3$ as the great circle.

Solution: The equation of the sphere through the given circle is

$$x^2+y^2+z^2+10y-4z-8+\lambda(x+y+z-3)=0.$$

i.e. $x^2+y^2+z^2+\lambda x+(10+\lambda)y+(-4+\lambda)z-8-3\lambda=0\cdots\cdots\cdots(1)$

Centre $=\left(\frac{-\lambda}{2}, \frac{-(10+\lambda)}{2}, \frac{-(-4+\lambda)}{2}\right)$

If the plane section of a given circle is a great circle

The centre of the sphere (1) lies on the plane $x+y+z-3=0$

i.e. $\frac{-\lambda}{2}-\frac{(10+\lambda)}{2}-\frac{(-4+\lambda)}{2}-3=0$

$$\Rightarrow -\lambda-10-\lambda+4-\lambda-6=0 \Rightarrow -3\lambda-12=0 \Rightarrow \lambda=-4$$

Required equation of the sphere is

$$x^2+y^2+z^2+(-4)x+(10-4)y+(-4-4)z-8-3(-4)=0$$

i.e. $x^2+y^2+z^2-4x+6y-8z+4=0$

4.8.4 Example: A sphere S has points $A(0,1,0)$, $B(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane

$$5x-2y+4z+7=0 \cdots\cdots\cdots (1)$$

as a great circle.

Solution: The equation of the sphere S with the AB as a diameter is

$$S \equiv (x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0$$

$$S \equiv x^2+y^2+z^2-3x+4y-2z-5=0 \cdots\cdots\cdots (2)$$

The equation of the sphere through the circle given by the equation (2) and (1) is

$$x^2+y^2+z^2-3x+4y-2z-5+\lambda(5x-2y+4z+7)=0 \ (\lambda \in \mathbb{R})$$

i.e. $$x^2+y^2+z^2 \ (-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z-5+7\lambda=0-(3)$$

Its centre $=\left(-\frac{(-3+5\lambda)}{2}, \frac{-(4-2\lambda)}{2}, \frac{-(-2+4\lambda)}{2}\right)$

Since the plane section is a great circle, the centre lies on the plane (1).

$$-5\left(\frac{-3+5\lambda}{2}\right)+2\left(\frac{4-2\lambda}{2}\right)-4\left(\frac{-2+4\lambda}{2}\right)+7=0$$

$$\Rightarrow -45\lambda+45=0 \qquad \Rightarrow \lambda=1$$

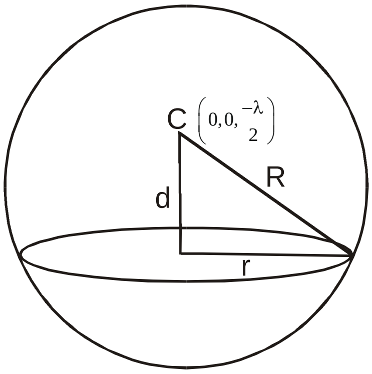
The required sphere is

$$x^2+y^2+z^2+2x+2y+2z+2=0$$

4.8.5 Example: Obtain the equation of the sphere which passes through the circle $x^2+y^2=4,\ z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

Solution: The equation of a sphere through the given circle is

$$x^2+y^2+z^2+\lambda z-4=0 \cdots\cdots\cdots (1) \qquad (\lambda \in \mathbb{R})$$



Its centre $=\left(0,0,\frac{-\lambda}{2}\right)$ and radius $=\mathbb{R}=\sqrt{\frac{\lambda^2}{4}+4}$

Given plane $x+2y+2z=0\cdots\cdots\cdots(2)$

The sphere cuts the plane (2)

$d=$ perpendicular distance from center $C\left(0,0,\frac{-\lambda}{2}\right)$ to the plane (2)

$$=\frac{|-\lambda|}{\sqrt{1+4+4}}=\frac{\lambda}{3}$$

Since the radius of the circle = 3.

We know that $R^2 = d^2+3^2$

$$\Rightarrow \frac{\lambda^2}{4}+4=\frac{\lambda^2}{9}+9$$

$$\Rightarrow \left(\frac{1}{4}-\frac{1}{9}\right)\lambda^2=5$$

$$\Rightarrow \frac{5\lambda^2}{36}=5 \qquad \Rightarrow \lambda^2=36 \Rightarrow \lambda=\pm 6$$

The equations of the required sphere is $x^2+y^2+z^2 \pm 6z-4=0$.

4.8.6 Example: Show that the two circles

$$2\left(x^2+y^2+z^2\right)+8x-13y+17z-17=0, \quad 2x+y-3z+1=0;$$

$$x^2+y^2+z^2+3x-4y+3z=0, \quad x-y-2z-4=0$$

lie on the same sphere and find its equation.

Solution: Equation of any sphere through the circle

$$x^2+y^2+z^2+4x-\frac{13}{2}y+\frac{17}{2}z-\frac{17}{2}=0,\ 2x+y-3z+1=0 \text{ is}$$

$$\left(x^2+y^2+z^2+4x-\frac{13}{2}y+\frac{17}{2}z-\frac{17}{2}\right)+\lambda_1(2x+y-3z+1)=0\cdots\cdots(1)$$

Equation of the sphere through the circle

$$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0 \text{ is}$$

$$x^2+y^2+z^2+3x-4y+3z+\lambda_2(x-y+2z-4)=0\cdots\cdots\cdots\cdots(2)$$

If the given circles lie on the same sphere, then (1) and (2) for some values of λ_1 and λ_2 must represent the same sphere.

By comparing the coefficients of x, y, z and constant terms in (1) and (2) we get

$$4+2\lambda_1=3+\lambda_2 \cdots\cdots\cdots(3)$$

$$\frac{-13}{2}+\lambda_1=-4-\lambda_2\cdots\cdots(4)$$

$$\frac{17}{2}-3\lambda=3+2\lambda_2\cdots\cdots\cdots(5)$$

$$\frac{-17}{2}+\lambda_1=-4\lambda_2\cdots\cdots(6)$$

By solving

(3) + (4) $\Rightarrow \frac{-5}{2}+3\lambda_1=-1 \quad \Rightarrow \lambda_1=\frac{1}{2}$

(3) $\Rightarrow 4+1=3+\lambda_2 \Rightarrow \lambda_2=2$

Clearly $\lambda_1 = \frac{1}{2},\ \lambda_2 = 2$ satisfy (5) and (6).

Given circles lie on the same sphere for $\lambda_1 = \frac{1}{2},\ \lambda_2 = 2$.

The equations of the sphere passing through the given circle is

$$x^2 + y^2 + z^2 + 3x - 4y + 3z + 2(x - y + 2z - 4) = 0$$

i.e., $x^2 + y^2 + z^2 + 5x - 6y + 7z - 8 = 0$

4.9 Summary:

After going through this lesson the student is expected to have a clear idea about the notions of sphere, equation of a sphere, plane section of a sphere, intersection of two sphers and others. The student shall also be able to appreciate the behaviour of the intersection of a sphere by a plane at a single point and in move than one point.

4.10 Technical Terms:

Plane section of a sphere

Great Circle

Small Circle

Diameter of a sphere

Radius of a sphere

Center of a sphere

4.11 Exercise:

1. Find the equation of the sphere through the points

   (i) $(0,0,0),\ (0,1,-1), (-1,2,0), (1,2,3)$

   (ii) $(4,-1,2),\ (0,-2,3), (1,5,-1,), (2,0,1)$

   (iii) $(0,0,0),\ (-a,b,c), (a,-b,c),\ (a,b,-c)$

2. Find the equation of the sphere through the points

   (i) $(1,-4,3),\ (1,-5,2),\ (1,-3,0)$ and whose centre lies on the piane $x + y + z = 0$.

   (ii) $(1,0,0), (0,1,0), (0,0,1)$ and having least radius.

   (iii) $(0,-2,-4),\ (2,-1,-1)$ and whose centre lies on the line $2x - 3y = 0 = 5y + 2z$.

3. Find the equation of the sphere on line joining the points

   (i) $(2,3,-1),\ (1,-2,-1)$ as a diameter.

   (ii) $(1,2,3),\ (2,3,4)$ as a diameter.

   (iii) $(1,-2,3),\ (-2,3,-1)$ as a diameter.

4. Find the equation of the sphere through the circle $x^2+y^2+z^2=9,\ 2x+3y+4z=5$ and the point $(1,2,3)$.

5. Find the centre and radius of the circle

   (i) $x^2+y^2+z^2=25,\ \ 2x+y+2z=9$

   (ii) $x^2+y^2+z^2+2x-2y-4z-19=0,\ x+2y+2z+7=0$

6. Find the equation of the sphere for which the circle

   (i) $x^2+y^2+z^2-4x+16y-8z+4=0,\ x+y+z=3$ as a great circle.

   (ii) $x^2+y^2+z^2+10y-4z=8,\ x+y+z=3$ as a great circle.

7. Prove that the plane $x+2y+2z=15$ cuts the sphere $x^2+y^2+z^2-2y-4z-11=0$ is a circle find center and radius of the circle. Also find the equation of the sphere which has the circle for one of the great circles.

8. Variable plane is parallel to the given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes in A, B, C respectively. Prove that the circle A, B, C lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

9. Prove that the two circles

   $x^2+y^2+z^2-y+2z=0,\quad x-y+z-2=0$ and

   $x^2+y^2+z^2+x-3y+z-5=0,\quad 2x-y+4z-1=0$

   lie on the same sphere and find its equation.

10. P is a variable point on a given line and A, B, C are its projections on the axis. Show that the sphere OABC passes through a fixed circle.

Answers to Exercise:

1. (i) $7\left(x^2+y^2+z^2\right)-15x-25y-11z=0$

   (ii) $x^2+y^2+z^2-4x-14y-22z+25=0$

   (iii) $\frac{1}{a^2+b^2+c^2}\left(x^2+y^2+z^2\right)-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$

2. (i) $x^2+y^2+z^2-4x+7y-3z+15=0$

   (ii) $3\left(x^2+y^2+z^2\right)-2(x+y+z)=1$

   (iii) $x^2+y^2+z^2-6x-4y+10z+12=0$

3. (i) $x^2+y^2+z^2-3x-y+2z-3=0$

   (ii) $x^2+y^2+z^2-3x-5y-7z+20=0$

   (iii) $x^2+y^2+z^2+x-y-2z-11=0$

4. $3\left(x^2+y^2+z^2\right)-2x-3y-4z-22=0$

5. (i) $(2,1,2)$, 4

   (ii) $\left(\frac{-7}{3},\frac{-5}{3},\frac{-2}{3}\right)$, 3

6. (i) $x^2+y^2+z^2-4x+6y-8z+4=0$

   (ii)

7. $(1,3,4)$, $\sqrt{7}$, $x^2+y^2+z^2-2x-6y-8z+19=0$

4.13 Model Examination Questions:

1. Define a sphere, center and radius.

2. Prove that the equation of a sphere with centre (x_1, y_1, z_1) and radius a > 0 is $(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=a^2$.

3. Find the equation of the sphere through the four points

$$(0,0,0),\ (-a,b,c),\ (a,-b,c),\ (a,b,-c)$$

4. Any plane section of a sphere is either circle or the emptyset.

5. P.T. The equation of the sphere on the join of $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ as diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$.

4.13 Model Practical Problem With Solution:

Problem: Show that the two circles

$$x^2+y^2+z^2-y+2z=0,\ x-y+z=2 \cdots\cdots\cdots\cdots\cdots (1)$$

and $$x^2+y^2+z^2+x-3y+z-5=0,\ 2x-y+4z-1=0 \cdots\cdots\cdots\cdots\cdots (2)$$

lie on the same sphere and find its equation.

AIM: To show that the given two circles lie on the same sphere and to find its equation.

Definitions and results used:

(1) The intersection of a sphere and a plane is called the plane section of the sphere.

(2) Any plane section of a sphere is either circle or the empty set.

(3) The equation of any sphere through a given circle which is the intersection of the sphere $S=0$, the plane $U=0$ is of the form $S+\lambda U=0$.

Solution:

A sphere through the circle (1) is

$$x^2+y^2+z^2-y+2z+\lambda\,(x-y+z-2)=0$$

i.e., $$x^2+y^2+z^2+\lambda x-(1+\lambda)y+(\lambda+2)z-2\lambda=0 \cdots\cdots\cdots\cdots (3)$$

A sphere through the circle (2) is

$$x^2+y^2+z^2+x-3y+z-5+\mu(2x-y+4z-1)=0$$

i.e., $$x^2+y^2+z^2+(2\mu+1)x-(3+\mu)y+(4\mu+1)z-5-\mu=0 \cdots\cdots\cdots\cdots\cdots (4)$$

But if the circles (1) and (2) lie on the same sphere then (3) and (4) represent the same sphere.

$\Rightarrow \lambda, \mu$ must satisfy the following equations.

$$\lambda = 2\mu + 1 \text{ i.e. } \lambda - 2\mu - 1 = 0 \cdots\cdots\cdots\cdots(i)$$

$$\lambda + 1 = 3 + \mu \text{ i.e. } \lambda - \mu - 2 = 0 \cdots\cdots\cdots\cdots(ii)$$

$$\lambda + 2 = 4\mu + 1 \text{ i.e. } \lambda - 4\mu + 1 = 0 \cdots\cdots\cdots\cdots(iii)$$

$$-2\lambda = -\mu - 5 \text{ i.e. } 2\lambda - \mu - 5 = 0 \cdots\cdots\cdots\cdots(iv)$$

from (iii) and (iv) $\lambda = 3, \mu = 1$

(i) and (ii) are satisfied by $\lambda = 3, \mu = 1$

$\therefore$ (3) and (4) represent the same sphere.

Equation to the required sphere is

$$x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$$

Lesson Writer

Dr. B. RAMI REDDY

Lesson - 5

SPHERE 2 - TANGENCY AND CONJUGANCY

5.1 Objective of the lesson:

After studying this lesson the student should be able to understand.

- Position of a point with respect to the sphere
- Tangent line and plane of a sphere
- Power of a point with respect to the sphere

5.2 Structure:

This lesson has the following components:

5.3 Introduction

5.4 Intersection of a sphere and a line

5.5 Tangent plane of a sphere

5.6 Touching spheres

5.7 Examples

5.8 Plane of contact

5.9 Pole and polar plane

5.10 Examples

5.11 Answers to SAQ

5.12 Summary

5.13 Technical Terms

5.14 Exercise

5.15 Answers to Exercise

5.16 Model Examination Questions

5.17 Model Practical Problem with Solution

5.3 Introduction:

In this lesson we continue our discussion on sphere, initiated in lesson 4. We concentrate here on tangent lines, tangent planes, planes of contract pole and polar. We also discuss about the conditions required for two spheres to "touch" each other.

5.3.1 Notations: We use the following no tations in this lesson.

$$S \equiv S(x,y,z) \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$$

$$S_i \equiv xx_i + yy_i + zz_i + zu(x + x_1) + 2v(y + y_1) + 2w(z + z_1) + d$$

$$S_{11} \equiv S(x_1, y_1, z_1) \equiv x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$$

$$S_{ij} \equiv x_i x_j + y_i y_j + z_i z_j + 2u(x_i + x_j) + 2v(y_i + y_j) + 2w(z_i + z_j) + d$$

If $\bar{r} = (x,y,z), \quad F(\bar{r}) = S = S(x,y,z)$

$$F(\bar{r}) = |\bar{r} - \bar{c}|^2 - a^2 = |\bar{r}|^2 - 2(\bar{r} \cdot \bar{c}) + |\bar{c}|^2 - a^2$$

where $\bar{c} = (u\ v\ w)$

5.4 Intersection of a sphere and a line:

5.4.1 Theorem: Let S be a sphere with centre $C(\bar{c})$ represented by the equation $F(\bar{r}) = 0$ and L be a line passing through a fixed point $D(\bar{d})$ and parallel to a unit vector $\bar{b}$ then.

(i) L intersects S in two distinct points if $|\bar{b}(\bar{c} - \bar{d})| > \sqrt{F(\bar{d})}$

(ii) L touches the sphere S if $|\bar{b} \cdot (\bar{c} - \bar{d})| = \sqrt{F(\bar{d})}$

(iii) L does not have a common points with S if $|\bar{b} \cdot (\bar{c} - \bar{d})| < \sqrt{F(\bar{d})}$

Proof: Let $a(\geq 0)$ be the radius of S. Then the vector equation of the sphere S is

$$F(\bar{r}) = |\bar{r}|^2 - 2\bar{r} \cdot \bar{c} + |\bar{c}|^2 - a^2 = 0 \cdots\cdots\cdots\cdots\cdots (1)$$

Let $\bar{r} = \bar{d} + t\bar{b} \cdots\cdots\cdots\cdots (2)$

be the straight line passing through a point $D(\bar{d})$ and parallel to unit vector $\bar{b}$.

(1) and (2) intersect at $\bar{r} = (\bar{d} + t\bar{b})$

$$\Leftrightarrow \left|\left(\bar{d}+t\bar{b}\right)\right|^2 - 2\left(\bar{d}+t\bar{b}\right)\cdot\bar{c} + \left|\bar{c}\right|^2 - a^2 = 0$$

$$\Leftrightarrow t^2 - 2t\left[\bar{b}\cdot\left(\bar{c}\cdot\bar{d}\right)\right] + F\left(\bar{d}\right) = 0 \cdots\cdots\cdots\cdots(3)$$

Equation (3) is a quadratic in t. Therefore it has two roots say t_1, t_2.

Since the discriminent of (3) is $\left|\bar{b}\cdot\left(\bar{c}-\bar{d}\right)\right|^2 - F\left(\bar{d}\right)$

Case (i): t_1, t_2 are real if $\left|\bar{b}\cdot\left(\bar{c}-\bar{d}\right)\right|^2 \geq F\left(\bar{d}\right) \cdots\cdots\cdots\cdots(4)$

In this case the line intersects the sphere in two points say P, Q. Where $P = \bar{d} + t_1\bar{d}, \quad Q = \bar{d} + t_2\bar{b}$. Then $DP = |t_1|, \quad DQ = |t_2|$.

From (3), $DP\cdot DQ = |t_1 t_2| = \left|F\left(\bar{d}\right)\right| \cdots\cdots\cdots\cdots(5)$

Thus the product $DP\cdot DQ$ is independent of the line i.e. $\bar{b}$, but depends on $D\left(\bar{d}\right)$ only.

Case (ii): If $\left|\bar{b}\cdot\left(\bar{c}-\bar{d}\right)\right|^2 = F\left(\bar{d}\right)$ then $t_1 = t_2$ and $\bar{d} + t_1\bar{b}$ is the only point of intersection of the sphere and the line.

From (3) $\quad DP^2 = t_1 t_2 = \left|F\left(\bar{d}\right)\right| \Rightarrow t_1^2 = \left|F\left(\bar{d}\right)\right|$

$$\Rightarrow DP = t_1 = t_2 = \sqrt{F\left(\bar{d}\right)}$$

Moreover $\left|\bar{b}\cdot\left(\bar{c}-\bar{d}\right)\right|^2 = F\left(\bar{d}\right)$.

Case (iii): If $\left|\bar{b}\cdot\left(\bar{c}-\bar{d}\right)\right|^2 < F\left(\bar{d}\right)$ then t_1, t_2 are complex numbers. In this case L does not have a common point with S.

5.4.2 Remark:

(1) If the point D lies on the sphere then $F\left(\bar{d}\right) = 0$ and $\bar{b}.\left(\bar{c}-\bar{d}\right) = 0$ i.e. CD is perpendicular to the given line.

(2) Any line through $D(\bar{d})$ meets the sphere in two points P and Q not necessarly distinct and $DP \cdot DQ = \left|F(\bar{d})\right|$ which is independent of the line.

5.4.3 Definition: If $D(x_1, y_1, z_1)$ is any point in the space and DPQ is a line intersecting the sphere $S = F(\bar{r}) = 0$ in P and Q, then the power of the point D with respect the sphere $S = 0$ is $DP \cdot DQ = \left|F(\bar{d})\right| = S_{11}$.

5.4.4 Tangent Line: Let $S = 0$ be a given sphere and $D(\bar{d})$ be any point. If a line through D has only one common point T with a given sphere, then $L(= DT)$ is called a tangent line to the sphere from D. T is called the point of contact of the tangent line $\overline{BT}$ with the sphere.

5.4.5 Note: If the line DT touches the sphere with centre C at T then

(i) $\overline{CT} \perp \overline{DT}$

(ii) If another line through D intersects, the sphere at P and Q then $DT^2 = DP \cdot DQ$.

5.4.6 Definition: Let S be a sphere having centre C and radius $a(> 0)$ then

(i) If $DC > a$ then D is called external point of the sphere S.

(ii) If $DC < a$ then D is called internal point of the sphere S.

5.4.7 S.A.Q.: If P, Q are points on a sphere then prove that any point on the line segment PQ is an internal point of the sphere.

5.4.8 Note: Let $F(\bar{r}) = 0$ be the equation of a sphere with centre $C(\bar{c})$ and radius $a(\geq 0)$. Let $D(\bar{d})$ be any point. Then

(i) D is an external point of the sphere $\Leftrightarrow DC > a \Leftrightarrow \left|\bar{c} - \bar{d}\right| > a$

$$\Leftrightarrow \left|\bar{d} - \bar{c}\right|^2 - a^2 > 0 \Leftrightarrow F(\bar{d}) > 0$$

$\Leftrightarrow$ Power of D is positive.

(ii) D is an internal point of the sphere $\Leftrightarrow F(\bar{d}) < 0$ or $S_{11} < 0$

$\Leftrightarrow$ Power of D is negative.

$\Leftrightarrow$ Power of D is zero.

(iii) D lies on the sphere $\Leftrightarrow F(\bar{d}) = 0$ $\Leftrightarrow$ Power of D is zero.

(2) If $\bar{e} = (-u, -v, -w)$, $a = \sqrt{u^2 + v^2 + w^2 - d}$, $\bar{d} = (x_1, y_1, z_1)$ then

$F(\bar{d}) = S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

(i) D is an external point $\Leftrightarrow F(\bar{d}) > 0 \Leftrightarrow S_{11} > 0 \Leftrightarrow$ power of D is positive

(ii) D is an internal point $\Leftrightarrow F(\bar{d}) < 0 \Leftrightarrow S_{11} < 0 \Leftrightarrow$ power of D is negative

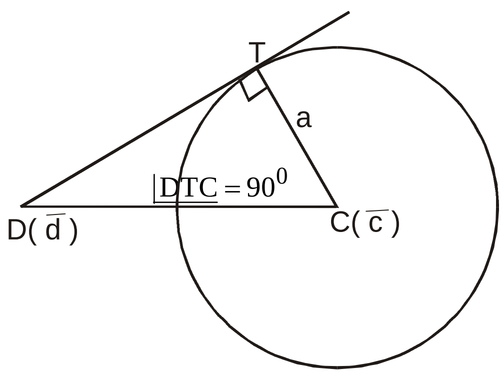
(iii) D lies on the sphere $\Leftrightarrow F(\bar{d}) = 0 \Leftrightarrow S_{11} = 0 \Leftrightarrow$ power of D is zero

5.4.9 S.A.Q.: Show that there exists a tangent line to the sphere from any external point.

5.4.10 S.A.Q.: From an internal point there exists no tangent line to the sphere.

Theorem: The length of the tangent from an external point $D(\bar{d})$ to the sphere $F(\bar{r}) = |\bar{r} - \bar{c}|^2 - a^2 = 0$ is $\sqrt{F(\bar{d})}$.

Proof: Since the centre is $C(\bar{c})$ and radius is a,



[DT is called length of the tangent to the sphere]

If DT is a tangent line from D tothe sphere, then $\underline{|DTC} = 90^0$ and

$$DT^2 = DC^2 - CT^2 = \left|\bar{d} - \bar{a}\right|^2 - a^2 = F\left(\bar{d}\right) > 0$$

$\Rightarrow$ The length of the tangent from $D\left(\bar{d}\right)$ to the sphere is

$$\boxed{DT = \sqrt{F\left(\bar{d}\right)}}$$

Note: Let $D = \bar{d} = \left(x_1, y_1, z_1\right)$ and the equation of the sphere be

$f\left(\bar{r}\right) = S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then its centre $C = \bar{c} = \left(-u, -v, -w\right)$ and radius $a = \sqrt{u^2 + v^2 + w^2 - \alpha}$

Let $\bar{r} = \left(x, y, z\right)$ then length of the tangent

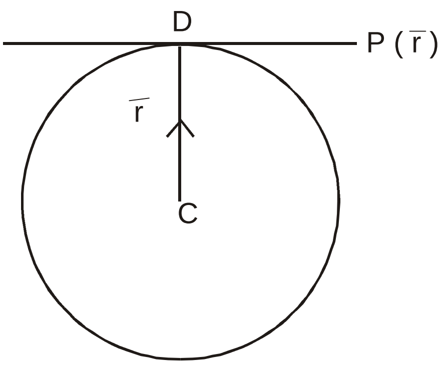
$$\boxed{DT = \sqrt{F\left(\bar{r}\right)} = \sqrt{S\left(x_1, y_1, z_1\right)} = \sqrt{S_{11}}}$$

5.5 Tangent Plane of a Sphere:

5.5.1 Theorem: The locus of the tangent line at a point on a sphere of non zero radius is a plane.

Proof: Let the equation of the sphere with centre $C\left(\bar{c}\right)$ and radius $a > 0$ be

$$F\left(\bar{r}\right) = \left|\bar{r}\right|^2 - 2\bar{r} \cdot \bar{c} + \left|\bar{c}\right|^2 - a^2 = 0$$



Let $D = \bar{d}$ be any point on the sphere. Then $F(\bar{d}) = 0 \cdots\cdots\cdots\cdots (1)$

Let $\bar{d} + t\bar{b}$ be any point on the line L through an external point D and parallel to a unit vector $\bar{b}$.

This point $\bar{d} + t\,\bar{b}$ lies on the sphere

$\Leftrightarrow \left|\bar{d} + t\,\bar{b}\right|^2 - 2\left(\bar{d} + t\,\bar{b}\right)\cdot\bar{c} + \left|\bar{c}\right|^2 - a^2 = 0$

$\Leftrightarrow t^2 - 2t\left[\bar{b}\cdot\left(\bar{c} - \bar{d}\right)\right] + F\left(\bar{d}\right) = 0$

$\Leftrightarrow t^2 - 2t\left[\bar{b}\cdot\left(\bar{c} - \bar{d}\right)\right] = 0$ (using (1))

$\Leftrightarrow t^2 = 0$ (since $\bar{b} \perp \overline{CD} \Rightarrow \bar{b}\cdot\left(\bar{c} - \bar{d}\right) = 0$)

$\Leftrightarrow t = 0$

$\Leftrightarrow \bar{d}$ is the only point common to the line L and the sphere

$\Leftrightarrow$ The line L touches the sphere at D

Thus L is a tangent line to the sphere

$\Leftrightarrow$ L passes through D and $L \perp CD$

Therefore, all the tangent lines at D passes through D and form a plane having CD as its normal.

5.5.2 Def: The locus of the tangent lines at a point D on a sphere $S = 0$ of non zero radius is a plane called the tangent plane to the sphere $S = 0$ at D. The point D is called the point of contact of the plane with the sphere $S = 0$.

5.5.3 Note:

(1) The tangent plane at a point D on the sphere is perpendicular to the diameter through D.

(2) The normal to the tangent plane of a sphere through the point of contact D passes through the centre of the sphere.

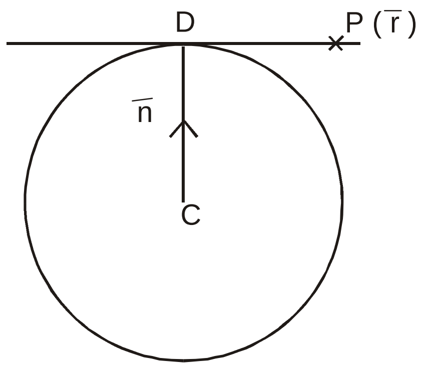
5.5.4 S.A.Q.: All points in the tangent plane of sphere expect the point of contact are external points of the sphere.

5.5.5 Theorem: The equation of the tangent plane at the point $D(x_1, y_1, z_1)$ of the sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $S_1 = 0$

Proof: Let $C(\bar{c})$ be the centre of the sphere $S = 0$ with radius $a > 0$ then



$$C = \bar{c} = (-u, -v, -w)$$

Let $\bar{n}$ be the unit vector in the direction of $\overline{CD}$. Then

$$\bar{n} = \frac{\overline{CD}}{|\overline{CD}|} = \left(\frac{x_1+u}{e}, \frac{y_1+v}{e}, \frac{z_1+w}{e}\right) \text{ where}$$

$$e = |\overline{CD}| = \sqrt{(x_1+u)^2 + (y_1+v)^2 + (z_1+w)^2}$$

A point $\bar{r}(x, y, z)$ lies on the tangent plane

$$\Leftrightarrow (\bar{r} - \bar{d}) \cdot \bar{n} = 0$$

$$\Leftrightarrow (x - x_1, y - y_1, z - z_1) \cdot \left(\frac{x_1+u}{e}, \frac{y_1+v}{e}, \frac{z_1+w}{e}\right) = 0$$

$$\Leftrightarrow (x - x_1)(x_1+u) + (y - y_1)(y_1+v) + (z - z_1)(z_1+w) = 0$$

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + z(z + z_1) + d - \left(x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d\right) = 0$$

$\Leftrightarrow\ S_1 - S_{11} = 0$

$\Leftrightarrow\ S_1 = 0$ (Since the point $D(x_1, y_1, z_1)$ lies on the sphere)

Conditions For Tangency:

5.5.6 Theorem: A necessary and sufficient condition that the plane

$$\ell x + my + nz = p \cdots\cdots\cdots(1)$$

is a tangent plane to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \cdots\cdots(2)$$

is that

$$(\ell u + mv + nw + p)^2 = (\ell^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$$

Proof: Center of the sphere (2) is $(-u, -v, -w)$ and radius $r = \sqrt{u^2 + v^2 + w^2 - d}$. If the plane (1) touches the sphere (2) then the perpendicular from the centre on the plane is equal to radius.

$$\Leftrightarrow \left| \frac{-\ell u - mv - nw - p}{\sqrt{\ell^2 + m^2 + n^2}} \right| = \sqrt{u^2 + v^2 + w^2 - d}$$

i.e., $(\ell u + mv + nw - p)^2 = (\ell^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$

which is the required condition.

5.6 Touching Spheres:

5.6.1 Def: Two spheres $S = 0, S^1 = 0$ are said to touch each other if $S = 0, S^1 = 0$ have onle one common point P. The common point P is called point of contact of the spheres $S = 0, S^1 = 0$.

5.6.2 Theorem: Two spheres $S = 0, S^1 = 0$ touch each other at a point P $\Leftrightarrow$ $S - S^1 = 0$ is the common tangent plane at P of the spheres $S = 0$ and $S^1 = 0$.

Proof: Let $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$$S^1 \equiv x^2 + y^2 + z^2 + 2u^1x + 2v^1y + 2w^1z + d^1 = 0$$

be two spheres touch each other at P.

Since $u \neq u^1$ or $v \neq v^1$ or $w \neq w^1$

$$S - S^1 = 2\left(u - u^1\right)x + 2\left(v - v^1\right)y + 2\left(w - w^1\right)z + d - d^1 = 0$$

represents a plane, say π.

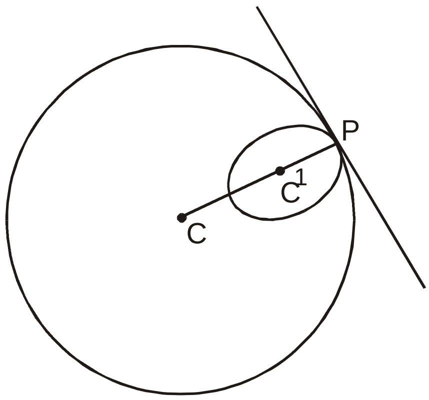
$$\{P\} = \left\{q/q \in S \cap S^1\right\} = \{q/q \in S,\ q \in \pi\} = \left\{q/q \in S^1,\ q \in \pi\right\}$$

$\therefore$ The plane π touches the sphere $S = 0$ at P and also $S^1 = 0$ at P.

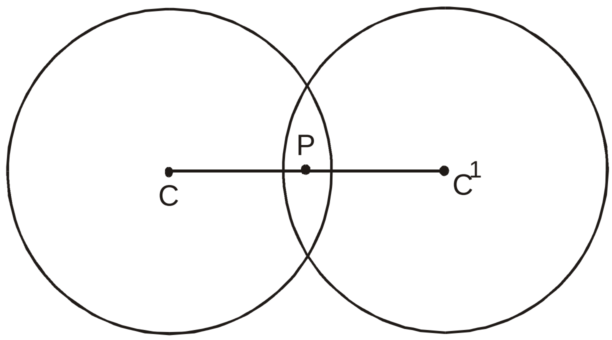
$\therefore$ π is the common tangent plane to the given two spheres at P.

5.6.3 **Def:** Let $S = 0, S^1 = 0$ be two spheres with centres C, C^1 and radii r, r^1 respectively and let π be the common tangent plane at P then

(i) If C, C^1 lie in the same side of the plane π then the two sphers $S = 0, S^1 = 0$ touch cach other internally.



(ii) If C, C^1 lie in the opposite side of the plane π, then the two spheres $S = 0, S^1 = 0$ touch each other externally.



Conditions for two spheres to touch:

5.6.4 Theorem: Let $S = 0, S^1 = 0$ be two spheres with centres C, C^1 and radii r, r^1 then the two spheres touch each other

(i) internally $\Leftrightarrow$ $CC^1 = \left|r - r^1\right|$

(ii) externally $\Leftrightarrow CC^1 = r + r^1$.

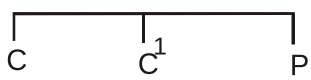
Proof: (i) Let the two spheres touch each other internally at a point P.

Let π be the common tangent plane for the spheres at P.

$\Rightarrow$ The points C, C^1 lie on same side of the plane π

$\Rightarrow C, P, C^1$ are collinear and C, C^1 lie in same side of P.

$\Rightarrow CC^1 = \left|CP - C^1P\right| = |r_1 - r_2|$



Conversely, suppose $r > r^1$ and $CC^1 = \left|r - r^1\right| = r - r^1$ then there exist a point P on the line CC^1 such that $CP = r, C^1P = r^1$.

Hence C^1 lies between C and P

Since $CP = r, C^1P = r^1, P$ is a common point of the two spheres.

Let π be the plane through the point P and perpendicular to CC^1

$CP \perp \pi \Rightarrow \pi$ is a tangent plane of the sphere $S = 0$

$C^1P \perp \pi \Rightarrow \pi$ is a tangent plane of the sphere $S^1 = 0$

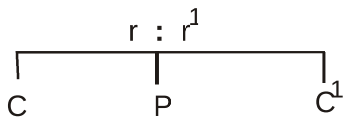
$\Rightarrow \pi$ is a common tangent plane of the sphers $S = 0, S^1 = 0$. Also C, C^1 lie on the same side of the plane π.

$\Rightarrow$ The two spheres $S = 0, S^1 = 0$ touch each other internally.

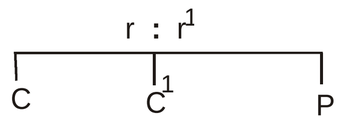
(ii) Similarly we can prove that the spheres $S = 0, S^1 = 0$ touch each other externally $\Leftrightarrow CC^1 = r + r^1$.

5.6.5 Note:

(i) If $CC^1 = r_1 + r_2$ then the two spheres touch each other externally the point of contact P divides CC^1 in the ratio r **:** r^1 internally



(ii) If $CC^1 = \left| r - r^1 \right|$ then the two spheres touch each other internally. The point of contact P divides CC^1 in the ratio r **:** r^1 externally



5.7 Examples:

5.7.1 Example: Find the equation of the tangent plane to the sphere $3\left(x^2 + y^2 + z^2\right) - 2x - 3y - 4z - 22 = 0$ at the point $(1,2,3)$.

Solution: Equation of the tangent plane at (x_1, y_1, z_1) is $S_1 = 0$

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$.

Here $(x_1, y_1, z_1) = (1,2,3)$

Equation of the tangent plane is

$$3\left(xx_1+yy_1+zz_1\right)-\left(x+x_1\right)-\frac{3}{2}\left(y+y_1\right)-2\left(z+z_1\right)-22=0$$

i.e., $$3\left(x+2y+3z\right)-\left(x+1\right)-\frac{3}{2}\left(y+2\right)-2\left(z+3\right)-22=0$$

$$\Rightarrow 4x+9y+14z-64=0$$

5.7.2 Example: Find the equation of the tangent line to the circle $x^2+y^2+z^2+5x-7y+2z-8=0,\ 3x-2y+4z+3=0$ at the point $\left(-3,5,4\right)$.

Solution: The tangent line to a circle is the line of intersection of the tangent plane to the sphere at the given point and the plane of the circle.

Given sphere $x^2+y^2+z^2+5x-7y+2z-8=0\cdots\cdots\cdots\cdots(1)$

Plane of the circle is $3x-2y+4z+3=0\cdots\cdots\cdots(2)$

Equation of the tangent plane to (1) at $\left(-3,5,4\right)$ is $S_1=0$

i.e., $$x\left(-3\right)+y\left(5\right)+z\left(4\right)+\frac{5}{2}\left(x-3\right)-\frac{7}{2}\left(y+5\right)+\left(z+4\right)-8=0$$

$$\Rightarrow x-3y-10z+58=0\cdots\cdots\cdots(3)$$

Equations of the tangent line to the circle at $\left(-3,5,4\right)$ is are (2) and (3)

Symmetric form of the tangent line: Let a, b, c be the dr's of the line. Then

$$3a-2b+4c=0$$

$$a-3b-10c=0$$

Solving $\dfrac{a}{32}=\dfrac{b}{34}=\dfrac{c}{-7}$

The equation of the tangent line to the circle at $\left(-3,\ 5,\ 4\right)$ is $\dfrac{x+3}{32}=\dfrac{y-5}{34}=\dfrac{z-4}{-7}$

5.7.3 Example: Find the value of a for which the plane $x+y+z=a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.

Solution: For the given sphere, $\left(1,1,1\right)$ is the centre and radius $r=\sqrt{1+1+1+6}=3$.

If the given plane touches the sphere then the distance from the centre of the sphere to the plane is equal to the radius of the sphere

$$\therefore \left| \frac{1+1+1-a\sqrt{3}}{\sqrt{1+1+1}} \right| = 3 \Rightarrow 3-a\sqrt{3} = \pm 3\sqrt{3} \Rightarrow a = \sqrt{3} \pm 3.$$

5.7.4 Example: Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact.

Solution: Given sphere equation is $x^2+y^2+z^2-2x-4y+2z-3=0 \cdots\cdots (1)$

Its centre $C=(1, 2, -1)$, radius $r=\sqrt{1+4+1+3}=3$

Given plane is $2x-2y+z+12=0 \cdots\cdots (2)$

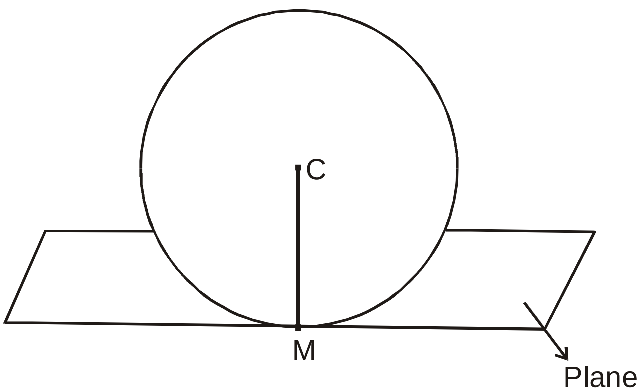
$\perp^{ler}$ distance from $(1, 2, -1)$ to the plane (2) $= \left| \dfrac{2(1)-2(2)+(-1)+12}{\sqrt{4+4+1}} \right|$

$$= \frac{9}{3} = 3 = \text{radius}$$

$\therefore$ The plane (2) touches the sphere (1).

The line through the centre and $\perp^{ler}$ to the plane (2) is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1} = r \text{ (say)}$$



Any point on this line is $(2r+1, -2r+2, r-1)$

If $M(2r+1, -2r+2, r-1)$ is the point of contact of (2) with (1). Then this point lies in the plane (2).

$$2(2r+1)-2(-2r+2)+r-1+12=0$$

$$\Rightarrow 9r+9=0 \Rightarrow r=-1$$

Thus, the point of contact $=(2(-1)+1, -2(-1)+2, -1-1)=(-1, 4, -2)$

5.7.5 Example: Find the coordinates of the points on the sphere

$$x^2+y^2+z^2-4x+2y=4 \text{ .... } (1)$$

the tangent planes at which are parallel to the plane is

$$2x-y+2z=1 \text{ .... } (2)$$

Solution: The centre of the sphere (1) $C=(2,-1,0)$. Radius $r=\sqrt{4+1+4}=3$

Let the equation of the tangent plane parallel to (2) be

$$2x-y+2z+k=0 \cdots\cdots\cdots (3)$$

The plane (3) is tangent to the sphere (1) $\Leftrightarrow$ $\perp$ distance from center of the sphere to the plane (3) is equal to the radius of the sphere.

$$\Leftrightarrow \left|\frac{2(2)-(-1)+2(0)+k}{\sqrt{4+1+4}}\right|=3 \Rightarrow k=\pm 9-5=4, -14$$

The equation of the line through the centre $(2,-1,0)$ and perpendicular to the plane (3) is

$$\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z}{2}=s \quad (\text{say})$$

Any point on this line is $(2r+2, -r-1, 2r)$

If this point is the point of contact of (3) with (1) then the point $(2r+2, -r-1, 2r)$ lies on the plane (3).

i.e., $2(2r+2)-(-2r-1)+2(2r)+k=0$

$$\Rightarrow \quad r = \frac{-5-k}{9} = \frac{-5-4}{9}, \quad \frac{-5+14}{9} \qquad (\because K = 4, \ -14)$$

$$= -1, \ 1$$

The coordinates of the points of contact of the plane (3) with the (1) are

$$\left(2(-1)+2, -(-1)-1, (-1)\right), \ \left(2(1)+2, -1-1, 2(1)\right)$$

$$= (0,0,-2), \quad (4,-2,2)$$

5.7.6 Example: Show that the equation of the sphere which touches the sphere

$$4\left(x^2+y^2+z^2\right)+10x-25y-2z=0 \cdots\cdots (1)$$

at the point $(1,2,-2)$ and passes through the point $(-1, 0, 0)$ is

$$x^2+y^2+z^2+2x-6y+1=0$$

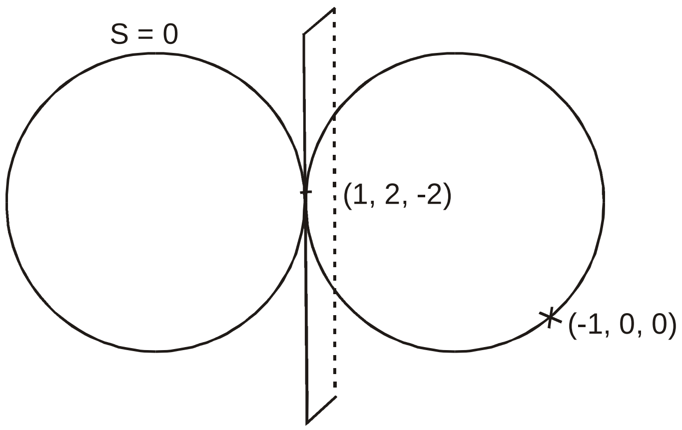
Proof:

Let the required equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \cdots\cdots (2).$$

It passes through $(-1, 0, 0)$ so $1-2u+d=0 \Rightarrow \boxed{d = 2u-1}$

The equation of the tangent plane at $(1,2,-2)$ of the sphere (2) is



$$x+2y-2z+u(x+1)+v(y+2)+w(z-2)+d=0$$

$$\Rightarrow (1+u)x+(2+v)y+(-2+w)z+3u+2v-2w-1=0 \cdots\cdots\cdots (3)$$

$$(\because d=2u-1)$$

The equation of the tangent plane at $(1,2,-2)$ of the sphere (1) is

$$4(x+2y-2z)+5(x+1)-\frac{25}{2}(y+2)-(z-2)=0$$

$$\Rightarrow 2x-y-2z-4=0 \cdots\cdots\cdots (4)$$

Since the two spheres touch each other at $(1,2,-2)$

Equations (3) and (4) represent the same plane.

$$\frac{1+u}{2}=\frac{2+v}{-1}=\frac{-2+w}{-2}=\frac{3u+2v-2w-1}{-4}=r \qquad (\text{say})$$

$\Rightarrow u = 2r-1,\ v=-r-2,\ w=-2r+2$ and

$$3u+2v-2w-1=-4r$$

$$\Rightarrow 3(2r-1)+2(-r-2)-2(-2r+2)-1=-4r$$

$$\Rightarrow 12r-12=0 \Rightarrow r=1$$

$$\therefore\ u=1,\ v=-3,\ w=0,\ d=1$$

Hence required equation is $x^2+y^2+z^2+2x-6y+1=0$

II Method: Given sphere quation is

$$S \equiv 4\left(x^2+y^2+z^2\right)+10x-25y-2z=0 \cdots\cdots\cdots (1)$$

Equation of the tangent plane at $(1,2,-2)$ is $S_1=0$

$$4(x+2y-2z)+5(x+1)-\frac{25}{2}(y+2)-(z-2)=0$$

$$\Rightarrow U \equiv 2x-y-2z-4=0 \cdots\cdots\cdots (2)$$

Equation of the sphere which touches the given sphere (1) at $(1,2,-2)$ is of the form

$$S + \lambda U = 0$$

$$\Rightarrow 4\left(x^2 + y^2 + z^2\right) + 10x - 25y - 2z + \lambda(2x - y - 2z - 4) = 0$$

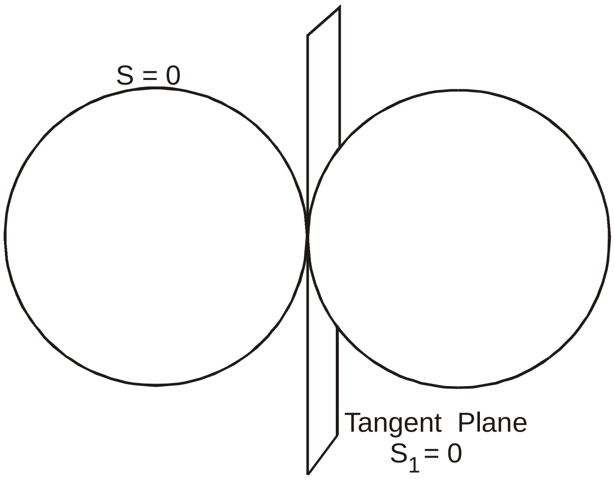
But this sphere passes through $(-1, 0, 0)$

$$\therefore 4 - 10 - 2\lambda - 4\lambda = 0 \Rightarrow \lambda = -1$$

Requirede sphere equation is

$$4\left(x^2 + y^2 + z^2\right) + 10x - 25y - 2z - 1(2x - y - 2z - 4) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$$



5.7.7 Example: Obtain the equation of the tangent planes to the sphere

$$x^2 + y^2 + z^2 + 6x - 2z + 1 = 0 \cdots\cdots (1)$$

Which pass through the line

$$3(16 - x) = 3z = 2y + 30 \cdots\cdots\cdots\cdots (2)$$

Solution: The centre of (1) $= (-3, 0, 1)$ and radius $r = \sqrt{9 + 1 - 1} = 3$

The equation of the line (2) is general form can be put as

$$U_1 \equiv 3(16 - x) = 3z \Rightarrow U_1 \equiv x + z - 16 = 0$$

$$U_2 \equiv 3z = 2y + 30 \Rightarrow U_2 \equiv 3z - 2y - 30 = 0$$

The equation of any plane through the line (2) given by

$U_1 = 0, U_2 = 0$ is

$$U_1 + \lambda U_2 = 0$$

i.e., $x + z - 16 + \lambda(3z - 2y - 30) = 0$

i.e., $x - 2\lambda y + (1 + 3\lambda)z - 16 - 30\lambda = 0 \cdots\cdots\cdots(3)$

If the plane (3) is a tangent plane to the sphere (1), then the perpendicular distance from centre $(-3,0,1)$ should be equal to radius 3

i.e., $\left|\dfrac{-3 - 2\lambda(0) + (1 + 3\lambda)1 - 16 - 30\lambda}{\sqrt{1 + 4\lambda^2 + (1 + 3\lambda)^2}}\right| = 3$

$$\Leftrightarrow \quad 9\left|\frac{3\lambda + 2}{\sqrt{(3\lambda^2 + 6\lambda + 2)}}\right| = 3$$

$$\Leftrightarrow \ 9(3\lambda + 2)^2 = 13\lambda^2 + 6\lambda + 2$$

$$\Leftrightarrow \ 68\lambda^2 + 102\lambda + 34 = 0 \Rightarrow 2\lambda^2 + 3\lambda + 1 = 0$$

$$\Leftrightarrow \ (2\lambda + 1)(\lambda + 1) = 0 \Leftrightarrow \lambda \in \left\{-1, \frac{-1}{2}\right\}$$

Thus the equations to ther equired planes are

$x + 2y - 2z + 14 = 0$ and $2x + 2y - z - 2 = 0$

5.7.8 Example: Obtain the equations of the spheres which passes through the circle $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0, \ 2x + y + z = 4$ and touch the plane $3x + 4y = 14$.

Solution: Any sphere through the given circle is represented by $S + \lambda U = 0$. $(\lambda \in \mathbb{R})$

i.e., $x^2+y^2+z^2-2x+2y+4z-3+\lambda(2x+y+z-4)=0$

i.e., $x^2+y^2+z^2+(-2+2\lambda)x+(2+\lambda)y+(4+\lambda)z-3-4\lambda=0 \cdots\cdots\cdots(1)$

$\Rightarrow$ Its centre is $\left(\frac{-(-2+2\lambda)}{2}, \frac{-(2+\lambda)}{2}, \frac{-(4+\lambda)}{2}\right)$

and radius is $\sqrt{\left(\frac{-2+2\lambda}{2}\right)^2+\left(\frac{2+\lambda}{2}\right)^2+\left(\frac{4+\lambda}{2}\right)^2+3+4\lambda}$

$$=\sqrt{\frac{6\lambda^2+20\lambda+36}{4}}$$

If $3x+4y+14=0$ is a tangent plane to the sphere (1) then the perpendicular from centre is equal to radius.

i.e., $$\left|\frac{-3\left(\frac{-2+2\lambda}{2}\right)-\frac{-4(2+\lambda)}{2}-14}{\sqrt{9+16}}\right|=\sqrt{\frac{6\lambda^2+20\lambda+36}{4}}$$

$$\Leftrightarrow \left|\frac{-5\lambda-15}{5}\right|=\sqrt{\frac{6\lambda^2+20\lambda+36}{4}}$$

Squaring

$$\Leftrightarrow 4\left(\lambda^2+9+6\lambda\right)=6\lambda^2+20\lambda+36$$

$$\Leftrightarrow 2\lambda^2-4\lambda=0 \Rightarrow 2\lambda(\lambda-2)=0 \Leftrightarrow \lambda\in\{0,2\}$$

Putting the values of λ in (1) we get equation to the spheres as

$x^2+y^2+z^2-2x+2y+4z-3=0$ for $\lambda=0$

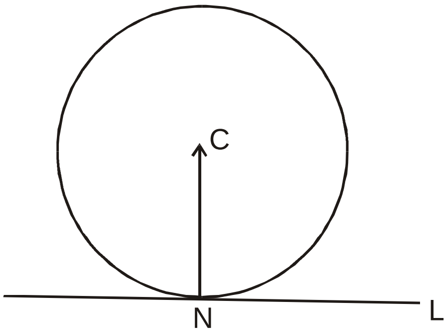
and $x^2+y^2+z^2+2x+4y+6z-11=0$ for $\lambda=2$

5.7.9 Example: Find the equation of the sphere which has its centre at the origin and which touches the line

$$2(x+1)=2-y=x+3$$

Solution: The radius r of the sphere is the distance of the centre from the given line L

$$\frac{x+1}{1}=\frac{y-2}{-2}=\frac{x+3}{2}=r\cdots\cdots(1)$$



Let N be the point where the line L touches the sphere.

Then $CN \perp L$

Any point on the line L is $N=(r-1,\ -2r+2,\ 2r-3)$

Since Dr's of CN are $r-1,\ -2r+2,\ 2r-3$ and DR's of L are $1,-2,2$

$$CN \perp L \Rightarrow 1(r-1)-2(-2r+2)+2(2r-3)=0$$

$$\Rightarrow 9r=11 \Rightarrow r=\frac{11}{9}$$

$$\Rightarrow N=\left(\frac{11}{9}-1,\ \frac{-22}{9}+2,\ \frac{22}{9}-3\right)=\left(\frac{2}{9},\frac{-4}{9},\frac{-5}{9}\right)$$

$$\Rightarrow\ r^2=CN=\frac{4}{9^2}+\frac{16}{9^2}+\frac{25}{9^2}=\frac{45}{9.9}=\frac{5}{9}$$

Equation of the sphere with centre $(0,0,0)$ and radius $r=\sqrt{\frac{5}{9}}$ is

$$x^2+y^2+z^2=\frac{5}{9}$$

i.e., $9\left(x^2+y^2+z^2\right)=5$

5.7.10 Example: Find the equation of the sphere of radius r which touches the three coordinate axes. How many such spheres are there?

Solution: Let the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

If the sphere touches x-axis at $(x_1, 0, 0)$. The x_1 must be a double root of

$$x^2 + 2ux + d = 0$$

This happens $\Leftrightarrow$ $4u^2 = 4d$ i.e. $u^2 = d$

The Y - axis and Z - axis touch the sphere $\Leftrightarrow$ $v^2 = d,\ w^2 = d$

In this case $r^2 = u^2 + v^2 + w^2 - d$

$$\Rightarrow r^2 = d + d + d - d = 2d \Rightarrow d = \frac{r^2}{2}$$

Thus the condiiton is

$$u^2 = v^2 = w^2 = d = \frac{r^2}{2}$$

$$\Rightarrow u = v = w = \pm\frac{r}{\sqrt{2}}$$

So the equations of the sphere are

$$x^2 + y^2 + z^2 + 2\left(\pm\frac{r}{\sqrt{2}}\right)x + 2\left(\pm\frac{r}{\sqrt{2}}\right)y + 2\left(\pm\frac{r}{\sqrt{2}}\right)z + \frac{r^2}{2} = 0$$

i.e., $2\left(x^2 + y^2 + z^2\right) + 2\sqrt{2}r(\pm x \pm y \pm z) + r^2 = 0$

If r is fixed then only eight such spheres exist.

5.7.11 Example:

(a) Prove that the equation of the sphere which lies in the octant $OXYZ$ and touches the coordinate planes is of the form

$$x^2 + y^2 + z^2 - 2\lambda(x + y + z) + 2\lambda^2 = 0$$

(b) Show that in general two spheres can be drawn through a given point to touch the coordinate planes and find what position of the point the spheres are (i) real, (ii) coincident.

Solution: **(a)** Let the radius of the sphere be λ, then the distance of its centre from coordinate planes is equal to radius λ.

The centre of the sphere lies in octant $OXYZ$. Hence center $=(\lambda,\lambda,\lambda)$.

So the equation of the sphere is

$$(x-\lambda)^2+(y-\lambda)^2+(z-\lambda)^2=\lambda^2$$

$\therefore\ x^2+y^2+z^2-2\lambda(x+y+z)+2\lambda^2=0$. This proves (a)

(b) Let $P(\alpha, \beta, \gamma)$ be the given point.

If the sphere passes through P then

$$\left(\alpha^2+\beta^2+\gamma^2\right)-2\lambda(\alpha+\beta+\gamma)+2\lambda^2=0\cdots\cdots\cdots(1)$$

This is a quadratic equation in λ.

Hence the discriminant of (1) is

$$4(\alpha+\beta+\gamma)^2-8\left(\alpha^2+\beta^2+\gamma^2\right)$$

Equation (1) has distinct roots if

$$2(\alpha\beta+\beta\gamma+\gamma\alpha)>\alpha^2+\beta^2+\gamma^2\cdots\cdots\cdots\cdots(2)$$

and the roots are equal if

$$2(\alpha\beta+\beta\gamma+\gamma\alpha)=\left(\alpha^2+\beta^2+\gamma^2\right)\cdots\cdots(3)$$

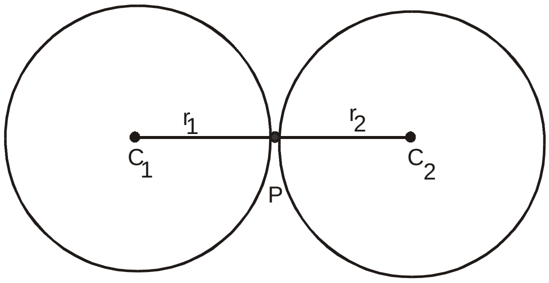
So, (i) there are two spheres if (2) holds and (ii) there is one sphere if (3) holds.

5.7.12 Example: Show that the spheres

$$x^2+y^2+z^2=25\cdots\cdots\cdots(1)$$

$$x^2+y^2+z^2-24x-4oy-18z+225=0\cdots\cdots(2)$$

touch externally and find the point of contact.



Solution: For the sphere (1), centre $C_1 = (0,0,0)$ and radius $r_1 = 5$

For the sphere (2), centre $C_2 = (12,20,9)$ radius $r_2 = \sqrt{144+400+81-225} = 20$

Now $C_1C_2 = \sqrt{12^2 + 20^2 + 9^2} = 25 = 20 + 5 = r_1 + r_2$

Thus the two spheres touch each other externally.

Let P be the point of contanct. Then P divides C_1C_2 in the ratio $r_1 : r_2 = 5 : 20 = 1 : 4$ internally.

Then $P = \left(\frac{60}{25}, \frac{100}{25}, \frac{45}{25}\right) = \left(\frac{12}{5}, \frac{20}{5}, \frac{9}{5}\right)$

5.7.13 Example : Find the centres of the two spheres which touch the plane

$$4x + 3y = 47$$

at the point $(8,5,4)$ and the sphere

$$x^2 + y^2 + z^2 = 1$$

Solution: Given tangent plane for the sphere is

$$4x + 3y = 47$$

Dr's of the normal are $(4,3,0) \Rightarrow DC's$ of the normal are $\left(\frac{4}{5}, \frac{3}{5}, 0\right)$

Let $P = (8,5,4)$. Then P is the point of contact.

Equations of the line through P and having DC's $\left(\frac{4}{5}, \frac{3}{5}, 0\right)$ are

$$\frac{x-8}{4/5}=\frac{y-5}{3/5}=\frac{z-4}{0}=r_1 \qquad (\text{say})$$

If r_1 is the radius of one sphere. Then the centre of the sphere is of the form

$$C_1=\left(\frac{4r_1}{5}+8, \frac{3r_1}{5}+5, 4\right)\cdots\cdots\cdots(1)$$

As second sphere is $x^2+y^2+z^2=1$

has centre $C_2=(0,0,0)$ and radius $r_2=1$.

The two spheres are touching $\Rightarrow$ distance between centres $=|r_1 \pm r_2|=|r_1 \pm 1|$

$$\Rightarrow \left(\frac{4}{5}r_1+8\right)^2+\left(\frac{3}{5}r_1+5\right)^2+16=(r_1 \pm 1)^2$$

$$\Rightarrow (4r_1+40)^2+(3r_1+25)^2+400=25(r_1 \pm 1)^2$$

$$\Rightarrow 16r_1^2+1600+320r_1+9r_1^2+625+150r_1+400=25r_1^2+25 \pm 50r_1$$

$$\Rightarrow 470r_1+2600=\pm 50r_1$$

$\Rightarrow 470r_1+2600=50r_1$ or $470r_1+2600=-50r_1$

$$\Rightarrow r_1=\frac{-2600}{420} \quad \text{or} \quad r_2=\frac{-2600}{520}$$

$\Rightarrow r_1=\dfrac{-130}{21}$ or $r_1=-5$

The coordinates of the centres are

$$\left(\frac{4}{5}\left(\frac{-130}{21}\right)+8, \frac{3}{5}\left(\frac{-130}{5}\right)+5, 4\right) \text{ and } \left(\frac{4}{5}(-5)+8, \frac{3}{5}(-5)+5, 4\right)$$

i.e., $\left(\dfrac{64}{21}, \dfrac{27}{21}, 4\right)$ and $(4,2,4)$

5.7.14 Example: Find the equations to the spheres through the points $(4,1,0)$, $(2,-3,4)$, $(1,0,0)$ and touching the plane $2x-2y-z=11$.

Solution: Let the sphere equation be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \cdots\cdots\cdots(1)$$

The sphere (1) passes through $(4,1,0),(2,-3,4),(1,0,0)$ then

$$16+1+0+84+2v+d=0 \Rightarrow 8u+2v+a=-17 \cdots\cdots\cdots\cdots(2)$$

$$4+9+16+44-6v+8w+d=0 \Rightarrow 4u-6v+8w+d=-29 \cdots\cdots(3)$$

The centre of the sphere $(-4,-u,-w)$, $\text{radius}=\sqrt{u^2+v^2+w^2-d}$

Since, the sphere tocuhes the plane $2x+2y-3=11$.

$$\left|\frac{-2u-2v+w-11}{\sqrt{4+4+1}}\right| = \sqrt{u^2+v^2+w^2-d}$$

$$\Rightarrow (2u+2v-w+11)^2 = 9\left(u^2+v^2+w^2-d\right)\cdots\cdots\cdots(5)$$

Now, $(4) \Rightarrow \boxed{d=-1-2u}$

$$(2) \Rightarrow 8u+2v+d=-17 \Rightarrow 2v=-8u-(-1-2u)-17=-16-6v$$

$$\Rightarrow \boxed{v=-8-3u}$$

$$(3) \Rightarrow 4u-6(-8-3u)+8w-1-2u=-29 \Rightarrow \boxed{w=\frac{-19-5u}{2}}$$

$$\therefore (5) \Rightarrow \left(2u+2(-8-3u)+\frac{19+5u}{2}+11\right)^2 = 9\left[u^2+(-8-3u)^2+\left(\frac{-19-5u}{2}\right)^2+1+2u\right]$$

$$\Rightarrow (-3u+9)^2 = 9\left[65u^2+39ou+621\right]$$

$$\Rightarrow 64u^2+396u+612=0$$

$$\Rightarrow 16u^2+99u+153=0 \quad \Rightarrow (u+3)(16v+51)=0$$

$$\Rightarrow u=-3, \frac{-51}{16}$$

If $u=-3$, then $v=1, w=-2, d=5$

If $u=\frac{-51}{16}$, then $v=\frac{25}{16}, w=\frac{-49}{32}, d=\frac{43}{8}$

So the equation of required spheres are

$x^2+y^2+z^2-6x+2y-4z+5=0$ and

$16\left(x^2+y^2+z^2\right)-102x+50y-49z+86=0$

5.7.15 Examples: Find the equation of the sphere inscribed in the tetrahedron whose faces are $x=0,\ y=0,\ z=0,\ x+2y+2z=1$.

Solution: Let the equation of the required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0\cdots\cdots\cdots(1)$$

Center $C=(-u,-v,-w)$, radius $r=\sqrt{u^2+v^2+w^2-d}$

Intercept form of the plane $x+2y+2z=1$ is

$$\frac{x}{1}+\frac{y}{1/2}+\frac{z}{1/2}=1$$

Since the sphere lies in side the tertrahedran,

$$0<-u<1,\ 0<-v<\frac{1}{2},\ 0<-w<\frac{1}{2}\cdots\cdots\cdots(2)$$

Since the plane $x=0$ touches the sphere (1)

$$|u|=r\ \Rightarrow\ r=-u$$

similarly the planes $y=0,\ z=0$ are touching the sphere (1) then $r=-v,\ r=-w$

Since $r^2=u^2+v^2+w^2-d$

$$\Rightarrow\ r^2=r^2+r^2+r^2-d\ \Rightarrow\ \boxed{d=2r^2}$$

$$\therefore u=v=w=-r,\ d=2r^2=2u^2$$

Since $x+2y+2z=1$ touches the sphere (1)

$$\left|\frac{-u-2v-2w-1}{\sqrt{1+4+4}}\right|=r \quad \Rightarrow |-54-1|=3r \quad \Rightarrow (54+1)^2=3r^2$$

$$\Rightarrow (5u+1)^2=3u^2 \Rightarrow 16\ u^2+10u+1=0$$

$$\Rightarrow 16u^2+10u+1=0 \ \Rightarrow (8u+1)(2u+1)=0$$

$$\Rightarrow u=\frac{-1}{2},\ \frac{-1}{8}$$

From (2) $u=v=w=\frac{-1}{8},\ d=2\frac{1}{64}=\frac{1}{32}$

Required equation of the sphere is

$$x^2+y^2+z^2-\frac{1}{8}(x+y+z)+\frac{1}{32}=0$$

or $$32\left(x^2+y^2+z^2\right)-(x+y+z)+1=0$$

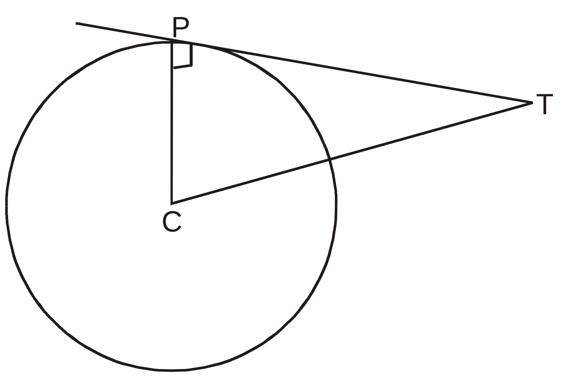
5.8 Plane of Contact:

5.8.1 Theorem: The set of points of contact of the tangent planes to the sphere. Which pass through an external point T is the empty set or a circle.

Proof: Let σ be a given sphere.

If σ is a point sphere then we know that the set of points of contact of the tangent planes to the sphere is a point circle.

Let the radius of the sphere σ be positive. Then P is a point of contact of the tangent plane through T $\Leftrightarrow$ $\underline{|CPT}=90^0$.



$\Leftrightarrow$ P lies on the sphere σ^1 with diameter CT.

$\Leftrightarrow P \in \sigma$ and $P \in \sigma^1 \Leftrightarrow P \in \sigma \cap \sigma^1$

But $\sigma \cap \sigma^1$ is a circle.

5.8.2 Note: If T is an internal point of the sphere σ, then there is no tangent plane through T. So the set of points of contact of the tangent planes is empty.

If T lies on the sphere, then the set of points of contact of the tangent planes is $\{T\}$.

5.8.3 Def: The plane containing the locus (which is a circle) of the points of contact of the tangent planes to the sphere σ which as through and external point T is called the plane of contact of the point T with respect the sphere σ.

5.8.4 Theorem: The equation of the plane of contact of an external point $T(x_1, y_1, z_1)$ with respect to the sphere $S = 0$ is $S_1 = 0$.

Proof: Let $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$.

Let $\left(x^1, y^1, z^1\right)$ be the point of contact of a tangent plane to the sphere which passes through the point (x_1, y_1, z_1).

The equation ofthe tangent plane at $\left(x^1, y^1, z^1\right)$ is

$$xx^1 + yy^1 + zz^1 + u\left(x + x^1\right) + v\left(y + y^1\right) + w\left(z + z^1\right) + d = 0 \cdots\cdots(1)$$

$\left(x^1, y^1, z^1\right)$ lies in the plane of contact

$\Leftrightarrow$ The plane (1) passes through T.

$$\Leftrightarrow x_1x^1 + y_1y^1 + z_1z^1 + u\left(x_1 + x^1\right) + v\left(y_1 + y^1\right) + w\left(z_1 + z^1\right) + d = 0$$

Thus $\left(x^1, y^1, z^1\right)$ satisifes the equation.

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

i.e. $S_1 = 0 \cdots\cdots(2)$

Since $(-u,-v,-w)$ is the centre of the sphere and T is an external point of the sphere,

$$(x_1, y_1, z_1) \neq (-u,-v,-w)$$

Hence $S_1 = 0$ represents a plane and $\left(x^1, y^1, z^1\right)$ belongs to this plane

Hence the plane of contact of T is $S_1 = 0$

5.8.5 Note:

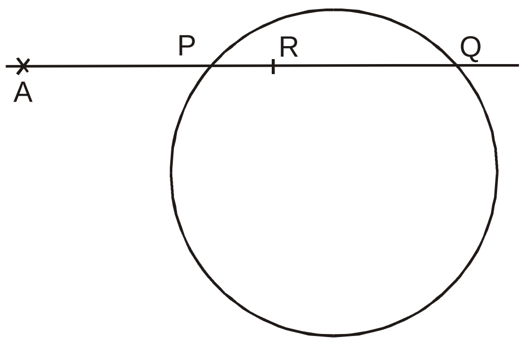
(1) The plane of contact is perpendicular to the line joining the given point with the centre of the sphere.

(2) The locus of the point of contact at any point with respect the sphere does not contain the centre of the sphere, hence the plane of contact does not pass through the centre of the sphere.

5.9 Pole and Polar Plane:

5.9.1 Def (1): The Polar Plane:

If a line drawn through a fixed point A meets a given sphere in points P, Q and a point R is taken on this line such that the segment AR is divided internally and externally by the points P, Q in the same ratio, then the locus of R is a plane called the polar plane of A w.r.t. the sphere. The fixed point A is called pole.



5.9.2 Def (2): If through a given point A, any transversal be drawn to meet the given sphere in P and Q and if R be a point on this line such that

$$\frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AR},$$ Then the locus of R,

is a plane, called the polar plane of A w.r.t the sphere.

5.9.3 Theorem: The above two definitions are equivalent.

Proof: Def (1) holds $\Leftrightarrow \dfrac{PR}{RQ} = \dfrac{PA}{AQ}$

$$\Leftrightarrow \frac{-PR}{AP} = \frac{-QR}{AQ}$$

$$\Leftrightarrow \frac{AP - AR}{AP \cdot AR} = \frac{AR - AQ}{AQ \cdot AR}$$

$$\Leftrightarrow \frac{1}{AR} - \frac{1}{AP} = \frac{1}{AQ} - \frac{1}{AR}$$

$$\Leftrightarrow \frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AR}$$

$$\Leftrightarrow \text{def } (2) \text{ holds}$$

5.9.4 Theorem: The polar plane of the point $A(x_1, y_1, z_1)$ w.r.t.the sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is $S_1 = 0$

Proof: Equations of any line through $A(x_1, y_1, z_1)$ are

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \cdots\cdots\cdots\cdots (1)$$

Any point on the line is $(x_1 + \ell r, y_1 + mr, z_1 + nr)$

This point is on the sphere $\Leftrightarrow$

$$(x_1 + \ell r)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 + 2u(x_1 + \ell r) + 2v(y_1 + mr) + 2w(z_1 + nr) + d = 0$$

i.e. $r^2 + 2r[\ell(u + x_1) + m(v + y_1) + n(w + z_1)] + s_{11} = 0$

This is a quadratic equation in r. This has two roots r_1, r_2 say.

If the line through A meets the sphere in P and Q then

$$AP = r_1, AQ = r_2$$

$$AP + AQ = -2[\ell(u + x_1) + m(v + y_1) + n(w + z_1)]$$

$(\because\ AP + AQ = r_1 + r_2 = \text{sum of the roots})$

$AP \cdot AQ = S_{11}$ $\quad (\because\ AP \cdot AQ = r_1 r_2 = \text{Product of the roots})$

Let $R(\alpha, \beta, \gamma)$ be a point on PQ such that

$$\frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AR} \Rightarrow \frac{2}{AR} = \frac{AP + AQ}{AP \cdot AQ}$$

$$\Rightarrow AR = \frac{2AP \cdot AQ}{AP + AQ} = \frac{2S_{11}}{-2\left[\ell(u + x_1) + m(v + y_1) + n(w + z_1)\right]} \cdots\cdots\cdots \quad (2)$$

Since R lies on line (1) $\dfrac{\alpha - x_1}{\ell} = \dfrac{\beta - y_1}{m} = \dfrac{\gamma - z_1}{n} = AR \cdots\cdots\cdots \quad (3)$

Eliminating ℓ, m, n from (2) & (3) we get

$$(\alpha - x_1)(u + x_1) + (\beta - y_1)(v + y_1) + (\gamma - z_1)(w + z_1) = -S_{11}$$

$$\Rightarrow \alpha x_1 + \beta y_1 + \gamma z_1 + u(\alpha + x_1) + v(\beta + y_1) + w(\gamma + z_1) + d = 0$$

locus of $R(\alpha, \beta, \gamma)$ is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

i.e., $\boxed{S_1 = 0}$

$\therefore$ The polar of (x_1, y_1, z_1) w.r.t the sphere $S = 0$ is $S_1 = 0$.

5.9.5 S.A.Q.: If O is centre of the sphere of radius a then

(i) the polar plane of the point P w.r.t. the sphere is perpendicular to OP.

(ii) $OP \cdot OQ = a^2$ where Q is the intersection of the polar plane with OP.

5.9.6 Note:

(i) When P lies out side the sphere $OP > 0$ and hence $OQ < a$ i.e. Q lies with in the sphere. The polar of P is the plane of contact of P.

(ii) When P lies on the sphere, $OP = OQ = a$.

Then P and Q are coincide and the polar plane of P is the tangent plane at P.

(iii) When P lies inside the sphere, $OP < a,\ OQ > a$. Then polar plane of P is simply a plane perpendicular to OP at Q which does not intersect the sphere.

5.9.7 Def: If π be the polar plane of a point P, Then P is called a pole of the plane π.

Pole of a plane:

5.9.8 Theorem: If $P \neq 0$, The plane $\ell x + my + nz = p \cdots\cdots\cdots(1)$ has a unique pole

$$\left(\frac{a^2\ell}{P}, \frac{a^2m}{P}, \frac{a^2n}{P}\right)$$ with respect to the sphere $x^2 + y^2 + z^2 = a^2 \cdots\cdots\cdots(2)$.

Proof: $T(x_1, y_1, z_1)$ is a pole of the plane (1)

$\Leftrightarrow$ The polar plane of T w.r.t. the sphere (2)

i.e., $xx_1 + yy_1 + zz_1 = a^2 \cdots\cdots(3)$

respresents the plane (1)

$$\Leftrightarrow \frac{x_1}{\ell} = \frac{y_1}{m} = \frac{z_1}{n} = \frac{a^2}{P}$$

i.e., $x_1 = \dfrac{a^2\ell}{P},\ y_1 = \dfrac{a^2m}{P},\ z_1 = \dfrac{a^2n}{P}$

Thus the plane (1) has a unique pole and the pole is

$$\left(\frac{a^2\ell}{P}, \frac{a^2m}{P}, \frac{a^2n}{P}\right)$$

Using the translation of axis (shiffting of the origin) we have the following.

5.9.9 Corollary: If $S = 0$ is a sphere and $\ell x + my + nz = p,\ p \neq 0$ is not passing through the centre of the sphere, then the pole of the plane is (x_1, y_1, z_1) where

$x_1 = -u + \ell t,\ y_1 = -v + mt,\ z_1 = -w + nt$ and

$$t = \left(\frac{u^2 + v^2 + w^2 - d}{\ell u + mv + nw - P}\right)$$

5.9.10 Theorem: The polar plane of a point P w.r.t. a sphere passes through another point Q, then the polar plane of Q w.r.t. the sphere pass through P.

Proof: Let the sphere be $x^2 + y^2 + z^2 = a^2$

Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ be the given points

The polar plane of P is

$$xx_1 + yy_1 + zz_1 = a^2$$

If this plane passes through Q then

$$x_2x_1 + y_2y_1 + z_2z_1 = a^2$$

This is also the condition for the polar plane of Q to pass through P.

Hence the polar plane of Q passes through Q.

5.9.11 Def: Conjugate Points: A point P is said to be a conjugate of Q w.r.t. a sphere $S = 0$ iff the polar plane of Q w.r.t. the sphere $S = 0$ passes through P.

5.9.12 Note:

(i) From the above theorem (5.9.10), it is clear that if P is a cojugate of Q then Q is a conjugate of P. Thus we may as well say that P and Q are conjugate w.r.t. a sphere $S = 0$ if the polar plane of either passes through the other. We may also note that P and Q are not necessary distinct. Infact P is always conjugate to itself with respect to the sphere.

(2) From the definition of conjugate points, conjugate to the centre of the sphere is not define.

5.9.13 Theorem: If the pole of a plane w.r.t. a sphere lies in another plane and the planes do not pass through the centre then the pole of the second plane lies on the first plane.

Proof: Since the sphere is invariant under shifiting of origin, we may assume that the sphere is

$$x^2 + y^2 + z^2 = a^2$$

and the planes π_1 and π_2 are

$$\pi_1 \equiv \ell_1 x + m_1 y + n_1 z - P_1 = 0$$

$$\pi_2 \equiv \ell_2 x + m_2 y + n_2 z - P_2 = 0$$

The pole of the plane π_1 w.r.t. the sphere is

$$\left(\frac{a^2\ell_1}{P_1}, \frac{a^2 m_1}{P_1}, \frac{a^2 n_1}{P_1}\right) \qquad \text{(by theorem 5.9.8)}$$

If this point lies on the second plane π_2 then

$$ {}_2\left(\frac{a^2\ell_1}{P_1}\right)+m_2\left(\frac{a^2m_1}{P_1}\right)+n_2\left(\frac{a^2n_1}{P_1}\right)=P_2 $$

i.e., $a^2\left(\ell_1\ell_2+m_1m_2+n_1n_2\right)=P_1P_2$

$$\Rightarrow\left(\frac{a^2\ell_2}{P_2},\frac{a^2m_2}{P_2},\frac{a^2n_2}{P_2}\right)$$ lies on the plane π_1.

$\Rightarrow$ The pole of the second plane π_2 to lie on the first plane π_1.

Hence the result.

5.9.14 Def: Conjugate Planes: Two planes none containing of the centre of the sphere such that the pole of either plane w.r.t. the sphere lies on the other plane are called conjugate planes.

5.9.15 Theorem: The polar planes of all points on a line L not passing through the centre of the sphere w.r.t. a sphere pass through line L.

Proof: Let the equation of the sphere be

$$S\equiv x^2+y^2+z^2-r^2=0$$

Let the line be $L\equiv\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=k$ (say)

Any point on this line is $P\left(\alpha+\ell k,\ \beta+mk,\ \gamma+mk\right)$

Polar plane of this point P w.r.t. the sphere is

$$x\left(\alpha+\ell k\right)+y\left(\beta+mk\right)+z\left(\gamma+nk\right)=r^2$$

i.e. $$\left(\alpha x+\beta y+\gamma z-r^2\right)+k\left(\ell x+my+nz\right)=0\cdots\cdots\cdots(1)$$

Since ℓ, m, n are DR's of the line L and $(\alpha,\beta,\gamma)\neq(0,0,0)$

The equation $\ell x+my+nz=0$ and $\alpha x+\beta y+\gamma z-r^2=0$

represent planes say π and π^1 respectively. Since $\ell:m:n\neq\alpha:\beta:\gamma$, π and π^1 intersect in a line L^1 (say) and the general form of all planes passing through L^1 is (1).

The polar planes of all points on L w.r.t. a sphere pass through the line L^1.

5.9.16 Definition:

Conjugate lines or polar lines: Two lines, such that the polar plane of either line w.r.t. a sphere passes through the other line are called conjugate lines or polar lines.

5.10 Examples:

5.10.1 Example: Find the plane of contact of the point $(3,-1,5)$ in the respect to the sphere $x^2+y^2+z^2-2x+4y+6z-11=0$.

Solution: The plane of contact of (x_1, y_1, z_1) w.r.t the sphere $S=0$ is $S_1=0$

$$(x_1, y_1, z_1)=(3,-1,5)$$

So the required plane is

$$3x-y+5z-(x+3)+2(y-1)+3(z+5)-11=0$$

i.e., $3x-y-5z-x-3+2y-2+3z+15-11=0$

i.e., $3x+y-2z-1=0$

5.10.2 Example: Find the polar plane of the point $(1,3,4)$ w.r.t. the sphere

$$x^2+y^2+z^2-6x-2z+5=0$$

Solution: The polar plane of the point (x_1, y_1, z_1) w.r.t. sphere $S=0$ is $S_1=0$

Required polar plane is

$$x\cdot 1+y(3)+z(4)-3(x+1)-(z+4)+5=0$$

$$-2x+3y+3z-2=0$$

i.e. $\boxed{2x-3y-3z+2=0}$

5.10.3 S.A.Q.: Find the pole of the plane $x-y+5z-3=0$ w.r.t. the sphere

$$x^2+y^2+z^2=9.$$

5.10.4 Example: Find the pole of the plane $x-y-z+9=0$ w.r.t. the sphere

$$x^2+y^2+z^2-2x+4y-6z+5=0.$$

solution: Given sphere is $(x-1)^2+(y+2)^2+(z-3)^2=9$

Shift the origin to the point $(1,-2,3)$ then

$$X^2+Y^2+Z^2=9\cdots\cdots\cdots\cdots\cdots(1)$$

Equation to the plane in the new system of axes is

$$(X+1)-(Y-2)-(Z+3)+9=0$$

i.e., $X-Y+Z+9=0\cdots\cdots(2)$

Let (X_1, Y_1, Z_1) be the pole, then the polar plane is

$$XX_1+YY_1+ZZ_1-9=0\cdots\cdots(3)$$

(2) & (3) represent the same plane.

$$\therefore\ \frac{X_1}{1}=\frac{Y_1}{-1}=\frac{Z_1}{-1}=\frac{9}{-9}\Rightarrow X_1=-1,\ Y_1=1,\ Z_1=1$$

Pole in the original system is $(0,-1,4)$

II method:

Given plane is $x-y-z+9=0\cdots\cdots\cdots\cdots(1)$

Let $P(x_1, y_1, z_1)$ be the pole of the plane (1) w.r.t. the sphere

$$S\equiv x^2+y^2+z^2-2x+4y-6z+5=0\cdots\cdots(2)$$

Polar plane of P w.r.t. (2) is $S_1=0$

i.e., $xx_1+yy_1+zz_1-(x+x_1)+2(y+y_1)-3(z+z_1)+5=0$

$$(x_1-1)x+(y_1+2)y+(z_1-3)z-x_1+2y_1-3z_1+5=0\cdots\cdots(3)$$

(1) & (3) represent the same plane

$$\frac{x_1-1}{1}=\frac{y_1+2}{-1}=\frac{z_1-3}{-1}=\frac{-x_1+2y_1-3z_1+5}{9}=r\ \text{(say)}$$

$x_1 = r+1,\ y_1 = -r-2,\ z_1 = -r+3$ and

$$-x_1 + 2y_1 - 3z_1 + 5 = 9r$$

$$\Rightarrow -(r+1) + 2(-r-2) - 3(-r+3) + 5 - 9r = 0$$

$$\Rightarrow -9r - 9 = 0 \Rightarrow r = -1$$

The pole is $(x_1, y_1, z_1) = (r+1, -r-2, -r+3)$

$$= (-1+1, +1-2, +1+3) = (0, -1, 4)$$

5.10.5 Example: Show that polar line of

$$\frac{x+1}{2} = \frac{y-2}{3} = \frac{z+3}{1} = r \cdots\cdots (1)$$

w.r.t. the sphere

$$x^2 + y^2 + z^2 = 1 \cdots\cdots (2)$$

is the line

$$\frac{7x+3}{11} = \frac{2-7y}{5} = \frac{z}{-1}$$

Solution: Any point on the line (1) $P(2r-1, 3r+2, r-3)$

The polar plane of P w.r.t the sphere (2) is

$$x(2r-1) + y(3r+2) + z(2r-3) - 1 = 0$$

$$\Rightarrow r(2x+3y+2z) - (x-2y+3z+1) = 0 \cdots\cdots (3)$$

For all values of r this plane (4) passes through the line

$$2x+3y+2z = 0$$

$$x-2y+3z+1 = 0$$

Symmetric form of the above line:

To get a point on the line, put $z = 0$ then

$$\left.\begin{array}{l} 2x+3y=0 \\ x-2y=-1 \end{array}\right\}$$ by solving $y = 2/7,\ x = -3/7$

$\left(-3/7,\ 2/7,\ 0\right)$ is a point on the line

Let ℓ, m, n be the DC's of the line then

$$2\ell + 3m + 2n = 0$$

$$\ell - 2m + 3n = 0$$

$$\Rightarrow \frac{\ell}{11} = \frac{m}{-5} = \frac{n}{-7} \quad \Rightarrow (\ell, m, n) = (11, -5, -7)$$

Equation to the line are

$$\frac{\left(x + \frac{3}{7}\right)}{11} = \frac{\left(y - \frac{2}{7}\right)}{-5} = \frac{z - 0}{-7}$$

or $$\frac{7x + 3}{11} = \frac{2 - 7y}{5} = \frac{z}{-1}$$

5.10.6 Example: If PA and QB be drawn perpendicular to the polars of Q and P respectively, with respect a sphere, with centre 0,

Then $\dfrac{PA}{QB} = \dfrac{OP}{OQ}$

Solution: Let the equation of the sphere be

$$x^2 + y^2 + z^2 = a^2,\ a \geq 0 \text{ with centre } (0,0,0)$$

Let $P = (x_1,\ y_1, z_1),\ Q = (x_2, y_2, z_2)$

Polar planes of P, Q w.r.t. the sphere are

$$xx_1 + yy_1 + zz_1 - a^2 = 0 \cdots\cdots (1)$$

$$xx_2 + yy_2 + zz_2 - a^2 = 0 \cdots\cdots\cdot (2)$$

$PA = \perp^{ler}$ distance from P to the polar plane of Q (2)

$$= \frac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{\sqrt{x_2^2 + y_2^2 + z_2^2}} = \frac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{OQ}$$

$PB = \perp^{ler}$ distance from Q to the polar plane of P (1)

$$= \frac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{\sqrt{x_1^2 + y_1^2 + z_1^2}} = \frac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{OP}$$

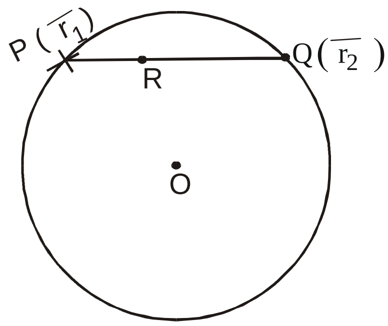
$$\frac{PA}{PB} = \frac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{OQ} \Big/ \frac{x_1x_2 + y_1y_2 + z_1z_2 - a^2}{OP} = \frac{OP}{OQ}$$

5.11 Answers to S.A.Q.'s:

5.11.1 Solution of 5.4.7:

Let S be a sphere having centre origin 'o' and radius a.

Let $P(\overline{r_1}),\ Q(\overline{r_2})$ be any two points on the sphere S. Then



$$\overline{OP} = \overline{r_1}, \quad \overline{OQ} = \overline{r_2}$$

and $\quad \left|\overline{OP}\right| = \left|\overline{r_1}\right| = a, \quad \left|\overline{OQ}\right| = \left|\overline{r_2}\right| = a$

Let $R(\bar{r})$ be any point on the line segment PQ.

Let R divides PQ in the ratio $\lambda:1$ then

$$\overline{OR} = \bar{r} = \frac{\lambda\overline{r_2} + \overline{r_1}}{\lambda + 1}$$

$$\left|\overline{OR}\right| = \left|\bar{r}\right| = \left|\frac{\lambda\overline{r_2} + \overline{r_1}}{\lambda + 1}\right|$$

$$\leq \frac{1}{\lambda+1}\left(\lambda\left|\overline{r_2}\right| + \left|\overline{r_1}\right|\right)$$

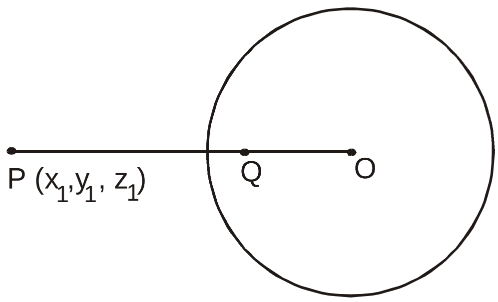
$$= \frac{1}{\lambda+1}(\lambda a + a) = a$$

$\Rightarrow \left|\overline{r}\right| < a \Rightarrow R$ is an internal point of the sphere

5.11.2 Solution of 5.9.5:

(i) Equations the sphere is

$$x^2 + y^2 + z^2 = a^2 \cdots\cdots\cdots\cdots (1)$$



Let $P(x_1, y_1, z_1)$ be any point.

The polar plane of P w.r.t to the sphere (1) is

$$xx_1 + yy_1 + zz_1 - a^2 = 0 \cdots\cdots\cdots\cdots (2)$$

Dr's of OP are $x_1, y_1, z_1 =$ Dr's of the normal to the polar plane (2)

$\Rightarrow$ The polar plane of P is perpendicular to OP.

(ii) $OQ =$ Perpendicular distance of the centre 0 form the polar plane (2)

$$= \frac{a^2}{\sqrt{x_1^2 + y_1^2 + z_1^2}} = \frac{a^2}{OP}$$

$$\Rightarrow OP \cdot OQ = a^2$$

5.12 Summary:

After going to this lesson the student is expected to have a clear idea about the notions of tangency, pole, polar plane, conjugacy and others. The student shall also be able to appriciate the behaviour of spheres intersecting in a single point and in more than one point.

5.13 Technical Terms:

Tangent line

Tangent Plane

Plane of Contact

Pole

Polar Plane

Conjugate Points

Conjugate lines or Polar lines

5.14 Exercise:

1. Find points of intersection of the line

$$\frac{x-8}{4}=\frac{y}{1}=\frac{z-1}{-1}$$

and the sphere

$$x^2+y^2+z^2-4x+6y-2z+5=0$$

2. Find the length of the tangent line from the point $(3,1,-1)$ to the sphere $x^2+y^2+z^2-3x+5y+7=0$.

3. Find the equation of the tangent plane to the sphere

$x^2+y^2+z^2-2x-4y+2z-3=0$ at $(-1,4,-2)$.

4. show that the plane $2x-2y+z+12=0$ touches the sphere

$x^2+y^2+z^2-2x-4y+2z-3=0$ and find the point of contact.

5. Find the equation of the sphere which touches the sphere $x^2+y^2+z^2+2x-6y+1=0$ at the point $(1,2,-2)$ and pass through the origin.

6. Find the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to $2x+2y-z=0$.

7. Find the equations of the spheres which pass through the circle.

(i) $x^2+y^2+z^2=5, \quad x+2y+3z-3=0$ and touch the plane $4x+3y-15=0$.

(ii) $x^2+y^2+z^2-6x-2z+5=0, \; y=0$ and touch the plane $3y+4z+5=0$.

(iii) $x^2+y^2+z^2=1, \; 2x+4y+5z=6$ and touching the plane $z=0$.

(iv) $x^2+y^2+z^2-2x+2y+4z-3=0, \; 2x+y+z=4$ and touch the plane

$3x+4y-14=0$.

8. Find the equation of the sphere which has its centre at the origin and touches the line

$$\frac{x+1}{1}=\frac{y-2}{-2}=\frac{z+3}{2}$$

9. Show that the locus of the centres of spheres which pass through the fixed point $(0,0,0)$ and touch the plane $z=0$ is $x^2+y^2-2az+a^2=0$.

10. Show that the spheres

(i) $x^2+y^2+z^2+2x-4y-6z+10=0, \; x^2+y^2+z^2-6x-4y-12z+40=0$ touch each other externally.

(ii) $x^2+y^2+z^2=25, \; x^2+y^2+z^2-24x-40y-18z+225=0$ touch externally. Find the point of contact.

(iii) $x^2+y^2+z^2=64, \; x^2+y^2+z^2-12x+4y-6z+48=0$ touch internally. Find the point of contact.

11. Find the pole of the plane $10x-2y-5z-2=0$ w.r.t. the sphere $x^2+y^2+z^2-6x+2y-3z+1=0$.

12. Prove that the polar plane of any point on the line $\frac{x}{2}=\frac{y-1}{3}=\frac{z+3}{4}$.

with respect to the sphere $x^2+y^2+z^2=1$ passes through the line

$$\frac{2x+3}{13}=\frac{y-1}{-3}=\frac{z}{-1}$$

13. Show that the planes $5x - y - 6z + 25 = 0,\ x - 2y - 3z + 25 = 0$ are conjugate w.r.t. the sphere $x^2 + y^2 + z^2 = 25$.

14. Find the polar line of

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ w.r.t. the sphere } x^2 + y^2 + z^2 = 16.$$

15. Findthe locus of the points whose polar planes w.r.t. the sphere $x^2 + y^2 + z^2 = a^2$ touch the sphere $(x-\alpha)^2 + (y-\beta)^2 + z^2 = r^2$

5.15 Answers to Exercise:

1. $(4,-1,2),\ (0,-2,3)$

2. $\sqrt{14}$

3. $2x - 2y + z + 12 = 0$

4. $(-1,4,-2)$

5. $4\left(x^2 + y^2 + z^2\right) + 10x - 25y - 2z = 0$

6. $2x + 2y - z + 10 = 0,\ 2x + 2y - z - 8 = 0$

7. (i) $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0,\ 5\left(x^2 + y^2 + z^2\right) - 4x - 8y - 12z - 13 = 0$

   (ii) $x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0,\ 4\left(x^2 + y^2 + z^2\right) - 24x - 11y - 8z + 20 = 0$

   (iii) $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0,\ 5\left(x^2 + y^2 + z^2\right) - 2x - 4y - 5z + 1 = 0$

   (iv) $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0,\ x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$

8. $9\left(x^2 + y^2 + z^2\right) = 5$

10. (i) $\left(\frac{12}{5}, 4, \frac{9}{5}\right)$

    (ii) $\left(\frac{48}{7}, \frac{-16}{7}, \frac{24}{7}\right)$

11. $(-2,0,4)$

14. $2x+3y+4z=0=x+2y+3z-16$

15. $\left(\alpha x+\beta y-a^2\right)^2=r^2\left(x^2+y^2+z^2\right)$

5.16 Model Examination Questions:

1. Prove that the locus of the tangent line at a point on a sphere of non zero radius is a plane.

2. Find the equation of the tangent plane to the sphere

   $3\left(x^2+y^2+z^2\right)-2x-3y-4z-22=0$ at the point $(1,2,3)$.

3. Find the equation of the sphere incribed in the tetrahedran whose faces are

$$x=0,\ y=0,\ z=0,\ x+2y+2z=1$$

5.17 Model Practical Problem with Solution:

Problem: Find the equation of the spheres which pass through the circle $x^2+y^2+z^2=5,\ x+2y+3z-3=0$ and touch the plane $4x+3y-15=0$.

AIM: To find the equation of the spheres which pass through the given circle and touch the given plane.

Definitions and results used:

(1) The equation of any sphere through a given circle which is the intersection of the sphere $S=0$, the plane $U=0$ is of the form $S+\lambda U=0$.

(2) The locus of the tangent lines at a point D on a sphere $S=0$ of non zero radius is a plane called the tangent plane to the sphere $S=0$ at D. The point D is called the point of contact of the plane with the sphere $S=0$.

(3) The equation of the tangent plane at the point $(x_1,\ y_1,\ z_1)$ of the sphere

$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $S_1=0$

i.e., $xx_1+yy_1+zz_1+u\left(x+x_1\right)+v\left(y+y_1\right)+w\left(z+z_1\right)+d=0$

Solution: Given circle is

$$x^2+y^2+z^2=5,\ x+2y+3z-3=0$$

Any sphere through the given circle is of the form

$$x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0 \cdots\cdots\cdots\cdots (1)$$

centre $\left(\frac{-\lambda}{2}, -\lambda, \frac{-3\lambda}{2}\right)$ and

radius $=\sqrt{\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}+5+3\lambda}=\sqrt{\frac{7\lambda^2}{2}+3\lambda+5}$

The sphere (1) touches the plane $4x+3y-15=0$ then the perpendicular distance from centre of the sphere to the plane is equal to radius of the sphere.

$$\therefore \left|\frac{4\left(\frac{-\lambda}{2}\right)+3(-\lambda)-15}{\sqrt{16+9}}\right|=\sqrt{\frac{7\lambda^2+6\lambda+10}{2}}$$

$$\therefore \lambda=2 \quad \text{or} \quad \lambda=\frac{-4}{5}$$

substitute $\lambda=2$ in (1) we get

$$x^2+y^2+z^2-5+2(x+2y+3z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+2x+4y+6z-11=0$$

substitute $\lambda=\frac{-4}{5}$ in (1) we get

$$x^2+y^2+z^2-5-\frac{4}{5}(x+2y+3z-3)=0$$

$$\Rightarrow 5\left(x^2+y^2+z^2\right)-4x-8y-12z-13=0$$

$\therefore$ The required spheres are

$x^2+y^2+z^2+2x+4y+6z-11=0$ and

$5\left(x^2+y^2+z^2\right)-4x-8y-12z-13=0$

Lesson Writer

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Lesson - 6

SPHERE 3 - SYSTEM OF SPHERES

6.1 Objective of the lesson:

After studying this lesson the student should be a able to gain working knowledge on

- Radical plane of two spheres
- Radical line of three spheres
- Radical centre of four spheres
- Coaxial system of spheres
- Limiting points of the coaxial system of spheres

6.2 Structure:

This lesson contains the following components:

6.3 **Introduction**

6.4 **Angle between spheres**

6.5 **Radical Plane and Line**

6.6 **Coaxial System of Spheres**

6.7 **Parametric forms of some Coaxial system of spheres**

6.8 **Limiting Points**

6.9 **Answers to S.A.Q.**

6.10 **Summary**

6.11 **Technical Terms**

6.12 **Exercise**

6.13 **Answers to Exercise**

6.14 **Model Examination Questions**

6.15 **Model Practical Problem with Solution**

6.3 Introduction:

In this last lesson on sphere we discuss about some deeper notions related to spheres such as Radical Planes, Radical lines, Coaxial systems and their limiting points.

6.4 Angles between Spheres:

6.4.1 Definition: An angle of intersection of two spheres is an angle between the tangent planes to them at a common point of intersection.

If α is one angle then $\pi-\alpha$ is another.

6.4.2 Theorem: If two spheres of radii r_1 and r_2 intersect an angle θ, then

$\operatorname{Cos}\theta = \pm\dfrac{r_1^2+r_2^2-d^2}{2r_1r_2}$ where d is the distance between the centres.

Proof:

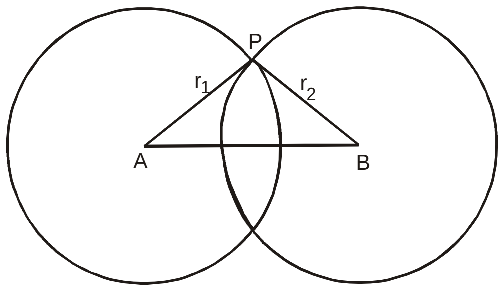
Let A and B be the centres of the spheres of radii r_1 and r_2.

Let P be a point on the section

(i.e., P is a common point of two spheres),

Let θ be an angle between two spheres at P.

Clearly $AP = r_1,\ BP = r_2$



Since PA and PB are the normals to the tangent planes at P.

$\theta =$ angle between the tangent planes at P.

= angle between the normals to the tangent planes at P.

= angle between $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$.

So $\theta = \underline{|APB}$ or $180^0 - \underline{|APB}$. If P does not lie on AB

From the ΔAPB, $AB^2 = PA^2 + PB^2 - 2PA\cdot PB \quad \cos \underline{|APB}$.

$$\Rightarrow d^2 = r_1^2 + r_2^2 \mp 2r_1r_2 \cos\theta$$

$$\Rightarrow \cos\theta = \pm\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

If P lies on AB, $AB = AP + PB \Rightarrow AB^2 = AP^2 + PB^2 + 2AP \cdot PB$

$$\Rightarrow d^2 = r_1^2 + r_2^2 + 2r_1r_2$$

In this case $\theta = 0$ so $\cos\theta = 1$ and we have

$$\cos\theta = 1 = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

6.4.3 Note: $|\cos\theta| < 1 \Leftrightarrow \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right| < 1$

$$\Leftrightarrow -1 < \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} < 1$$

$\Leftrightarrow (r_1 - r_2)^2 < d^2$ and $(r_1 + r_2)^2 > d^2$

$\Leftrightarrow |r_1 - r_2| < d$ and $r_1 + r_2 > d$

$\Leftrightarrow$ there exist a point P such that OPO^1 is a triangle with

$OP = r_1$ and $O^1P = r_2$

$\Leftrightarrow$ Given spheres are intersecting.

6.4.4 Definition: Two intersecting spheres are said to be orthogonal or cut orthogonally if the angle of intersection of the spheres is $\frac{\pi}{2}$.

The condition for two spheres to cut orthogonally is that the sum of the squares on the radii is equal to the square on the distance between their centres.

i.e., $r_1^2 + r_2^2 = d^2$

6.4.5 Theorem: The intersecting spheres

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$S^1 \equiv x^2 + y^2 + z^2 + 2u^1x + 2v^1y + 2w^1z + d^1 = 0$$

Cut orthogonally

$$\Leftrightarrow 2uu^1 + 2vv^1 + 2ww^1 = d + d^1$$

Proof: Let A, B be the centres and r_1, r_2 be the radii of the spheres $S = 0, S^1 = 0$ then

$$A = (-u, -v, -w), B = \left(-u^1, -v^1, -w^1\right) \text{ and}$$

$$r_1 = \sqrt{u^2 + v^2 + w^2 - d}, \; r_2 = \sqrt{u^{1^2} + v^{1^2} + w^{1^2} - d^1}$$

Spheres $S = 0, S^1 = 0$ cut orthogonally

$\Leftrightarrow$ Given spheres $S = 0, S^1 = 0$ are intersecting and angle between them is 90^0.

$\Leftrightarrow \exists$ a point P such that $PA = r$, $PB = r_2$ and $\underline{|APB} = 90^0$

$\Leftrightarrow AB^2 = r_1^2 + r_2^2$

$$\Leftrightarrow \left(u - u^1\right)^2 + \left(v - v^1\right)^2 + \left(w - w^1\right)^2 = u^2 + v^2 + w^2 - d + u^{1^2} + v^{1^2} + w^{1^2} - d$$

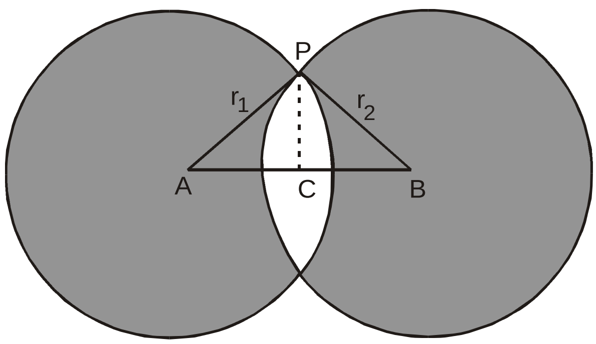
$$\Leftrightarrow 2uu^1 + 2vv^1 + 2ww^1 = d + d^1$$

6.4.6 Theorem: If r_1, r_2 are the radii pf two orthogonal spheres, then the radius of the circle of their intersection is $\dfrac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

Proof: Let A, B be the centres of two orithogonal spheres and P be a common point.

Let $PA = r_1$, $PB = r_2$ and $AB = d$

Let the centre and radius of circle of intersection of spheres be C and a respectively.



Given two spheres cut orthogonally

$$\Rightarrow r_1^2 + r_2^2 = d^2 \cdots\cdots\cdots\cdots(1)$$

$$CP = a \cdots\cdots\cdots(2)$$

$$CP \perp AB \cdots\cdots\cdots\cdots(3)$$

$$AC + CB = d$$

$$\Rightarrow \sqrt{r_1^2 - a^2} + \sqrt{r_2^2 - a^2} = d$$

Squaring on both sides

$$r_1^2 - a^2 + r_2^2 - a^2 + 2\sqrt{\left(r_1^2 - a^2\right)\left(r_2^2 - a^2\right)} = d^2$$

$$\Rightarrow 2\sqrt{\left(r_1^2 - a^2\right)\left(r_2^2 - a^2\right)} = 2a^2 \qquad \left(\because\ r_1^2 + r_2^2 = d^2\right)$$

$$\Rightarrow \left(r_1^2 - a^2\right)\left(r_2^2 - a^2\right) = a^4$$

$$r_1^2 r_2^2 - \left(r_1^2 + r_2^2\right) a^2 + a^4 = a^4$$

$$a = \frac{r_1\ r_2}{\sqrt{r_1^2 + r_2^2}}$$

6.4.7 Example: Find the equation of the sphere that passes through the circle.

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0, \quad 3x - 4y + 5z - 15 = 0$$

and cuts the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$$

Orthogonally

Solution: The general form of the equation of the sphere through the given circle is

$$\left(x^2 + y^2 + z^2 - 2x + 3y - 4z + 6\right) + \lambda\left(3x - 4y + 5z - 15\right) = 0$$

i.e., $x^2 + y^2 + z^2 + (3\lambda - 2)x + (3 - 4\lambda)y + (5\lambda - 4)z + 6 - 15\lambda = 0 \cdots\cdots\cdots(1)$

The condition that (1) cuts the sphere $x^2+y^2+z^2+2x+4y-6z+4=0\cdots\cdots\cdots\cdots(2)$ orthogonally is

$$2\frac{(3\lambda-2)}{2}\cdot 1+2\cdot\frac{(3-4\lambda)}{2}\cdot 2+2\cdot\frac{(5\lambda-4)}{2}(-3)=6-15\lambda+11$$

i.e., $3\lambda-2+6-8\lambda-15\lambda+12=17-15\lambda$

i.e., $-5\lambda=1$

i.e, $\lambda=\frac{-1}{5}$

Putting $\frac{-1}{5}$ for λ in (1) the equation to sphere becomes

$$x^2+y^2+z^2-\frac{13}{5}x+\frac{19}{5}y-5z+9=0$$

i.e.,

$$5\left(x^2+y^2+z^2\right)-13x+19y-25z+45=0$$

6.4.8 **Example:** Find the equation of the sphere that passes through the two points $(0,3,0)$, $(-2,-1,-4)$ and cuts, orthogonally the two spheres

$$x^2+y^2+z^2+x-3z-2=0,\ 2\left(x^2+y^2+z^2\right)+x+3y+4=0$$

Solution: Let the equation of the required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0\cdots\cdots\cdots(1)$$

Since it passes through $(0,3,0)$ and $(-2,-1,-4)$ we get

$$9+6v+d=0\cdots\cdots\cdots(2)$$

and $4+1+16-4u-2v-8w+d=0$

i.e., $4u+2v+8w-d-21=0\cdots\cdots\cdots\cdots(3)$

Sphere (1) is orthogonal to $x^2+y^2+z^2+x-3z-2=0$

$$\Leftrightarrow\ 2u\frac{1}{2}+2v(0)+2w\left(\frac{-3}{2}\right)=d-2 \qquad \Rightarrow\ u-3w-d+2=0\cdots\cdots\cdots(4)$$

Sphere (1) is orthogonal to $x^2+y^2+z^2+\frac{x}{2}+\frac{3}{2}y+2=0$

$$\Leftrightarrow 2u\frac{1}{4}+2v\frac{3}{4}+2w(0)=d+2$$

$$\Leftrightarrow\ u+3v-2d-4=0\cdots\cdots\cdots(5)$$

From (2) $\boxed{v=\frac{-9-d}{6}}$

From (5) $u+\cancel{3}\left(\frac{-9-d}{\cancel{6}_{2}}\right)-2d-4=0 \Rightarrow 2u-9-d-4d-8=0$

$$\Rightarrow \boxed{u=\frac{5d+17}{2}}$$

From (4) $\frac{5d+17}{2}-3w-d+2=0 \Rightarrow 5d+17-6w-2d+4=0$

$$\Rightarrow\ w=\frac{3d+21}{6}=\frac{d+7}{2}$$

From (3) $4\left(\frac{5d+17}{2}\right)+2\left(\frac{-9-d}{6}\right)+8\left(\frac{d+7}{2}\right)-d-21=0$

$$\Rightarrow 38d+114=0 \Rightarrow d=\frac{-114}{38}=-3$$

$\therefore\ \ u=1,\ v=-1,\ \ w=2$

$\therefore$ The required, equation is

$$x^2+y^2+z^2+2x-2y+4z-3=0$$

6.4.9 Example: Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere

$$x^2+y^2+z^2-4x+6y+4=0$$

Solution: Let centre of the required sphere be C_1 and radius be r_1.

Given sphere is $x^2+y^2+z^2-4x+6y+4=0 \cdots\cdots(1)$

with centre $C_2=(2,-3,0)$, and radius $r_2=\sqrt{4+9-4}=3$

Required sphere touches the plane

$$3x+2y-z+2=0 \cdots\cdots(2)$$

at the point $(1,-2,1)$.

Hence centre C_1 lies on the line through $(1,-2,1)$ and perpendicular to the plane (2), its equation is

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-1}{-1}=r \qquad (say)$$

C_1 can be taken as $C_1=(3r+1,\ 2r-2,\ -r+1)$ and radius r_1 is the distance of the centre C_1 from the point of contact $(1,-2,1)$.

$$r_1^2=(3r+1-1)^2+(2r-2+2)^2+(-r+1-1)^2$$

$$=9r^2+4r^2+r^2=14r^2$$

$$r_1=\pm\sqrt{14}\ r=\sqrt{14}\ |r|$$

Since the required sphere and the sphere (1) cut orthogonally

$$\therefore\ C_1C_2 = r_1^2+r_2^2$$

$$\Rightarrow (3r-1)^2+(2r+1)^2+(-r+1)^2=14r^2+9$$

$$\Rightarrow 9r^2+1-6r+4r^2+1+4r+r^2+1-2r-14r^2-9=0$$

$$\Rightarrow -4r-6=0 \Rightarrow r=-3/2 \ \Rightarrow r_1=\frac{3}{2}\sqrt{14}$$

$\therefore$ Centre of therequired sphere $C_1 = \left(3\left(\frac{-3}{2}\right)+1,\ 2\left(\frac{-3}{2}\right)-2,\ \frac{3}{2}+1\right)$

$$= \left(\frac{-7}{2},\ -5,\ 5/2\right)$$

and radius $r_1 = \frac{3}{2}\sqrt{14}$

Required equation of the sphere is

$$\left(x+\frac{7}{2}\right)^2 + (y+5)^2 + \left(z-\frac{5}{2}\right)^2 = \left(\frac{3}{2}\sqrt{14}\right)^2$$

$$\Rightarrow x^2 + \frac{49}{4} + 7x + y^2 + 25 + 10y + z^2 + \frac{25}{4} - 5z = \frac{63}{2}$$

$$\Rightarrow\ x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$$

6.4.10 Example: Show that every sphere through the circle

$$x^2 + y^2 - 2ax + r^2 = 0, \quad z = 0$$

cuts orthogonally every sphere through the circle

$$x^2 + z^2 = r^2,\ \ y = 0$$

Solution: The two spheres through the given circles are

$$x^2 + y^2 + z^2 - 2ax + r^2 + 2\lambda z = 0 \text{ for some } \lambda \in \mathbb{R}$$

and $$x^2 + y^2 + z^2 - r^2 + 2\mu y = 0 \text{ for some } \mu \in \mathbb{R}$$

These can be re-written as

$$x^2 + y^2 + z^2 - 2ax + 2\lambda z + r^2 = 0$$

$$x^2 + y^2 + z^2 + 2\mu y - r^2 = 0$$

The condition of orthogonal intersection is of the 6.4.5 holds

Since $2(-a)0 + 2\cdot 0\cdot \mu + 2\lambda\cdot(0) = r^2 - r^2$ or $0 = 0$

Hence the two spheres intersect orthogonally.

6.4.11 S.A.Q.: If two points P and Q are conjugate with respect to a sphere S. Prove that the sphere with diameter PQ cuts S orthogonally.

6.4.12 S.A.Q.: Two spheres $S_1 = 0$ and $S_2 = 0$ are orthogonal show that the polar plane of any point P on S_1 with respect to S_2, passes through the other end of the diameter of S_1 through P.

6.5 Radical Plane and Line:

Let S be a sphere with centre $C(\bar{c}) = (-u, -v, -w)$ and radius a. The vector equation of the sphere is given by

$$F(\bar{r}) = |\bar{r} - \bar{c}|^2 - a^2 = 0$$

and cartician form of the equation is

$$S(x, y, z) = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$$

where $u^2 + v^2 + w^2 - d = a^2$

For any $D = (x_1, y_1, z_1)$, The power of D is defined to be S_{11} from def (5.4.3). If a line through D meets the sphere S in P and Q then $DP \cdot DQ = S_{11}$ is independent of the line.

6.5.1 Theorem: The locus of points of whose powers w.r.t. two non-concentric spheres are equal is the plane perpendicular to the line of centeres of the two spheres.

Proof: Let the equations of two spheres be

$$|\bar{r} - \overline{c_1}|^2 = a_1^2, \quad |\bar{r} - \overline{c_2}|^2 = a_2^2$$

$P(\bar{r})$ lies on the locus $\Leftrightarrow$ powers of the point $P(\bar{r})$ w.r.t. two spheres are equal

$$\Leftrightarrow |\bar{r} - \bar{c}_2|^2 - a_1^2 = |\bar{r} - \bar{c}_2|^2 - a_2^2$$

$$\Leftrightarrow 2\bar{r} \cdot (\overline{c_1} - \overline{c_2}) + |\overline{c_2}|^2 - |\overline{c_1}|^2 + a_2^2 - a_1^2 = 0$$

This is of the form $\bar{r} \cdot \bar{n} = p$.

Hence the above equation represents the plane perpendicular to $(\overline{c_1} - \overline{c_2})$ i.e., the line of centres of the given two spheres.

6.5.2 Def: The locus of points each of whose powers w.r.t. two non - concentric spheres are equal, is a plane called the radical plane of the two spheres.

6.5.3 Theorem: The equation of the radical plane of the spheres

$$S=0,\ S^1=0 \text{ is } S-S^1=0$$

Proof: Let $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$S^1 \equiv x^2+y^2+z^2+2u^1x+2v^1y+2w^1z+d^1=0$

$P(x_1, y_1, z_1)$ is a point on the radical plane of $S=0, S^1=0$

$\Leftrightarrow \quad S_{11}=S_{11}^1$

$\Leftrightarrow \quad x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=x_1^2+y_1^2+z_1^2+2u^1x+2v^1y+2w^1z+d^1=0$

$\Leftrightarrow 2\left(u-u^1\right)x_1+2\left(v-v^1\right)y_1+2\left(w-w^1\right)z_1+d-d^1=0$

The locus of P is

$2\left(u-u^1\right)x+2\left(v-v^1\right)y+2\left(w-w^1\right)z+d-d^1=0$

$\Leftrightarrow S-S^1=0$ Since at least one of $u-u^1, v-v^1, w-w^1$ is non zero this represents a plane.

6.5.4 Note:

1) If two spheres intersect then their radical plane is the plane of their circle of intersection.

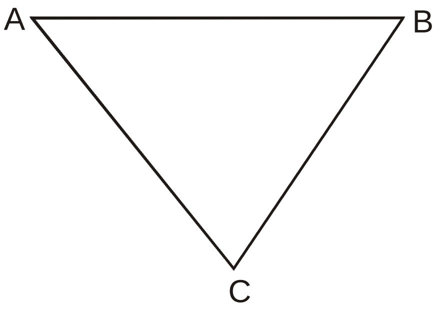
2) If two spheres touch, their radical plane is the common tangent plane at the point of contact.

3) The radical plane of two spheres is perpendicular to their line of centres.

6.5.5 Theorem: If $S_1=0, S_2=0, S_3=0$ are three spheres whose centres are non - collinear, then the three radical planes of the spheres taken in pairs pass through a line.

Proof: Let A, B, C be the centres of the spheres $S_1=0, S_2=0, S_3=0$.

The radical plane (π_1) of $S_2=0, S_3=0$ is $S_2-S_3=0$ and π_1 is perpendicular to $BC(6.5.1)$.



The radical plane (π_2) of $S_3 = 0, S_1 = 0$ is $S_3 - S_1 = 0$ and π_2 is perpendicular to AC.

The radical plane (π_3) of $S_1 = 0, S_2 = 0$ is $S_1 - S_2 = 0$ and π_3 is perpendicular to AB.

Since AB, BC are intersecting lines, the planes π_3, π_1 have a common line L (say) (Ref -)

Since the general equation of the plane passing through the intersection of $S_1 - S_2 = 0$ and $S_2 - S_3 = 0$ is

$$(S_1 - S_2) + \lambda(S_2 - S_3) = 0$$

It follows by taking $\lambda = 1$, that the plane $S_1 - S_3 = 0$ that is π_2 passes through L.

So π_1, π_2, π_3 pass through the common line L.

6.5.6 Def: It $S_1 = 0, S_2 = 0, S_3 = 0$ a three spheres whose centres are non - collinear then the common line of the three radical planes of the spheres taken in pairs is called radical line of the spheres $S_1 = 0, S_2 = 0, S_3 = 0$.

6.5.7 Theorem: If $S = 0, S^1 = 0, S^{11} = 0, S^{111} = 0$ are four spheres whose centres are non - coplanar, then the four radical lines of these four spheres taken three by three intersect at a unique point.

Proof: Let $S \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

$$S^1 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

$$S^{11} \equiv x^2 + y^2 + z^2 + 2u_3x + 2v_3y + 2w_3z + d_3 = 0$$

$$S^{111} \equiv x^2 + y^2 + z^2 + 2u_4x + 2v_4y + 2w_4z + d_4 = 0$$

be the equations of the given four spheres.

The radical plane of $S=0, S^1=0$ is $S-S^1=0$

$$\Rightarrow 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0\cdots\cdots\cdots(1)$$

The radical plane of $S=0, S^{11}=0$ is $S-S^{11}=0$

$$\Rightarrow 2(u_1-u_3)x+2(v_1-v_3)y+2(w_1-w_3)z+d_1-d_3=0\cdots\cdots\cdots\cdot(2)$$

The radical plane of $S=0, S^{111}=0$ is $S-S^{111}=0$

$$\Rightarrow 2(u_1-u_4)x+2(v_1-v_4)y+2(w_1-w_4)z+d_1-d_4=0\cdots\cdots\cdot(3)$$

Since the centres $(-u_1,-v_1,-w_1)$, $(-u_2,-v_2,-w_2)$, $(-u_3,-v_3,-w_3)$, $(-u_4,-v_4,-w_4)$ are non coplanar.

$$\Delta=\begin{vmatrix} u_1 & v_1 & w_1 & 1 \\ u_2 & v_2 & w_2 & 1 \\ u_3 & v_3 & w_3 & 1 \\ u_4 & v_4 & w_4 & 1 \end{vmatrix}\neq 0$$

$$\Leftrightarrow\begin{vmatrix} u_1 & v_1 & w_1 & 1 \\ u_2-u_1 & v_2-v_1 & w_2-w_1 & 0 \\ u_3-u_1 & v_3-v_1 & w_3-w_1 & 0 \\ u_4-u_1 & v_4-v_1 & w_4-w_1 & 0 \end{vmatrix}\neq 0$$

$$\Leftrightarrow\begin{vmatrix} u_1-u_2 & v_1-v_2 & w_1-w_2 \\ u_1-u_3 & v_1-v_3 & w_1-w_3 \\ u_1-u_4 & v_1-v_4 & w_1-w_4 \end{vmatrix}\neq 0$$

$\Leftrightarrow$ The radical planes (1), (2), (3) pass through a unique point P.

Since P lies on the radical plane of $S=0, S^1=0\,;\, S^1=0, S^{11}=0\,;$

$\Leftrightarrow$ P lies on the radical line of $S=0, S^1=0, S^{11}=0$

Similarly, P lies on the radical line of $S=0, S^1=0, S^{111}=0$

P lies on the radical line of $S=0, S^{11}=0, S^{111}=0$

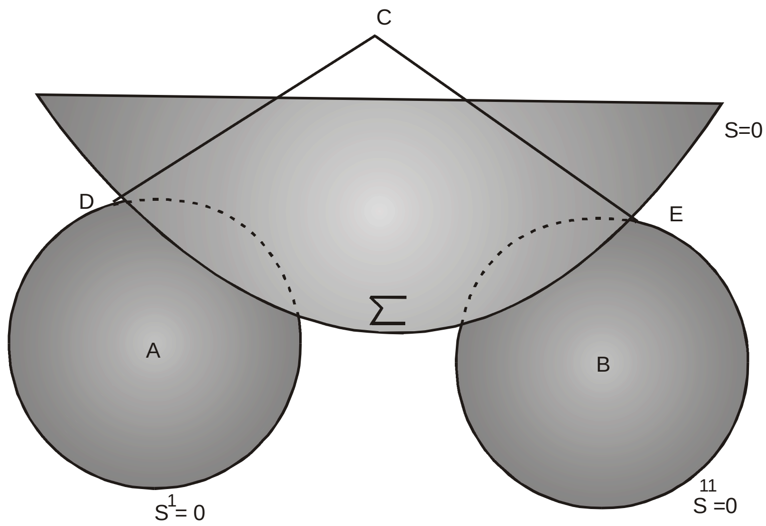
P lies on the radical line of $S^{1}=0, S^{11}=0, S^{111}=0$

$\Rightarrow$ The four radical lines of the four spheres taken three by three intersect at a unique point.

6.5.8 Definition: The four radical lines of four spheres with non - coplanar centres taken three by three intersect at a unique point, called the radical centre of the spheres.

6.5.9 Theorem: The centre of a sphere S = 0 which intersects two spheres $S^{1}=0, S^{11}=0$ orthogonally lies on the radical plane of the spheres $S^{1}=0, S^{11}=0$.

Proof: Let he centres of the spheres $S^{1}=0, S^{11}=0, S=0$ be A, B, C respectively.



Let D be the common point of the spheres $S=0, S^{1}=0$ and E be the common point of the spheres $S=0, S^{11}=0$.

Since $S=0$ intersects the two spheres $S^{1}=0, S^{11}=0$ orthogonally, CD, CE are tangent lines of the spheres $S^{1}=0, S^{11}=0$ respectively. (by theorem)

$\therefore$ CD^{2}, CE^{2} are powers ofthe point C with respect the spheres $S^{1}=0, S^{11}=0$.

Since $CD=CE$ (because radius of the spheres $S=0$)

The power of C w.r.t. to the spheres $S^1 = 0, S^{11} = 0$ are equal.

$\Rightarrow$ C lies on the radical plane of $S^1 = 0, S^{11} = 0$

6.6 Coaxal System of Spheres:

6.6.1 Def: A system of spheres is said to be a coaxal system of spheres if any two spheres of the system have the same radical plane.

6.6.2 Theorem: If $S = 0$ is a sphere and $U = 0$ is a plane then the equation.

$S + \lambda U = 0$, λ being the positive constatnt, represents a coaxal system of spheres with radical plane $U = 0$.

Proof: Let $S + \lambda_1 U = 0,\ S + \lambda_2 U = 0$ be two distinct spheres of the system of spheres $S + \lambda U = 0$. Then $\lambda_1 \neq \lambda_2$.

Radical plane of these two spheres is

$$(S + \lambda_1 U) - (S + \lambda_2 U) = 0$$

i.e., $(\lambda_1 - \lambda_2) U = 0$

i.e., $U = 0$

Therefore every two spheres of the system $S + \lambda U = 0$ have the same radical plane $U = 0$.

The system of spheres $S + \lambda U = 0$ represents a coaxal system of spheres with radical plane $U = 0$.

6.6.3 Corollary: If $S = 0, S^1 = 0$ are two non-concentric spheres then $\lambda_1 S + \lambda_2 S^1 = 0$ where $\lambda_1 + \lambda_2 \neq 0$ represents a coaxal system with the same radical plane as that of $S = 0$ and $S^1 = 0$.

6.6.4 Theorem: The centres of the spheres of a coaxal system of spheres are collinear and the line of centres is perpendicular to the radical plane.

Proof: Let C_1, C_2, C_3 be the centres of three spheres $S_1 = 0, S_2 = 0, S_3 = 0$ of coaxal system of spheres with radical plane $U = 0$.

Since the radical plane of two spheres is $\perp$ to the line of their centres, the radical plane of any two spheres is $U = 0$.

$C_1 C_2$ and $C_2 C_3$ are perpendicular to the plane $U = 0$

C_1, C_2, C_3 are collinear.

Since $C_1 C_2$ and $C_2 C_3$ are not parallel.

The centres of the spheres of a coaxal system of spheres are collinear and clearly their line is $\perp^{lar}$ to $U=0$.

6.7 Parametric Forms of Some Coaxal System of Spheres:

6.7.1 Theorem: The parametric equation fo a coaxal system **K** of spheres with YOZ as radical plane and x axis as the line of centres, is of the form

$$x^2+y^2+z^2+2\lambda x+d=0$$

where d is a fixed number and λ is the parametric constant.

Proof: If the sphere represented by $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

belongs to **K** then

The centre $(-u,-v,-w)$ lies on X - axis

$$\Rightarrow v=w=0$$

Then the equation of the sphere becomes

$$x^2+y^2+z^2+2ux+d=0 \cdots\cdots\cdots\cdots\cdots (1)$$

Since the origin lies on the radical plane YOZ $(\text{i.e.,}\ \ x=0)$

The power of the point 'O' w.r.t. (1) is d.

The power of the point O w.r.t. the coaxal system of spheres is same

Therefore d is fixed constant.

$\therefore$ The equation to the coaxal system of spheres in its simplest form is

$x^2+y^2+z^2+2\lambda x+d=0$ where $\lambda=u$ is a parametric constant and d is a fixed number.

6.7.2 Nature of the coaxal system of spheres of the form $x^2+y^2+z^2+2\lambda x+d=0$:

The radical plane of the coaxal system of spheres

$$x^2+y^2+z^2+2\lambda x+d=0 \cdots\cdots\cdots (1)$$

where λ is a parametric constant and d is a fixed constant is the YZ plane

$$\text{i.e., } x = 0 \cdots\cdots\cdots (2)$$

The points of intersection of the sphere (1) and the plane (2) must satisfy the equations

$$x = 0,\ y^2 + z^2 + d = 0$$

or $$x = 0,\ y^2 + z^2 = \left(\sqrt{-d}\right)^2$$

Case (1): If d < 0 then the poitns of intersection lie on the circle $x = 0,\ y^2 + z^2 = \left(\sqrt{-d}\right)^2$ and every sphere of the system passes through the circle.

Case (2): If $d = 0$ then $x = 0,\ y^2 + z^2 = 0 \Rightarrow x = 0,\ y = z = 0$ and hence $(0,0,0)$ is the only point common to (1) and (2).

Case (3): If $d > 0$ then there exits no points common to both (1) & (2) and hence no two spheres of the system have common points.

6.8 Limiting Points:

6.8.1 Def: The point spheres in a coaxal system of spheres are called limiting points of the coaxal system.

6.8.2 Example: Find the limiting points of the coaxal system of spheres.

$S_\lambda : x^2 + y^2 + z^2 + 2\lambda x + d = 0$ (d is fixed) .... (1).

The centre of S is $(-\lambda, 0, 0)$ and radius $r = \sqrt{\lambda^2 - d}$

Clearly S_λ is a point sphere iff $\sqrt{\lambda^2 - d} = 0 \Leftrightarrow \lambda^2 = d$.

(i) If $d > 0$ then, any two spheres in the coaxal system does not intersect and this system (1) has two limiting points $\left(-\sqrt{d}, 0, 0\right), \left(\sqrt{d}, 0, 0\right)$

(ii) If $d = 0$ then the coaxal system is a touching coaxal system of spheres at origin; and origin is the only limiting point of the system.

(iii) If $d < 0$ then the coaxal system of spheres has no limiting points. The system has intersecting spheres.

6.8.3 Example:

(i) Show that the sphere if $d > 0$

$$x^2 + y^2 + z^2 + 2vy + 2wz - d = 0 \cdots\cdots\cdots (1)$$

pases through the limiting points of the coaxal system

$$x^2 + y^2 + z^2 + 2kx + d = 0 \cdots\cdots\cdots (2)$$

and cuts every member the system orthoganally,

whatever be the values of v and w.

(ii) Hence deduce that every sphere that passes through the limiting points of the coaxal system (2) cuts every sphere of the system orthogonally.

Proof:

(i) From the sphere (2), centre $= (-k, 0,0)$, radius $r = \sqrt{k^2 - d}$

If $r = 0$ then $k^2 - d = 0 \Rightarrow k = \pm\sqrt{d}$

$\Rightarrow$ The limiting points of (2) are $\left(\sqrt{d}, 0,0\right)$ and $\left(-\sqrt{d}, 0,0\right)$ substituting these values is (1) we get

$d + 0 + 0 + 0 + 0 - d = 0$ and $d + 0 + 0 + 0 + 0 - d = 0$

$\Rightarrow$ The sphere (1) passes through the limiting points of the system (2)

Now, Apply orthogonality condition $2uu^1 + 2vv^1 + 2ww^1 = d + d^1$ on the sphere (1), (2) we get

$$2(0)k + 2v(0) + 2w(0) = -d + d$$

$\Rightarrow 0 = 0$

The sphere (1) cuts every member of the system (2) orthogonally.

(ii) Let the coaxal system of spheres be

$$x^2 + y^2 + z^2 + 2\lambda x + d = 0 \cdots\cdots\cdots (3)$$

Limiting points of the system are $\left(\sqrt{d}, 0, 0\right)$ and $\left(-\sqrt{d}, 0, 0\right)$

Let the equation of the sphere passing through these limiting points be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0\cdots\cdots(4)$$

$\Rightarrow$ $d-2u\sqrt{d}+c=0$ and $d+2u\sqrt{d}+c=0$

$\Rightarrow$ By solving $u=0$ and $c=-d$

So the sphere (4) raduces to $x^2+y^2+z^2+2vy+2wz-d=0\cdots\cdots(5)$

Clearly the spheres (3) and (5) are orthogonal.

Hence every sphere passes through the limiting points of coaxal system of spheres cuts every sphere of the system orthogonally.

6.8.4 Example: Show that the locus of the point spheres of the system

$$S\equiv x^2+y^2+z^2+2vy+2wz-d=0\cdots\cdots(1)$$

is the common circle of the system

$$S^1\equiv x^2+y^2+z^2+2ux+d=0\cdots\cdots(2)$$

u, v, w being the parameters and d a constant.

Solution: Limiting points of the system (1) are

$(0,-v,-w)$ with radius $r=\sqrt{v^2+w^2+d}=0$

$$\therefore\ v^2+w^2+d=0\cdots\cdots(3)$$

The common circle of (1) and (2) lies on the radical plane $S-S^1=0$

i.e., $-2ux+2vy+2wz-2d=0$

i.e., $-ux+vy+wz-d=0\cdots\cdots(4)$

The limiting points lie on the common circle of (1) and (2)

$\Leftrightarrow$ $(0,-v,-w)$ lies on the plane (4)

$\Leftrightarrow -v^2-w^2-d=0$

i.e., $v^2+w^2-d=0$

This is same as equation (3)

6.8.5 Example: Find the liminting points of the coaxzal system of spheres

$$x^2+y^2+z^2-20x+30y-40z+29+\lambda(2x-3y+4z)=0$$

Solution: The equation of coaxal system of spheres is

$$x^2+y^2+z^2+(-20+2\lambda)x+(30-3\lambda)y+(-40+4\lambda)z+29=0 \cdots\cdots\cdots(1)$$

centre = $\left(10-\lambda, \dfrac{3\lambda-30}{2}, 20-2\lambda\right)$

Radius = $\sqrt{(10-\lambda)^2+\left(\dfrac{3\lambda-30}{2}\right)^2+(20-2\lambda)^2-29}$

If (1) is a point sphere, then radius is zero

$$\Rightarrow (10-\lambda)^2+\left(\frac{3\lambda-30}{2}\right)^2+(20-2\lambda)^2-29=0$$

$$\Rightarrow 100+\lambda^2-20\lambda+\frac{9\lambda^2+900-180\lambda}{4}+400+4\lambda^2-80\lambda-29=0$$

$$\Rightarrow 29\lambda^2-580\lambda+2784=0$$

$$\Rightarrow \lambda^2-20\lambda+96=0$$

$$\Rightarrow (\lambda-8)(\lambda-12)=0$$

$$\Rightarrow \lambda=8,12$$

Limiting points are $\left(10-8, \dfrac{3(8)-30}{2}, 20-2(8)\right)$ and

$$\left(10-12, \frac{3(12)-30}{2}, 20-2(12)\right)$$

i.e., $(2,-3,4),(-2,3,-4)$

6.8.6 Example: Find the equation to radical line of three spheres

$$S_1 \equiv x^2+y^2+z^2+2x+2y+2z+2=0 \cdots\cdots\cdots\cdots(1)$$

$$S_2 \equiv x^2 + y^2 + z^2 + 4y = 0 \cdots\cdots\cdots\cdots(2)$$

$$S_3 \equiv x^2 + y^2 + z^2 + 3x - 2y + 8z + 6 = 0 \cdots\cdots\cdots\cdots(3)$$

Solution: Radical plane of (1) and (2) is $S_1 - S_2 = 0$ i.e., $2x - 2y + 2z + 2 = 0$ or $x - y + z + 1 = 0$.

Radical plane of (2) and (3) is $S_2 - S_3 = 0$ i.e., $3x - 6y + 8z + 6 = 0$

Radical line is $x - y + z + 1 = 0 = 3x - 6y + 8z + 6$.

6.8.7 Example: Find the equation to radical centre of the three spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0 \cdots\cdots\cdots(1)$$

$$S_2 \equiv x^2 + y^2 + z^2 + 4y = 0 \cdots\cdots\cdots(2)$$

$$S_3 \equiv x^2 + y^2 + z^2 + 3x - 2y + 8z + 6 = 0 \cdots\cdots\cdots(3)$$

$$S_4 \equiv x^2 + y^2 + z^2 - x + 4y - 6z - 2 = 0 \cdots\cdots\cdots(4)$$

Solution:

Radical plane of (1) & (2) is $S_1 - S_2 = 0 \Rightarrow x - y + z + 1 = 0$

Radical plane of (1)& (3) is $S_1 - S_3 = 0 \Rightarrow 3x - 6y + 8z + 6 = 0$

$\therefore$ Radical line of (1), (2), (3) is

$$x - y + z + 1 = 0 = 3x - 6y + 8z + 6 \cdots\cdots\cdots\cdots\cdots\cdots(5)$$

Radical plane of (3) & (4) is $S_3 - S_4 = 0 \Rightarrow 4x - 6y + 14z + 8 = 0$

$$\Rightarrow 2x - 3y + 7z + 4 = 0$$

Radical line of (1), (3), (4) is

$$3x - 6y + 8z + 6 = 0 = 2x - 3y + 7z + 4 = 0 \cdots\cdots\cdots\cdots(6)$$

The point of intersection of (5) & (6) is given by

$$x - y + z + 1 = 0$$

$$3x - 6y + 8z + 6 = 0$$

$$2x - 3y + 7z + 4 = 0$$

Solving these quation we get $x = \frac{-1}{5},\ y = \frac{1}{2},\ z = \frac{3}{10}$

$\therefore$ Radical centre is $\left(\frac{-1}{5}, \frac{1}{2}, \frac{3}{10}\right)$

6.8.8 Example: Three spheres of radii r_1, r_2, r_3 have their centres at

$A(a,0,0),\ B(0,b,0),\ C(0,0,c)$ and

$$r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (*)$$

A fourth sphere passes through the origin and the points A, B, C. Show that the radical centre of the four spheres lie on the plane $ax + by + cz = 0$.

Proof: The three given spheres are

$$S_1 \equiv x^2 + y^2 + z^2 - 2ax + a^2 - r_1^2 = 0 \cdots\cdots\cdots (1)$$

$$S_2 \equiv x^2 + y^2 + z^2 - 2by + b^2 - r_2^2 = 0 \cdots\cdots\cdots (2)$$

$$S_3 \equiv x^2 + y^2 + z^2 - 2cz + c^2 - r_3^2 = 0 \cdots\cdots\cdots (3)$$

Sphere of OABC is

$$S_4 \equiv x^2 + y^2 + z^2 - ax - by - cz = 0 \cdots\cdots\cdots (4)$$

Radical plane of (1) and (2) is $S_1 - S_2 = 0$

i.e., $-2ax + 2by + a^2 - b^2 + r_2^2 - r_1^2 = 0 \cdots\cdots\cdots (5)$

Radical plane of (1) and (3) is $S_1 - S_3 = 0$

$$-2ax + 2cz + a^2 - c^2 + r_3^2 - r_1^2 = 0 \cdots\cdots\cdots (6)$$

Radical plane of (1) and (4) is $S_1 - S_4 = 0$

i.e., $-ax + by + cz + a^2 - r_1^2 = 0$

or $-2ax + 2by + 2cz + 2a^2 - 2r_1^2 = 0 \cdots\cdots\cdots (7)$

By solving (5), (6) and (7) we get radical centre

(7) - (5) gives $2cz + a^2 + b^2 - r_1^2 - r_2^2 = 0$ .... (8)

Similarly (7) - (6) gives $2by + a^2 + c^2 - r_1^2 - r_3^2 = 0$ .... (9)

(7) - [(5) + (6)] gives $2ax + b^2 + c^2 - r_2^2 - r_3^2 = 0$ .... (10)

(8) + (9) + (10) $\Rightarrow 2ax + 2by + 2cz + 2\left(a^2 + b^2 + c^2 - 2\left(r_1^2 + r_2^2 + r_3^2\right)\right) = 0$

$\Rightarrow 2ax + 2by + 2cz = 0$ (using (*))

$\Rightarrow ax + by + cz = 0$

$\therefore$ The radical centre of the four spheres lies on the plane $ax + by + cz = 0$.

6.8.9 Example: Show that the locus of a point from which equal tangents may be drawn to the three spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0 \cdots\cdots\cdots(1)$$

$$S_2 \equiv x^2 + y^2 + z^2 + 4x + 4z + 4 = 0 \cdots\cdots\cdots(2)$$

$$S_3 \equiv x^2 + y^2 + z^2 + x + 6y - 4z - 2 = 0 \cdots\cdots\cdots(3)$$

is the straight line $\frac{x}{2} = \frac{y-1}{5} = \frac{z}{3}$

solution: The required locus is the radical line of the given spheres.

The radical plane of (1) & (2) is $S_1 - S_2 = 0 \Rightarrow 2x - 2y + 2z + 2 = 0$

$\Rightarrow x - y + z + 1 = 0$

Theradical plane of (1) and (3) is $S_1 - S_3 = 0 \Rightarrow x - 4y + 6z + 4 = 0$

The equation of the radical line is

$$x - y - z + 1 = 0 = x - 4y + 6z + 4 \cdots\cdots\cdots\cdots\cdots\cdots(4)$$

Let ℓ, m, n be the D.C's of the line (4). Then the line (4) is $\perp^{lar}$ to both the normals of the planes in (4).

$\ell - m + n = 0$ and $\ell - 4m + 6n = 0$

$$\begin{array}{ccccccc} & l & & m & & n & \\ -1 & & 1 & & 1 & & -1 \\ -4 & & 6 & & 1 & & -4 \end{array}$$

Solving $\dfrac{\ell}{2} = \dfrac{m}{5} = \dfrac{n}{3}$

Putting $z = 0$ in (4) we get

$$x - y + 1 = 0$$

$$x - 4y + 4 = 0$$

By solving $x = 0,\ y = 1$

$\therefore (0,1,0)$ lies on the line (4).

The equation of the radical line (4) in symmetric form is

$$\frac{x-0}{2} = \frac{y-1}{5} = \frac{z-0}{3}$$

or $\dfrac{x}{2} = \dfrac{y-1}{5} = \dfrac{z}{3}$

Which is the required locus.

6.9 Answers to S.A.Q.:

6.9.1 Solution of 6.4.11:

Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \cdots\cdots\cdots(1)$

Let $P = (x_1,\ y_1, z_1)$, $Q = (x_2, y_2, z_2)$

Polar plane of P w.r.t. the sphere (1) is

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0 \cdots\cdots(2)$$

Since P, Q are conjugate Q lies on the plane (2) we have

$$x_1x_2 + y_1y_2 + z_1z_2 + u(x_1 + x_2) + v(y_1 + y_2) + w(z_1 + z_2) + d = 0 \cdots\cdots\cdots(3)$$

Sphere on PQ as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

or

$$x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z + x_1x_2 + y_1y_2 + z_1z_2 = 0 \cdots\cdots\cdots\cdots(4)$$

Spheres (1) & (3) cut orthogonall

$$\Leftrightarrow -2u\frac{(x_1+x_2)}{2}-2v\frac{(y_1+y_2)}{2}-2w\frac{(z_1+z_2)}{2}$$

$$=d+x_1x_2+y_1y_2+z_1z_2$$

$$\Leftrightarrow x_1x_2+y_1y_2+z_1z_2+u(x_1+x_2)+v(y_1+y_2)+w(z_1+z_2)+d=0$$

Which is true by (3)

Hence the result.

6.9.2 Solution of 6.4.12:

Let $P=(x_1, y_1, z_1)$, $Q=(x_2, y_2, z_2)$ be two points on the sphere $S_1=0$ such that PQ as diaimeter.

$\therefore$ Equation of the sphere with PQ as diameter is

$$S_1 \equiv (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$\Rightarrow S_1 \equiv x^2+y^2+z^2-(x_1+x_2)x-(y_1+y_2)y-(z_1+z_2)z+x_1x_2+y_1y_2+z_1z_2=0\cdots\cdots(1)$$

Let $S_2=x^2+y^2+z^2+2u^1x+2v^1y+2w^1z+d^1=0\cdots\cdots(2)$

Given that $S_1=0, S_2=0$ are orthogonal.

$$\therefore -2u^1\left(\frac{x_1+x_2}{2}\right)-2v^1\left(\frac{y_1+y_1}{2}\right)-2w^1\left(\frac{z_1+z_2}{2}\right)=d^1+x_1x_2+y_1y_2+z_1z_2$$

$$\Rightarrow x_1x_2+y_1y_2+z_1z_2+u^1(x_1+x_2)+v^1(y_1+y_2)+w^1(z_1+z_2)+d^1=0\cdots\cdots\cdots(3)$$

The polar of P w.r.t. the sphere $S_2=0$ is

$$xx_1+yy_1+zz_1+u^1(x+x_1)+v^1(y+y_1)+w^1(z+z_1)+d^1=0$$

This plane passes through Q

$$\therefore x_1x_2+y_1y_2+z_1z_2+u^1(x_1+x_2)+v^1(y_1+y_2)+w^1(z_1+z_2)+d^1=0$$

This is true by (3)

$\therefore$ The polar plane of P on $S_1=0$ w.r.t. the sphere $S_2=0$ pass through the other end Q of the diameter of $S=0$ through P.

6.10 Summary:

After going to this lesson the student is expected to have a clear idea about the notion of angle between two spheres, orthogonal spheres, Radical plane, Radical line, Radical centre, coaxial system of spheres and limiting points.

6.11 Technical Terms:

Orthogonal

Radical Plane and Line

Radical Centre

Coaxial System of Spheres

Limiting Points

6.12 Exercise:

1. Find the angle of intersection of the spheres

$$x^2+y^2+z^2-4=0$$

and $x^2+y^2+z^2-2x-2y=0$

2. Prove that the spheres

$$x^2+y^2+z^2+6y+2z+8=0$$

$x^2+y^2+z^2+6x+8y+4z+20=0$ are orthogonal

3. Find the equation to the sphere which cuts orthogonally each of the spheres.

$$x^2+y^2+z^2-14=0$$

$$x^2+y^2+z^2+4y-4=0$$

$$x^2+y^2+z^2+6z-9=0$$

4. Find the equation to the sphere which touches the plane

$$3x+2y-z+2=0$$

at the point $(1,-2,1)$ and cuts orthogonally the sphere

$$x^2+y^2+z^2-4x+6y+4=0$$

5. Find the equation to radical line of the three spheres

$$x^2+y^2+z^2+2x+2y+2z+2=0$$

$$x^2+y^2+z^2+4y=0$$

$$x^2+y^2+z^2+3x-2y+8z+6=0$$

6. Find the radical line of the spheres

$$x^2+y^2+z^2+2x+2y+2z+2=0$$

$$x^2+y^2+z^2+4x+4z+4=0$$

$$x^2+y^2+z^2+x+6y-4z-2=0$$

in symmetric form

7. If A and B are two fixed points and P moves so that $PA=nPB$. Show that the locus of P is a sphere. Show also that all such spheres, for different values of n have a common radical plane.

8. Show that the spheres which cut two given spheres along great circle, all pass through two fixed points

9. Show that the spheres

$$x^2+y^2+z^2+4x+8y+12z+22=0$$

$$x^2+y^2+z^2+3x+6y+9z+18=0$$

$x^2+y^2+z^2+5x+10y+15z+26=0$ are coaxial

10. Find the equation ofthe sphere through the point $(0,1,2)$ and belonging to the co axial system defined by

$$x^2+y^2+z^2+3x-3y+2z=0$$

$$x^2+y^2+z^2+2x-y-z+10=0$$

11. Find the limiting points of coaxial system of spheres determined by

(i) $x^2+y^2+z^2+3x-3y+6=0$

$x^2+y^2+z^2-6y-6z+6=0$

(ii) $x^2+y^2+z^2+3x-3y+6=0$

$x^2+y^2+z^2+4x-2y+2z+6=0$

(iii) $x^2+y^2+z^2-8x+2y-2z+32=0$

$x^2+y^2+z^2-7x+z+23=0$

12. Find the equation of the coaxial system of spheres whose limiting points are $(-1,2,1), (-2,1,-1)$ and also find the equation of the common radical plane.

13. $(-2,1,-1)$ is a limiting point of a coaxial systems for which $(x+y+2z=0)$ is the radical plane. Find the other limiting point.

14. Find the equations of the spheres of the coaxial system $\left(x^2+y^2+z^2-5\right)+\lambda(2x+y+3z-3)=0$ which touch the plane $3x+4y=15$.

15. Show that the radical planes of the spheres of a coaxial system and of a given sphere pass through a line.

6.13 Answers to Exercise:

1) $\frac{\pi}{4}$

3) $3\left(x^2+y^2+z^2\right)+39x+15y+5z+4z=0$

4) $x^2+y^2+z^2+7x+10y-5z+12=0$

5) $x-y+z+1=0=3x-6y+8z+2$

6) $\frac{x}{2}=\frac{y-1}{5}=\frac{3}{5}$

10) (i) $x^2+y^2+z^2+4x-5y-5z+10=0$

11) (i) $(-1,2,1), (-2,1,1)$

(ii) $(-1,2,1), (-1,1,-1)$

(iii) $(3,1,-2), (5,3,4)$

12) $x^2+y^2+z^2+2x-4y-2z+6+\lambda(x+y+2z)=0,\ x+y+2z=0$

13) $(-1,2,1)$

14) $5\left(x^2+y^2+z^2\right)-8x-4y-12z-13=0$

6.14 Model Examination Questions:

1. If r_1, r_2 are the radii of two orthogonal spheres then the radius of the circle of their intersection is $\frac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$.

2. The centre of a sphere $S=0$ which intersects two spheres $S^1=0, S^{11}=0$ orthogonally lies on the radical plans of the spheres $S^1=0, S^{11}=0$.

3. Find the limiting point of Coaxial system of spheres determined by

$$x^2+y^2+z^2+3x-3y+6=0$$

$$x^2+y^2+z^2-6y-6z+6=0$$

6.17 Model Practical Problem with Solution:

Problem: Find the limiting points of coaxial system of spheres determined by

$$x^2+y^2+z^2+3x-3y+6=0 \cdots\cdots\cdots\cdots\cdots\cdots(1)$$

$$x^2+y^2+z^2-6y-6z+6=0 \cdots\cdots\cdots\cdots\cdots\cdots(2)$$

Aim: To find the limiting points of coaxial system of spheres determined by given spheres (1) & (2).

Definitions and Results used:

(1) A system of spheres is said to be a coaxal system of spheres if any two spheres of the system have the same radical plane.

(2) The locus of points each of whose powers w.r.t. two non - concentric spheres are equal, is a plane called the radical plane of the two spheres.

(3) The point spheres in a coaxal system system of spheres are called limiting points of the coaxal system.

Solution: Radical plane of (1) and (2) is

$$3x+3y+6z=0 \Rightarrow x+y+2z=0 \cdots\cdots\cdots(3)$$

The equation of coaxal system of spheres is

$$x^2+y^2+z^2+3x-3y+6+\lambda(x+y+2z)=0, \lambda \text{ is a parameter}$$

$$\Rightarrow x^2+y^2+z^2+(3+\lambda)x+(\lambda-3)y+2\lambda z+6=0 \cdots\cdots\cdots\cdots(4)$$

center of $(4)=\left(-\left(\frac{3+\lambda}{2}\right),-\left(\frac{\lambda-3}{2}\right),-\lambda\right)$

Radius $=\sqrt{\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6}$

If (4) is a point sphere then radius is zero.

$$\therefore \left(\frac{3+\lambda}{2}\right)^2 + \left(\frac{\lambda-3}{2}\right)^2 + \lambda^2 - 6 = 0$$

$$\Rightarrow\ 9+\lambda^2+6\lambda+\lambda^2+9-6\lambda+4\lambda^2-24=0$$

$$\Rightarrow 6\lambda^2-6=0 \Rightarrow \lambda^2=1 \Rightarrow \lambda=\pm 1$$

Substitute the value of λ in the center of the sphere (4)

We get

$$\left(-\frac{3+1}{2}, -\frac{1-3}{2}, -1\right),\ \left(-\frac{3-1}{2},\ -\frac{-1-3}{2},\ -(-1)\right)$$

i.e., $(-2, 1, -1), (-1, 2, 1)$

$\therefore$ The required limiting points are

$(-2, 1, -1), (-1, 2, 1)$.

Lesson Writer

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Lesson - 7

CONE - I

7.1 Objective of the lesson:

In this lesson the student is introduced to various aspects of the cone such as equation of a cone, vertex, generator, quadratic cones, tangent lines and tangent planes, reciprocal cones.

7.2 Structure:

This lesson contains the following components:

7.3 Introduction

7.4 Definition and Examples

7.5 Some special types of cones

7.6 General Second Degree Equation

7.7 Tangent Lines and Tangent Planes

7.8 Reciprocal Cones

7.9 Summary

7.10 Technical Terms

7.11 Exercise

7.12 Answers to Exercise

7.13 Model Examination Questions

7.14 Model Practical Problem with Solution

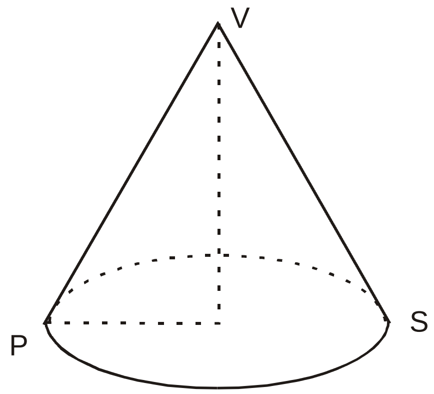
7.3 Introduction:

In this lesson we define a second degree cone and concentrate on derivation of the equations that represent certain types of cones including those passing through the origin and the general equation of the cone, cones with a guiding curve, and we also derive equations for the tangent lines and tangent planes of a cone. The reciprocal cone of a cone is defined and its equation in term of the coefficients of the equation of the cone is obtained. The properties of the right circular cone is also studied.

7.4 Definitions and Examples:

7.4.1 Definition: Let S be a set of points in the space. It there existes a point $V \in S$, such that $P \in S \Rightarrow \overline{VP} \subseteq S$. Then the surface S is called a cone, V is said to be the vertex of the cone, $\overline{VP}$ is called the generator of the cone.

7.4.2 Examples of a Cone:



(1) Any straight line is a cone with every point on it as a vertex.

(2) A pair of intersecting lines is a cone, with the point of intersection as the vertex.

(3) A plane is a cone with every point on it as a vertex.

(4) If π_1, π_2 are two intersecting planes, they together represent a cone with every point on their line of intersection as a vertex.

7.4.3 Def: A cone represented by a second degree polymonnial $f(x,y,z)=0$ is called a Quadric cone. Here f is of the form.

$$f(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2cone+2y+2wz+d$$

where at least one of the constants a,b,c,f,g,h is non zero.

All the cones having more than one vertex are called degeneratege cones.

7.4.4 Non Degenerate Cone: A cone which has only one vertex is called a non - degenerate cone.

In this lesson we deal with non - degenerate cones only.

7.5 Some Special Types of Cones:

7.5.1 Theorem: If $f : R^3 \to R$ is a homogeneous Polynomial of degree two, then the surface $S=\{(x,y,z)/f(x,y,z)=0\}$ is a cone with vertex at the origin. conversely, a cone with vertex at the origin, is represented by an equation $f(x,y,z)=0$, where f is a homogeneous polynomial of degree 2.

Proof: The if part is valid for any homogeneous polynomial of degree 2 let $f(x,y,z)$ be a homogeneous polynomial of degree 2 in x, y, z. Then for $\lambda \in R$

$$f(\lambda x, \lambda y, \lambda z) = \lambda^2 \cdot f(x,y,z) \cdots\cdots\cdots\cdots (1)$$

Suppose that $f(x,y,z) = 0$ represents a surface S.

Clearly origin $O \in S$.

If $P(x,y,z) \in S$ then $f(x,y,z) = 0 \cdots\cdots (2)$

The d.r's of the line OP are (x_1, y_1, z_1)

Equations to $\overline{OP}$ are $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1} \cdots\cdots\cdots (3)$

If Q is any point on the line OP, then it is of the form $(\lambda x_1, \lambda y_1, \lambda z_1)$

Now $f(\lambda x_1, \lambda y_1, \lambda z_1) = \lambda^2 \; f(x_1, y_1, z_1)$ by (1)

$= \lambda^2 \cdot 0$ by (2)

$= 0$

Hence every point Q on the line OP lies on S. Hence by definition of a cone the surface S represented by the equation $f(x,y,z) = 0$ is a cone with the vertex at '0'.

Converse: We prove the converse for a second degree polynomial. Let $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ represent a cone with vertex at 0. Let $P \in S$, where $P = (x,y,z)$. Then $S(x,y,z) = 0$.

Since $OP \subseteq S$, any Pt on $OP \in S$. i.e. $(\lambda x, \lambda y, \lambda z) \in S, \forall \lambda \in R$

If $\lambda = -1$ then we have $S(x_1, y_1, z_1) = S(-x_1, -y_1, -z_1)$

$$\Rightarrow S(x_1, y_1, z_1) - S(-x_1, -y_1, -z_1) = ux_1 + vy_1 + wz_1 = 0$$

The point P lies on the plane $ux + uy + wz = 0 \cdots\cdots\cdots\cdots (5)$

Also $d = 0$ $\qquad (\because O \in S) \cdots\cdots\cdots\cdots (6)$

We show that $u = v = w = 0$.

All the points on S satisfy the equation (5)

If $(u, v, w) \neq (0,0,0)$ then (5) represents a plane through 'O' so that $S \subseteq \pi$. But S is a non degenerate cone.

$\therefore\ S \not\subseteq \pi$. which is a contradiction.

Hence $u = v = w = 0$

Hence $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2fzx + 2hxy$, which is homogeneous polynomial of degree 2.

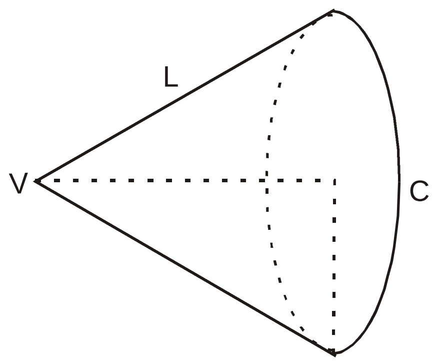
7.5.2 **Cor:** If $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \cdots\cdots (1)$ is a generator of a cone $f(x, y, z) = 0$ with vertex at the origin then $f(\ell, m, n) = 0$.

Proof: Any point on the line (1) is $(\lambda\ell, \lambda m, \lambda n)$

$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ is generator of $f(x, y, z) = 0 \Leftrightarrow (\lambda\ell, \lambda m, \lambda n)$ satisfies $f(x, y, z) = 0$

$$\Leftrightarrow f(\lambda\ell, \lambda m, \lambda n) = 0 \Leftrightarrow \lambda^2 f(\ell, m, n) = 0 \Leftrightarrow f(\ell, m, n) = 0$$

7.5.3 **Theorem:** Given a conic C in the plane $z = 0$ and a point V not in the plane of the conic, then there is a unique cone S with vertex at V such that a line L through V is a generator for 'S' iff L touches C.



Cone S

Proof: Let the equation to the curve C in the plane $z=0$. be $f(x,y)=0\cdots\cdots(1)$

Let V be the point (x_1, y_1, z_1) Any line L through V may be given by

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \qquad (\text{say})\cdots\cdots\cdots(2)$$

The set of common points of (2) and (3) is given by

$$f(\ell r+x_1, \quad mr+y_1)=0, \quad nr+z_1=0 \Rightarrow r=\frac{-z_1}{n} \text{ and}$$

$$f\left(x_1-\frac{\ell}{n}\cdot z_1,\ y_1-\frac{m}{n}\cdot z_1\right)=0\cdots\cdots\cdots\cdots(3)$$

$$(2)\Rightarrow \frac{\ell}{n}=\frac{x-x_1}{z-z_1}, \quad \frac{m}{n}=\frac{y-y_1}{z-z_1}\cdots\cdots(4)$$

Let S be the cone, with vetex at V. Let P (x,y,z) be any point $Q\in C$ and $V\neq P \Leftrightarrow V$ is a generator of S.

$\Leftrightarrow$ VP is a line which intersects XY plane and P lies on C.

$$\Leftrightarrow z\neq z_1 \text{ and } \left(x_1-z_1\cdot\frac{x-x_1}{z-z_1},\ y_1-z_1\cdot\frac{y-y_1}{z-z_1}\right)\in C$$

$$\Leftrightarrow z\neq z_1 \text{ and } f\left(x_1 - z_1\cdot\frac{x-x_1}{z-z_1}, \quad y_1-z_1\cdot\frac{y-y_1}{z-z_1}\right)=0$$

On expansion, we get a second degree polynormial in $(x-x_1, y-y_1, z-z_1)$ satisfying (x_1, y_1, z_1).

So it represents a cone with vertex at (x_1, y_1, z_1)

7.5.4 Guiding Curve (or) Base Curve: Let $V(x_1, y_1, z_1)$ be a point. Let C be a plane not containing V. If there is a cone with vertex at V, whose generators intersect the curve C, then C is called a guiding curve to the cone S.

7.5.5 Procedure for problems with vertex at 'O': Since the equation of a cone curve is second degree homogeneous in (x,y,z) homogenize the given cone as in the chapter "pair of lines" of Intermediate course.

7.5.6 Examples: Find the equation to the cone with the vertex at the origin and with the following curve as a base curve.

(1) $f(x,y) = ax^2 + 2hxy + by^2 + 2gn + 2fy + c = 0, \quad z = k$

(2) $x^2 + y^2 = 4, \quad z = 2$

(3) $ax^2 + by^2 + cz^2 = 1, \quad \ell x + my + nz = p$

(4) $x^2 + y^2 + z^2 + x - 2y + 3z = 4, \quad x^2 + y^2 + z^2 + 2x \cdot 3y + 4z = 5$

Solution:

(1) Since 'O' is the vertex the required cone eqation is obtained by homogenizing the base curve equations. (Common for all 4 examples.)

Given curve is $ax^2 + by^2 + 24xy + 2gx + 2fy + c = 0 \cdots\cdots (1)$

$$z = k \cdots\cdots (2)$$

$(2) \Rightarrow \frac{z}{k} = 1$ rewrite (1)

$$ax^2 + 2hxy + by^2 + (2gx + 2fy)\ (1)\ + c(1)^2\ = 0$$

Substitute $1 = \frac{z}{k}$ in above equation. The required cone equation is given by

$$ax^2 + 2bxy + by^2 + 2(gx + fy)\frac{z}{k} + x\frac{z^2}{k^2} = 0$$

i.e., $ak^2\lambda^2 + 2hk^2xy + bk^2y^2 + 2gkxz + kyz + cz^2 = 0$

or $\quad f\left(\frac{x}{z}k, \frac{y}{z}k\right) = 0$

Solution for (2): given curve is $x^2 + y^2 = 4 \cdots\cdots\cdots (1)$, $\quad z = 2 \cdots\cdots (2)$

$$(2) \Rightarrow\ z^2 = 4$$

Substisting in (1) the revised cone is $x^2 + y^2 = z^2$

Solution for (3): such that the given base curve is

$$ax^2 + by^2 + cz^2 = 1 \cdots\cdots\cdots (1)$$

$$\ell x + my + nz = p \cdots\cdots (2)$$

$$(2) \quad \Rightarrow \quad \frac{\ell x + my + nz}{p} = 1 \cdots\cdots\cdots\cdots (3)$$

Rewriting (1) $ax^2 + by^2 + cz^2 = 1^2$

Substituting (3) the revised cone is

$$ax^2 + by^2 + cz^2 = \frac{(\ell x + my + nz)^2}{P^2}$$

i.e. $P^2\left(ax^2 + by^2 + cz^2\right) = (\ell x + my + nz)^2$

Solutin for (4): Given surfaces are

$$x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0 \cdots\cdots (1)$$

$$x^2 + y^2 + z^2 + 2x - 3y + 4z - 5 = 0 \cdots\cdots (2)$$

Common plane of the two surfaces is obtained by

$$(1) - (2) = 0 \Rightarrow x - y + z = 1 \cdots\cdots\cdots (3)$$

Rewriting (1) $x^2 + y^2 + z^2 + (x - 2y + 3z)(1) = 4 \cdot (1)^2$

$\therefore$ The required cone curve is

$$x^2 + y^2 + z^2 + (x - 2y + 3z)(x - y + z) = 4(x - y + z)^2$$

7.5.7 Show that the general equation of the cone which passes through the coordinate axes is given by

$$fyz + gzx + hxy = 0, \quad \text{where } (f, g, h) \neq (0, 0, 0)$$

Solution: Since the generators intersect at the vertex and the axes intersect at 'O', origin is the vertex. Equation to the cone may be written as

$$ax^2 + by^2 + cz^2 + 2fyz + 2fzx + 2fxy = 0 \cdots\cdots\cdots (1)$$

X axis is a genertor $\Rightarrow (1,0,0)$ lies on (1) $\Rightarrow$ $a=0$

Similarly Y axis is a generator $\Rightarrow$ $b=0$ Z axis is a generator $\Rightarrow$ $c=0$

The required cone equation is given by $fyz + fzx + hxy = 0$

7.5.8 The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Prove that the equation to the cone generated by the lines joining the origin to the circle ABC is given by

$$\left(\frac{a}{b}+\frac{b}{a}\right)xy+\left(\frac{b}{c}+\frac{c}{b}\right)yz+\left(\frac{c}{a}+\frac{a}{c}\right)zx=0$$

Solution: Equation to the given plane in $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\cdots\cdots\cdots(1)$

It meets the X axis in $y=0,\ z=0 \Rightarrow \frac{x}{a}=1 \Rightarrow x=a$

$\therefore$ A is $(a, 0, 0)$

Similarly B is $(0,b,0)$, $C(0,0,c)$

The circle through A, B, C is P the plane of intersection of the sphere OABC and the plane of ABC. Equation of the sphere through the points O, A, B, C is given by

$$x^2+y^2+z^2-ax-by-cz=0\cdots\cdots\cdots\cdots\cdots(2)$$

The circle ABC is the section of the sphere (2) and the plane (1).

Making (2) homogeneous with the help of (1), the required equation of the cone whose vertex is at the origin and whose generators meet the circle ABC is given by

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

i.e., $yz\left(\frac{b}{c}+\frac{c}{a}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$

7.5.9 Find the equation of the cone with vetex at (x_1, y_1, z_1) $z_1 \neq 0$ and base curve in the xy plane given by $f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c=0,\quad z=0\cdots\cdots\cdots\cdots(1)$

Solution: As in the theorem 7.5.3 the cone equation in given by

$$(z-z_1)^2 \cdot f\left(\frac{x(z-z_1)-x_1(x-x_1)}{z-z_1}, \quad \frac{y(z-z_1)-y_1(y-y_1)}{z-z_1}\right)=0$$

i.e, $a(x_1z-z_1x)^2+2h(x_1z-z_1x)(y_1z-yz_1)+b(y_1z-yz_1)^2+2g(x_1z-xz_1)$

$$(z-z_1)+2f(y_1z-yz_1)(yy_1)+c(z-z_1)^2=0 \cdots\cdots\cdots\cdots (2)$$

Conversely if $P(x_0, y_0, z_0)$ is a point and $V(x_1, y_1, z_1)$ is the vertex then the line VP in given by

$\dfrac{x-x_1}{x_0-x_1}=\dfrac{y-y_1}{y_0-y_1}=\dfrac{z-z_1}{z_0-z_1}=r$. if this line meets the xy plane in $Q(x,y,0)$ then

$$x=x_1-\frac{z_1(x_0-x_1)}{z_0-z_1}, \; y=y_1-\frac{z_1(y_0-y_1)}{z_0-z_1}.$$

This point satisfies $ax^2+2hxy+by^2+2gx+2fy+c=0$

$\therefore$ The generator through VQ meeets the curve in P. Hence the cone with vertex at V and base curve C is given by (2).

7.5.10 Find the equation of the cone with vertex at (x_1, y_1, z_1) and the base curve satisfying.

$$f(x,y)=ax^2+2hxy+by^2+2gx+2fy+c=0, \; L \equiv px+qy+rz+s=0$$

where (x_1, y_1, z_1) does not lie in the plane L.

Solution: The given vertex is $V(x,y,z)$

The base curve is $ax^2+2hxy+by^2+2gx+2fy+c=0 \cdots\cdots\cdots (1)$

$$px+qy+rz+s=0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

Shift the origin to the point (x_1, y_1, z_1) we have

$$x=X+x_1, \; y=Y+y_1, \; z=Z+z$$

Now (1) vertex is the origin

(2) Base curve is $f(X+x_1, Y+y_1)$

$$= ax^2 + 2hxy + by^2 + 2x\left(ax_1 + by_1 + 2y\left(hx_1 + by_1 + f\right) + f\left(x_1y_1\right)\right) = 0$$

$$L \equiv px + qy + rz + L_{11}$$

Let $\Delta \equiv L - s = px + qy + rz + px_1 + qy_1 + rz_1$

The equation to the required cone in given by

$$L_{11}^2 : ax^2 + 2hxy + by^2 - 2L_{11} \cdot \Delta\left[\left(ax_1 + hy_1 + f\right)X + \left(hx_1 + hy_1 + f\right)Y\right] + f\left(x_1, y_1\right) \cdot \Delta^2 = 0$$

Again shift the origin to the original point. Equation to the cone is given by

$$L_{11}^2\left[a\left(x - x_1\right)^2 + b\left(y - y_1\right)^2 + c\left(z - z_1\right)^2\right]$$

$$-2L_{11} \cdot \Delta\left[\left(ax_1 + hy_1 + f\right)\left(x - x_1\right) + \left(hx_1 + by_1 + f\right)\left(y - y_1\right)\right] + f\left(x_1, y_1\right) \cdot \Delta^2 = 0,$$

Where $\Delta = px + qy + rz + px_1 + qy_1 + rz_1$

7.5.11 Find the equation of the cone with vertex at $(1, 2, 3)$, and whose guiding curve is represented by the equations, $x^2 + y^2 + z^2 = 4,\ x + y + z = 1$

Solution: The given vertex is $(1, 2, 3)$. Any line through it is

$$\frac{x-1}{\ell} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{x+y+z-6}{\ell+m+n} \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

The given curve is $x^2 + y^2 + z^2 = 4, \quad x + y + z = 1 \cdots\cdots\cdots\cdots (2)$

Substitue (2) in (1) we have

$$\frac{x-1}{\ell} = \frac{y-2}{m} = \frac{z-3}{n} = \frac{1-6}{\ell+m+n} = \frac{-5}{\ell+m+n}$$

$$\Rightarrow x - 1 = \frac{-5\ell}{\ell+m+n},\ y - 2 = \frac{-5m}{\ell+m+n},\ z - 3 = \frac{-5n}{\ell+m+n}$$

$$\Rightarrow x = \frac{m+n-4\ell}{\ell+m+n},\ y = \frac{2\ell+2n-3m}{\ell+m+n}, \quad z = \frac{3\ell+3m-2n}{\ell+m+n}$$

Substituting in $x^2 + y^2 + z^2 = 4$

$$\left(m+n-4\ell\right)^2+\left(2\ell+2n-3m\right)^2+\left(3\ell+3m-2n\right)^2=4\left(\ell+m+n\right)^2 \cdots\cdots\cdots\cdots\cdots(3)$$

The required cone curve is obtained by writting

$\ell=K(x-1),\ \ m=K(y-2),\ n=K(z-3)$ in (3)

$\therefore$ The required cone equation is

$$\left(y+z-4x-1\right)^2+\left(2x+2z-3y-2\right)^2+\left(3x+3y-2z-3\right)^2=4\left(x+y+z-6\right)^2$$

$$\Rightarrow\ 5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y+10z-26=0$$

7.5.12 Find the equation of the cone with vertex at $(2, \beta, \gamma)$ and base curve, $ax^2+by^2=1,\ \ z=0$.

Solution: Given vertex is (α, β, γ) Any line through (α, β, γ) is $\dfrac{x-\alpha}{\ell}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}\cdots\cdots\cdots(1)$

Any point on (1) is $(\ell r+\alpha,\ mr+\beta,\ nr+\gamma)$ The base curve is $ax^2+by^2=1,\ z=0\cdots\cdots\cdots(2)$

(1) will be a generator of the required curve if it cuts the curve (2)

i.e., $a(\ell r+\alpha)^2+b(mr+\beta)^2=1,$

$$nr+\gamma=0,\ \text{ i.e. } \gamma=\frac{-r}{n}$$

$$\Rightarrow\ a\left(\alpha-\frac{\ell}{n}\cdot\gamma\right)^2+b\left(\beta-\frac{m}{n}\gamma\right)^2=1\cdots\cdots\cdots\cdots(3)$$

$$(1)\Rightarrow\frac{\ell}{m}=\frac{x-\alpha}{z-\gamma},\qquad \frac{m}{n}=\frac{y-\beta}{z-\gamma}$$

Substituting in (3), the required cone equation is

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+b\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2=1$$

$$\text{i.e. } (\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2$$

7.6 Second Degree Equation x, y, z:

If a, b, c, f, g, h are not zero simultaneously, then the equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$, is called second degree general equation in x, y, z.

7.6.1 We use the following notation:

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d$$

$$E \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$U = ax + hy + gz + u$$

$$V = hx + by + fz + v$$

$$W = gx + fy + cz + w$$

$$D = ux + vy + wz + d$$

$$U_1 = ax_1 + hy_1 + gz_1 + u$$

$$V_1 = hx_1 + by_1 + fz_1 + v$$

$$W_1 = gx_1 + fy_1 + cz_1 + w$$

$$D_1 = ux_1 + vy_1 + cz_1 + d$$

$$S_1 = ux_1 + vy_1 + wz_1 + D$$

$$S_{11} = u_1x_1 + v_1y_1 + w_1z_1 + D_1$$

$$\Delta = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

7.6.2 We have the following consequences:

(1) $S_{12} = S_{21}$

(2) $S_{11} \equiv S(x_1, y_1, z_1)$

(3) $S = Ux + Vy + Wz + D$

7.6.3 Condition for second degree equation to represent a cone: The conditions for the equation $S(x,y,z)=0$ to represent a cone with vertex at (x_1, y_1, z_1) are $U_1=0, V_1=0, W_1=0$ and $\Delta=0$

Proof: Assume that $S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ represents a cone with vetex at (x_1, y_1, z_1). We shift the origin to the point P by translation of axes. the cone now has the vertex at the origin. The equation $S \equiv 0$ becomes

$$\begin{aligned} S \equiv\ & a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2+2f(y+y_1)(z+z_1)+2g(z+z_1)(x+x_1)+ \\ & 2h(x+x_1)(y+y_1)+2u(x+x_1)+2v(y+y_1)+2w(z+z_1)+d=0 \end{aligned}$$

i.e. $$\begin{aligned} S \equiv\ & ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2(ax_1+hy_1+gz_1+u)x+ \\ & 2(hx_1+by_1+fz_1+v)y+2(gx_1+fy_1+cz_1+w)z+ax_1^2+by_1^2+cz_1^2+ \\ & 2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d=0 \end{aligned}$$

i.e. $S \equiv E(x,y,z)+2U_1x+2V_1y+2W_1z+S_{11}=0$

$$\equiv E(x,y,z)+2U_1x+2V_1y+2w_1z+U_1x_1+V_1y_1+W_1z_1+D_1=0$$

This cone is having vertex at 0. It is homogeneous in x, y, z.

$$\left.\begin{array}{llll} \Rightarrow & \text{Coeff of } x=0 & \Leftrightarrow & U_1=0 \\ & \text{Coeff of } y=0 & \Leftrightarrow & V_1=0 \\ & \text{Coeff of } z=0 & \Leftrightarrow & W_1=0 \\ \text{and} & \text{Constant term } =0 & \Leftrightarrow & D_1=0 \end{array}\right\} \quad (A)$$

hence $\Delta=0$

7.6.4 Procedure to find the vertex of the cone $S \equiv 0$

Introduce the fourth variable 't' and write

$$F(x,y,z,t)=ax^2+by^2+cz^2+2fyz+2fzx+2hxy+2uxt+2vyt+2wtz+dt^2$$

Find $U=\dfrac{dF}{du},\ V=\dfrac{dF}{dv}, W=\dfrac{dF}{dw}, D=\dfrac{dF}{dt}$

Put $t=1$ and solve the above equation to get x,y,z. Substitute in the fourth equation. If it is satisfied then $S=0$ represents a cone with vetex at (x_1, y_1, z_1).

7.6.5 Find the vertex of the cone $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$

write $F(x,y,z,t) = 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0$

$$U_1 = \frac{\partial F}{\partial x} = 8x + 2y + 12t \qquad V_1 = \frac{\partial F}{\partial y} = 2x - 2y - 3z - 11t$$

$$W_1 = \frac{\partial F}{\partial z} = 4z - 3y + 6t \qquad D_1 = \frac{\partial F}{\partial t} = 12x - 11y + 6z + 8t$$

Put $t = 1$ and solve $U_1 = 0,\ V_1 = 0,\ W_1 = 0,\ D_1 = 0$

$$4x + y + 6 = 0 \cdots\cdots\cdots (1)$$

$$2x - 2y - 3z - 11 = 0 \cdots\cdots\cdots (2)$$

$$3y - 4z - 6 = 0 \cdots\cdots\cdots (3)$$

$$12x - 11y + 6z + 8 = 0 \cdots\cdots\cdots (4)$$

solving (1), (2) and (3)

$$(2) \times 2 - (1) \Rightarrow 5y + 6z + 28 = 0 \cdots\cdots\cdots (5)$$

$$(3) \times 3 + (5) \times 2 \Rightarrow 19y + 38 = 0 \Rightarrow y = -2$$

$$(3) \Rightarrow -6 - 4z - 6 = 0 \Rightarrow 4z = -12 \Rightarrow z = -3$$

$$(1) \Rightarrow 4x - 2 + 6 = 0 \Rightarrow 4x = -4 \Rightarrow x = -1$$

Substituting the values $x = -1,\ y = -2,\ z = -3$ in (4)

$$12(-1) - 11(-2) + 6(-3) + 8 = -30 + 30 = 0$$

$\therefore$ The given equation represents a cone with vertex at $(-1, -2, -3)$.

7.6.6 Find the vertex of the cone

$$2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$$

Solution: Write $F(x,y,z,t) = 2y^2 - 8yz - 4zx - 8xy + 6xt - 4yt - 2zt + 5t^2$

Consider $U = \frac{dF}{dx} \equiv -4z - 8y + 6t \quad = 0$ $\qquad V = \frac{dF}{dy} \equiv 4y - 8z - 8x - 4t = 0$

$W = \frac{df}{dz} \equiv -8y - 4x - 2t = 0$ $\qquad D = \frac{dF}{dt} \equiv 6x - 4y - 2z + 10t = 0$

When $t = 1$ these equations become

$4y + 2z - 3 = 0 \cdots\cdots\cdots (1)$

$2x - y + 2z + 1 = 0 \cdots\cdots\cdots\cdots (2)$

$2x + 4y + 1 = 0 \cdots\cdots\cdots\cdots (3)$

$3x - 2y - z + 5 = 0 \cdots\cdots\cdots\cdots\cdots (4)$

Solving for (x, y, z)

$(2) - (1) \Rightarrow \; 2x - 5y + 4 = 0 \cdots\cdots\cdots (5)$

$(3) \Rightarrow 2x + 4y + 1 = 0$

$(-) \qquad \Rightarrow 9y - 3 = 0 \Rightarrow y = \frac{1}{3}$

$(3) \Rightarrow 2x + \frac{4}{3} + 1 = 0 \Rightarrow 2x = \frac{-7}{3} \Rightarrow x = \frac{-7}{6}$

$(6) \Rightarrow \frac{4}{3} + 2z - 3 = 0 \; \Rightarrow 2x = \frac{5}{3} \; \Rightarrow z = \frac{5}{6}$

Substituting the values of (x, y, z) in (4) $3\left(\frac{-7}{6}\right) - 2\left(\frac{1}{3}\right) - \frac{5}{6} + 5$

$$= \frac{-21 - 4 - 5 + 30}{6} = 0$$

The required vertex is $\left(\frac{-7}{6}, \frac{1}{3}, \frac{5}{6}\right)$

7.7 Tangent Lines and Tangent Planes:

7.7.1 Theorem: A line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\cdots\cdots\cdots\cdots\cdots(1)$

meets the cone $S(x,y,z)=0$ at $(x_1+\ell t,\ y_1+mt,\ z_1+nt)$ iff $At^2+Bt+C=0\cdots\cdots(2)$

where $A=E(\ell,m,n),\ B=\ell U_1+mV_1+nW_1,\ C=S_{11}$

Proof: For any $t\in R$ $S(x_1+\ell t,\ y_1+mt,\ z_1+nt)$

$$=a(x_1+\ell t)^2+b(y_1+mt)^2+c(z_1+nt)^2+2f(y_1+mt)(z_1+nt)$$

$$+2g(z_1+nt)(x_1+\ell t)+2h(x_1+\ell t)(y_1+mt)+2u(x_1+\ell t)$$

$$+2v(y_1+mt)+2w(z_1+nt)+d$$

$$=\left(a\ell^2+bm^2+cn^2+2fmn+2g\ell n+2h\ell m\right)t^2+$$

$$2\left(a\ell x_1+bmy_1+cnz_1+fmz_1+fny_1+gnx_1+g\ell z_1+h\ell y_1+hmx_1+u\ell m+vmn+w\ell n\right)t$$

$$+ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d$$

$=At^2+2Bt+C$ where $A=E(\ell,m,n),\ B=\ell U_1+mV_1+nW_1,\quad C=S_{11}$

Since the general point on the line (1) is of the form $(x_1+\ell t,\ y_1+mt,\ z_1+nt)$, It follows that points of intersection of (1) and the cone S. Correspond precisely to t satisfying $At^2+Bt+C=0\cdot$ Thus the line (1) meets the curve at $(x_1+\ell t,\ y_1+mt,\ z_1+nt)$ iff $At^2+Bt+C=0\cdot$

7.7.2 Consequences:

(1) If $E(\ell,m,n)\neq 0,\ (\ell u_1+mv_1+nw_1)^2>S_{11}\cdot E(\ell,m,n)$ then the equation (2) has different roots say $t_1,\ t_2$ we get two points $P_1,\ P_2$. The line $P_1\ P_2$ is called a chord to the cone.

(2) If $E(\ell,m,n)\neq 0,\ (\ell u_1+mv_1+nw_1)^2<S_{11}\cdot E(\ell,m,n)$ the equation (2) has no real roots. the line (1) does not intersect the cone.

(3) If $E(\ell,m,n)\neq 0$ and $(LU_1+mV_1+nW_1)^2=S_{11}\cdot E(\ell,m,n)$ then cone and line intersect at only one point since (1) has double root. We say that the line is a tangent line at that point, which is called the point of contact.

(4) If $A=B=C=0$ then the line becomes a generator to the cone.

7.7.3 Example: Find whether the line $\frac{x-1}{1}=\frac{y}{0}=\frac{z}{1}$ is a tangent to the cone $x^2+y^2=z^2$.

Solution: The given cone is ..... $x^2+y^2=z^2 \cdots\cdots(1)$

The given line is $\frac{x-1}{1}=\frac{y}{0}=\frac{z}{1}=(r) \quad (\text{say})\cdots\cdots\cdots(2)$

Any point on (2) is $(r+1,0,r)$. If (1) and (2) intersect then

$$(r+1)^2+0^2=r^2 \quad\Rightarrow\quad 2r+1=0 \;\Rightarrow\; r=-\frac{1}{2}$$

The only common point is $\left(\frac{1}{2},0,\frac{-1}{2}\right)$. But the line is not a tangent line.

Since $A=E(\ell,m,n)=0$, $S_{11}\neq 0$ in this case line is a parellel line to a generator.

7.7.4 Tangent Plane: Definition: If $P(x_1,y_1,z_1)$ is a point on the cone $S=0$ and atleast one of U_1,V_1,W_1 is non zero, then the locus of all tangent lines at P is called the tangent plane at P for the cone $S=0$, If $U_1=V_1=W_1=0$ then P is called a singular point of the cone.

7.7.5 Theorem: The tangent plane of the cone $E(x,y,z)\equiv 0$ at $P(x_1,y_1,z_1)$ on it which is not a singular point is the plane represented by $U_1x+V_1y+W_1z=0$.

Proof: Let $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\cdots\cdots\cdots\cdots(1)$ be a line through a point (x_1,y_1,z_1) on the cone $E(x,y,z)=0\cdots\cdots\cdots(2)$

$\therefore\ E(x_1,y_1,z_1)=0\cdots\cdots\cdots(3)$

$\therefore\ At^2+Bt=0$ (7.6.2) and (7.7.1) ....(4)

(1) Will be tangent line at P it the roots of (4) are equal for which the condition is $\ell U_1 + mV_1 + nW_1 = 0 \cdots\cdots (5)$.

The locus of the tangent lines at $P(x_1, y_1, z_1)$ is the tangent plane, obtained from (1) and (5)

$$(1) \Rightarrow \ell = (x - x_1)K, \; m = (y - y_1)K, \quad n = (z - z_1)K$$

Substiing in (5) we have

$$(x - x_1)U_1 + (y - y_1)V_1 + (z - z_1)W_1 = 0$$

$$\Rightarrow U_1x + V_1y + W_1z = U_1x_1 + V_1y_1 + W_1z_1 = 0 \qquad \text{by (3)}$$

$\therefore$ The tangent plane at $P(x_1, y_1, z_1)$ on the cone $E(x, y, z)$ may be represented by $U_1x + V_1y + W_1z = 0$.

i.e., $(ax_1 + hy_1 + gz_1)x + (hx_1 + by_1 + fz_1)y + (gx_1 + fy_1 + cz_1)z = 0$

7.7.6 Theorem: (Conditions for tangency of a plane.) A necessary and sufficient condition for a plane $(\ell x + my + nz) = 0 \cdots\cdots\cdots\cdots\cdots (1)$ to be a tangent plane to the cone $E(x, y, z) = 0 \cdots\cdots\cdots\cdots (2)$ is

$$K = \begin{vmatrix} a & h & g & \ell \\ h & b & f & m \\ g & f & c & n \\ \ell & m & n & o \end{vmatrix} = 0$$

Proof:

Necessity: Equation to the tangent plane at a point (x_1, y_1, z_1) on the cone $E(x, y, z) = 0$ $U_1x + V_1y + W_1z = 0 \cdots\cdots\cdots\cdots\cdots (3)$. since (1) is also a tangent is given by plane at (x_1, y_1, z_1). (1) and (3) represents the same plane by comparing coefficients of x, y, z in (1) and (3).

$$\frac{U_1}{\ell} = \frac{V_1}{m} = \frac{W_1}{n} = -\lambda \qquad (\text{say})$$

$$\Rightarrow U_1 = -\lambda\ell \quad , \; V_1 = -\lambda m, \quad w_1 = -\lambda n$$

$$\left.\begin{aligned} \Rightarrow \quad & ax_1 + hy_1 + gz_1 + \lambda\ell = 0 \\ & hx_1 + by_1 + fz_1 + \lambda m = 0 \\ & gx_1 + fy_1 + cz_1 + \lambda n = 0 \\ \text{Also} \quad & \ell x_1 + my_1 + nz_1 + \lambda \cdot 0 = 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (A)$$

Since the system (A) possesses a non zero solution (x, y, z, t)

$$K = \begin{vmatrix} a & h & g & \ell \\ h & b & t & m \\ g & t & c & n \\ \ell & m & n & 0 \end{vmatrix} = 0$$

Sufficiency: If K = 0 then the system (A) possess non - zero solution. If $\lambda = 0$, then $(x_1, y_1, z_1) \neq (0,0,0)$ and $U_1 = 0, V_1 = 0, W_1 = 0$ which is a contradiction to "$(0,0,0)$ is the only vertex". Set (x, y, z, λ) be a non zero solution for the system (A) We have $U_1 = -\lambda\ell, \quad V_1 = -\lambda m, \quad W_1 = -\lambda n$ and $\ell x_1 + my_1 + nz_1 = 0$.

Also $E(x_1, y_1, z_1) = U_1x_1 + V_1y_1 + W_1z_1 = -\lambda(\ell x_1 + my_1 + nz_1) = 0$

$\Rightarrow (x_1, y_1, z_1)$ lies on the cone $E(x, y, z)$

and $U_1x + V_1y + W_1z = \ell x + my + nz = 0$

$\therefore \ell x + my + nz = 0$ is a tangent plane to $E(x, y, z)$

Note: If $A, , B, C, F, G, H$ are the cofactors of a, b, c, f, g, h in

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

then the required condition becomes

$$A\ell^2 + Bm^2 + Cn^2 + 2Fmn + 2Gn\ell + 2H\ell m = 0$$

7.7.7 Example: Are the lines $x = 0, y = 1$ and $x = 1, y = 0$ tangents to the cone $x^2 + y^2 = z^2$?

Solution: The given cone is $x^2 + y^2 = z^2 \cdots\cdots\cdots\cdots (1)$

The given line is $\frac{x-1}{0}=\frac{y}{0}=\frac{z}{1}\cdots\cdots\cdots\cdots(2)$ $\quad (x=1,\ y=0)$

Any point on (2) is $(1,0,r)$ Replace by : This lies on (1) if

$1+0=r^2 \Rightarrow r=\pm 1$ So the points of intersection are $(1,0,\pm 1)$

$\therefore$ The given line is not a tangent line. Similarly the second line is not a tangent line to the given cone.

7.7.8 Example: Show that the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ is the line of intersection of the tangent planes to the cone $E(x,y,z)=0$ along the line which is cut by the plane

$$(a\ell+hm+gn)x+(h\ell+bm+fn)y+(g\ell+fm+cn)z=0$$

Solution: The given cone is $E(x,y,z)=0\cdots\cdots\cdots(1)$

given line is $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}\cdots\cdots\cdots(2)$

any point on (2) is $(\ell r,\ mr,\ nr)$.

The tangent plane at the point $(x_1,\ y_1,\ z_1)$ on (1) is given by $U_1x+V_1y+W_1z=0$ ....(3)

The line (2) lies on (3) iff $U_1\ell r+V_1mr+W_1nr=0$

$$\Leftrightarrow\ \ell(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0$$

$$\Leftrightarrow (a\ell+hm+gn)x_1+(h\ell+bm+fn)y_1+(g\ell+fm+cn)z_1=0$$

Thus the point $(x_1,\ y_1,z_1)$ lies on the plane

$$(a\ell+hm+gn)x+(h\ell+bm+fn)y+(g\ell+fm+cn)z=0$$

7.7.9 Example: Show that the locus of intersection of the tangent planes to the cone $ax^2+by^2+cz^2=0$ which touch along the perpendicular generators is the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0$$

Solution: The given cone is $ax^2 + by^2 + cz^2 = 0 \cdots\cdots\cdots\cdots (1)$

Let the tangent plane along two perpendicular generaters of the cone meet in the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \cdots\cdots\cdots\cdots (2)$

Then the equation to the plane containing two perpendicular generators is given by $a\ell x + bmy + cnz = 0 \cdots\cdots\cdots (3)$.

The condition for a plane $\ell x + my + nz = 0$ to intersect the cone $E(x, y, z) = 0$ in two perpendicular generators is

$$(a+b+c)\left(\ell^2 + m^2 + n^2\right) = E(u, v, w)$$

Here (1) and (3) intersect in two perpendicular generators

$$\therefore (a+b+c)\left(a^2\ell^2 + b^2m^2 + c^2n^2\right) = a(a\ell)^2 + b(bm)^2 + c(cn)^2$$

$$\Rightarrow a \cdot a^2\ell^2 + (b+c)a^2\ell^2 + b \cdot b^2m^2 + (c+a)b^2m^2 + c \cdot c^2n^2 + (a+b)c^2n^2$$

$$= a \cdot a^2\ell^2 + b \cdot b^2m^2 + c \cdot c^2n^2$$

$$\Rightarrow a^2(b+c)\ell^2 + b^2(c+a)m^2 + c^2(a+b)n^2 = 0$$

locus of the tangent line (2) is

$$a^2(b+c)x^2 + b^2(c+a)y^2 + c^2(a+b)z^2 = 0$$

Note: Equation to the pair of planes normal to the generator of the cone $E(x, y, z) = 0$, which are intersected by the plane $\ell x + my + nz = 0$ is given by

$$\begin{vmatrix} a & h & g & \ell & x \\ h & b & f & m & y \\ g & f & c & n & z \\ \ell & m & n & o & o \\ x & y & z & o & o \end{vmatrix} = 0$$

i.e. $E(yn - zm,\ z\ell - xn,\ mx - y\ell) = 0$

If θ is the angle between the normals of the planes then we have

$$\frac{\cos\theta}{(a+b+c)\left(\ell^2+m^2+n^2\right)-E(\ell,m,n)}=\frac{\sin\theta}{2\sqrt{P\left(\ell^2+m^2+n^2\right)}}$$

where $P=\begin{vmatrix} a & h & g & \ell \\ h & b & f & m \\ g & f & c & n \\ \ell & m & n & o \end{vmatrix}$

7.7.10 Example: If the lines of intersection of $ax+by+cz=0$ and the cone $xy+yz+zx=0$ are at right angles show that $ab+bc+ca=0$ or $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

The condition for the plane $\ell x+my+nz=0$ to cut the cone $E(x,y,z)=0$ in two perpendicular generators is $(a+b+c)\left(\ell^2+m^2+n^2\right)=E(\ell,m,n)$

Here $a=b=c=0,\quad f=g=h=\frac{1}{2}\ \ \ell=a,\ m=b,\ n=c$

$\therefore$ The required condition is $o=E(\ell,m,n)=ab+bc+ca$

i.e., $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

7.8 Reciprocal Cones:

7.8.1 Theorem: If $E(x,y,z)=0$ is a cone then the locus of the normals to the tangent planes at the vertex is the cone given by

$$E^*(x,y,z)\equiv\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & o \end{vmatrix}=0$$

Also: (i) The locus of the normals to the tangent planes of

$E^*(x,y,z)=0$ is the cone $E(x,y,z)=0$

(ii) If $E(x,y,z)=0$ is the point $(0,0,0)$ then $E^*(x,y,z)=0$ is also the point $(0,0,0)$.

Proof: Let $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}\cdots\cdots\cdots(1)$ be a normal to the cone $E(x,y,z)=0$. Then we have the condition

$A\ell^2+Bm^2+Cn^2+2fmn+2g\ell n+2H\ell m=0$ .... (7.7.6)

The locus of the line (1) is given by $Ax^2+By^2+Cz^2+2H\ell y+2Fyz+2Gzx=0$

i.e., $E^*(x,y,z)=\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & o \end{vmatrix}=0$

7.8.2 Reciprocal Cones:

Definition: If two cones are such that the locus of the normals drawn through vertex to the tangent plane of one is the other then they are called reciprocal cones.

7.8.3 Example: Show that the equation of the cone which touches the co-ordinate planes is given by $\sqrt{fx}\pm\sqrt{gy}\pm\sqrt{hz}=0,\ f,g,h$ being parameters

Solution: If a cone has three coordinate axes as genarators then its reciprocal cone touches the three coordinate planes.

We know that the cone which is having the axes as genarators is given by $2hxy+2fyz+2gzx=0\cdots\cdots(1)$ we determine its reciprocal cone, which is given by $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0\cdots\cdots\cdots(2)$.

Here $\Delta=\begin{vmatrix} o & h & g \\ h & o & f \\ g & f & o \end{vmatrix}$

$A=-f^2\quad F=hg\quad B=-g^2\quad G=fh\quad C=-h^2\quad H=fg$

The required cone is given by

$$f^2x^2+g^2y^2+h^2z^2-2ghyz-2fhzx-2fgxy=0$$

i.e. $f^2x^2+g^2y^2+h^2z^2-2ghyz-2fhzx+2fgxy=4fgxy$

i.e. $(fx+gy-hz)^2=\left(2\sqrt{fgxy}\right)^2$

i.e. $fx + gy - hz = \pm 2\sqrt{fgxy}$

i.e. $fx + gy \pm 2\sqrt{fgxy} = hz$

i.e. $\left(\sqrt{fx} \pm \sqrt{gy}\right)^2 = \left(\sqrt{hz}\right)^2$

i.e. $\left(\sqrt{fx} \pm \sqrt{gy}\right) = \pm \sqrt{hz}$

i.e. $\sqrt{fx} \pm \sqrt{gh} \pm \sqrt{hz} = 0$

7.8.4 Example: Show that the cones $fyz + fzx + hxy = 0$ and $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ are reciprocal.

solution: If a cone has three coordinate axes of the genarators then its reciprocal cone touches the three coordinate planes.

We know that the cone which is having the axes any genarators is given by $2hxy + 2fyz + 2gzx = 0 \cdots\cdots(1)$ we determine its reciprocal cone, which is given by $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \cdots\cdots\cdots(2)$.

Here $\Delta = \begin{vmatrix} o & h & g \\ h & o & f \\ g & f & o \end{vmatrix}$

$A = -f^2 \qquad F = hg$

$B = -g^2 \qquad G = fh$

$C = -h^2 \qquad H = fg$

Substistuting in (2), required cone is given by

$$f^2x^2 + f^2y^2 + h^2z^2 - 2ghyz - 2fhzx - 2fgxy = 0$$

$$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2fhzx + 2fgxy = 4fgxy$$

$$\Rightarrow (fx + gy - hz)^2 = \left(2\sqrt{fgxy}\right)^2$$

$$\Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$$

$$\Rightarrow fx + gy \pm 2\sqrt{fgxy} = hz$$

$$\Rightarrow \left(\sqrt{fx} \pm \sqrt{gy}\right)^2 = \left(\sqrt{hz}\right)^2$$

$$\Rightarrow \left(\sqrt{fx} \pm \sqrt{gy}\right) = \pm\sqrt{hz}$$

$$\Rightarrow \sqrt{fx} \pm \sqrt{gh} \pm \sqrt{hz} = 0$$

7.8.5 Example: Show that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal.

solution: The given cones are represented by

$$ax^2 + by^2 + cz^2 = 0 \cdots\cdots\cdots(1)$$

$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0 \cdots\cdots\cdots(2)$, we show that reciprocal cone of (1) is (2)

Reciprocal cone of (1) is given by

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

where $\Delta = \begin{vmatrix} a & o & o \\ o & b & o \\ o & o & c \end{vmatrix}$

$$A = bc \quad F = 0 \quad B = ca \quad G = 0 \quad C = ab \quad H = 0$$

$\therefore$ reciprocal cone is $bcx^2 + cay^2 + abz^2 = 0$

i.e. $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

7.8.6 Example: Show that the locus of the line of intersection of the perpendicular tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ is the cone given by

$$a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = 0$$

Solution: If two tangents planes of a cone are perpendiculars then their normals are perpendicular generators to its reciprocal cone. We find the locus of the normals through the origin, to the planes which cut the reciprocal cone along perpendicular genatators.

The given cone is $ax^2+by^2+cz^2=0 \cdots\cdots\cdots\cdots\cdots (1)$.

Its reciprocal cone is $bcx^2+cay^2+abz2=0 \cdots\cdots\cdots\cdots (2)$

Let the plane $\ell x+my+nz=0 \cdots\cdots\cdots\cdots\cdots (3)$ cut the cone (2) in two perpendicular generators. For that the condition is $a(b+c)\ell^2+b(c+a)m^2+c(a+b)n^2=0 \cdots\cdots\cdots (4)$

Equations to the normal to the plane (3) through the origin are $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n} \cdots\cdots\cdots (5)$

To find the locus of the normal (5), we eliminate ℓ, m, n from (4) and (5) and get

$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0$ which is the required locus.

7.9 Summary:

After reading this lesson the student should be able to gain working knowledge on various types of equations of cone, equations of generators, whether the second degree equation in x, y, z represents a cone and the equation of a cone with a given guiding curve. The student should also have thorough understanding of tangent lines and planes, condition for tangency.

7.10 Technical Terms:

Cone

Vertex

Generator

Guiding curve

Receprocal cone.

7.11 Exercise:

1. Find the equation to the cone whose generators passes through the point (α, β, γ) and have their directional cosines satisfying $a\ell^2+bm^2+cn^2=0$.

2. Find the equation of the cone whose vertex is $(1,1,0)$ and whose guiding curve is $y=0,\ x^2+z^2=4$.

3. Find the equation to the cone with vertex at 'O' and directional cosines of the generators satisfying $3\ell^2 - 4m^2 + 5n^2 = 0$.

4. Find the equation to the cone with vertex at 'O' and passing through the curves given by $ax^2 + by^2 = 2z, \ \ell x + my + nz = p$

5. Find the equation to the lines of intersection of the planes $3x + 4y + z = 0$ and the cone $15x^2 - 32y^2 - 7z^2 = 0$.

6. Find the angle of intersection of the cone $20x^2 + 7y^2 - 108z^2 = 0$ and the plane $10x + 7y - 6z = 0$.

7. Find the condition that the plane $\ell x + my + nz = 0$ may touch the cone $4x^2 - y^2 + 3z^2 = 0$.

8. Show that the equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represents a cone touching the coordinate planes. Find the equation to its reciprocal cone.

7.11.1 Answers to Exercise:

1. $a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$
2. $x^2 - 3y^2 - 2xy + 8y - 4 = 0$
3. $3x^2 - 4y^2 + 5z^2 = 0$
4. $P\left(ax^2 + by^2\right) = 2z(\ell x + my + nz)$
5. $\frac{x}{-3} = \frac{y}{2} = \frac{z}{1}$; $\frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$
6. $\cos^{-1}\left(\frac{16}{21}\right)$
7. $3\ell^2 - 12m^2 + 4n^2 = 0$
8. $fyz + gzx + hxy = 0$

7.12 Model Examination Questions:

1. a) Write the definition of a cone and give an example.

   b) The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meet the axes in A, B, C. Prove that the equation to the cone generated by the line joining the origin to the circle ABC is given by

$$\left(\frac{a}{b} + \frac{b}{a}\right)xy + \left(\frac{b}{c} + \frac{c}{a}\right)yz + \left(\frac{c}{a} + \frac{a}{c}\right)zx = 0.$$

c) Find the equation of the cone with vertex at $(1,2,3)$ and whose guiding curve is represented by the equations

$$x^2+y^2+z^2=4,\ x+y+z=1.$$

2. a) Define tangent line and tangent plaens of a cone.

b) The tangent plane of the cone $E(x,y,z)=0$ at $P(x_1, y_1, z_1)$ on it which is not a singular point in the plane represented by $U_1x+V_1y+W_1z=0$.

c) Are the line $x=0,\ y=1$ and $x=1,\ y=0$ are tangents to the cone $x^2+y^2=z^2$?.

7.14 Model Practical Problem With Solution:

Problem: Show that the equation $F(x,y,z)\equiv x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z+17=0$ represents a cone with vertex $(1,-2,2)$.

Definitions: Let S be a set of points in the space. If there exists a point $V\in S$, such that $P\in S\Rightarrow \overline{VP}\subseteq S$. Then the surface S is called a cone. V is said to be the vertex of the cone, $\overline{VP}$ is called a generator of the cone.

Result: $F(x,y,z)\equiv ax^2+by^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ represents a cone with vertex (x_1, y_1, z_1) if

$$\begin{vmatrix} a & h & b \\ h & b & f \\ g & f & c \end{vmatrix}=0 \qquad \text{and} \qquad \begin{vmatrix} a & h & b & u \\ h & b & f & v \\ g & f & c & w \\ n & b & e & 1 \end{vmatrix}=0$$

$$\frac{\partial F}{\partial x_1}=\frac{\partial F}{\partial y_1}=\frac{\partial F}{\partial z_1}=0 \text{ and } ux_1+vy_1+wz_1+d=0$$

Stepwise division of the solution

Step 1: Introduce a new variable t to make F homogeneous:

i.e. consider

$$F(x,\ y,\ z,\ t)\equiv ax^2+by^2+cz^2+2gzx+2fyz+hxy+zuxt+2vyt+2wzt+dt^2=0$$

Step 2: Compute and simplify equations $\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0,\ \frac{\partial F}{\partial t}=0$.

Step 3: Put $t=1$ in the above equations and first solve for $x=x_1, y=y_1, z=z_1$ from the first three equations.

Step 4: Verify (x_1, y_1, z_1) satisfies $\frac{\partial F}{\partial t}=0$

Conclusion: The given equation represents a cone with vertex (x_1, y_1, z_1).

Solution of the given problem:

Introduce new variable t and write

Step 1: $F(x, y, z, t)=7x^2+2y^2+2z^2-10zx+10xy-26xt-2yt+2zt-17t^2=0$

Step 2: $\frac{\partial F}{\partial x}=14x-10z+10y-26t \qquad \frac{\partial F}{\partial y}=10x+4y-2t$

$\frac{\partial F}{\partial z}=4z+10x+2t \qquad \frac{\partial F}{\partial t}=26x-2y+2z-34t$

Step 3: put $t=1$ and solve for x, y, z from the first three equations: namely

$$7x+5y-5z=13 \rightarrow (1)$$

$$5x+2y=1 \rightarrow (2)$$

$$5x-2z=1 \rightarrow (3)$$

$(3)\times 5-(1)\times 3$ gives $11x-10y-31=0$

$(2)\times 1$ gives $5x+2y=1$

$$\begin{array}{cccc} & x & y & 1 \\ -10 & -31 & 11 & -10 \\ 2 & -1 & 5 & 2 \end{array}$$

From these we solve for x & y in the routine way

$x_1=1,\ y_1=2$ substituting $x_1=1$ in (3) we get $z_1=2$

So the solution of (1), (2) and 3 is $(1,\ -2,\ 2)$

Substituting these values for x, y, z respectively and putting $t=1$ in

$\frac{\partial\ F}{\partial\ t}=0$ we get $26\times(1)-2\times(-2)+2(2)-34(1)=0$

Conclusion: Hence the given equation represents a cone with vertex $(1,-2,\ 2)$

Lesson Writer

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Lesson - 8

CONE - II

8.1 Objective of the lesson:

In this lesson the student is introduced to various types of cones such as envoloping cone and right circular cone and some of their properties.

8.2 Structure:

This lesson contains the following components:

8.3 Introduction

8.4 Enveloping cone

8.5 Three Mutually Perpendicular Generators

8.6 Right Circular Cone

8.7 Summary

8.8 Technical Terms

8.9 Exercise

8.10 Answers to Exercise

8.11 Model Examination Questions

8.12 Model Practical Problem With Solution

8.3 Introduction:

In this lesson we continue the study of the cone. The tangents from an external point of a sphere form a cone called envoloping cone of the sphere. We study some interesting properties of the envoloping cone of a sphere and extend these ideas for a general surface. It is not necessary that the generators of a cone must have perpendiculars among themselves. It is interesting to note that in certain types of cones every generator has a perpendicular generator. We prove in this lesson that the necessary and sufficient condition for a cone $f(x,y,z)=0$ to have three mutually perpendicular generators is that the sum of the coefficients of x^2, y^2 and z^2 is zero. We then concentrate on cones whose guiding curve is a circle and the vertex is on the normal at the centre to the circle, such cones are called right circular cones and possess some nice geometric properties we present a few of them.

8.4 Enveloping Cone:

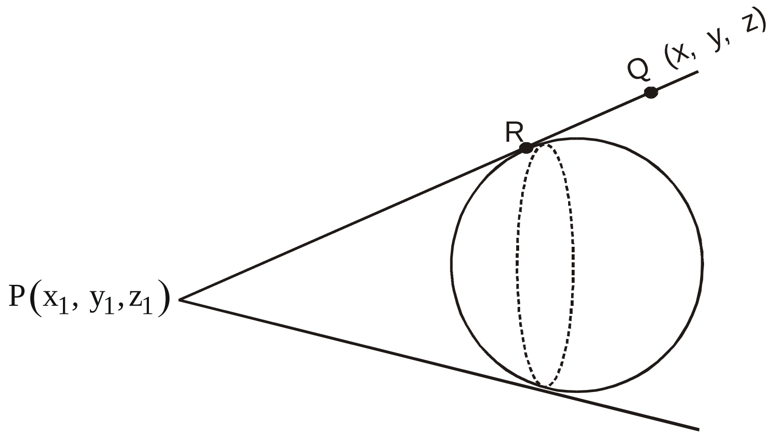
8.4.1 Def: Tangent line to a surface from a point (x_1, y_1, z_1): Let $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be a line through (x_1, y_1, z_1). It is said to be a tangent line to the surface $S(x, y, z)=0$ if and only if, the line meets the surface at one and only one point.

8.4.2 Theorem: The locus of the tangent lines through the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2=p^2$ is a cone represented by

$$\left(x\cdot x_1+y\cdot y_1+z\cdot z_1-p^2\right)^2=\left(x^2+y^2+z^2-p^2\right)\left(x_1^2+y_1^2+z_1^2-p^2\right)$$

Proof: The given sphere is $x^2+y^2+z^2=p^2 \cdots\cdots\cdots(1)$

Let $P(x_1, y_1, z_1)$ be a point not on (1) and $\overline{PQ}$ be a tangent line to (1) and $Q=(x, y, z)$ Let R be the point of contact. Suppose R divides PQ in theratio $K:1$. Then R is given by



Enveloping Cone

$$R=\left(\frac{Kx+x_1}{K+1}, \frac{Ky+y_1}{K+1}, \frac{Kz+z_1}{K+1}\right) \text{ R lies on (1) } \Rightarrow$$

$$\left(\frac{Kx+x_1}{K+1}\right)^2+\left(\frac{Ky+y_1}{K+1}\right)^2+\left(\frac{Kz+z_1}{K+1}\right)^2=P^2$$

$$\Rightarrow (Kx + x_1)^2 + (Ky + y_1)^2 + (Kz + z_1)^2 = (K+1)^2 P^2$$

$$\Rightarrow K^2\left(x^2 + y^2 + z^2 - P^2\right) + 2K\left(x \cdot x_1 + y \cdot y_1 + z \cdot z_1 - P^2\right) + \left(x_1^2 + y_1^2 + z_1^2 - P^2\right) = 0$$

$$\Rightarrow K^2 \cdot S + 2K \cdot S_1 + S_{11} = 0 \cdots\cdots\cdots\cdots (2)$$

Since PQ is a tangent line the roots of (2) are equal $\Leftrightarrow \Delta = 0$

$\Leftrightarrow b^2 - 4ac = 0 \;\Leftrightarrow\; 4 \cdot S_1^2 - 4S \cdot S_{11} = 0 \;\Leftrightarrow S_1^2 = S \cdot S_{11} \cdots\cdots\cdots\cdots (3)$

Consequently any point R satisfying (3) lies on the tangent line PQ.

By the definition of the cone the equation $S_1^2 = S \cdot S_{11}$ i.e.,

$\left(x \cdot x_1 + y \cdot y_1 + z \cdot z_1 - P^2\right) = \left(x^2 + y^2 + z^2 - P^2\right)\left(x_1^2 + y_1^2 + z_1^2 - P^2\right)$ represent a cone.

8.4.3 The locus of the tangent line through the point $P(x_1, y_1, z_1)$ to the surface $S(x, y, z) = 0$ is a cone represented by $S_1^2 = S \cdot S_{11}$.

Proof: The given surface is $S(x, y, z) = 0 \cdots\cdots\cdots (1)$ Let a line through (x_1, y_1, z_1) be L given by

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \cdots\cdots\cdots (2)$$

If (2) is a tangent line or to (1) then we have

$$(\ell U + mV + nW)^2 = S_{11} \cdot E(\ell, m, n) \cdots\cdots\cdots\cdots (3)$$

Conversely if (3) is true and $H(\ell, m, n) \neq 0$. Then L is tangent to (1).

$\Leftrightarrow (\ell U + mV + nW)^2 = S_{11} \cdot E(\ell, m, n)$ (see 7.6.1 for notation)

$$\Leftrightarrow \left((x - x_1)U + (y - y_1)V + (z - z_1)W\right)^2 = S_{11} \cdot E(x - x_1,\, y - y_1,\, z - z_1) \cdots\cdots (A)$$

$\therefore$ L is generator to the cone represented by (A) Cone (A) is given by

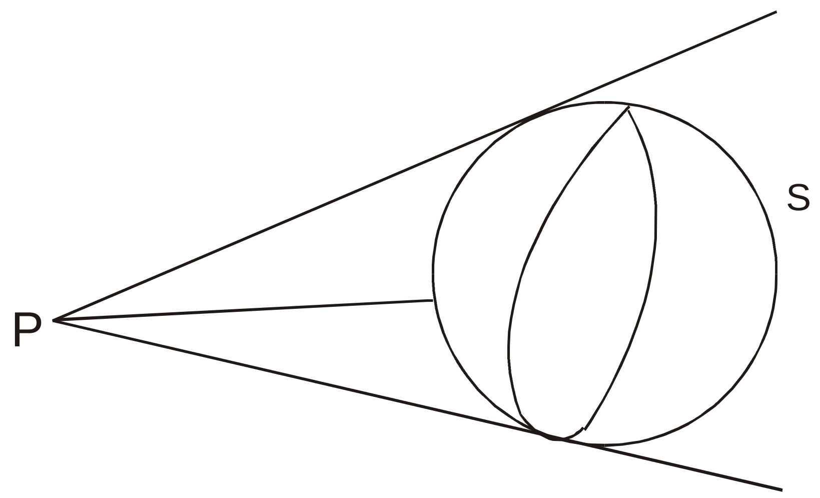
$$(S_1 - S_{11})^2 = S_{11}\left[(S - S_1) - (S_1 - S_{11})\right]$$

$$= S_{11}(S + S_{11} - 2S_1)$$

i.e. $S_1^2 - 2S_1 \cdot S_{11} + S_{11}^2 = S \cdot S_{11} + S_{11}^2 - 2S_1 \cdot S_{11}$

i.e. $S_1^2 = S \cdot S_{11}$

8.4.4 Def: enveloping cone: Let S be a surface in the space. Let P be a point not on 'S'. The locus of tangent lines through the point P to the surface S is a cone, which is called the enveloping cone of 'S', with the vertex at 'P'.



8.4.5 Example: Find the enveloping cone of the sphere $x^2 + y^2 + z^2 = 11$ with vertex at $(2,4,1)$.

Solution: Equation to the given surface is $x^2 + y^2 + z^2 = 11 \cdots\cdots\cdots(1)$

The given point is $P(2,4,1)$ Equation to the enveloping cone is given by

$S \cdot S_{11} = S_1^2$. Here $S \equiv x^2 + y^2 + z^2 - 11$

$S_1 \equiv 2x + 2y + z - 11,\ S_{11} = 2^2 + 4^2 + 1^2 - 11 = 21 - 11 = 10$

$\therefore$ Enveloping cone is $10\left(x^2 + y^2 + z^2 - 11\right) = (2x + 4y + z - 11)^2$

8.4.6 Example: Prove that the lines drawn from the origin so as to touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ lie on the cone $d\left(x^2 + y^2 + z^2\right) = (ux + vy + wz)^2$

Solution: The given sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \cdots\cdots\cdots(1)$

The locus of the tangent lines to (1), through 'O' is the enveloping cone of (1), given by $S \cdot S_{11} = S_1^2$.

Since $S_{11} = d, S_1 = ux + vy + wz + d$ the required equation is

$$d\left(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d\right) = (ux + vy + wz + d)^2$$

$$\Rightarrow d\left(x^2 + y^2 + z^2\right) + 2d(ux + vy + wz) + d^2 = (ux + vy + wz)^2 + 2d(ux + vy + wz) + d^2$$

$$\Rightarrow d\left(x^2 + y^2 + z^2\right) = (ux + vy + wz)^2$$

8.5 Three Mutually Perpendicular Generators:

8.5.1 Theorem: A necessary and sufficient condition for a cone $E(x, y, z) = 0$ to have three mutually perpendicular generators is $a + b + c = 0$.

Proof: **Nessary Condittion:** The given cone equation is $E(x, y, z) \equiv 0$ i.e.

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \cdots\cdots\cdots\cdots(1)$$

Let it have three mutually perpendicular generators say ox^1, oy^1, oz^1. Rotate the coordinate axes ox, oy, oz to the generators ox^1, oy^1, oz^1. Then the equation (1) changes to form $a^1x^2 + b^1y^2 + c^1z^2 + 2f^1yz + 2g^1zx + 2h^1xy = 0 \cdots\cdots\cdots(2)$.

ox^1 is a generator at (2) $\Rightarrow E^1(x, 0, 0) = 0 \Rightarrow a^1x^2 = 0 \Rightarrow a^1 = 0$

Similarly $b^1 = 0, c^1 = 0 \Rightarrow a^1 + b^1 + c^1 = 0$ we know that for rotation of axes $a^1 + b^1 + c^1 = a + b + c \quad \therefore a + b + c = 0$.

Sufficient Condition: Let us suppose that $a + b + c = 0$, we show that the cone has three mutually perpendicular generators.

Let us rotate the axes, so that one generator is x axis then $a = 0$ since $a + b + c$ is unaltered by rotation of axes $a + b + c = 0 \Rightarrow b + c = 0 \Rightarrow b = -c$.

Then the equation (1) becomes $b\left(y^2 - z^2\right) + 2fyz + 2gzx + 2hxy = 0 \cdots\cdots\cdots(2)$

If the plane section of the cone with yz plane's the entire yz plane, then any two perpendicular lines through 'o' on yz plane are two generators which are perpednccicular. Thus the cone has three mutually perpendicular generators.

If the plane section is not the entire YZ plane, then the cone (2) becomes $b\left(y^2 - z^2\right) + 2fyz = 0,\ x = 0 \cdots\cdots\cdots (3)$

In (3) coeff of $x^2 + \text{coeff of } y^2 = 0$ and $f \overset{2}{\geq} -b^2$ (3) represents two perpendicular lines. Since XOX^1 is a generator, the given cone has three muchually perpendicular generators.

Note: (1) If $a + b + c = 0$ and L is a generator of the cone $E(x, y, z) = 0$, then there exist two more lines M, N such that L, M, N are three mutually perpendicular generators.

(2) The same condition for the general cone $S = 0$ is applied.

8.5.2 Example: Prove that the cone $xy + yz + zx = 0$ and the plane $ax + by + cz = 0$ cut in two perpendicular generators if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

Solution: The given cone is $xy + yz + zx = 0 \cdots\cdots (1)$

Comparing with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

$a = 0,\ b = 0,\ c = 0 \Rightarrow a + b + c = 0 \Rightarrow$ The cone has three mutually perpendicular generators.

The given plane is $ax + by + cz = 0 \cdots\cdots\cdots (2)$

(1) and (2) intersect in two perpendicular lines if the normal to the plane (2) is a generator of (1)

i.e. $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ is a generator $\Rightarrow (ar, br, cr)$ lies on (1)

$\Rightarrow (ab + bc + ca)r^2 = 0 \Rightarrow \frac{1}{c} + \frac{1}{a} + \frac{1}{b} = 0$

The given cone (1) and the plane (2) intersect in two perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.

8.5.3 Example: Prove that the plane $\ell x+my+nz=0$ cuts the cone

$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$ in two perpendicular lines if

$$(b-c)\ell^2+(c-a)m^2+(a-b)n^2+2fmn+2gn\ell+2h\ell m=0$$

Solution: Given cone is

$$(b-c)x^2+(c-a)y^2+(a-b)z+2fyz+2gzx+2hxy=0\cdots\cdots\cdots(1)$$

Comparing with $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

$A=b-c,\ B=c-a,\ C=a-b,\ F=f,\ G=g,\ H=h$

$A+B+C=b-c+c-a+a-b=0$.

$\therefore$ Cone (1) has three mutually perpendicular generators

The given plane $\ell x+my+nz=0\cdots\cdots\cdots(2)$

(1) and (2) intersect in two perpendicular lines if the normal to (2) is a generator of (1)

i.e., $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ is a generator

$\Rightarrow(\ell r,\ mr,\ nr)$ lies on (1)

$$\Rightarrow(b-c)\ell^2+(c-a)m^2+(a-b)n^2+2\ell mn+2gn\ell+2h\ell m=0$$

8.5.4 Example: $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpenducular generators of the cone $5yz-8zx-3xy=0$ find the other two generators.

Solution: The given line is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}\cdots\cdots\cdots\cdots(1)$

given cone is $5yz-8zx-3xy=0\cdots\cdots\cdots\cdots(2)$

If (1) is of the three mutually perpendicular generators of (2), the other two are, perpendicular lines of intersection of (2) and the normal plane to (1) whose d.r.s are (1,2,3).

The plane is given by $x+2y+3z=0 \cdots\cdots\cdots (3)$

If (2) and (3) intersect in the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n} \cdots\cdots\cdots (4)$

Since line (4) lies on (2) and (3), we have $\ell+2m+3n=0 \cdots\cdots\cdots (5)$

i.e. $5mn-8\ell n-3\ell m=0 \cdots\cdots\cdots (6)$

(5) $\Rightarrow \ell=-(2m+3n)$, substituting in (6)

$$5mn+(8n+3m)(2m+3n)=0$$

$$\Rightarrow 5mn+16mn+24n^2+6m^2+9mn=0$$

$$\Rightarrow 6m^2+30mn+24n^2=0$$

$$\Rightarrow 6\left(m^2+4mn+mn+4n^2\right)=0$$

$$\Rightarrow m(m+4n)+n(m+4n)=0$$

$$\Rightarrow (m+n)(m+4n)=0$$

D.r's of the lines of intersection of (2) and (3) are given by

(1) $\ell+2m+3n=0$ $\qquad$ 2 3 1 2

$0\cdot\ell+1\cdot m+1\cdot n=0$ $\qquad$ 1 4 0 1

$$\Rightarrow \frac{\ell}{8-3}=\frac{m}{0-4}=\frac{n}{1-0}$$ (By cross multiplicative method)

$$\Rightarrow \frac{\ell}{5}=\frac{m}{-4}=\frac{n}{1}$$

(2) $\ell+2m+3n=0$ $\qquad$ 2 3 1 2

$0.\ell+1\cdot m+1\cdot n=0$ $\qquad$ 1 1 0 1

$$\frac{\ell}{2-3}=\frac{m}{0-1}=\frac{n}{1}$$

$$\Rightarrow \frac{\ell}{1} = \frac{m}{1} = \frac{n}{-1}$$

$\therefore$ The other two genrators are $\frac{x}{5} = \frac{y}{-4} = \frac{3}{1}$ and $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$

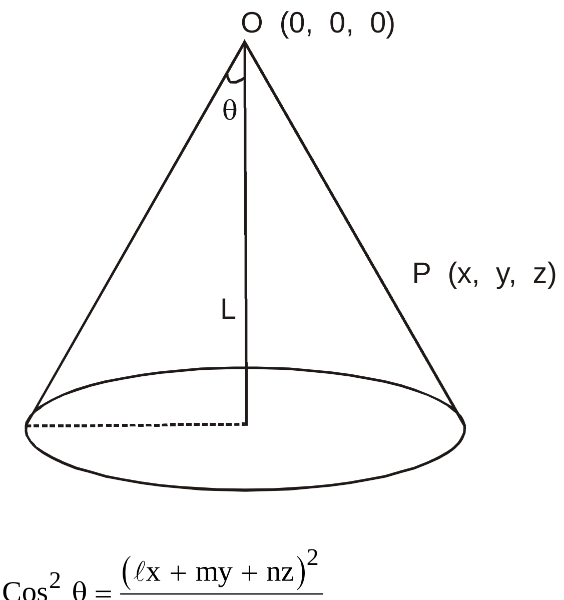
8.6 Right Circular Cone:

8.6.1 Theorem: Given a line L, a point O on L and angle θ in $\left(0, \frac{\pi}{2}\right)$. The locus of the point 'P' such that OP makes an angle θ with L is a non degenerate cone with vertex at '0'.

Proof: **Case I:** Assume that 0 is the origin. Let ℓ, m, n be the d.c's of L so that $\ell^2 + m^2 + n^2 = 1$ and L be represented by $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \cdots\cdots\cdots (1)$

Let S be the required locus. For any point $p(x, y, z)$ different from 'o' the d.rs. of OP are (x, y, z). Thus $p \in S$ iff OP makes an angle θ, with L

$$\Leftrightarrow \text{Cos}\ \theta = \frac{\ell x + my + nz}{\sqrt{x^2 + y^2 + z^2} \cdot \sqrt{\ell^2 + m^2 + n^2}}$$



$$\Leftrightarrow \text{Cos}^2\ \theta = \frac{(\ell x + my + nz)^2}{x^2 + y^2 + z^2} \qquad \left(\because \ell^2 + m^2 + n^2 = 1\right)$$

$$\Leftrightarrow \left(x^2+y^2+z^2\right)\cos^2\theta = (\ell x + my + nz)^2$$

$$\Leftrightarrow \left(\ell^2-\cos^2\theta\right)x^2+\left(m^2-\cos^2\theta\right)y^2+\left(n^2-\cos^2\theta\right)z^2+2\ell mxy+2mnyz+2\ell nzx=0 \cdots\cdots\cdots(2)$$

Now (2) is a homogeneous equation of degree (2) with $(a,b,c)\neq(0,0,0)$ other wise $\ell^2=m^2=n^2=\cos^2\neq 0$ then $\ell m, mn, \ell n$ are non zero and (2) represents a cone.

Also

$$\Delta = \begin{vmatrix} \ell^2-\cos^2\theta & \ell m & n\ell \\ \ell m & m^2-\cos^2\theta & mn \\ n\ell & mn & n^2-\cos^2\theta \end{vmatrix} = \begin{vmatrix} \ell^2 & \ell m & \ell n \\ \ell m & m^2 & mn \\ n\ell & mn & n^2 \end{vmatrix} - \cos^2\theta \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\ell mn)^2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} - \cos^2\theta\cdot 1 = -\cos^2\theta \neq 0$$

$\therefore$ Locus of P is a non degenerating cone with vertex at '0'.

Case - II : Let L be any line and (α, β, γ) be any point on L. Shifting the origin to the point (α,β,γ) we have $X=x-\alpha,\ Y=y-\beta,\ Z=z-\gamma$

As above we have the locus as

$$\left(x^2+y^2+z^2\right)\cos^2\theta = (\ell x+my+nz)^2$$

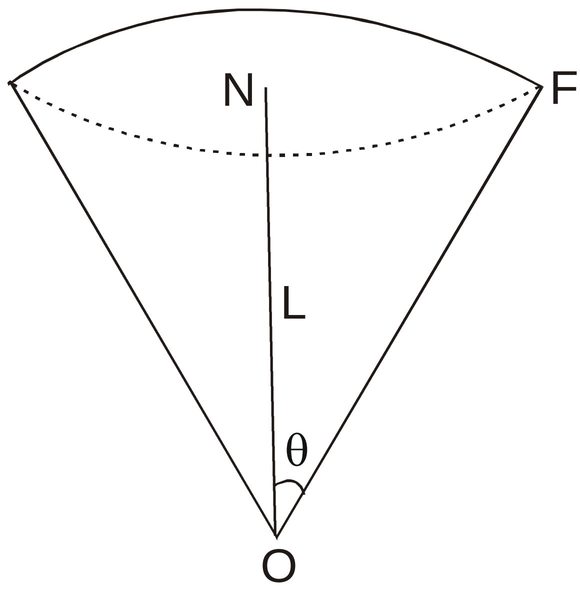
$$\text{i.e.}\quad \left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\cos^2\theta = \left[\ell(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2$$

which represents a non degenerate cone with vertex at (α,β,γ).

8.6.2 Definition: The cone which is the locus of 'P' such that the line OP makes an angle $\theta\in\left(0,\frac{\pi}{2}\right)$ with a fixed line L, passing through 0 is called a right circular cone with vertex at '0'. L is called the axis of the cone and θ is called the semi vertical angle.

8.6.3 Theorem: Any section of a right circular cone by a plane perpendicular to the axis is a circle.

Proof: Let 'o' be the vertex of the cone, L the axis of the cone and θ be the semivertical angle. If C is the section of the cone with a plane π, perpendicular to the axis, let P be any point an C and N be the point of intersection of π and L. Then ON is perpendicular to π. We have



$$\frac{PN}{ON} = \tan\theta$$

$$\Rightarrow NP^2 = ON^2 \cdot \tan^2\theta$$

Thus locus of P represents a circle with centre at N and radius $ON \ \tan\theta$.

8.6.4 Theorem: The equation to the right circular cone with vertex at 'o', with semivertical angle 'θ' and the axis being the Z axis is

$$x^2 + y^2 = z^2 \ \tan^2\theta$$

Proof: We know that the equation to the cone with vertex at 'o' and the axis as $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$, with semi veertical angle θ is given by $\left(x^2 + y^2 + z^2\right)Cos^2\theta = (\ell x + my + nz)^2$

Here $(\ell, m, n) = (0,0,1)$ ($\because$ z axis is the axis)

The cone is $\left(x^2 + y^2 + z^2\right) = z^2 Sec^2\theta$

$$\Rightarrow x^2 + y^2 = z^2 \left(Sec^2\theta - 1\right)$$

$$\Rightarrow x^2 + y^2 = z^2 \tan^2 \theta$$

8.6.5 The semi vertical angle of a right circular cone admitting three mutually perpedndicular generators is $\tan^{-1}\sqrt{2}$.

Proof: We know that the equation to the right circular cone with semivertical angle θ is given by

$$(\ell x + my + nz)^2 = \left(x^2 + y^2 + z^2\right)Cos^2\theta, \qquad \ell^2 + m^2 + n^2 = 1$$

If the cone admits three mutually perpendicular generators then $a + b + c = 0$

$$\Leftrightarrow \ell^2 + m^2 + n^2 = (1+1+1)\ Cos^2\theta \ \Leftrightarrow 1 \cdot Sec^2\theta = 1 + 2 \ \Leftrightarrow \tan^2\theta = 2 \ \Leftrightarrow \tan\theta = \sqrt{2}$$

$$\Leftrightarrow \theta = \tan^{-1}\sqrt{2}$$

8.6.6 The semi vertical angle of a right circular cone having a set of three mutually perpendicular tangent planes is $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Proof: Without loss of generality, let the vertex be 0 and the axis be z axis. Then the cone equation is given by $x^2 + y^2 = z^2 \tan^2\theta \cdots\cdots\cdots (1)$

If this posses three mutually perpendicular tangent planes, then its reciprocal cone will have three mutually perpendicular generators

Reciprocal cone of (1) is given by

$$\frac{x^2}{1} + \frac{y^2}{1} - \frac{z^2}{\tan^2\theta} = 0$$

$$\Rightarrow x^2 + y^2 - z^2 \cot^2\theta = 0 \cdots\cdots\cdots\cdots (2)$$

(2) is having three mutually perpendicular generators

$$\Leftrightarrow a + b + c = 0$$

$$\Leftrightarrow 1 + 1 - Cot^2\theta = 0 \ \Leftrightarrow Cot^2\theta = 2$$

$$\Leftrightarrow \theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

8.6.7 Example: Find the equation to the right circular cone which has the coordinate axes as generators.

Solution: Since the generators are the co ordinate axes and they intersect at 0. 0 is the vertex. Let the axis be given by

$$\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n} \quad \text{where} \quad \ell^2+m^2+n^2=1.$$

since the axis makes equal angle with the generators (OX, OY, OZ)

$$\ell=m=n \quad \Rightarrow 3\ell^2=1 \quad \Rightarrow \ell=\text{Cos} \quad \alpha=\pm\frac{1}{\sqrt{3}}$$

Equations to the axes are

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$$

Cone equation is given by

$$(\ell x+my+nz)^2=\left(\ell^2+m^2+n^2\right)\left(x^2+y^2+z^2\right)\text{Cos}^2\,\alpha$$

$$\Rightarrow \left(\frac{x\pm y\pm z}{\sqrt{3}}\right)^2=1\left(x^2+y^2+z^2\right)\cdot\frac{1}{3}$$

$$\Rightarrow x^2+y^2+z^2+2(xy\pm yz\pm zx)=x^2+y^2+z^2$$

$$\Rightarrow xy\pm yz\pm zx=0.$$

8.6.8 Example: if α is the semivertical angle of the cone which passes through OX, OY and $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ show that $\alpha=\text{Cos}^{-1}\left(9-4\sqrt{3}\right)^{\frac{-1}{2}}$

Solution: Since the generators $OX, OY,$ $x=y=z$ meet at 'o' os is the vertex and d.c's of the generators are $(1,0,0)\,(0,1,0)$ and $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Let the equation to the axis be $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n},\quad \ell^2+m^2+n^2=1\cdots\cdots\cdots\cdots(1)$

Since α is the semivertical angle

$$(0x, (1)) = (oy, (1)) = \alpha$$

$$\therefore (ox, (1)) = \alpha \Rightarrow \text{Cos } \alpha = \frac{1 \cdot \ell + 0 \cdot m + 0 \cdot n}{\sqrt{1^2 + 0 + 0} \cdot \sqrt{\ell^2 + m^2 + n^2}} = \ell$$

$$(ox, (1)) = \alpha \Rightarrow \text{Cos } \alpha = \frac{0 \cdot \ell + 1 \cdot m + 0.1}{\sqrt{\ell^2 + m^2 + n^2} \cdot \sqrt{0 + 1 + 0}} = m$$

Also

$$\text{Cos } \alpha = \frac{1 \cdot \ell + 1 \cdot m + 1 \cdot n}{\sqrt{1+1+1} \cdot \sqrt{\ell^2 + m^2 + n^2}} = \frac{\ell + m + n}{\sqrt{3}}$$

We have $\ell^2 + m^2 + n^2 = 1 \cdots\cdots\cdots\cdots (3)$

$$\ell = \text{Cos}\,\alpha,\ m = \text{Cos } \alpha,\ \ell + m + n = \sqrt{3} \text{ Cos } \alpha$$

$$\Rightarrow n = \left(\sqrt{3} - 2\right) \text{Cos } \alpha$$

(3) $\Rightarrow \text{Cos}^2\ \alpha + \text{Cos}^2\ \alpha + \left(\sqrt{3} - 2\right)^2 \text{Cos}^2\alpha = 1 \Rightarrow \text{Cos}^2\ \alpha \left(1+1+4+3-4\sqrt{3}\right) = 1$

$$\Rightarrow \text{Cos}^2\ \alpha \left(9 - 4\sqrt{3}\right) = 1 \Rightarrow \text{Cos}^2\ \alpha = \left(9 - 4\sqrt{3}\right)^{-1}$$

$$\Rightarrow \text{Cos } \alpha = \left(9 - 4\sqrt{3}\right)^{\frac{-1}{2}} \Rightarrow \alpha = \text{Cos}^{-1} \left(9 - 4\sqrt{3}\right)^{-\frac{1}{2}}$$

8.7 Summary:

After reading this lesson the student should be able to gain knowledge on envoloping cones of a sphere and a given surface. The student gains knowledge for a cone to have three mutually perpendicular generators. The student gains knowledge about a right circular cones.

8.8 Technical Terms:

Enveloping Cone

Right Circular Cone

8.9 Exercises:

1. Find the enveloping cone of the sphere

$$x^2+y^2+z^2+2x-2y=2 \text{ with vertex at } (1,1,1)$$

2. Prove that the locus of tangent lines from the origin to the sphere $(x-a)^2+\left(y-b^2\right)+(z-c)^2=k^2$ (lie. on the cone given by the equation $\left(a^2+b^2+c^2-k^2\right)\left(x^2+y^2+z^2\right)=(ax+by+cz)^2$

3. Find the locus of the point from which three mutually perpendicular generators (tangent lines) can be drawn to the surface $ax^2+by^2+2cz=0$.

4. Find the enveloping cone of the sphere

$$x^2+y^2+z^2-2x+4z=1 \text{ with vertex at } (1,1,1).$$

5. If $x=\frac{y}{2}=z$ represents of the three mutually perpendicular generators of the cone $11yz+6zx-14xy=0$. Find the other two generators.

6. If $x=y=\frac{z}{2}$ is one of the three muctually perpendicular generators of the cone $3yz-2zx-2xy=0$, find the other two.

7. Find the locus of the points from which three muctually perpendicular generators can be drawn to intersect the conic $z=0, \quad ax^2+by^2=1$.

8. Find the equation to the right circular cone whose vertex is $P(2,-3,5)$ axis PQ, which makes equal angles with the axes and semivertical angle $\frac{\pi}{6}$.

9. Prove that $x^2-y^2+z^2-2x+4y+6z+6=0$ represents a right circular cone, whose vertex is $(1,2,-3)$ and axis parallel to OY, semivertical angle $\frac{\pi}{4}$.

10. Find the equation to the right circular cone. Whose vertex is $(2,-3,5)$ axis PQ, which makes equal angles with the axes and passes through the point $(1,-2,3)$.

11. The axis of a right cone, vertex at 0, makes equal angles with the coordinate axes, the cone passes through the line drawn from 0 with d.rs. proportional to $(1,-2,2)$. Find the equation of the cone.

12. Find the equation to the right cone, which passes through $(1,1,2)$, has vertex at $(0,0,0)$ with axis $\frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$.

8.10 Answers:

1. $3x^2-y^2+4zx-10x+2y-4z+6=0$

3. $ab\left(x^2+y^2\right)+2c(a+b)z-c^2=0$

4. $4x^2+3y^2-5z^2-6yz-8x+16z-4=0$

5. $\frac{x}{2}=\frac{y}{-3}=\frac{z}{4}$; $\frac{x}{-11}=\frac{y}{2}=\frac{z}{7}$

6. $\frac{x}{2}=\frac{y}{-4}=\frac{z}{1}$; $\frac{x}{3}=\frac{y}{1}=\frac{z}{-2}$

7. $ax^2+by^2+(a+b)z^2=1$

8. $5\left(x^2+y^2+z^2\right)-8(xy+yz+zx)-4x+86y-58z+278=0$

10. $x^2+y^2+z^2+6(xy+yz+zx)-4(4x+9y+z+7)=0$

11. $4\left(x^2+y^2+z^2\right)+9(xy+yz+zx)=0$

12. $4x^2+40y^2+19z^2-12(4xy+6yz-3zx)=0$

8.11 Model Exam Questions:

1. (a) Define enveloping cone

   (b) Prove that the locus of the tangent line through the point $V(x_1, y_1, z_1)$ to the surface $S(x,y,z)=0$ is a cone represented by $S_1^2=SS_{11}$.

   (c) Find the envoloping cone of the sphere

$$x^2+y^2+z^2=11$$

with vertex at $(2,4,1)$. Show that the plane $z=0$ cuts it in a rectangular hyperbola.

2. (a) Define right circular cone

(b) Prove that every section of a right circular cone by a plane perpendicular to the axis is a circle.

(c) If X is the semi vertical angle of the cone which passes through Ox, OY and $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$. Show that $\alpha = \mathrm{Cos}^{-1}\left(9-4\sqrt{3}\right)^{-\frac{1}{2}}$.

8.12 Model Practical Problem with Solution:

Lines are drawn from the origin with direction cosines propogitional to $(1,2,2)$, $(2,3,6)$ and $(3, 4, 12)$. Find the direction cosines of the axes of the right circular cone through them and prove that the semivertical angle of the cone is $\mathrm{Cos}^{-1}\frac{1}{\sqrt{3}}$.

Definitions:

A right circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line through the fixed point.

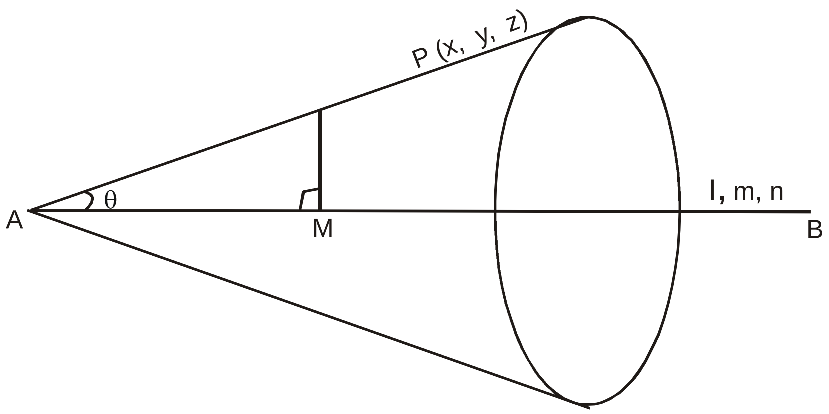
The fixed point is called the vertex, the fixed line is called the axis and the fixed angle is called the semivertical angle.

Note:

(1) If the vertex is the horizon then the right circular cone is given by

$$(\ell x + my + nz)^2 = \left(x^2+y^2+z^2\right)\cos^2\theta .$$

Fig - A



Solution:

Let ℓ, m, n be the d.c.'s of the axis of the right circular cone through three lines say OA, OB, OC whose d.c.'s are propotional to $(1, 2, 2)$, $(2, 3, 6)$ and $(3, 4, 12)$ respectively.

Since the axis makes the same angle (say α) with the genarators OA, OB, OC.

$$\therefore \frac{\ell(1)+m(2)+n(2)}{\sqrt{1+4+4}} = \frac{\ell(2)+m(3)+n(6)}{\sqrt{4+9+36}} = \frac{\ell(3)+m(4)+n(12)}{\sqrt{9+16+144}} = \cos\alpha$$

$$\Rightarrow \frac{\ell+2m+2n}{3} = \frac{2\ell+3m+6n}{7} = \frac{3\ell+4m+12n}{13} = \cos\alpha \cdots\cdots\cdots\cdots(1)$$

From first and second members of (1)

$$7\ell+14m+14n = 6\ell+9m+18n \Rightarrow \ell+5m-4n=0 \cdots\cdots\cdots\cdots(2)$$

From (1) $26\ell + 39m \;\; 78n = 21\ell + 28m + 84n \Rightarrow 5\ell + 11m - 6n = 0 \cdots\cdots\cdots\cdots(3)$

Solving (2) and (3) by cross multiplication,

$$\begin{array}{cccc} & \ell & m & n \\ 5 & -4 & 1 & 5 \\ 11 & -6 & 5 & 11 \end{array}$$

we have $\dfrac{\ell}{-30+44} = \dfrac{m}{-20+6} = \dfrac{n}{11-25}$

$$\Rightarrow \frac{\ell}{-1} = \frac{m}{+1} = \frac{n}{+1}$$

Hence d.c.'s of axes are propotional to -1, 1, 1. Directional cosines are $\dfrac{-1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

$$\therefore \operatorname{Cos}\alpha = \frac{\ell+2m+2n}{3} = \frac{\frac{-1}{\sqrt{3}}+\frac{2}{\sqrt{3}}+\frac{2}{\sqrt{3}}}{3} = \frac{\frac{3}{\sqrt{3}}}{3} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \operatorname{Cos}^{-1}\frac{1}{\sqrt{3}}$$

Conclusion: The semivertical angle of the cone whose generators have d.c.'s propotional to $(1, 2, 2)$, $(2, 3, 6)$ and $(3, 11, 12)$ is $\operatorname{Cos}\dfrac{1}{\sqrt{3}}$. The d.c.'s are proportional to -1, 1, 1.

Lesson Writer

G. NARAYANA

Lesson - 9

CYLINDER

9.1 Objective of the lesson:

We start with the geometric definition of a cylinder and concentrate on cylinders being represented by a quadratic equation. We study about cylinders with guiding curve enveloping cylinders and also right circular cylinders.

9.2 Structure:

This lesson contains the following components:

9.3 Definitions and Examples

9.4 Equations of Some Cylinders

9.5 Enveloping Cylinders of a Sphere

9.6 Right Circular Cylinder

9.7 Summary

9.8 Technical terms

9.9 Exercises

9.10 Answers to Exercises

9.11 Model Exam Questions

9.12 Model Practical Problem With Solution

9.3 Definitions and Examples:

9.3.1 Definition: A subset 'S' of the three dimensional Euclidean space R^3 is a cylinder, if there exists a line L such that if $P \in S$ then the line through P and parallel to L is contained in 'S'. Equivalently, there exists ℓ, m, n not all zero such that $(x, y, z) \in S \Rightarrow (x + \ell r,\ y + mr,\ z + nr) \in S$ for all $r \in S$. Any such line L is called an axis of the cylinder and any line in the cylinder is called a generator of it.

9.3.2 Examples: We mention a few examples of cylinders and their axes.

	Cylinder	**Axis**
(a)	Empty set	Any line
(b)	Line L	Any line parallel to L

(c)	A collection 'C' of lines	Any line parallel to a line in C
(d)	A plane π	Any line parallel to π
(e)	A collection 'P' of parallel planes	Any line parallel to any plane of 'P'
(f)	A collection 'P' of planes containing a line 'L'	Any line parallel to 'L'

9.3.3 Remark: As in the case of cones we may call a cylinder of one of the above types as a degenerate cylinder and the others as non degenerate cylinders, that are quadratic surfaces. We use the same notation of in the case of cones.

9.4 Equations of Some Cylinders:

9.4.1 Theorem: Condition for a second degree surface to represent a cylinder:

The equation $f(x,y,z) \equiv E(x,y,z) + 2ux + 2vy + 2wz + d = 0$ where

$E(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ represents a cylinder iff there exist ℓ, m, n not all zero such that (1) $E(\ell,m,n) = 0$ (2) $\ell U + mV + nW = 0$ whenever $f(x,y,z) = 0$ where U, V, W are as in 7.6.1.

Proof: By definition $f(x,y,z) = 0$ represents cylinder S iff, there exists a line with directional ratios, say ℓ, m, n such that $(x,y,z) \in S \Rightarrow (x + \ell r,\ y + mr,\ z + nr) \in S$. This is equivalent to the condition that there exists (1) $E(\ell,m,n) = 0$ and (2) whenever $f(x,y,z) = 0$ $\ell U + mV + nW = 0$.

9.4.2 Find the equation to the cylinder, whose generators are parallel to X - axis and cut the curve $ax^2 + by^2 + cz^2 = 1 \cdots\cdots\cdots\cdots (1)$ and $\ell x + my + nz = p \cdots\cdots\cdots\cdots (2)$.

Solution: If $P(x_1, y_1, z_1)$ is a point on the cylinder, then the equations to the generators through P are given by

$$\frac{x - x_1}{1} = \frac{y - y_1}{0} = \frac{z - z_1}{0} = r, \qquad r \in R \cdots\cdots\cdots\cdots (3)$$

The coordinates of the any point on the cylinder are $(x_1 + r,\ y_1,\ z_1)$

P lies on the cylinder iff

$$a(x_1 + r)^2 + dy_1^2 + cz_1^2 = 1 \text{ and } \ell(x_1 + r) + my_1 + nz_1 = p$$

posseses a unique solution for r

$$\Leftrightarrow\ r = \frac{-(\ell x_1 + my_1 + nz_1 - p)}{\ell} \text{ and } a(x_1 + r)^2 + by_1^2 + cz_1^2 = 1$$

$$\Leftrightarrow\ a\frac{(my_1 + nz_1 - p)^2}{\ell^2} + by_1^2 + cz_1^2 = 1$$

The locus of the point P is the equation of the required cylinder, which is given by

$$\left(am^2 + b\ell^2\right)y^2 + \left(an^2 + c\ell^2\right)z^2 + 2amnyz - 2ampy - 2anpz + ap^2 - \ell^2 = 0$$

9.4.3 The equation to the cylinder whose axis is the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ where $x \neq 0$, and whose generators touch the conic $f(x,y) = 0,\ z = 0$ is $f\left(x - \frac{\ell z}{n},\ y - \frac{mz}{n}\right) = 0$

Proof: The given axis is $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n} \cdots\cdots\cdots(1)$

The given conic is $f(x,y) = 0, \quad z = 0 \cdots\cdots\cdots\cdots\cdots(2)$

Let $P(x_1, y_1, z_1)$ be a point on the cylinder

Equation to the line throught P, parallel to (1) is given by $\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \cdots\cdots\cdots(3)$

P lies on S $\Leftrightarrow$ every point on (3) lies on S

$\Leftrightarrow\ (x_1 + lr,\ y_1 + mr,\ z_1 + nr)$ lies on (2)

$\Leftrightarrow\ f(x_1 + \ell r,\ y_1 + mr) = 0, \quad z_1 + nr = 0$

$\Leftrightarrow\ r = -\frac{z_1}{n}$ and $f\left(x_1 - z_1\frac{\ell}{n},\ y_1\frac{m}{n}\right) = 0$

Hence the equation to the cylinder is given by

$$f\left(x - z\frac{\ell}{n},\ y_1 - z\frac{m}{n}\right) = 0$$

9.4.4 Find the cylinder whose generators are parallel to the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ and touch the plane curve. $f(x,y,z)=0$ and $ax+by+cz+d=0$, where $a\ell+bm+cn\neq 0$.

Solution: The given line is $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}\cdots\cdots(1)$

The given curve is $f(x,y,z)=0, \quad ax+by+cz+d=0\cdots\cdots\cdots(2)$

Let $P(x_1, y_1, z_1)$ be any point in the space. The line through P and parallel to (1) is

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r\cdots\cdots\cdots\cdots(3)$$

So that a point on it is $(\ell r+x_1,\ mr+y_1,\ nr+z_1),\ r\in R\cdots\cdots\cdots\cdots\cdots(4)$

The line (3) touches (2) iff there exists a unique 'r', such that

$$a(\ell r+x_1)+b(mr+y_1)+c(nr+z_1)+d=0\cdots\cdots\cdots(5)$$

and $f(\ell r+x_1,\ mr+y_1,\ nr+z_1)=0\cdots\cdots\cdots\cdots\cdots(6)$

$(5)\Rightarrow r=-\left(\frac{ax_1+by_1+cz_1+d}{a\ell+bm+cn}\right)$ substituting in (6)

$$f\left(x_1-\frac{\ell(ax_1+by_1+cz_1+d)}{a\ell+bm+cn},\ \ y_1-\frac{m(ax_1+by_1+cz_1+d)}{a\ell+bm+cn},\ z_1-\frac{n(ax_1+by_1+cz_1+d)}{a\ell+bm+cn}\right)=0$$

$\therefore$ The equation to the cylinder is given by

$$f\left[\frac{(bm+cn)x-\ell(by+cz+d)}{(a\ell+bm+cn)},\ \frac{(a\ell+cn)y-m(ax+cz+d)}{(a\ell+bm+cn)},\ \frac{(a\ell+bm)z-n(ax+by+d)}{(a\ell+bm+cn)}\right]=0$$

9.4.5 Find the equation to the cylinder whose generators are parallel to z - axis and are tangents to the curve $ax^2+by^2=2z\cdots\cdots\cdots\cdots(1)$ and $\ell x+my+nz=p\cdots\cdots\cdots(2)$ where $a^2+b^2\neq 0$, $n\neq 0$.

Solution: Since the generators are parallel to z - axis equation of a generator of cylinder is given by

$$\frac{x-x_1}{0}=\frac{y-y_1}{0}=\frac{z-z_1}{1}=r, \quad r\in R \cdots\cdots\cdots\cdots(3)$$

where (x_1, y_1, z_1) is any point on the cylinder. Thus (x_1, y_1, z_1) lie on the cylinder iff

(1) $x=x_1,\ y=y_1,\ z=z_1+r$

(2) $ax_1^2+by_1^2=2(z_1+r)$

(3) $\ell x_1+my_1+nz_1+nr=p$

$$\Leftrightarrow\ r=-\frac{(\ell x_1+my_1+nz_1-p)}{n}$$

and $ax_1^2+by_1^2=2z_1+2r=2z_1-2\dfrac{(\ell x_1+my_1+nz_1-p)}{n}$

$\therefore$ The locus of the point (x_1, y_1, z_1) is the cylinder equation which is given by

$$(ax^2+by^2)n-2(\ell x+my-p)=0$$

9.4.6 Find the equation to the cylinder, having generators parallel to the line with directional ratios $(1,1,1)$ and touch the curve $xy=1$ and $z=1\cdots\cdots(1)$

Solution: For any point $P(x_1, y_1, z_1)$ on the cylinder, equations to the generator are given by

$$\frac{x-x_1}{1}=\frac{y-y_1}{1}=\frac{z-z_1}{1}=r, \quad r\in R \cdots\cdots\cdots\cdots\cdots(2)$$

The points on the cylinder, lying on the generator are given by $(x_1+r, y_1+r, z_1+r)\cdots\cdots\cdots\cdots(3)$

In order that (2) is a tangent line to (1), the equations $(x_1+r)(y_1+r)=1$ and $z_1+r=1$ have a unique solution for r.

$\Leftrightarrow\ x_1y_1+r(x_1+y_1)+r^2=z_1+r$ have a unique solution

$\Leftrightarrow\ r^2+(x_1+y_1-1)r+x_1y_1-z_1=0$ has a unique solution

$$\Leftrightarrow (x_1 + y_1 - 1)^2 = 4(x_1y_1 - z_1)$$

The locus of the point (x_1, y_1, z_1) is the equation to the cylinder, which is given by

$$x^2 + y^2 - 2xy - 2x - 2y + 4z + 1 = 0$$

9.4.7 find the cylinder whose genarators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and which touches the curve $x^2 + 2y^2 = 1,\ z = 0 \cdots\cdots\cdots(1)$

Solution: For any point $P(x_1, y_1, z_1)$ on the cylinder, the equations to the generator through P are given by

$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{z - z_1}{3} = r \qquad r \in R \cdots\cdots\cdots\cdots(2)$$

Any point on the cylinder and also on (2) is

$$(x_1 + r,\ y_1 - 2r,\ z_1 - 3r)$$

(x_1, y_1, z_1) lies on the cylinder iff

$(x_1 + r)^2 + 2(y_1 - 2r)^2 = 1$ ....(3) and $z_1 + 3r = 0 \cdots\cdots\cdots\cdots\cdots(4)$ have a unique solution for r

$$\Leftrightarrow \left(x_1 - \frac{z_1}{3}\right)^2 + 2\left(y_1 + \frac{2z_1}{3}\right)^2 = 1$$

$$\Leftrightarrow \frac{(3x_1 - z_1)^2}{9} + \frac{2(3y_1 + 2z_1)^2}{9} = 1$$

The locus of the point $P(x_1, y_1, z_1)$ is the equation to the cylinder, which is given by

$$9x^2 + z^2 - 6xz + 18y^2 + 8z^2 + 24zy = 9$$

i.e., $3\left(x^2 + 2y^2 + z^2\right) - 2(zx - 4zy) = 3$

9.5 The Enveloping Cylinder of a Sphere:

9.5.1 Theorem: The locus of the lines touching the sphere $x^2+y^2+z^2=a^2 \cdots\cdots\cdots\cdots(1)$ and parallel to the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}\cdots\cdots\cdots(2)$ is the cylinder representes by

$$\left(\ell^2+m^2+n^2\right)\left(x^2+y^2+z^2-a^2\right)=\left(\ell x+my+nz\right)^2$$

Proof: We first notice that the locus is a cylinder since it contains a point iff it containing the line through the point which is parallel to the line.

$$\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(L)$$

For any point $P(x_1, y_1, z_1)$ equation of the line through P, parallel to the line (2) are

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \quad r\in R\cdots\cdots\cdots\cdots(3)$$

any point P lies on the line (3)

If $x=x_1+\ell r,\ y=y_1+mr,\ z=z_1+nr$ for then the line touches the sphere iff the equation.

$$\left(x_1+\ell r\right)^2+\left(y_1+mr\right)^2+\left(z_1+nr\right)^2=a^2 \text{ has a unique solution for r}$$

$$\Leftrightarrow r^2\left(\ell^2+m^2+n^2\right)+2r\left(\ell x_1+my_1+nz_1\right)+x_1^2+y_1^2+z_1^2-a^2=0 \text{ has equal roots}$$

$$\Leftrightarrow 4\left(\ell x_1+my_1+nz_1\right)^2=4\left(\ell^2+m^2+n^2\right)\left(x_1^2+y_1^2+z_1^2-a^2\right)$$

The locus of the point (x_1, y_1, z_1) is the cylinder equation which is given by

$$\left(\ell^2+m^2+n^2\right)\left(x^2+y^2+z^2-a^2\right)=\left(\ell x+my+nz\right)^2\cdots\cdots\cdots\cdots(A)$$

9.5.2 Definition: The cylinder formed by the tangent lines to the sphere $x^2+y^2+z^2=a^2$, that are parallel to the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$, is called the enveloping cylinder of the sphere with axis $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$.

9.5.3 find the enveloping cylinder of the sphere $x^2+y^2+z^2-2x+4y=1 \cdots\cdots\cdots (1)$ with axis $\frac{x}{1}=\frac{y}{1}=\frac{z}{1} \cdots\cdots\cdots (2)$

Solution: Let $P(x_1, y_1, z_1)$ be any point on the sphere. Since the axis is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$, equation to the generator through P is $\frac{x-x_1}{1}=\frac{y-y_1}{1}=\frac{z-z_1}{1}=r, \quad r \in R \cdots\cdots\cdots\cdots (3)$

Any arbitrary point on (3) is $(x_1+r,\ y_1+r,\ z_1+r)$

Thus a point lies on the enveloping cylinder, iff (3) is a tangent line to (1)

$\Leftrightarrow (x_1+r)^2+(y_1+r)^2+(z_1+r)^2-2(x_1+r)+4(y_1+r)=1$ has a unique solution for r

$\Leftrightarrow 3r^2+2r(x_1+y_1+z+1)+x_1^2+y_1^2+z_1^2-2x_1+4y_1-1=0$ has a unique solution for r

$\Leftrightarrow\ 4(x_1+y_1+z_1+1)^2 = 4\cdot 3\cdot\left(x_1^2+y_1^2+z_1^2-2x_1+4y_1-1\right) \quad \left(\because B^2=4AC\right)$

$\Leftrightarrow\ x_1^2+y_1^2+z_1^2+2x_1y_1+2y_1z_1+2z_1x_1+2x_1+2y_1+2z_1+1$

$= \quad 3x_1^2+3y_1^2+3z_1^2-6x_1+12y_1-3$

$\Leftrightarrow\ 2x_1^2+2y_1^2+2z_1^2-2x_1y_1-2z_1y_1-2x_1z_1-8x_1+10y_1-2z_1-4=0$

The locus of the point $P(x_1, y_1, z_1)$ is the equation to the enveloping cylinder, which is given by

$$x^2+y^2+z^2-xy-yz-zx-4x+5y-z-2=0$$

9.6 The Right Circular Cylinder:

9.6.1 Definition: A right circular cylinder is a cylinder whose generators are parallel to the normal of a circle at its centre. equivalently, a right circular cylinder whose generators are tangents to a fixed circle.

9.6.2 Theorem: The equation of a right circular cylinder whose axis is the line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \cdots\cdots\cdots (1)$ and guideing circle has the centre at (x_1, y_1, z_1) with radius 'r' is

$$\left(\ell^2+m^2+n^2\right)\left((x-x_1)^2+(y-y_1)^2-r^2\right)=\left[\ell(x-x_1)+m(y-y_1)+n(z-z_1)\right]^2$$

Proof: The axis of the cylinder is $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\cdots\cdots\cdots\cdot(1)$

The guiding curve is a circle with centre at (x_1, y_1, z_1), and radius r lies in the plane, normal to (1)

Its equations are given by

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$$

and $\ell(x-x_1)+m(y-y_1)+n(z-z_1)=0$

The required cylinder is the enveloping cylinder is the enveloping cylinder of the sphere

$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=r^2$ with axis parallel to the line $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, which is given by 9.5.1.

i.e.,

$$\left(\ell^2+m^2+n^2\right)\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r^2\right)=\left[\ell(x-x_1)+m(y-y_1)+n(z-z_1)\right]^2$$

9.6.3 Example: Find the equation to the right circular cylinder whose axis is $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$, having the radius '2' units.

Solution: Here the axis is the line

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}\cdots\cdots\cdots(1)$$

Radius of the cylinder is '2' units

Equation to the right circular cylinder with radius r and axis as $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ is given by

$$\left(\ell^2+m^2+n^2\right)\left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r^2\right]=\left[\ell(x-x_1)+m(y-y_1)+n(z-z_1)\right]^2$$

Hence the required cylinder is

$$\left(2^2+1^2+2^2\right)\left((x-1)^2+(y-2)^2+(z-3)^2-4\right)=\left(2(x-1)+1(y-2)+2(z-3)\right)^2$$

$$\Rightarrow\ 9\left(x^2+y^2+z^2-2x-4y-6z+10\right)=\left(2x+y+2z-10\right)^2$$

$$\Rightarrow\ 9x^2+9y^2+9z^2-18x-36y-54z+90$$

$$=4x^2+y^2+4z^2+100+4xy+4yz+8xz-40x-20y-40z$$

$$\Rightarrow\ 5x^2+8y^2+5z^2-4yz-8zx-4xy+22x-16y-14z-10=0$$

9.6.4 Example: find the equation to the circular cylinder whose axis is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ with radius 2 units.

solution: The axis of the circular cylinder is

$$\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}\cdots\cdots\cdots(1)$$

radius is 2 units

We know that the equation to the circular cylinder whose axis is $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, with radius 2 units is

$$\left(\ell^2+m^2+n^2\right)\left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r^2\right]=\left[\ell(x-x_1)+m(y-y_1)+n(z-z_1)\right]^2$$

$\therefore$ Required cylinder's equation is

$$\left(2^2+1^2+3^2\right)\left((x-1)^2+y^2+(z-3)^2-2^2\right)=\left(2(x-1)+3y+1(z-3)\right)^2$$

$$\Rightarrow 14\left(x^2+y^2+z^2-2x-6z+1+9-4\right)=\left(\overline{2x+3y+z}-5\right)^2$$

$$\Rightarrow\ 14x^2+14y^2+14z^2-28x-84z+84=4x^2+9y^2+z^2+12xy+6yz+4zx+25-10(2x+3y+z)$$

$$\Rightarrow\ 10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0$$

This is the equation to circular cylinder.

9.6.5 Example: Find the equation to the circular cylinder whose guiding circle is

$$x^2+y^2+z^2=9, \quad x-y+z=3$$

Solution: We find the (1) radius of the circle (2) axis of the cylinder and apply the formula.

Here the circle is $x^2+y^2+z^2=9 \cdots\cdots\cdots(1)$ and

$$x-y+z=3 \cdots\cdots\cdots(2)$$

centre of (1) is $(0,0,0)$, $\quad r=3$

The distance from 0 to (2) is $=\dfrac{|3|}{\sqrt{3}}=\sqrt{3}$

$\therefore$ Radius of the circle $=\sqrt{R^2-d^2}=\sqrt{9-3}=\sqrt{6}$

Equation to the axis : Axis the line through 0 and normal to the plane (2) i.e., it given by

$$\frac{x}{1}=\frac{y}{-1}=\frac{z}{1} \cdots\cdots\cdots\cdots(3)$$

Hence the circular cylinder is

$$\left(\ell^2+m^2+n^2\right)\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r^2\right)=\left[\ell(x-x_1)+m(y-y_1)+n(z-z_1)\right]^2$$

$$\Rightarrow (1+1+1)\left(x^2+y^2+z^2-6\right)=(x-y+z)^2$$

$$\Rightarrow 3x^2+3y^2+3z^2-18=x^2+y^2+z^2-2xy-2yz+2zx$$

$$\Rightarrow 2\left(x^2+y^2+z^2+xy+yz-zx-9\right)=0$$

$\therefore$ Equation to the circular cylinder is

$$x^2+y^2+z^2+xy+yz-zx-9=0$$

9.6.6 Example: Find the equation to the right circular cylinder whose guiding curve passes through the points $(1,0,0),(0,1,0),(0,0,1)$.

Solution: Let the given points be $A(1,0,0)$, $B(0,1,0)$, $C(0,1,0)$ and 0 is $(0,0,0)$.

(1) We find a circle through A, B, C axis of the cylinder

(2) Its radius and apply the standard formula

(1) Sphere through O, A, B, C is given by

$$x^2+y^2+z^2-x-y-z=0 \cdots\cdots (1)$$

Plane ABC in the intercept from is

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=1 \Rightarrow x+y+z=1 \cdots\cdots (2)$$

centre of (1) is $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$

$\therefore$ Axis is gieven by

$$\frac{x-\frac{1}{2}}{1}=\frac{y-\frac{1}{2}}{1}=\frac{z-\frac{1}{2}}{1} \qquad (\because \text{it is normal to } (2))$$

(2) radius of the circle

ΔABC is equilateral, centre = G $\qquad \therefore$ centre is P $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

radius is $PA=\sqrt{\left(1-\frac{1}{3}\right)^2+\left(0-\frac{1}{3}\right)^2+\left(0-\frac{1}{3}\right)^2}=\sqrt{\frac{4+1+1}{9}}=\frac{\sqrt{2}}{\sqrt{3}}$

Equation to the circular cylinder is given by

$$\left(\ell^2+m^2+n^2\right)\left((x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r^2\right)=\left[\ell(x-x_1)+m(y-y_1)+n(z-z_1)\right]^2$$

$$\Rightarrow (1+1+1)\left(\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+\left(z-\frac{1}{2}\right)^2-\frac{2}{3}\right)=\left(x-\frac{1}{2}+y-\frac{1}{2}+z-\frac{1}{2}\right)^2$$

$$\Rightarrow 3\left(x^2+y^2+z^2-x-y-z+\frac{3}{4}-\frac{2}{3}\right)$$

$$= \left(x+y+z-\frac{3}{2} \right)^2$$

$$\Rightarrow 3x^2+3y^2+3z^2-\cancel{3x-3y-3z}-2+\cancel{\frac{9}{4}}$$

$$= (x+y+z)^2 - \cancel{3(x+y+z)} + \cancel{\frac{9}{4}}$$

$$\Rightarrow 2x^2+2y^2+2z^2-2xy-2yz-2zx-2=0$$

The equation to the circular cylinder is

$$x^2+y^2+z^2-xy-yz-zx=1$$

9.7 Summary:

Starting with the generator definition of a cylinder we have observed conditions for second degree equation to represent a cylinder; we derived equations for all types of cylinders such as those with guiding curves, enveloping cylinder for a sphere and right circular cylinder.

We may note that, we have not gone deeper in the topic when compared to the sphere and cone.

9.8 Technical Terms:

Cylinder

Axis

Generator

Guiding curve

Enveloping cylinder

Right Circular Cylinder

9.9 Exercises:

1. Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and passes through the curve $x^2+y^2=16, \; z=0$.

2. A cylinder cuts the plane $z=0$ in the curve $x^2+\frac{y^2}{4}=\frac{1}{4}$ and has its axis parallel to $3x=-6y=2z$. Find its equation.

3. Find the equation of the quadratic cylinder whose generators intersect the curve $ax^2+by^2=2z$ and $\ell x+my+nz=P$ and parallel to the z - axis.

4. Find the equation fo the cylinder whose generators are parallel to the x - axis and pass through the curve $x^2+y^2+2z^2=1, \quad x-3y+z=4$.

5. Find the equation of the right circular cylinder whose axis is $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{2}$ and radius 2.

6. Find the equation to the right circular cylinder whose guiding circle is $x^2+y^2+z^2=9,\ x-y+z=3$.

7. A right circular cylinder is such that its section by the plane $x+y-z=0$ is a circle of radius 3 and centre $(1,2,3)$. Find its equation.

8. Find the equation of the cylinder whose generators are parallel to the line $x=\frac{-y}{2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1$ and $z=3$.

9. Find the cartesian equation of the right circular cylinder whose axis is the z - axis and radius a.

9.10 Answers:

1. $9x^2+9y^2+5z^2-6zx-12yz-144=0$

2. $36x^2+9y^2+17z^2+6yz-48zx=9$

3. $n\left(ax^2+by^2\right)+2\ell x+2my-2p=0$

4. $10y^2-6yz+3z^2+24y-8z+15=0$

5. $10x^2+5y^2+13z^2-6yz-4zx-12xy-8x+30y-74z+59=0$

6. $x^2+y^2+z^2+xy+yz-zx-9=0$

7. $2\left(x^2+y^2+z^2-xy+yz+zx\right)-6x-12y-18z+15=0$

8. $3x^2+6y^2+3z^2+8yz-2zx+6x-24y-18z+24=0$

9. $x^2+y^2=a^2,\ \ z=0$

9.11 Model Examination Questions:

1. (a) Definition of cylinder, generator and axis

   (b) Find the equation of the cylinder whose generators are parallel to X - axis and cut the curve $ax^2+by^2+cz^2=1$ and $\ell x+my+nz=p$.

   (c) Find the equation to the cylinder, having generators parallel to the line with directional ratios $(1,1,1)$ and touch the curve $xy=1$ and $z=1$.

2. (a) Define enveloping cylinder of a sphere.

   (b) The locus of the lines touching the sphere $x^2+y^2+z^2=a^2$ and parallel to the line $\frac{x}{\ell}=\frac{y}{m}=\frac{z}{n}$ is the cylinder represented by

$$\left(\ell^2+m^2+n^2\right)\left(x^2+y^2+z^2-a^2\right)=(\ell x+my+nz)^2$$

   (c) Find the equation to the right circular cylinder whose axis is $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$ having the radius 2 units.

9.12 Model Practical Problem With Solution:

Find the equation of the right circular cylinder whose axis is the line $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$ and radius is 2.

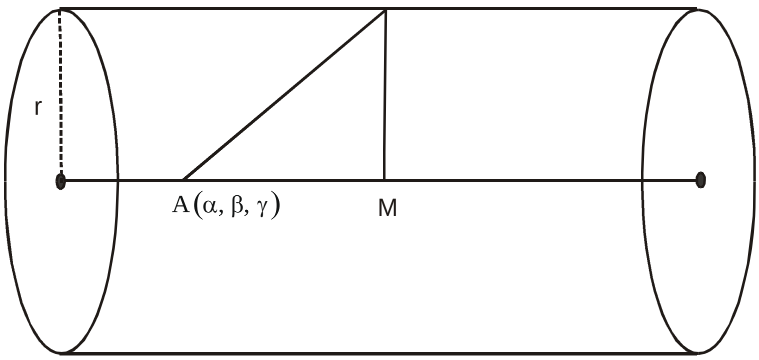
Definition:

Let C be a circle of radius r in a plane π and L be the normal to the plane π through the centre of C. The surface Y consisting of lines parallel to L equivalently $\perp$ to π and intersecting π on C is called right circular cylinder with centre 0 and radius r.

S = { $L \mid L$ is perpendicular to π and L intersects π on C}

Result:

The equation to the right circular cylinder whose radius is the line $\frac{x-\alpha}{\ell}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is given by $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\frac{\{\ell(x-\alpha)+m(y-\beta)+n(z-\gamma)\}^2}{\ell^2+m^2+n^2}+r^2$.



$$AP^2=AM^2+MP^2=AM^2+r^2$$

Solution:

Let AB be the axis of the cylinder,

$A=(1, 0, 3)$ then the equations of AB are

$$\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$$

So that the direction ratio's of AB are 2, 3, 1.

Dividing each by $\sqrt{14}=\sqrt{2^2+3^2+12}$ gives the d.c.'s of AB to be $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

Let $P(x, y, z)$ be any point on the cylinder. Join PA and draw PM $\perp$ on the axis so that PM = radius of cylinder = 2.

Then in rt $\underline{|d}$ Δ AMP

$$AP^2=AM^2+MP^2=AM^2+4 \cdots\cdots\cdots\cdots (1) \qquad \because MP=2$$

Now $AP^2 = (x-1)^2 + (y-0)^2 + (z-3)^2$

and AM = projection of AP on the line AB whose actual d.c.'s are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$

$$= \frac{2}{\sqrt{14}}(x-1) + \frac{3}{\sqrt{14}}(y-0) + \frac{1}{\sqrt{14}}(z-3)$$

$$= \frac{2x+3y+z-5}{\sqrt{14}}$$

Putting these values of AP^2 and AM^2 in (1) we have

$$(x-1)^2 + y^2 + (z-3)^2 = \frac{(2x+3y-z-5)^2}{14} + 4$$

$$\Rightarrow 14\left[(x-1)^2 + y^2 + (z-3)^2\right] = (2x+3y-z-5)^2 + 56$$

$$= (2x+3y+z)^2 + 25 - 10(2x+3y+z) + 56$$

$$\Rightarrow 14\left(x^2 - 2x + 1 + y^2 + z^2 + 9 - 6z\right)$$

$$= 4x^2 + 9y^2 + z^2 + 12xy + 6yz + 4zx + 25 - 20x - 30y - 10z + 56$$

$$= 10x^2 + 5y^2 + 13z^2 - 6yz - 4zx - 12xy - 8x + 30y - 74z + 59 = 0$$

which is the required equation.

Lesson Writer

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Lesson - 10

CENTRAL CONICOIDS

10.1 Objective of the lesson:

In this lesson the student is introduced to the concept of a central conicoid, some special types of conicoids and a few basic properties of a central conicoid.

10.2 Structure:

This lesson contains the following components:

10.3 Definitions and Examples of a Conicoid

10.4 The Ellipsoid

10.5 The Hyperboliod of One Sheet

10.6 The Hyperboloid of Two Sheets

10.7 Central Conicoid

10.8 The Enveloping Cone

10.9 The Enveloping Cylinder

10.10 Summary

10.11 Technical Terms

10.12 Exercises

10.13 Answers

10.14 Model Examination Questions

10.3 Definitions and Examples:

Let us recall that if $f: R^3 \to R$ is a polynomial of degree $n > 0$ then the set $S = \{(x, y, z) / f(x, y, z) = 0\}$ is a surface of degree 'n'.

Examples of first degree surfaces are planes a second degree surface is called a quadric, some examples of quadrics are pair of lines, pair of planes, cones, cylinders spheres and so on.

A quadric is said to be degenarate if it reduces to one of the following (1) the empty set ϕ; (2) a single point set; (3) a line; (4) a pair of lines in a plane; (5) a plane.

10.3.1 Definition: A non degenerate quadric is called a conicoid.

10.3.2 Definition: A quadric is called a central conicoid if its defining equation is of the form $ax^2 + by^2 + cz^2 = 1$, where a, b, c are real numbers.

10.3.3 Definition: Centre: A point $V(\alpha, \beta, \gamma)$ is said to be the centre of a conicoid 'S' if $(x, y, z) \in S \Rightarrow (2\alpha - x,\ 2\beta - y,\ 2\gamma - z) \in S$.

i.e., The point which bisectes all the chords through it is called the centre of the conicoid.

10.3.4 Definition: Principle Planes: A plane through the centre of a conicoid is called a principle plane if it bisects all the chords of the conicoid, perpendicular to it.

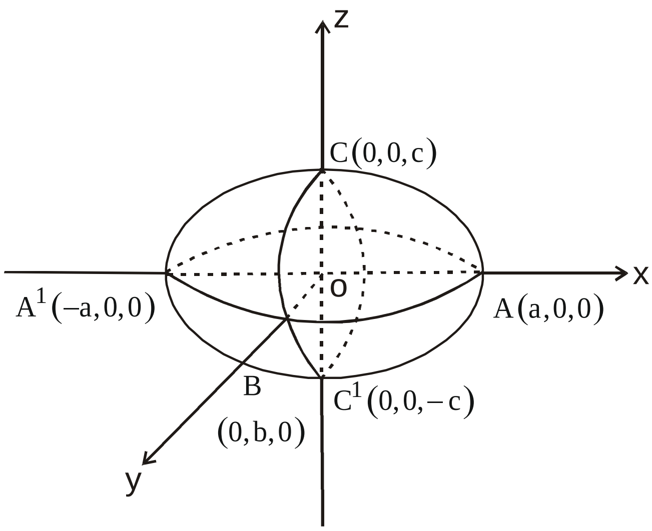
10.3.5 Definition: Principle Axes: The lines of intersections of the principle planes taken in pairs are called the principle axes:

10.4 The Ellipsoid:

10.4.1 The surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\ a,\ b,\ c \in R$$ is called an ellipsoid

10.4.2 Properties of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



(a) The points $(\pm a, 0, 0)$, $(0, \pm b, 0)$, $(0, 0 \pm c)$ lie on 'S' and also on the principle axes. These are called the vertices of S. Also $(x, y, z) \in S \Rightarrow |x| \le |a|, \ |y| \le |b|, \ |z| \le |c|$. Thus the surface S is a closed surface.

(b) Origin is the centre of the ellipsoid 'S' since $(x, y, z) \in S \Rightarrow (0 \pm x, 0 \pm y, 0 \pm z) \in S$

(c) Ellipsoid is symmytrical about the co ordinate planes. Also every co ordinate plane bisects the chords of S, perpendicular to the plane, for example $(x, y, z) \in S \Rightarrow (x, y, -z) \in S$. So xy plane bisects all the chords perpendicular to it

So the co ordinate planes are principle planes of 'S'.

(d) The co ordinate axes are the principle axes, since they are the lines of intersection of the principle planes.

(e) The sections of the coordinate planes with S are ellipses,

(1) Section with XOY plane is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$

(2) Section with YOZ plane is $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad x = 0$

(3) Section with XOZ plane is $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \quad y = 0$

(f) For any curve $z = k, \quad |k| \le |c|$ the plane section of the ellipsoid 'S' is the ellipse S_k given by $S_k : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, \ z = k$. Thus $S = \underset{|k| < c}{U} S_k$

(g) Length of the principle axes: $AA' = 2a, \ BB' = 2b, \ CC' = 2c$

10.4.3 Note (1): The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$, represents a vacuous ellipsoid. Here $S = \phi$.

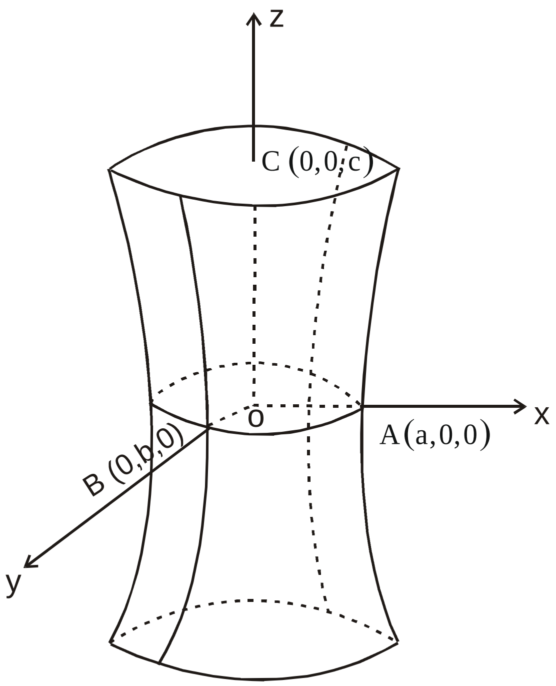
Note (2): If in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$a^2 = b^2$ or $b^2 = c^2$ or $c^2 = a^2$ then it is called ellipsoid of revolution.

10.5 The Hyperboloid of One Sheet:

10.5.1 Definition: The hyperboloid of one sheet is the surface 'S' represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \cdots\cdots\cdots\cdots (5)$$



(a) Origin is the centre, since it bisects all chords through it.

(b) Symmetry: Since $(x,y,z)\in S \Rightarrow (\pm x, \pm y, \pm z)\in S$. Hence S is symmetric about the coordinate planes.

Also the coordinate planes bisects all chords perpendicular to it. Hence they are the principle planes of S.

The lines of intersection of the principle planes are the coordinate axes therefore coordinate axes are the principle axes.

(c) Intercepts on axes: The surface S meets X axis in

$$y=0,\ z=0 \text{ i.e. } \frac{x^2}{a^2}=1 \Rightarrow x=\pm a$$

$\therefore A(a,0,0),\ A^1(-a,\ 0,\ 0)$ are the points of intersection on X axis

If $x=0,\ z=0 \Rightarrow y^2=b^2$ $\therefore B(0,b,0)$ $B^1(0,-b,0)$ are the points of intersection on y axis.

$x=0,\ y=0 \Rightarrow z^2=-c^2$ $\therefore$ The surface S does not meet the z axis at all.

(d) Section by the coordinate planes

(1) with XY plane in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse in XY plane

(2) with YZ plane is $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is a hyperbola in YZ plane

(3) with ZX plane is $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ is a hyperbola in ZX plane

(e) Sections by the planes

(1) with the plane $z=k,\ \ (k \in R)$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$$

$$\text{i.e. } \frac{x^2}{a_k^2}+\frac{y^2}{b_k^2}=1$$

where $a_k^2=\left(1+\frac{k^2}{c^2}\right)a^2$ and $b_k^2=\left(1+\frac{k^2}{c^2}\right)b^2$

(2) These are ellipses. With the plane $x=k,\ |k|\le a$ are hyperbolas

$$S_k:\ \frac{y^2}{b_k^2}-\frac{z^2}{c_k^2}=1,$$

where $b_k^2=\left(1-\frac{k^2}{a^2}\right)b^2,\quad c_k^2=\left(1-\frac{k^2}{a^2}\right)c^2$

(3) With plane $y = k, \ |k| \le b$, are hyperbolas

$$T_k : \frac{x^2}{a_k^2} - \frac{z^2}{c_k^2} = 1, \text{ where } a_k^2 = \left(1 - \frac{k^2}{b^2}\right) a^2$$

$$c_k^2 = \left(1 - \frac{k^2}{b^2}\right) c^2$$

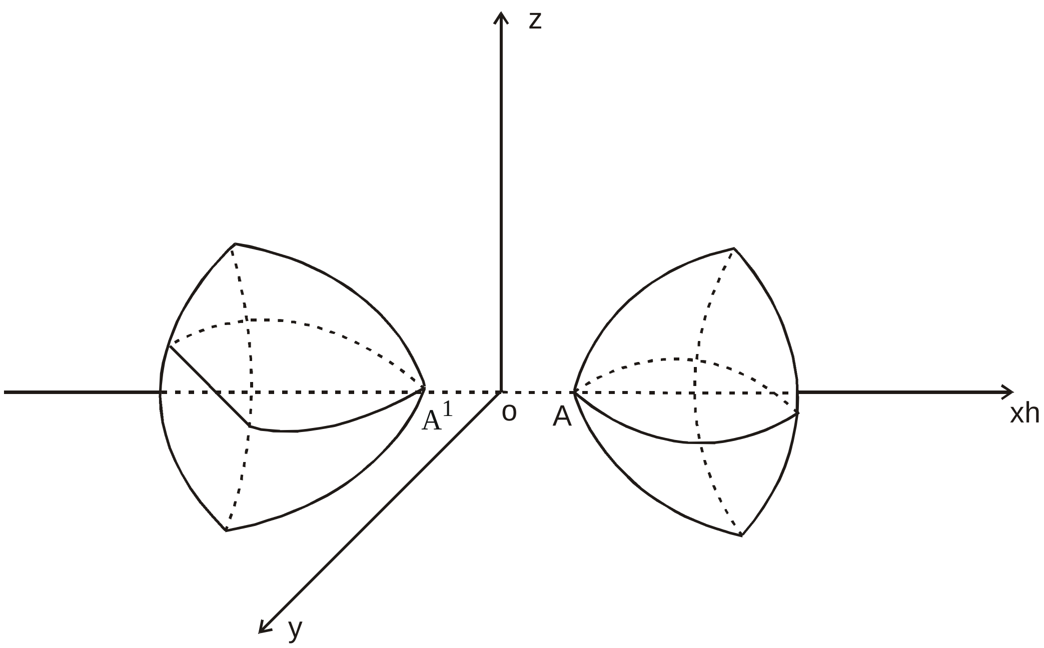
Also

$$S = \bigcup_{|k| \le a} S_k = \bigcup_{|k| \le b} T_k$$

10.6 Hyperbolid of Two Sheets:

10.6.1 Def: A hyperboloid of two sheets is the surface S represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \text{ where } a > 0, \ b > 0, \ c > 0$$



10.6.2 Properties of Hyperboloid of two sheets:

(i) The surface intersects the x-axis is $(a,0,0)$ and $(-a,0,0)$ but does not meet the y and z - axes.

$(x,y,z)\in S \Rightarrow (\pm x, \pm y, \pm z)\in S$, hence S is symmetric about the origin as well as coordinate planes.

(ii) It can be shown that if $b \neq c$ then there are no other planes of symmetry except the coordinate planes. While when $b = c$, in addition to the coordinate planes all other planes of symmetry are the planes passing through the x axis are planes of symmetry.

(iii) The plane $x = k$ does not cut S if $|k| < a$. If $|k| > a$ the plane $x = k$ cuts the surface in the ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1, \quad x = k$$

Thus the hyperboloid of two sheets is the surface generated by the ellipses

$$\frac{x^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1, \quad x = k \qquad (|k| > a)$$

(iv) The plane sections corresponding to $y = k$ are the hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1+\frac{k^2}{b^2}, \qquad y = k$$

and those corresponding to $z = k$ are

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1+\frac{k^2}{c^2}, \qquad z = k$$

10.7 Cental Conicoids:

We now discuss some properties of central conicoids. Let us write S for the central conicoid defined by the equation.

$$S(x,y,z) \equiv ax^2+by^2+cz^2-1=0 \cdots\cdots\cdots (1)$$

We know that when at the three coefficients a, b, c are +ve, S is an ellipsoid.

(1) $a > 0,\ b > 0,\ c < 0 \Rightarrow S$ is hyperboliod of one sheet with X axis as axis

(2) $a > 0,\ b < 0, c > 0 \Rightarrow S$ is hyperboliod of one sheet with Y axis as axis.

(3) $a<0,\ b >0,\ c > 0 \Rightarrow S$ is hyperboliod of one sheet with Z axis as axis

If two of the three coefficients are -ve then S represents hyperboloid of two sheets.

In in all these cases

(1) 0 is the centre

(2) Coordinate planes are the planes of symmetry

(3) Coordinate axes are the principle axes

Notation: We use the following notation in this lesson

$$S \equiv ax^2 + by^2 + cz^2 - 1$$

$$S_1 = ax \cdot x_1 + by \cdot y_1 + cz \cdot z_1 - 1$$

$$S_{11} = ax_1^2 + by_1^2 + cz_1^2 - 1$$

$$S_{ij} = ax_i x_j + by_i y_j + cz_i z_j - 1, \qquad (x_i, y_j, z_j) \in R^3$$

Definitions: As in the case of sphere, we define the following with respect to a central conical conicoid S.

(1) Tangent planes - Tangent Line (2) Polar Plane of a point

(3) The pole of a plane (4) Conjugate lines

(5) Conjugate planes (6) Conjugate points etc.

One can prove the following results for a central conicoid in the lines of their anologues for sphere.

As the proofs are completely similar, we merely state the results, without proofs.

(1) Equation to the tangent plane at (x_1, y_1, z_1) on S is $S_1 \equiv 0$

(2) A necessary and sufficient condition for a plane $\ell x + my + nz = p$ to be a tangent plane is

$$\frac{\ell^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

Also the point of contact is $\left(\dfrac{\ell}{ap},\ \dfrac{m}{bp},\ \dfrac{n}{cp}\right)$

(3) The polar plane of the point (x_1, y_1, z_1) with respect to S is $S_1 = 0$

(4) The pole of the plane $\ell x + my + nz = p$ with respect to S is $\left(\frac{\ell}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$.

(5) The plane of contact of the point (x_1, y_1, z_1) with respect to S is $S_1 = 0$

(6) A necessary and sufficient condition for two points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ to be congugate with repect to S is $S_{12} = 0$.

10.8 Enveloping Cone:

10.8.1 Theorem: The locus of a tangent line to a central conicoid $ax^2 + by^2 + cz^2 = 1$ from a point (x_1, y_1, z_1) is a cone.

Proof: The given central conicoid is $ax^2 + by^2 + cz^2 - 1 = 0 \cdots\cdots\cdots\cdots (1)$.

A line through the given point (x_1, y_1, z_1) is of the form

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r, \quad r \in R \cdots\cdots\cdots (2)$$

Any point on this line is of the form

$$(\ell r + x_1, mr + y_1, nr + z_1) \cdots\cdots\cdots (3)$$

Thus the line (2) is a tangent to (1) iff the quadratic in r

$a(\ell r + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 = 1$, has a unique root.

$$\Leftrightarrow \left(a\ell^2 + bm^2 + cn^2\right)r^2 + 2(a\ell x_1 + bmy_1 + cnz_1)r + \left(ax_1^2 + by_1^2 + cz_1^2 - 1\right) = 0$$

has a double root, and $a\ell^2 + bm^2 + cn^2 \neq 0$

$$\Leftrightarrow (a\ell x_1 + bmy_1 + cnz_1)^2 = \left(a\ell^2 + bm^2 + cn^2\right)\left(ax_1^2 + by_1^2 + cz_1^2 - 1\right) \cdots\cdots\cdots\cdots\cdots (4)$$

form (2) and (4)

We have the revised condition as

$$\left[ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)\right]^2=\left[a(x-x_1)^2+b(y-y_1)^2+c(z-z_1)^2\right]$$

$$\left[ax_1^2+by_1^2+cz_1^2-1\right]\cdots\cdots\cdots\cdots(5)$$

Shifting the origin to the point (x_1, y_1, z_1) the above condition becomes

$$(axx_1+byy_1+czz_1)^2=\left(ax_1^2+by_1^2+cz_1^2-1\right)\left(ax^2+by^2+cz^2\right)$$

Which is homogenous in x, y, z hence represents a cone with vertex at (x_1, y_1, z_1).

Hence the locus of the tangent line is a cone.

10.8.2 Definition: Enveloping Cone: The locus of the tangent lines to quadric through any point is a cone, which is called on enveloping cone to the central conicoid.

10.8.3 Note: Notation for enveloping Cone:

The above equation (5) may be written as

$$(S_1-S_{11})^2=[S-2S_1+S_{11}]\cdot S_{11}$$

i.e. $S_1^2-2S_1\cdot S_{11}+S_{11}^2=S\cdot S_{11}-2S_1\cdot S_{11}+S_{11}^2$

i.e. $S_1^2=S\cdot S_{11}$

Thus the equation to the enveloping cone to the conicoid $S\equiv 0$ is given by $S\cdot S_{11}=S_1^2$ for which the vertex is (x_1, y_1, z_1).

10.8.4 Example: A point P moves so that the section of the enveloping cone of the ellipsiod $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with P as the vertex, by the plane $z=0$ is a circle. Find the locus of 'P'.

Solution: The given conicoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\cdots\cdots\cdots(1)$

Let the given point be $P(x_1, y_1, z_1)$. Then the equation to the enveloping cone of (1), with vertex at P is given by $S\cdot S_{11}=S_1^2$.

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\cdot\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left[\frac{x\cdot x_1}{a^2}+\frac{y\cdot y_1}{b^2}+\frac{z\cdot z_1}{c^2}-1\right]^2 \cdots\cdots\cdots(2)$$

It meets the plane $z=0$ in

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left[\frac{x\cdot x_1}{a^2}+\frac{y\cdot y_1}{b^2}-1\right]^2, \qquad z=0\cdots\cdots\cdots(3)$$

The curve (3) represents a circle if

(1) Coefficient of a^2 = Coefficient of y^2

(2) Coefficient of $xy=0$

i.e., if $\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)\cdots\cdots\cdots(4)$

and $\frac{2x_1y_1}{a^2b^2}=0\cdots\cdots\cdots(5)$

$(5)\Rightarrow x_1=0 \quad$ or $\quad y_1=0$

Case (1) $x_1=0$ and $\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$

$$\Rightarrow \frac{y_1^2}{a^2b^2}+\frac{z_1^2}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)=\frac{1}{a^2}-\frac{1}{b^2}$$

$$\Rightarrow \frac{y_1^2}{a^2-b^2}=\frac{z_1^2}{c^2}-1$$

$\Rightarrow$ locus of $P(x_1, y_1, z_1)$ is given by

$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1, \qquad x=0$$

Case (2) $y_1 = 0$ and $\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$

$$\Rightarrow \frac{z_1^2}{c^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+\frac{x_1^2}{a^2b^2}=\frac{1}{b^2}-\frac{1}{a^2}$$

$$\Rightarrow \frac{z_1^2}{c^2}+\frac{x_1^2}{a^2-b^2}=1,$$

$\therefore$ locus of $P(x_1, y_1, z_1)$ is given by

$$\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1, \qquad y=0$$

10.8.5 Example: The section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, with vertex at P by the plane $z=0$ is (1) a parabola (2) a rectangular hyperbola, find the locus of P.

solution: Given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \cdots\cdots\cdots\cdots(1)$

Given point is $P(x_1, y_1, z_1)$

Equation to the enveloping cone of (1) with vertex at P is given by $S\cdot S_{11}=S_1^2$

$$\Rightarrow\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left[\frac{x\cdot x_1}{a^2}+\frac{y\cdot y_1}{b^2}+\frac{z\cdot z_1}{c^2}-1\right]^2 \cdots\cdots\cdots\cdots(2)$$

This meets the plane $z=0$ in

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left[\frac{x\cdot x_1}{a^2}+\frac{y\cdot y_1}{b^2}-1\right]^2 \cdots\cdots\cdots\cdots(3)$$

$$\Rightarrow \frac{x^2}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{y^2}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= + \frac{2xy \cdot x_1 \cdot y_1}{a^2 + b^2} + \cdots\cdots\cdots\cdots\cdots$$

(Remaining terms are lefttout as they do not occur in our computation)

Comparing with $Ax^2 + By^2 + 2Hxy + 2Gx + 2Fy + C$

We have

$$A = \frac{1}{a^2}\left(\frac{x_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right), \qquad B = \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1\right),$$

$$H = \frac{x_1 y_1}{a^2 b^2}$$

Case (1), (3) represents a parabola if $H^2 = AB$

i.e., $\frac{1}{\cancel{a^2 b^2}}\left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1\right) = \frac{x_1^2 \cdot y_1^2}{\cancel{a^2 b^2} \cdot a^2 b^2}$

$$\Rightarrow \frac{x_1^2 \cancel{y_1^2}}{\cancel{a^2} \cdot b^2} + \frac{z_1^2}{c^2}\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) - 1\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \frac{x_1^2 \cancel{y_1^2}}{\cancel{a^2} b^2}$$

$$\Rightarrow \left(\frac{z_1^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = 0$$

$\because$ $S_{11} \neq 0$, the locus is given by $z^2 = c^2$, a pair of planes

Case (2) If (3) represents a rectangular hyperbola in XY palne then

$$A + B = 0 \Rightarrow \frac{1}{a^2}\left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) + \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1\right) = 0$$

$$\Rightarrow \frac{x_1^2 + y_1^2}{a^2 b^2} + \frac{z_1^2}{c^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \frac{1}{a^2} + \frac{1}{b^2}$$

$$\Rightarrow x_1^2 + y_1^2 + \frac{z_1^2}{c^2}\left(a^2 + b^2\right) = a^2 + b^2$$

$$\Rightarrow \frac{x_1^2 + y_1^2}{a^2 + b^2} + \frac{z_1^2}{c^2} = 1$$

Locus of the point $P(x_1, y_1, z_1)$ is

$$\frac{x^2 + y^2}{a^2 + b^2} + \frac{z^2}{c^2} = 1$$

10.8.6 find the locus of the point P from which three mutually perpendicular tangents can be drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$.

Solution: The given conicoid is $ax^2 + by^2 + cz^2 = 1 \cdots\cdots (1)$

Let $P(x_1, y_1, z_1)$ be eny point from which three mutually perpendicular tangents can be drawn to (1). Then the enveloping cone is given by $S \cdot S_{11} = S_1^2$.

i.e., $\left(ax^2 + by^2 + cz^2 - 1\right)\left(ax_1^2 + by_1^2 + cz_1^2 - 1\right) = \left(axx_1 + byy_1 + czz_1 - 1\right)^2 \cdots\cdots\cdots (2)$

(2) posseses three muctually perpendicular generators if $A + B + C = 0$

i.e., coeff of $x^2 + \text{coeff} \quad \text{or} \quad y^2 + \text{coeff} \quad \text{of} \quad z^2 = 0$

$$\Rightarrow (a + b + c)\left(ax_1^2 + by_1^2 + cz_1^2 - 1\right) = a^2x_1^2 + b^2y_1^2 + c^2z_1^2$$

$$\Rightarrow a\left(by_1^2 + cz_1^2 - 1\right) + b\left(zx_1^2 + cz_1^2 - 1\right) + c\left(ax_1^2 + by_1^2 - 1\right) = 0$$

$$\Rightarrow (ab + ac)x_1^2 + (bc + ba)y_1^2 + (ca + cb)z_1^2 = a + b + c$$

locus of the point (x_1, y_1, z_1) is

$$a(b + c)x^2 + b(c + a)y^2 + c(a + b)z^2 = a + b + c$$

10.9 Enveloping Cylinder:

10.9.1 **Theorem:** The locus of the tangent line to a central conicoid and parallel to a given line is a cylinder.

Proof: Let $S \equiv ax^2 + by^2 + cz^2 - 1 = 0 \cdots\cdots\cdots\cdots (1)$ represents a central conicoid and (ℓ, m, n) be the directional ratios of a line L and $P(x_1, y_1, z_1) \in R^3$. Then equation of any line through P which is parallel to L are

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r, \qquad r \in R \cdots\cdots\cdots (2)$$

The line (2) is a tangent to (1) if there is a unique $r \in R$ such that

$$a(\ell r + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 = 1$$

i.e., $\left(a\ell^2 + bm^2 + cn^2\right)r^2 + 2(a\ell x_1 + bmy_1 + cnz_1)r + \left(ax^2 + by^2 + cz^2 - 1\right) = 0$ has a double root.

i.e., if $(a\ell x_1 + bmy_1 + cnz_1)^2 = \left(a\ell^2 + bm^2 + cn^2\right)\left(ax_1^2 + by_1^2 + cz_1^2 - 1\right)$

The locus of P is

$$(a\ell x + bmy + cnz)^2 = \left(a\ell^2 + bm^2 + cn^2\right)\left(ax^2 + by^2 + cz^2 - 1\right) \cdots\cdots\cdots (3)$$

By the definition of a cylinder (3) is a cylinder whose generators are parallel to the line with d.rs. (ℓ, m, n).

Hence the theorem.

10.9.2 Definition: Given a central conicoid S and a line L. The locus of the tangent line to S, parallel to L is a cylinder, which is called the enveloping cylinder of the quadric S.

10.9.3 Example: Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, whose generators are parellel to the lines $\frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c}$, meet the plane $z = 0$ in circles.

Solution: The given ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \cdots\cdots\cdots (1)$$

The given lines are

$$\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}\cdots\cdots\cdots\cdots(2)$$

We know that the enveloping cylinder of $ax^2+by^2+cz^2=1$

Whose generators are parallel to the line with d.rs. ℓ, m, n is given by

$$(a\ell x+bmy+cnz)^2=\left(a\ell^2+bm^2+cn^2\right)\left(ax^2+by^2+cz^2-1\right)$$

Here $(\ell, m, n)=\left(0,\ \pm\sqrt{a^2-b^2},\ c\right)$

$$(a,b,c)=\left(\frac{1}{a^2},\frac{1}{b^2},\frac{1}{c^2}\right)$$

$\therefore$ Enveloping Cylinder is given by

$$\left[0\pm\frac{\sqrt{a^2-b^2}\,y}{b^2}+\frac{cz}{c^2}\right]^2=\left(0+\frac{a^2-b^2}{b^2}+\frac{1}{c^2}\cdot c^2\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\Rightarrow\left[\frac{z}{c}\pm\frac{\sqrt{a^2-b^2}\;y}{b^2}\right]^2=\left(\frac{a^2\cancel{-b^2}\cancel{+b^2}}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\cdots\cdots(3)$$

(3) meet the plane $z=0$ is

$$\frac{a^2-b^2\;y^2}{\left(b^2\right)^{\cancel{2}}}=\frac{a^2}{b^{\cancel{2}}}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

$$\Rightarrow\cancel{\frac{a^2}{b^2}y^2}-y^2=x^2+\cancel{\frac{a^2}{b^2}y^2}-a^2$$

$\Rightarrow z=0$, $x^2+y^2=a^2$ which is a circle.

10.9.4 Example: Find the locus where the enveloping cylinder of $ax^2+by^2+cz^2-1=0$ with generators perpendicular to z -axis meet the plane $z=0$.

Solution: The given conicoid is $ax^2 + by^2 + cz^2 - 1 = 0 \cdots\cdots\cdots (2)$

The enveloping cylinder in generated by the lines perpendicular to the z axis. Therefore its drs are $(1,1,0)$ equation to the enveloping cylinder is given by

i.e. $(a\ell x + bmy + cnz)^2 = \left(a\ell^2 + bm^2 + cn^2\right)\left(ax^2 + by^2 + cz^2 - 1\right)$

Here $(\ell, m, n) = (1,1,0)$

$\therefore$ Enveloping cylinder meet z = 0 in

$$(ax + by)^2 = (a + b)\left(ax^2 + by^2 - 1\right)$$

$$\Rightarrow \cancel{a^2x^2} + \cancel{b^2y^2} + 2abxy = \cancel{a^2x^2} + \cancel{b^2y^2} + ab\left(x^2 + y^2\right) - (a + b)$$

$$\Rightarrow ab\left(x^2 + y^2 - 2x + y\right) = a + b$$

$\Rightarrow (x - y)^2 = \dfrac{(a + b)}{ab}$, which is the required locus.

10.10 Summary:

In this lesson we learnt how to draw a rough sketch of the ellipsoid, hyperboloids of one sheet as well as of two sheets using some of their salient features. We also learnt some properties of a centrel conicoid, its enveloping cone and enveloping cylinder.

10.11 Technical Terms:

Central Conicoid

Ellipsoid

Hyperboloids of one and two sheets

10.12 Exercise:

1. Find the equations to the tangent planes to the conicoid $3x^2 - 6y^2 + 9z^2 = 17$ parallel to the plane $x + 4y - 2z = 8$.

2. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$. Find the point of contact.

3. Find the equations to the tangent planes which pass through the line

$$\frac{x}{3}=\frac{y-3}{-3}=\frac{z}{1} \text{ and touching } \frac{x^2}{6}+\frac{y^2}{3}+\frac{z^2}{2}=1$$

4. A tangent plane to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ makes equal angles with the coordinate planes. Show that it forms with them a tetrahedron of volume $\frac{1}{6}\left(a^2+b^2+c^2\right)^{3/2}$.

5. Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, whose generators are parallel to the line $\frac{x}{0}=\frac{y}{\sqrt{a^2-b^2}}=\frac{z}{c}$ meets the plane $z=0$ in a circle.

6. Find the enveloping cylinder of the sphere $x^2+y^2+z^2-2x+4y-1=0$ having its generators parallel to the line $x=y=z$.

7. Find the locus of the feet of the perpendiculars from the origin to the tangent planes to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which cut off from the axis, intercepts the sum of whose reciprocals is equal to $\frac{1}{k}$.

8. Show that the plane $x+2y+3z=2$ touches the conicoid $x^2-2y^2+3z^2=2$ and find the point of contact.

10.13 Answers:

1. $3x+12y-6z=\pm 17$

2. $\left(-1, 2, \frac{2}{3}\right)$

3. $x+y=3, \quad x+2y+3z=6$

6. $x^2+y^2+z^2-yz-zx-4x+5y-z-2=0$

8. $(1, -1, 1)$

10.14 Model Examination Questions:

1. (a) Define ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets.

   (b) A point P moves so that the section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with P as the vertex by the plane $z=0$ in a circle. find the locus of P.

   (c) Find the locus of the point P from which three mutually perpendicular tangents can be drawn to the conicoid $ax^2+by^2+cz^2=1$.

2. (a) Define hyperboloid of one sheet

   (b) The section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ with vertex at P by the plane $z=0$ in a parabola. Find the locus of P.

   (c) Find the locus where the enveloping cylinder $ax^2+by^2+cz^2=1$ with generators perpendicular to Z - axis meet the plane $z=0$.

Lesson Writer

G. NARAYANA

Bolzano (1781 - 1848)

Bolzano successfully freed calculus from the concept of the infinitesimal. He also gave examples of 1 - 1 correspondences between the elements of an infinite set and the elements of a proper subset.

Lesson - 11

SEQUENCES - I

11.1 Objective of the lesson:

The objective of the lesson is to acquaint the student with sequences, limits of sequences and some properties of the limits.

11.2 Structure:

This lesson contains the following components:

11.3 Introduction

11.4 Sequences

11.5 The limit of a Sequence

11.6 Limit Theorems

11.7 Answers to SAQ's

11.8 Summary

11.9 Technical Terms

11.10 Exercises

11.11 Answers

11.12 Model Examination Questions

11.13 Model Practical Problem with Solution

11.3 Introduction:

The analysis of a real valued function of real variable is based on continuity, differentiation, integration and so on. All these notions, even though involve the continuous real variable, can be explained better in terms of a positive integer variable. The irrational numbers which encounter very frequently are usually approximated by rational numbers with which one is more comfortable. Several considerations, including those mentioned above prompt the beginner to gain acquaintance with sequences.

This section is an introduction to sequences, their limits and properties of the limits.

11.4 Sequences:

11.4.1 Functions whose domain is the set of natural numbers are of significant importance. If A is any set, by a sequence in A we mean a function $f : \mathbb{N} \to A$. The value of a natural number n under f is denoted by $f(n)$ or f_n. The sequence f itself is denoted by $f = (f_n)$; $\{f_n / n \in \mathbb{N}\}$ or $\{f_1, f_2, \cdots\cdots\cdots, f_n, \cdots\cdots\}$. f_n is called the n^{th} term of the sequence f. We mostly use capital letters $A, B, C, \cdots\cdots\cdots X, Y, Z \cdots\cdots$ and so on for sequences.

A sequence, being a function has range, the collection of all the values. This range may be finite or infinite. For example the range of the sequence f defined by $f(n) = (-1)^n$ is the set $\{-1, 1\}$. However we specify a sequence by its terms but not by the range. We arrange the terms in the order of the natural numbers and represent the sequence $\{a_n\}$ by the arrangement $A = \{a_1, a_2, \cdots\cdots, a_n \cdots\cdots\}$.

We donot ignore any term even if it is repeated several times. Thus for a sequence the function or its range are not important but the placement of the values is significant.

11.4.2 Example:

List the first four terms of the sequence $X = (x_n)$ when

$$x_1 = 2 \quad \text{and} \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

$$n=1\ :\ x_2=\frac{1}{2}\left(x_1+\frac{2}{x_1}\right)=\frac{1}{2}(2+1)=\frac{3}{2}$$

$$n=2: x_3=\frac{1}{2}\left(x_2+\frac{2}{x_2}\right)=\frac{1}{2}\left(\frac{3}{2}+2\cdot\frac{2}{3}\right)=\frac{17}{2}$$

$$n=3:\quad x_4=\frac{1}{2}\left(\frac{17}{12}+2\cdot\frac{2}{17}\right)=\frac{577}{408}$$

Inductive (recursive) definition: A sequence may be defined by specifying a finite number of terms and expressing the general term, in terms of the preceding terms. This definition is called the inductive or recursive definition of the sequence.

Example: The famous Fibonacci sequence is inductively defined as follows:

$F_1=1,\ \ F_2=2$ and $F_n=F_{n-1}+F_{n-2}$ $\qquad (n>2)$

From this definition the first few terms can be seen to be 1, 2, 3, 5, 8, 13, 21, 34 and so on.

Caution: It is in bad taste to just specify a few terms of a sequence and leave with dots without specifying the general terms.

For example 1, 2, 3, 5, 8 could be the first five terms of the sequence (x_n) defined by $x_1=1, x_2=2$ and $x_{n+1}=x_n+(n-1)$ for $n>2$. These terms are the first five terms of the Fibonacci sequence (F_n) as well.

However $x_6=x_5+4=12$ while $F_6=13$

Example: Assuming natural pattern indicated by the terms provided, find the general term of the following sequences.

(a) 5, 7, 9, 11, ...

Here starting with the first term 5 we are adding 2 successively. Thus

$$x_1=5,\ x_2=5+2,\ x_3=5+2+2,\ x_4=5+2+2+2$$

Thus $x_3=5+2.2,\ \ x_4=5+3.2$ and so on that $x_{n+1}=5+2n$

We may also describe the sequence inductively by setting $x_1=5$ and $x_{n+1}=x_n+2$.

(b) $\frac{1}{2},\ -\frac{1}{4},\frac{1}{8},-\frac{1}{16}.............$

Here $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{4}$, $x_3 = \frac{1}{8}$, $x_4 = -\frac{1}{16}$ and so on.

Since $x_2 = -\frac{1}{2}x_1$, $x_3 = -\frac{1}{2}x_2$, $x_4 = -\frac{1}{4}x_3$ we may define $x_{n+1} = -\frac{1}{2}x_n$

We may also define $x_{n+1} = (-1)^{n-1} \cdot \left(\frac{1}{2}\right)^n$ for $n \geq 1$.

11.4.3 Defintion: Let $X = (x_n)$ be a sequence in $\mathbb{R}$. We say that

(i) X is monotonically increasing or simply increasing if $x_n \leq x_m$ whenever $n \leq m$, $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

i.e., $x_1 \leq x_2 \leq x_3 \leq \cdots\cdots\cdots$

(ii) X is monotonically decreasing or simply decreasing if $x_n \geq x_m$ whenever $n \leq m$, $m \in \mathbb{N}$ and $m \in \mathbb{N}$.

i.e., $x_1 \geq x_2 \geq x_3 \geq \cdots\cdots\cdots \geq x_n \geq \cdots\cdots\cdots$

(iii) X is stritly (monotonically) increasing if $x_n < x_m$ whenever $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $n \leq m$.

(iv) X is strictly (monotonically) decreasing if $x_n > x_m$ whenever $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

(v) X is monotone if X is either monotonically increasing or X is monotonically decreasing.

11.4.4 Example: Any constant sequence, i.e., for any $k \in \mathbb{R}$ the sequence (x_n) defined by $x_n = k$ for all $n \in \mathbb{N}$, is monotonic.

Reason: If $k \in \mathbb{R}$ the sequence $\{k, k, \cdots\cdots\cdots, k \cdots\cdots\cdots\}$ is monotonically increasing as well as decreasing because for any n, m in $\mathbb{N}$

$x_n = x_m = k$ and hence $x_n \leq x_m$ if $m \geq n$ as well as $x_n \leq x_m$ if $n \geq m$.

11.4.5 Definition: Let $X = (x_n)$ be a sequence of real numbers and $n_1 < n_2 < n_3 < \cdots\cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. The sequence $Y = (y_k \; : \; k \in \mathbb{N})$ where $y_k = x_{n_k}$ $\forall\, k \in \mathbb{N}$ is called a subsequence of X.

Sequence X : $x_1, x_2, x_3, \cdots\cdots\cdots, x_n, \cdots\cdots\cdots$

Strictly increasing sequence in $\mathbb{N}$ $1 \le n_1 < n_2 < n_3 < \cdots\cdots\cdots n_k < \cdots\cdots\cdots$

Subsequence y: $x_{n_1}, x_{n_2}, x_{n_3} \cdots\cdots\cdots x_{n_k} \cdots\cdots\cdots$

$y_1 \quad y_2 \quad y_2 \cdots\cdots\cdots y_k \cdots\cdots$

11.4.6 Example: If $X = (x_n)$ is any sequence in $\mathbb{R}$ and $Y = (y_n)$, $Z = (z_n)$ are defined by $y_n = x_{2n}$ and $z_n = x_{2n-1}$ $\forall\, n \in \mathbb{N}$ then Y and Z are subsequences of X, Y is the sub-sequence of even terms in X and Z is the subsequence of odd terms in X.

$$\begin{array}{lllllllll} X & = & x_1 & x_2 & x_3 & x_4 & \cdots\cdots & x_n & \cdots\cdots \\ Y & = & & x_2 & & x_4 & x_6 \cdots\cdots & x_{zn} & \cdots\cdots \\ Z & = & x_1 & & x_3 & & x_5 & x_7 \cdots\cdots & x_{zn-1} \cdots\cdots \end{array}$$

11.4.7 S.A.Q.: Show that if $n_1 < n_2 < \cdots\cdots < n_k < \cdots\cdots$ is a strictly increasing sequence of natural numbers then $n_k \ge k$ $\forall\, k \in \mathbb{N}$.

Definition: If $X = (x_n)$ is a sequence of real numbers and m is any natural number then the m - tail of X is the sequence $X_m = (x_{m+n} / n \in N)$.

Then if $X = (x_1, \cdots\cdots, x_m, \cdots\cdots\cdots)$, $X_m = (x_{m+1}, x_{m+2}, \cdots\cdots\cdots)$

Note: Any m - tail of a sequence $X = (x_n)$ is a subsequence of X.

11.4.8 S.A.Q.: Show that (x_n) is monotonically increasing in $\mathbb{R}$ iff $(-x_n)$ is monotonically decreasing.

11.4.9 S.A.Q.: Let (x_n) be a decreasing sequence of positive terms and

$$s_n = x_1 - x_2 + x_3 - \cdots\cdots + (-1)^{n-1} x_n$$

Show that for every $n \in \mathbb{N}$ (1) $s_{2n+1} > s_{2n} > 0$, (2) $0 < s_{2n-2} < s_{2n} < s_{2n+1} \le x_1$

11.4.10 Bounded Sequence: Definition: Let $X = (x_n)$ be a sequence of real numbers. X is said to be bounded iff there exists a real number M such that $|x_n| \le M$ for all $n \in \mathbb{N}$.

Note: $X = (x_n)$ is bounded $\Leftrightarrow \exists\, M \in \mathbb{R} \ni |x_n| \le M\ \forall\, n \in \mathbb{N}$

$\Leftrightarrow \exists\, M \in \mathbb{R} \ni -M \le x_n \le M\ \forall\, n \in \mathbb{N}$ $\Leftrightarrow$ The set $\{x_n / n \in \mathbb{N}\}$ is bounded.

11.4.11 S.A.Q.:

a) Show that $X = (x_n)$ is bounded iff $\exists$ α, β in $\mathbb{R} \ni \alpha \le x_n \le \beta$ for every $n \in \mathbb{N}$.

b) Show that if X is increasing, then $x_1 \le x_n \ \forall\ n \in \mathbb{N}$

c) Show that if X is decreasing then $x_1 \ge x_n \ \forall\ n \in \mathbb{N}$

11.4.12 Algebraic Operations of Sequences:

Let $X : (x_n)$ and $Y = (y_n)$ be sequences of real numbers

(i) The sequence $Z = X + Y$ is defined by $Z = X + Y = (z_n)$ where $z_n = x_n + y_n$ for every $n \in \mathbb{N}$.

(ii) The sequence $Z = XY$ is defined by $Z = XY = (z_n)$ where $z_n = x_n y_n$.

(iii) If $c \in \mathbb{R}$ the sequence cX is defined by $cX = (z_n)$ where $z_n = cx_n$

(iv) If $y_n \ne 0\ \forall\ n$, the sequence $\frac{X}{Y} = (z_n)$ is defined by $z_n = \frac{x_n}{y_n}$

11.4.13 S.A.Q.: Show that if X and Y are increasing sequences in $\mathbb{R}$ so are $X + Y$ and cX where $c > 0$.

11.4.14 S.A.Q.: If X and Y are increasing sequences in $\mathbb{R}$ does it follow that XY is monotonically increasing? Justify.

11.4.15 S.A.Q.: If X and Y are monotone, does it follow that $X + Y$, $X - Y$ and cX where $c \in \mathbb{R}$ are monotone? Justify.

11.4.16 S.A.Q.: Show that if X and Y are bounded sequences in $\mathbb{R}$ so are $X + Y$, $X - Y$, cX where $c \in \mathbb{R}$.

11.4.17 S.A.Q.: If X, Y are bounded sequences in $\mathbb{R}$ does it imply that XY is bounded? Justify.

11.4.18 Examples:

If $b \in \mathbb{R}$ then $B = (b^n)$, is the sequence

$$B = \left(b,\ b^2, b^3, \cdots\cdots b^n, \cdots\cdots\cdots\cdots\right)$$

$$bB = \left(b^2, b^3, b^4, \cdots\cdots\cdots\cdots\right)$$

bB is the 1 tail of B.

$b^2B = \left(b^3, b^4, b^5, \cdots\cdots\cdots\cdots\right)$ is the 2 tail of B.

For any $m \in \mathbb{N}$,

$b^mB = \left(b^{m+1}, b^{m+2}, \cdots\cdots\cdots\cdots\right)$ is the m- tail of B.

If $|b| \le 1$, for every $n \ge 1$, $|b|^{n+1} = |b|^n \cdot |b| \le |b|^n$

Hence if $|b|^n \le 1$, $|b|^{n+1} \le |b|^n \le 1$

Thus by induction if $|b| \le 1$ B is bounded. The inductive definition of B is given by $b_1 = b$ and $b_{n+1} = b_n b$ Also if $0 \le b \le 1$, $|b| = b$ and since $|b|^{n+1} \le |b|^n$, $b^{n+1} \le b^n \le 1$.

Thus the sequence B is monotonically decreasing.

11.4.19 Let $x_n = 1 + (-1)^n$ and $X = (x_n)$. The terms of X are

$(1-1,\ 1+1,\ 1-1, 1+1, 1-1, 1+1, \cdots\cdots\cdots\cdots)$ i.e., $(0,2,0,2,0,2\cdots\cdots\cdots\cdots)$

If n is odd $(-1)^n = -1$ so $x_n = 0$ and if n is even $(-1)^n = 1$ so $x_n = 2$

Thus X is neither monotonically increasing nor decreasing. However X is bounded.

11.5 The Limit of a Sequence:

11.5.1 Definition: A sequence $X : (x_n)$ in $\mathbb{R}$ is said to converge to x in $\mathbb{R}$ and x is said to be a limit of $X = (x_n)$ if for every $\in > 0$ there corresponds a natural number $k(\in)$ such that for all $n \ge k(\in)$ $(n \in \mathbb{N})$ the terms x_n satisfy $|x_n - x| < \in$.

If a sequence has a limit we say that the sequence is convergent. If a sequence has no limit we say that the sequence is divergent. When (x_n) has limit x we use the notation $\ell im\ X = x$, $\ell im\,(x_n)$ or $x_n \to x$ (reading tends to for "$\to$").

Note: the notation $k(\in)$ is used to emphasize that the choice of k depends upon $\in$. However it is customary to write k instead of $k(\in)$.

11.5.2 Search for $k(\in)$: To show that $\ell im\,(x_n) = x$ we have to prove existence of a natural number $k(\in)$ corresponding to every $\in > 0$ such that for all $n \geq k(\in)$, $|x_n - x| < \in$. Naturally $k(\in)$ depends upon $\in$.

(i) It is clear that once we are able to find one such $k(\in)$ then every $k \geq k(\in)$ also serves the purpose since $n \geq k \Rightarrow n \geq k(\in)$.

(ii) Moreover in our attempt to find a natural number $k(\in)$ such that $|x_n - x| < \in$ holds for all $n \geq k(\in)$ we may end up with a positive real number A instead of a natural number $k(\in)$ with the above property. This is enough for us because we may use Archemedian principle and choose any natural number greater than A as our $k(\in)$.

(iii) We may restrict $\in$ it self to the open interval $(0,1)$ and find $k(\in)$ easily. This $k(\in)$ works out for all $\in > 0$ as well because $|x_n - x| < \in < 1$ implies that $|x_n - x| < \in < \in'$ whenever $\in' \geq 1 > \in$.

11.5.3 S.A.Q.: Show that $\ell im\ (x_n) = x$ iff for every $\in > 0$ there is a natural number $k(\in)$ such that if $n \geq k(\in)$ $|x_n - x| \leq \in$.

Uniqueness of limits:

11.5.4 Theorem: A sequence in $\mathbb{R}$ can have atmost one limit.

Proof: Let (x_n) be a sequence in $\mathbb{R}$ and x, x^1 be limits of (x_n). We show that $x = x^1$ by proving that for every $\in > 0$, $0 \leq |x - x^1| < \in \to (*)$

If $\in > 0$ so is $\frac{\in}{2}$. Since lim $(x_n) = x$ there is a natural number k_1 corresponding to $\frac{\in}{2}$ such that if $n \geq k_1$, $|x_n - x| < \frac{\in}{2} \cdots\cdots\cdots (1)$

Since $\ell im\ (x_n) = x^1$, there is a natural number k_2 such that if $n \geq k_2$, $|x_n - x^1|, \frac{\in}{2} \to (2)$

If $m = \max\{k_1, k_2\}$, $m \geq k_1$ as well as $m \geq k_2$ so that both the inequalities (1) and (2) hold for $n = m$.

Now $0 \le \left|x - x^1\right| = \left|x - x_n + x_m - x^1\right| \le \left|x - x_m\right| + \left|x_m - x^1\right| < \frac{\in}{2} + \frac{\in}{2} = \in$

Thus $(*)$ holds good for every $\in > 0$. This inplies that $x = x^1$

11.5.5 Thumb Rule - I for non existence of ℓimit: From theorem 11.5.4 $\ell\text{im}\,(x_n) = \ell$ and $\ell\text{im}\,(x_n) = \ell'$ implies that $\ell = \ell'$. This implication helps us more in establishing non convergence (i.e. divergence) of a sequence. For example consider.

(i) $X = (x_n)$ where $x_n = (-1)^{n-1}$

Suppose X converges and $\ell\text{im}\,(x) = \ell$. Then for every $\in > 0$ there corresponds a natural number $k(\in)$ such that $n \ge k(\in)$

$$\Rightarrow \left|x_n - \ell\right| = \left|(-1)^n - \ell\right| < \in$$

If we choose $n = 2k(\in) + 1,$ n is odd so $n-1$ is even,

$$x_n = +1, \text{ hence } \left|1 - \ell\right| < \in.$$

Since $\in > 0$ is arbitrary and $0 < \left|1 - \ell\right| < \in$, it follows that $\ell = 1$.

If we take $n = 2k(\in)$, $n-1$ is odd so $x_n = -1$ and we get $0 < \left|-1 - \ell\right| = \left|1 + \ell\right| < \in$ for every $\in > 0$. Thus as above $\ell = -1$.

Thus lim $X = 1$ and $\ell\text{im}\ X = -1$. This cannot happen. Hence (x_n) does not converge.

(ii) Let $x_n = \sin\frac{n\pi}{2} \cdot X = (x_n)$ does not converge.

Suppose $\ell\text{im}\,(x_n) = \ell$. Then for every $\in > 0$ there is a natural number $k(\in)$ such that if $n \ge k(\in)$ $\quad 0 \le \left|\sin\frac{n\pi}{2} - \ell\right| < \in$

If $n = 2\,k(\in)$ $\sin\frac{n\pi}{2} = \sin k(\in)\,\pi = 0$ so $0 \le \left|\ell\right| < \in$

If $n = 4k(\in)+1 \quad \frac{n\pi}{2} = 2k(\in)+\frac{\pi}{2} \Rightarrow \sin\frac{n\pi}{2} = 1 \Rightarrow |1-\ell| < \in$

$0 \le |\ell| < \in \; \forall \in > 0 \Rightarrow \ell = 0$. $0 \le |\ell+1| < \in \; \forall \in > 0 \Rightarrow \ell = -1$

Hence the limit of (x_n) does not exist.

11.5.6 Equivalent Forms:

Theorem: Let $X = (x_n)$ be a sequence in $\mathbb{R}$ and Let $x \in \mathbb{R}$. the following statements are equivalent.

(a) X converges to x.

(b) For every $\in > 0$ there exists a natural number $k(\in)$ such that for all $n \ge k(\in)$ the terms x_n satisfy $|x_n - x| < \in$.

(c) For every $\in > 0$ there exists a natural number $k(\in)$ such that for all $n \ge k(\in)$ the terms $x_n \in V_\in(x)$ where $V_\in(x)$ is the neighbourhood.

$$V_\in(x) = \{y \in \mathbb{R} / |y - x| < \in\}$$

Proof: $(a) \Leftrightarrow (b)$ by definition

For any real numbers u, v. $|u-v| < \in \Leftrightarrow u - v < \in$ and $-(u-v) < \in$

$\Leftrightarrow u - v < \in$ and $-\in < u - v$

$\Leftrightarrow -\in < u - v < \in \;\Leftrightarrow\; v - \in < u < v + \in$

Thus $|x_n - x| < \in \;\Leftrightarrow\; x - \in < x_n < x + \in$

Now (b) holds $\Leftrightarrow \forall \in > 0$ there exists a natural number k such that for all natural numbers $n \ge k$ $|x_n - x| < \in$ i.e., $x - \in < x_n < x + \in \Leftrightarrow$ (c) holds. Hence $(b) \Leftrightarrow (c)$.

Since $x_n \in V_\in(x) \Leftrightarrow |x_n - x| < \in$

$(b) \Leftrightarrow \forall \in > 0$ there exists a natural number K

such that $x_n \in V_\in(x)$ for all natural numbers $n \ge K$.

$\Leftrightarrow$ (d) holds Hence $(b) \Leftrightarrow (d)$

Thus $a \Leftrightarrow (b) \Leftrightarrow (c)$

$$\Updownarrow$$

$$(d)$$

Hence (a), (b), (c) and (d) are equivalent.

11.5.7 Theorem: Let $x = (x_n)$ be a sequence of real numbers and x be a real number. Then the following are equivalent.

(i) The sequence $X = (x_n)$ does not converge to x in $\mathbb{R}$.

(ii) There exists an $\epsilon_0 > 0$ such that for any $k \in \mathbb{N}$ there exists $n_k \geq k$ in $\mathbb{N}$ such that $\left|x_{n_k} - x\right| \geq \epsilon_0$.

(iii) There exists an $\epsilon_0 > 0$ and a subsequence $X' = \left(x_{n_k}\right)$ of X such that $\left|x_{n_k} - x\right| \geq \epsilon_0$ for all $k \in \mathbb{N}$.

Proof: $(i) \Rightarrow (ii)$**:** Suppose (i) holds. Then for some $\epsilon_0 > 0$ there does not exist $k(\epsilon_0)$ in $\mathbb{N}$ satisfying $|x_n - x| < \epsilon_0$ for all $n \geq k(\epsilon_0) \cdots\cdots\cdots\cdots\cdots\cdots (A)$

If k is a natural number, k cannot be $k(\epsilon_0)$. This means that this k does not satisfy (A). This implies that for atleast one $n \geq k$ the first part of (A) does not hold

i.e., $|x_n - x| \geq \epsilon_0$ for atleast one $n \geq k$. Thus there exists $\epsilon_0 > 0$ such that for any $k \in \mathbb{N}$ there exists at least one n_k in $\mathbb{N}$ such that $n_k \geq k$ and $\left|x_{n_k} - x\right| \geq \epsilon_0$. Thus (ii) holds. Hence $(i) \Rightarrow (ii)$

$(ii) \Rightarrow (iii)$: Suppose (ii) holds. For $k = 1$ there is $n_1 \in \mathbb{N}$ such that $n_1 \geq 1$ and $\left|x_{n_1} - x\right| \geq \epsilon_0$. Since $n_2 \geq k \geq \max\{2, n_1 + 1\}$, $n_2 > n_1$.

Assuming that $1 \leq n_1 < \cdots\cdots\cdots\cdots < n_k$ are such that $n_k \geq k$ and $\left|x_{n_k} - x\right| \geq \epsilon_0$

Let $k^1 = \max\{n_k + 1 \quad \text{and} \quad k + 1\}$ as (ii) there is a natural number $n_{k+1} \geq k'$ such that $\left|x_{n_{k+1}} - x\right| \geq \epsilon_0$. Clearly $n_{k+1} \geq n_k + 1 > n_k$ and $n_{k+1} \geq k + 1$. Thus by induction for every $k \in \mathbb{N}$ there is a natural number $n_k \ni k \leq n_k < n_{k+1}$

$\left|x_{n_k} - x\right| \geq \epsilon_0$ for all k

The subsequence $\left(x_{n_k}\right)$ satisfies the condition (iii). Then $(ii) \Rightarrow (iii)$

$(iii) \Rightarrow (i)$: Suppose (iii) holds but (i) does not. Then $\lim(x_n) = x$

Then for the ϵ_0 in (iii) there corresponds $k(\epsilon)$ in $\mathbb{N}$ such that $|x_n - x| < \epsilon_0$ for all $n \geq k(\epsilon)$

Since (n_k) is a strictly increasing subsequence of $\{1,2,3,...............,n..........\}$ there is a n_{k_0} such that $n_{k_0} \geq k(\epsilon)$. So for $n_k \geq n_{k_0}$, $n_k \geq k(\epsilon)$ hence $\left|x_{n_k} - x\right| < \epsilon_0$ for all $n_k \geq n_{k_0}$.

But this contradicts (iii)

Hence (i) must hold if (iii) holds $(iii) \Rightarrow (i)$

11.5.8 Thumb rule - 2 for nonexistence of limit:

From theorem 11.5.7 it is clear that to prove that lim $(x_n) \neq x$ it is enough to find a sub sequence $\left(x_{n_k}\right)$ of (x_n) and $\epsilon_0 > 0$ such that $\left|x_{n_k} - x\right| \geq \epsilon_0$ for all n_k.

11.5.9 Example: Let $x_n = \dfrac{(-1)^n 2n}{n^2+1}$ show that lim $(x_n) \neq 1$

If n is even $|x_n - 1| = |x_n - 1| = \left|\dfrac{2n}{n^2+1} - 1\right| = \left|\dfrac{(n-1)^2}{n^2+1}\right| = \dfrac{(n-1)^2}{n^2+1} = \dfrac{\left(1-\dfrac{1}{n}\right)^2}{1+\dfrac{1}{n^2}}$

If $n \geq 2$ $\quad 0 < \dfrac{1}{n} \leq \dfrac{1}{2} \quad \Rightarrow \quad 1 > 1 - \dfrac{1}{n} \geq \dfrac{1}{2} \quad \geq \quad \left(1-\dfrac{1}{n}\right)^2 \geq \dfrac{1}{4}$

Also if $n \geq 2$ $\quad \dfrac{1}{n^2} \leq \dfrac{1}{4} \Rightarrow 1 + \dfrac{1}{n^2} \leq 1 + \dfrac{1}{4} \leq 2 \Rightarrow \dfrac{1}{1+\dfrac{1}{n^2}} \geq \dfrac{1}{2}$

$\Rightarrow |x_n - 1| \geq \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$ if n is even

Thus the subsequence (x_{2n}) satisfies $|x_{2n} - 1| \geq \frac{1}{8}$ for all n. Hence lim $x_{2n} \neq 1$.

11.5.10 Theorem:

(a) If $\ell im\ (x_n) = x$ then $\ell im\ |x_n| = |x|$

(b) $\ell im\ (x_n) = 0$ if and only if $\ell im\ (|x_n|) = 0$

(c) If $\ell im\ (x_n) > 0$ then there exists $N \in \mathbb{N}$ such that $x_n > 0$ if $n \geq \mathbb{N}$

Proof: (a) We use the in equality $||a| - |b|| \leq |a - b| \cdots\cdots\cdots (1)$

Since $\ell im\ x_n = x$, given $\in > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - x| < \in$ if $n \in \mathbb{N}$ and $n \geq N$. From (1) $||x_n| - |x|| \leq |x_n - x| < \in$ if $n \in \mathbb{N}$ and $n \geq N$. This is true for every $\in > 0$, hence $\ell im\ |x_n| = |x|$.

(b) We use the equality $||a|| = |a| \cdots\cdots\cdots\cdots (2)$

$\ell im\ (x_n) = 0 \Leftrightarrow \forall \in > 0$ there corresponds $N \in \mathbb{N}_\in$ such that $|x_n| < \in$ whenever $n \in \mathbb{N}$ and $n \geq N$

$\Leftrightarrow \forall \in > 0$ there corresponds $N \in \mathbb{N}_\in$ such that $||x_n|| < \in$ whenever $n \in \mathbb{N}$ and $n \geq N_\in \Leftrightarrow \ell im\ (|x_n|) = 0$

(c) Let $x = \ell im\ (x_n)$ and assume that $x > 0$.

Taking $\in = x$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < x$ if $n \in \mathbb{N}$ and $n \geq N$.

$\Rightarrow 0 = x - x < x_n < x + x$ if $n \in \mathbb{N}$ and $n \geq N \Rightarrow x_n > 0$ if $n \geq N$

Example: That the converse of 11.5.10 (a) is false is evident from the following example.

Let $x_n = (-1)^n \cdot |x_n| = 1$ for even $n \in \mathbb{N}$. Hence $\ell im\ (|x_n|) = 1$. But (x_n) is not convergent.

11.5.11 Theorem: Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers such that $\ell im\ (a_n) = 0$ and if for some constant C and some $m \in \mathbb{N}$, $|x_n - x| \leq a_n$ for $n \geq m$ then $\ell im\ (x_n) = x$.

Proof: We are given (x_n), x in $\mathbb{R}$, (a_n) in $\mathbb{R} \ni a_n > 0\ \forall\ n$ and $\ell im\ a_n = 0$. We are also given $c \in \mathbb{R}$ and $m \in \mathbb{N} \ni |x_n - x| \leq x\ a_n$ for $n \geq m$. We prove that $\ell im\ (x_n) = \chi$

Let $\in > 0$. Since $\ell im\ (a_n) = 0$, there exists a natural number k such that

$$|a_n| < \frac{\in}{c} \text{ for } n \geq k. \text{ Since } a_n > 0,\ a_n < \frac{\in}{c} \text{ for } n \geq k$$

If $k(\in) = \max\{k\ ,\ m\}$, $n \geq k(\in) \Rightarrow n \geq k$ and $n \geq m$, $\Rightarrow a_n < \frac{\in}{c}$ and $|x_n - x| < c a_n$,

$$\Rightarrow |x_n - x| < c\ a_n < c\frac{\in}{c} = \in \text{ for } n \geq k(\in).$$

Since this holds forevery $\in > 0$, $\ell im\ (x_n) = x$.

11.5.12 Theorem: If $X = (x_n)$ is a sequence and $Y = (y_k)$ is any subsequence of X and $\ell im\ (X) = x$ then $\ell im\ (Y) = x$

Proof: Since $\ell im\ x_n = x$, if $\in > 0$ there is a natural number $k(\in)$ such that if $n \geq k(\in)$ $|x_n - x| < \in$

Since Y is a subsequence of X; there is a strictly increasing subsequence $1 \leq n_1 < \cdots\cdots\cdots < n_k < \cdots\cdots\cdots\cdots$ such that $y_k = x_{n_k}$ for every k, from S.A.Q. 11.4.8 $n_k \geq k$ for every k. Thus if $k \geq k(\in)$, $n_k \geq k \geq k(\in)$.

Hence $|y_k - x| = |x_{n_k} - x| < \in$ if $k \geq k(\in)$. Since $\in > 0$ is arbitrary $\ell im\ y_k = x$.

11.5.13 Theorem: Let $X = (x_n)$, $\ell im\ X = x$ iff for some natural number m, $\ell im = X_m = x$ where X_m is the m - tail of X. In this case $\ell im\ X_m = x$ for every m.

Proof: Since X_m is a subsequence of X. $\ell im\ X = x \Rightarrow \ell im\ X_m = x$

Conversely suppose $\ell im\ X_m = x$. Since $X = (x_n), X_m = (y_n)$ where $Y_n = x_{m+n}$ for $n \in \mathbb{N}$.

If $\in > 0$ there is a natural number k such that if $n \geq k$, $|y_n - x| < \in$

$\Rightarrow |x_{m+n} - x| < \in$ if $n \geq k$. $n \geq k \Leftrightarrow m+n \geq m+k$ Thus $|x_{m+n} - x| < \in$ if $m+n \geq m+k$ i.e. $|x_r - x| < \in$ if $r \geq m+k$ we choose $k(\in) = m+k$ and get $|x_r - x| < \in$ if $r \geq k(\in)$. Since $\in > 0$ is albitrary $\ell im\ X = x$.

11.5.14 S.A.Q.: Let $X = (x_n)$ be a sequence. $X_1 = (x_{2n-1})$ the subsequence of odd terms and $X_2 = (x_{2n})$ the subsequence of even terms of X. Show that the sequence X converges if and only if the subsequences X_1 and X_2 converge and have the same Limit. In this case show that $\ell im\,(X) = \ell im(X_1) = \ell im(X_2)$.

11.5.15 S.A.Q.: Let $X = (x_n)$ be a sequence in $\mathbb{R}$ and $x_n \geq 0$ for every n Show that if $\ell im\left((-1)^n\ x_n\right)$ exists then $\ell im\ X$ exists and $\ell im\ X = 0$.

11.5.16 S.A.Q.: Suppose (x_n) (y_n) are sequences such that (i) $\ell im\,(x_n) = x$ and

(ii) for every $\in > 0$ there is a $k(\in)$ such that $|x_n - y_n| < \in$ if $n \geq k(\in)$ show that $\ell im\,(y_n) = x$.

11.5.17 S.A.Q.: Let $X = (x_n)$ bea sequence in $\mathbb{R}$ and $\ell \in \mathbb{R}$. Show that lim $X = \ell$ if and only if every subsequence of X has a subsequence whose limit is ℓ.

11.5.18 Examples: **1.** $\ell im\left(\frac{1}{n}\right) = 0$

Solution: Let $\in > 0$. We have to find a natural number $k(\in)$ such that

$$\frac{1}{n} = \left|\frac{1}{n} - 0\right| < \in \text{ if } n \geq k(\in)$$

$$\frac{1}{n} < \in \Leftrightarrow n \in > 1 \Leftrightarrow n > \frac{1}{\in}$$

So if $k(\in)$ is any natural number such that $k(\in) > \frac{1}{\in}$ and n is a natural number such that $n \geq k(\in)$ then $\frac{1}{n} \leq \frac{1}{k(\in)} < \in$. Since $\in > 0$ is arbitary $\ell im \left(\frac{1}{n}\right) = 0$

2. $\ell im \left(\frac{3n+2}{n+1}\right) = 3$.

Solution: $\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right| = \frac{1}{n+1}$

If $\in > 0$ as in example 1 above we choose $k(\in)$ in $\mathbb{N}$ such that $k(\in) > \frac{1}{\in}$

If $n \geq k(\in),\ n+1 > n \geq k(\in) \Rightarrow \frac{1}{n+1} < \in$

Hence $\left|\frac{3n+2}{n+1} - 3\right| < \in$ if $n \geq k(\in)$. Since $\in > 0$ is arbitary $\ell im \frac{3n+2}{n+1} = 3$

3. If $0 < b < 1$ then $\ell im \left(b^n\right) = 0$

Solution: We have to show that for every $\in > 0$ there exists $k \in \mathbb{N}$ such that for $n \geq k$ $\left|b^n - 0\right| < \in$. It is enough if we find $k \in \mathbb{R} \ni k > 0$ corresponding to $\in$ when $0 < \in < 1$.

So we choose $0 < \in < 1$. Then log $\in < 0$ and also log $b < 0$. Hence $\log \frac{1}{\in} > 0$ and $\log \frac{1}{b} > 0$

Further $\left|b^n - 0\right| = b^n$

Now $b^n < \in \Leftrightarrow \left(\frac{1}{b}\right)^n > \frac{1}{\in} \Leftrightarrow n \log \frac{1}{b} > \log \frac{1}{\in}$

$$\Leftrightarrow n > \frac{\log \frac{1}{\in}}{\log \frac{1}{b}} = \frac{\log \in}{\log b} > 0 \quad \left(\because \log \frac{1}{b} > 0\right)$$

We thus choose $k = \frac{\log \in}{\log b}$. g for $n > k$. Then from the above it is clear that

$\left|b^n - 0\right| < \in$ if $n \geq k$. Hence $\ell im\ b^n = 0$.

4. If $a > 0$, $\ell im\left(\frac{1}{1+na}\right) = 0$

Solution: We consider $\in \ni 0 < \in < 1$. $\left|\frac{1}{1+na} - 0\right| = \frac{1}{1+na} < \in$

$\Leftrightarrow 1 < \in (1+na)$ $\Leftrightarrow 1 + na > \frac{1}{\in}$

$\Leftrightarrow na > \frac{1}{\in} - 1$ $\Leftrightarrow n > \frac{1}{a}\left(\frac{1}{\in} - 1\right)$

Hence $\ell im \frac{1}{1+na} = 0$

5. If $k \in \mathbb{R}$ and $x_n = k\ \forall\ n$ then $\ell im\left(x_n = k\right)$.

Solution: $\left|x_n - k\right| = \left|k - k\right| = 0\ \forall\ n$

$\Rightarrow$ If $\in > 0$, $\left|x_n - k\right| = 0 < \in\ \forall\ n \geq 1$

$\Rightarrow \ell im\left(x_n\right) = k$

6. If $c > 0$ $\ell im\left(c^{1/n}\right) = 1$

Solution: If $c > 0$ by definition $c^{\frac{1}{n}}$ is the unique $b > 0 \ni c = b^n$

Case (i): If $c = 1$ $c^{\frac{1}{n}} = 1\ \forall\ n$ By (5) $\ell im\left(c^{\frac{1}{n}}\right) = 1$

Case (ii): If $c > 1$ and $c^{\frac{1}{n}} = b$ then $b^n = c > 1$ so $b > 1$.

This implies that $c^{\frac{1}{n}} > 1$. Let $h = c^{\frac{1}{n}} - 1$

Then $h > 0$ and $1+h = c^{\frac{1}{n}}$ so that $(1+h)^n = c$.

By the Binomial Theorem

$$c = (1+h)^n = 1+\binom{n}{1}h + ... + h^n > 1+\binom{n}{1}h = 1+nh$$

$$\Rightarrow c > 1+nh \Rightarrow c-1 > nh \Rightarrow \frac{c-1}{n} > h$$

Thus $\left|c^{\frac{1}{n}} - 1\right| = |h| = h < \frac{c-1}{n}$

If $\in > 0$, $\frac{c-1}{n} < \in \Leftrightarrow (c-1) < n\in$

$$\Leftrightarrow n > \frac{c-1}{\in}$$

So that $\left|c^{\frac{1}{n}} - 1\right| < \in \Leftrightarrow n > \frac{c-1}{\in}$

Hence in this case $\ell im\left(n^{1/n}\right) = 1$.

Case (iii) If $0 < c < 1$ then $\frac{1}{c} > 1$. We write $\left(\frac{1}{c}\right)^{\frac{1}{n}} - 1 = h$

Then $h > 0$ and $\frac{1}{c^{1/n}} = 1+h$.

$$\Rightarrow c^{\frac{1}{n}} = \frac{1}{1+h} \quad \Rightarrow c = \frac{1}{(1+h)^n}$$

Since $(1+h)^n = 1+\binom{n}{1}h+\binom{n}{2}h^2 + ... + h^2$

$$> 1+\binom{n}{1}h = 1+nh > nh$$

$$\Rightarrow\ c = \frac{1}{(1+h)n} < \frac{1}{nh} \cdot \Rightarrow h < \frac{1}{nc}$$

If $\in > 0$, $\frac{1}{n}\ c < \in \Leftrightarrow nc \in > 1 \quad \Leftrightarrow\ n > \frac{1}{c\in}$

and this implies that if $\in > 0$ $\left|c^{\frac{1}{n}} - 1\right| = h < \frac{1}{nc} < \in$ when ever $n > \frac{1}{c\in}$

This implies that $\ell im \left(c^{\frac{1}{n}}\right) = 1$ if $0 < c < 1$.

7. $\ell im \left(n^{\frac{1}{n}}\right) = 1$

Solution: We adopt the method of example (6).

For $n > 1$ $n^{\frac{1}{n}} > 1$ write $h = n^{\frac{1}{n}} - 1$. Then $h > 0$.

$$n = (1+h)^n = 1+\binom{n}{1}h + \binom{n}{2}h^2 + \ldots\ldots\ldots\ldots + h^n \text{ (by Binomial Theorem)}$$

$$\Rightarrow\ n > 4\binom{n}{2}h^2 = 1+\frac{n(n-1)}{1\cdot 2}h^2$$

$$\Rightarrow n\cdot 1 > \frac{n(n-1)}{2}h^2$$

$$\Rightarrow\quad 1 > \frac{n}{2}h^2 \text{ i.e. } h^2 < \frac{2}{n} \text{ i.e., } h < \sqrt{\frac{2}{n}}$$

$\sqrt{\frac{2}{n}} < \in$ if $\frac{2}{n} < \in^2$ i.e., $n > \frac{2}{\in^2}$. Then if $n > \frac{2}{\in^2}$, $0 < h < \sqrt{\frac{2}{n}} < \in$

So that $0 < \left(n^{\frac{1}{n}} - 1\right) < \in$ for $n > \frac{2}{\in^2}$

Hence $\ell im \left(n^{1/n}\right) = 1$

8. $\ell im \dfrac{n^2-1}{2n^2+3} = \dfrac{1}{2}$

Solution: $\left|\dfrac{n^2-1}{2n^2+3} - \dfrac{1}{2}\right| = \left|\dfrac{2n^2-2-2n^2-3}{2\left(2n^2+3\right)}\right| = \left|\dfrac{-5}{2\left(2n^2+3\right)}\right| = \dfrac{5}{2\left(2n^2+3\right)}$

If $\in > 0$ $\quad \dfrac{5}{2\left(2n^2+3\right)} < \in \quad \Leftrightarrow \in (2)\left(2n^2+3\right) > 5$

$$\Leftrightarrow\ 2n^2+3 > \frac{5}{2\in} \quad \Leftrightarrow\ n^2 > \frac{1}{2}\left(\frac{5}{2\in} - 3\right)$$

$$\Leftrightarrow\ n > \sqrt{\frac{1}{2}\left(\frac{5}{2\in} - 3\right)} \quad \left(\text{if} \quad \frac{5}{2\in} - 3 > 0\right)$$

$$\frac{5}{2\in} - 3 > 0$$

$$\Leftrightarrow\ \frac{5}{2\in} > 3 \text{ i.e., } 5 > 6\in \text{ i.e., } 0 < \in < \frac{5}{6}$$

Thus if $0 < \in < \dfrac{5}{6}$ and $n > \sqrt{\dfrac{1}{2}\left(\dfrac{5}{2\in} - 3\right)}$

Then $\left|\dfrac{n^2-1}{2n^2+3} - \dfrac{1}{2}\right| < \in$

If $\in \geq \dfrac{5}{6}$ the above inequality holds good for all n because $\dfrac{5}{2\in} - 3 \leq 0$ and so for all $n \in \mathbb{N}$, $\quad n^2 > \dfrac{1}{2}\left(\dfrac{5}{2\in} - 3\right)$ from which the required inequality follows

Thus $\ell im \dfrac{n^2-1}{2n^2+3} = \dfrac{1}{2}$

11.6 Limit Theorems:

11.6.1 Theorem: A convergent sequence of real numbers is bounded.

Proof: Let (x_n) be a converegent sequence of real numbers and

$\ell im\ (x_n) = x$. Then corresponding to $\in = 1$ there is a natural number k,

such that $|x_n - x| < 1$ for $n \geq k$. Since $\left||x_n| - |x|\right| \leq |x_n - x|$ for all n,

$$\left||x_n| - |x|\right| < 1 \text{ for } n \geq k \Rightarrow 1 - |x| < |x_n| < 1 + |x| \text{ for } n \geq k$$

If $M = \max \{|x_1|, \ldots\ldots\ldots\ldots, |x_{k-1}|, 1 + |x|\}$ $|x_n| \leq M$ for $1 \leq n < k$ and $|x_n| \leq 1 + |x| \leq M$ for $n \geq k$

so that $|x_n| \leq M$ for all $n \in \mathbb{N}$, hence (x_n) is bounded.

Divergence criterian I: If (x_n) is unbounded, then (x_n) is divergent.

Remark: It is not true that a bounded sequence is convergent. For example the sequence $\left((-1)^{n-1}\right)$ is bounded but not convergent (see 11.5.5(i)).

11.6.2 Theorem: If $\ell im\ x_n = x$ and $m \leq x_n \leq M$ for all $n \in \mathbb{N}$ then $m \leq x \leq M$.

Proof: Suppose $m > x$. Then by taking $\in = m - x > 0$ we get a natural number k such that $|x_n - x| < m - x$ for $n \geq k$.

$$\Rightarrow m - x - x < x_n < m - x + x = m - x \text{ for } n \geq k.$$

$$\Rightarrow x_n < m \text{ for } n \geq k \text{, contrary to the hypothesis } \Rightarrow m \leq x$$

For the second part of the inequality i.e., to prove that $M \geq x$, we assume the contrary, i.e. $M < x$ and take $\in = x - M$. As above, there exists a natural number k_1, such that if $n \geq k_1$

$|x_n - x| < x - M$ from which it would follow that $x_n > M$ for $n \geq k_1$ which is a contradiction. Thus $x \leq M$ and hence $m \leq x \leq M$.

11.6.3 Corollary: (i) If $\ell im\,(x_n) = x$ and $x_n \geq 0$ if $n \in \mathbb{N}$ then $x \geq 0$.

(ii) If $\ell im\; x_n = x$ and $a \leq x_n \leq b$ for all $n \geq m$

where m is a fixed natural number then $a \leq x \leq b$.

11.6.4 Theorem: Suppose $x_n \leq y_n$ for $n \in N$. $\ell im(x_n) = x$ and $\ell im\,(y_n) = y$. Then $x \leq y$.

Proof: Suppose on the contrary that $x > y$. Corresponding to $\in = \dfrac{x-y}{2}$ there is a natural number

k such that $|x_n - x| < \dfrac{x-y}{2}$ for $n \geq k$

$$\Rightarrow y = x - \frac{(x-y)}{2} < x_n < x + \frac{(x-y)}{2} \text{ for } n \geq k$$

$$\Rightarrow x_n > \frac{x+y}{2} \text{ for } n \geq k$$

$$\Rightarrow y_n \geq x_n > \frac{x+y}{2} \text{ for } n \geq k.$$

From the first part of the inequality in 11.6.2 it follows that $y = \ell im\,(y_n) \geq \dfrac{x+y}{2}$

$$\Rightarrow 2y - x - y \geq 0$$

$$\Rightarrow y - x \geq 0$$

$\Rightarrow y \geq x$, a contradiction.

Hence our assumption tht $x > y$ is false and so $x \leq y$.

11.6.5 Theorem: Let $X = (x_n), Y = (y_n)$ be sequences in $\mathbb{R}$.

$\ell im\,(X) = x$ and $\ell im(Y) = y$. Then

(a) $\ell im\,(X+Y) = x + y$,

(b) $\ell im(X - y) = x - y$ and

(c) for any $c \in \mathbb{R}$ $\ell im\; cX = cx$

Proof: (a) Let $\in > 0$ since $\ell im(X) = x$ and $\ell im(Y) = y$, there exist natural numbers k_1, k_2 such that $|x_n - x| < \frac{\in}{2}$ if $n \geq k_1 \cdots\cdots(1)$

and $|y_n - y| < \frac{\in}{2}$ if $n \geq k_2$ $\cdots\cdots\cdots\cdots(2)$

Let $k(\in) = \max\{k_1, k_2\}$. If $n \geq k(\in)$ then (1) and (2) hold.

So that for $n \geq k(\in)$ $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$

$$\leq |x_n - x| + |y_n - y|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence $\ell im\,(X + Y) = x + y$

(b) Also $|(x_n - y_n) - (x - y)| = |(x_n - x) - (y_n - y)|$

$$\leq |x_n - x| + |y_n - y|$$

$$< \frac{\in}{2} + \frac{\in}{2} \text{ for } n \geq k(\in)$$

Hence $\ell im\,(X - Y) = x - y$

(c) If $c = 0$ $cX = (0)$ and $cx = 0$. Since $\ell im\,(0) = 0, \ell im\; cX = 0 = 0 \quad \ell im\; x = c\; \ell im\; x$. Suppose $c \neq 0$ and $\in > 0$. Since $\ell im\; x_n = x$

There corresponds $k(\in) \in \mathbb{N}$ such that for $n \geq k(\in)$ $|x_n - x| < \frac{\in}{|c|}$

Hence for $n \geq k(\in)$ $|cx_n - cx| = |c|\; |x_n - x| < |c| \frac{\in}{|c|} = \in$

Hence (cx_n) converges to cx.

11.6.6 Corollary: If $A, B, \cdots\cdots, Z$ are any finite number of sequences in $\mathbb{R}$ each one of which converges in $\mathbb{R}$, then the sequence $A + B + \ldots\ldots + Z$ is convergent in $\mathbb{R}$ and

$$\ell im\,(A + B + C + \ldots\ldots\ldots + Z) = \ell im\ A + \ell im\ B + \ldots\ldots + \ell im\ Z$$

Proof: The proof is by induction on the number of sequences when $n = 2$ and A, B are sequences such that $\ell im\ A = a$ and $\ell im\ B = b$

By theorem 11.6.5. A+B converges and $\ell im\,(A + B) = a + b$.

Assume that the statement is true for any n sequences.

Let $A_1, \cdots\cdots A_{n+1}$ be any $(n+1)$ sequences each of which converges and $\ell im\ A_i = a_i$ for $1 \le i \le n+1$.

By induction hypothesis $A_1 + A_2 + \cdots\cdots + A_n$ converges and

$$\ell im\,(A_1 + \cdots\cdots + A_n) = \ell im\ A_1 + \cdots\cdots + \ell im\ A_n = a_1 + \cdots\cdots + a_n \cdots\cdots\cdots (1)$$

Let $B = A_1 + \cdots\cdots + A_n$. Then B converges.

Again A_{n+1} is a sequence with $\ell im\ A_{n+1} = a_{n+1}$

Hence $B + A_{n+1}$ converges and $\ell im\,(B + A_{n+1}) = \ell im\ B + \ell im\, A_{n+1}$

By (1) $\ell im\ B = \ell im\ A_1 + \cdots\cdots\cdots + \ell im\ A_n$

Hence $\ell im\,(A_1 + \cdots\cdots + A_n) = \ell im\ A_1 + \cdots\cdots\cdots + \ell im\ A_n$

Hence $A_1 + \cdots\cdots + A_{n+1} = B + A_{n+1}$ converges and

$$\ell im\,(A_1 + \cdots\cdots + A_{n+1}) = \ell im\,(B + A_{n+1}) = \ell im\ B + \ell im\ A_{n+1}$$

$$= \ell im\, A_1 + \cdots\cdots + \ell im\ A_n + \ell im\ A_{n+1}$$

Hence the statement holds for all n.

11.6.7 Theorem: If $X = (x_n)$ converges to x and $Y = (y_n)$ converges to y then XY converges to xy.

Proof: Since a convergent sequence in $\mathbb{R}$ is bounded and X is convergent there exists a $M \in \mathbb{R}$ such that $|x_n| < M$ for all $n \in \mathbb{N}$ and $k_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\in}{2(1+|y|)} \text{ if } n \geq k_1 \cdots\cdots\cdots\cdots (2)$$

Since $\ell im\,(y_n) = y$ there exists $k_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\in}{2M} \text{ if } n \geq k_2 \cdots\cdots\cdots (3)$$

Let $k(\in) = \max\{k_1, k_2\}$ and $n \geq k(\in)$

Then $|x_n\, y_n - xy| = |x_n y_n - x_n y + x_n y - xy|$

$$= |x_n\,(y_n - y) + (x_n - x)\,y|$$

$$\leq |x_n\,(y_n - y)| + |(x_n - x)\,y|$$

$$= |x_n|\,|y_n - y| + |x_n - x|\,|y|$$

$$\leq M\frac{\in}{2M} + |y|\frac{\in}{2(1+|y|)} \text{ by (1) (2) and (3)}$$

$$\leq \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence XY converges and $\ell im\ XY = xy$.

11.6.8 Corollary: Let $X_{1,}X_2, \cdots\cdots\cdots, X_k$ be a finite number of convergent sequences and X be the "product" sequence whose n^{th} term is the product of the n^{th} terms of the sequences $X_1, X_2, \cdots\cdots\cdots, X_k$. Then X is convergent and

$$\ell im(X) = \ell im\,(X_1)\ \ell im(X_2) \cdots\cdots\cdots\cdots\cdots \ell im(X_k)$$

Proof: We prove that statement by induction on k.

When $k = 2, X = X_1, X_2$ from 11.6.7. $X_1\, X_2$ is convergent and $\ell im(X) = \ell im(X_1)\ \ell im(X_2)$. Assume that the statement holds for any k sequences and X be the product of $(k+1)$ sequences say $X_1, \cdots\cdots\cdots, X_k,\ X_{k+1}$. Let Y be the product of the k sequences $X_2, \cdots\cdots\cdots\cdots\cdots, X_{k+1}$. Then $X = X_1 X_2 \cdots\cdots\cdots X_{k+1} = X_1\, Y$. Since Y is the product of k convergent sequences, by induction hypothesis Y is convergent and $\ell im(Y) = \ell im(X_2)\, \ell im(X_3) \cdots\cdots\cdots \ell im(X_{k+1})$.

Since X_1 and Y are convergent, by 11.6.7 $X = X_1 Y$ is convergent and $\ell im(X) = \ell im(X_1 Y)$

$$= \ell im(X_1)\, \ell im(Y)$$

$$= \ell im(X_1)\, \ell im(X_2) \cdots\cdots \ell im(X_{k+1})$$

As the statement holds for $k = 2$ and for $k+1$ whenever it holds for k, the statement holds good for all k.

11.6.9 Theorem: If $Z = (z_m)$ is a sequence of non zero real numbers. Z converges to $z \neq 0$ then the sequence $\frac{1}{Z} = \left(\frac{1}{z_n}\right)$ converges to $\frac{1}{z}$.

Proof: Let $\in = \frac{|z|}{2}$. Then $\in > 0$. Since $\ell im\, Z_n = z$ there exists k_1 in $\mathbb{N}$

such that $|z_n - z| < \frac{|z|}{2}$ for $n \geq k_1 \cdots\cdots (1)$

$$\Rightarrow \| z_n | - | z \| \leq |z_n - z| < \frac{|z|}{2} \text{ for } n \geq k_1$$

$$\Rightarrow |z| - \frac{|z|}{2} < |z_n| < |z| \mid \frac{|z|}{2} \text{ for } n \geq k_1$$

$$\Rightarrow |z_n| > \frac{|z|}{2} \text{ for } n \geq k_1 \cdots\cdots\cdots\cdots (2)$$

Hence for $n \geq k_1$, $\left|\frac{1}{z_n} - \frac{1}{z}\right| = \frac{|z_n - z|}{|z_n|\,|z|} < |z_n - z| \cdot \frac{2}{|z| \cdot |z|} \cdots\cdots\cdots (3)$

Since $\ell im\, z_n = z$ if $\in > 0$ $\exists\, k_2 \in \mathbb{N}$ such that

$$|z_n - z| < \frac{\in |z|^2}{2} \text{ for } n \geq k_2 \text{ Let } k(\in) = \max\{k_1, k_2\}$$

by (3) $\left|\frac{1}{z_n}-\frac{1}{z}\right|<\frac{|z_n-z|}{|z|^2}<\frac{\in \cancel{|z|^2}}{2\cancel{|z|^2}}=\frac{\in}{2}<\in \text{ for } n \geq k(\in)$

Thus $\left(\frac{1}{n}\right)$ converges and $\ell\text{im}\left(\frac{1}{z_n}\right)=\frac{1}{z}$

11.6.10 Corollary: If $X=(x_n)$ converges to x, $Y=(y_n)$ is a convergent sequence of non zero terms converging to anon zero number y then $\frac{X}{Y}=\left(\frac{x_n}{y_n}\right)$ converges to $\frac{x}{y}$.

Proof: The corollary follows from theorems 11.6.7 and 11.6.9.

Since $\ell\text{im } X = x$ and $\ell\text{im } \frac{1}{Y}=\frac{1}{y}$,

$$\ell\text{im}\left(\frac{X}{Y}\right)=\ell\text{im}\left(X\cdot\frac{1}{Y}\right)=\ell\text{im } X \ \ell\text{im } \frac{1}{Y}=\frac{x}{y}$$

11.6.11 S.A.Q.: If $\ell\text{im } (x_n)=\ell$ are $r>0$ $\ell\text{im}\left(x_n^r\right)=\ell^r$

11.6.12 S.A.Q.: If $b_0 \neq 0$ and $n \in \mathbb{N}$ $\ell\text{im}\left(\frac{a_0n^r+a_1n^{r-1}+\cdots\cdots\cdots+a_r}{b_0n^r+b_1n^{r-1}+\cdots\cdots\cdots+b_r}\right)=\frac{a_0}{b_0}$ whenever $r \in \mathbb{N}$.

11.6.13 S.A.Q.: If $r \in \mathbb{N}$ and $s \in \mathbb{N}$ and $r<s$ and $b_0 \neq 0$.

$$\ell\text{im}\left(\frac{a_0n^r+\cdots\cdots\cdots+a_r}{b_0n^s+\cdots\cdots\cdots+b_s}\right)=0$$

Let us recall that if a, b are real numbers $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$

11.6.14 Theorem: If (X_n) and (y_n) are convergent sequences of real numbers then the sequence $(\max(x_n, y_n))$ converges and $\ell\text{im}(\max\{x_n, y_n\})=\max\{\ell\text{im}(x_n), \ \ell\text{im}(y_n)\}$

Proof: Given $\in>0$ there exist natural numbers K_1, K_2 such that

$x-\in < x_n < x+\in$ for $n \geq K_1 \cdots\cdots(1)$

and $y-\in < y_n < y+\in$ for $n \geq K_2 \cdots\cdots\cdots(2)$

If $K = \max\{k_1, K_2\}$ then (1) and (2) hold good for $n \geq K$. Without loss of generality suppose $x \vee y = x$. From (1) and (2) we have for $n \geq k$ $x_n > x-\in = (x \vee y)-\in$

$\Rightarrow x_n \vee y_n \geq x_n > (x \vee y) - \in$ for $n \geq K \cdots\cdots(3)$

Also $x_n < x+\in = (x \vee y)+\in$ for $n \geq K$ and $y_n < y+\in \leq (x \vee y) + \in$ for $n \geq k$

so that $x_n \vee y_n < (x \vee y)+\in$ for $n \geq k \cdots\cdots(4)$

From (3) and (4) it follows that

$(x \vee y)-\in < x_n \vee y_n < x \vee y + \in$ for $n \geq k$

Since $\in > 0$ is arbitrary it follows that

$\ell im\,(x_n \vee y_n) = x = x \vee y$

11.6.15 Theorem: Squeeze (Sandwich) Theorem: Suppose that $X = (x_n)$, $Y = (y_n)$ and $Z = (Z_n)$ are sequences of real numbers such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and that $\ell im\,(x_n) = \ell im(z_n)$.

Then the sequence Y is convergent and $\ell im(x_n) = \ell im(y_n) = \ell im(z_n)$.

Proof: Let $w = \ell im\, x_n = \ell im\, z_n$. If $\in > 0$ then since $\ell im\,(x_n) = w$ there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$, $|x_n - w| < \in$

and since $\ell im\, z_n = 0$ there exists $k_2 \in \mathbb{N}$ such that for all $n \geq k_2$ $|z_n - w| < \in$

Let $k(\in) = \max\{k_1, k_2\}$. Then for all $n \geq k(\in)$

$|x_n - w| < \in$ and $|z_n - w| < \in$

$\Rightarrow w-\in < x_n < w+\in$ and $w-\in < z_n < w+\in$ for $n \geq k(\in)$

$\Rightarrow w-\in < x_n \leq y_n \leq z_n < w+\in$ for $n \geq k(\in)$

$\Rightarrow w - \in \leq y_n \leq w + \in$ for $n \geq k(\in)$

$\Rightarrow |y_n - w| < \in$ for $n \geq k(\in)$

Since $\in > 0$ is arbitrary, it follows that $\ell im(y_n) = w$.

11.6.16 Corollary: Let $X = (x_n)$, $Y = (y_n)$, $Z = (z_n)$ be sequences of real numbers such that for some $m \in \mathbb{N}$.

$$x_n \leq y_n \leq z_n \quad \text{for all } n \geq m$$

If $\ell im\,(x_n) = \ell im(z_n)$ then the sequence y converges and

$$\ell im\,(x_n) = \ell im(y_n) = \ell im(z_n)$$

Proof: From hypothesis it follows that the m - tails $X_{m,} Y_{m,} Z_m$ satisfy $x_{m+n} \leq y_{m+n} \leq z_{m+n}$ for all $n \in \mathbb{N}$.

Also $\ell im\, X_m - \ell im\, Z_m$. Hence by the squeeze theorem the sequence Y_m converges and $\ell im\,(X_m) = \ell im\,(Y_m) = \ell im(Z_m)$.

Since $\ell im(X) = \ell im(X_m)$, $\ell im(Y) = \ell im(Y_m)$ and $\ell im(Z) = \ell im(Z_m)$.

$$\ell im(X) = \ell im(Y) = \ell im(Z)$$

11.6.17 Example: Use squeeze theorem to determine the limit of the sequence $\left(\frac{1}{n^{n^2}}\right)$.

Solution: Since $n^2 \geq n \geq 1$, $\frac{1}{n^2} \leq \frac{1}{n} \leq 1$ Hence $1 \leq n^{\frac{1}{n^2}} \leq n^{\frac{1}{n}}$ $\quad \forall\, n \in \mathbb{N}$

Since $\ell im\,(1) = \ell im\left(\frac{1}{n^n}\right) = 1$ it follows by the squeeze theorem that $\ell im\left(\frac{1}{n^{n^2}}\right) = 1$.

11.6.18 Example: Show that if $0 < a < b$ and $Z_n = \left(a^n + b^n\right)^{\frac{1}{n}}$ then $\ell im\,(Z_n) = b$.

Solution: since $0 < a < b$, $\;0 < a^n < b^n$ for $n \in \mathbb{N}$

$$\Rightarrow b^n < a^n + b^n < 2b^n \quad \text{for} \quad n \in \mathbb{N}$$

$$\Rightarrow b < \left(a^n + b^n\right)^{\frac{1}{n}} < 2^{\frac{1}{n}} b \quad \text{for } n \in \mathbb{N}$$

The constant sequence (b) converges to b and the sequence $\left(2^{\frac{1}{n}}\right)$ converges to 1 so that the sequence $\left(2^{\frac{1}{n}} b\right)$ converges to b.

Hence by the squeeze theorem, the sequence $\left(\left(a^n + b^n\right)^{\frac{1}{n}}\right)$ converges to b.

11.6.19 Examples:

(a) Show that $\ell\text{im } x_n = 0$ when $= \dfrac{1}{(n+1)^2} + \cdots\cdots\cdots + \dfrac{1}{(n+n)^2}$

$$\frac{1}{(2n)^2} \le \frac{1}{(n+k)^2} \le \frac{1}{n^2} \quad \text{for } 1 \le k \le n\,.$$

$$\Rightarrow \frac{n}{(2n)^2} \le x_n \le \frac{n}{n^2}$$

Since $\ell\text{im } \dfrac{1}{4n} = \ell\text{im } \dfrac{1}{n} = 0$, it follows by the squeere theorem that $\ell\text{im } x_n = 0$

(b) Use the squeeze theorem to prove that $\ell\text{im}\left(\sqrt{n+1} - \sqrt{n}\right) = 0$

$$\sqrt{n+1} - \sqrt{n} = \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Since $\sqrt{n} < \sqrt{n+1}$, $\sqrt{n} + \sqrt{n} < \sqrt{n+1} + \sqrt{n} < \sqrt{n+1} + \sqrt{n+1}$

$$\Rightarrow \frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$$

Since $\ell im \frac{1}{\sqrt{n+1}} = \ell im \frac{1}{\sqrt{n}} = 0$ it follows by the squeeze theorem that $\ell im \sqrt{n+1} - \sqrt{n} = 0$.

(c) Show that $\ell im \left(x_n\right) = 1$ when $x_n = \frac{1}{\sqrt{n^2+1}} + \cdots\cdots\cdots + \frac{1}{\sqrt{n^2+n}}$

Since $n < \sqrt{n^2+k} < (n+1)$ if $1 \le k \le n$, $\frac{1}{n+1} < \frac{1}{\sqrt{n^2+k}} < \frac{1}{n}$ if $1 \le k \le n$

$$\Rightarrow \frac{n}{n+1} < x_n < \frac{n}{n} = 1$$

Since $\ell im \frac{n}{n+1} = 1$, by the squeeze theorem it follows that $\ell im\, x_m = 1$.

11.6.20 Theorem: Let $X = \left(x_n\right)$ be a sequence of real numbers that converges to x and suppose $x_n \ge 0$. Then the sequence $\left(\sqrt{x_n}\right)$ of positive square roots converges and $\ell im \sqrt{x_n} = \sqrt{x}$.

Proof: Since $x_n \ge 0$ for every n, by 11.6.3. $x \ge 0$.

Case (i): If $x = 0$, $\ell im\, x_n = 0$ so if $\in > 0$ there is a natural number k such that $0 \le x_n = \left|x_n - 0\right| < \in^2$ if $n \ge k$

Therefore $0 \le \sqrt{x_n} < \in$ if $n \ge k$. Since $\in > 0$ arbitary it follows that $\ell im\left(\sqrt{x_n}\right) = 0$

Case (ii): If $x > 0$ then there is a natural number k_1 such that

$$\left|x_n - x\right| < x \text{ for } n \ge k_1$$

$$\Rightarrow \quad x - x < x_n < x + x \quad \text{for } n \ge k_1$$

$$\Rightarrow x_n > 0 \text{ for } n \ge k_1$$

If $\in > 0$ there is natural number k_2 such that for all $n \ge k_2$

$\left|x_n - x\right| < \in \sqrt{x} \cdots\cdots\cdots\cdots (1)$ If $k(\in) = \max\left\{k_1, k_2\right\}$ and $n \ge k(\in)$

$$\sqrt{x_n} - \sqrt{x} = \frac{\left(\sqrt{x_n} - \sqrt{x}\right)\left(\sqrt{x_n} + \sqrt{x}\right)}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Hence $\left|\sqrt{x_n} - \sqrt{x}\right| = \dfrac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} < \dfrac{\in\sqrt{x}}{\sqrt{x}} = t$ by (1) (since $\sqrt{x_n} + \sqrt{x} > \sqrt{x}$)

Hence $\left|\sqrt{x_n} - \sqrt{x}\right| < \in$ if $n \geq k(\in)$ Since $\in > 0$ is arbitary it follows that $\ell im \sqrt{x_n} = \sqrt{x}$

11.6.21 Ratio Test: Let (x_n) be a sequence of positive real numbers and $\ell im \left(\dfrac{x_{n+1}}{x_n}\right) = L$, If $L < 1, (x_n)$ converges and $\ell im\,(x_n) = 0$.

Proof: Since $\dfrac{x_{n+1}}{x_n} > 0$ for every natural number n, $L \geq 0$. Let $L < r < 1$ and $\in = r - L$

There exists a natural number k such that for $n \geq k$

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \in \cdots\cdots\cdots\cdots (1)$$

$$\Rightarrow L - \in < \frac{x_{n+1}}{x_n} < L + \in \text{ for } n \geq k$$

$$\Rightarrow 2L - r < \frac{x_{n+1}}{x_n} < r \text{ for } n \geq k$$

$$\Rightarrow 0 < \frac{x_{k+1}}{x_k} \cdot \frac{x_{k+2}}{x_{k+1}} \cdots\cdots\cdots\cdots \frac{x_n}{x_n} < (r)^{n-k} \qquad \text{if } n \geq k$$

$$\Rightarrow 0 < \frac{x_n}{x_k} < \frac{r^n}{r^k} \text{ if } n \geq k$$

$$\Rightarrow 0 < x_n < \left(\frac{x_k}{r^k}\right) r^n \text{ if } n \geq k$$

Since $\ell im\, r^n = 0$, by the corollary 11.6.16 to squeeze theorem

$\ell im\,(x_n) = 0$. This completes the proof.

11.6.22 S.A.Q.: Show that if (x_n) is a sequence of positive real numbers such that $\ell im\left(\frac{x_{n+1}}{x_n}\right) = L > 1$ then (x_n) is unbounded and hence (x_n) does not converge.

11.6.23 Theorem: If $\ell im\,(s_n) = \ell$ then $\ell im\left(\frac{s_1 + \cdots\cdots + s_n}{n}\right) = \ell$

Proof: Write $x_n = s_n - \ell$. Then $\ell im\,(x_n) = 0$.

Since $\frac{x_1 + \cdots\cdots + x_n}{n} = \frac{s_1 + \cdots\cdots + s_n}{n} - \ell$

it suffices to shows that $\ell im\left(\frac{x_1 + \cdots\cdots + x_n}{n}\right) = 0$ when $\ell im\; x_n = 0$

Let $\in > 0$. Since $\ell im(x_n) = 0$ there is a natural number k such that if $n \geq k$, $|x_n| < \frac{\in}{2}$.

Also since (x_n) is convergent, (x_n) is bounded. So there is a positive real number A such that $|x_n| < A$ for all $n \in \mathbb{N}$.

If $n \geq k$, $\left|\frac{x_1 + \cdots\cdots + x_n}{n}\right| = \left|\frac{(x_1 + \cdots\cdots + x_{k-1})}{n} + \frac{(x_k + \cdots\cdots + x_n)}{n}\right|$

$$\leq \frac{|x_1| + \cdots\cdots + |x_{k-1}|}{n} + \frac{|x_k| + \cdots\cdots + |x_n|}{n}$$

$$\leq (k-1)\frac{A}{n} + (n-k+1)\frac{\in}{2n}$$

Let $k(\in)$ be a natural number $\geq k$ and also $2(k-1)\frac{A}{\in}$. If $n \geq k(\in)$ then $n \geq k(\in) \geq \frac{2(k-1)A}{\in} \Rightarrow (k-1)\frac{A}{n} < \frac{\in}{2}$. Also $\frac{n-k+1}{n} < 1$ so that $\left(\frac{n-k+1}{n}\right)\frac{\in}{2} < \frac{\in}{2}$

Hence $\left|\frac{x_1 + \cdots\cdots + x_n}{n}\right| < \frac{\in}{2} + \frac{\in}{2} = \in$ if $n \geq k(\in)$

Since $\in > 0$ is arbitrary $\ell im\left(\frac{x_1+\cdots\cdots+x_n}{n}\right)=0$

Note: This theorem is referred to as Cauchy's first theorem.

11.6.24 Theorem: If $x_n > 0$ for all n and $\ell im(x_n)=\ell$ then $\ell im\left((x_1\cdots\cdots\cdots x_n)^{\frac{1}{n}}\right)=r$.

Proof: Clearly $\ell \geq 0$. We use the $A \cdot M, GM$ inequality: (10A. 11)

If $a_1, \cdots\cdots\cdots\cdots, a_n$ are any positive numbers,

$$\frac{a_1+\cdots\cdots\cdots+a_n}{n} \geq (a_1\cdots\cdots\cdots\cdots a_n)^{\frac{1}{n}} \qquad (\because\ AM \geq GM)$$

Case (i): If $\ell = 0$ then $\ell im\left(\frac{x_1+\cdots\cdots\cdots+x_n}{n}\right)=0$

we apply squeeze theorem to the $AM - GM$ inequality.

$$0<(x_1\cdots\cdots\cdots x_n)^{1/n} \leq \frac{x_1+\cdots\cdots\cdots+x_n}{n} \quad \text{and get } \ell im\left((x_1\cdots\cdots\cdots x_n)^{\frac{1}{n}}\right)=0$$

Case (ii): If $\ell > 0$ by 11.6.9. $\ell im\frac{1}{x_n}=\frac{1}{\ell} \Rightarrow \ell im\left(\frac{\frac{1}{x_1}+\cdots\cdots\cdots+\frac{1}{x_n}}{n}\right)=\frac{1}{\ell}$ (by 11.6.23.)

$$\Rightarrow \ell im\left(\frac{n}{\frac{1}{x_1}+\cdots\cdots\cdots+\frac{1}{x_n}}\right)=\frac{1}{\left(\frac{1}{\ell}\right)}=\ell \text{ (by 11.6.9.)}$$

Since $\frac{\frac{1}{x_1}+\cdots\cdots\cdots\cdots+\frac{1}{x_n}}{n} \leq \left(\frac{1}{x_1}\cdots\cdots\cdots\cdots\frac{1}{x_n}\right)^{\frac{1}{n}}$,

$$(x_1\, x_2\cdots\cdots\cdots x_n)^{\frac{1}{n}} \leq \frac{n}{\frac{1}{x_1}+\cdots\cdots\cdots+\frac{1}{x_n}}$$

$$\Rightarrow \frac{n}{\frac{1}{x_1}+\cdots\cdots\cdots+\frac{1}{x_n}} \geq (x_1\cdots\cdots\cdots x_n)^{\frac{1}{n}} \geq \frac{x_1+\cdots\cdots\cdots+x_n}{n}$$

By Cauchy's first theorem $\ell im\left(\frac{x_1+\cdots\cdots\cdots+x_n}{x}\right)=\ell$

and $\ell im\left(\frac{n}{\frac{1}{x_1}+\cdots\cdots\cdots+\frac{1}{x_n}}\right)=\ell$ Hence by squeeze theorem $\ell im\left((x_1\cdots\cdots\cdots x_n)^{\frac{1}{n}}\right)=\ell$

11.6.25 Example: If (b_n) is a bounded sequence and $\ell im\ a_n=0$ show that $\ell im\ a_n\ b_n=0$, Explain why theorem 11.6.7 cannot be used.

Solution: We cannot apply theorem 11.6.7 as it requires convergence of both (a_n) and (b_n). However in the present case (b_n) is given to be bounded. As boundedness of a sequence does not guarantee convergence, theorem 11.6.7 cannot be applied. Since (b_n) is bounded there exists M in $\mathbb{R}$ such that $|b_n|<M$ for all n. since (a_n) converges to zero, if $\in>0$ there exists a natural number k such that $|a_n|<\frac{\in}{M}$ for $n\geq k$.

For $n\geq k$ $|a_n\ b_n|=|a_n|\,|b_n|<M|a_n|<M\frac{\in}{M}=\in$

Since this holds for all $n\geq k$ and $\in>0$ is arbitrary and k depends on $\in$ it follows that $\ell im\,(a_n\ b_n)=0$.

11.6.26 Example: Let $y_n=\sqrt{n+1}-\sqrt{n}$. Show that (y_n) and $\left(\frac{\sqrt{n}}{y_n}\quad y_n^{-1}\right)$ converge. Find their limits.

Solution: (i) $$y_n=\sqrt{n+1}-\sqrt{n}=\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Since $\sqrt{n+1}>\sqrt{n}$, $\sqrt{n+1}+\sqrt{n}>2\sqrt{n}$

Therefore $0\leq y_n<\frac{1}{2\sqrt{n}}$ for every n

Since the constant sequence (0) and the sequence $\left(\frac{1}{2\sqrt{n}}\right)$ both converge to zero it follows by the squeere theorem that (y_n) converges and $\ell im\,(y_n) = 0$.

(ii) $$\sqrt{n}\;\, y_n^{-1} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{\frac{n+1}{n}}+\sqrt{1}} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$\ell im\,\frac{1}{n} = 0 \Rightarrow \ell im\left(1+\frac{1}{n}\right) = 1 \Rightarrow \ell im\left(\sqrt{1+\frac{1}{n}}\right) = 1$$

So $\ell im\left(\sqrt{1+\frac{1}{n}}+1\right) = \ell im\left(\sqrt{1+\frac{1}{n}}\right) + \ell im(1) = 1+1 = 2$

11.6.27 Example: Show that $\ell im\,(\sin n)$ does not exist in $\mathbb{R}$.

Solution: Suppose $\ell im\,(\sin n)$ exists and $\ell im\,(\sin n) = \ell$

$$\sin\,(2n+2)+\sin(2-2n) = 2\cos\,2n\,\sin\,2$$

Since

$$\ell im\,\sin\,(1-n) = \ell im\,(-\sin\,(n-1)) = -\,\ell im\,\sin\,(n-1)$$

we get by taking limits $2\sin 2\ \mathrm{Lim}\cos\ 2n = 0 \Rightarrow \mathrm{Lim}\ \cos\ 2n = 0$

III $\ell im\,\cos\,(2n+1) = 0$

Hence $\ell im\,\cos\ n = 0$

Since $\sin\,2n = 2\sin n\cos n,$ we get $\ell = \ell im\,\sin n = \ell im\,\sin 2n = 0$

Since $1 = \sin^2 n + \cos^2 n\ \forall\ n,$ we get

$$1 = \ell im\left(\sin^2 n + \cos^2 n\right) = \ell im\left(\sin^2 n\right) + \ell im\left(\cos^2 n\right)$$

$$= (\ell im\,\sin\,n)^2 + (\ell im\,(\cos\,n))^2$$

$= 0+0$, a contradiction

Hence $\ell im\ \sin$ n does not exist.

11.6.28 Example: Suppose that (x_n) is a bounded sequence of real numbers. Write

$s_n = \sup(x_k : k \geq n)$ and $S = \inf(s_n : n \in \mathbb{N})$

Show that there is a subsequence of (x_n) that converges to S.

Solution: Since $S-1 < S \leq s_1 = \text{Sup}(x_k : k \geq 1)$ there is a $n_1 \geq 1$ such that $S-1 < x_{n_1}$

since $S - \frac{1}{2} < S \leq s_{n_1+1} = \sup(x_k : k \geq n_1 + 1)$

there is a $n_2 \geq n_1 + 1 > n_1$ such that $S - \frac{1}{2} < x_{n_2}$

Continuing this proces it follows that there is a subsequence (x_{n_k}) of x_n such that

$S - \frac{1}{k} < x_{n_k}$ for very k in $\mathbb{N}$.

If $\in > 0$ $S+\in$ is not a lower bound of $(s_n : n \in \mathbb{N})$

$\Rightarrow$ if $h \geq m$ there is a natural number m such that $s_m < S+\in$.

Since $n_k \geq k$, $S - \frac{1}{k} < x_{n_k} \leq s_k < S+\in$

Choose $k(\in)$ to be any natural number > m and $\frac{1}{\in}$. For $k > k(\in)$, $k > \frac{1}{\in}$ so that $\frac{1}{k} < \in$

$\Rightarrow S-\in < S - \frac{1}{k} < x_{n_k} \leq s_k < S+\in \Rightarrow \left|x_{n_k} - S\right| < \in$ for $k \geq k(\in) \Rightarrow \ell\text{im}\left(x_{n_k}\right) = S$

11.7 Solutions to Short Answer Questions:

11.4.8 Show that if $n_1 < n_2 < \cdots\cdots < n_k < \cdots\cdots$ is a strictly increasing sequence of natural numbers then $n_k \geq k\ \forall\ k \in \mathbb{N}$.

solution: We prove this statement by induction.

When $k=1$ $n_1 \geq 1$ since $n_1 \in \mathbb{N}$ assume that $k \in \mathbb{N}$ and $n_k \geq k$. Then $n_{k+1} > n_k \Rightarrow n_{k+1} > k$. Since $n_{k+1} \in \mathbb{N}$, n_{k+1} is a natural number $> k$ so that $n_{k+1} \geq k+1$ Thus $n_1 \geq 1$ and $n_k \geq k \Rightarrow n_{k+1} \geq k+1$. So by induction $n_k \geq k\ \forall\ k \in \mathbb{N}$.

11.4.9 Show that (x_n) is monotonically increasing iff $(-x_n)$ is monotonically decreasing.

Solution:

(x_n) is monotonically increasing $\Leftrightarrow x_n \geq x_m$ whenever $n \geq m$

$\Leftrightarrow -x_n \leq -x_m$ whenever $n \geq m$ $\Leftrightarrow (-x_n)$ ismonotonically decreasing.

11.4.10 S.A.Q.: Let (x_n) be a decreasing sequence of positive terms and $s_n = x_1 - x_2 + x_3 - \cdots\cdots + (-1)^{n-1} x_n$.

Solution: (i) $s_{2n+1} > s_{2n} > 0$ for every $n \in \mathbb{N}$

$$s_{2n} = (x_1 - x_2) + \cdots\cdots\cdots + (x_{2n-1} - x_{2n}) = \sum_{i=1}^{n} (x_{2i-1} - x_{2i})$$

Since (x_n) is decreasing , $x_{n-1} > x_n$ for every $n \in \mathbb{N}$. So $s_{2n} > 0$

$s_{2n+1} = s_{2n} + x_{2n+1} > s_{2n}$ since $x_{2n+1} > 0$

(ii) $0 \leq s_{2n-1} \leq s_{2n} \leq s_{2n+1} \leq s_{2n-1} \leq x_1$ for $n \in \mathbb{N}$

$$s_{2n} = \sum_{i=1}^{n} (x_{2i-1} - x_{2i})$$

$$= \sum_{i=1}^{n-1} (x_{2i-1} - x_{2i}) + (x_{2n-1} - x_{2n})$$

$$\geq s_{2(n-1)} \qquad \because x_{2n-1} \geq x_{2n}$$

That $s_{2n} \leq s_{2n+1}$ follows from (i) above

$$s_{2n+1} = x_1 - x_2 + x_3 - x_4 + \cdots\cdots + x_{2n-1} - x_{2n} + x_{2n+1}$$

$$= x_1 - x_2 + x_3 - x_4 + \cdots\cdots\cdots + x_{2n-1} - (x_{2n} - x_{2n+1})$$

$$= s_{2n-1} - (x_{2n} - x_{2n+1}) < s_{2n-1} \text{ since } x_{2n} \geq x_{2n+1}$$

$\Rightarrow s_{2n+1} \leq s_{2n-1}$ for $n \in \mathbb{N}$

Hence $s_{2n-1} \le s_1$ for $n \in \mathbb{N}$. Since $s_1 = x_1$, $s_{2n-1} \le x_1$ for $n \ge N$

The inequality is proved.

11.4.11(a): By definition $X = (x_n)$ is bounded if and only if there exists $M \in \mathbb{R}$ such that $|x_n| \le M \ \forall\ n \in N$ iff $\exists\ M \in \mathbb{R} \ni -M \le x_n \le M \ \forall\ n \in N$.

Thus if X is bounded and $|x_n| \le M$ for all $n \in \mathbb{N}$

We may choose $\alpha = -M$ and $\beta = M$

Conversely if $\alpha \le x_n \le \beta$ for all $n \in \mathbb{N}$

We choose $M \in \mathbb{R} \ni\ -M \le \alpha \le \beta \le M$ i.e., $M \ge \max\{\beta \text{ and } -\alpha\}$

Then $-M \le \alpha \le x_n \le \beta \le M$ for all $n \in \mathbb{N}$

i.e., $|x_n| \le M$ for all $n \in \mathbb{N}$ This implies that X is bounded.

(b) If $X = (x_n)$ is increasing, $x_n \le x_m$ whenever $m \ge n$ put $n = 1$.

(c) If $X = (x_n)$ is decreasing $x_n \ge x_m$ whenever $m \ge n$ put $n = 1$.

11.4.13 Let $X = (x_n), Y = (y_n)$. By hypothesis $x_n \ge x_m$ if $n \ge m$ and $y_n \ge y_m$ if $n \ge m$. Hence if $n \ge m$, $x_n + y_n \ge x_m + y_m$ Hence $X + Y$ is increasing

If $c > 0$ and $n \ge m$ $cx_n \ge cx_m$. Hence cx is monotonically increasing

11.4.14 Let $x_n = n$ and $X = (x_n)$

Since $x_n < x_m$ whenever $n < m, X$ is monotonically (strictly) increasing.

Let $y_n = -1$ for all n, and $Y = (y_n)$. Then Y is monotonically increasing.

$XY = (x_n\ y_n) = (-n)$ clearly XY is not monotonically increasing.

11.4.15 Let $x_n = n$ for $n \in \mathbb{N}$ and $X = (x_n)$ Then X is monotonically increasing

Let $y_n = -1$ if $n < 3$

$= -4$ if $n \ge 3$

and $Y = (y_n)$

Then Y is monotonically decreasing

$$X+Y=(1-1,\ 2-1,\ 3-4,\ 4-4,\ 5-4,\cdots\cdots\cdot)$$

$$=(0,1,-1,0,1,2,\cdots\cdots\cdots\cdot)$$

Clearly $x_1 < x_2$ and $x_3 < x_2$

Thus $X+Y$ is neither increasing nor decreasing.

Let $Y^{'}=-Y=(-y_n)$. Then $Y^{'}$ is increasing since Y is decreasing

Hence X and $Y^{'}$ are monotone. However $X-Y^1=X+Y$ is not monotone, as seen above.

If X is increasing and $X=(x_n)$ then $x_n \geq x_m$ if $n>m$

If $C\geq 0,\ Cx_n \geq Cx_m$ if $n>m$

So if $C\geq 0$ CX is increasing, hence monotone

If $c<0,\ cx_n \leq cx_m$ if $n>m$

So if $C<0$ CX is decreasing, hence monotone.

In either case if X is increasing CX is monotone.

Similarly if X is decreasing CX is monotone.

Then if X is monotone CX is monotone.

11.4.16 Let $X=(x_n),\ Y=(y_n),\ Z=X+Y=(Z_n)$ so that $Z_n=x_n+y_n$.

Since X and Y are bounded there are M_1 and M_2 in the such that $|x_n|\leq M_1$ and $|y_n|\leq M_2$ for $n\in\mathbb{N}$. If $n\in\mathbb{N}$ $|Z_n|=|z_n+y_n|\leq|x_n|+|y_n|\leq M_1+M_2$. Since $|Z_n|\leq M_1+M_2$ for every $n\in\mathbb{N}$ Z is bounded. In a similar way we can prove that X - Y is bounded. If $c\in\mathbb{R}$ $|cx_n|=|c|\,|x_n| \leq |c|\,M_1$ for every $n\in\mathbb{N}$ so that cx is bounded.

11.4.17 Let X, Y, M_1, M_2 be as in 11.3.16 above and $Z=XY=(Z_n)$ where $Z_n=x_n y_n$.

Then $|Z_n|=|x_n\ y_n|=|x_n|\,|y_n|\leq M_1\ M_2$ for $n\in\mathbb{N}$

Since this holds for all $n\in\mathbb{N}$ it follows that XY is bounded.

11.5.3 S.A.Q.: Show that $\ell im\left(x_n\right)=x$ if an only if for every $\in>0$ there corresponds $k(\in)\in\mathbb{N}$ such that for all $\mathbb{N}\geq k(\in)$ $|x_n-x|\leq\in$.

Solution: If $\operatorname{Lim}\left(x_n\right)=x$ then for every $\in>0$ there corresponds $k(\in)\in\mathbb{N}$ such that for all $n\geq k(\in)$ $|x_n-x|<\in$

Since $|x_n-x|<\in\Rightarrow|x_n-x|\leq\in$ it follows that for every $\in>0$ there corresponds $k(\in)\in\mathbb{N}$ such that for $n\geq k(\in)$ $|x_n-x|\leq\in$.

If for every $\in>0$ there corresponds $k(\in)$ in $\mathbb{N}$ such that for $n\geq k(\in)$ $|x_n-x|\leq\in$

starting with $\in>0$ we apply the above condition for $\frac{\in}{2}$ so that there exists a natural number $k_1=k\left(\frac{\in}{2}\right)$ satisfying

$$|x_n-x|\leq\frac{\in}{2} \text{ for } n\geq k_1$$

Then for $n\geq k_1,\ |x_n-x|\leq\frac{\in}{2}<\in\Rightarrow\ell im\left(x_n\right)=X$

Thus for every $\in>0$ there corresponds a natural number k_1 such that for all $n\geq k_1$ $|x_n-x|\leq\in$ if and only if $\ell im\left(x_n\right)=x$

11.5.14 S.A.Q.: Suppose (x_n) is a sequence such that $\ell im(x_{2n})=\ell im\left(x_{2n-1}\right)$ show that $\ell im(x_n)$ exists and $\ell im(x_n)=\ell im(x_{2n})$.

Let $=\ell im\left(x_{2n}\right)=\ell im\left(x_{2n-1}\right)$. If $\in>0$ there exist natural numbers v such that

$$|x_{2n}-\ell|<\in \text{ if } 2n\geq k_1\cdots\cdots\cdots(1)$$

and $$|x_{2n-1}-\ell|<\in \text{ if } 2n-1\geq k_2\cdots\cdots\cdots\cdots(2)$$

Let $k(\in)=\max\{k_1,k_2\}$. If $n\geq k(\in),\ n\geq k_1$ as well as k_2.

If n is even $|x_n-\ell|<\in$ by (1)

and if n is odd $|x_n - \ell| < \in$ by (2)

Hence $|x_n - \ell| < \in$ for all $n \geq k(\in)$

Since $\in > 0$ is arbitary, $\text{Lim}\,(x_n)$ exists and $\ell\text{im}(x_n) = \ell$ converse is trivial.

11.5.15 S.A.Q.: Suppose that $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\ell\text{im}\,(-1)^n x_n$ exists. Show that $\ell\text{im}\ x_n$ exists and $\ell\text{im}\ x_n = 0$.

Let $\ell\text{im}\,(-1)^n x_n = \ell$. Then $\ell\text{im}\left(1|(-1)^n x_n|\right) = |\ell|$

Since $x_n \geq 0$ for all n, $\ell\text{im}\ x_n = |\ell|$

Since $(-1)^{2n} x_{2n} = x_{2n}$ and $(-1)^{2n-1} x_{2n-1} = -x_{2n-1}$ for $n \in \mathbb{N}$

(x_{2n}) and $(-x_{2n-1})$ are subsequences of the sequence $\left((-1)^n x_n\right)$

Hence $\ell\text{im}\,(x_{2n}) = \ell\text{im}\,(-x_{2n-1}) = \ell \Rightarrow \ell\text{im}\,(x_{2n}) = \ell = -\ell\text{im}(x_{2n-1})$

Since $x_{2n} \geq 0\ \forall\ n,\ \ell \geq 0 \cdots\cdots\cdots\cdots(1)$

Since $x_{2n-1} \geq 0\ x_{2n-1} \geq 0\ \forall\ n\quad \ell\text{im}\,(x_{2n-1}) = -\ell \geq 0 \cdots\cdots\cdots\cdots(2)$

From (1) and (2) it follows that $\ell = 0$, hence $\ell\text{im}\ x_n = |\ell| = \ell = 0$

11.5.16 S.A.Q.: Suppose (x_n) converges and $\ell\text{im}(x_n) = x$. If for every $\in > 0$ there corresponds natural number $k(\in)$ such that $|x_n - y_n| < \in$ for $n \geq k(\in)$ show that (y_n) converges and $\ell\text{im}\ y_n = x$.

solution: For any $n \in \mathbb{N}$ $|y_n - x| = |y_n - x_n + x_n - x| \leq |y_n - x_n| + |x_n - x|$

If $\in > 0$ there exist natural numbers k_1, k_2 such that

$$|y_n - x_n| < \frac{\in}{2} \text{ for } n \geq k_1 \text{ and } |x_n - x| < \frac{\in}{2} \text{ for } n \geq k_2$$

If $k(\in) = \max\{k_1, k_2\}$ and $n \geq k(\in)$ then $n \geq k_1$ and $n \geq k_2$

so that $|y_n - x| \leq |y_n - x_n| + |x_n - x| < \frac{\in}{2} + \frac{\in}{2} = \in$

Since $\in > 0$ is arbitrary it follows that (y_n) converges and $\ell\text{im}(y_n) = X$

11.5.17 S.A.Q.: Show that $\ell\text{im}(x_n) = L$ if and only if every subsequence of (x_n) has a subsequence that covnerges to L.

Suppose $\ell\text{im}\, x_n = L$ by 11.5.12 every subsequence of (x_n) converges to L. Hence every subsequence of (x_n) has a convergent subsequence (it self) that converges to L. Conversely suppose $\ell\text{im}(x_n) \neq L$. Then there exists $\in_0 > 0$ such that for every natural number k there is a natural number n_k such that $n_k > n_{k-1} \geq k$ and $|x_{n_k} - \ell| \geq \in_0$

If some subsequence of (x_{n_k}) converges to L

It must hold that $|x_{n_k} - \ell| < \in_0$ for infinitely many k. This is a contradiction.

11.6.11 S.A.Q.: Let $X = (x_n)$, $X_1 = \cdots\cdots\cdots X_r = X$ and $X^r = X_1 \cdots\cdots\cdots X_r$.

Then $X^r = (x_n^r)$. Since $\ell\text{im}(X_1) = \ell\text{im}(X_2) = \cdots\cdots\cdots \ell\text{im}(X_r) = \ell\text{im}\, X = x$ from 11.6.8 it follows that

$$\ell\text{im}(x_n^r) = \ell\text{im}\, X^r = \ell\text{im}(X_1) \cdot \ell\text{im}(X_2) \cdots\cdots\cdots \ell\text{im}(X_r) = (\ell\text{im}(X))^r = x^r$$

11.6.12 S.A.Q.: If r is a natural number and $b_0 \neq 0$

$$\frac{a_0 n^r + \cdots\cdots + a_r}{b_0 n^r + \cdots\cdots + b_r} = \frac{a_0 + \frac{a_1}{n} + \cdots\cdots\cdots + \frac{a_r}{n^r}}{b_0 + \frac{b_1}{n} + \cdots\cdots\cdots + \frac{b_r}{n^r}}$$

Since $\ell\text{im}\, \frac{1}{n} = 0$, $\ell\text{im}\, \frac{1}{n^k} = 0$ for every $k \in \mathbb{N}$ (by 11.6.8)

Hence $\ell\text{im}\left(a_0 + \frac{a_1}{n} + \cdots\cdots\cdots + \frac{a_r}{n^r}\right) = \ell\text{im}(a_0) + \ell\text{im}\left(\frac{a_1}{n}\right) + \cdots\cdots\cdots + \ell\text{im}\left(\frac{a_r}{n^r}\right)$

$$= a_0 + a_1 \ \ell im\left(\frac{1}{n}\right) + \cdots\cdots\cdots + a_r \ \ell im\left(\frac{1}{n^r}\right)$$

$$= a_0$$

Similarly $\ell im\left(b_0 + \frac{b_1}{n} + \cdots\cdots\cdots + \frac{b_r}{n^r}\right) = b_0$

Hence $\ell im \dfrac{a_0 n^r + \cdots\cdots\cdots + a_r}{b_0 n^r + \cdots\cdots\cdots + b_r} = \dfrac{a_0}{b_0}$

11.6.13 S.A.Q.: $a_0 n^r + \cdots\cdots\cdots\cdots + a_r = n^r\left(a_0 + \frac{a_1}{n} + \cdots\cdots\cdots + \frac{a_r}{n^r}\right)$

$$b_0 n^s + \cdots\cdots\cdots + b_s = n^s\left(b_0 + \frac{b_1}{n} + \cdots\cdots\cdots\cdots + \frac{b_s}{n^s}\right)$$

$$\Rightarrow \frac{a_0 n^r + \cdots\cdots\cdots + a_r}{b_0 n^s + \cdots\cdots\cdots\cdots + b_s} = \frac{1}{n^{s-r}} \cdot \left(\frac{a_0 + \frac{a_1}{n} + \cdots\cdots\cdots + \frac{a_r}{n^r}}{b_0 + \frac{b_1}{n} + \cdots\cdots\cdots + \frac{b_s}{n^s}}\right)$$

$$\Rightarrow \ell im\left(\frac{a_0 n^r + \cdots\cdots\cdots + a_r}{b_0 n^s + \cdots\cdots\cdots + b_s}\right) = \ell im \frac{1}{n^{s-r}} \cdot \ell im\left(\frac{a_0 + \frac{a_1}{n} + \cdots\cdots\cdots + a_r\left(\frac{1}{n}\right)}{b_0 + \frac{b_1}{n} + \cdots\cdots\cdots + b_s\left(\frac{1}{n^s}\right)}\right)$$

$\ell im \dfrac{1}{n^{s-r}} = 0$ and

$$\ell im\left(\frac{a_0 + \frac{a_1}{n} + \cdots\cdots\cdots + \frac{a_r}{n^r}}{b_0 + \frac{b_1}{n} + \cdots\cdots + \frac{b_s}{n^s}}\right) = \frac{a_0}{b_0}$$

Hence $\ell im \dfrac{a_0 n^r + \cdots\cdots\cdots + a_r}{b_0 n^s + \cdots\cdots\cdots + b_s} = 0 \cdot \dfrac{a_0}{b_0} = 0$

11.6.22 S.A.Q.: Show that if (x_n) is a sequence of posotive real numbers such that $\ell im\left(\frac{x_{n+1}}{x_n}\right) = L > 1$ then (x_n) is not bounded hence not convergent.

Solution: Since $\ell im \frac{x_{n+1}}{x_n} = L > 1$ we can choose $L' \in \mathbb{R}$

Such that $L > L' > 1$. Corresponding to $\in = L - L' > 0$ there exists a natural number k such that for $n \geq k$.

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \in \text{ for } n \geq k$$

$$\Rightarrow L - \in < \frac{x_{n+1}}{x_n} < L + \in \text{ for } n \geq k$$

$$\Rightarrow \frac{x_{n+1}}{x_n} > L' \text{ for } n \geq k$$

If $n > k$ then $x_n = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots\cdots\cdots\cdots \frac{x_{k+1}}{x_k} \cdot x_k$

$$> \left(L'\right)^{n-k-1} x_k = \left(L'\right)^n \frac{x_k}{\left(L'\right)^{k+1}}$$

Since $L' > 1, \left(\left(L'\right)^n\right)$ is unbounded and $\frac{x_k}{\left(L'\right)^{k+1}}$ is a positive constant. Hence (x_n) is unbounded, hence not convergent.

11.8 Summary:

After carefully working on this lesson the student should get familiarity with convergence and divergence of sequences including working knowledge on convergence or otherwise of numerical sequences.

11.9 Technical Terms:

Sequence

Convergence

Convergent Sequence

Limit of a Sequence

Divergence

11.10 Exercises:

1. Write the first five terms of the sequence $X=(x_n)$ where

(a) $x_n=\frac{1}{n(n+1)}$ (b) $x_n=\frac{1}{n^2+1}$ (c) $x_n=\frac{1}{n^2+n+1}$

(d) $x_1=1,\ x_n=\frac{x_{n+1}}{x_n}\ n\in\mathbb{N}$ (e) $x_{n+2}=x_nx_{n+1},\ \ x_1=x_2=1$

(f) $y_1=1,\ y_{n+1}=3,\ y_n+1$ (g) $Z_1=1, Z_2=2, Z_{n+2}=\frac{z_{n+1}+Z_n}{Z_{n+1}-Z_n}$

(h) $t_1=3,\ t_2=5$ and $t_{n+2}=t_n+t_{n+1}$

2. Which of the following sequences are monotone? Which of them are bounded?

(a) $x_n=\frac{n}{n+1}$ (b) $x_n=\frac{(-1)^n n}{n+1}$

(c) $x_n=\frac{n^2}{n+1}$ (d) $x_n=\frac{(-1)^n}{n+1}$

3. (a) Define the sequence of even numbers in $\mathbb{N}$ inductively.

(b) Define the sequence of odd numbers in v inductively.

4. Arrange the set of integers in a sequence.

5. Find a formula for the general term of the sequence

(a) 0, 4, 0, 8, 0, 12, ...

(b) 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, ...

6. Let (f_n) be the Fibonacci sequence and α be the "Golden number" $\alpha = \frac{1+\sqrt{5}}{2}$

Show that (1) $1.5 < \alpha < 2$ (2) $\alpha^2 = \alpha + 1$ (3) $\alpha^n < f_{n+1} \ \forall \ n \in \mathbb{N}$

7. If $b \in \mathbb{R}$ show that $\ell im \left(\frac{b}{n}\right) = 0$

8. Prove the following using definition of a Limit.

(a) $\ell im \frac{n}{n^2+1} = 0$ (b) $\ell im \frac{2n}{n+1} = 2$

(c) $\ell im \frac{\sqrt{n}}{n+1} = 0$ (d) $\ell im \frac{2n}{n+2} = 2$

(e) $\ell im (-1)^n \frac{n}{n^2+1} = 0$ (f) $\ell im \frac{1}{\sqrt{n+5}} = 0$

9. Let $x_n = \frac{1}{1n(n+1)}$ show that $\ell im (x_n) = 0$. Find a $k(\in)$ for $\in = \frac{1}{2}$ such that $x_n < \in$ for $n \geq k(\in)$. (Ans: n > 8)

10. Show that $\ell im (x_n) = \ell \Rightarrow \ell im |x_n| = |\ell|$

11. Show that $\ell im_{n} (-1)^n$ does not exist.

12. Show that if $x_n > 0$ and $\ell im (x_n) = 0 \Rightarrow \ell im \left(\sqrt{x_n}\right) = 0$

13. Show that $\ell im (x_n) = 0 \Rightarrow \ell im \left(x_n^2\right) = 0$

14. Show that $\ell im \left(\frac{1}{3^n}\right) = 0$

15. Show that $\ell im \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$

16. If $\ell im\left(a_n\right)=\ell$ and $b_n=a_n-a_{n+1}$ showthat $\ell im\left(b_n\right)=0$

17. Find $\ell im_n\left(2n\right)^{\frac{1}{n}}$

18. Find $\ell im_n\left(n^2 a^n\right), 0< a <1$

19. Show that $\ell im \frac{n^2}{n!}=0$

20. Determine the smallest natural number n such that

(a) $\frac{n}{n^2+1}< 0\cdot 01$ (Hint: $\left(n-50\right)^2 \geq 50^2$)

(b) $\frac{1}{n}+\frac{(-1)^n}{n^2}<\cdot 0001$ (c) $n^2+\left(-1\right)^n < n >1000$

21. If $\left(x_n\right)$ is a sequence such that $\ell im\left(x_{n+1}-x_n\right)=\ell$ show that $\ell im \frac{x_n}{n}=\ell$

22. Let $x_n=\frac{1}{2}\left(1-\left(-1\right)^n\right)$ show that $\ell im \frac{x_1+\cdots\cdots+x_n}{n}=\frac{1}{2}$

23. (a) If a, b are real numbers show that

$$a \wedge b=-\left(-a \vee -b\right)$$

(b) If $\ell im\left(x_n\right)=x$ and $\ell im\left(y_n\right)=y$ Prove that $\ell im\left(x_n \wedge y_n\right)=x\wedge y$

24. Establish the convergence or divergence of $\left(x_n\right)$ where $x_n=$

(i) $\frac{n}{n+n^2}$ (ii) $\frac{(-1)^n n^2}{n+1}$ (iii) $\frac{n^2}{n+n^2}$ (iv) $\frac{2n^2+3}{n^2+1}$

25. Let $x_n=n$ for $n\in\mathbb{N}$ and $X=\left(x_n\right), Y=\left(-x_n\right)$

Show that X and Y are divergent but $x+y$ is convergent

26. Let $x_n = \frac{(-1)^n}{\sqrt{n}} = y_n$ for $n \ge 1$, $X = (x_n)$ and $Y = (y_n)$

Show that $x = (x_n)$ and $y = (y_n)$ are divergent.

Determine the convergence or divergence of XY.

27. Show that if X and $X + Y$ are convergent sequences then so is Y.

28. Show that if the sequences X, Y are such that X and XY are convergent and if $\ell\text{im}(X) \neq 0$ then Y is convergent.

29. Show that (a) (2^n) is not bounded hence not convergent

and (b) $((-1)^n n^2)$ is not convergent.

30. Find the limits of the following sequences.

(a) $\left(2 + \frac{1}{n^2}\right)$ (b) $\frac{(-1)^n}{n+2}$ (c) $\frac{\sqrt{n} - 1}{\sqrt{n} + 1}$ (d) $\frac{n+1}{n\sqrt{n}}$

31. (a) Show that $\ell\text{im}\left(3\sqrt{n}\right)^{1/2n} = 1$ Hint: $(3\sqrt{n})^{\frac{1}{n}} = (\sqrt{3})^{\frac{1}{n}} \cdot \left(n^{\frac{1}{n}}\right)^{\frac{1}{2}}$

(b) Show that $\ell\text{im}\ (n+1)^{1/1n(n+1)} = e$ (Hint: $\ell n(n+1) = x_n$)

32. If $0 < a < 1$ and find $\ell\text{im}\left(\frac{x_{n+1}}{x_n}\right)$ and deduce that $\ell\text{im}\left(a^n\right) = 0$.

33. Find the limits of the following sequences when $1 < b$

(a) $\frac{b^n}{2^n}$ (b) $\frac{n}{b^n}$ (c) $\frac{2^{3n}}{3^{2n}}$

34. If $x_n = \frac{1}{n}$ find $\ell\text{im}\left(\frac{x_{n+1}}{x_n}\right)$ and $\ell\text{im}\,(x_n)$.

35. If $x_n = n$ find $\ell\text{im}\left(\frac{x_{m+1}}{x_n}\right)$ and $\ell\text{im}\,(x_n)$.

36. (a) If $\ell im\,(x_n) = L < 1$ show that there exists a $k \in \mathbb{N}$ such that $x_n < 1$ for $n \geq k$.

(b) If $0 < r < 1$ and $\ell im\left(x_n^{\frac{1}{n}}\right) = L < 1$ where $x_n > 0$ for all n show that there exists a natural number k such that $0 < x_n < r^n$ for $n \geq k$.

(c) Deduce that if (x_n) is a sequence of positive real numbers such that $\ell im\left(\frac{1}{x_n}\right)^n = L < 1$ then $\ell im\, x_n = 0$.

37. Show that in 13(c) above if $L = 1$ it cannot be concluded that $\ell im\,(x_n) = 0$ by considering the following.

(a) If $x_n = n$ $\ell im\left(\frac{1}{x_n}\right)^n = 1$ but (x_n) diverges

(b) If $x_n = \frac{1}{n}$ $\ell im\left(x_n^{\frac{1}{n}}\right) = 1$ but $\ell im\,(x_n) = 0$

38. Prove that $\ell im_n \left(\frac{1}{(n+1)^r} + \cdots\cdots\cdots + \frac{1}{(n+n)^r}\right) = 0$ when $r > 0$.

(Hint: Try squeeze theorem.)

39. If $P(x)$ is a polynomial and $s_n = P\left(\frac{1}{n}\right)$ show that $\ell im\,(s_n) = 0$

11.11 Answers:

1. (a) $\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{1}{4\cdot 5}, \frac{1}{5\cdot 6}$ i.e. $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$

(b) $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \frac{1}{26}$

(c) $\frac{1}{3}, \frac{1}{7}, \frac{1}{13}, \frac{1}{21}, \frac{1}{31}$

(d) $x_{n+1} = x_n^2 \Rightarrow x_1 = x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 1$

(e) $x_3 = 1, x_4 = 1, x_5 = 1$

(f) $y_1 = 1, y_2 = 4, y_3 = 13, y_4 = 40, y_5 = |2|$

(g) $Z_3 = \frac{1+2}{2-1} = 3, Z_4 = \frac{3+2}{3-2} = 5, Z_5 = \frac{5+3}{5-3} = 4.$

(h) $t_3 = 8, t_4 = 13, t_5 = 21.$

2. (a) monotonically increasing

(b) not monotonic

(c) monotonically increasing $\left(\frac{x_{n+1}}{x_n} > 1\right)$

(d) not monotonic

3. (a) $x_1 = 2, x_{n+1} = x_n + 2$ for $n \in \mathbb{N}$

(b) $x_1 = 1, x_{n+1} = x_n + 2$ for $n \in \mathbb{N}$

4. $\{0, 1, -1, 2, -2, 3, -3, \cdots\cdots\}$

$x_1 = 0;\ x_{2n} = n$ for $n \in \mathbb{N}$

$x_{2n+1} = -x_{2n}$

5. (a) $x_{2n} = 4n\ ;\ x_{2n-1} = 0$

(b) $x_{3n+1} = 1$ for $n = 0$ or $n \in \mathbb{N}$, $x_n = 0$ otherwise

17 1

20 (a) 100 (b) 1 (c) 32

24 (i) 0 (ii) diverges (iii) 1 (iv) 2

26 converges to zero

30 (a) 2 (b) 0 (c) 1 (d) 0

32 a

33 (a) 0 (b) 0 (c) 0

34 1,000

35 1,000

11.12 Model Examnination Questions:

1. Let (x_n) be a sequence of real number and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\ell im\,(a_n) = 0$ and if for some constant $C > 0$ and some $m \in N$ we have

   $$|x_n - x| \le Ca_n \text{ for all } n \ge m$$

   Then show that $\ell im\,(x_n) = x$

2. Show that $\ell im\left(n^2/n!\right) = 0$

3. Suppose that $X = (x_n)$, $Y = (y_n)$ and $Z = (z_n)$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in N$, and that $\ell im\,(x_n) = \ell im(Z_n)$. Then show that $Y = (y_n)$ is convergent and $\ell im\,(x_n) = \ell im(y_n) = \ell im(z_n)$.

4. Show that $\ell im\left(\dfrac{\sin\, n}{n}\right) = 0$

5. Let (x_n) be a sequence of positive real numbers such that $L = \ell im\left(x_{n+1}/x_n\right)$ exists. If $L < 1$, then show that (x_n) converges and $\ell im\,(x_n) = 0$.

6. If $a > 0,\ b > 0$ show that $\ell im\left(\sqrt{(n+a)\,(n+b)} - n\right) = (a+b)/2$

7. Show that if $Z_n = \left(a^n + b^n\right)^{1/n}$ where $0 < a < b$, then $\ell im\,(Z_n) = b$.

11.13 Model Practical Problem with Solution:

Problem: Let $x_n = \dfrac{1}{2}\left(1 + (-1)^n\right)\,(n \in \mathbb{N})$ and

$$s_n = \frac{x_1 + \cdots\cdots\cdots + x_n}{n}$$

(i) Does Limit (x_{2n}) exist?

(ii) Does Limit (x_{2n-1}) exist?

(iii) Does Limit (x_n) exist?

(iv) Does Limit (s_n) exist?

Aim: To determine existence of the above limits.

Definition: We say that a sequence (a_n) converges in $\mathbb{R}$ if there exists $a \in \mathbb{R}$ such that for every positive number $\in$ there corresponds $N_\in$ in $\mathbb{N}$ satisfying.

$$|a_n - a| < \in \text{ if } n \in \mathbb{N} \text{ and } n \geq \mathbb{N}_\in .$$

In this case we say that (a_n) converges to a call a the limit of (a_n) and write $\ell\text{im}\,(a_n) = a$. Thus if there exists $a \in \mathbb{R}$ such that $\ell\text{im}\,(a_n) = a$ we say that $\ell\text{im}\, a_n$ exists.

Result: (i) If (a_n) is a seuence in $\mathbb{R}$ such that $\ell\text{im}(a_n) = a$ then for every sub sequence (b_n) of (a_n) $\ell\text{im}\,(b_n)$ exists and $\ell\text{im}\,(b_n) = 0$.

(ii) If $a_n = k$ for all $n \in \mathbb{N}$ where $k \in \mathbb{R}$, $\ell\text{im}\, a_n = k$.

(iii) $\ell\text{im}\,\dfrac{1}{n} = 0$

(iv) If $\ell\text{im}\,(a_{2n}) = \ell\text{im}\,(a_{2n-1}) = a$ then $\ell\text{im}\,(a_n) = a$

Solution:

(i) $x_{2n} = \dfrac{1}{2}\left(1 + (-1)^{2n}\right) = \dfrac{1+1}{2}$ for $n \in \mathbb{N}$

(ii) $x_{2n-1} = \dfrac{1}{2}\left(1 + (-1)^{2n-1}\right) = \dfrac{1-1}{2} = 0$ for $n \in \mathbb{N}$

From result (ii) $\ell\text{im}\,(x_{2n}) = 1$ and $\ell\text{im}(x_{2n-1}) = 0$

(iii) From result (i) $\ell\text{im}\,(x_n)$ does not exist.

$$s_{2n} = \frac{x_1 + x_2 + \cdots\cdots\cdots + x_{2n}}{2n} = \frac{0+1+0+1+\cdots\cdots+0+1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

$$s_{2n-1} = \frac{x_1 + x_2 + \cdots\cdots\cdots + x_{2n-1}}{2n-1} = \frac{0+1+0+1+\cdots\cdots+0}{2n-1} = \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n}$$

From (ii) $\ell im\, s_{2n} = \frac{1}{2}$

From (i) and (iii) $\ell im \frac{1}{2n} = 0$ so $\ell im\, s_{2n-1} = \frac{1}{2}$

From result (iv) $\ell im\, s_n = \frac{1}{2}$.

To show that $\ell im \quad s_n = \frac{1}{2}$

Let $\in > 0$. There exists $N_1 \in \mathbb{N} \ni \left| s_{2n} - \frac{1}{2} \right| < \in$ if $n \geq N_1$.

There exists $N_2 \in \mathbb{N} \ni \left| s_{2n-1} - \frac{1}{2} \right| < \in$ if $n \geq N_2$

Let $N = \max \left\{ 2(N_1) \text{ and } 2(N_2) \right\}$

If $n \geq N$ then $n \geq 2N_1$ and $n \geq 2N_2$

If n is even $\frac{n}{2} \geq N_1 \Rightarrow \left| s_n - \frac{1}{2} \right| < \in$

If n is odd $8n = 2m - 1$ then $2m-1, m \geq N_2$ so $\left| s_n - \frac{1}{2} \right| < \in$

$\Rightarrow \left| s_n - \frac{1}{2} \right| < \in$ if $n \geq N$

Hence $\ell im\, s_n = \frac{1}{2}$

Lesson Writer

I. RAMABHADRA SARMA

Augusting Louis Cauchy (1789 - 1857)

Cauchy pioneered the study of analysis, both real and complex and the theory of permutation groups. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.

Lesson - 12

SEQUENCES - II

12.1 Objective of the lesson:

To introduce the student some important classes of sequences such as monotone sequences, bounded sequences and the notion of Cauchy convergence and proper divergence. Bolzano weierstrass property is also studied.

12.2 Structure:

This lesson contains the following components:

12.3 Introduction

12.4 Monotone Sequences

12.5 Bolzano Weierstrass Property

12.6 Cauchy Convergence

12.7 Proper Divergence

12.8 Solutions to S.A.Q.'s

12.9 Summary

12.10 Technical Terms

12.11 Exercises

12.12 Model Examination Questions

12.13 Model Practical Problem with Solution

12.3 Introduction:

In this lesson we initiate our study with properties of a simple but powerful type of sequences - namely monotone sequences and characterize them into two classes convergent and property divergent.

Monotonicity is a very restrictive condition. Sequences of this type form a very small class of all sequences. But we come across several sequences whose convergence is to be decided if not the limit be found. Cauchy's criterion becomes handy in such cases. We make study of Cauchy convergence. We also establish Bolzano Weierstrass property for bounded sequences which helps in finding a convergent subsequence of a bounded sequence. After citing the example of contractive sequences which make effective use of Cauchy's criterion, we discuss divergence properties of sequences.

12.4 Monotone Sequences:

We recall the definition of monotone sequences. A sequence $X = (x_n)$ is said to be increasing if $x_n \geq x_m$ whenever $n \geq m$. X is said to be decreasing if $x_n \leq x_m$ whenever $n \geq m$. X is monotone if X is either increasing or decreasing.

12.4.1 Monotone Convergence Theorem: A monotone sequence of real numbers $X = (x_n)$ is convergent if and only if it is bounded. Further if

(a) $X = (x_n)$ is bounded and increasing, $\ell im\,(X_n) = \sup\{x_n : n \in \mathbb{N}\}$

(b) $X = (x_n)$ is bounded and decreasing $\ell im\,(x_n) = \inf\,\{x_n / n \in \mathbb{N}\}$

Proof: Let $X = (x_n)$ be a monotone sequence of real numbers. If X is convergent then X is bounded. Conversely suppose that X is monotone and bounded

(a) We first consider the case where X is increasing. Since X is bounded there exists a positive real number M such that $x_1 \leq x_n \leq M$ for all n. sine the set $E = \{x_n / n \geq 1\}$ is bounded above, E has supremum. Let $x^* = \sup\ E$. We show that X is convergent and $\ell im\,(x_n) = x^*$. If $\in > 0$, $x^* - \in$ is not an upper bound of E. So there exists a natural number k such that $x^* - \in < x_k$. Since X is increasing, $x_k \leq x_n$ if $n \geq k$

Hence $x^{*}-\in < x_n$ if $n \geq k$. This implies that $x^{*}-\in < x_n < x^{*}+\in$ for $n \geq k$. Since this holds for every $\in > 0$, $\ell im\, x_n = x^{*}$.

(b) If X is a bounded and decreasing sequence there exists a real number M^1 such that $M^1 \leq x_n \leq x_1$ for all n. The set $E^1 = \{x_n / n \in \mathbb{N}\}$ is bounded below and hence has infimum, Let $\ell = \inf E^1$.

We show that the sequence X is convergent and $\ell im(x_n) = \ell$. If $\in > 0$, $\ell + \in$ is not a lower bound of E^1, so there is a natural number k^1 such that $x_{k^1} < \ell + \in$

Since X is decreasing $x_n \leq x_{k^1}$ if $n \geq k^1$. So $\ell - \in < \ell \leq x_n \leq x_{k^1} < \ell + \in$ if $n \geq k^1$

$\Rightarrow \ell - \in < x_n \quad \ell + \in$ if $n \geq k^1$

Since $\in > 0$ is arbitrary it follows that X is convergent and $\ell im\,(x_n) = \ell$.

12.4.2 Examples:

(1) $\ell im \dfrac{1}{\sqrt{n}} = 0$

This can be proved directly follows:

If $\in > 0$, $\left|\dfrac{1}{\sqrt{n}}\right| = \dfrac{1}{\sqrt{n}} < \in \Leftrightarrow \dfrac{1}{n} < \in^2 \Leftrightarrow \dfrac{1}{\in^2} < n$

Thus if k is a positive integer $> \dfrac{1}{\in^2}$, for $n \geq k$ $\quad \left|\dfrac{1}{\sqrt{n}}\right| < \in$

Since $\in$ is arbitrary & $\in > 0$, $\ell im\left(\dfrac{1}{\sqrt{n}}\right) = 0$

How ever we can use monotone convergence theorem as well.

If $n \geq 1, \sqrt{n}+1 > \sqrt{n} \geq 1$, so $0 < \dfrac{1}{\sqrt{n}+1} < \dfrac{1}{\sqrt{n}} \leq 1$. Thus $\left(\dfrac{1}{\sqrt{n}}\right)$ is monotonically decreasing and bounded. Hence $\left(\dfrac{1}{\sqrt{n}}\right)$ converges and $\ell im \dfrac{1}{\sqrt{n}} = \inf \left\{\dfrac{1}{\sqrt{n}} \Big/ n \in \mathbb{N}\right\}$. We now show that inf $\left\{\frac{1}{\sqrt{n}} \middle/ n \in \mathbb{N}\right\} = 0$. Clearly $0 \le \frac{1}{\sqrt{n}}\ \forall\ n \in \mathbb{N}$

If $\in > 0 \quad \in +0 = \in > 0$. If $n > \frac{1}{\in^2}$, $\in^2 > \frac{1}{n} \Rightarrow \in > \frac{1}{\sqrt{n}}$

This shows that $\in$ is not a lower bound of E and hence inf $E = \ell$.

We can also prove that $\ell = 0$ as follows.

Since $\ell im\left(\frac{1}{\sqrt{n}}\right) = \ell$, $\ell m\left(\frac{1}{n}\right) = \ell im\left(\frac{1}{\sqrt{n}}\ \frac{1}{\sqrt{n}}\right) = \ell im\left(\frac{1}{\sqrt{n}}\right)\ \ell im\left(\frac{1}{\sqrt{n}}\right) = \ell \cdot \ell = \ell^2$

But $\ell im\, \frac{1}{n} = 0$. Hence $0 = \ell im\left(\frac{1}{n}\right) = \ell^2 \Rightarrow \ell = 0$

12.4.3 Example: Let $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots\cdots + \frac{1}{n}$. The sequence $X = (x_n)$ is monotonically increasing and unbounded.

Solution: Clearly $x_{n+1} = x_n + \frac{1}{n+1} > x_n$. Hence (x_n) is increasing. To prove that (x_n) is unbounded we consider x_{2^n}.

$$x_{2^n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots\cdots\cdots\cdots + \frac{1}{8}\right) + \cdots\cdots\cdots\cdots + \left(\frac{1}{z^{n-1}_{+1}} + \cdots\cdots\cdots + \frac{1}{2^n}\right)$$

$$= s_0 + s_1 + s_2 + \cdots\cdots\cdots\cdots\cdots + s_n$$

$$= \sum_{j=0}^{n} s_j$$

where $s_0 = 1$, $s_1 = \frac{1}{2}$ and $s_j = \frac{1}{2^{j-1}_{+1}} + \cdots\cdots\cdots\cdots\cdots + \frac{1}{2^j}\ (2 \le j \le n)$

For $2 \le j \le n$, $\quad s_j = \sum_{k=1}^{2^{j-1}} \frac{1}{2^{j-1} + k}$ (since $z^j = 2^{j-1} + 2^{j-1}$)

Since $2^{j-1} + k < 2^{j-1} + 2^{j-1} = 2^j$ for $1 \le k \le 2^{j-1}$, $\quad \frac{1}{2^{j-1} + k} > \frac{1}{2^j}$

$$\text{so } s_j > \sum_{x=1}^{2^{j-1}} \frac{1}{2^j} = \frac{1}{2^3} \cdot 2^{j-1} = \frac{1}{2}$$

This is true for $2 \le j \le n$ and $s_2 = \frac{1}{2}$ so

$$x_{2^n} = 1 + \frac{1}{2} + \sum_{j=2}^{n} s_j > 1 + \frac{1}{2} + \sum_{j=2}^{n} \frac{1}{2} = 1 + \sum_{j=1}^{n} \frac{1}{2} = 1 + \frac{n}{2} > \frac{n}{2}$$

Hence for every $n \ge 1$, $x_{2^n} > \frac{n}{2}$

Since the sequece $\left(\frac{n}{2}\right)$ is unbounded above, it follows that $\left(x_{2^n}\right)$ is unbounded above.

Since $\left(x_{2^n}\right)$ is an unbounded sequence in $\left(x_n\right)$

The sequence $\left(x_n\right)$ is not bounded.

Hence $\left(x_n\right)$ is not convergent.

12.4.4 Example: Let $x_n = \frac{1}{n+1} + \cdots\cdots\cdots\cdots + \frac{1}{n+n}$

Show that $\left(x_n\right)$ is increasing and bounded and hence convergent.

$$x_{n+1} = \frac{1}{n+2} + \cdots\cdots\cdots\cdots + \frac{1}{2n+2} = x_n + \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$$

$$\Rightarrow x_{n+1} - x_n = \frac{1}{2n-1} - \frac{1}{2n+2} > 0$$

Hence $\left(x_n\right)$ is increasing

Also $\frac{1}{n+k} < \frac{1}{n}$ for $1 \le k \le n$; $\Rightarrow 0 < x_n < \frac{n}{n} = 1$

Hence $\left(x_n\right)$ is bounded. Since $\left(x_n\right)$ is increasing and bounded $\left(x_n\right)$ is convergent.

12.4.5 S.A.Q.: Show that the sequence (x_n) defined by

$x_n = \frac{n!}{n^n}$ is monotonically decreasing and $0 < x_n < \frac{1}{n}$ for all n.

Find $\ell\text{im } x_n$.

12.4.6 Example: If $0 < b < 1$ show that $\ell\text{im}\left(b^n\right) = 0$

By induction $0 < b^{n+1} < b^n < 1$. Hence $\left(b^n\right)$ is decreasing and bounded.

By monotone convergence theorem $\left(b^n\right)$ is convergent. Let $\ell = \ell\text{im } b^n$.

The sequence $\left(b^{2n}\right)$ is a subsequence of the sequence $\left(b^n\right)$.

Hence $\ell = \ell\text{im}\left(b^{2n}\right)$

$$b^{2n} = \left(b^n\right)^2 \Rightarrow \ell = \ell\text{im}\left(b^{2n}\right) = \ell\text{im}\left(b^n\right)^2 = \ell\text{im}\left(b^n\right)\cdot\ell\text{im}\left(b^r\right) = \ell^2$$

$$\Rightarrow \ell(\ell - 1) = 0 \quad \Rightarrow \ell = 0 \text{ or } \ell = 1$$

$\ell = 1$ cannot happen since by the monotone convergence theorem $\ell = \inf\left\{b^n / n \geq 1\right\} < 1$. Hence $\ell = 0$

12.4.7 Example: The sequence (e_n) where $e_n = \left(1 + \frac{1}{n}\right)^n$ is increasing.

solution: We compare the terms in the binomial expansion of e_n and e_{n+1}

$$e_{n+1} = 1 + \binom{n+1}{1}\frac{1}{n+1} + \cdots\cdots + \binom{n+1}{r}\frac{1}{(n+1)^n} + \cdots\cdots + \frac{1}{(n+1)^{n+1}} \text{ and}$$

$$e_n = 1 + \binom{n}{1}\frac{1}{n} + \cdots\cdots + \binom{n}{r}\frac{1}{n^r} + \cdots\cdots + \frac{1}{n^n}$$

The $(r+1)$th terms in the r.h.s. are respectively

$$\binom{n+1}{r}\frac{1}{(n+1)^n} \text{ and } \binom{n}{r}\frac{1}{n^r}; \text{ for } 0 \le r \le n$$

e_{n+1} has an extra term $\dfrac{1}{(n+1)^{n+1}}$

For $1 \le r \le n$ $\binom{n}{r}\dfrac{1}{n^r} = \dfrac{(n-r+1)!}{r!} \quad \dfrac{1}{n^r}$

$$= \frac{1}{r!}\ \frac{n(n-1)\cdots\cdots\cdots(n-r+1)}{n^r}$$

$$= \frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\cdots\left(1-\frac{r-1}{n}\right)$$

$$< \frac{1}{r!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\cdots\cdots\left(1-\frac{r-1}{n+1}\right)$$

(since $n+1 > n,\ \dfrac{k}{n+1} < \dfrac{k}{n}$ and $\left(\dfrac{k}{n}\right) < \left(1-\dfrac{k}{n+1}\right)$

$$= \binom{n+1}{r}\frac{1}{(n+1)^r} \qquad \text{(as bove)}$$

Thus for $\le r \le n$ the $(r+1)$ term in the expansion of e_{n+1} is greater than the corresponding term for e_n. Further ℓ_{n+1} has an extra term.

Hence $e_{n+1} > e_n$ for $n \in \mathbb{N}$

12.4.8 S.A.Q.: Show that $2 < e_n < 3$ for $n \in \mathbb{N}$ using $2^{n-1} < n_1$ for $n > 2$

12.4.9 S.A.Q.: If $a > 0$ and $s_1 > 0$ define $s_{n+1} = \dfrac{1}{2}\left(s_n + \dfrac{a}{s_n}\right)$

show that $s_{n+1} < s_n$ for $n > 2$ and (s_n) converges to $s > 0$

where $s^2 = a$

12.4.10 Example: Let $y_1 = 1, y_{n+1} = \frac{1}{4}(2y_n + 3)$ for $n \geq 1$. Show that (y_n) is strictly monotonically increasing and bounded.

Solution: Clearly $y_2 = \frac{5}{4}$. Then $y_1 < y_2 < 2$. and $y_n > 0$ for all n

(i) We show by induction that $0 < y_k < 2$ for all $k \in \mathbb{N}$ Assume that $0 < y_n < 2$

Then $0 < 2y_n < 4 \Rightarrow 0 < 2y_n + 3 < 7 \Rightarrow 0 < \frac{1}{4}(2y_n + 3) < \frac{7}{4} < 2 \Rightarrow 0 < y_{n+1} < 2$

Since $0 < y_1 < 2$ and $0 < y_{n+1} < 2$ whenever $0 < y_n < 2$,

It follows by mathematical induction that $0 < y_n < 2$ for all n.

(ii) We show by induction that $y_n < y_{n+1}$ for all n.

Clearly $0 < y_1 < y_2$. Assume that $0 < y_n < y_{n+1}$ Then $0 < 2y_n + 3 < 2y_{n+1} + 3$

$\Rightarrow 0 < \frac{1}{4}(2y_n + 3) < \frac{1}{4}(2y_{n+1} + 3) \Rightarrow 0 < y_{n+1} < y_{n+2}$

Thus assuming that $0 < y_n < y_{n+1}$ we have proved that $0 < y_{n+1} < y_{n+2}$

Thus by mathematical induction

$$0 < y_n < y_{n+1} \text{ for all n}$$

$\text{Hence}(y_n)$ is strictly monotonically increasing.

12.4.11 Example: Let $Z = (z_n)$ where $z_1 = 1$ and $z_{n+1} = \sqrt{2z_n}$ show that (x_n) is strictly monotonically increasing and bounded.

Solution: We prove strictly monotonic increasing property and boundedness by using mathematical induction.

(a) $1 < z_n < z_{n+1}$ for all n

Clearly $1 < z_1$ and $z_2 = \sqrt{2z_1} = \sqrt{2} > 1 = z_1$ so $1 \leq z_1 < z_2$

Assume that $b < z_k < z_{k+1}$

$$\Rightarrow 2 < 2z_k < 2z_{k+1}$$

$$\Rightarrow 1 < \sqrt{2} < \sqrt{2z_k} < \sqrt{2z_{k+1}}$$

$$\Rightarrow 1 < z_{k+1} < z_{k+2}$$

Hence $1 < z_{k+1} < z_{k+2}$ whenever $1 < z_k < z_{k+1}$

Thus by mathematical induction $1 < z_n < z_{n+1}$ for all $n \in \mathbb{N}$

(b) We show that $1 \le z_n < 2$ for all n. Then clearly holds when $n = 1$

Assume that $1 \le z_k < 2 \Rightarrow 2 \le 2z < 4 \Rightarrow 1 < \sqrt{2} \le \sqrt{2z_k} < 2$

$\left(\because 0 < a < b \Rightarrow 0 < \sqrt{a} < \sqrt{b}\right) \Rightarrow 1 \le z_{k+1} < 2$

Thus $1 \le z_k < 2 \Rightarrow 1 \le z_{k+1} < 2$

Hence by mathematical induction $1 \le z_n < 2$ for all $n \in \mathbb{N}$.

12.4.12 Example: Let $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$ and $x_1 = 8$ Show that (x_n) is bounded and monotone $x_2 = \frac{1}{2}x_1 + 2 = \frac{8}{2} + 2 = 6 < x_1 \Rightarrow x_1 - x_2 > 0$ Since $x_{n+1} = \frac{1}{2}x_n + 2$ and $x_{n+2} = \frac{1}{2}x_{n+1} + 2$ $x_{n+1} - x_{n+2} = \frac{1}{2}(x_n - x_{n+1}) \cdots\cdots\cdots(1)$

We show by induction on n that $x_{n+1} - x_{n+2} = \frac{1}{2^n}(x_1 - x_2) = \frac{1}{2^{n-1}} \cdots\cdots\cdots(2)$

Replacing n by $n+1$ in (1) we get $x_{n+2} - x_{n+3} = \frac{1}{2}(x_{n+1} - x_{n+2})$

If we assume (2) for n we get $x_{n+2} - x_{n+3} = \frac{1}{2} \cdot \frac{1}{2^n}(x_1 - x_2) = \frac{1}{2^{n+1}}(x_1 - x_2)$

Since $x_2 - x_3 = \frac{1}{2}(6) - 2 = 6 - 3 - 2 = 1 = \frac{1}{2} \cdot (x_1 - x_2)$ It follows by mathematical induction that

$$x_{n+1} - x_{n+2} = \frac{1}{2^n}(x_1 - x_2) = \frac{1}{2^{n-1}} > 0 \text{ for all } n \in \mathbb{N}$$

Hence $x_{n+1} > x_{n+2}$ for $n \in \mathbb{N}$ i.e., (x_n) is monotonically decreasing.

Since $x_{n+1} = \frac{x_n}{2} + 2$ and $x_1 > 2$ It follows that $x_{n+1} > 2 \ \forall\ n$

Hence (x_n) is bounded below by 2. Also $x_n \le x_1 = 8 \ \forall\ n$, so $x_n \le 8 \ \forall\ n$

Hence (x_n) is bounded.

12.4.13 Example: Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$

Showthat (x_n) is bounded and monotone

$$x_2 = 2 - \frac{1}{x_1} \Rightarrow x_2 - x_1 = 2 - \frac{1}{x_1} - x_1 = -\left(x_1 - 2 + \frac{1}{x_1}\right) = -\left(\sqrt{x_1} - \frac{1}{\sqrt{x_1}}\right)^2 < 0 \Rightarrow x_2 < x_1$$

We show by induction that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. When $n = 1$ we have proved above that $x_2 < x_1$.

Assume that $k \in \mathbb{N}$ & $x_{k+1} < x_k$, so that $\frac{1}{x_k} < \frac{1}{x_{k+1}}$

$$x_{k+2} - x_{k+1} = \left(2 - \frac{1}{x_{k+1}}\right) - \left(2 - \frac{1}{x_k}\right) = \frac{1}{x_k} - \frac{1}{x_{k+1}} < 0 \quad \Rightarrow x_{k+2} < x_{k+1}$$

Thus $x_{k+2} < x_{k+1}$ whenever $x_{k+1} < x_k$. Since $x_2 < x_1$ it follows by induction that $x_{n+1} < x_n$ for $n \in \mathbb{N}$. Hence (x_n) is monotonically decreasing. To show that (x_n) is bounded we show that $x_k > 1 \ \forall\ n$. Since $x_1 > x_n$ for all n it follows that $x_1 > x_n > 1 \ \forall n \in \mathbb{N}$ and hence (x_n) is bounded. Clearly $x_2 = 2 - \frac{1}{x_1} > 2 - 1 = 1$

Assume that $x_k > 1$ then $\frac{1}{x_k} < 1$ so $2 - \frac{1}{x_k} > 1$ i.e. $x_{k+1} > 1$. Thus by induction $x_n > 1 \ \forall\ n \in \mathbb{N}$ Hence $1 < x_n < x_1 \ \forall\ n \in \mathbb{N} \ \Rightarrow (x_n)$ is bounded.

12.4.14 Example: Let a be an infinite set in $\mathbb{R}$ which is bounded above and $u = \sup A$. Show that there is an increasing sequence (x_n) in A such that $\ell im(x_n) = u$.

If $u \in A$ we choose $x_n = u$ for all $n \in \mathbb{N}$. Clearly (x_n) is increasing and $\ell im(x_n) = u$.

Assume that $u \notin A$. Since $y_1 = u - 1 < u$, y_1 is not an upper bound of A so that there is x_1 in A such that $y_1 < x_1 \le u$. Since $u \notin A$ and $x_1 \in A$, $y_1 < x_1 < u$. Let $y_2 = \max\left\{u - \frac{1}{2}, x_1\right\}$. Then $y_2 < u$. So there is $x_2 \in A$ such that $y_2 < x_2 < u$. Repeating this process we define $y_n = \max\left\{u - \frac{1}{n}, x_{n-1}\right\}$ where $x_1, \cdots\cdots x_{n-1}$ are chosen as above. Since $u - \frac{1}{n} < u$ & $x_{n-1} < u$, $y_n < u$. So as above, there is $x_n \in A$ such that $y_n < x_n < u$. Since A is infinite this process does not stop yielding an infinite sequence (x_n) in $A \ni y_n < x_n < u$.

Since $u - \frac{1}{n} \le y_n, u - \frac{1}{n} < x_n < u$ for all $n \in \mathbb{N}$ $0 < u - x_n < \frac{1}{n}$ for all n. Since $\ell im\left(\frac{1}{n}\right) = 0$, by Squeeze theorem $\ell im(u - x_n) = 0$. Hence $\ell im(x_n) = u$.

12.4.15 Limit Superior and Limt Inferior of a Sequence:

Let (x_n) be a bounded sequence. For each n let $s_n = \sup\{x_k / k \ge n\}$ and $t_n = \inf\{x_k / k \ge n\}$. (1) Then (t_n) is increasing

Proof: By definition $t_n = \inf\{x_k / k \ge n\} \Rightarrow t_m \le x_k$ if $k \ge n \Rightarrow t_n \le x_k$ if $k \ge n+1$

This is true for every $k \ge n+1$

$\Rightarrow$ t_n is lower bounded for $\{x_k / k \ge n+1\}$

Since $t_{n+1} = g\ell b\{x_k / k \ge n+1\}$, $t_n \le t_{n+1}$. Thus (t_n) is increasing.

(2) (s_n) is decreasing:

Proof: By definition $s_n = \sup\{x_k / k \geq n\} \Rightarrow s_n \geq x_k$ if $k \geq n \Rightarrow s_n \geq x_k$ if $k \geq n+1$

$\Rightarrow s_n$ is an upper bound of $\{x_k / k \geq n+1\}$

Since $s_{n+1} = \sup\{x_k / k \geq n+1\}$

$s_{n+1} \leq s_n$ for all n

Then (s_n) is increasing.

(3) (s_n) and (t_n) are convergent and $\ell im(t_n) \leq \ell im(s_n)$:

Proof: Since (x_n) is bounded there are $m \in \mathbb{R}$ and $M \in \mathbb{R}$ such that

$m \leq x_n \leq M$ for all $n \in \mathbb{N}$

$\Rightarrow m \leq x_n \leq M$ for all $n \geq k$ and $k \in \mathbb{N}$

$\Rightarrow m \leq t_k \leq s_k \leq M$ for all $k \in \mathbb{N}$ $\{\text{Since } \inf\{x_n\} \leq \sup\{x_n\}\}$

$\Rightarrow (s_n)$ is bounded and increasing and (t_n) is bounded and decreasing

$\Rightarrow (s_n)$ and (t_n) are convergent. Since $t_n \leq s_n$ for all $n \in \mathbb{N}$, $\ell im\ t_n \leq \ell im\ s_n$

(4) If $\ell im\ s_n = \ell im\ t_n = \ell$, then (t_n) is convergent and $\ell im\ x_n = \ell$

Proof: Let (x_{n_k}) be any subsequence of (x_n). We show that $\ell im(x_{n_k}) = d$.

Let $\in > 0$ since $\ell im\ s_n = \ell$ there is a natural number N_1 such that $\ell + \in > s_{N_1}$. Since (s_n) is decreasing $\ell + \in > s_n$ for $n \geq N_1$.

Since $\ell im\ t_n = \ell$, there is a natural number N_2 such that $\ell - \in < t_{N_2}$

Since (t_n) is increasing, $\ell - \in < t_{N_2} \leq t_n$ for $n \geq N_2$

Let $N(\in) = \max\{N_1, N_2\}$. Then $\ell - \in < t_{N_2} \leq t_{N(\in)} \leq s_{N(\in)} < \ell + \in$

$\Rightarrow \ell - \in < t_n \leq x_n \leq s_n < \ell + \in$ for $n \geq N(\in)$

$\Rightarrow \ell - \in < t_n \leq x_k \leq s_N < \ell + \in$ for $n \geq N(\in)$

$\Rightarrow \ell - c < x_k < \ell + \in$ for $k \geq N\ (\in)$

This is true for every $k \geq N(\in)$, hence

$\ell - \in < x_{n_k} < \ell + \in$ for $k \geq N(\in) \quad (\because N_k \geq k)$

Hence $\left| {}^{x}n_k - \ell \right| < \in$ for $k \geq N\ (\in) \Rightarrow \ell im\ {}^{x}n_k = \ell$

The number $\ell im\left(s_n\right)$ is called that limit superior of $\left(x_n\right)$ and $\ell im\left(t_n\right)$ is called the limit inferior of $\left(x_n\right)$.

12.5 Bolzano - Weiestrass' Theorem:

We say that a sequence $\left(I_n\right)$ of intervals is nested if $I_n \supseteq I_{n+1}$ for every natural number n.

i.e., $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots\cdots\cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots\cdots\cdots$

12.5.1 Theorem: If $\left(I_n\right)$ is a sequence of nested intervals and each I_n is closed and bounded then there exists a real number α such that $\alpha \in I_n$ for every n.

Proof: Let $I_n = \left[a_n , b_n\right]$. Since $I_n \supseteq I_{n+1}$ for every n We have $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$.

Also if $m \geq n \cdot I_n \supseteq I_n$ so that $a_n \leq a_m \leq b_m \leq b_n \cdots\cdots\cdots (1)$

Hence $\left(a_n\right)$ is increasing and $\left(b_n\right)$ is decreasing .... (2)

Further if $m < n \quad a_m < a_n \leq b_n \leq b_m \cdots\cdots\cdots (3)$

Thus from (1) & (2) it follows that $a_m \leq b_n$ for all m and n ....(4)

From (4) we conclude that every b_n is an upper bound for $\left(a_n\right)$ and every a_n is a lower bound for $\left(b_n\right)$.

Consequently if $a = \sup\left(a_n\right)$ and $b = \inf\ a_n\ a_n \leq a \leq b \leq b_n$ for every n.

If $a \leq \propto \leq b$ then $a_n \leq a \leq \propto\ b \leq b_n\ \forall\ n$

So that $\propto \in \left[a_n, b_n\right]$ for every n.

12.5.2 Theorem: If $I_n = [a_n, b_n],\ n \in \mathbb{N}$ is a nested sequence of closed bounded intervals and the lengths $b_n - a_n$ of I_n satisfy $\inf(b_n - a_n / n \in \mathbb{N}) = 0$ then the number α such that $\alpha \in I_n$ for all n, is unique.

Proof: Since the sequence (a_n) is increasing, (b_n) is decreasing and every b_n is an upper bound of (a_n) and every a_n is a lower bound of (b_n) it follows as in the above theorem that if $a = \sup\{a_n / n \in \mathbb{N}\}$ and $b = \inf\{b_n / n \in \mathbb{N}\}$, $a_n \le a \le b \le b_n$ for all $n \in \mathbb{N}$.

$$\Rightarrow 0 \le b - a \le b_n - a_n \text{ for all } n \in \mathbb{N} \Rightarrow 0 \le b - a \le \inf(b_n - a_n / n \in \mathbb{N}) = 0$$

$$\Rightarrow b - a = 0 \quad \text{i.e., } b = a \Rightarrow a \le \alpha \le b \Rightarrow \alpha = a = b$$

Also if $\alpha \in I_n$, $a_n \le \alpha \le b_n$ for all n

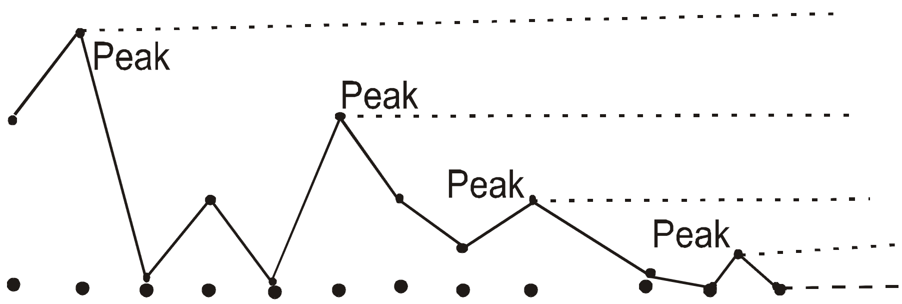
$\Rightarrow \alpha$ is an upper bound of (a_n) and a lowe r bound of $(b_n) \Rightarrow a \le \alpha \le b \Rightarrow \alpha = a = b$

Thus the number α such that $\alpha \in I_n$ for every n, is unique.

12.5.3 Monotone Subsequence Theorem:

If $X = (x_n)$ is a sequence of real numbers then X has a subsequence that is monotone.

Proof: We call a term x_m a peak of the sequence (x_n) if $x_m \ge x_n$ for all $n \ge m$.



Case (1): If X does not have peaks, x_1 is not a peak of X. So there is natural number $n_1 > 1$ such that $x_1 < x_{n_1}$. Again x_{n_1} is not peak of X, so there is $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$. We repeat this process. Assuming that $1 < n_1 < n_2 < \cdots\cdots n_k$ are already chosen such that $x_1 < x_{n_1} < x_{n_2} < \cdots\cdots < x_{n_k}$, we use the fact that x_{n_k} is a peak of X and choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} > x_{n_k}$. By induction we get a sub-sequence $\left(x_{n_k}\right)$ of (x_n) such that $x_1 < x_{n_1} < x_{n_2} < \cdots\cdots < x_{n_k} < \cdots\cdots$

This is a monotonically increasing subsequence of X.

Case (ii): If X has a finite number of peaks, say $x_{n_1} > x_{n_2} > \cdots\cdots > x_{n_r}$ and $m_1 = n_r + 1$. Since $m_1 > n_r$, x_{m_1} is not a speak, so there is $m_2 > m_1$ such that $x_{m_1} < x_{m_2}$ is not a peak, so there is $m_2 > m_1$ such that $x_{m_1} < x_{m_2}$. Again $m_2 > m_1 > n_r$ imples that x_{m_2} is not a peak, so there is $m_3 > m_2$ such that $x_{m_3} < x_{m_2}$. We repeat the argument of case (i) and get an increasing sequence. $x_{m_1} < x_{m_2} < \cdots\cdots < x_{m_k} < \cdots\cdots$

which is a subequence of X.

Case (iii): If X has infinitely many peaks, Let $P = \{m \in \mathbb{N},\ x_m \text{ is a peak of } X\}$. P is an infinite subset of $\mathbb{N}$. So P has a first element, say m_1. Then x_m, is the "first" peak. Remove m_1 from P, $P - \{m_1\}$ is infinite, so as above has first element say m_2. then $m_1 < m_2$ and since x_{m_1} is a peak $x_{m_1} > x_{m_2}$. Since P is infinite $P - \{m_1, m_2\}$ is infinite. We choose the first element, say m_3 of $P - \{m_1, m_2\}$.

Since $m_1 < m_2 < m_3$ and x_{m_2} is apeak so $x_{m_2} > x_{m_3}$.

This implies that $x_{m_1} \geq x_{m_2} \geq x_{m_3}$ and each x_{m_1} is a peak. We repeat this process and at the $(k+1)$ th stage we pick up the first element m_{k+1} of $P - \{m_1, \cdots\cdots, m_k\}$ so that as above $m_{k+1} > m_k$ and being a peak $x_{m_k} \geq x_{m_{k+1}}$.

Since the set P does not exharust at any stage we get a decreasing sequence of peaks.

$$x_{m_1} > x_{m_2} > x_{m_3} > \cdots\cdots > x_{m_k} > \cdots\cdots$$

which form a subsequence of X.

12.5.4 Bolzano - Weirestrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

We present two proofs of this theorem. The first is based on monotone subsequence theorem and the second proof makes use of the theorem on nested intervals.

First Proof: Let $X=(x_n)$ be a bounded sequence of real numbers. By the Monotone subsequence theorem X has a monotone sub sequence $Y=\left(x_{n_k}\right)$. Since X is bounded so is y. Since Y is bounded and monotone, Y is convergent.

Second Proof: Since X is bounded there exist a, b in $\mathbb{R}$ such that $a \le x_n \le b$ for all n. We obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots\cdots \supseteq I_k \supseteq \cdots\cdots$

such that length $I_k = \frac{b-a}{2^k}$ for every k and a sub sequence $\left(x_{n_k}\right)$ of X such that $x_{n_k} \in I_k \ \forall \ k$. First Let $n_1 = 1$ Let $I=[a, b]$, $I' = [a, c], I'' = [c, b]$ where $c = \frac{a+b}{2}$ is the midpoint of I. Thus I', I'' are obtained by bisecting I into equal parts. Since I contains x_n for all n, $x_n \in I''$ for every n.

Let $A_1 = \left\{n \in \mathbb{N} : n > n_1 = 1 \quad x_n \in I'\right\}$ $B_1 = \left\{n \in \mathbb{N}/n > n_1 = 1 \ \& \ x_n \in I''\right\}$

Clearly either A_1 or B_1 is an infinite set.

If A_1 is infinite we write $I_1 = I^1$., A_1 has the first element, say n_2.

If A_1 is finite the B_1 must be infinite. In this case we write $I_1 = I''$ and choose n_2 to be the first element in B_1. We now bisect I_1 isnto two equal parts I_1', I_1'' as above

Write $A_2 = \left\{n \in \mathbb{N} \ : \ n > \mathbb{N}_2, \ x_n \in I_1'\right\}$ and $B_2 = \left\{n \in \mathbb{N} : n > n_2, x_n \in I_1''\right\}$

and obtain n_2 as the first element of A_2 if A_2 is infinite and as the first element of B_2, if A_2 is finite. We continue this process to obtain a sequence (I_n) of nested intervals such that length of $I_n = \frac{1}{2}$ length I_{n-1} and a sub sequence $n_1 < n_2 < \cdots\cdots < n_k \ \cdots\cdots$ in $\mathbb{N}$ such that $x_{n_k} \in I_k$ for every k.

By the theorem on nested intervals, there is a unique α such that $\alpha \in I_k$ for all k. Since $x_{n_k} \in I_k$ it follows that

$$0 \le \left|x_{n_k} - \alpha\right| \le \text{length } I_k = \frac{b-a}{2^k}$$

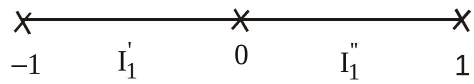
Since $\ell im \left(\frac{1}{2^k}\right) = 0$ it follows by the squeze theorem that $\ell im \left|x_{n_k} - \alpha\right| = 0$

This implies that $\ell im \ x_{n_k} = \alpha$.

12.5.5 Example: Apply the second proof of Bolzano weierestarss theorem to the sequence $X = (x_n)$ where $x_n = \frac{(-1)^n}{n}$ and $I_1 = [-1,\ 1]$ $x_{2n} = \frac{1}{2n}$ and $x_{2n-1} = \frac{1}{2n-1}$ for $n \in \mathbb{N}$.

Step 1: $I_1 = I = [-1, 1] = \text{Bisect } I_1 = I_1^{'} \cup I_1^{''}$

$n_1 = 1$

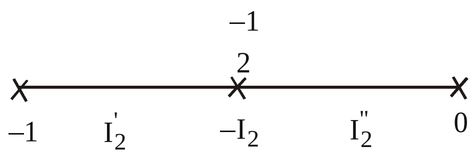


$$A_1 = \{n \in \mathbb{N}/n > 1,\ x_n \in I_1{'}\}$$

$$= \{n > 1 \quad x_n \le 0\} = \{1, 3, 5, 7, 9, \cdots\cdots\cdots\}$$

$$B_1 = \left\{n \in \mathbb{N}/n > 1\ x_{\mathbb{N}} \in I_1^{''}\right\} = \{2, 4, 6, 8, \cdots\cdots\cdots\cdots\}$$

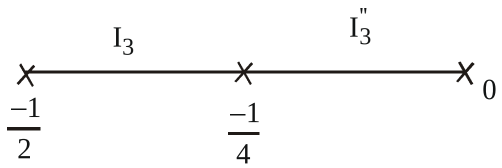
$$n_2 = 3, \qquad I_2 = [-1, 0] \qquad I_2 = I_2^{'} \cup I_2^{''}$$



$$A_2 = \left\{n \in \mathbb{N}\big/n > 3 \quad x_{\mathbb{N}} \in I_2^{'}\right\} = \left\{n > 3\Big/x_n \le \frac{-1}{2}\right\} = \phi$$

$$B_2 = \left\{n \in \mathbb{N}\big/x > 3,\ x_n \in I_2^{''}\right\} = \left\{n > 3\Big/0 > x_n > \frac{-1}{2}\right\} = \{5, 7, 9, \cdots\cdots\cdots\}$$

$n_3 = 5 \qquad I_3 = \left[\frac{-1}{2}, 0\right] = I_3^{'} \cup I_3^{''}$



$A_3 = \left\{n > 5 / x_n \in I_3^1\right\} = \phi$

$B_3 = \left\{n > 5 / x_n \in I_3^{''}\right\} = \{5, 7, 9, \cdots\cdots\cdots\}$

Repeating this process we get

$n_k = 2k-1,\ A_k = \phi,\ B_k = \{2k+1,\ 2k+3 \cdots\cdots\cdots\}$

and $\quad I_k = \left[\frac{-1}{2(k-1)}, 0\right] \qquad n_k = 2k-1$

$x_{n_k} = \frac{-1}{2k-1} \in I_k$

$\ell im \quad x_{n_k} = \ell im \ \frac{-1}{2k-1} = 0$

12.5.6 Example: Let (x_n) be a bounded sequence and $s = \sup(x_n / n \in \mathbb{N})$.

If $s \notin \{x_n / n \in \mathbb{N}\}$ show that there is a sub sequence $\left(x_{n_k}\right)$ of (x_n) such that $\ell im \ x_{n_k} = s$.

Solution: Since s is the least upper bound and $y_1 = s - 1 < s$ there is a smallest natural number n_1 such that $y_1 < x_{n_1} \le s$ clearly $s \ne x_1$ so $y_1 < x_{n_1} < s$

Let $y_2 = \max\left\{s - \frac{1}{2}, x_{n_1}\right\}$. Then $y_2 < s$. So as above there is $n_2 > n_1$ such that and n_2 is the smallest natural number such that $y_2 < x_{n_2} < s$

Repeating this process we get a sub sequence x_{n_k} of X and a sequence (y_k) such that

$$y_k = \max\left\{s-\frac{1}{k}, x_{n_{k-1}}\right\} \text{ and } y_k < x_{n_k} < s$$

$$\Rightarrow s-\frac{1}{k} < x_{n_k} < s \quad \Rightarrow 0 \le s - x_{n_k} < \frac{1}{k}$$

By Squeeze theorem $\ell im\left(s - x_{n_k}\right) = 0$ so that $\ell im\left(x_{n_k}\right) = s$.

12.6 Cauchy Convergence:

The applicability of monotone convergence theorem is restricted to monotone sequences alone which are considerably very small in the set of all sequences. The Cauchy criterion, which we discuss now covers the entire class of sequences without any restriction.

Definition: A sequence $X = (x_n)$ is said to be a Cauchy sequence if for every $\in > 0$ there is a natural number $H(\in)$ such that for all natural numbers $n \ge H(\in)$ and $m \ge H(\in)$ the terms x_n, x_m satisfy $|x_n - x_m| < \in$.

(Here after we write this condition as "$\left(|x_n - x_m| < \in \text{ for } n, m \ge H(\in)\right)$"

Our goal is to show that in $\mathbb{R}$ Cauchy sequences are precisely convergent sequences. We first present a couple of examples.

12.6.1 Examples: (a) The sequence $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

If $\in > 0$ we choose using Archemedian principle a natural number $H(\in) > \frac{2}{\in}$. If m, n are natural numbers such that $m \ge H(\in)$ and $n \ge H(\in)$.

$$\frac{1}{m} < \frac{\in}{2} \text{ and } \frac{1}{n} < \frac{\in}{2} \text{ so } \left|\frac{1}{m} - \frac{1}{n}\right| \le \frac{1}{m} + \frac{1}{n} < \frac{\in}{2} + \frac{\in}{2} = \in$$

This implies that $\left(\frac{1}{n}\right)$ is a Cauchy sequence.

Example: **(b)** The sequence $\left((-1)^n\right)$ is not a Cauchy sequence.

If this were a Cauchy sequence, for $\in = \frac{1}{2}$ t here would exist a natural number H such that

$$\left|(-1)^n - (-1)^m\right| < \frac{1}{2} \text{ for } m \geq H \text{ and } n \geq H.$$

In particular it would follow that $\left|(-1)^{2H} - (-1)^{2H+1}\right| < \frac{1}{2}$

i.e., $2 < \frac{1}{2}$ which is absend

So our assumption is false. Hence $\left((-1)^n\right)$ is not a Cauchy sequence.

12.6.2 Lemma: If $X = (x_n)$ is a convergent sequence of real numbers then X is a Cauchy sequence.

Proof: Let $\ell im\ X = x$. Given $\in > 0$ there is a natural number $K(\in)$ that corresponds to $\frac{\in}{2}$ so that for all $n \geq K(\in)$ $|x_n - x| < \frac{\in}{2}$ For $n \geq K(\in)$ and $m \geq K(\in)$

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq |x_n - x| + |x - x_m| < \frac{\in}{2} + \frac{\in}{2} = \in$$

This is true for every $\in > 0$. So (x_n) is a Cauchy sequence.

12.6.3 Lemma: A Cauchy sequence of real numbers is bounded.

Proof: Let $X = (x_n)$ be a Cauchy sequence of real numbers. For $\in = 1$ there corresponds a natural number k such that $|x_n - x_m| < 1$ if $n \geq k$ and $m \geq k$

In particular $|x_n - x_k| < 1$ if $n \geq k$, so that $|x_n| - |x_k| < |x_n - x_k| < 1$ if $n \geq k$

$\Rightarrow |x_n| < 1 + |x_k|$ if $n \geq k$ If $M = \max\left\{|x_1|, \cdots\cdots\cdots |x_{k-1}| \text{ and } 1 + |x_k|\right\}$

then $|x_n| \leq M$ if $1 \leq n \leq k - 1 \leq 1 + |x_k| \leq M$ if $n \geq k$

so that $|x_n| \leq M$ for all n. Hence $X = (x_n)$ is bounded.

12.6.4 Example: A bounded sequence is not necessarily a Cauchy sequence.

Let $x_n = -1$ for all n, $y_n = +1$ for all n

and $Z = (Z_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \cdots\cdots\cdots) = (-1, +1, -1, 1, -1, 1, -1, \cdots\cdots\cdots)$

so that $Z_n = (-1)^n$

It is clear that Z is not a Cauchy sequence but range $Z = \{-1, 1\}$ is bounded so Z is bounded.

12.6.5 Lemma: If a Cauchy sequence $X = (x_n)$ contains a subsequence $X' = (x_{n_k})$ that is convergent then X is convergent, hence $\ell im\ X = \ell im\ X^1$.

Proof: Since (x_n) is a Cauchy sequence, gvien $\in > 0$ there is a natural number k_1 such that for $n \geq k_1$ and $m \geq k_1$ $|x_n - x_m| < \frac{\in}{2}$

Since (x_{n_k}) converges, corresponding to the above $\in > 0$ there is a natural number k_2.

such that $|x_{n_k} - x| < \frac{\in}{2}$ for all $n_k \geq k_2$ where $x = \ell im\ (x_{n_k})$ Let $k(\in) = \max\{k_1, k_2\}$

Since $1 \leq n_1 < n_2 < \cdots\cdots\cdots\cdots < n_k < \cdots\cdots\cdots$

is a strictly increasing sub sequence of $\{1, 2, 3, \cdots\cdots\cdots\}$

$n_k \geq k$ for every $k \in \mathbb{N}$ so that

if $k_0 \geq k(\in)$, and $k \geq k_0, n_k \geq n_{k_0} \geq k_0 \geq k(\in)$ so that $|x_{n_k} - x| < \frac{\in}{2}$

If $n \geq k_0$ $|x_n - x| - |(x_n - x_{n_k}) + (x_{n_k} - x)| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\in}{2} + \frac{\in}{2}$

Since $\in > 0$ is arbitrary

$\ell im\ (x_n) = x = \ell im\ (x_{n_k})$

i.e., $\ell im\ X = \ell im\ X^1$

12.6.6 Cauchy Convergence Criterion: A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof: Let $X = (x_n)$ be any sequence of real numbers.

If X is convergent then X is a Cauchy sequence (by 12.6.2) conversely suppose X is a Cauchy sequence. By lemma 12.6.3 X is bounded

By Bolzano weierestrass theorem X has a covnergent sub sequence say $X^1 = (x_{n_k})$

By lemma 12.6.5 the sequence X is convergent and

$$\ell im\ X = \ell im\ X^1$$

12.6.7 Application to contractive sequences:

Definition: A sequence (x_n) of real numbers is said to be contractive if there is a real number C such that $0 < C < 1$ and

$$|x_{n+2} - x_{n+1}| \le C\,|x_{n+1} - x_n| \text{ for all } n \in \mathbb{N}$$

Any such C is called a constant of the contractive sequence (x_n) or simply a contractive constant.

12.6.8 Theorem: Every contractive sequence is a Cauchy sequence and hence is covergent.

Proof: Let (x_n) be a contractive sequence with contraction constant C. Then $|x_{n+2} - x_{n+1}| \le C\,|x_{n+1} - x_n|$ for all n. This implies that $|x_{n+2} - x_{n+1}| \le C^n\,|x_2 - x_1|$ for all n. because this holds for $n = 1$.

$$|x_3 - x_2| \le C\,|x_2 - x_1|$$

when this inequality holds for n,

$$|x_{n+3} - x_{n+2}| \le C\,|x_{n+2} - x_{n+1}| \le C \cdot C^n\,|x_2 - x_1| = C^{n+1}\,|x_2 - x_1|$$

so that the inequality holds for n+1.

Consequently $|x_{n+2} - x_{n+1}| < C^n\,|x_2 - x_1|$ for all $n \in \mathbb{N}$.

We now show that (x_n) is a Cauchy sequence.

If $m > n$

$$|x_m - x_n| = |x_m - x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{m+1} - x_n|$$

$$\leq \left(C^{m-2} + C^{m-3} + \cdots + C^{n-1}\right)|x_2 - x_1|$$

$$= C^{n-1}\left(1 + C + \cdots + C^{m-n-1}\right)|x_2 - x_1|$$

$$= C^{n-1}\left(\frac{1-C^{m-n}}{1-C}\right)|x_2 - x_1|$$

$$< \frac{C^{n-1}}{1-C} \cdot |x_2 - x_1| \qquad \left(\text{Since } 0 < C^{m-n} < 1\right)$$

Since $0 < C < 1$, $\ell im\, C^n = 0$ so given $\in > 0$ there is a natural number $k(\in)$

such that $X^{n-1} < \dfrac{(1-C)\in}{1+|x_2 - x_1|}$ for $n \geq k(\in)$

$$\Rightarrow \frac{c^{n-1} \cdot |x_2 - x_1|}{1-C} < C^{n-1}\frac{(1+|x_2 - x_1|)}{1-C} < \in \quad \text{for } n \geq k(\in)$$

Hence if $m > n \geq k(\in)$ $|x_m - x_n| < \in$

Hence if $m \geq k(\in)$ and $n \geq k(\in)$ $|x_m - x_n| < \in$

This implies that (x_n) is a Cauchy sequence, hence convergent.

12.6.9 Corollary: If (x_n) is a contractive sequence of real numbers with contractive constant C and $x^* = \ell im\,(x_n)$ then

(i) $|x^* - x_n| \leq \dfrac{C^{n-1}}{1-C}|x_2 - x_1|$ and (ii) $|x^* - x_n| \leq \dfrac{C}{1-C}|x_n - x_{n-1}|$

Proof: (i) From 12.6.8, $\left|x_m - x_n\right| \le \dfrac{C^{n-1}}{1-C}\left|x_2 - x_1\right|$ if $m > n \cdots\cdots\cdots\cdots\cdot(A)$

Since $\lim\limits_{m} x_m = x^*$, for any fixed $n \in \mathbb{N}$ $\ell im\left(x_m - x_n\right) = x^* - x_n$

$\Rightarrow \ell im\left|x_m - x_n\right| = \left|x^* - x_n\right|$ Since (A) holds for $m > n$,

$$\left|x^* - x_n\right| = \lim_{m}\left|x_m - x_n\right| \le \frac{C^n}{1-C}\left|x_2 - x_1\right|$$

(ii) If $m > n$, $\left|x_m - x_n\right| \le \left|x_m - x_{m-1}\right| + \cdots\cdots\cdots\cdots + \left|x_{n+1} - x_n\right|$

Also for all $k \in \mathbb{N}$,

$$\left|x_{n+k} - x_{n+k-1}\right| \le C^k\left|x_n - x_{n-1}\right|$$

Hence $\left|x_m - x_n\right| \le \left(C^{m-n} + C^{m-n-1} + \cdots\cdots + C^2 + C\right)\left|x_n - x_{n-1}\right|$

$$= C\left(1 + C + \cdots\cdots\cdots\cdot + C^{m-n-1}\right)\left|x_n - x_{n-1}\right|$$

$$= C\frac{\left(1 - C^{m-n}\right)}{1-C}\left|x_n - x_{n-1}\right|$$

$$< \frac{C}{1-C}\left|x_n - x_{n-1}\right|$$

12.6.10 Example: Define the sequence $\left(x_n\right)$ by $x_1 = 1, x_2 = 2$ and $x_{n+1} = \dfrac{1}{2}\left(x_n + x_{n-1}\right)$.

(i) Show that $1 \le x_n \le 2$ for all $n \in \mathbb{N}$.

The proof is by induction. The inequality holds good when $n = 1$ and $n = 2$. Assuming that $1 \le x_k \le 2$ for $1 \le k \le n$ we get that x_{n+1}, being the arithmetic mean of x_n and x_{n-1} lies between x_{n-1} and x_n. Since $1 \le x_{n-1} \le 2$ and $1 \le x_n \le 2$ it follows that $1 \le x_{n+1} \le 2$.

Hence by induction the inequality holds for all $n \in \mathbb{N}$.

(ii) $1 \le x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n} \le 2$ fro all $n \in \mathbb{N} \cdots\cdots\cdots\cdots (2)$

The proof is induction. The inequality holds when $n = 1$ since

$$x_1 = 1,\ x_2 = 2,\ x_3 = \frac{3}{2} \text{ and } x_4 = \frac{1}{2}\left(\frac{3}{2} + 2\right) = \frac{7}{4}$$

Assume that (2) holds for $n = k \in \mathbb{N}$

$$x_{2k+3} = \frac{x_{2k+1} + x_{2k+2}}{2} \Rightarrow x_{2k+1} < x_{2k+3} < x_{2k+2}$$

$$x_{2k+4} = \frac{x_{2k+3} + x_{2k+2}}{2} \Rightarrow x_{2k+3} < x_{2k+4} < x_{2k+2}$$

$\Rightarrow x_{2k+1} < x_{2k+3} < x_{2k+4} < x_{2k+2}$ so that (2) holds for $K = k + 1$

By induction (2) holds for all $n \in \mathbb{N}$

(iii) (x_{2n-1}) is increasing and (x_{2n}) is decreasing. Moreover $[x_{2n-1}, x_{2n}] \supseteq [x_{2n+1}, x_{2n+2}]$ so that $x_{2n+1} < x_{2m}$ for all n, m in $\mathbb{N}$. Follows from (ii)

(iv) $|x_{n+1} - x_n| = \dfrac{1}{2^{n-1}}$ for $n \in \mathbb{N} \cdots\cdots\cdots\cdots (3)$

When $n = 1, |x_2 - x_1| = |2 - 1| = 1 = \dfrac{1}{20}$ so (3) holds when $n = 1$

Assume that (3) holds for k

$$|x_{k+1} - x_k| = \left|\frac{1}{2}(x_k + x_{k-1}) - x_k\right| = \frac{1}{2}|x_k - x_{k-1}| = \frac{1}{2} \cdot \frac{1}{2^{k-2}} = \frac{1}{2^{k-1}}$$

So (3) holds for $k + 1$ when it holds for k. By induction (3) holds for all n.

(v) $x_{2n+1} = 1 + \dfrac{1}{2} + \dfrac{1}{2^3} + \cdots\cdots + \dfrac{1}{2^{2n-1}}$ for $n \in \mathbb{N} \cdots\cdots\cdots\cdots (4)$

$x_3 = \dfrac{1}{2}(x_2 + x_1) = 1 + \dfrac{1}{2} \Rightarrow (4)$ holds when $n = 1$ assume that (4) holds for k. From

(iv) $|x_{2k+2} - x_{2k+1}| = \dfrac{1}{2^{2k}}$ and $|x_{2k+3} - x_{2k+2}| = \dfrac{1}{2^{2k+1}}$.

From (ii) $x_{2k+1} < x_{2k+3} < x_{2k+2} < x_{2k} \Rightarrow 0 < x_{2k+2} - x_{2k+1} = |x_{2k+2} - x_{2k+1}| = \frac{1}{2^{2k}}$

and similarly $0 < x_{2k+2} - x_{2k+3} = |x_{2k+3} - x_{2k+2}| = \frac{1}{2^{2k+1}} \Rightarrow x_{2k+3} = x_{2k+2} - \frac{1}{2^{2k+1}}$

$$= x_{2k+1} + \frac{1}{2^{2k}} - \frac{1}{2^{2k+1}} = x_{2k+1} + \frac{1}{2^{2k}}\left(1 - \frac{1}{2}\right) = x_{2k+1} + \frac{1}{2^{2k+1}}$$

$$= 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots\cdots\cdots + \frac{1}{2^{2k+1}}$$

By induction (4) holds for all n.

(vi) (x_n) is Cauchy sequence and hence converges.

For $m > n$ $|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots\cdots + (x_{n+1} - x_n)|$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots\cdots\cdots + |x_{n+1} - x_n|$$

$$= \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots\cdots\cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \cdots\cdots + \frac{1}{2^{m-n-1}}\right)$$

$$= \frac{1}{2^{n-1}}\left(\frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}}\right)$$

$$= \frac{1}{2^{n-2}}\left(1 - \frac{1}{2^{m-n}}\right) < \frac{1}{2^{n-2}}$$

Since $\ell im \frac{1}{2^{n-2}} = 0$, given $\in > 0$ there is a $k(\in) \in \mathbb{N}$ such that $\frac{1}{2^{\mathbb{N}-2}} < \in$ for $n \geq k(\in)$ so for $m > n \geq k(\in)$, $|x_m - x_n| < \in$ this implies that (x_n) is a Cauchy sequence. By Cauchy's criterion, (x_n) converges.

(vii) $\ell im\ x_n = \frac{5}{3}$

By step 5 $\quad x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots\cdots\cdots + \frac{1}{2^{2n-1}}$

$$= 1 + \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots\cdots + \frac{1}{2^{2n}}\right)$$

$$= 1 + \frac{1}{2}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\cdots + \left(\frac{1}{4}\right)^n\right)$$

$$= 1 + \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - \frac{1}{4}}$$

$$= 1 + \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^{n+1}\right)$$

Since $0 < \frac{1}{4} < 1 \qquad \ell im \left(\frac{1}{4}\right)^{n+1} = 0$

$\therefore$ Hence $\ell im\ (x_{2n+1}) = 1 + \frac{2}{3} = \frac{5}{3}$

Since (x_n) is a Cauchy sequence with convergent subsequence (x_{2n+1}) it follows that $\ell im\,(x_n) = \ell im\,(x_{2n+1}) = \frac{5}{3}$.

12.6.11 Example: Consider the cubic equation $x^2 - 7x + 2 = 0$. It is given that this equation has a root between 0 and 1. We find out an approximation for this root. Towords this end we make use of contractive sequences.

Write $x = \frac{x^3 + 2}{7}$

Choose any x_1 such that $0 < x_1 < 1$ Define $x_{n+1} = \frac{x_n^3 + 2}{7}$

Clearly $0 < x_2 = \frac{x_1^3 + 2}{7} < \frac{1+2}{7} < 1$

Assuming that $0 < x_n < 1$ we get $0 < x_{n+1} = \frac{x_n^3 + 2}{7} < \frac{1+2}{7} < 1$

so that by induction it follows that $0 < x_n < 1$ for all n. Moreover for $n \in \mathbb{N}$

$$|x_{n+2} - x_{n+1}| = \left|\frac{1}{7}\left\{\left(x_{n+1}^3 + 2\right) - \left(x_n^3 + 2\right)\right\}\right| = \frac{1}{7}\left|\left(x_{n+1}^3 - x_n^3\right)\right|$$

$$= \frac{1}{7}|x_{n+1} - x_n|\ |x_{n+1}^2 + x_{n+1}\ x_n + x_n^2|$$

$$< \frac{1}{7} \cdot |x_{n+1} - x_n| \cdot \left(|x_{n+1}|^2 + |x_{n+1}\ x_n| + |x_n|^2\right)$$

$$< \frac{3}{7}|x_{n+1} - x_n| \qquad \left(\text{Since } |x_n| < 1 \text{ for all } n \in \mathbb{N}\right)$$

Thus (x_n) is a contractive sequence with $\frac{3}{7}$ as constatnt of contrction. By 12.6.8 there is r such that $\ell im(x_n) = r$. From the equation $x_{n+1} = \frac{1}{7}\left(x_n^3 + 2\right)$ we have

$$r = \ell im\ x_{n+1} = \frac{1}{7}\ \ell im\left(x_n^3 + 2\right) = \frac{1}{7}\left(r^3 + 2\right)$$

so that $r^3 - 7r + 2 = 0$. Also $0 \le r \le 1$

Since $r \ne 0$ and $n \ne 1$, r is the solution of the given equation in $(0, 1)$.

12.7 Proper Divergence:

Earlier we defined a sequence (x_n) of real numbers to be divergent if the sequence is not convergent. A divergent sequence may be bounded or unbounded - unbounded above or unbounded below or both. We now concentrate on unbounded divergent sequences.

12.7.1 Defintion: Let (x_n) be a sequence of real numbers. We say that (x_n) tends to $+\propto$ and write $\ell im\,(x_n) = +\propto$ if for every $\alpha \in \mathbb{R}$ there exists a natural number $k(\alpha)$ such that if $n \geq k(\alpha),\ x_n > \alpha$.

Definition: We say that (x_n) tends to $-\propto$ and write $\ell im\,(x_n) = -\propto$ if for every $\beta \in \mathbb{R}$ there exists a natural number $k(\beta)$ such that if $n \geq k(\beta)\ x_n < \beta$.

We say that (x_n) is properly divergent if $\ell im\, x_n = \pm\propto$

From the definitions it is clear that $\ell im\,(x_n) = +\propto$ if and only if for every β at most a finite number of the points (n, x_n) lie with in the region $\{(x, y)/y \leq \beta\}$ of the plane. Likewise $\ell im\,(x_n) = -\propto$ if and only y for every β atmost a finite number of the points (n, x_n) lie in the region $\{(x, y)/y \geq \beta\}$

12.7.2 Theorem: Let (x_n) and (y_n) be two sequences of real numbers and suppose that $x_n \leq y_n$ for all $n \in \mathbb{N}$

(a) If $\ell im\, x_n = +\propto$ then $\ell im\ y_n = +\propto$

(b) If $\ell im\, y_n = -\propto$ then $\ell im\ x_n = -\propto$

Proof: (a) Suppose $\ell im\ x_n = +\propto$. If $\alpha \in \mathbb{R}$ there is a natural number $k(\alpha)$ such that if $n \geq k(\alpha), x_n > \alpha$ since $y_n \geq x_n$ for every n it follows that if $n \geq k\,(\alpha)$

$y_n \geq x_n > \alpha$ so that $y_n > \alpha$ if $n \geq k(\alpha)$

Hence $\ell im\ y_n = +\propto$

(b) Suppose $\ell im\ y_n = -\propto$ then if $\beta \in \mathbb{R}$ there is a natural number $k(\propto)$ such that $y_n < \beta$ if $n \geq k(\beta)$. Since $x_n \leq y_n$ for all n, $x_n \leq y_n < \beta$ if $n \geq k(\beta)$. Since $x_n < \beta$ for every $\beta \in \mathbb{R}$ and $n \geq k(\beta)$ it follows that $\ell im\,(x_n) = -\propto$.

12.7.3 S.A.Q.: Show that if $x_n > 0$ for every n, $\ell im\,(x_n) = 0$ iff and only if $\ell im\, \frac{1}{x_n} = +\propto$

12.7.4 S.A.Q.: Show that if (x) and (y_n) are properly divergent and $\ell im(x_n) = \ell im(y_n)$ there $(x_n + y_n)$ is properly divergent and $\ell im(x_n + y_n) = \ell im(x_n) = \ell im(y_n)$.

12.7.5 Theorem: A monotone sequence of real numbers is properly divergent if and only if it is unbounded.

(a) If (x_n) is an unbounded increasing sequence then $\ell im(x_n) = \propto$

(b) If (x_n) is an unbounded decreasing sequence then $\ell im(x_n) = -\propto$

Proof: **(a)** Let (x_n) be a monotonically increasing sequence. Then (x_n) is bounded below and x_1 is a lower bound. If (x_n) is bounded above then (x_n) is convergent and $\ell im(x_n) = \sup(x_n \; n \in \mathbb{N})$ If (x_n) is unbounded above, for every $\alpha \in \mathbb{R}$ there is a natural number such that $x_{k(\alpha)} > \alpha$. Since (x_n) is increasing.

If $k > k(\propto)$, $x_k \geq x_{k(\propto)} > \propto$ This implies that $\ell im \; x_n = +\propto$ so that (x_n) is properly divergent.

12.7.6 Theorem: Let (x_n) and (y_n) be sequences of positive real numbers and suppose

$$\ell im\left(\frac{x_n}{y_n}\right) = x_n > 0 \quad \text{Then } \ell im(x_n) = +\propto \text{ if and only if } \ell im(y_n) = +\propto$$

Proof: Let $\ell im\left(\frac{x_n}{y_n}\right) = L$. Since $L > 0$ there is a natural number k such that if $n \geq k$

$$\left|\frac{x_n}{y_n} - L\right| < \frac{L}{2}$$

This implies that $L - \frac{L}{2} < \frac{x_n}{y_n} < L + \frac{L}{2} \quad \text{if} \quad n \geq k$

$$\Rightarrow \frac{3L}{2} > \frac{x_n}{y_n} > \frac{L}{2} \quad \text{if} \quad n \geq k$$

$$\Rightarrow x_n > \frac{L}{2}(y_n) \quad \text{if} \quad n \geq k \quad \text{and} \quad x_n < \frac{3L}{2} y_n \quad \text{if} \quad n \geq k$$

Suppose $\ell im\ y_n = +\infty$. If β is any real number there is a natural number k_1 such that

if $n \geq k_1$; $x_n > \frac{2}{3L}\beta$. If $k(\beta) = \max\{k, k_1\}$ and $n \geq k(\beta)$ then $n \geq k$ as well as $n \geq k_1$ so that

$$\beta < y_n < \frac{2}{3L} x_n$$

$$\Rightarrow x_n > \frac{3L}{2} y_n > \frac{3L}{2} \cdot \frac{2}{3L} \cdot \beta = \beta$$

$$\Rightarrow \ell im\ x_n = +\infty$$

Suppose $\ell im\ x_n = +\infty$ If $\alpha \in \mathbb{R}$ there is natural number k_2 such that $x_n > \frac{3L}{2}\alpha$ if $n \geq k_2$

If $k(\alpha) = \max\{k, k_2\}$ and $n \geq k(\alpha)$ then $n \geq k \Rightarrow \frac{3L}{2} y_n > x_n$ $n \geq k_2 \Rightarrow x_n > \frac{3L}{2} \cdot \alpha$

So if $n \geq k(\in)$, $y_n > \alpha$

This implies that $\ell im\,(y_n) = +\infty$

12.8 Solutions to S.A.Q.'s:

12.4.5 S.A.Q.:

$$x_n = \frac{n!}{n^n} \Rightarrow \frac{x_{n+1}}{x_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n < 1$$

$$\Rightarrow 0 < x_{n+1} < x_n \text{ for all } n \in \mathbb{N}$$

Also $x_n = \frac{1}{n} \cdot \frac{2}{n} \cdots\cdots\cdots\cdots \frac{n}{n} \leq \frac{1}{n}$ Hence $0 < x_n \leq \frac{1}{n}$

Since $\ell im \frac{1}{n} = 0$ $\ell im\ x_n = 0$, by Squeeze theorem

Remark: For convergence, monotonicity is not required! Squeeze theorem is enough.

12.4.8 S.A.Q.: For $n > 2$ $2^{n-1} < n!$

when $n = 3$, $\ell hs = 2^2$ and rhs $= 3! = 6$ so that the inequality holds good when $n = 3$.

Assume that $Z^{n-1} < n!$ where $n > 2$. Then $2^n = 2^{n-1} \cdot 2 < n!(n+1) = (n+1)!$ $(\because 2 < n < n+1)$. By induction it follows that $2^{n-1} < n!$ for all $n \in \mathbb{N}$

Since $e_n = \left(1 + \frac{1}{n}\right)^n$

$$= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots\cdots\cdots + \binom{n}{n}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots\cdots\cdots + \frac{1}{n^n}$$

and $\frac{1}{n^n} = \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\cdots\cdots\left(1 - \frac{n-1}{n}\right)$

We have for $1 \le r \le n$ the r-th term in the rhs

$$= \frac{1}{r!}\left(1 - \frac{1}{r}\right) \cdot \left(1 - \frac{r-1}{n}\right)$$

$$< \frac{1}{r!} < \frac{1}{2^{r-1}}$$

Hence $e_n < 1 + 1 + \frac{1}{2} + \cdots\cdots\cdots + \frac{1}{2^{n-1}}$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 + 2\left(1 - \frac{1}{2^n}\right) < 3$$

It is clear that $e_n > 2$

Hence $2 < e_n < 3$

12.4.9 S.A.Q.: Given $a > 0, s_1 > 0$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$

we show that (s_n) is decreasing for sufficiently large n and $\ell im\ s_n = \sqrt{2}$

Step (1): $s_n > 0$ for $n \in \mathbb{N}$

Proof: By hypothesis $s_1 > 0$. If we assume that $s_n > 0$ then $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) > 0$

Hence by induction, $s_n > 0$ for all n

Step (2): $s_{n+1}^2 \geq a$ for all $n \in \mathbb{N}$

from $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$ it follows that $s_n^2 - 2s_n s_{n+1} + a = 0$

so that s_n is a root of the quadratic equation $x^2 - 2s_{n+1}x + a = 0$ since this equation has real root, the discriminant is non negative.

Hence $s_{n+1}^2 - a \geq 0$ so that $s_{n+1}^2 = a$ for $n \in \mathbb{N}$

Step (3): $s_{n+1} \leq s_n$ for $n \geq 2$

Since $s_{n+1}^2 \geq a$ and $s_{n+1} > 0$ $s_{n+1} > \frac{a}{s_{n+1}}$ for $n \in \mathbb{N}$

$$\Rightarrow s_{n+1} > \frac{1}{2}\left(s_{n+1} + \frac{a}{s_{n+1}}\right) > \frac{a}{s_{n+1}}$$

$$\Rightarrow s_{n+1} > s_{n+2} \text{ for } n \in \mathbb{N}$$

$$\Rightarrow s_{n+1} \leq s_n \text{ for } n \geq 2$$

Step (4): By monotone convergence theorem (s_n) is convergent

If $\ell im\ s_n = s$ then $\ell im\ s_{n+1} = s$ and $s > 0$ so that $s = \ell im\ s_{n+1} = \ell im\ \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$

$$\Rightarrow s - \frac{r}{2} = \frac{a}{2s} \Rightarrow \frac{s}{2} = \frac{a}{s} \Rightarrow s^2 = a$$

12.7.3 S.A.Q.: Assume that $x_n > 0$ for every n and $\ell im\ x_n = \infty$ then,

If $\in > 0$ there is a natural number $k(\in)$ such that

$x_n > \frac{1}{\in}$ for $n \geq k(\in)$. Then if $n \geq k(\in)$, $\frac{1}{x_n} < \in$

Since $0 < \frac{1}{x_n} < \in$ for $n \geq k(\in)$ and $\in$ is arbitrary it follows that $\ell im \frac{1}{x_n} = 0$

Conversely suppose $\ell im \frac{1}{x_n} = 0$. Given $\in > 0$ there is a positive integer $k(\in)$ such that $\frac{1}{x_n} < \frac{1}{\in}$ if $n \geq k(\in)$

Then if $n \geq k(\in)$ $x_n > \in$. Hence $\ell im\ x_n = +\infty$

12.7.4 S.A.Q.: **(a)** Assume that $\ell im\ (x_n) = \ell im\ (y_n) = +\infty$

(The proof for the case $-\infty$ is similar)

If $\in > 0$ there are natural numbers k_1, k_2 such that if $n \geq k_1, x_n > \frac{\in}{2}$ and if $n \geq k_2, y_n > \frac{\in}{2}$

If $k(\in) = \max\{k_1, k_2\}$ and $n \geq k(\in)$ then $x_n > \frac{\in}{2}$ and $y_n > \frac{\in}{2}$ so that $x_n + y_n > \in$

(b) Since $\ell im\ x_n = \ell im\ y_n = +\infty$ there are natural numbers k_1, k_2 such that

$x_n > 0$ if $n \geq k_1$ and $y_2 > 0$ if $n \geq k_2$

If $\in > 0$ there are natural numbers k_1 and k_2 such that

$x_n > \sqrt{\in} > 0$ if $n \geq k_1$ and $y_n > \sqrt{\in} > 0$ if $n \geq k_2$

If $k(\in) = \max\{k_1, k_2\}$ and $n \geq k(\in)$

and $n \geq k_1,\ n \geq k_2 \Rightarrow x_n > \sqrt{\in} > 0,\ y_n > \sqrt{\in} > 0 \Rightarrow x_n\ y_n > \in$ if $n \geq k(\in)$

Hence $\ell im\ (x_n\ y_n) = +\infty$

12.9 Summary:

A careful study of this lesson should enable the student to realize the importance of monotone sequences and their applications and limitations, the role of sub sequences in determining the nature of the sequence, the importance of Bolzano Weierstrass theorem on sequences and finally the utitlity of the Cauchy criterion for convergence. The student should also notice that contractive sequences are very useful in locating the real roots of polynomial equations with real coefficients and approximating them within the desired error.

12.10 Technical Terms:

Monotone Sequence

Peak

Subsequence

Bounded Sequence

Contractive Sequence

12.11 Exercises:

1. If (a_n) increases show that $\left(\frac{a_1+\cdots\cdots+a_n}{n}\right)$ increases.

2. If (a_n) decreases show that $\left(\frac{a_1+\cdots\cdots+a_n}{n}\right)$ decreases.

3. If (a_n) increases and $a_n > 0$ show that $(\log a_n)$ increases.

4. Show that if $a_n > 0$ for every n and (a_n) increases $\left[(a_1 a_2 \cdots\cdots\cdots a_n)^{\frac{1}{n}}\right]$ increases.

5. Do exercise 4 when (a_n) decreases

6. Let $x_1 > 0$ and $x_{n+1} = \frac{2}{1+x_n}$. Show that one of the sequences (x_{2n-1}) and (x_{2n}) is increasing and the other one is decreasing. Show also that the two sequences are convergent and have the same limit. Deduce that (x_n) converges.

7. Let $k > 0$ and $x_1 > 0$. Define $x_{n+1} = \sqrt{k + x_n}$

Show that (x_n) is monotone and bounded

Show that $\ell im\ x_n = a$ where $a > 0$ and $a^2 = a + k$

8. Let $x_1 = 2$ and $x_{n+1} = \frac{x}{x_n}$ for $n \in \mathbb{N}$. Prove

(a) $x_n > 0$ for $n \in \mathbb{N}$

(b) $(x_{n+1} - x_n)$ and $x_n - x_{n-1}$ have same sign

(c) (x_{2n-1}) is increasing and bounded

(d) (x_{2n}) is decreasing and bounded

9. Let $c > 1$ and $x_n = c^{\frac{1}{n}}$

(a) Show that $1 < x_{n+1} < x_n \le c$ for $n \in \mathbb{N}$

(b) Show that $\ell im\ x_n = 1$

10. Let $x_1 > 1$ and $x_{n+1} = 2 - \frac{1}{x_n}$. Show that $k < x_n < 2$ and (x_n) is decreasing. Find $\ell im\ x_n$. (Ans: 1)

11. Let $x_1 \ge 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$ Show that (x_n) is monotonically decreasing and $2 \le x_n \le k_1$ for all find $\ell im\ x_n$. (Ans: 1)

12. Let $x_1 \ge 2$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$ show that (x_n) converges. Find limit (x_n). (Ans: 2)

13. Let $P > 0,\ y_1 = \sqrt{P}$ and $y_{n+1} = \sqrt{P + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find $\ell im\ (y_n)$ (See Ex.7)

14. Let (a_n) be an increasing sequence, (b_n) be decreasing and $a_n \le b_n$ for all $n \in \mathbb{N}$.

    (a) Show that $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all n

    (**Hint:** obtain a contradiction if $b_m < a_m$ for some m)

    (b) Deduce that $\ell im\,(a_n) \le \ell im\,(b_n)$

    (c) Show that if $b_{n+1} - a_{n+1} = \dfrac{b_n - a_n}{2}$ for all $n \in \mathbb{N}$ $\ell im(b_n) = \ell im\,(a_n)$.

15. Deduce from (14) above that if $I_n = [a, b_n], I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, then there is $\alpha \in I_n$ for all $n \in \mathbb{N}$. If $b_{n+1} - a_{n+1} = \dfrac{b_n - a_n}{2}$ for all n. Show that there is a unique α which belongs to every I_n.

12.11.B Exercise:

1. Let $I_n = \left[0, \dfrac{1}{n}\right]$. Show that $I_n \supseteq I_{n+1}$ for every $n \in \mathbb{N}$. Find $\bigcap\limits_{n=1}^{\infty} I_n$ (Ans: {0})

2. Let $I_n = \left(0, \dfrac{1}{n}\right)$ show that $I_n \supset I_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcap\limits_{n=1}^{\infty} I_n = \phi$

3. Let $k_n = [n, \infty] = \{x/x \in \mathbb{R} \text{ and } x \ge n\}$

   show that $k_n \ge k_{n+1}$ for $n \in \mathbb{N}$ and $\bigcap\limits_{n=1}^{\infty} k_n = \phi$

4. Give an example of an unbounded sequence which contains a convergent sub sequence.

5. Show that if $x = (x_n)$ is unbounded there is a sub sequence (x_{n_k}) of X such that $\ell im\left(\dfrac{1}{x_{n_k}}\right) = 0$. (Hint: Find $n_k > k$ such that $x_{n_k} > k$).

6. If every term in (x_n) is a peak of (x_n) showthat (x_n) is decreasing.

7. Find all peaks of the sequence $\left((-1)^n\right)$ (Ans: ϕ)

8. Call x_n a foot of (x_n) if and only if $-x_n$ is a peak of $(-x_n)$.

   Imitate the proof of monotone sub sequence theorem to show that a monotone sub sequence of a given sequence can be picked by using "feet".

9. Apply the procedure in method of Bolzano - Weierestrass theorem for the sequences $\left(\frac{1}{n}\right)$ and $\left(\frac{n}{n+1}\right)$.

12.11.C Exercise:

1. Give an example of a bounded sequence that is not a Cauchy sequence.

2. Show directly from the definition that the following are Cauchy sequences.

   (a) $\left(\frac{n+1}{n}\right)$ (b) $\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots\cdots+\frac{1}{n!}\right)$

3. Show directly from the definition that the following are not Cauchy sequences.

   (a) $\left((-1)^n\right)$ (b) $\left(n+\frac{(-1)^n}{n}\right)$ (c) $(\ell n\ n)$

4. Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences then so are (x_n+y_n) and $(x_n\ y_n)$

5. If (x_n) is a sequence of integers, show that $|x_n-x_m|=0$ or $|x_n-x_m|\geq 1$

   Deduce that if (x_n) is Cauchy sequence of integers then there is an integer k such that $x_n=k$ for sufficiently large n.

6. Let P be any natural number and $x_n=\sqrt{n}$ showthat $|x_{n+p}-x_n|<\frac{P}{2\sqrt{n}}$ deduce that $\ell im\,|x_{n+p}-x_n|=0$ Is (x_n) a Cauchy sequence? Find $\ell im\,(x_{n+1}-x_n)$

7. Establish the convergence of the following sequences.

   (a) $\left(1+\frac{1}{n^2}\right)^{n^2}$ (b) $\left(1+\frac{1}{2n}\right)^{n}$ (c) $\left(1+\frac{1}{n^2}\right)^{2n^2}$ (d) $\left(1+\frac{1}{2n}\right)^{3n}$

Hint: Let $e_n = \left(1+\frac{1}{n}\right)^n$ express the general term in each case in terms of the elements of e_n. For example in (a) The general term is e_{n^2} and in (b), $e_{2n}^{1/2}$

8. Find (a) $\ell\text{im}\left(1+\frac{2}{n}\right)^n$, (b) $\ell\text{im}\,(3n)^{\frac{1}{2n}}$

9. Let (x_n) be a bounded sequence and $s = \sup(x_n : n \in \mathbb{N})$. If $s \neq x_n$ for any n show that for every $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}\ \forall\ k$ and $x - \frac{1}{k} < x_{n_k}$. Deduce that $\ell\text{im}\left(x_{n_k}\right) = s$.

10. If x_1, x_2 are real numbers and $x_1 < x_2$ show that the sequence (x_n) defined by $x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$ is conergent. Find $\ell\text{im}\,(x_n)$ (Hint: 12.6.11 may be helpful)

11. If x_1, x_2 are real numbers and $x_1 < x_2$ define $x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}x_{n-1}$ show that (x_n) is a contractive sequence and hence converges. Find $\ell\text{im}\,(x_n)$.

12. Let $x_1 > 0$ and $x_{n+1} = \frac{1}{2+x_n}$ for $n \in \mathbb{N}$

Show that $\frac{1}{2} > x_n > 0$ for all n and $|x_{n+2} - x_n| < \frac{1}{4}|x_{n+1} - x_n|$

Deduce that (x_n) is a contractive sequence find $\ell\text{im}\,(x_n)$. (Ans: $\frac{-1 \pm \sqrt{2}}{2}$)

13. Let $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$. Show that $x_{n+1} > 2$ for all $n \in \mathbb{N}$ and $|x_{n+2} - x_{n+1}| < \frac{1}{4}|x_{n+1} - x_n|$

Deduce that (x_n) is a contractive sequence. Find $\ell\text{im}\,(x_n)$ (Ans: $\frac{1 \pm \sqrt{2}}{2}$)

14. Let $0 < x_1 < 1$ and $x_{n+1} = \frac{1}{5}\left(x_n^3 + 1\right)$

Show that $\left|x_{n+2} - x_{n+1}\right| < \frac{3}{5}\left|x_{n+1} - x_m\right|$

Deduce that (x_n) converges to a solution of the cubic equation $x^3 - 5x + 1 = 0$

If $x_1 = \frac{1}{2}$ find x_2, x_3, x_4 and x_5.

12.11.D Exercise:

1. Showthat if (x_n) is unbounded there exists a properly divergent subsequence.

   Hint: If (x_n) is unbounded above there exists $\left(x_{n_k}\right) \ni n_{k+1} \geq k$ and $x_{n_k} > k$.

2. Let $(x_n) = n$. (a) Showthat (x_n) & $\left(x_n^2\right)$ are properly divergent

   (b) Show that $\left(\frac{x_n}{x_n^2}\right) = \left(\frac{1}{x_n}\right)$ is convergent and $\left(\frac{x_n^2}{x_n}\right)$ is properly divergent.

3. Show that if (x_n) is properly divergent so is $\left(x_n^2\right)$.

4. If (x_n) and $(x_n\ y_n)$ are properly divergent, does it follow that

   (a) (y_n) is properly divergent? (Hint: Let $y_n = n$ for every n)

   (b) (y_n) is convergent ? (Hint: $y_n = 1$ for every n)

5. Establish proper divergence of

   (a) $\sqrt{n}$ (b) $\sqrt{n+1}$ (c) $\sqrt{n-1}$ (d) $\frac{n}{\sqrt{n+1}}$

6. If (x_n) is properly divergent and $\ell im\,(x_n\ y_n)$ exists in $\mathbb{R}$, show that $\ell im\ y_n = (0)$

7. Given that $x_n > 0$ and $y_n > 0$ for every natural number n

   show that if $\ell im\ \frac{x_n}{y_n} = 0$ and

(a) If $\ell im\ x_n = +\infty$ then $\ell im\ y_n = \infty$

(b) If (y_n) is bounded $\ell im\ x_n = 0$

8. Showthat if $(x_n), (y_n)$ are positive sequences such that $\ell im \dfrac{x_n}{y_n} = +\infty$

(a) $\ell im\ x_n = +\infty$ if $\ell im\ y_n = +\infty$

(b) $\ell im\ y_n = 0$ if $\ell im\ (x_n)$ is bounded

9. Show that if $a_n > 0$ and $\ell im \dfrac{a_n}{n} = L > 0$ then $\ell im\ a_n = +\infty$

10. Give an example of an unbounded sequence that has convergent sub sequence.

Show that if $0 < C < 1$ Limit $C^{1/n} = 1$

11. Let $x_n = 1 - (-1)^n + \dfrac{1}{n}$ for $n \in \mathbb{N}$

Show that $\ell im\ (x_{2n})$ and $\ell im\ (x_{2n+1})$ exist Does $\ell im\ (x_n)$ exist? why?

12. Let $x_n = \sin \dfrac{n\pi}{4}$

Find $\ell im\ (x_{4n})$ and $\ell im (x_{8n+2})$, Does $\ell im\ (x_n)$ exist? why?

13. (a) Show that $(n+1)^n < n^{n+1} \Leftrightarrow (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}}$

(b) Show that for $n \geq 3$ $\left(1 + \dfrac{1}{n}\right)^n < 3 \leq n$

(c) Deduce that the sequence $\left(n^{\frac{1}{n}}\right)$ satisfies

$1 < m^{\frac{1}{m}} < n^{\frac{1}{n}}$ if $m > n \geq 3$

and hence conclude that $\ell im \left(n^{\frac{1}{n}}\right)$ exists.

(d) If $\ell im\left(\frac{1}{n^n}\right)=\ell$, by considering the subsequence of even terms, show that $\ell=\sqrt{\ell}$. Deduce that $\ell=1$.

15. Let $\ell im\left(x_n\right)=\ell=\ell im\left(y_n\right)$ and $Z_n=\begin{cases} x_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ y_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$

Show that $\ell im\left(Z_n\right)=\ell$.

12.12 Model Examination Questions:

1. Show that a monotone sequence of real numbers is convergent if and only if it is bounded. Further

   (a) If $X=\left(x_n\right)$ is a bounded increasing sequence, then show that

   $\ell im\left(x_n\right)=\text{Sup}\left\{x_n : n \in N\right\}$

   (b) If $Y=\left(y_n\right)$ is a bounded decreasing sequence, then show that

   $\ell im\left(y_n\right)=\inf\left\{y_n : n \in N\right\}$

2. Let $a>0$; construct a sequence $\left(s_n\right)$ of real numbers that converges to $\sqrt{a}$ and $s_n < s_{n+1}$ for every n.

3. Let $e_n=\left(1+\frac{1}{n}\right)^n$ for $n \in N$. Show that the sequence $E=\left(e_n\right)$ is bounded and increasing and hence is convergent. Show that the limit of this sequence is Euler number e.

4. Let $x_1=a>0$ and $x_{n+1}=x_n+\frac{1}{x_n}$ for $n \in \mathbb{N}$. Determine if $\left(x_n\right)$ coverges or diverges.

5. Establish the convergeence or divergence of the sequence $\left(y_n\right)$, where $y_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots\cdots+\frac{1}{2n}$ for $n \in \mathbb{N}$.

6. If $X=(x_n)$ is a sequence of real numbers, then show that there is a sub sequence of X that is monotone.

7. State and prove Bolzano - Weierstass theorem on sequences.

8. Let $X=(x_n)$ be a bounded sequence of real numbers and Let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then show that the sequence X converges to x.

9. Let $X=(x_n)$ be defined by

   $x_1=1, x_2=2$ and $x_n=\frac{1}{2}(x_{n-2}+x_{n-1})$ for $n>2$.

   It can be shown by Induction that $1 \le x_n \le 2$ for all $n \in \mathbb{N}$.

   Some calcultaion shows that the sequence X is not monotone

   since the terms are formed by averaging, show that

   $$|x_n - x_{n+1}| = \frac{1}{2^{n-1}} \quad \text{for } n \in N$$

10. Show that every contractive sequence is a Cauchy sequence and that it is convergent.

12.13 Model Practical Problem:

Given $0 < x_1 < 1$ show that the sequence (x_n) defined by $x_{n+1}=\sqrt{1+x_n}$ converges to the positive root of the equation $x^2-x-1=0$.

Definition: (x_n) converges to x in $\mathbb{R}$ if for every $\in > 0$ here corresponds $N_\in \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \ge N_\in$ $|x_n - x| < \in$.

Results:

1. If (x_n) is monotonically increasing and bounded then (x_n) converges.
2. If (x_n) converges to x, every subsequence of (x_n) converges to x.
3. Principle of mathematical induction: If a property $P(n)$ is such that $P(1)$ is true and $P(n+1)$ is true whenever $P(1)$ is true then $P(n)$ is true for all $n \in \mathbb{N}$.
4. If $x_n > 0$ for all n and $\ell im\ x_n = x$ then $\ell im \sqrt{x_n} = \sqrt{x}$.

Division into steps: (a) $0 < x_n < 2$ for all n

(b) (x_n) is monotonically increasing

(c) If $x^2 = \ell im\ x_n,\ x^2 = x + x$

Step wise solution:

(a) Proof by induction (1) clearly $0 < x_1 < 1 < 2$. Moreover $x_{n+1} = \sqrt{1+x_n} > 0$ for all n

If $0 < x_n < 2, 0 < x_{n+1} = \sqrt{1+x_n} < \sqrt{2+1} < \sqrt{2+2} = 2$ Hence bu induction $0 < x_n < 2$ for all n

(b) $x_{n+1}^2 = 1 + x_n \Rightarrow x_{n+1}^2 - x_n^2 = x_n - x_{n-1}$ since $x_{n+1} + x_n > 0, x_{n+1} - x_n > 0$ or < 0 according as $x_n - x_{n-1} > 0$ or < 0 $x_2 - x_1 = \sqrt{1+x_1} - x_1 > \sqrt{1+1} - x_1 > 1 - x_1 > 0$ since $x_2 - x_1 > 0, x_n - x_{n-1} > 0$ for all n. Hence (x_n) is increasing.

(c) Since (x_n) is increasing and bounded (by (a)) (x_n) converges by (by result 1) If $\ell im\ x_n = x,$ $\ell im\ x_{n+1} = x$ (by result 2) and $\ell im \sqrt{1+x_n} = \sqrt{1+x}$. Hence $x = \ell im\ x_{n+1} = \ell im \sqrt{1+x_n} = \sqrt{1+x}$ $\Rightarrow x^2 = x + 1$. Since $x_n > 0$ for all n, $\ell im\ x_n = x \geq 0$. Since $x^2 = x + 1, x > 0$

Hence $\ell im\ x_n$ is the positive root of the equation $x^2 - x - 1 = 0$.

Lesson Writer

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Lesson - 13

INFINITE SERIES

13.1 Objective of the lesson:

Here our aim is to provide working knowledge to the student problems connected with infinite series. To this end a few characterization theorems and some important convergence tests are discussed.

13.2 Structure:

This lesson contains the following components:

13.3 Introduction

13.4 Definition and Some Elementary Results

13.5 Comparison Tests

13.6 Applications

13.7 Solutions to S.A.Q.'s

13.8 Summary

13.9 Technical Terms

13.10 Exercises

13.11 Model Examination Questions

13.12 Model Practical Problem with Solution

13.3 Introduction:

This lesson on infinite series is merely a brief introduction to a vast theory which is useful in Real Analysis. However because of our limitations and compulsions we have to be content with a very short discussion of the topic.

A few important results such as the Cauchy criterion, convergence or divergence criteria are discussed first and some useful tests including comparision test, limit comparision test, condenation test and so on are presented.

13.4 Definition and Some Elementary Results:

13.4.1 Definition: If $X = (x_n)$ is a sequence of real numbers, the infinite series generated by X is the sequence $S = (s_n)$ define inductively by

$$s_1 = x_1 \text{ and for } n > 1, \quad s_n = s_{n-1} + x_n$$

The n^{th} term x_n of the sequence is called the n^{th} term of the series and s_n is called the n^{th} partial sum of the series. If the sequence S converges we say that the series converges and call lim (s), the sum of the series generated by X. If $\text{Lim}\,(S)$ does not exist we say that the series diverges.

Notation: If $X = (x_n)$ is a sequence in $\mathbb{R}$, $S = (s_n)$ is the sequence of series generated by X , we use any one of the following symbols to denote both the series generated by X and the sum: Σx_n ; $\Sigma(x_n)$, $\sum\limits_{n=1}^{\infty} x_n$. It is easy to show by induction on n that $s_n = x_1 + \cdots\cdots\cdots + x_n = \sum\limits_{i=1}^{n} x_i$. If the terms of the sequence $X = (x_n)$ start with x_0 we write $\sum\limits_{n=0}^{\infty} x_n$ for $\sum\limits_{n=1}^{\infty} x_n$. We also use the symbols.

$$\sum_{n=k}^{\infty} x_n \text{ and } \sum_{n \geq k} x_n \quad \text{when } X = (x_k,\ x_{k+1},\ \cdots\cdots\cdots\cdots)$$

13.4.2 Example: The series $\sum\limits_{n=0}^{\infty} r^n$ associated with the sequence $X = \left(r^{n-1}\right)$ where $r \in \mathbb{R}$ is called a geometric series. We discuss the convergence of the geometric series without any restriction on $r \in \mathbb{R}$. The sequence (s_n) of partial sums s_n is defined by

$$s_n = 1 + r + \cdots\cdots\cdots\cdots + r^{n-1}$$

$$\Rightarrow rs_n = r + \cdots\cdots\cdots\cdots\cdots\cdots + r^{n-1} + r^n \Rightarrow s_n\ (1-r) = 1 - r^n. \text{ If } r \neq 1,\ s_n = \frac{1-r^n}{1-r}.$$

Case (1): If $|r| < 1$, $\ell\text{im}\left(r^n\right) = 0$ so that $\ell\text{im}\ (s_n) = \ell\text{im}\left(\frac{1-r^n}{1-r}\right) = \frac{1-\ell\text{im}\ r^n}{1-r} = \frac{1}{1-r}$

Thus if $|r| < 1$ the series $\sum\limits_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$.

Case (2): If $r = 1,\ s_n = n$ and $\ell\text{im}\,(s_n) = +\infty$. In this case the series diverges to $+\infty$

Case (3): If $r = -1,\ s_{2n} = 0$ and $s_{2n+1} = 1$ for $n \in \mathbb{N}$. So (s_n) does not converge

Hence the geometric series diverges if $r=-1$. We discuss the case $|r|>1$ at a latter stage.

13.4.3 Example: The series $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

$$x_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

$$\Rightarrow s_n = x_1+\cdots+x_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots\cdots\cdots\cdots\cdots\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{n+1}$$

$\ell im\ (s_n)=\ell im\left(1-\frac{1}{n+1}\right)=1-\ell im\left(\frac{1}{n+1}\right)=1-0=1$. Hence $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}=1$

We now prove a simple result, called the n-th term test which is more useful in proving non - convergence of a series.

13.4.4 Theorem: If the series $\sum x_n$ converges then $\ell im\,(x_n)=0$.

Proof: Let (s_n) be the sequence of partial sums of $\sum x_n$ so that $s_n=x_1+x_2+\cdots\cdots\cdots+x_n$. Clearly $s_{n+1}-s_n=x_{n+1}$ for $n\geq 1$. Since $\sum x_n$ converges, $\ell im\,(s_n)$ exists.

Since (s_{n+1}) is a subsequence of (s_n); $\ell im\ s_{n+1}=\ell im\ s_n$

Hence $\ell im\ s_{n+1}-\ell im\ s_n=0$. Thus $\ell im\ x_{n+1}=\ell im\,(s_{n+1}-s_n)=0$

Since the sequence (x_n) has just one additional term x_1 in the first place, it follows that $\ell im\ x_n=0$.

Remark: The n^{th} term test gives only a necessary condition for convergence. This is not a sufficient condition as is evident from the following example 13.4.5 (iii).

13.4.5 Examples: (i) $x_n=\frac{n}{n+1}$, $\ell im\ (x_n)=\ell im\left(1-\frac{1}{n+1}\right)$

$$=1-\ \ell im\ \frac{1}{n+1}=1$$

Hence $\sum\limits_{n=1}^{\infty}\frac{n}{n+1}$ does not converge.

Example: **(ii)** Consider the geometric series $\sum_{n=0}^{\infty} r^n$. We proved earlier that the series converges if $|r|<1$ and diverges when $r=-1$. Also if $|r|>1$, $\ell im\left(\left|r^n\right|\right)=\ell im\left(|r|^n\right)=+\infty$.

Since $\ell im\,|r|^n \neq 0$ it follows that $\sum_{n=0}^{\infty} r^n$ does not converge when $|r|>1$ i.e., when $r<-1$ or $r>1$.

Example: **(iii)** The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ satisfies the condition $\ell im\ x_n = 0$ we have already proved that the sequence (s_n) of partial sums of the series diverges. (see 12.4.3, 12.6.2 and 12.6.3) if $\ell im\ x_n = 0$ it does not necessarily hold that $\sum_{n=1}^{\infty} x_n$ is convergent.

Example: **(iv)** Another equally interesting example is the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\cdots\cdots\cdots$$

in which the general term is given by $x_n = \frac{(-1)^{n-1}}{n}$. As in the case of the harmonic series, here also $\ell im\,(x_n)=0$ since $|\,x_n\,|=\frac{1}{n}$ for every n and $\ell im\left(\frac{1}{n}\right)=0$. If (s_n) is the sequence of partial sums,

$$s_{2n} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right)$$

and $$s_{2n+1} = 1-\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)\cdots\cdots\cdots\cdots\left(\frac{1}{2n}-\frac{1}{2n+1}\right)$$

$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\cdots\cdots\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\frac{1}{2n+1}$$

$$= s_{2n} + \frac{1}{2n+1}$$

Clearly $0 < s_{2n} < s_{2n} + \frac{1}{2n+1} = s_{2n+1} \le 1$

Moreover $s_{2n+2} = s_{2n} + \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) > s_{2n}$

and $s_{2n+3} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) \cdots\cdots\cdots\cdots \left(\frac{1}{2n} - \frac{1}{2n+1}\right) - \left(\frac{1}{2n+2} - \frac{1}{2n+3}\right) < s_{2n+1}$

Thus $0 < s_{2n} < s_{2n+2} < 1, \quad 0 < s_{2n+3} < s_{2n+1} < 1$

Thus (s_{2n}) is increasing and bounded and (s_{2n+1}) is decreasing and bounded. Hence (s_{2n}) and (s_{2n+1}) converge.

Since $s_{2n+1} = s_{2n} + \frac{1}{2n+1}$

$$\ell\text{im}\,(s_{2n+1}) = \ell\text{im}\,(s_{2n}) + \ell\text{im}\left(\frac{1}{2n+1}\right) = \ell\text{im}\,(s_{2n})$$

Hence (s_n) converges and $0 \le \ell\text{im}\, s_n \le 1$

Thus the series $1 - \frac{1}{2} + \frac{1}{3} + \cdots\cdots\cdots\cdots + \frac{(-1)^{n-1}}{n} + \cdots\cdots\cdots$ converges

13.4.6 The p - series: If $p \in \mathbb{R}$ the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the p - series. We prove that if $p \in \mathbb{R}$, the p - series converges if $p > 1$ and diverges if $p \le 1$.

We write $x_n = \frac{1}{n^p}$ and $s_n = 1 + \frac{1}{2^p} + \cdots\cdots\cdots\cdots + \frac{1}{n^p}$.

Proof:

(i) When $p = 0$ the p - series diverges properly to $+\infty$ because $x_n = 1$ for every n and $s_n = n$ so that $\ell\text{im} = s_n = +\infty$.

(ii) When $p < 0 \quad x_n = n^{-p} \ge 1$ for every $n \in \mathbb{N}$, hence $s_n \ge n$ so that $\ell\text{im}\, s_n = +\infty$. Thus the p - series diverges properly to $+\infty$.

(iii) When $p=1$ the 1 - series is the harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ which properly diverges to $+\infty$.

(iv) when $p=2$ the 2 - series converges: For every $p\in\mathbb{R}$, $x_n=\frac{1}{n^p}>0$

$\Rightarrow s_{n+1}=s_n+x_n>s_n$

Thus (s_n) is increasing

We prove convergence of $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ by showing that the sequence (s_n) is bounded

For each j let $k_j=2^j-1$ we have

$k_1=1$ and $s_{k_1}=x_1=1$

$k_2=3$ and $s_{k_2}=1+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)<1+\frac{1}{2^2}+\frac{1}{2^2}=1+\frac{2}{2^2}=1+\frac{1}{2}$

$k_3=7$ and $s_{k_3}=s_{k_2}+\left(\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right)$

$$<s_{k_2}+\frac{4}{4^2}<1+\frac{1}{2}+\frac{1}{2^2}$$

Assuming that $s_{k_{j-1}}<1+\frac{1}{2}+\cdots\cdots\cdots\cdots+\frac{1}{2^{j-2}}$

we show that $s_{k_j}<1+\frac{1}{2}+\cdots\cdots\cdots\cdots+\frac{1}{2^{j-1}}$

$$s_{k_j}=s_{2^j-1}=1+\frac{1}{2}+\cdots\cdots\cdots\cdots\cdots+\frac{1}{\left(2^j-1\right)^2}$$

$$=1+\frac{1}{2^2}+\cdots\cdots\cdots\cdots+\frac{1}{\left(2^{j-1}\right)^2}+\cdots\cdots\cdots\cdots+\frac{1}{\left(2^j-1\right)^2}=s_{k_{j-1}}+\sum_{k=2^{j-1}}^{2^j-1}\frac{1}{k^2}$$

By induction assumption $s_{k_{j-1}} = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots\cdots+\frac{1}{2^{j-2}}$

Each of the terms $\left(2^{j-1}\right)^2, \left(2^{j-1}+1\right)^2, \cdots\cdots\cdots\cdots\cdots\cdots\cdots\left(2^j\right)^2$ is greater than $\left(2^{j-1}\right)^2$ hence $\frac{1}{\left(2^{j-1}+k\right)^2} < \frac{1}{\left(2^{j-1}\right)^2}$ for $0 \le k \le 2^{j-1} - 1$

$$\Rightarrow \sum_{k=2^{j-1}}^{2^j-1} \frac{1}{k^2} < \frac{2^{j-1}}{\left(2^{j-1}\right)^2} = \frac{1}{2^{j-1}}$$

Hence $s_{k_j} < 1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots\cdots\cdots+\frac{1}{2^{j-2}}+\frac{1}{2^{j-1}}$

Further the geometric series $1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots\cdots\cdots$ is convergent and has sum 2, this sum being the supremum of the sequence of partial sums $\left(t_j\right)$ where $t_j = 1+\frac{1}{2}+\cdots\cdots\cdots\cdots+\frac{1}{2^{j-1}}$ so that $t_j < 2$ for all j. Hence $s_{k_j} < 2$ for all j. Since (s_n) is monotonically increasing, if $n \in \mathbb{N}$ and $n < 2^j$, $n \le k_j$ so taht $s_n \le s_{k_j} < 2$ for all n. The sequence (s_n) being increasing and bounded (s_n) is convergent. thus the 2 - series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(v) The p - series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$

The proof is similar to the case $p = 2$. The sequence (s_n) of partial sums is increasing. For each $j \in \mathbb{N}$ we write $k_j = 2^j - 1$ and show by induction that

$$s_{k_j} < 1+r+r^2+\cdots\cdots\cdots+r^{j-1} < \frac{1}{1-r} \text{ where } r = \frac{1}{2^{p-1}}$$

When $j=1,\ k_1 = 1$ and $s_1 = 1 < \frac{1}{1-r}$

We assume that $s_{k_{j-1}} < 1 + r + r^2 + \cdots\cdots\cdots + r^{j-2}$ and as in the case $p = 2$, write

$$s_{k_j} = s_{k_{j-1}} + \sum_{k=0}^{2^j-1} x_{2^{j-1}+k} \quad \text{where } x_k = \frac{1}{k^p} < s_{k_{j-1}} + r^{j-1} = 1 + r + r^2 + \cdots\cdots\cdots + r^{j-1}$$

Since $0 < r < 1$ the geometric series $\sum_{j=0}^{\infty} \frac{1}{r^j}$ converges.

Using increasing property of (s_n) we get convergence of the p - series.

(vi) The p - series diverges if $0 < p \le 1$. As the case $p = 1$ is already considered, we consider the case $0 < p < 1$. Then $1 - p > 0$ so $n^{1-p} > 1 \Rightarrow n > n^p \Rightarrow \frac{1}{n^p} > \frac{1}{n}$.

If $t_n = 1 + \cdots\cdots\cdots\cdots + \frac{1}{n}$ it follows that $s_n > t_n$

Since the harmonic series is divergent, (t_n) is unbounded hence (s_n) is unbounded.

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges properly to $+\infty$.

13.4.7 S.A.Q.: Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of real numbers show that given $\in > 0$ there is a $N_\in \in \mathbb{N}$ such that $\left| \sum_{n=k+1}^{\infty} a_n \right| < \in$ whenever $n \ge N_\in$.

13.4.8 Algebraic Properties: Suppose $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are convergent. Then

(i) $\sum_{n=1}^{\infty} x_n + y_n$ converges and $\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$.

(ii) For any $C \in \mathbb{R}$, $\sum_{n=1}^{\infty} Cx_n$ converges and $\sum_{n=1}^{\infty} Cx_n = C \sum_{n=1}^{\infty} x_n$

Proof: (i) Let $x = \sum_{n=1}^{\infty} x_n$, $y = \sum_{n=1}^{\infty} y_n$, $s_n = x_1 + x_2 + \cdots\cdots\cdots\cdots + x_n$ and

$t_n = y_1 + y_2 + \cdots\cdots\cdots + y_n$. Then $\ell im\,(s_n + t_n) = \ell im\,(s_n) + \ell im\,(t_n)$

$$= x + y$$

Further if the sequence of partial sums for $\sum_{n=1}^{\infty}(x_n+y_n)$ is u_n then

$u_n=(x_1+y_1)+\cdots\cdots+(x_n+y_n)=(x_1+\cdots\cdots+x_n)+(y_1+\cdots\cdots+y_n)=s_{n+t_n}$.

Thus the sequence (u_n) converges and $\ell\text{im}(u_n)=\ell\text{im}(s_n)+\ell\text{im}(t_n)=x+y$.

Hence $\sum_{n=1}^{\infty} x_n+y_n$ converges and $\sum_{n=1}^{\infty}(x_n+y_n)=\sum_{n=1}^{\infty} x_n+\sum_{n=1}^{\infty} y_n$.

(ii) If (v_n) is the sequence of partial sums of $\sum_{n=1}^{\infty} Cx_n$ then

$$v_n=Cx_1+\cdots\cdots+Cx_n=C(x_1+\cdots\cdots+x_n)=Cs_n$$

Since $\ell\text{im}(s_n)=x$ it follows that (v_n) converges and

$$\ell\text{im}(v_n)=\ell\text{im}(Cs_n)=C\,\ell\text{im}(s_n)=Cx$$

Thus $\sum_{n=1}^{\infty} Cx_n$ converges and $\sum_{n=1}^{\infty} Cx_n=Cx=C\sum_{n=1}^{\infty} x_n$

13.4.9 Cauchy Criterion for Series:

Theorem: The series $\sum_{n=1}^{\infty} x_n$ converges if and only if for every $\in>0$ there is a natural number $M(\in)$ such that if $m>n\geq M(\in)$, then $|s_m-s_n|=|x_{n+1}+\cdots\cdots+x_m|<\in$ where (s_n) is the sequence of partial sums of the series.

Proof: The series $\sum x_n$ converges if and only if (s_n) converges in $\mathbb{R}$ iff the Cauchy criterion holds good for (s_n). Thus $\sum x_n$ converges $\Leftrightarrow$ for every positive real number $\in$ there exists a natural number $M(\in)$ such that if $m\geq M(\in)$ and $n\geq M(\in)$, $|s_m-s_n|<\in$.

Thus $\sum x_n$ converges if and only if for every $\in>0$ there is a natural number $M(\in)$ such that if $m>n\geq N(\in)$, $|s_m-s_n|=|x_{n+1}+\cdots\cdots+x_m|<\in$.

Remark: The importance of Cauchy criterion in infinite series lies in its application to convergence by comparision, one of the most fundamental techniques in the theory of infinite series.

We first prove a simple but important lemma. When each $x_n > 0$ we call the series Σx_n , a series of postive terms. When each $x_n \geq 0$. We call Σx_n a series of non negative terms.

13.4.10 Lemma: A series Σx_n of positive (non negative) terms either converges or diverges properly to $+\infty$.

Proof: Let (s_n) be the sequence of partial sums of Σx_n , $s_n = x_1 + \cdots\cdots + x_n$. Clearly $s_{n+1} = s_n + x_{n+1} \geq s_n$ for every $n \in \mathbb{N}$. Moreover $s_n \geq 0$ for all n.

Thus (s_n) is monotonically increasing sequence in $\mathbb{R}$. Hence $\sum_{n=1}^{\infty} x_n$ converges if and only if (s_n) converges if and only if (s_n) is bounded. If (s_n) is unbounded then $\ell im\ s_n = +\infty$ and in this case Σx_n diverges properly to $+\infty$.

13.4.11 Absolute Convergence: A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem: If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof: Let $s_n = a_1 + a_2 + \cdots\cdots + a_n$ and $t_n = |a_1| + \cdots\cdots + |a_n|$ since $\sum_{n=1}^{\infty} |a_n|$ is convergent, given $\in > 0$ there is $N_\in \in \mathbb{N}$ such that $\sum_{k=n+1}^{m} |a_k| < \in$ for $m > n \geq N_\in$.

$$\Rightarrow |t_m - t_n| = (t_m - t_n) < \in \text{ for } m > n \geq N_\in$$

For such $m, n, |s_m - s_n| = |a_{n+1} + \cdots + a_m| \leq |a_{n+1}| + \cdots\cdots + |a_m| = t_m - t_n < \in$

$$\Rightarrow \sum_{n=1}^{\infty} an \text{ is convergent.}$$

Remark: The converse of 13.4.10 is false.

The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$ converges by $13.4.5\ (v)$ but the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots$ diverges by 13.4.5 (iv)

13.4.12 Alternating Series - Leibnitz's Theorem:

Theorem: Let (x_n) be a decreasing sequence of positive terms such that $\ell im\,(x_n) = 0$. Then the series (called alternating series) $\sum_{n=1}^{\infty}(-1)^{n-1} x_n$ converges.

Proof: Write $s_n = \sum_{i=1}^{n}(-1)^{n-1} x_n = x_1 - x_2 + \cdots\cdots + (-1)^{n-1} x_n$

$$0 < s_n \le x_1 \text{ for } n \in N \quad \text{and } s_{2n+1} = s_{2n} + x_{2n+1} \cdots\cdots\cdots\cdots (1)$$

Also (s_{2n}) is increasing, (s_{2n+1}) is decreasing and in fact

$$0 < s_{2n-2} < s_{2n} < s_{2n+1} < s_{2n-1} \le x_1 \text{ for } n \in \mathbb{N}.$$

Hence $\ell im\,(s_{2n})$ and $\ell im\,(s_{2n-1})$ exists. Since $\ell im\; x_n = 0$, $\ell im\; x_{2n+1} = 0$ by (1) $\Rightarrow \ell im\; s_{2n} = \ell im\, s_{2n-1} \Rightarrow \ell im\; s_n = \ell im\; s_{2n} = \ell im\, s_{2n-1}$

Hence the series $\sum_{n=1}^{\infty}(-1)^{n-1} x_n$ converges.

Remark: Compare with 13.4.5 (iii) and (iv)

13.5 Comparison Tests:

13.5.1 Let $X = (x_n)$ and $Y = (y_n)$ be real sequences and suppose that for some natural number k, $0 \le x_n \le y_n$ for $n \ge k$. Then (a) the convergence of $\sum y_n$ implies the convergence of $\sum x_n$ and (b) the divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

Proof: Let $s_n = x_1 + x_2 + \cdots\cdots\cdots + x_n$ and $t_n = y_1 + \cdots\cdots + y_n$ for $n \in \mathbb{N}$.

If $m > n$, $\quad s_m - s_n = x_{n+1} + \cdots\cdots\cdots\cdots + x_m \le y_{n+1} + \cdots\cdots + y_m = t_m - t_n \cdots\cdots (1)$

(a) Suppose $\sum y_n$ converges and $\in > 0$. By Cauchy's criterion there is a natural number $M(\in)$ such that if $m > n \ge M(\in)$, $|t_m - t_n| = t_m - t_n < \in$.

From (1), for $m > n \ge N(\in)$, $\quad |s_m - s_n| = s_m - s_n \le t_m - t_n < \in$

Again by Cauchy's criterion $\sum x_n$ converges.

(b) We prove this part by contrapositive. That is we analyse the situation when Σx_n diverges but Σy_n does not diverge. Since Σy_n is a series of positive terms either Σy_n converges or properly dvierges to $+\infty$. If we assume that Σy_n does not diverge the series must converge and hence by (a) Σx_n must converge. Thus under the assumption that Σx_n diverges it must necessarily happen that Σy_n diverges.

13.5.2 S.A.Q.: Show that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms then $\sum_{n=1}^{\infty} a_n^2$ is convergent.

13.5.3 S.A.Q.: If $a_n > 0$ show that for every n & $b_n = \dfrac{a_1 + a_2 + \cdots\cdots + a_n}{n}$ then $\sum_{n=1}^{\infty} b_n$ is divergent.

13.5.4 S.A.Q. (Ratio test): Let (x_n) be a sequence in $\mathbb{R}$, $r > 0,\ k \in \mathbb{N}$. If $0 < r < 1$ and $\left|\dfrac{x_{n+1}}{x_n}\right| \le r$ for $n \ge k,\ n \in \mathbb{N}$. Then the series $\sum_{x=1}^{\infty} x_n$ converges absolutely. If $\left|\dfrac{x_{n+1}}{x_n}\right| \ge r$ for infinitely many n, $\sum_{n=1}^{\infty} x_n$ diverges.

13.5.5 S.A.Q. (Root Test): Let (x_n) be a sequence in $\mathbb{R}$, $r > 0$ and $k \in \mathbb{N}$. Show that

(i) If $r < 1$ and $|x_n|^{1/n} \le r$ for all $n \ge k,\ n \in \mathbb{N}$ then the series $\sum_{n=1}^{\infty} x_n$ converges absolutely.

(ii) If $|x_n|^{1/n} \ge 1$ for infinitely many n show that $\sum_{n=1}^{\infty} x_n$ diverges.

13.5.6 Limit Comparison Test:

Let (x_n) and (y_n) be strictly positive sequences i.e., $x_n > 0$ and $y_n > 0$ and suppose that $\ell im\left(\dfrac{x_n}{y_n}\right) = r \in \mathbb{R}$

(a) If $r \ne 0$ Σx_n converges if and only if Σy_n converges.

(b) If $r = 0$ and Σy_n converges then Σx_n converges.

Proof: (a) Asume that $r \neq 0$. Since $x_n > 0, y_n > 0, \frac{x_n}{y_n} > 0$ for every n. Since $r \neq 0$

$r = \ell im\left(\frac{x_n}{y_n}\right) > 0$. For $\in = \frac{r}{2}$ there corresponds a natural number K such that $\left|\frac{x_n}{y_n} - r\right| < \frac{r}{2}$ if $n \geq k$

$$\Rightarrow \frac{r}{2} = r - \frac{r}{2} < \frac{x_n}{y_n} < \frac{3r}{2} \text{ if } n \geq k \Rightarrow \frac{r}{2}\, y_n < x_n < \frac{3r}{2}\, y_n \text{ if } n \geq k$$

By comparision test, Σy_n converges $\Rightarrow \frac{2}{3r}\Sigma x_n$ converges and hence Σx_n converges. Similarly if Σx_n converges, Σy_n converges.

(b) If $r = 0$ corresponding to $\in = 1$ there is a natural number k such that $\frac{x_n}{y_n} = \left|\frac{x_n}{y_n} - 0\right| < 1$ if $n \geq k$.

$$\Rightarrow x_n < y_n \text{ if } n \geq k$$

Hence if Σy_n converges, Σx_n converges.

13.5.7 Cauchy's Condensation Test: Let $a(n)$ be a strictly decreasing sequence of positive terms and $s(n)$ be the sequence of partial sums of the series $\sum_{n=1}^{\infty} a(n)$. Then

(i) $\frac{1}{2}\left(a(1) + 2 \cdot a(2) + \cdots + 2^n a\left(2^n\right)\right) < s\left(2^n\right) < a_1 + 2a(2) + \cdots + 2^{n-1} a\left(2^{n-1}\right) + a\left(2^n\right)$ for $n \in \mathbb{N}$

(ii) The series $\sum_{n=1}^{\infty} a(n)$ converges if any only if the series $\sum_{n=1}^{\infty} 2^n\, a\left(2^n\right)$ converges.

Proof: We write $t_n = a(1) + 2(a(2)) + \cdots + 2^{n-1}\, a\left(2^{n-1}\right)$

Then (t_n) is the sequence of partial sums of the series $\sum_{n=1}^{\infty} 2^n\, a\left(2^n\right)$ and the inequality (i) becomes $\frac{1}{2}\, t_{n+1} < s\left(2^n\right) < t_n + a\left(2^n\right)$. Since $a(n) > a(n+1)$ for every n.

$$2\,a\left(2^2\right) = 2a(4) = a(4) + a(4) < a(3) + a(4)$$

$$2^2\,a\left(2^3\right) = 4\left(a(8)\right) = a(8) + a(8) + a(8) < a(5) + a(6) + a(7) + a(8)$$

Similarly for $n \in \mathbb{N}$ and $1 \le k \le 2^{n-1}$, $a\left(2^n\right) < a\left(2^{n-1} + k\right)$

so that $2^{n-1}\,a\left(2^n\right) < a\left(2^{n-1} + 1\right) + \cdots\cdots + a\left(2^{n-1} + 2\right) + \cdots\cdots + a\left(2^n\right)$

Adding the terms on both sides we get

$$2a\left(2^2\right) + 2^2 a\left(2^3\right) + \cdots\cdots + 2^{n-1} a\left(2^n\right) < a(3) + a(4) + \cdots\cdots + a\left(2^n\right)$$

$$\Rightarrow \frac{a(1)}{2} + a(2) + 2a\left(2^2\right) + \cdots\cdots + 2^{n-1} a\left(2^n\right) < a(1) + a(2) + \cdots\cdots + a\left(2^n\right)$$

$$\Rightarrow \frac{1}{2}\left\{ a(1) + 2a(2) + 2^2 a\left(2^2\right) + \cdots\cdots + 2^n a\left(2^n\right)\right\} < a(1) + a(2) + \cdots\cdots + a\left(2^n\right)$$

$$\Rightarrow \frac{1}{2}\, t_{n+1} < s\left(2^n\right)$$

Since $a(3) < a(2)$, $\quad a(2) + a(3) < 2a(2)$

Since each of $a(5),\ a(6), a(7)$ is less than $a(4)$

$$a(4) + a(5) + a(6) + a(7) < 4a(4)$$

Similarly for any n and k such that $0 < k < 2^{n-1}$

$a\left(2^{n-1} + k\right) < a\left(2^{n-1}\right)$ hence

$$a\left(2^{n-1}\right) + a\left(2^{n-1} + 1\right) + \cdots\cdots + a\left(2^n - 1\right) < 2^{n-1} a\left(2^{n-1}\right)$$

Hence $a(2) + a(3) + \cdots\cdots + a\left(2^n - 1\right) < 2a(2) + 4a(4) + \cdots\cdots + 2^{n-1} a\left(2^{n-1}\right)$

$$\Rightarrow s\left(2^n\right) = a(1) + a(2) + a(3) + \cdots\cdots + a\left(2^n - 1\right) + a\left(2^n\right)$$

$$< a(1) + 2a(2) + \cdots\cdots\cdots + 2^{n-1}a\left(2^{n-1}\right) + a\left(2^n\right)$$

$$= t_n + a\left(2^n\right)$$

Thus $\frac{1}{2}t_{n+1} < s\left(2^n\right) < t_n + a\left(2^n\right)$

This complete the proof of (i)

To prove (ii), it is enough to prove that s_n and (t_n) both converge or both diverge together. Since the series under consideration have positive terms the sequences $(s_n), (t_n)$ of partial sums are increasing. Hence convergence of (s_n) and (t_n) holds iff (s_n) and (t_n) are bounded, infact, bounded above. Since (s_n) is increasing it is true that (s_n) is bounded if and only if $(s(2^n))$ is bounded.

Now suppose $(s(n))$ is bounded. Then $\left(s\left(2^n\right)\right)$ is bounded. Hence there is $M > 0$ such that $s\left(2^n\right) < M$ for $n \in N$. Since $\frac{1}{2}t_{n+1} < s\left(2^n\right) < M$ for every $n \in \mathbb{N}$ it follows that (t_{n+1}), hence (t_n) is bounded.

Conversely suppose (t_n) is bounded. Then there is $M^1 > 0$. Such that $t_n < M^1$ for every $n \in \mathbb{N}$. Since $a\left(2^n\right) < a(1)$. Since $s\left(2^n\right) < t_n + a\left(2^n\right) < t_n + a(1)$ for every n; $\left(s\left(2^n\right)\right)$ is bounded. Hence $(s(n))$ is bounded. This completes the proof.

13.5.8 Cauchy's Integral Test: This test for convergence of infinite series is linked with the notion of convergence of infinite (improper) integrals. The student is familiar with the notion of the Riemann integral of a bounded function defined on a closed and bounded interval and will be exposed to a detailed account of this theory in lessons 18, 19 and 20.

If $f : [a, \infty) \to \mathbb{R}$ satisfies (i) f is Riemann integrable on $[a, b]$ for every $b \in \mathbb{R},\ b > a$ and (ii) for some $A \in \mathbb{R},\ \lim\limits_{b \to +\infty} \int_a^b f = A$ then we say that the infinite integral $\int_a^\infty f dx$ converges and write $A = \int_a^\infty f d_x$.

Theorem: Let $f:(1,\infty)\to\mathbb{R}$ be a positive decreasing function such that $\lim\limits_{x\to+\infty} f(x)=0$. Then the infinite series $\sum\limits_{n=1}^{\infty} f(n)$ converges if and only if the infinite integral $\int\limits_{a}^{\infty} fd_x$ converges.

Proof: For $r\in\mathbb{N}$ write $s_n=\sum\limits_{k=1}^{n} f(k)=f(1)+\cdots\cdots\cdots\cdots+f(n)$ and $t_n=\int\limits_{1}^{n+1} f(t)dt$. Since $f(x)>0$ for $x\in[1,\alpha]$, (s_n) is monotonically increasing and $s_n>0$ and $t_n\geq 0$ for all n.

If $n\leq b\leq n+1,\ 0\leq \int\limits_{1}^{n} fd\int\limits_{0}^{b} tdt\leq \int\limits_{0}^{n+1} fdt$. Hence $\lim\limits_{b\to\infty}\int\limits_{1}^{b} fdt$ exists in $\mathbb{R}$ if and only if (t_n) converges. Again if $k\in\mathbb{N}$ and $k\leq x\leq k+1$, $f(k+1)\leq f(x)\leq f(k)$.

$$\Rightarrow \int\limits_{k}^{k+1} f(k+1)dx\leq \int\limits_{k}^{k+1} f(x)dx\leq \int\limits_{k}^{k+1} f(k)dx$$

$$\Rightarrow f(k+1)\leq \int\limits_{k}^{k+1} fdx\leq f(k)$$

Taking the summation from $k=1$ to n we get

$$f(2)+\cdots\cdots\cdots\cdots+f(n+1)\leq \int\limits_{1}^{n+1} fdx\leq f(1)+\cdots\cdots\cdots+f(n)$$

$$\Rightarrow 0\leq s_{n+1}-s_1\leq t_n\leq s_n$$

If (s_n) is convergent, (s_n) is bounded hence (t_n) is bounded, hence convergent. Conversely if (t_n) is convergent (t_n) is bounded, hence (s_n) is bounded and hence convergent.

13.6 Applications:

13.6.1 Discuss the convergence of the p - series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ using Cauchy's condensation test.

Let $a(n)=\frac{1}{n^p}$. $(a(n))$ decreases strictly if $p\geq 1$. By the condensation test $\sum\limits_{n=1}^{\infty} a(n)$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n\, a(2^n)$ converges.

$$a\left(2^n\right)=\frac{1}{\left(2^n\right)^p}\Rightarrow b_n=2^n\ a\left(2^n\right)=\left(\frac{1}{2^{p-1}}\right)^n$$

If $p=1,\ 2^n\ a\left(2^n\right)=1\ \Rightarrow \sum\limits_{n=1}^{\infty} b_n$ diverges.

If $p>1,\quad b_n=r^n$ where $r=\frac{1}{2^{p-1}}$ satisfies $0<r<1$

As the geometric series converges $\sum b_n$ converges $\sum\limits_{n=1}^{\infty} a(n)$ converges if $p>1$.

If $0<p<1,\ r=\frac{1}{2^{p-1}}>1$. The geometric series $\sum\limits_{n=0}^{\infty} r^n$ is divergent. Hence $\sum\limits_{n=1}^{\infty} a\left(2^n\right)$ is divergent.

13.6.2 Discuss the convergence of the series $\sum\limits_{n=2}^{\infty}\frac{1}{n\ \log n}$

Solution: The sequence (x_n) where $x_n=\frac{1}{n\log n}\ (n>1)$ is strictly decreasing sequence of positive terms because $\log n<\log(n+1)$ and hence

$$n\log n<n\log(n+1)<n+1\ \log(n+1)$$

$$2^n\ x\left(2^n\right)=\frac{2^n}{2^n\ \log 2^n}=\frac{1}{\log\left(2^n\right)}=\frac{1}{n\ \log 2}$$

The series $\sum\limits_{n=2}^{\infty}\frac{1}{n}$ is divergent hence $\sum\limits_{n=2}^{\infty} 2^n\ x\left(2^n\right)$ is divergent hence $\sum\limits_{n=2}^{\infty} x(n)$ diverges by Cauchy's condensation test.

13.6.3 Discuss the convergence of $\sum\limits_{n=2}^{\infty} x_n$ where $x_n=\frac{1}{n\log n\log\ \log n}$

Solution: Clearly $x_n>0$ and since $\log(n+1)>\log\ n$, $\log\ \log(n+1)>\log\ \log\ n$, hence $x_{n+1}<x_n$ for every n. Thus x_n is strictly decreasing and positive

$$2^n x_{2^n} = \frac{1}{n \log 2 \log(n \log 2)}$$

If $n \geq 2, \log n \geq \log 2 \Rightarrow \log(n \log 2) = \log n + \log 2 \leq 2 \log n$

$$\Rightarrow 2^n x_{2^n} \geq \frac{1}{(n \log 2)\ 2 \log n} = \frac{1}{2 \log 2 \cdot n \log n}$$

The series $\sum\limits_{n=2}^{\infty} \frac{1}{n \log n}$ diverges by 13.6.2

Hence $\sum\limits_{n=2}^{\infty} 2^n\ x(2^n)$ diverges.

Hence by Cauchy's condensation test $\sum\limits_{n=2}^{\infty} x_n$ diverges.

13.7 Answers To S.A.Q.'s:

13.4.7 **S.A.Q.:** Let $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series of real numbers with sum s. Then for every $k \in \mathbb{N}$ the series $\sum\limits_{n=k+1}^{\infty} a_n$ converges and has sum $s - s_k$ where $a_1 + \cdots\cdots\cdots + a_k$.

Let $t_n = a_{k+1} + \cdots\cdots\cdots\cdots + a_{k+n}$. Then (t_n) is the sequence of partial sums for $\sum\limits_{j=k+1}^{\infty} a_j$ and $t_n = S_{k-n} - s_k$. Since $\ell im(s_n) = s$ and (s_{k+n}) is a subsequence of (s_n).

$$\ell im_n (t_n) = \ell im_n (s_{k+n} - s_k) = \ell im_n (s_{k+n}) - s - s_k$$

Hence $\sum\limits_{j=k+1}^{\alpha} a_j = \ell im_n\, t_n = s - s_k$ since $\ell im\, s_k = s, \quad \ell im_k (s - s_k) = 0$

Hence given $\in > 0$ there exists $N_\in$ in $\mathbb{N}$ such that $|s - s_k| < \in$ for $n \geq N_\in$

$$\Rightarrow \left| \sum_{n=k+1}^{\infty} a_k \right| < \in \text{ for } k \geq N_\in$$

13.5.2 Since $\sum a_n$ is convergent, $\ell im\ a_n = 0$

Hence for $\in = 1$ there is a natural number N such that $|a_n| = a_n < 1$ if $n \geq N$. Since $1 > a_n > 0$ it follows that $a_n^2 < a_n$ for $n \geq N$.

Hence by comparision test $\sum\limits_{n=N}^{\infty} a_n^2$ and hence $\sum\limits_{n=1}^{\infty} a_n^2$ are convergent.

Converse is false: for example the 2 - series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent but $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.

13.5.3 Since $b_n = \frac{a_1 + a_2 + \cdots\cdots + a_n}{n}$ and $a_k > 0$ for all k, $b_n > \frac{a_1}{n}$ for all n. The series $\sum \frac{1}{n}$ is divergent. Hence by comparision test $\sum b_n$ is divergent.

13.5.4 **Ratio Test:** Let (x_n) be a sequence of real numbers $0 < r < 1$ and there is a natural number N such that for $n \geq k$ $\left|\frac{x_{n+1}}{x_n}\right| \leq n$ for $n \geq k$. Then the series $\sum\limits_{n=1}^{\infty} |x_n|$ is convergent.

If $\left|\frac{x_{n+1}}{x_n}\right| \geq 1$ for infinitely many n then the series $\sum\limits_{n=1}^{\infty} x_n$ diverges.

Proof: (1) If $\left|\frac{x_{n+1}}{x_n}\right| \leq r$ for $n \geq k$ then for $n > k$,

$$x_n = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots\cdots\cdots\cdots\cdots \frac{x_{k+1}}{x_k} \cdot x_k$$

$$\Rightarrow |x_n| \leq r^{n-k} |x_k| = \left|\frac{x_k}{r^k}\right| r^n$$

Since $0 < r < 1$ the series $1 + r + r^2 + \cdots\cdots\cdots\cdots$ is convergent

$\Rightarrow r^k + r^{k+1} + \cdots\cdots\cdots\cdots$ is convergent

$\Rightarrow \left|\frac{x_k}{r^k}\right| \cdot r^k + \left|\frac{x_k}{r^k}\right| r^{k+1} + \cdots\cdots$ is convergent.

Hence by comparision test $|x_k| + |x_{k+1}| + \cdots\cdots\cdots$ is convergent.

Thus $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

(ii) If $n_1 < n_2 < \cdots\cdots\cdots < n_k \cdots\cdots\cdots$ be an infinite sequence in $\mathbb{N}$ such that $\left|\frac{x_{n_{k+1}}}{x_{n_k}}\right| \geq 1$ for all n_k then $|x_{n_k}| \geq |x_{n_1}| > 0$ for all n_k.

$\Rightarrow$ either $\ell im\ x_{n_k}$ does not exist or $\ell im\ x_{n_k} \neq 0$

Hence $\ell im\ x_n \neq 0$ if the limit exists.

so the series $\sum_{n=1}^{\infty} x_n$ diverges by 13.4.4

13.5.5 Root Test:

Let (x_n) be a sequence in $\mathbb{R}$.

(i) If there exists a real number r and $N \in N$ such that $0 < r < 1$ and $|x_n|^{1/n} \leq r$ for $n \geq N$ then the series $\sum_{n=1}^{\infty} x_n$ converges absolutely.

(ii) If $|x_n|^{1/n} \geq 1$ for infinitely many $n \in \mathbb{N}$ then the series diverges.

Proof: If (i) holds then $| x_n \leq r^n$ for $n \geq N$,

Since $0 < r < 1$ the geometric series $\sum_{n=1}^{\infty} r^n$ converges, $\sum_{n=1}^{\infty} |x_n|$ converges hence $\sum_{n=1}^{\infty} x_n$ converges absolutely.

If (ii) holds, $|x_n| \geq 1$ for infinitely many n hence either (x_n) many not converge even if (x_n) converges $\ell im\ (x_n) \neq 0$. Hence the series $\sum_{n=1}^{\infty} x_n$ diverges.

13.8 Summary:

After a brief introduction on infinite series Cauchy's criterion for convergence of an infinite series is established. A few important tests such as the comparison test, limit comparision test, condensation test and Leibnitz's result on convergence of alternating series are established. A good number of examples and applications are also presented for the benefit of the student.

13.9 Technical Terms:

Infinite Series

Sequence of Partial Sums

Convergence and divergence of infinite series

Comparision Test

Condensation Test

Alternating series and

Absolute Convergence of Series

13.10 Exercises:

1. Let (a_n) be a sequence of real numbers and $a_n = 0$ if and only if $n = n_k$ where $1 \le n_1 < n_2 < \cdots\cdots < n_k \cdots\cdots$ Let (b_n) be the subsequene of (a_n) obtained by removing the subsequence (a_{n_k}) from (a_n).

   Compare the sequence of partial sums of Σa_n and Σb_n and prove that Σa_n converges if and only if Σb_n converges.

2. Show that if (a_n) is a sequence of real numbers $N \in \mathbb{N}$ and $b_n = a_n$ for $n \ge N$ then Σa_n converges iff Σb_n converges.

3. Using partial fractions show that

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+\alpha)(n+\alpha+1)} = \frac{1}{\alpha}$ if $\alpha > 0$, (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

4. Show by means of an example that if Σx_n and Σy_n are divergent $\Sigma(x_n + y_n)$ is not necessarily divergent $\left(x_n = -y_n = \frac{1}{n}\right)$

5. Show that $\Sigma \cos n$ is divergent (**Hint:** $\ell\text{im} \cos n$ does not exist)

6. Show that $\Sigma \frac{\cos\ n}{n^2}$ is convergent (compare with $\Sigma \frac{1}{n^2}$)

7. Show that $\Sigma \frac{(-1)^n}{\sqrt{n}}$ is convergent (alternating Series)

8. Show that if $a_n > 0$ and Σa_n is convergent then $\Sigma \frac{\sqrt{a_n}}{n}$ is conergent

(Hint: $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$)

9. (a) Determine convergence of

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\cdots\cdots\cdot+\frac{1}{3n+1}+\frac{1}{3n+2}-\frac{1}{3n+3}+\cdots\cdots$$

Hint $(s_3 n)$ is unbounded

(b) Show that $1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\cdots\cdots\cdots\cdot$ is convergent.

(c) Show that if (a_n) and (b_n) are decreasing sequence of positive terms and $\ell\text{im}\,(a_n) = \ell\text{im}\,(b_n) = 0$ then

$a_1 - a_2 - b_1 + a_3 - a_4 + b_2 + a_5 - a_6 - b_3 + \cdots\cdots\cdots + a_{2n-1} - a_{2n-2} - (-1)^n b_n + \cdots\cdot$
is convergent.

Hint $a_1 - a_2 + a_3 - a_4$.................. converges and

$b_1 - b_2 + b_3 - b_4$...................... converges

11. Show that if $a_n > 0$ and Σa_n is convergent then $\Sigma \sqrt{a_n a_{n+1}}$ is convergent

(Hint $\left(\sqrt{a_n} - \sqrt{a_{n+1}}\right)^2 \geq 0$)

12. Given an example to show that if Σa_n^2 is convergent, Σa_n may be convergent but not necessarily absolutely.

$$\left(a_n = \frac{(-1)^n}{n} \right)$$

13. If $a_n > 0$ for $n \in \mathbb{N}$ and $b_n = \left(\frac{a_1 + \cdots\cdots\cdots + a_n}{n}\right)^{\frac{1}{2}}$ show that Σb_n is divergent.

(Hint: $b_n > \frac{\sqrt{a_1}}{\sqrt{n}}$ for $n \in \mathbb{N}$)

14. Apply Cauchy's condensation test to the series $\sum x_n$ where

(a) $x_n = \dfrac{1}{n(\log n)c}$ $(n > 1)$ and $c > 1$ (Ans: convergent)

(b) $x_n = \dfrac{1}{n \log n(\log \log n)(\log \log \log n)}$ (Ans: convergent)

15. Discuss te convergence of the series

(a) $(-1)^n \dfrac{n^n}{(n+1)^{n+1}}$ (convergent)

(b) $\dfrac{n^n}{(n+1)^{n+1}}$ (diverges)

(c) $(-1)^n \dfrac{(n+1)^n}{n^n}$ (diverges)

(d) $\dfrac{(n+1)^n}{n^{n+1}}$ (converges)

16. Apply the limit comparison test and show that

(a) $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^2+1}$ diverges (Hint: Consider $\left(1/n\right)$)

(b) $\sum \dfrac{1}{n(n+1)}$ converges (Hint: Consider $1/n^2$)

17. Apply Leibnitz's theorem and discuss the convergence of the following alternating series.

(a) $1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} \cdots\cdots\cdots\cdots\cdots\cdots$ (Converges)

(b) $1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} - \dfrac{1}{4\sqrt{4}} + \cdots\cdots\cdots\cdots$ (Converges)

(c) $1 - \dfrac{3}{2} + \dfrac{4}{3} - \dfrac{5}{4} + \cdots\cdots\cdots\cdots$ (Diverges)

(d) $1 - \dfrac{2}{3} + \dfrac{3}{5} - \dfrac{4}{7} + \cdots\cdots\cdots\cdots$ (Diverges)

18. (a) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and (b_n) is a bounded sequence then $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent .

(b) Deduce that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent then $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

(c) Show by means of an example that the converse of 18(a) does not hold if absolute convergence is replaced by convergence.

19. Prove Abel's lemma: Let $(x_n),(y_n)$ be sequences in $\mathbb{R}$, $s_0 = 0$ and $s_n = x_1 + x_2 + \cdots\cdots\cdots\cdots + x_n$ for $n \in \mathbb{N}$ show that if

$$m > n \qquad \sum_{k=n+1}^{m} x_k\ y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m} (x_k - x_{k+1}) s_k$$

20. Prove Dirichlet's Test. If (x_n) is a decreasing sequence of positive terms, $\ell\text{im}\ x_n = 0$ and (y_n) is such that the sequence (s_n) is bounded then $\sum_{n=1}^{\infty} x_n y_n$ is convergent.

Hint: Apply Abel's lemma and derive $\left|\sum_{k=n+1}^{m} x_k y_k\right| \leq 2x_{n+1}B$ where $|s_n| \leq B$ for all n.

21. Let (x_n) be a sequence of positive real numbers and $\ell\text{im}\left(\frac{x_{n+1}}{x_n}\right) = r$

(a) If $r < 1$ show that $\sum_{n=1}^{\infty} x_n$ converges and

(b) If $r \geq 1$ show that $\sum_{n=1}^{\infty} x_n$ diverges

22. Show by means of examples that the ratio test and the root test fail to determine the convergence of the series if $r = 1$. $\left(\text{Try}: \ \frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots\cdots\right)$

23. Let (x_n) be a sequence of positive real numbers and $\ell\text{im}\ x_n^{1/n} = r$ show that

(a) If $r < 1$ $\sum_{n=1}^{\infty} x_n$ converges and (b) If $r \geq 1$ $\sum_{n=1}^{\infty} x_n$ diverges

24. Show that root test and ratio test fail so determine convergence of the p - series $\sum_{n=1}^{\infty} \frac{1}{n^p} (P > 0)$.

25. Show that root test and ratio test fail so determine the convergence of $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{4^5} + \cdots\cdots\cdots$ which is infact convergent.

26. Show that $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$ converges if $p > 1$ (where $a > 0$, $b > 0$) and diverges if $0 < p \leq 1$.

27. Let $0 < a < 1$ and consider the series

$$a^2 + a + a^4 + a^3 + \cdots\cdots\cdots + a^{2n} + a^{2n-1} + \cdots\cdots\cdots$$

(a) Show directly that the series converges by considering the series of odd terms and the series of even terme separatily.

(b) Show that convergence can be established by using root test.

(c) show that ratio test fails to establish convergence.

28. Examine the convergence of the series $\sum x_n$ where $x_n =$

(a) $\frac{1}{2^n}$ (Geometric Series, Converges)

(b) $\frac{n+1}{n-1} (n > 1)$ ($\ell\text{im}\ x_n \neq 0$, diverges)

(c) $\left(1 + \frac{1}{n}\right)^n$ ($\ell\text{im}\ x_n \neq 0$, diverges)

(d) $\frac{1}{2n^2+1}$ (Compare with $1/n^2$)

(e) $\frac{1}{\log n}\left(\log n < n\,;\text{diverges}\right)$

(f) $\frac{1}{n^n}$ $(n^n \geq 2^n$ for $n \geq 2$, converges)

(g) $\frac{\sqrt{n}}{n^2+1}$ $(x_n < \frac{1}{n^{3/2}}$, converges)

(h) $\sin n$ ($\ell im \sin n \neq 0$, diverges)

(i) $\sin 1/n$ ($\ell im\ n \sin 1/n = 1$, diverges)

(j) $\frac{1}{2^n+3^n}$ $(x_n < \frac{1}{2^{n+1}}$, converges)

(k) $\log\left(1+1/n\right)$ $\left(x_n = a_{n+1-a_n}\right.$ where $\left. a_n = \log n\right)$

(l) $\frac{1}{n!}$

(m) $\frac{3^n\ n!}{n^n}$ ((n) and (0) converge)

(n) $\sqrt{n^4+1}-\sqrt{n^4-1}$ $(n>1)$ (Rationalise $\sqrt{n^4 \pm 1} > n^2 +$ converges$)$

29. Test the series $\sum_{n=1}^{\infty} a_n$ for convergence and absolute convergence where a_n is given below.

(a) $\frac{(-1)^{n+1}}{n^2+1}$ (Absolutely Convergent)

(b) $\frac{(-1)^n n}{n+2}$ (Diverges)

(c) $\frac{(-1)^{n+1}}{n+1}$ (Converges but not absolutely.)

(d) $\frac{(-1)^{n+1}}{n} 1n(n)$ (Converges but not absolutely)

13.11 Model Examination Questions:

1. Show that if $\sum_{n=1}^{\infty}(a_n)$ is convergent then $\ell im_n (a_n) = 0$. Is the converse true? Justify.

2. Show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

3. Discuss the convergence of the series

$$1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \cdots\cdots\cdots + \frac{(-1)^{n-1}}{3^{n-1}} + \cdots\cdots\cdots$$

4. If $\sum_{n=1}^{\infty} a_n$ converges show that $\sum_{n=1}^{\infty} a_n\, a_{n+1}$ and $\sum_{n=1}^{\infty} a_n^2$ are convergent.

5. (a) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$ converges

   (b) Discuss the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + n}}$

6. Show that $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

7. Find the sum of the series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots\cdots\cdots$

8. State and prove limit comparison test.

13.12 Model Practical Problem with solution:

Examine the convergence of the infinite series $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots\cdots$

Aim: To examine the convergence or divergence of the above series.

Definition: If $X = (x_n)$ is a sequence of real numbers the sequence (s_k) defined by $s_1 = x_1,\ s_k = s_{k-1} + x_k\ (k > 1)$ is called the infinite series generated by (x_n) and is denoted by $\sum_{n=1}^{\infty} x_n$ or $x_1 + x_2 + x_3 + \cdots\cdots + x_n + \cdots\cdots\ s_n$ is called te n^{th} partial sum and x_n the n^{th} term of the infinite series. If $\ell im_n (s_n)$ exists in $\mathbb{R}$ we say that the infinite series converges and otherwise we say that the series diverges.

Results used:

(i) Limit comparison test: If $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are infinite series of positive terms and

$\ell im\left(\frac{x_n}{y_n}\right) = r \in \mathbb{R}$ then

(a) if $r \neq 0$ $\sum_{n=1}^{\infty} x_n$ converges $\Leftrightarrow$ $\sum_{n=1}^{\infty} y_n$ converges and

(b) if $r = 0$ $\sum_{n=1}^{\infty} y_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ converges

(ii) The infinite series $\sum_{n=1}^{\alpha} \frac{1}{n^p}$

(i) Converges if $p > 1$ and (ii) diverges if $p \leq 1$.

Solution: The n^{th} term of the given series is $x_n = \frac{n+1}{n^p}$.

$x_n = \frac{n+1}{n^p} = \frac{n\left(1+\frac{1}{n}\right)}{n^p} = \frac{1+\frac{1}{n}}{n^{p-1}}$. If $y_n = \frac{1}{n^{p-1}}$ $\ell im\left(\frac{x_n}{y_n}\right) = \ell im\left(1+\frac{1}{n}\right) = 1$. Since $\ell im\left(\frac{x_n}{y_n}\right) \neq 0$ the series $\sum_{n=1}^{\alpha} x_n$ and $\sum_{n=1}^{\alpha} y_n$ behave in the same way. By result (ii) $\sum_{n=1}^{\alpha} y_n$ diverges if $p-1 \leq 1$ and converges if $p-1 > 1$. Hence $\sum_{n=1}^{\alpha} x_n$ diverges if $p \leq 2$ and converges if $p > 2$.

(Method II: Hint $x_n = \frac{1}{n^{p-1}} + \frac{1}{n^p}$)

Lesson Writer
C. SANDHYA

Gottfried Wilhelm von Leibniz 1646 – 1716

Gottfried Leibniz was a German mathematician who developed the present day notation for the differential and integral calculus though he never thought of the derivative as a limit. His philosophy is also important and he invented an early calculating machine.

LESSON –14

LIMITS OF FUNCTIONS

14.1 Objective

This lesson is introduced to introduce the concept of a limit of a function at a cluster point of the domain and study various properties of limits. The notions of one sided limits, infinite limits and limits at infinity are also introduced.

14.2 Structure:

14.3 Introduction
14.4 Cluster points and limits
14.5 Limit theorems
14.6 Extension of limit concept
14.7 Infinite limits and limits at infinity
14.8 Solutions to SAQs
14.9 Summary
14.10 Technical terms
14.11 Exercises
14.12 Model Examination Questions
14.13 Model practical problem with solution

14.3 Introduction:

In this lesson the student is introduced to the notions of the limit of a function of a real variable, extending earlier notions of limit of a sequence – i.e., a function of integer variable. In section 4, the limit of a function $f:\rightarrow \mathbb{R}$ is considered at a "cluster point" c of A which is close to A, though not in A in the sense that every neighborhood of c contains infinitely many points of A. The $\in,\delta$ definition for the limit is shown to be equivalent to the sequential approach. Section 5 is devoted to derive results that are useful in calculating limits of functions. Our basic tool is the sequential approach which simplifies the arguments when compared with $\in,\delta$ arguments. In section 6 we discuss a few important features of one sided limits. Finally in section 7 we discuss the notion of infinite limits and limits of infinity.

14.4 Limits:

14.4.1 We recall the definitions δ - neighborhood. If c is a real number and $\delta > 0$ the δ - neighborhood (hereafter abrivated δ - nbd) of c is the set.

$V_\delta(c) = (c - \delta, c + \delta) = \{x / x \in \mathbb{R} \text{ and } |x - c| < \delta\}$

14.4.2 Remark: (1) If $c \neq c^1$ there is a δ - nbd of c not containing c^1. Any δ such that $0 < \delta < |c - c^1|$ satisfies $V_\delta(c) \cap \{c^1\} = \phi$. Further if $0 < \delta' < \delta$ $V_{\delta^1}(c) \subseteq V_\delta(c^1)$:

$c-\delta \quad c \quad c+\delta \quad c'$ when $c < c'$ $\qquad c' \quad c-\delta \quad c \quad c+\delta$ when $c' < c$

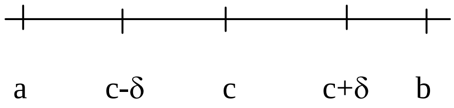
$a \quad c-\delta \quad c \quad c+\delta' \quad c+\delta$

$v_{\delta'}(c) \subseteq v_\delta(c)$

(2) If $F = \{d_1, d_2 \text{ -------- } d_n\}$ is a finite set and $c \in \mathbb{R}$

we can choose a small δ - nbd of c that does not contain any points of F, other than c.

(3) If c $\in$ (a,b) there is a $\delta > 0$ such that V_δ (c) $\subset$ (a,b) this holds for all δ such that $0 < \delta < \min \left\{ \frac{b-c}{2}, \frac{c-a}{2} \right\}$



14.4.3 Definition: cluster point:

If A is a subset of $\mathbb{R}$ and c $\in$ $\mathbb{R}$. c is called a cluster point of A if for every $\delta > 0$ there exists at least one x in A such that $x \neq c$ and $| x - c | < \delta$

Equivalently c is a cluster point of A if and only if every δ nbd of c contains a point of A other than c. i.e for every δ > o V_δ (c) $\cap$ (A – {c}) $\neq \phi$

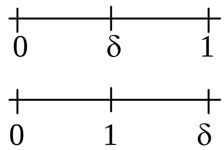
14.4.4 Notes:

1. The point c may or may not be a member of A. Even if c $\in$ A the definition is not influenced by he fact that c $\in$ A as c is excluded from consideration, the requirement being V_δ (c) $\cap$ A – {c} is non-empty. Thus c is a cluster point of A if and only if c is a cluster point of A – {c}
2. To show that c is a cluster point of A it may be useful to exhibit one $\delta_0 > 0$, such that V_{δ_0} (c) $\cap$ (A – {c}) $\neq \phi$

14.4.5 Examples:

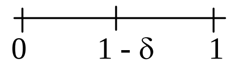
(a) Every point of [0,1] is a cluster point of (0,1). If 0 < c < 1 we can choose $\delta_0 > 0$ such that $0 < c - \delta_0 < c < c + \delta_0 < 1$ so V_{δ_0} (c) $\subseteq$ (0,1). From remark (1) above c is a cluster point of (0,1)

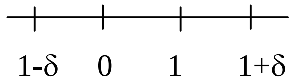
If c = 0 and $\delta > 0$, V_δ (c) $\cap$ (0,1) = (0, δ) $\cap$ (0,1) = $\begin{cases} (0, \delta) \text{ if } \delta < 1 \\ (0, 1) \text{ if } \delta \geq 1 \end{cases}$



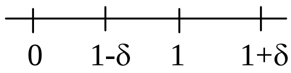
So 0 is a cluster point of (0,1)

If $c = 1$ and $\delta > 0$, $V_\delta(1) \cap (0,1) = \begin{cases} (1-\delta, 1) \text{ if } 0 < \delta < 1 \\ (0,1) \text{ if } \delta \geq 1 \end{cases}$





So 1 is a cluster point of (0,1)



(b) Let $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$

Zero is a cluster point of A. If $c \neq 0$, c is not a cluster point of A.

Solution: If $\delta > 0$, there is a natural number K such that $K > \frac{1}{\delta}$

Since $0 < \frac{1}{K} < \delta$, $\frac{1}{K} \in A \cap (0, \delta) \subseteq A \cap (-\delta, \delta)$ Since every nbd $(-\delta, \delta)$ of 0 contains some member $\frac{1}{K}$ of A which is clearly $\neq 0$, so 0 is a cluster point of A.

(c) The set $\mathbb{N}$ of natural numbers does not have cluster points.

Solution: From note 2 it sufficis to show that if $c \in \mathbb{R}$ there is a $\delta_0 > 0$ such that $(c - \delta_0, c + \delta_0) \cap N = \phi$ or $\{c\}$

If $c < 0$, let $\delta_0 = -c/2$, Then $(c - \delta_0, c + \delta_0) = \left(\frac{3c}{2}, \frac{c}{2}\right)$. Since $\frac{c}{2} < 0$, $\left(\frac{3c}{2}, \frac{c}{2}\right) \cap \mathbb{N} = \phi$

If $c = 0$ let $\delta_0 = \frac{1}{2}$ and $(-\frac{1}{2}, \frac{1}{2}) \cap \mathbb{N} = \phi$

If $c > 0$ but $c \notin N$ there is a natural number K such that $K - c < K$. If $0 < \delta_0 <$ (c-K+1, K–c) then $K - 1 < c - \delta_0 < c < c + \delta_0 < K$. So that $(c - \delta_0, c + \delta_0)$ does not contain any element of $\mathbb{N}$

$\Rightarrow (c - \delta_0, c + \delta_0) \cap \mathbb{N} = \phi$.

If $c \in N$ and $\delta_0 = \frac{1}{2}$ then $(c - \delta_0, c + \delta_0)$ contains one and only one natural number namely c so that $(c - \delta_0, c + \delta_0) \cap N = \{c\}$. Since $(c - \delta_0, c + \delta_0)$ has no element of $\mathbb{N}$ other than c, c is not a cluster point of N. Thus N does not have a cluster point.

14.4.6 SAQ:

If $c \neq 0$, show that c is not a cluster point of the set A = {1, ½, 1/3, ….. 1/n ……..}

14.4.7 Theorem: If A is a set of real numbers and $c \in \mathbb{R}$, c is a cluster point of A if and only if there is a sequence (a_n) in A such that $a_n \neq c$ for every n and lim (a_n) = c

Proof: If c is a cluster point of A then for every natural number n, $V_{1/n}(c) \cap A - \{c\} \neq \phi$. Choose $a_n \in V_{1/n}(c) \cap A - \{c\}$. Clearly $0 < |a_n - c| < \frac{1}{n}$ so that $a_n \neq c$ for every n since $\lim (\frac{1}{n}) = 0$, by squeeze theorem lim (a_n) = 0

Conversely suppose that there is a sequence (a_n) in A such that $a_n \neq c$ for every n and

lim (a_n) = c. If $\delta > 0$ there is a $k \in N$ such that $0 < |a_n - c| < \delta$ for $n \geq k$.

$\Rightarrow a_n \in V_\delta(c) \cap A$ for $n \geq k$ clearly $a_n \neq c$ for each n.

since $\delta > 0$ is arbitary and $V_\delta(c) \cap (A - \{c\}) \neq \phi$; it follows that c is a cluster point of A.

14.4.8 SAQ:

(a) If c is a cluster point of $A \cup B$ there c is a cluster point of either A or B.

(b) c is a cluster point of A if and only if for every open interval (a, b) containing c, $(a, b) \cap A - \{c\} \neq \phi$

(c) A finite set has no cluster points

(d) Let $f: \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and $g(x) = f(x + c)$ for $x \in \mathbb{R}$. Show that $\lim_{x \to c} f(x) = l$ if and only if $\lim_{x \to 0} g(x) = l$.

(e) Suppose $f: \mathbb{R} \to \mathbb{R}$ has limit l at 0. let $a > 0$. If $g: \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(ax)$ show that $\lim_{x \to 0} g(x) = l$.

(f) If $f: \mathbb{R} \to \mathbb{R}^1$ is defined by f(x) = x if x is rational, 0 if x is irrational show that (a) $\lim_{x \to 0} f(x) = 0$ and (b) $\lim_{x \to c} f(x)$ does not exist if $c \neq 0$

(g) Let $f: \mathbb{R} \to \mathbb{R}$ I be any open interval in $\mathbb{R}$. If f_1 is the restriction of f to I and $c \in I$. Show that f_1 has a limit at c if and only if f has a limit at c. show also that if the limits exist, $\lim_{x \to c} f(x) = \lim_{x \to c} f_1(x)$

14.4.9 Theorem: C is a cluster point of A if and only if every neighbourhood of c contains infinitely many points of A.

Proof: Suppose c is a cluster points of A and $\delta > 0$. If V_δ (c) $\cap$ A is a finite set say F = $\{a_1, a_2 \ldots. A_n\}$ then there is a $\delta_1 > 0$ such that V_{δ_1} (c) $\cap$ F = ϕ. If $\delta_2 = \min\{\delta_1, \delta\}$,

V_{δ_2} (c) $\subseteq$ V_{δ_1} (c) as well as V_δ (c) so that V_{δ_2} (c) $\subseteq V_\delta$ (c)

$\Rightarrow V_{\delta_2}(c) \cap A \subseteq V_\delta(c) \cap A = F \Rightarrow V_{\delta_2}(c) \cap A \subseteq V_\delta(c) \cap F = \phi$

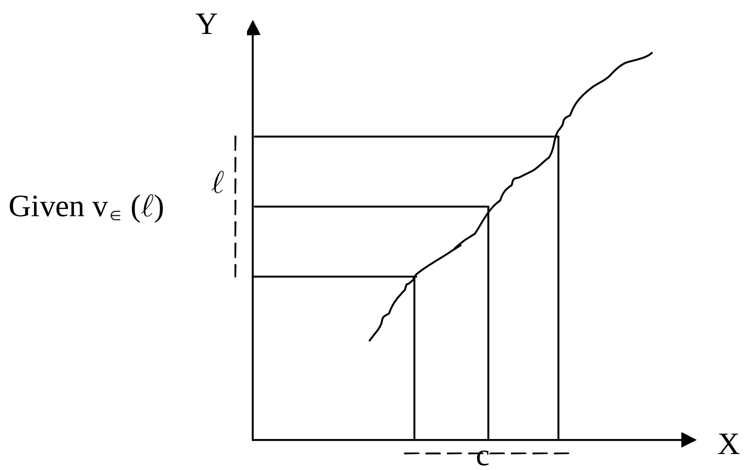
This would imply that the δ_2 – nbd of c does not contain points of A so that c cannot be a cluster point of A, which is a contradiction. Hence every nbd of c contains infinitely many points of A.

Conversely, suppose that every nbd of c contains infinitely many points of A. even if this infinite set in A includes c we can choose a point $c^1 \neq c$ in A in the nbd. So every nbd of c contains atleast one point of A other than c. so c is a cluster point of A.

Corallary: A finite set has no cluster points.

14.4.10 Definition of the limit:

Let $A \subseteq \mathbb{R}$ and let c be a cluster point of A For a function $f : A \to \mathbb{R}$, a real number l is said to be a limit, of f at c if given $\in > 0$, there is a positive real number $\delta(\in)$ such that if $x \in A$ and $0 < |x - c| < \delta(\in)$ then $|f(x) - l| < \in$.



These exists v_δ (c)

If l is a limit of f at c, then we also say that (1) f converges to l at c (ii) f (x) approaches l as x approaches c. We express this in symbols by (1) $\lim\limits_{x\to c} f(x) = l$ and (ii) $f(x)\to l$ as $x\to c$.

If the limit of f at c does not exist, we say that f diverges at c.

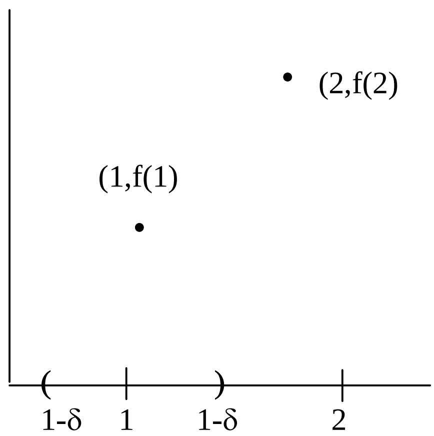
14.4.11 Why cluster points?

Why should c be a cluster point of A rather than a point of A.

Consider A = {1,2}, $1 \in A$ but 1 is not a cluster point of A. Let $f: A \to \mathbb{R}$.

As per the definition of limit (not necessarily at a cluster point), the limit l must satisfy $|f(x) - l| < \in$ if $0 < |x - 1| < \delta$ and $x \in A$ for given $\in > 0$ and suitable $\delta > 0$.

What ever δ be and what so ever l be the inequality $|f(x) - l| < \in$ holds vacuously for every $\in > 0$ and $\delta > 0$ such that there is no x in A satisfying $0 < |x - d| < \delta$. This leads to the unwanted conclusion that every $l, \in \mathbb{R}$ is limit of every function $f: A \to \mathbb{R}$.



No. other points of A

14.4.12 Theorem: If $f: A \to \mathbb{R}$ and c is a cluster point of A, then f can have only one limit at c.

Proof: Suppose that l and l^1 are limits of f at c. Corresponding to any $\in > 0$, there exists $\delta(\frac{\in}{2}) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta\,(\frac{\in}{2})$, then $|f(x) - l| < \frac{\in}{2}$ ------- (1)

Also there exists $\delta^1\left(\frac{\in}{2}\right) > 0$ such that if $x \in A$ and $0 < |x-c| < \delta^1\left(\frac{\in}{2}\right)$, then $|f(x)-l| < \frac{\in}{2}$ --- (2)

Now let $\delta = \min\{\delta, \delta^1\}$. If $x \in A$ and $0 < |x - c| < \delta$, then x satisfies (1) and (2)

So $\quad 0 \leq |l - l^1| = |l - f(x) + f(x) - l^1|$

$\leq |l - f(x)| + |f(x) - l^1| < \frac{\in}{2} + \frac{\in}{2} = \in$

Since $\in > 0$ is arbitary, we conclude that $l - l^1 = 0$ so that $l = l^1$.

Thus f can have only one limit at c.

14.4.13 Theorem: Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.

(1) $\lim\limits_{x \to c} f(x) = l$

(2) Given any $\in$ - nbd $V_\in(l)$ of l, there is a δ - nbd $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \wedge A$ then f(x) belongs $V_\in(l)$.

Proof: Assume (1). To prove (2) let $\in > 0$. By (1) there is $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in A \Rightarrow |f(x) - l| < \in$.

If $x \neq c$ and $x \in V_\delta(c) \cap A$ then $0 < |x - c| < \delta$ and $x \in A$ so $|f(x) - l| < \in$ i.e. $f(x) \in V_\in(l)$ This proves (2).

Assume (2) To prove (1) let $\in > 0$

By (2) there is $\delta > 0$ such that $x \neq c$, $x \in V_\delta(c) \wedge A \Rightarrow f(x) \in V_\in(c)$

LHS means that $0 < |x - c| < \delta$

RHS means that $|f(x) - l| < \in$

Since LHS $\Rightarrow$ RHS, we get that $0 < |x-c| < \delta$ and $x \in A \Rightarrow |f(x) - l| < \in$

Since this is true for every $\in > 0$, $\lim\limits_{x \to c} f(x) = l$.

14.4.14 Examples: (a) $\lim\limits_{x \to c} b = b$

Solution: Let f(x) = b for all $x \in \mathbb{R}$ (Here $A = \mathbb{R}$). Let $\in > 0$ and take $\delta = 1$

Then if $0 < |x-c| < 1$, we have $|f(x) - b| = |b-b| = 0 < \in$. Since $\in > 0$ is arbitary we conclude that $\lim\limits_{x \to c} f(x) = b$.

(b) $\lim_{x\to c} x = c$

Let $g(x) = x$ for all $x \in \mathbb{R}$. If $\in > 0$ we choose $\delta(\in) = \in$. Then if $0 < |x - c| < \delta(\in)$ we have $|g(x) - c| = |x - c| < \delta(\in) = \in$. Since $\in > 0$ is arbitary, we get $\lim_{x\to c} g(x) = c$.

(c) $\lim_{x\to c} x^2 = c^2$

Let $h(x) = x^2$ for all $x \in \mathbb{R}$

If $0 <|x - c| < 1$ we have $|x| = |x - c + c| \le |x - c| + |c| < 1 + |c|$

$\Rightarrow |x+c| \le |x| + |c| < 1 + |c| + |c| = 1 + 2|c|$

Now $|h(x) - c^2| = |x^2 - c^2| = |x-c|\,|x+c| \le (2|c| + 1)\,|x - c|$ ----------- (1)

If $\in > 0$ we choosse $\delta(\in) = \min\{1, \frac{\in}{2|c|+1}\}$ If $0 < |x - c| < \delta(\in)$ then from (1)

$|x^2 - c^2| < (2|c| + 1)\,|x - c| < \in$

since this holds for any arbitary choice of $\in > 0$, we infer that $\lim_{x\to c} h(x) = c^2$

(d) $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ if $c \neq 0$

Solution: The function $\phi(x) = \frac{1}{x}$ has domain $\mathbb{R} - \{0\}$. We consider the case $c > 0$. The proof, when $c < 0$ is similar. As we consider the limit as $x \to c$ we may restrict to x in a small neighborhood of c, say $V_{\frac{c}{2}}(c)$.

$x \in V_{\frac{c}{2}}(c) \Leftrightarrow |x - c| < \frac{c}{2} \Leftrightarrow \frac{c}{2} < x < \frac{3c}{2}$

For $x \in V_{\frac{c}{2}}(c)$, $|\phi(x) - \phi(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{x-c}{cx}\right| < \frac{|x-c|}{c}.\frac{2}{c}$ $\left(\because x > \frac{c}{2}\right)$

$|\phi(x) - \phi(c)| < \in$ if $|x - c| < \frac{\in c^2}{2}$.

As we are considering $x \in V_{\frac{c}{2}}(c)$ i.e. $\{x\}\ |x - c| < \frac{c}{2}\}$

We will have $|\phi(x) - \phi(c)| < \in$ if $0 < |x - c| < \delta$ where $\delta = \min\left\{\frac{\in c^2}{2}, \frac{c}{2}\right\}$

As this holds for all $\in > 0$, it follows that $\lim\limits_{x \to c} \frac{1}{x} = \frac{1}{c}$ when c > 0.

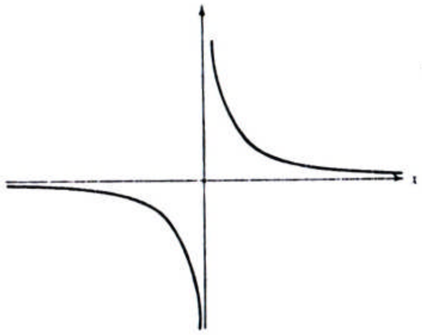


Fig. 14.4.14(d) Graph of g(x) = 1/x (x ≠ 0)

(e) $\underset{x \to 2}{Limit} \; \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$

Solution: Even though the function $\phi(x) = \frac{x^3 - 4}{x^2 + 1}$ is defined for all $x \in \mathbb{R}$ we may restrict x to a small neighborhood around 2 at which the limit is under consideration, say V_a (2) where we fix a soon.

$$|\phi(x) - \frac{4}{5}| = \frac{|5x^3 - 4x^2 - 24|}{5(x^2+1)} = \left|\frac{(x-2)\,(5x^2 + 6x + 12)}{5(x^2+1)}\right|$$

We have to ultimately find $\delta > 0$ and consider x in V_δ (2) only.

If 0 < a < 2 and |x – 2| < a then 0 < 2 – a < x < 2 + a)

For convenience we choose a = 1 so that 1 < x < 3 and hence $5x^2 + 6x+12 < 5(3)^2 + 6(3) + 12 = 75$.

So if 0 < |x – 2| < 1, $|\phi(x) - \frac{4}{5}| < \frac{|x-2|> 75}{5(x^2+1)}$ If $\in > 0$, 15 |x-2| < $\in$ if |x-2| < $\frac{\in}{15}$

So we choose $\delta(\in) = \min\left\{\frac{\in}{15}, 1\right\}$ If 0 < |x-2| < δ ($\in$), $|\phi(x) - \frac{4}{5}| < 75\,|x-2| < \in$

Hence $\underset{x \to 2}{Lim}\; \phi(x) = \frac{4}{5}$

(f) Let I be an interval in $\mathbb{R}$, let f: I → $\mathbb{R}$ and let c ∈ I. Suppose there exist constants K and L such that |f(x) – L| ≤ K |x-c| for x ∈ I. Show that $\lim\limits_{x \to c}$ f(x) = L.

Solution: For given $\in > 0$, choose $\delta(\in) = \dfrac{\in}{K}$.

If $0 < |x-c| < \delta(\in)$ then $|f(x) - L| \le K\,|x-c| < \in$. Since $\in > 0$ is arbitary the $\lim\limits_{x\to c} f(x) = L$

14.4.14 – A. SAQ's

(a) Let $f: \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and $g(x) = f(x+c)$ for $x \in \mathbb{R}$.
show that $\lim\limits_{x\to c} f(x) = l$ if and only if $\lim\limits_{x\to c} g(x) = l$.

(b) Suppose $f: \mathbb{R} \to \mathbb{R}$ has limit l at 0. If $g: \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = f(ax)$ show that $\lim\limits_{x\to 0} g(x) = l$.

(c) If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x$ if x is rational and 0 if x is irrational show that $\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to c} f(x)$ does not exist if $c \neq 0$.

(d) Let $f: \mathbb{R} \to \mathbb{R}$ and I be any open internal. If the restriction of f to I is f_1 and $c \in I$ show that f_1 has limit at c if and only if f has limit at c. show also tha if the limits exist then $\lim\limits_{x\to c} f_1(x) = \lim\limits_{x\to c} f(x)$.

14.4.15 Theorem: (Sequential criterion)

Let $f : A \to \mathbb{R}$ and let c be an cluster print of A. Then the following are equivalent.

(1) $\lim\limits_{x\to c} f(x) = l$

(2) For any sequence (x_n) in A that converges to c, such that $x_n \neq c$ for all $n \in N$, the sequence $(f(x_n))$ converges to l.

Proof: (1) $\Rightarrow$ (2) Assume that $\lim\limits_{x\to c} f(x) = l$ and suppose (x_n) is a sequence in A with $\lim\limits_{n\to\infty}(x_n) = c$ and $x_n \neq c$ for all n. Let $\in > 0$ be given. Then there exists $\delta > 0$ such that if $x \in A$, and $0 < |x - c| < \delta$, then $f(x)$ satisfies $|f(x) - l| < \in$. Since (x_n) is a convergent sequence and $\delta > 0$ there is a natural number $K(\delta)$ such that if and $n \ge K(\delta)$ then $|x_n - c| < \delta$. Since each such $x_n \in A$, we have $|f(x_n) - l| < \in$ Thus if $n \ge K(\delta)$ then $|f(x_n) - l| < \in$. Hence the sequence $(f(x_n))$ converges to l.

(2) $\Rightarrow$ (1) The proof is a contrapositive argument. Suppose (I) is not true. Then there is $\in_0 > 0$ such that for every $\delta > 0$ the δ - nbd of c contains atleast one number x_δ in $A \cap V_\delta(c)$ with $x_\delta \neq c$ and $f(x_\delta) \notin V_{\delta_0}(l)$.

For every $n \in N$, we take $\delta = \frac{1}{n}$ so that $\frac{1}{n}$ nbd of c contains atleast one x_n in A such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - l| \geq \in_0$. As this is true for every $n \in N$, we conclude that the sequence $(x_n) \subseteq A - \{c\}$ and the sequence $(f(x_n))$ does not converge to l.

Therefore we have shown that if (1) is not true then (ii) is not true. Hence (2) $\Rightarrow$ (1).

14.4.16 Examples: Use sequence criterion to establish that

(i) $\lim\limits_{x\to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$ (ii) $\lim\limits_{x\to 2} \frac{1}{1-x} = -1$ (iii) $\lim\limits_{x\to 0} \frac{x^2}{|x|} = 0$

(i) Solution:(1) Let (x_n) be any sequence in $\mathbb{R} - \{2\}$ such that $\lim (x_n) = 2 \Rightarrow \lim(x_n^2) = 4$ and $\lim(x_n^3) = 8 \Rightarrow \lim(x_n^2+1) = \lim(x_n^2) + 1 = 4 + 1 = 5$ and $\lim(x_n^3-4) = \lim(x_n^3) - 4 = 8 - 4 = 4$.

Hence, $\lim \frac{x_n^3-4}{x_n^2+1} = \frac{5}{4}$.

(ii) $\lim\limits_{x\to 2} \frac{1}{1-x} = -1$

Solution: Domain of $\frac{1}{1-x}$ is $R - \{1\}$

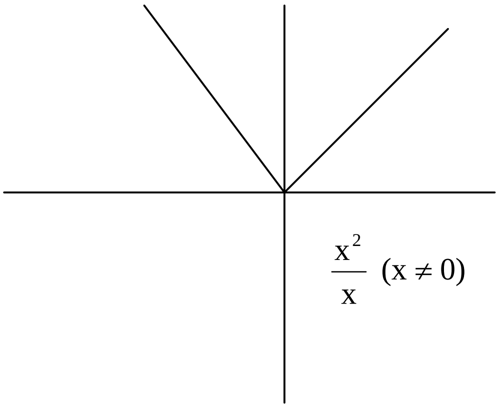
Let (x_n) be a sequence in $R - \{1\}$ such that $\lim x_n = 2$. Then $\lim (1 - x_n) = 1 - \lim (x_n) = 1 - 2 = -1$

$\lim\left(\frac{1}{1-x_n}\right) = \frac{1}{-1} = -1$. Hence $\lim\limits_{x\to 2} \frac{1}{1-x} = -1$.

(iii) Solution: Domain of $\frac{x^2}{|x|}$ is $R - \{0\}$

If (x_n) is any sequence in $R - \{0\}$ such that $\lim (x_n) = 0$ then $\lim |x_n| = 0$

$\lim\limits_{n}$ Since $\dfrac{x_n^2}{|x_n|} = \dfrac{|x_n|^2}{|x_n|} = |x_n|$, it follows that $\lim \dfrac{|x_n|^2}{|x_n|} = 0$. Hence $\lim\limits_{x\to 0} \dfrac{x^2}{|x|} = 0$



14.4.17 Divergence Criterion: Let $A \subseteq \mathbb{R}$, c cluster point of A, $f : A \to \mathbb{R}$, $l \in R$. We derive criterion for (1) $\lim\limits_{x\to c} f(x) \neq l$ and (2) $\lim\limits_{x\to c} f(x)$ does not exist.

Theorem: If $A \subseteq \mathbb{R}$, c is a cluster point of A, f: $A \to R$ and $l \in R$ the following are equivalent.

A – 1 : $\lim\limits_{x\to c} f(x) \neq l$

A – 2 : There is a sequence (x_n) in A – {c} such that $\lim (x_n) = c$ and $\lim f(x_n)) \neq l$.

Proof: Suppose A – 1 does not hold. Then $\lim\limits_{x\to c} f(x) = l$. For every (x_n) in A – {c} such that $\lim (x_n) = c$, $\lim (f(x_n)) = l$. So A – 2 does not hold.

Hence A – 2 $\Rightarrow$ A – 1.

Conversely suppose that A-2 does not hold. Then for every (x_n) in A-{c} with $\lim(x_n)= c$, $\lim f(x_n) = l$. $\Rightarrow \lim\limits_{x\to c} f(x) = l$. So A – 1 does not hold. Hence A – 1 $\Rightarrow$ A – 2.

14.4.18 Theorem: If $A \subseteq R$, c is a cluster point of A and $f : A \to$!R, the following are equivalent.

B – 1 : f(x) does not converge at c.

B – 2: There is a sequence (x_n) in A – {c} such that $\lim (x_n) = c$ and $\{f(x_n)\}$ does not converge.

Proof: Suppose B – 1 does not hold. Then f(x) converges at c. Let $\lim\limits_{x \to c} f(x) = l$

Then for every sequence (x_n) in A – {c} such that lim (x_n) = c and lim $f(x_n)$ = l. So B – 2 does not hold.

Thus B – 2 $\Rightarrow$ B – 1.

Conversely suppose that B – 2 does not hold, then for every (x_n) in A – {c} such that lim (x_n) = c; $(f(x_n))$ converges.

We show that B – 1 does not hold. For this it is enough to show that there is $l \in \mathbb{R}$ such that $\lim\limits_{x \to c} f(x) = l$. Equivalently it is enough to show that there is l in $\mathbb{R}$ such that whenever (x_n) in A – {c} converges to c, $(f(x_n))$ converges to l.

By assumption for any such (x_n), lim $f(x_n)$ exists any way. All that we have to show is that lim $f(x_n)$ is the same for all such (x_n). On the contrary, if lim (x_n) = lim (y_n) = c, where $(x_n \in A - \{c\}$ and $(y_n) \in A - \{c\}$ and lim $f(x_n) \neq$ lim $f(y_n)$. We form a new sequence z = (z_n) in A – {c} by defining $Z_{2n-1} = x_n$ and $Z_{2n} = y_n$ for all $n \in N$.

Since lim x_n = c. Since lim x_n = c = lim y_n then lim Z_{2n-1} = lim Z_{2n} = c hence Z_n = c. Also lim $f(Z_{2n-1})$ = lim $f(x_n)$ and lim $f(Z_{2n})$ = lim $f(y_n)$.

$\Rightarrow$ lim $f(Z_{2n-1}) \neq$ lim $f(Z_{2n}) \Rightarrow$ lim $f(Z_n)$ does not exist.

This contradicts our assumption that B – 2 does not hold. Thus when B – 2 does not hold, $\lim\limits_{x \to c} f(x)$ exists; i.e. B – 1 does not hold. Hence B – 1 $\Rightarrow$ B – 2.

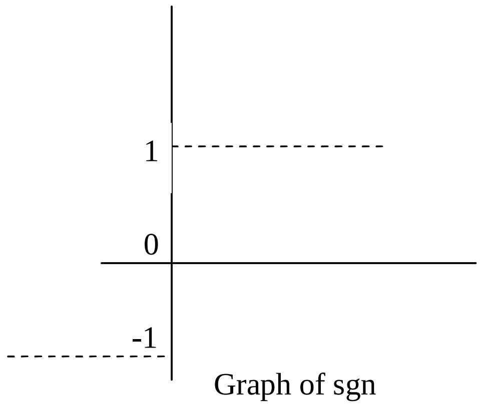
14.4.19 Examples:

a) $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist in R.

Solution: Let f(x) = $\frac{1}{x}$ for x > 0. 0 is cluster point of A = {x / x $\in$ $\mathbb{R}$, x > 0}. If we take the sequence (x_n) with $x_n = \frac{1}{n}$ for n $\in$ N, then lim (x_n) = 0, But $f(x_n) = \mathbb{N}$. We know that the sequence $(f(x_n))$ = (n) is not convergent in R. Hence $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist.

(b) The sigmum function (sgn) is defined by

$$\text{sgn}(n) = \begin{cases} 1 \ \ for\ \ x > 0 \\ 0 \ \ for\ \ x = 0 \\ -1 \ \ for\ \ x < 0 \end{cases}$$



Graph of sgn

Note that sgn (x) = $\frac{x}{|x|}$ for x $\neq$ 0.

We show that $\lim\limits_{x \to 0}$ sign (x) does not exist.

Let $x_n = \frac{(-1)^n}{n}$ for n $\in$ N so that lim (x_n) = 0.

But sgn $(x_n) = (-1)^n$ for n $\in$ N.

$\Rightarrow$ The sequence (sgn (x_n)) = $((-1)^n)$ is not convergent. Hence $\lim\limits_{x \to 0}$ sgn (x) does not exist.

(c) $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist in $\mathbb{R}$. (see ex. 14.4.14 (b))

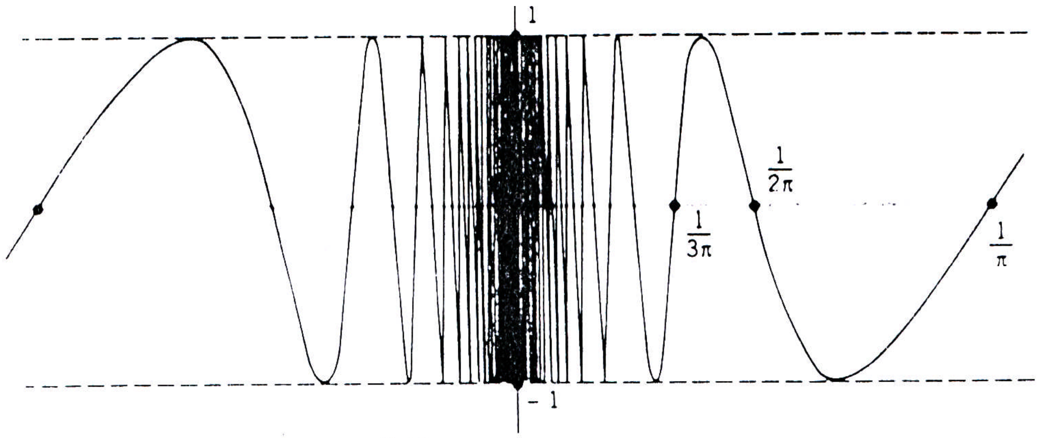
Solution: Let g(x) = $\sin\left(\frac{1}{x}\right)$ for x $\neq$ 0. (0 is a cluster point of $\mathbb{R}$ – {0}). We exhibit sequences (x_n) and (y_n) in R – {0} such that lim (x_n) = lim (y_n) = 0, but lim $g(x_n) \neq$ lim $g(y_n)$.

Let $x_n = \dfrac{1}{n\pi}$ for $n \in \mathbb{N}$. Then $\lim (x_n) = 0$ and $g(x_n) = \sin(n\Pi) = 0$ for all $n \in \mathbb{N}$; so that

$\lim g(x_n) = 0$ can the other hand, let $y_n = \left(\dfrac{\pi}{2} + 2n\pi\right)^{-1}$ for $n \in N$ then $\lim (y_n) = 0$ and

$g(y_n) = \sin\left(\left(\dfrac{\pi}{2} + 2n\pi\right)\right) = 1$ for all $n \in N$ so that $\lim g(y_n) = 1. \Rightarrow \lim g(x_n) \neq \lim g(y_n)$.

Hence $\lim\limits_{x \to 0} g(x)$ does not exist.



14.4.9 (c)

Figure The function $g(x) = \sin(1/x)$ $(x \neq 0)$.

14.5 Limit Theorems:

14.5.1 Theorem: Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, c a cluster point of A

$\lim\limits_{x \to c} f(x) = l$ and $\lim\limits_{x \to c} g(x) = m$. Then

(i) $\lim\limits_{x \to c} (f + g)(x) = l + m = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$ and

(ii) $\lim\limits_{x \to c} (f - g)(x) = l - m = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$

Proof: Let (x_n) be a sequence in $A - \{c\}$ such that $\lim (x_n) = c$.

$\Rightarrow \lim f(x_n) = l$ and $\lim g(x_n) = m$

$\Rightarrow \lim (f+g)(x_n) = \lim (f(x_n) + g(x_n)) = l + m = \lim f(x_n) + \lim g(x_n)$

This is true for every sequence (x_n) in A – {c} with lim x_n = c; hence $\lim\limits_{x\to c}$ (f + g) (x) =

$\lim\limits_{x\to c}$ f(x) + $\lim\limits_{x\to c}$ g(x) = l + m.

Also lim (f – g) (x_n) = lim $(f(x_n) - g(x_n))$ = l – m

Hence $\lim\limits_{x\to c}$ (f – g) (x) = l – m = $\lim\limits_{x\to c}$ f(x) - $\lim\limits_{x\to c}$ g(x).

14.5.2 Theorem: Let $A \subseteq \mathbb{R}$, c cluster point of A, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be such that $\lim\limits_{x\to c}$ f(x) = l and $\lim\limits_{x\to c}$ g(x) = m. Then

(i) $\lim\limits_{x\to c}$ (af) (x) = a l, for $a \in \mathbb{R}$ and $\lim\limits_{x\to c}$ (fg) (x) = lm

Proof: Let (x_n) be any sequence in A – {c} such that lim (x_n) = c. Then lim $f(x_n)$ = l and lim $g(x_n)$ = m.

Hence lim $f(x_n)\, g(x_n)$ = l m and for every $a \in \mathbb{R}$ lim a $f(x_n)$ = a l. Since this holds for every (x_n) in A-{c} with lim(x_n) = c, it follows that lim (fg) (x) = $\lim\limits_{x\to c}$ f(x) $\lim\limits_{x\to c}$ g(x) and a f(x) = a $\lim\limits_{x\to c}$ f(x).

14.5.3 Corollary : Under the above hypothesis on A and c if $1 \le i \le n$, f_c; : $A \to \mathbb{R}$ is such f_i that lim f_i (x) = l_i and b_1, b_{r1} - - - - b_n all real members then.

(i) $\lim\limits_{x\to c}$ $(b_1 f_1 + b_2 f_2 - - - + b_n f_n)$ (x) = $b_1\ l_1 + b_2\ l_2 + - - - + b_n\ l_n$

(ii) $\lim\limits_{x\to c}$ $(f_1 + f_2\ - - - f_n)$ (x) = $l_1 + l_2\ - - - l_n$

(iii) $\lim\limits_{x\to c}$ $(f_1 + f_2 + \ldots\ldots + f_n)$ (x) = $l_1 + l + \ldots\ldots + l_n$

Proof: by induction

14.5.4 SAQ: Prove that $\lim\limits_{x\to c}$ $(f_1 + f_2)$ (x) = $\lim\limits_{x\to c}$ f(x) + $\lim\limits_{x\to c}$ f_2(x) by using $\in$, δ definition.

14.5.5 Theorem : Let A ⊆ IR, it f and g functions on A to IR and let c ∈ IR be a cluster point of A. If g (x) ≠ 0 for all x ∈ A and lim g(x) = m ≠ 0, the lim $\left(\frac{f}{g}\right)$ (x) = $\left(\frac{1}{m}\right)$.

Proof :- Let (x_n) be a sequence in A such that $x_n \neq c$ for n ∈ N and lim (x_n) = c.

$\Rightarrow \lim f(x_n) = l$ and $\lim g(x_n) = m$

$$\Rightarrow \lim \left(\frac{f}{g}\right)(x_n) = \lim \frac{f(x_n)}{g(x_n)} = \frac{l}{m}$$

Hence $\lim\limits_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$.

14.5.6 Examples :

(a) $\lim\limits_{x \to c} x^k = c^k$ for $K \in N$ and if $c > 0$

then $\lim\limits_{x \to c} \frac{1}{x} = \frac{1}{c}$

(b) $\lim\limits_{x \to c} (x^2+1)(x^3-4) = 20$

solution : $\lim\limits_{x \to 2} (x^2+1)(x^3-4) = \lim\limits_{x \to 2} ((x^2+1) \lim\limits_{x \to 2} (x^3-4)$

$= (2^2+1)(2^2-4) = 5 . 4 = 20$

(c) $\lim\limits_{x \to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$

By theorem 14.5.5 $\lim\limits_{x \to 2} \frac{x^3-4}{x^2+1} = \frac{\lim\limits_{x \to 2}(x^3-4)}{\lim\limits_{x \to 2}(x^2+1)} = \frac{4}{5}$

(d) $\lim\limits_{x \to 2} \frac{x^2-4}{3x-6} = \frac{4}{3}$.

Let $f(x) = x^2 - 4$, $g(x) = 3x-6$ for $x \in \mathbb{R} - \{2\}$.

We cannot use 14.5.5 because $\lim g(x) = 0$.

However if $x \neq 2$, then $\frac{x^2-4}{3x-6} = \frac{(x-2)(x+2)}{3(x-2)} \ \frac{x+2}{3}$. Hence $\lim\limits_{x \to 2} \frac{x^2-4}{3x-6} = \frac{4}{3}$

14.5.7 Example: Let $A \subseteq IR$, $f : A \to IR$ and let $c \in R$ be a cluster point of A. suppose $f(x) \geq 0$ for $x \in A$. defines $\sqrt{f} : A \to IR$ by $(\sqrt{f})(x) = \sqrt{f(x)}$ for $x \in \mathbb{R}$. if $\lim f(x) = \gamma$, show that $\lim (\sqrt{f})(x) = \sqrt{\gamma}$.

Solution : Case (1) : $\gamma = 0$ If $\in > 0$, there is $\delta > 0$ such that $f(x) = |f(x)| < \in^2$

If $0 < |x - c| < \delta$ and $x \in A$.

$\Rightarrow \sqrt{f(x)} = = |\sqrt{f(x)}| < \in$ if $0 < |x - c| < \delta$ and $x \in A$. Since this holds for all $\in > 0$,

$\lim \sqrt{f(x)} = \sqrt{l}$.

Case (ii): $l > 0$ There is $\delta_1 > 0$ such that $|f(x) - l| < l/2$ if $x \in A$ and $0 < |x - c| < \delta_1$.

$\Rightarrow |\ l\ | - |f(x)| < l/2$ if $x \in A$ and $0 < |x - c| < \delta_1$.

$\Rightarrow l/2 < f(x)$ if $x \in A$ and $0 < |x - c| < \delta_1$.

$\Rightarrow \sqrt{f(x)} > \sqrt{l/2}$ if $x \in A$ and $0 < |x - c| < \delta_1$.

Also if $\in > 0$ there is $\delta_2 > 0$ such that if $0 \in A$ and $0 < |x - c| < \delta_2$, $|f(x) - l| <$ $\left(\sqrt{l/2} + \sqrt{l}\right) \in$

Let $\delta(\in) = \min\{\delta_1, \delta_2\}$. Now if $0 < |x - c| < \delta(\in)$ and $x \in A$, Then

$$\left|\sqrt{f(x)} - \sqrt{l}\right| = \frac{|f(x) - l|}{\sqrt{f(x)} + \sqrt{l}} < \frac{|f(x) - l|}{\sqrt{l/2} + \sqrt{l}} < \frac{\sqrt{l} + \sqrt{l/2}}{\sqrt{l} + \sqrt{l/2}} \in = \in$$

Hence $\lim\limits_{x \to c} \sqrt{f(x)} = \sqrt{l}$

14.5.8 SAQ: Find $\lim\limits_{x \to 0} \dfrac{\sqrt{1+2x} - \sqrt{1+3x}}{x + x^2}$

14.5.9 SAQ: (a) If P is a polynomial function then show that $\lim\limits_{x \to c} p(x) = p(c)$.

(b) If p and q are polynomial functions on $\mathbb{R}$ and if $q(c) \neq 0$ then show that

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

14.5.10 Theorem: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If $a \leq f(x) \leq b$ for all $x \in A$, $x \neq c$ and if $\lim\limits_{x \to c} f(x)$ exists, then $a \leq \lim\limits_{x \to c} f(x) \leq b$.

Proof: If $\lim\limits_{x \to c} f(x) = l$, then if (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in N$ and if (x_n) converges to c, then the sequence $(f(x_n))$ converges to l.

Since $a \leq f(x_n) \leq b$ for all $x \in b$, then $a \leq \lim f(x_n) \leq b \Rightarrow a \leq l \leq b$.

14.5.11 Squeeze theorem for functions: Let $A \subseteq \mathbb{R}$, let f, g, h : $A \to \mathbb{R}$ and let c be a cluster point of A. If $f(x) \le g(x) \le h(x)$ for all $x \in A$, $x \ne c$ and if $\lim\limits_{x\to c} f(x) = l = \lim\limits_{x\to c} h(x)$ then $\lim\limits_{x\to c} g(x) = l$.

Proof: First proof: To show that $\lim\limits_{x\to c} g(x) = l$, let (x_n) be a sequence in $A - \{c\}$ such that $\lim x_n = 0$ since $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} h(x) = l$, then $\lim\limits_{x\to c} f(x_n) = l = \lim\limits_{x\to c} h(x_n)$.

Since $f(x_n) \le g(x_n) \le h(x_n)$ for every n, it follows by the squeeze theorem for sequences that $\lim g(x_n) = l$. Since this holds for every seqeunce (x_n) in $A - \{c\}$ such that $\lim x_n = c$, it follows that $\lim\limits_{x\to c} g(x) = l$.

Second proof: Given $\in > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f(x) - l| < \in$ for $(0<|x - c| < \delta_1$ and $|h(x) - l| < \in$ for $0 < |x - c| < \delta_2$ and $x \in A$

$\Rightarrow l - \in < f(x) < l + \in$ for $0 < |x - c| < \delta_2$ and $x \in A$ and

$l - \in < h(x) < l + \in$ for $0 < |x - c| < \delta_2$ and $x \in A$

Let $\delta = \min\{\delta_1, \delta_2\}$, then $|x - c| < \delta$. $x \in A$, $x \ne c$

$\Rightarrow l - \in < f(x) < l + \in$ and $l - \in < h(x) < l + \in$

For $0 < |x - c| < \delta$

Thus $l - \in < f(x) \le g(x) \le h(x) < l + \in$

$\Rightarrow l - \in < g(x) < l + \in$

$\Rightarrow |g(x) - l| < \in$ for $0 < |x - c| < \delta$ and $x \in A$. Hence $\lim\limits_{x\to c} g(x) = l$.

14.5.12 Examples:

a) $\lim\limits_{x\to 0} x^{3/2} = 0$

Domain of the function $x^{3/2}$ is $(0, \infty)$

If $0 < x < 1$ then $x < x^{1/2} < 1 \Rightarrow x^2 < x^{3/2} < x$ for $0 < x < 1$

But $\lim\limits_{x\to 0} x^2 = 0 = \lim\limits_{x\to 0} x$

Hence by squeeze theorem $\lim\limits_{x\to 0} x^{3/2} = 0$

(b) $\lim\limits_{x\to 0} \sin x = 0$

We use the fact, - x ≤ sin x ≤ x for all x > 0 since $\lim_{x\to 0}$ x = 0 = $\lim_{x\to 0}$ (- x), by squeeze theorem $\lim_{x\to 0}$ sin x = 0.

(c) $\lim_{x\to 0} \cos x = 1$

We use the fact that $1 - \frac{1}{2}x^2 \le \cos x \le 1$ for all $x \in \mathbb{R}$. Since $\lim_{x\to 0} (1-\frac{1}{2}x^2) = 1 = \lim_{x\to 0} 1$, by squeeze theorem $\lim_{x\to 0} \cos x = 1$.

(d) $\lim_{x\to 0} \left(\frac{\cos x - 1}{x}\right) = 0$

We use the fact that $1 - \frac{1}{2}x^2 \le \cos x \le 1$ for all $x \in R$= $-\frac{1}{2}x^2 < \cos x - 1 < 0$ for $x \in \mathbb{R}$

$\Rightarrow -\frac{1}{2}x \le \frac{\cos x - 1}{x} \le 0$ for x > 0 and $0 \le \frac{\cos x - 1}{x} \le -\frac{1}{2}x$ for x < 0.

$0 \le \frac{\cos x - 1}{x} \le -\frac{1}{2}|x|$ for $x \in \mathbb{R}$.

Now let f(x) = -x/2 for x ≥ 0

= 0 for x < 0

let h(x) = 0 for x ≥ 0

= x/2 for x < 0

Then $f(x) \le \left|\frac{\cos x - 1}{x}\right| \le h(x)$ for $x \ne 0$ and $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x)$

Hence by squeeze theorem $\lim_{x\to 0} \left|\frac{\cos x - 1}{x}\right| = 0$, hence $\lim_{x\to 0} \left|\frac{\cos x - 1}{x}\right| = 0$.

(e) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

We use the fact: $x - \frac{1}{6}x^3 \le \sin x \le x$ for $x \ge 0$ and $x \le \sin x \le x - \frac{1}{6}x^3$ for x < 0.

$\Rightarrow 1 - \frac{1}{6}x^2 \le \frac{\sin x}{x} \le 1$ for all $x \ne 0$. But $\lim_{x\to 0} (1 - \frac{1}{6}x^2) = 1 = \lim_{x\to 0} 1$

Hence $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(f) $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

Let f(x) = x sin (1/x) for x ≠ 0

Since –1 ≤ sin $\mathbb{R}$ ≤ 1 for all $\mathbb{R} \in \mathbb{R}$, we have - |x| ≤ x. sin (1/x) ≤ |x| for all x ≠ 0. Since $\lim\limits_{x\to 0}$ |x| = 0, squeeze theorem $\lim\limits_{x\to 0}$ x. sin (1/x) = 0.

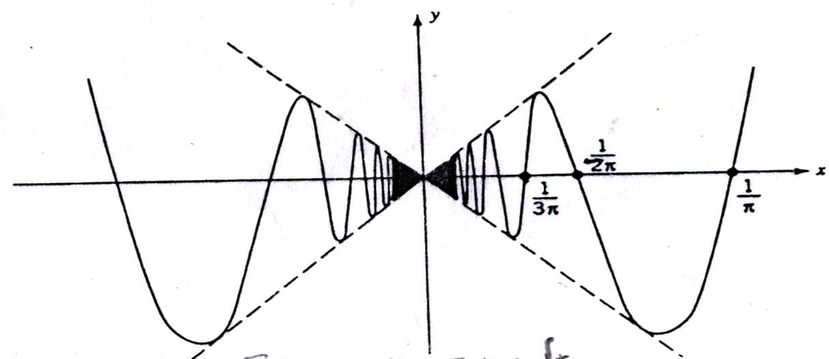


Figure 14.5.12(b)
Graph of $f(x) = x\sin(1/x)$ $(x \neq 0)$.

14.5.13 Theorem: Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim\limits_{x\to c}$ f(x) > 0 ($\lim\limits_{x\to c}$ f(x) < 0) then there exists a nbd V_δ (c) of c such that f(x) > 0 (f(x) < 0) for all x ∈ V_δ (c) ∩ A – {c}.

Proof: Let $\lim\limits_{x\to c}$ f(x) = l and suppose that l > 0. Corresponding to $\in = \frac{l}{2}$ there is δ > 0 such that if 0 < |x – c| < δ and x ∈ A then |f(x) – l | < $\frac{l}{2}$ so x ∈ V_δ (c) and x ≠ c ⇒ $\frac{l}{2}$ = $l - \frac{l}{2} < f(x) < l + \frac{l}{2}$. Hence if x ∈ V_δ (c) and x ≠ C, f(x) > $\frac{l}{2}$ > 0.

14.5.14 SAQ: If f : $\mathbb{R} \to \mathbb{R}$ satisfies f(x+y) = f(x) + f(y) for all x, y in $\mathbb{R}$, show that f(x) = x.f(1) for all rational numbers x.

14.5.15 SAQ: If f : $\mathbb{R} \to \mathbb{R}$ satisfies f(x+y) = f(x) + f(y) for all x,y in $\mathbb{R}$ and $\lim\limits_{x\to 0}$ f(x) = L then

(i) L = 0 (ii) $\lim\limits_{x\to c}$ f(x) = f(c) for all c and (iii) f(x) = x.f (1) for all x

14.6 Extensions of limit concept

We begin with an example. Consider the signnun function defined on $\mathbb{R}$ by

$g(x) = \frac{x}{|x|}$ if $x \neq 0$ and of $g(0) = 1$ equivalently $g(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$

Let $A_1 = \{x/x>0\} = (0, \infty)$ and $A_2 = \{x/x < 0\} = (-\infty, 0)$ clearly $A_1 \cup A_2 = \mathbb{R} - \{0\}$ and 0 is a cluster point of A_1 as well as A_2.

We consider the restrictions g_1, g_2 of g on each of A_1 and A_2. $g_1 = A_1 \to \mathbb{R}$ is defined by $g_1(x) = -1$ for $x \in A_1$ and $g_2 : A_2 \to \mathbb{R}$ is defined by $g_2(x) = 1$ for $x \in A_2$

If $\delta > 0$, $A_1 \cap V_\delta(0) = (0, \delta)$ and $A_2 \cap V_\delta(0) = (-\delta, 0)$ So if $x \in A_1 \cap V_\delta(0)$, $g_1(x) = 1$ so that $|g_1(x) - 1| = 0 < \in$ for every $\in >0$. Thus $\lim\limits_{x\to 0} g(x) = 1$. Similarly $\lim\limits_{x\to 0} g_2(x) = -1$

However $\lim\limits_{x\to 0} g(x)$ does not exist.

This example places before us a situation where a function defined on a set A has no limit at a cluster point c while its restriction to $A \cap (c, \infty)$ and $A \cap (-\infty, c)$ do have limits.

We are thus head to define the concept of one sided limits at a cluster point.

14.6 One sided limits

14.6.1 Definitions:

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A/a > c\}$ then we say that $l \in \mathbb{R}$ is a right hand limit of f at c and we write $\lim\limits_{x\to c+} f(x) = l$.

i e, if given $\in > 0$ then exists $\delta(\in) > 0$ such that for all $x \in A$ with $0 < |x-c| < \delta(\in)$, $|f(x) - l| < \in$

(ii) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A / x < c\}$ then we say that $l \in \mathbb{R}$ is a left hand limit of f at c and we write $\lim\limits_{x\to c-} f(x) = l$.

if given $\in > 0$ then exists $\delta(\in) > 0$ such that for all $x \in A$ with $0 < c - x < \delta(\in)$, $|f(x) - l| < \in$.

14.6.2 Note: 1. The limits $\lim\limits_{x\to c+} f(x)$ and $\lim\limits_{x\to c-} f(x)$ are called one sided limits of f at c.

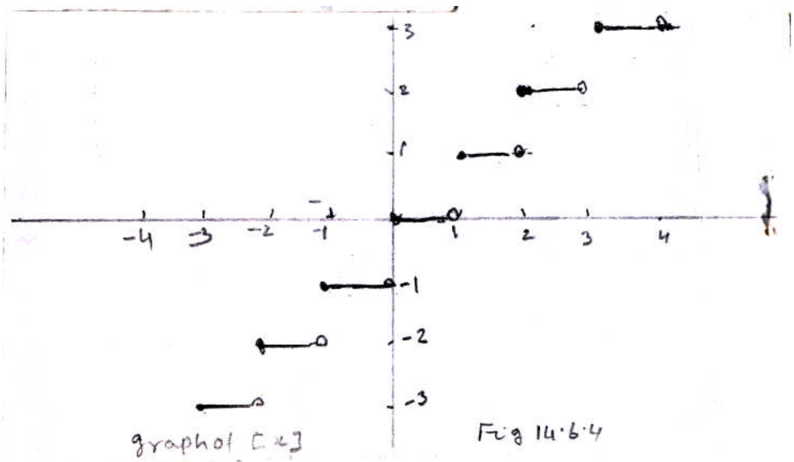
2. It is possible that neither one sided limit may exist Also one of them may exist without the other existing. Similarly as in the case g(x) = sgn (x) at c = 0, they may both exist and be different.

3. If A is an interval with left end point c, then it is seen that f: A $\to \mathbb{R}$ has a limit at c if and only if it has a right hand limit at c. Moreover, in this case $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c+} f(x)$ are equal. (A similar situation occurs for the left hand limit when A is an interval with right end point c.)

14.6.3 SAQ: For f; (a, b) $\to \mathbb{R}$, $\lim\limits_{x\to a} f(x)$ exists if and only if $\lim\limits_{x\to a+} f(x)$ exists. In this case the two limits are equal.

14.6.4 Example: (a) The greatest integer function [x] : Let ϕ (x) = [x] = n where n is an integer and $n \le x < n + 1$ We show that

(i) if c is not an integer $\lim\limits_{x\to c} \phi(x) = \phi(c)$ (ii) if c is an integer $\lim\limits_{x\to c+} \phi(x) = c$ and $\lim\limits_{x\to c-} \phi(x) = c-1$



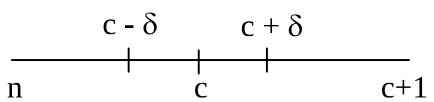
Case (i): When c is not an integer, there is a integer n such that $n < c < n + 1$

$\phi(x) = n$ in (n, n+1) so if $0 < \delta < \min\{c-n, n+1-c\}$

and $0 < |x-c| < \delta$, $n < c - \delta < x < c + \delta < n + 1$ so that

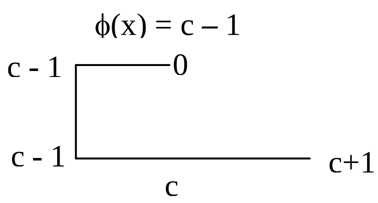
$\phi(x) = n$ and hence if $\in > 0$, $|\phi(x) - n| = 0 < \in$.

This shows that $\lim\limits_{x\to c+} \phi(x) = n = \phi(c)$



$c < \delta < c - n$, $0 < \delta < x + 1 - c$

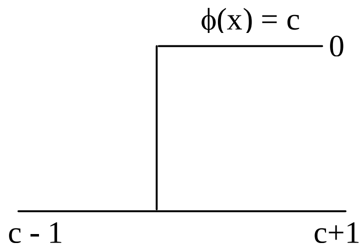
Case (ii): If c is an integer, $\phi(x) = c-1$ if $c-1 < x < c$.



If $0 < \delta < 1$, $c - 1 < c - \delta < c$ so that if $c - \delta < x < c$

$\phi(x) = c - 1$ if $\in > 0$, $|\phi(x) - \phi(c)| = 0 < \in$

This shows that $\lim\limits_{x \to c-} \phi(x) = c - 1$



Case (iii): If c is an integer and $c < x < c + 1$, $\phi(x) = c$

As above we se that if $0 < \delta < 1$ and $c < x < c + \delta < c + 1$

$\phi(x) = c$ so that for any $\in > 0$, $|\phi(x) - c| = 0 < \in$

This shows that $\lim\limits_{x \to c+} \phi(x) = c$

However $\lim\limits_{x \to c} \phi(x)$ does not exist when c is an integer as $\phi(c + \frac{1}{n}) = c$ for $n \in N$ and

$\phi\left(c - \frac{1}{n}\right) = c - 1$ and hence

$\lim\limits_{n} \left(\phi\left(c + \frac{1}{n}\right) \neq \lim\limits_{n} \left(\phi\left(c - \frac{1}{n}\right)\right)\right.$.

(b) Let f (x) = sgn (x) we know that sgn does not have a limit at 0. It is clear that

$\lim\limits_{x \to 0+} \operatorname{sgn}(x) = 1$ and $\lim\limits_{x \to 0-} \operatorname{sgn}(x) = -1$

Since these one sided limits are different, it follows that sgn (x) does not have a limit at 0.

(c) Let $g(x) = e^{1/x}$ for $x \neq 0$.

We first show that does not have a finite right hand limit at c = 0, since it is not bounded on any right nbd $(0, \delta)$ of 0. We make use of the inequality $0 < \frac{1}{x} < e^{1/x}$ for $x > 0$.

If we take $x_n = \frac{1}{n}$ then $g(x_n) > n$ for all $n \in \mathbb{N}$. Therefore $\lim\limits_{x \to 0} e^{1/x}$ does not exist in $\mathbb{R}$.

However $\lim\limits_{x \to 0} e^{1/x} = 0$. Indeed if $x < 0$ then $0 < -\frac{1}{x} < e^{-1/x}$ so $0 < e^{1/x} < -x$ for all $x < 0$.

$\lim\limits_{x \to 0-} e^{1/x} = 0 \neq \lim\limits_{x \to 0+} e^{-1/x}$ Hence $\lim e^{1/x}$ does not exist.

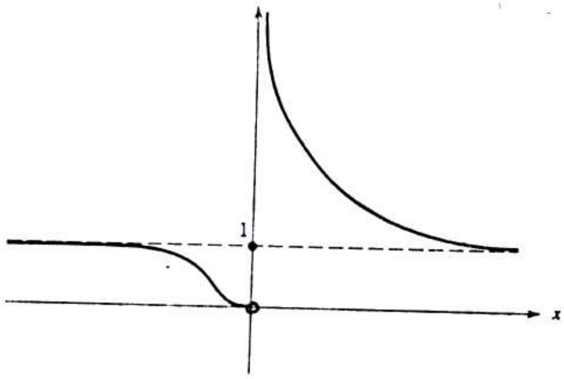


Figure 14.6.4(c) Graph of $g(x) = e^{1/x}$ $(x \neq 0)$.

(d) Let $h(x) = \dfrac{1}{e^{1/x}+1}$ for $x \neq 0$.

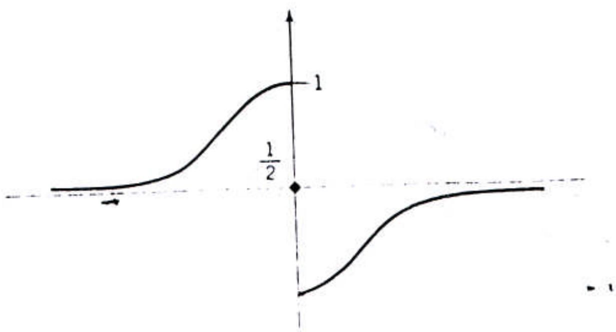
If $0 < x \;\; 0 < \dfrac{1}{x} < c^{1/x}$

then $0 < \dfrac{1}{e^{1/x}+1} < \dfrac{1}{e^{x}} < x$ which implies that

$\lim\limits_{x\to 0+} h(x) = 0$. From (c) above $\lim\limits_{x\to 0-} e^{1/x} = 0$. So

$\lim\limits_{x\to 0-} h(x) = \lim\limits_{x\to 0-} (\dfrac{1}{e^{1/x}+1}) = \dfrac{1}{0+1} = 1$

Here both one sided limits exists $\mathbb{R}$ but they are different and hence $\lim\limits_{x\to 0} h(x)$ does not exist.



Graph of $h(x) = 1/(e^{1/x}+1)$ $(x \neq 0)$. 14.6.4(d)

14.6.5 Theorem:

Let $A \subseteq \mathbb{R}$, f: $A \to R$ and c be a cluster point of $A \cap (c, \infty)$. Then the following are equivalent.

(i) $\lim\limits_{x \to c+} f(x) = l$

(ii) For every sequence (x_n) that converges to c such that $x_n > c$ for all $n \in N$; the sequence $(f(x_n)$ converges to l.

Proof: Suppose $\lim\limits_{x \to c+} f(x) = l$. Given $\in > 0$ there is $\delta(\in) > 0$ such that $0 < x - c < \delta(\in)$ and $x \in A$, $(f(x) - l| < \in$. Let (x_n) be any sequence such that $x_n > c$ and $\lim (x_n) = c$.

Since $\delta > 0$, there is a natural number $K(\delta)$ such that $|x_n - c| = x_n - c < \delta$ for $n \geq K(\delta)$

So if $n \geq K(\delta)$, $|f(x) - l| < \in$. Since $\in > 0$ is arbitary, $\lim f(x_n) = l$ whenever $x_n > 0$, $x_n \in A$ and $\lim x_n = C$. Conversely suppose $\lim\limits_{x \to c+} f(x) \neq l$. Then there is $\in_0 > 0$ such that for every $\delta > 0$ then exists x depending on δ such that $x \in A$, $0 < x - c < \delta$ and $|f(x) - l| > \in_0$.

In particular if $x \in N$ and $\delta = \frac{1}{n}$, there is $x_n \in A$ such that $0 < x_n - c < \frac{1}{n}$ and $|f(x_n) - l| > \in_0$. Clearly $\lim x_n = c$ but $\lim f(x_n) \neq l$.

14.6.6 SAQ: Show that c is a cluster point of A if and only if c is a cluster point of $A \cap (C, \infty)$ or $A \cap (-\infty, C)$.

14.6.7 Example: Consider the Euler function ϕ defined on $\mathbb{R}$ by

$$\phi(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{cases}$$

We show that $\lim\limits_{x \to c+} \phi(x)$ and $\lim\limits_{x \to c-} \phi(x)$ do not exist for every $c \in \mathbb{R}$

Solution: Let c be any rational number. Then for $x \in \mathbb{N}$, $c + \frac{1}{n} > c$, $c + \frac{1}{n}$ is a rational number and $\phi\left(c + \frac{1}{n}\right) = 0$, $\lim\limits_{n} \left(c + \frac{1}{n}\right) = C$ and $\lim\limits_{n} \phi\left(c + \frac{1}{n}\right) = 0$

Also $c + \frac{\sqrt{2}}{n} > c$ and $c + \frac{\sqrt{2}}{n}$ is an irrational number for $n \in \mathbb{N}$ and $\lim_{n} (c + \frac{\sqrt{2}}{n}) = c$.

$\phi (c + \frac{\sqrt{2}}{n}) = 1$ for every n, so $\lim_{n} \phi (c + \frac{\sqrt{2}}{n}) = 1$. Hence $\lim_{x\to c+} \phi(x)$ does not exist.

In a similar way we can show that $\lim_{x\to c-} \phi(x)$ does not exist when $c \in Q$. The proof when c is an irrational number is similar.

14.6.8 Example:

Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$\phi (x) \quad = 0$ if $x < 0$

$\qquad = 0$ if $x \geq 0$ and x is rational

$\qquad = 1$ if $x \geq 0$ and x is irrational

we show that $\lim_{x\to 0-} \phi (x) = 0$ and $\lim_{x\to 0+} \phi(x)$ does not exist.

Solution:

If $x \in (-\infty, 0)$, $\phi (x) = 0$ so if $\in > 0$ and $\delta > 0$ for $x \ni - \delta < x < 0$, $|\phi(x) – 0| = 0 < \in$.

Hence $\lim_{x\to 0-} \phi(x) = 0$. $\lim (\frac{1}{n}) = 0 = \lim (\frac{\sqrt{2}}{n})$ So $\lim \phi(\frac{1}{n}) = 0 \neq 1 = \lim \phi (\frac{\sqrt{2}}{n})$`

Hence $\lim_{x\to 0+} \phi(x)$ does not exist.

14.6.9 Theorem: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of both the sets $A \cap (c, \infty,)$ and $A \cap (-\infty,c)$. Then $\lim_{x\to c} f(x) = l$ if and only if $\lim_{x\to c+} f(x) = l = \lim_{x\to c-} f(x)$

Proof: If c is a cluster point of $A \cap (c, \infty) \subseteq A$, c is a cluster point of A.

Assume that lim $\lim_{x\to c} f(x) = l$. Given $\in > 0$ there is $\delta (\in) > 0$ such that if $0 < |x-c| < \delta(\in)$ and $x \in A$, $|f(x) – l| < \in$.

If $0 < x-c < \delta(\in)$ and $x \in A$ then $0 < |x-c| < \delta(\in) \Rightarrow |f(x) – l| < \in \Rightarrow \lim_{x\to c+} f(x) = l$. Also since c is a cluster point of $A \cap (-\infty, c)$. $0 < c – x < \delta(\in)$ and $x \in A \Rightarrow 0 < |x-c| < \delta(\in)$

and $x \in A \Rightarrow |f(x) - l| < \in$. $\Rightarrow \lim_{x \to c-} f(x) = l$.

Conversely suppose that $\lim_{x \to c+} f(x) = l. = \lim_{x \to c-} f(x)$. If $\in > 0$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(x) - l| < \in$ for all $x \in A$ with $0 < x\text{-}c < \delta_1$ and for all $x \in A$ with $0 < c\text{-}x < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x \in A$ and $o < |x\text{-}c| < \delta$, then either $0 < x\text{-}c < \delta$ or $0 < c\text{-}x < \delta$.

If $0 < x\text{-}c < \delta$ then $0 < x - c < \delta_1$ so that $|f(x) - l| < \in$ If $0 < x\text{-}c < \delta$ then $0 < c - x < \delta_1$ so that $|f(x) - l| < \in$ Thus if $x \in A$ and $0 < |x - c| < \delta$, $|f(x) - l| < \in$ Since $\in > 0$ is arbitary, $\lim_{x \to c} f(x) = l$.

14.7 Infinite limits and limits at infinity

14.7.1 Infinite limits:

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$, be a cluster point of A

(i) We say that f tends to ∞ as $x \to c$ and write $\lim_{x \to c} f(x) = \infty$ if for every $\alpha \in \mathbb{R}$ there exists $\delta(\alpha) > 0$ Such that for all $x \in A$ with $0 < |x - c| < \delta(\alpha)$, $f(x) > \alpha$

(ii) We say that f tends to $-\infty$ as $x \to c$ and write $\lim_{x \to c} f(x) = -\infty$, if for every $\beta \in \mathbb{R}$ there exists $\delta(\beta) > 0$ such that for all $x \in A$ with $0 < |x\text{-}c| < \delta(\beta)$, $f(x) > \beta$.

14.7.2: Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A \,/\, x > c\}$. Then we say that f tends to ∞ (respectively $-\infty$) as $x \to c+$ and write $\lim_{x \to c+} f(x) = \infty$ (respectively $\lim_{x \to c+} f(x) = -\infty$) if for every $\alpha \in \mathbb{R}$, there exists $\delta(\alpha) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$ then $f(x) > \alpha$ (respectively $f(x) < \alpha$).

14.7.3 Examples:

(a) $\lim_{x \to 0} \frac{1}{x^2} = \infty$ For if $\alpha > 0$ is given, let $\delta = \frac{1}{\sqrt{\alpha}}$. If follows that if $0 < |x| < \delta$ then $x^2 < \frac{1}{\alpha}$ so that $\frac{1}{x^2} > \alpha$. Hence $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

(b) Let $g(x) = \frac{1}{x}$ for $x \neq 0$. $\lim\limits_{x\to 0+} g(x) = +\infty$ and $\lim\limits_{x\to 0-} g(x) = -\infty$

If $x > 0$ and $\in > 0$ $|g(x)| > \in \Leftrightarrow \frac{1}{x} > \in. \Leftrightarrow 0 < x < \frac{1}{\in}$. So $\lim\limits_{x\to 0+} g(x) = \infty$. $x < 0$ and $\beta \in \mathbb{R}$,

$\beta < 0$ $g(x) = \frac{1}{x} < 0$ so $g(x) < \beta \Leftrightarrow \frac{1}{x} < \beta \Leftrightarrow -\frac{1}{x} > -\beta > 0 \Leftrightarrow -x < \frac{1}{\beta} \Leftrightarrow \frac{1}{\beta} < x < 0$

If $\beta > 0$, $g(x) < \beta$ for all $x < 0 \Rightarrow \lim\limits_{x\to 0-} g(x) = -\infty$.

However $\lim\limits_{x\to 0} g(x)$ does not exist in $\mathbb{R}$ as well as $\lim\limits_{x\to 0} g(x) \neq \infty$ and $\lim\limits_{x\to 0} g(x) \neq -\infty$.

(c) $\lim\limits_{x\to 1+} \frac{x}{x-1}$

If $\alpha > 1$ and $1 < x < \frac{\alpha}{\alpha - 1}$ then $\alpha < \frac{x}{x-1}$ hence we have $\lim\limits_{x\to 1+} \frac{x}{x-1} = \infty$.

(d) $\lim\limits_{x\to 0+} \frac{x+2}{\sqrt{x}} = \infty$

Let $g(x)=\frac{x+2}{\sqrt{x}}$ $(x > 0)$ Since $x > 0$ then $\frac{x+2}{\sqrt{x}} > \frac{2}{\sqrt{x}}$ If $\alpha > 0$ then $\frac{x+2}{\sqrt{x}} > \alpha$ if $\frac{4}{\alpha^2} > x > 0$

So $0 < x < \frac{4}{\alpha^2} \Rightarrow g(x) > \alpha$ since $\alpha > 0$ is arbitary $\lim\limits_{x\to 0+} g(x) = \infty$.

14.7.4 Limits at infinity:

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that $l \in \mathbb{R}$ is a limit of f as $x \to \infty$ and write $\lim\limits_{x\to\infty} f(x) = l$ if given $\in > 0$ there exists $\delta(\in) > 0$ such that for any $x > \delta(\in)$, $|f(x) - l| < \in$.

14.7.5 Theorem:

Let $a \in \mathbb{R}$, $A \subseteq \mathbb{R}$, $(a, \infty) \subseteq A$ and $f: A \to \mathbb{R}$. The following are equivalent.

(i) $\lim\limits_{x\to\infty} f(x) = l$

(ii) for every sequence (x_n) in $A \cap (a, \infty)$ such that $\lim x_n = \infty$, $\lim f(x_n) = l$.

Proof: (i) $\Rightarrow$ (ii): Assume (i) Let $\in > 0$, there is $\delta > 0$ such that if $x \in A \cap (a, \infty)$, $x > \delta$ then $|f(x) - l| < \in$.

If (x_n) is a sequence in $A \cap (a, \infty)$ and $\lim (x_n) = \infty$, corresponding to $\delta > 0$ there is a natural number $K(\delta)$ such that if $n \geq k(\delta)$, $x_n > \delta$ so that $|f(x_n) - l| < \in$ if $n \geq k(\delta)$. Since $\in > 0$ is arbitary, $\lim f(x_n) = l$ Hence (ii) holds.

(ii) $\Rightarrow$ (i)

Assume (ii): Suppose (i) is false, then there is $\in > 0$ such that if $\delta > 0$ there corresponds atleast one x_δ in $A \cap (a, \infty)$ such that $|f(x_n) - l| \geq \in$. By choosing $\delta = \frac{1}{n}$ where $n \in N$, we get $x_n \in A \cap (a, \infty)$ such that $\lim x_n = \infty$.

But $\lim f(x_n) \neq l$. So (ii) does not hold: Thus if (i) is false then (ii) is false. Hence (ii) $\Rightarrow$(i)

14.7.6 Examples:

(a) Let $f(x) = \frac{1}{x}$ for $x \neq 0$

$f(x) = \frac{1}{x}$ $(x > 0)$

From the graph $\lim\limits_{x\to\infty} \frac{1}{x} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

Further if $\in > 0$ then $0 - \frac{1}{|x|} < \in$ whenever $\frac{1}{\in} < |x|$ i.e. $x > \frac{1}{\in}$ and $x < -\frac{1}{\in}$.

(b) Let $g(x) = \frac{1}{x^2}$ for $x \neq 0$

$|g(x)| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} < \in \Leftrightarrow \frac{1}{\in} < x^2 \Leftrightarrow |x| > \frac{1}{\in}$ i.e. $x > \frac{1}{\in}$ or $-x < -\frac{1}{\in}$

From the graph $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0 = \lim\limits_{x\to-\infty} \frac{1}{x^2}$

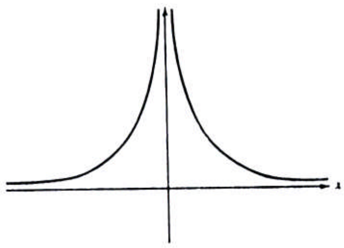


Fig 14.7.6(c)

Graph of
$f(x) = 1/x^2$ $(x \neq 0)$

14.7.7 Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that f trends to ∞ (respectively $-\infty$) as $x \to \infty$ and write $\lim\limits_{x\to\infty} f(x) = \infty$ (respectively $\lim\limits_{x\to\infty} f(x) = -\infty$) if given any $\alpha \in \mathbb{R}$ there exists $K(\alpha) > 0$ such that for any $x > k(\alpha)$ for k. $f(x) > \alpha$ (respectively $f(x) < \alpha$)

14.7.8 Theorem: Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. then the following statements are equivalent.

(i) $\lim\limits_{x\to\infty} f(x) = \infty$ (respectively $\lim\limits_{x\to\infty} f(x) = -\infty$)

(ii) for every sequence (x_n) in (a, ∞) such that $\lim (x_n) = \infty$ $\lim f(x_n) = \infty$. (respectly $\lim f(x_n) = -\infty$).

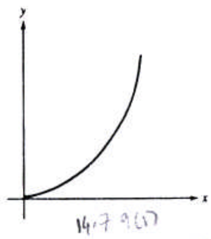
Proof: Assume. Given $\in > 0$ there is $\delta > 0$ such that $f(x) > \in$ for $x \geq \delta$.
If (x_n) is any sequence such that $\lim (x_n) = \infty$, then corresponding to the above δ, there is a natural number k such that $x_n > \delta$ if $n \geq k$. Hence if $n \geq k$, $x_n > \delta$ and so $f(x_n) > \in$. Since $\in > 0$ if arbitary, $\lim f(x_n) = \infty$.

Conversely suppose that $\lim\limits_{x\to\infty} f(x) \neq \infty$. Then there is $\in_0 > 0$ such that for every positive number δ, there corresponds atleast one x depending to δ such that $x > \delta$ but $f(x) > \in_0$.
In particular if $x \in N$ and $\delta = n$ there is x_n corresponding to n such that $x_n > n$ but $f(x_n) \leq \in_0$.
Clearly $\lim (x_n) = \infty$ but $\lim f(x_n) \neq \infty$ so that (ii) does not hold.

14.7.9 Examples:

(1) $\lim\limits_{x\to\infty} x^n = \infty$ for $n \in \mathbb{N}$.

Let $g(x) = x^n$ for $x \in (0, \infty)$. Given $\alpha \in R$, let $k = \sup \{1, \alpha\}$ then for all $x > k$, we have $x^n \geq n > \alpha \Rightarrow g(x) > \alpha$. Since $\alpha \in \mathbb{R}$ is arbitary, $\lim\limits_{x\to\infty} g(x) = \infty$.



14.7.9(1)

Figure Graph of $f(x) = x^n$ $(x \geq 0, n$ even).

(2) Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Prove that $\lim\limits_{x\to\infty} f(x) = l$ if and only if $\lim\limits_{x\to 0+} f\left(\frac{1}{x}\right) = l$.

Solution: $\lim\limits_{x\to\infty} f(x) = l \Rightarrow$ for given $\in > 0$ there is $K > 0$ such that $|f(x) - l| < \in$ for $x > K$.

$\Leftrightarrow |f\left(\frac{1}{z}\right) - l| < \in$ for $0 < Z < \frac{1}{K}$.

$\Leftrightarrow |f\left(\frac{1}{n}\right) - l\} < \in$ for $0 < x < \frac{1}{K}$

$\Leftrightarrow \lim\limits_{x\to 0+} f\left(\frac{1}{x}\right) = l$.

(3) Show that $f: (a, \infty) \to \mathbb{R}$ is such that $\lim\limits_{x\to\infty} x.f(x) = l$ where $l \in \mathbb{R}$ then $\lim\limits_{x\to\infty} f(x) = 0$

Solution: $\lim\limits_{x\to\infty} xf(x) = l \Rightarrow$ for given $\in > 0$ there is $\alpha > 0$ such that $|x.f(x) - l| < \in$ whenever $x > \alpha$.

$\Rightarrow |x.f(x)| - |l| < \in$ is $x > \alpha$

$\Rightarrow |x.f(x)| < \in + | l |$

$\Rightarrow +f(x)| < \frac{\in + |l|}{x} < \in$ for $x > \frac{\in + |l|}{\in}$

Hence $\lim\limits_{x\to\infty} f(x) = 0$

(4) Let f and g be defined on (a, ∞) and suppose $\lim\limits_{x\to\infty} f(x) = l$ and $\lim\limits_{x\to\infty} g(x) = \infty$. Prove that $\lim\limits_{x\to\infty} (f \text{ o } g)(x) = l$.

Solution: $\lim\limits_{x\to\infty} f(x) = l \Rightarrow$ for given $\in > 0$ there is $K > 0$ such that $|f(x) - l| < \in$ for $x > K$.

$\lim\limits_{x\to\infty} g(x) = \infty$ and $K > 0$ there is $H > 0$ such that $g(y) > K$ for $y > H$. If $y > H$, $g(y) > K$ so $|(fog)(y) - l| = |f(g(y)) - l| < \in$ if $g(y) > k$ and hence $y > H$. $\Rightarrow |(fog)(y) - l| < \in$ for $y > H$.

Hence $\lim\limits_{x\to\infty} (fog)(x) = l$.

(5) Let $A \subseteq \mathbb{R}$, c a cluster point of A, $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be such that $\lim\limits_{x \to c} f(x) = \ell$ where $\ell > 0$ and $\lim\limits_{x \to c} g(x) = \infty$. Show that $\lim\limits_{x \to c} f(x) \,.\, g(x) = \infty$.

If $\ell = 0$, show by example that this conclusion may fail.

Since $\lim\limits_{x \to c} f(x) = \ell > 0$ (for $\in = \frac{\ell}{2}$) there is $\delta_1 > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_1$, $\frac{\ell}{2} < f(x) < \frac{3}{2}$. Since $\lim\limits_{x \to c} g(x) = \infty$ if $\in > 0$ there is $\delta_2 > 0$ such that if $x \in A$ and $0 < |x - c| < \delta_2$, $g(x) > \frac{2}{\ell} \in$.

If $\delta = \min \{\delta_1, \delta_2\}$ and $x \in A$, $0 < |x - c| < \delta$, $f(x) > \frac{\ell}{2}$ and $g(x) > \frac{2}{\ell} \in$; so that $f(x).\, g(x) > \in$. Since $\in > 0$ is arbitary it follows that $\lim\limits_{x \to c} f(x) \,.\, g(x) = \infty$.

Example: $\lim\limits_{x \to \infty} x = \infty$, $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

But $\lim\limits_{x \to 0} (x \,.\, \frac{1}{x}) = 1$. So the statement is false when $\ell = 0$.

6) (i) Find functions f and g defined on $(0, \infty)$ such that $\lim\limits_{x \to \infty} f(x) = \infty$, $\lim\limits_{x \to \infty} g(x) = \infty$ and $\lim\limits_{x \to \infty} (f - g)(x) = 0$. (ii) Can you find such functions with $g(x) > 0$ for all $x \in (0, \infty)$ such that $\lim\limits_{x \to \infty} \left(\frac{f}{g}\right)(x) = 0$?

(i) Let f be any function defined on $(0, \infty)$ such that $\lim\limits_{x \to \infty} f(x) = \infty$ and $g(x) = f(x)$.

Then $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} g(x) = \infty$ but $f - g = 0$ so $\lim\limits_{x \to \infty} (f - g)(x) = 0$.

(ii) $f(x) = x$ and $g(x) = x^2$

Then $\lim\limits_{x \to \infty} f(x) = \infty$, $\lim\limits_{x \to \infty} g(x) = \infty$

$\frac{f(x)}{g(x)} = \frac{1}{x}$ so that $\lim\limits_{x \to \infty} \left(\frac{f}{g}\right)(x) = 0$.

14.7.10 Theorem: Let $A \subseteq \mathbb{R}$, $f,g : A \to \mathbb{R}$ and c be a cluster point of A. Suppose $f(x) \le g(x)$ for all $x \in A$, $x \neq c$.

(a) If $\lim\limits_{x\to c} f(x) = \infty$ then $\lim\limits_{x\to c} g(x) = \infty$

(b) If $\lim\limits_{x\to c} g(x) = -\infty$ then $\lim\limits_{x\to c} f(x) = -\infty$

Proof: Suppose $\lim\limits_{x\to c} f(x) = \infty$ and $\alpha \in \mathbb{R}$. Then there is $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$ then $f(x) > \alpha$.

As $g(x) \ge f(x)$, we get $g(x) > \alpha$ if $x \in A$ and $0 < |x - c| < \delta$ so $\lim\limits_{x\to c} g(x) = \infty$.

Suppose $\lim\limits_{x\to\infty} g(x) = -\infty$. Then if $\beta \in \mathbb{R}$ there is $\delta > 0$ such that $g(x) < \beta$ if $x \in A$ and $0 < |x - c| < \delta$. As $f(x) \le g(x)$,

Hence $f(x) < \beta$ if $x \in A$ and $0 < |x - c| < \delta$ So $\lim\limits_{x\to c} f(x) = -\infty$.

14.7.11 Theorem: Let $A \subseteq \mathbb{R}$, f, g: $A \to \mathbb{R}$ and suppose $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$ for $x \in A$ and that for some $L \in \mathbb{R}$, $L \neq 0$ we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

(1) If $L > 0$, $\lim\limits_{x\to\infty} f(x) = \infty$ if and only if $\lim\limits_{x\to\infty} g(x) = \infty$.

(2) If $L < 0$, $\lim\limits_{x\to\infty} f(x) = -\infty$ if and only if $\lim\limits_{x\to\infty} g(x) = -\infty$.

Proof: If $L < 0$, $-L$ is positive so we get (ii) from (i) by considering $-f$ instead of f, we thus prove (i) only. Suppose $L > 0$. Since $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$ there is $a_1 > a$ such that

$$0 < \frac{L}{2} < \frac{f(x)}{g(x)} < \frac{3L}{2} : g(x) \text{ for } x > a_1 \text{ ------ (1)}$$

since $g(x) > 0$, $\frac{L}{2} g(x) < f(x) < \frac{3L}{2} g(x)$ for $x > a_1$

Hence if $\lim\limits_{x\to\infty} f(x) = \infty$ and $\in > 0$ there is $\delta > 0$ such that $f(x) > \frac{3L}{2} \in$ for $x > \delta$.

So that if x > max {δ and a_1} $g(x) > \frac{2}{3L} f(x) > \in$. Hence $\lim_{x\to\infty} g(x) = \infty$

Conversely suppose $\lim_{x\to\infty} g(x) = \infty$. If $\in > 0$ there is $\delta > 0$ such that $g(x) > \frac{2}{L} \in$ if $x > \delta^1$.

If $x > \max \{\delta^1, a_1\}$, $f(x) > \frac{L}{2} \cdot \frac{2}{L} \in = \in$. Hence $\lim_{x\to\infty} f(x) = \infty$.

14.8 Solutions SAQ's

14.4.6 (i) If $c < 0$, $-c > 0$ and the nbd $V_{-c/2}(c) = (\frac{3c}{2}, \frac{c}{2}) \cap A - \{c\} = \phi$, since

$x \in (\frac{3c}{2}, \frac{c}{2}) \Rightarrow x < \frac{c}{2} < 0$ so $x \notin \cap = A - \{c\}$ so c is not a cluster of A.

(ii) If $c > 1$, choose $\delta > 0$ such that $(c - \delta, c + \delta)$ does not contain 1. Then $(c - \delta, c+\delta)$ does not contains any point of A. So c is not a cluster point of A.

(iii) If $0 < c < 1$, by Archimedean principle, there is $n \in N$ such that $n < \frac{1}{c} < n + 1$.

Since $\frac{1}{n+1} < c < \frac{1}{n}$ we can choose $\delta > o$ such that $V_\delta(c) \cap (\frac{1}{n+1}, \frac{1}{n}) = \phi$. As there are no other elements of A in $V_\delta(c)$, $V_\delta(c) \cap A = \phi$. So c is not a cluster point of A if $c \notin A$ but $0 < c \le 1$.

(iv) If $c = \frac{1}{n}$ for some $n > 1$, choose $\delta > 0$ such that $(c - \delta, c + \delta) \cap (\frac{1}{n+1}, \frac{1}{n-1}) = \{\frac{1}{n}\}$

clearly $V_\delta(c) \cap A = \{\frac{1}{n}\}$ so $V_\delta(c) \cap A - \{c\} = \phi$. So $\frac{1}{n}$ is not a cluster point of A if $n \in \mathbb{N}$ and $n > 1$.

(v) If $c = 1$, $\left(\frac{3}{4}, \frac{5}{4}\right)$ is $\frac{1}{4}$ - nbd of 1 and $\left(\frac{3}{4}, \frac{5}{4}\right) \cap A - \{1\} = \phi$. So 1 is not a cluster point of A.

Thus the only cluster point of A is 0.

14.4.8 (a) Suppose c is not a cluster point of A as well as B.

Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$V_{\delta_1}(c) \cap A - \{c\} = \phi$ and $V_{\delta_2}(c) \cap B - \{c\} = \phi$

If $0 < \delta < \min\{\delta_1, \delta_2\}$, $V_\delta(c) \subseteq V_{\delta_1}(c)$ and $V_\delta(c) \subseteq V_{\delta_2}(c)$

Hence $V_\delta(c) \cap (A - \{c\}) = \phi$ and $V_\delta(c) \cap (B - \{c\}) = \phi$

$\Rightarrow V_\delta(c) \cap (A \cup B - \{c\}) = (V_\delta(c) \cap (A - \{c\})) \cup (V_\delta(c) \cap (B - \{c\})) = \phi$.

As there is a nbd of c which has no common points with $A \cup B - \{c\}$, c is not a cluster point of $A \cup B$.

14.4.8 (b): If c is a cluster point of A and $c \in (a, b)$ choose $\delta > 0$ so that $a < c - \delta < c < c + \delta < b$. This is possible when $0 < \delta < \min\{c - a, b - c\}$.

By definition there is $x \in (c - \delta, c + \delta) \cap (A - \{c\}) \subseteq (a, b) \cap A - \{c\}$

The converse is clear since for every $\delta > 0$, $(c - \delta, c + \delta) \cap (A - \{c\}) \neq \phi$.

14.4.8 (c) Let F be a finite set. We may assume that $F \neq \phi$

If $c \notin F$, by 14.4.2 (2) there exists $\delta > 0$ $V_\delta(c) \cap F = \phi$. If $c \in F$ there exists $\delta > 0$ such $V_\delta(c) \cap F = \{c\}$

So c is not a cluster point of F.

(d) Let f: $\mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and $g(x) = f(x + c)$ show that $\lim\limits_{x\to c} f(x) = l$ if and only if $\lim\limits_{x\to 0} g(x) = l$.

SAQ 14.4 A

Solution: (a) Suppose $\lim\limits_{x\to c} f(x) = l$.

If $\in > 0$ there is $\delta > 0$ such that $|f(x) - l| < \in$ if $0 < |x - c| < \delta$. If $0 < |y| < \delta$ and $x = y + c$ then $0 < |y| = |x - c| < \delta$,

So that $|g(y) - l| = |f(y + c) - l| = |f(x) - l| < \in$.

Thus $0 < |y| < \delta \Rightarrow |g(y) - l| < \in$

This is true for every $\in > 0$ so $\lim\limits_{x\to c} f(x) = l \Rightarrow \lim\limits_{x\to 0} g(x) = l$

Proof of the converse part is similar.

If $\in > 0$ there is $\delta_1 > 0$ such that $|f(x) ; - l| < \in$

If $0 < |x| < \frac{\delta}{a}$, $0 < |a x| < \delta$ so $|f(ax) - l| < \in$

i e $|g(x) - l| < \in$. Thus $|g(x) - l| < \in$ if $0 < |x| < \frac{\delta}{a}$

Since $\in > 0$ is arbitary $\lim\limits_{x \to 0} g(x) = l$.

Solution (c): (a) For $x \in \mathbb{R}$, $|f(x)|$ is either 0 or $|x|$ so that if $\in > 0$, $|f(x) - 0| = |f(x)| \le |x| < \in$ if $0 < |x| < \in$ so (a) holds.

(b) Suppose $c \neq 0$. By the "Density theorem", for every natural number n, there are x_n, y_n such that x_n is rational number and y_n is an irrational number,

$c < x_n < c + \frac{1}{n}$ and $c < y_n < c + \frac{1}{n} \Rightarrow \lim x_n = \lim y_n = c$

Since $f(x_n) = x_n$ and $f(y_n) = 0$, $\lim f(x_n) = \lim (x_n) = c$ and $\lim f(y_n) = 0$.

Since $c \neq 0$, by the Divergence cisteria, $\lim\limits_{x \to c} f(x)$ does not exist.

(d) Solution: Since I is open and $c \in I$, we can find a δ_0 such that $(c - \delta_0, c + \delta_0) \subseteq I$

Suppose $\lim\limits_{x \to c} f_1(x) = l$. Given $\in > 0$ there is $\delta_1 > 0$ such that $|f_1(x) - l| < \in$ if

$0 < |x - c| < \delta_1$ and $x \in I$. If $\delta(\in) = \min\{\delta_0, \delta_1\}$, $\delta(\in) \le \delta_0$ and δ_1 so $(c - \delta(\in), c + \delta(\in)) \subseteq (c - \delta_0, c + \delta_0) \subseteq I$ and $(c - \delta_0, c + \delta_0) \subseteq (c - \delta_0, c + \delta_0)$. So if $0 < |x - c| < \delta(\in)$, $x \in I$, we get $|f(x) - l| = |f_1(x) - l| < \in$

Thus for every $\in > 0$ there is $\delta(\in) > 0$ such that if $0 < |x - c| < \delta(\in)$, $|f(x) - l| < \in$

Hence $\lim\limits_{x \to c} f(x) = l$

Conversely suppose that $\lim\limits_{x \to c} f(x) = l$

If $\in > 0$ there is $\delta_1 > 0$, such that if $0 < |x - c| < \delta_1$, $|f(x) - l| < \in$

Again let $\delta(\in) = \min\{\delta_1, \delta_0\}$ where δ_0 is chosen as above. If $0 < |x - c| < \delta(\in)$, then $x \in I$ and $0 < |x - c| < \delta_1$ so that $|f_1(x) - l| = |f(x) - l| < \in$ Hence $\lim\limits_{x \to c} f_1(x) = l$.

14.5.4 SAQ

Let $\in > 0$. Since $\lim_{x\to c} f(x) = l$, there is $\delta_1 > 0$ such that $0 < |x - c| < \delta_1$ and $x \in A \Rightarrow$

$|f(x) - l| < \frac{\in}{2}$ ________ (1)

Similarly there is $\delta_2 > 0$ such that $0 < |x - c| < \delta_2$ and $x \in A \Rightarrow |g(x) - m| < \frac{\in}{2}$ _____ (2)

Let $\delta(\in) = \min\{\delta_1, \delta_2\}$. If $0 < |x - c| < \delta(\in)$ and $x \in A$, x satisfies the conditions in (1) and (2) so that $|f(x) - l| < \frac{\in}{2}$ and $|g(x) - m| < \frac{\in}{2}$. Thus if $0 < |x - c| < \delta(\in)$ and $x \in A$,

$|(f+g)(x) - (l+m)| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m| < \frac{\in}{2} + \frac{\in}{2} = \in$ _______ (3)

This is true for every $\in > 0$, so $\lim_{x\to c}(f + g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = l - m$, we need only to replace (3) by $|(f-g)(x) - (l - m)| = |(f(x) - l) - (g(x) - m)| \leq |f(x) - l| + |g(x) - m| < \frac{\in}{2} + \frac{\in}{2} = \in$.

14.5.8 SAQ:

If $f(x) = \sqrt{1+2x} - \sqrt{1+3x}$ and $g(x) = x + x^2$; then $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$. So the quotient formula does not work here.

However for $x > 0$, $\frac{f(x)}{g(x)} = \frac{(1+2x)-(1+3x)}{(x+x^2)\left\{\sqrt{1+2x}+\sqrt{1+3x}\right\}}$

$$= \frac{-1}{(1+x)\{\sqrt{1+2x}+\sqrt{1+3x}\}}$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{-1}{(1+x)\left\{\sqrt{1+2x}+\sqrt{1+3x}\right\}}$$

$$= \frac{-1}{(1+0)\,(1+1)} = \frac{1}{2}$$

14.5.9 (a) Let $p(x) = a_0 + a_1x + \ldots\ldots\ldots\ldots + a_n x^n$

We know that $\lim\limits_{x\to c} x^k = c^k$ if $1 \le k \le n$.

Hence $\lim\limits_{x\to c} p(x) = a_0 + a_1 \lim\limits_{x\to c} x + \ldots\ldots + a_n \lim\limits_{x\to c} x^n$

$= a_0 + a_1c + \ldots\ldots\ldots\ldots + a_n c^n$

(b) By SAQ 14.5.9 (a) $\lim\limits_{x\to c} p(x) = p(c)$ and $\lim\limits_{x\to c} q(x) = q(c)$

since $q(c) \ne 0$, $\lim\limits_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$

(Here we note that there is a nbd V_δ (c) of such that for $x \in V_\delta$ (c) and $x \ne c$, $q(x) \ne 0$ so that both p(x) and q(x) are defined and $q(x) \ne 0$ in V_δ (c).)

14.5.14 (i) $f(0) = f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0$

(ii) $0 = f(0) = f(x + (-x)) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$ for $x \in \mathbb{R}$.

(iii) If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $f(nx) = n. f(x)$

Reason $f(1. x) = f(x) = 1. f(x)$

$f(2. x) = f(x + x) = f(x) + f(x) = 2. f(x)$

If $K \in N$ and $f(Kx) = K. f(x)$ then

$f((K + 1) x) = f(Kx + x) = f(Kx) + f(x) = K. f(x) + f(0) = (K + 1) f(x)$.

Since the required holds for K + 1 whenever it holds for K, by mathematical introduction, equality holds for all $n \in N$.

If n is a negative integer $f(nx) = f(-(-n)x) = f(-nx) = -(-n) f(x) = nf(x)$ (since- $n \in N$).

So that $f(nx) = n. f(x)$ if n is any integer.

14.5.15 By 14.5.4 $f(x) = x. f(1)$ for every rational number x.

(i) If (x_n) is a sequence of rational numbers such that $\lim x_n = 0$.

$L = \lim\limits_{x\to 0} f(x) = \lim f(x_n) = \lim x_n f(1)$

$= f(1). \lim x_n = f(1) . 0 = 0.$

(ii) if $c \in \mathbb{R}$, $f(x) = f(x - c) + f(c) \Rightarrow |f(x) - f(c)| = |f(x - c)|$

Since $\lim\limits_{x\to 0} f(x) = 0$, given $\epsilon > 0$ there is $\delta > 0$ such that $|f(t)| < \epsilon$ if $0 < |t| < \delta$.

Hence if $0 < |x-c| < \delta$, $|f(x) - f(c)| = |f(x-c)| < \epsilon$.

This implies that $\lim_{x\to c} f(x) = f(c)$ for every $c \in \mathbb{R}$.

(iii) If x is any real number, choose a sequence of rational numbers (x_n) such that $\lim x_n = x$ and $x_n = x$ and $x_n \neq x$ for $n \in \mathbb{N}$. Since $\lim_{t\to x} f(t) = f(x)$, $\lim f(x_n) = f(x)$ So $f(x) = \lim f(x_n) = \lim x_n f(1) = f(1) \lim x_n. = f(1) . x \Rightarrow f(x) = x . f(1)$.

14.5.16: Since $\lim_{x\to c} f(x) = l$, given $\in > 0$ there is $\delta > 0$ such that $|f(x) - l| < \in$ if $0 < |x-c| < \delta$ and $x \in A$.

If $0 < |x-c| < \delta$, $|\,|f|(x) - |l|\,| = |\,|f(x)| - |l|\,| \le |f(x) - l| < \in$. Hence $\lim |f(x)| = |l|$.

14.5.17: f is bounded $\Rightarrow$ there is $\delta_1 > 0$ and $M > 0$ such that $|f(x)| \le M$ if $x \in A$ and $0 < |x-c| < \delta_1$. If $\in>0$ there is $\delta_2 >0$ such that $|g(x)|<\frac{\in}{M}$ such that if $x \in A$ and $0<|x-c|< \delta_2$. Let $\delta = \min \{\delta_1, \delta_2\}$ If $x \in A$ and $0 < |x-c| < \delta$ then $0 < |x-c| < \delta_1$ and $0 < |x-c| < \delta_2 \Rightarrow$ $|f(x)| \le M$ and $|g(x)| < \frac{\in}{M}$ $|(f g)(x) \sim 0| = |f(x) . g(x)| = |f(x) . |g(x)|. < M. \frac{\in}{M} = \in$. Hence $\lim_{x\to c} (f g)(x) = 0$.

14.6.3 Consider $f : (a, b) \to \mathbb{R}$, $\lim_{x\to a} f(x) = l$.

$\Leftrightarrow$ for every $\in > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ and $x \in (a, b)$, $|f(x) - l| < \in$.

$\Leftrightarrow$ for every $\in > 0$ there is a $\delta > 0$ such that $0 < \delta < b - a$ and whenever $a < x < a + \delta < b$, $|f(x) - l| < \in$.

$\Leftrightarrow \lim_{x\to a+} f(x) = l$.

SAQ 14.6.6: c is a cluster point of A iff c is a cluster point of $A \cap (c, \infty)$ or c is a cluster point of $A \cap (-a, c)$.

If c is not a cluster point of $A \cap (c, \infty)$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $V_{\delta_1} \cap A \cap (c, \infty) = \phi = V_{\delta_2} \cap A \cap (-\infty, c)$. If $\delta = \min \{\delta_1, \delta_2\} V_\delta \cap A \cap (c, \infty) =$

$V_\delta \cap A \cap (-\infty, c) = \phi$, so that $V_\delta \cap A - \{c\} = V_\delta \cap \{A \cap (c, \infty) \cup A \cap (-\infty, c)\} = \phi$. Conversely if c is a cluster point of one of $A \cap (-\infty, c)$ and $A \cap (c, \infty)$ then V is a cluster point of A since $A \cap (-\infty, c)$ as well as $A \cap (c, \infty)$ is a subset of $A - \{c\}$.

14.9 Summary

In this lesson the notions of cluster point of a set, limit of a real valued function at a cluster point of the domain are introduced and the equivalence of "$\in$, δ" definition and sequential approach is established. Discussion on onesided limits, limits
at ∞ and infinite limits is made in some detail.

14.10 Technical Terms:

Cluster point, limit of a function, right limit and left limit, infinite limit, limit at infinity

14.11 Exercise

1. Determine a condition on $|x-1|$ that will assure that

   (a) $|x^2 - 1| < \frac{1}{2}$ (b) $|x^2 - 1| < \frac{1}{10^{+3}}$

   (c) $|x^2 - 1| < \frac{1}{n}$ for given $n \in N$

2. Determine a condition on $|x-4|$ that will assure that (a) $|\sqrt{x} - 2| < \frac{1}{2}$ (b) $|\sqrt{x} - 2| < 10^{-2}$

3. If c is a cluster point of A then show that c is a cluster point of $A - \{c\}$.

4. If $\lim\limits_{n} (a_n) = a$ and the set $A = \{a_n / n \in \mathbb{N}\}$ is infinite; then prove that a is the only cluster point of A.

5. If $A \subset B$ and c is a cluster point of A, show that c is a cluster point of B.

6. Find the cluster points of the set $\{\pm 1 + \frac{1}{n} / n \in \mathbb{N}\}$.

7. Show that any cluster point c of A is a cluster point of $A \cap \{x / x < c\}$ or of $A \cap \{x / x > c\}$.

8. Let c be a cluster point of $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ prove that $\lim_{x\to c} f(x) = l$ if and only if $\lim_{x\to c} |f(x) - l| = 0$.

9. Let f: $\mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x\to c} f(x) = L$. if and only if $\lim_{x\to 0} f(x + c) = L$.

10. Determine the following limits:

(a) $\lim_{x\to 1} (x+1)(x+20)$

(b) $\lim_{x\to 1} \dfrac{x^2+2}{x^2-2}$ (x > 0)

(c) $\lim_{x\to 2} \left(\dfrac{1}{x+1} - \dfrac{1}{2x}\right)$ (x>0)

(d) $\lim_{x\to 0} \dfrac{x+1}{x^2+2}$

11. Show that

(a) $\lim_{x\to 2} \sqrt{\dfrac{2x+1}{x+3}} = \sqrt{\dfrac{7}{5}}$

(b) $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = 4$

(c) $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} = \dfrac{1}{2}$

(d) $\lim_{x\to 0} \dfrac{(1+x)^2-1}{x} = 2$

12. $\lim_{x\to 0} \sin \dfrac{1}{x^2}$ does not exist (let $x_n = \dfrac{1}{x^2}$ and $y_n = \dfrac{1}{\sqrt{(4n+1)\Pi/2}}$

13. $\lim_{x\to 0} x \cdot \sin\left(\dfrac{1}{x^2}\right) = 0$ (Hint : $|\sin t| \le 1$)

14. $\lim_{x\to 0} \operatorname{sgn}\left(\sin \dfrac{1}{x}\right)$ does not exist (Hint : $x_n \dfrac{1}{(4n+1)\Pi/2}$, $y_n = \dfrac{1}{\sqrt{(4n-1)\Pi/2}}$)

15. $\lim_{x\to 0} \sqrt{x} \sin\left(\dfrac{1}{x^2}\right) = 0$ (x > 0)

16. $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist ($x_n = \frac{1}{2n\Pi}$, $y_n = \frac{1}{(2n+1)\Pi}$

17. $\lim_{x \to 0} x.\cos (\frac{1}{x}) = 0$

18. Suppose f(x) = f(-x) for $x \geq 0$, show that $\lim_{x \to 0+} f(x) = l$. If and only if $\lim_{x \to 0-} f(x) = l$.

19. Suppose f(x) = - f(-x) for $x \geq 0$, show that if $\lim_{x \to 0} f(x) = l$ then $l = 0$.

20. Find $\lim_{x \to 1+} [2x]$ and $\lim_{x \to 1-} [2x]$.

Answers: (1) $|x-1| < \alpha$ where α is on Rhs of a, b, c; for (d) $|x-1| < \frac{1}{10^n}$

(2) (a) $|x-4| < 1$ (b) $|x-4| < 0.001$

(6) {-1, 1}

(10) (a) 0, (b) -3 (c) $\frac{1}{12}$ (d) $\frac{1}{2}$

(20) 2, 1

14.12 Model Examination Questions

1. Find the cluster points of $\{\frac{1}{n} / n \in N\}$

2. If C is a cluster point of $A \cup B$ show that c is cluster point of either A or B.

3. State and prove squeeze theorem for functions

4. Find $\lim_{x \to 0} \sqrt{x}\ \sin(\frac{1}{x^2})$

5. Find $\lim_{x \to 1+} [2x]$ and $\lim_{x \to 1-} [2x]$

6. If f(x) = f(-x) for $x \geq 0$, $x \in \mathbb{R}$ show that $\lim_{x \to 0+} f(x) = l$ if and only if $\lim_{x \to 0-} f(x) = l$.

14.13 Model Practical Problem with solution:

Discuss the existence of $\lim_{x \to 0} f(x)$ where $f(x) = \sqrt{|x|}\ (\text{sgn } x) \sin \frac{1}{x}$ if $x \neq 0$ and f(0) = 0.

Aim: To decide whether $\lim_{x \to 0} f(x)$ exists or not and find the limit if it exists.

Definitions:

(1) The function sgn $\mathbb{R} \to \mathbb{R}$ is defined by sgn x = $\begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$

(2) If E $\subset \mathbb{R}$, c is a cluster point of E, f: E $\to \mathbb{R}$ and $l \in \mathbb{R}$, we say that f(x) converges to l as x $\to$ c, in symbols $\lim\limits_{x \to c} f(x) = l$ if for every $\in > 0$ there corresponds $\delta(\in) > 0$ such that if x $\in$ E and $0 < |x-c| < \delta(\in)$, $|f(x) - l| < \in$.

Results used: $|\sin\theta| \le 1$ for $\theta \in \mathbb{R}$.

Solution: Clearly 0 is a cluster point of the domain $\mathbb{R}$ of f.

If $x \neq 0$ $|f(x)| = |\sqrt{|x|}|\,|\text{sgn } x|\,|\sin \frac{1}{x}|$

$\le |\sqrt{|x|}|.\ 1.\ 11 = |\sqrt{|x|}|$

also $|f(0)| = |0| = 0$.

If $\in > 0$ and $0 < |x| < \in^2$, $|f(x) - 0| = |f(x)| \le |\sqrt{|x|}| < \in$.

Hence by definition $|f(x) - 0| = |f(x)| \le |\sqrt{|x|}| < \in$ if $0 < |x| < \in^2$

Then $\lim\limits_{x \to 0} f(x)$ exists and $\lim\limits_{x \to 0} f(x) = 0$.

K. SAMBASIVA RAO

Karl Theodor Wilhelm Weierstrass (1815-1897)

Weierstrass is best known for his construction of the theory of complex functions by means of power series.

LESSON – 15

CONTINUOUS FUNCTIONS

15.1 Objective:

To obtain the student to get familiarity with the notion of jcontinuity – especially some properties of the range of a continuous function on a closed and bounded interval.

15.2. Structure:

15.3 Introduction

15.4 Continuous functions

15.5 Combinations of continuous functions

15.6 Continuous functions on an interval

15.7 Uniform continuity

15.8 Solutions to SAQ's

15.9 Summary

15.10 Technical terms

15.11 Exercises

15.12 Model examination Questions

15.13 Model practical problem with solution

15.3 Introduction:

In this lesson the student is introduced to the most important notion in analysis – namely continuity motivated by the most natural description of a curve as any unbroken paths.

Beginning with the notion continuity of a function at a point of the domain we define continuity on a set and define conditions for continuity as well as discontinuity. Various theorems concerning the algebraic properties and order properties will be established. We then study properties of a continuous function on a closed and bounded interval including boundedness, attaining the maximum and minimum values, intermediate value property and so on.

The notion of uniform continuity is also introduced and its equivalence with continuity when the domain is closed and bounded interval are desired.

This is the most appropriate time that the student comes to know that though Sir Issac Newton used the notion of continuity to explain the motion of bodies, it was Bolzano and Cauchy who had identified continuity as an important concept in analysis while Karl Weierstress was responsible for making a careful study of this topic.

15.4 Continuous functions:

15.4.1 Definition: Let $A \subseteq \mathbb{R}$ and $c \in A$. A function $f : A \to \mathbb{R}$ is said to be continuous at 'c' if given $\in > 0$ there is a $\delta > 0$ depending on $\in$ such that

$x \in A, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \in$

If f is not continuous at c, we say that f is discontinuous at c. If f I continuous at every point of A, we that f is continuous on A.

Example: (1) The constant function f(x) = K is continuous on $\mathbb{R}$.

Sol: Let $\in > 0$ and take $\delta = \in$ and let $n_, c \in \mathbb{R}$, $|x - c| < \delta$. Now $|f(x) - f(c)| = |K-K| = 0$ if $0 < |x - c| < \delta = \in$. $\Rightarrow$ f is continuous at c, where $c \in \mathbb{R}$. Hence f is continuous on $\mathbb{R}$.

(2) The function f(x) = x for all $x \in \mathbb{R}$ is continuous on $\mathbb{R}$.

Sol: Let $\in > 0$ and take $\delta = \in$ and let $c \in \mathbb{R}$, $|n - c| < \delta$. Now $|f(x) - f(c)| = |x-c| < \delta = \in$

$\Rightarrow$ f is continuous at c whenever $c \in \mathbb{R}$. Hence f is continuous on $\mathbb{R}$

15.4.2 Remark: Given f: $A \to \mathbb{R}$, last us compare continuity and existence of the limit of f(x) at c

1.	For continuity at c, c must be a point of A	1.	For $\lim_{x\to c} f(x) = \ell$, to exist c need not be in A, but c must be a cluster point of A.
2.	For $\in > 0$ there exist $\delta > 0$ such that $\lvert f(x) - f(c) \rvert < \in$ if $x \in A$ and $\lvert x - c\rvert < \delta$ (includes x = c as well)	2.	For $\in > 0$ there is $\delta > 0$ such that $\lvert f(x) - \ell\rvert < \in$ if, $x \in A$ and $0<\lvert x-c\rvert<\delta$ The value x = c is ignored and ℓ is not necessarily equal to f(c).
3.	If $c \in A$ but not a cluster point f is trivially continuous at c as there is $\delta>0$ such that $V_\delta(c) \cap A = \{c\}$ and so $\lvert f(x) - f(c)\rvert = 0$ if $x \in V_\delta(c)(\cap A$	3.	If $c \in A$ but c is not a cluster point of A. The, question of existence of ℓim f(x) does not arise $x \to c$

We have the following theorem when $c \in A$ and c is a cluster point of A.

15.4.3 Theorem: If $c \in A$ and c is a cluster point of A then $f : A \to \mathbb{R}$ is continuous at c if and only if $\lim_{x\to c} f(x)$ exists and the limit is f(c).

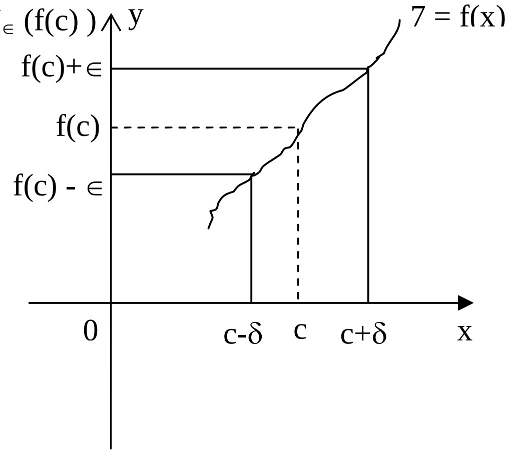
Proof: f is continuous at c $\Leftrightarrow$ for every $\in > 0$, there is $\delta > 0$ such that $\lvert f(x) - f(c)\rvert < \in$ if $x \in A$ and $\lvert x - c\rvert < \delta$.

$\Leftrightarrow$ for every $\in > 0$ there is $\delta > 0$ such that $\lvert f(x) - f(c)\rvert < \in$ if $x \in A$ and $0 < \lvert x - c\rvert < \delta$ (as when x = c, f(x) = f(c))

$\Leftrightarrow \lim_{x\to c} f(x)$ exists and is equal to f(c).

A condition for continuity of a function at any given point of A in terms of neighbourhoods is explained in the following result.

15.4.4. Theorem: Let f: $A \to \mathbb{R}$ and $c \in A$. then f is continuous at c iff for every $\in > 0$ there is a $\delta > 0$ such that for all $x \in A \cap V_\delta(c)$, $f(x) \in V_\in(f(c))$



Proof: f is continuous at c $\Leftrightarrow$ for every $\in > 0$, there is $\delta > 0$ such that $|f(x)-f(c)| < \in$ if $x \in A \cap V_\delta(c)$. Since $|f(x) - f(c)| < \in \Leftrightarrow f(x) \in V_\in(f(c))$; f is continuous at c.

$\Leftrightarrow$ for every $\in > 0$ there is $\delta > 0$ such that for every $x \in V_\delta(c) \cap A$, $f(x) \in V_\in(f(c))$.

15.4.5 Theorem: Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be continuous at a point $c \in A$. Then for any $\in > 0$, there exists a neighbourhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$ then $|f(x) - f(y)| < \in$.

Proof: $f : A \to \mathbb{R}$ is continuous at c $f:A \to \mathbb{R}$ is continuous at c $\Rightarrow$ given $\in > 0$ there exists $\delta > 0$ such that $x \in A \wedge v_\delta(c) \Rightarrow f(x) \in V_\delta(f(c))$. i.e. if $x \in A \cap V_\delta(c)$ then $|f(x) - f(c)| < \in/2$.

Let $x, y \in A \cap V_\delta(c) \Rightarrow |f(x) - f(c)| < \in/2$ and $|f(y) - f(c)| < \in/2 \Rightarrow$

$$
\begin{aligned}
|f(x) - f(y)| &= |f(x) - f(c) + f(c) - f(y)| \\
&\le |f(x) - f(c)| + |f(c) - f(y)| \\
&< \in/2 + \in/2 = \in
\end{aligned}
$$

Hence if $x, y \in A \cap V_\delta(c)$ then $|f(x) - f(y)| < \in$.

Sequential criterion for continuity:

15.4.6 Theorem: A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

Proof: Let f be continuous at c and let (x_n) be a sequence in A such that $\lim (x_n) = c$.
f is continuous at c $\Rightarrow$ for given $\in > 0$ there is $\delta > 0$ depending on $\in$ such that $|f(x)-f(c)| < \in$ whenever $|x - c| < \delta$ and $x \in A$.

Since (x_n) converges to c, corresponding to δ, there is a positive integer m such that $|x_n - c| < \delta$ whenever $n > m$.

So when $n > m$, $|x_n - c| < \delta$, hence $|f(x_n) - f(c)| < \in$

Since $\in > 0$ is arbitary, $\lim f(x_n) = f(c)$.

$\Rightarrow (f(x_n))$ converges to f (c).

Conversely suppose that for every sequence (x_n) in A that converges to c, $(f(x_n))$ converges to f(c).

Suppose, if possible f is not continuous at c.

$\Rightarrow$ there is $\in_0 > 0$ such that for every $\delta > 0$, there corresponds $x \in A$ such that $|x - c| < \delta$, but $|f(x) - f(c)| \geq \in_0$.

If $n \in \mathbb{N}$ and $\delta = \frac{1}{n}$, there is $x_n \in A$ such that $|x_n - c| < \frac{1}{n}$, but $|f(x_n) - f(c)| \geq \in_0$.

Then (x_n) converges to c, but $(f(x_m))$ does not converge to f(c), which is a contradiction.

Hence f is continuous at c.

15.4.7 Corollary (Discontinuity Criterion):

$f : A \to \mathbb{R}$ is discontinuous at $c \in A$ iff there is a sequence (x_n) in A such that $\lim(x_n)=c$, but $\lim(f(x_n)) \neq f(c)$.

15.4.8 SAQ: (Extension of a continuous function) Let $A \subseteq \mathbb{R}$, c cluster point of A, $c \notin A$ $f: A \to \mathbb{R}$ be continuous $\ell \in \mathbb{R}$, $\lim\limits_{x \to c} f(x) = \ell$. Show that f can be extended to $A \cup \{c\}$ by defining $F(x) = f(x)$ if $x \in A$ and, ℓ if $x = c$ and the function F defined as above is continuous on $A \cup \{c\}$.

15.4.9. Remarks: If $f : A \to \mathbb{R}$ is continuous on A, c a cluster point of A but $\lim\limits_{x \to c} f(x)$ does not exist, then it is possible to extend f to a continuous function F on $A \cup \{c\}$, because for any such F, it must hold that $\lim\limits_{x \to c} F(x) = F(c)$. Since $F(x) = f(x)$ for $x \in A$ and $\lim\limits_{x \to c} f(x)$ does not exist, the above requirement does not hold good.

15.4.10 Examples:

(a) The function f(x) defined on $(0, \infty)$ by $f(x) = \sin \frac{1}{x}$ is continuous.

If $x_n = \frac{1}{n\pi}$, $\lim (x_n) = 0$ but $\lim f(x_n) = \lim \sin n\pi = 0$

If $x_n = \frac{1}{(4n+1)\frac{\pi}{2}}$, $\lim (x_n) = 0$ and $f(x_n) = 1$ for every n.

So $\lim f(x_n) = 1$; existence of two such sequences implies that $\lim\limits_{x \to 0} \sin \frac{1}{x}$ does not exist. Hence by the above remark, there is no number ℓ such that the function F defined by

$F(x) = \begin{cases} \sin 1/x \text{ if } x \neq 0 \\ \ell \text{ if } x = 0 \end{cases}$ is continuous.

(b) The constant function defined by f(x) = a (when $a \in \mathbb{R}$) for all x is continuous on every subset A of $\mathbb{R}$.

Reason: If $\in > 0$ and $\delta > 0$ then for $c \in A$ and $x \in A$ is such that $|x - c| < \delta$, $|f(x) - f(c)| = |a - a| = 0 < \in$

So f is continuous at c for every $c \in A$.

(c) The function of defined on $\mathbb{R}$ by of g(x) = x in continuous on $\mathbb{R}$.

Reason: If $c \in \mathbb{R}$ and $\in > 0$, let $\delta = \in$

If $|x - c| < \delta$, then $|g(x) - g(c)| = |x - c| < \delta = \in$

So g is continuous at c for every $c \in \mathbb{R}$.

(d) For any $k \in \mathbb{N}$, the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^k$ is continuous on $\mathbb{R}$.

Reason: If $c \in \mathbb{R}$, $|x^k - c^k| = |(x - c)(x^{k-1} + n^{k-2} c + \ldots\ldots + c^{k-1}]$

$\leq |x - c| (|x|^{k-1} + |x|^{k-2} |c| + \ldots\ldots\ldots\ldots.. + |c|^{k-1})$

If $|x - c| < 1$, $(|x| - |c|) \leq |x - c| < 1 \Rightarrow |x| < 1 + |c|$

$\Rightarrow |x|^r < (1+|c|)^r$ for $1 \leq r \leq k$.

$\Rightarrow |x|^{k-1} + |x^{k-2}| |c| + \ldots\ldots\ldots\ldots\ldots. + |c|^{k-1}$

$\leq (1+|c|)^{k-1} + (1+|c|)^{k-2}\,|c| + \ldots. + |c|^{k-1} = M$ (say)

$\Rightarrow |g(x) - g(c)| \leq |x - c|\, M$ if $|x - c| < 1$.

If $\in > 0$ and $|x - c| < \delta = \min\{1, \frac{\in}{M}\}$, $|g(x) - g(c)| < M|x - c| < \in$

Hence g is continuous at c. This is true for every $c \in \mathbb{R}$. So g is continuous on $\mathbb{R}$.

Note: this can be proved by sequential method also.

(e) Define $g : (0, \infty) \to \mathbb{R}$ by $g(x) = \frac{1}{n}$ if $x \in (0, \infty)$

if $0 < c < \infty$ and $\frac{c}{2} < x < \frac{3c}{2} \Rightarrow \frac{2}{3c} < \frac{1}{x} < \frac{2}{c}$

$|g(x) - g(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{c\,x} < \frac{2\,|x-c|}{c^2}$

If $\in > 0$ and $\frac{c}{2} < x < \frac{3c}{2}$, choose $\delta > 0$ such that $\delta < \min\left\{\frac{c^2 \in}{2}, \frac{c}{2}\right\}$.

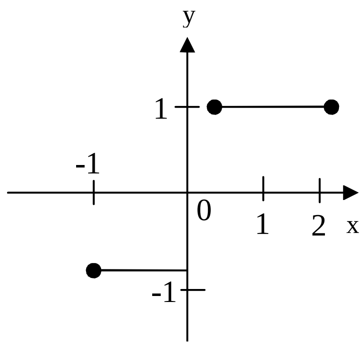
If $|x-c|<\delta$ then $|x-c| < c/2$ i.e. $\frac{c}{2} < x < \frac{3c}{2}$ and $|x-c| < \frac{c^2 \in}{2} \Rightarrow |g(x)-g(c)| < \frac{2}{c^2}\,|x - c| < \in$

Hence g s continuous at c.

(f) The greatest integer function defined on $\mathbb{R}$ by $g(x) = [x] = n$ where n is the integer and $n \leq x < (n+1)$ is discontinuous at every integer values of x and continuous otherwise.

Reason: Let n be any integer and $x_k = n - \frac{1}{k+1}$, $x_k \neq$ n for every n. $\lim_k (x_k) = n$, since $\lim\left(\frac{1}{k}\right) = 0$. Since $n - 1 < x_k < n$ for $k \in N$ $g(x_k) = n - 1$ so that $\lim g(x_k) = n - 1$. Thus we have a sequence, (x_k) such that $x_k \neq n$ for every n, $\lim x_k = n$, but lim of $(x_k) \neq g(n)$. Hence g is discontinuous at n. If x is not an integer there is a unique integer n such that $n-1<x<n$.

For y in this interval of $g(y) = n - 1$



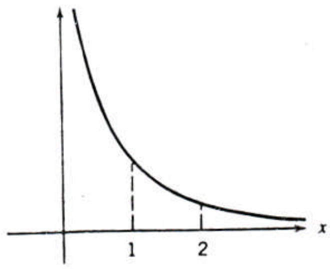
Graph of [x] function

Hence of g(y) – g (x) = 0

Thus if $\in > 0$ and δ is any positive number such that $n - 1 < n - \delta < x < n + \delta < n$, ($0 < \delta < \min\{ n - 1 - \delta_1\ n - \delta\}$ for every y is $(x - \delta, x + \delta)$, $|g(y) - g(x)| = 0 < \in$.

This shows that g is continuous at x if x is not an integer.

(g) Let $g = [0, \infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ if $x > 0$, $g(0) = K$ where $K \in \mathbb{R}$. Then g is not continuous at 0.



The function $f(x) = 1/x\ (x > 0)$.

Let $x_n = \frac{1}{n}$ for $x \in N$, $\lim(x_n) = 0$ and

$\lim(g(x_n)) = \lim(n) = \infty$

Since $K < \infty$, $\lim(x_n) \neq K$. thus there is a sequence (x_n) in $[0, \infty)$ such that $\lim(x_n) = 0$ but $\lim g(x_n) \neq K$.

This implies that g is discontinuous at 0.

(h) The Dirichlet's function is defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } n \text{ is an irrational} \end{cases}$

This function was introduced by P.G.L. Dirichlet in 1829. this is also called the ruler function. We show that f is discontinuous at every point of $\mathbb{R}$.

If c is any number, we choose sequence (x_n) and (y_n)such that each x_n is a rational number and each y_n is an irrational number.

$c - \frac{1}{n} < x_n < c$ and $c - \frac{1}{n} < y_n < c$

So $0 < |x_n - c| < \frac{1}{n}$ and $0 < |y_n - c| < \frac{1}{n}$, by squeeze theorem $\lim(x_n) = c = \lim(y_n)$.

However for every n, $f(x_n) = 1$ and $f(y_n) = 0$ so that $\lim f(x_n)=1$ while $\lim f(y_n)=0$

Hence by 14.4.7 f is not continuous at c.

(i) The function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\frac{1}{x} & \text{if } x \neq 0, \end{cases}$$ is contunous at 0

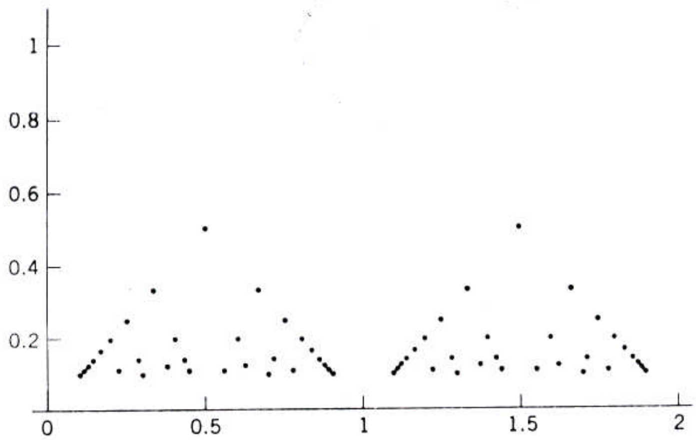
Reason: For all $x \in \mathbb{R}$, $|f(x)| \leq \left|x \sin \frac{1}{x}\right| \leq |x|$ If $\in > 0$ and $|x| < \in = \delta$,

$|f(x) - f(0)| = |x \sin y_n| \leq |x| < \in$ Hence f is continuous at 0.

(j) **Thomae's function:** Define $h : (0\ \infty) \to (0, \infty)$ as follows

If x is irrational h(x) = 0, h(0) = 1 and if $x \neq 0$ and is x is rational and $x = \frac{p}{q}$ where p, q are natural numbers without common factors other than ± 1, define $h(x) = \frac{1}{q}$. The function h is called Thomae function.

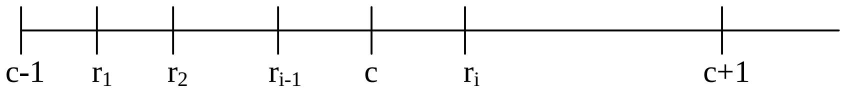
We show that h is continuous at c if c is any irrational and discontinuous at c, if c is any rational number.



Thomae's function.

Case (i) Let c be any irrational number and $\in > 0$. The number $q \in N$ such that $\frac{1}{q} < \in$ is finite. For each such q the number of rational numbers in $(c - 1, c + 1)$ is finite. Then the interval $(c - 1, c + 1)$ contains atmost a finite number of rationals $\frac{p}{q}$ in their simplist from with $\frac{1}{q} < \in$.

We may arrange them in an increasing order $r_1 < r_2 < \ldots\ldots\ldots < r_{i-1} < c < r_i$.



Choose $\delta > 0$ so that $(c - \delta, c + \delta) \subseteq (r_{i-1}, r_i)$ It is clear that for any rational number $\frac{p}{q}$ (in the simplest form) in $(c - \delta, c + \delta)$, $\frac{1}{q} < \in$; so that $|f(\frac{p}{q}) - f(c)| = |\frac{1}{q} - 0|$ $= \frac{1}{q} < \in$. Thus h is continuous at the irrational number c.

Also for any irrational number x in $(c - \delta, c + \delta)$, $f(x)=f(c)=0$ so $|f(x)–f(c)| = 0 < \in$

This gives $\in > 0$ there is $\delta > 0$ such that if $c - \delta < x < c + \delta$, $|f(x) - f(c)| < \in$

This implies that f is continuous at c.

(ii) Let c be any rational number, $c = \frac{p}{q}$, p, q natural numbers without common factors others than ± 1. We consider $\in$ satisfying $0 < \in < \frac{1}{q}$. For any $\delta > 0$, for any irrational number x such that $|x - \frac{p}{q}| < \delta$, $f(x) = 0$, $f(\frac{p}{q}) = \frac{1}{q}$ so that $|f(x) - f(\frac{p}{q})| = |0 - \frac{1}{q}| = \frac{1}{q} > \in$

Thus there is $\in > 0$ for which we cannot find δ satisfying the condition in the definition of continuity so f is not continuous at c.

(k) **Define g:** $\mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x+3 & \text{if } x \text{ is irrational} \end{cases}$

Final all x at which g is continuous.

Let c be a rational number. Then $g(c) = 2c$.

Choose sequence (x_n) of irrational numbers such that $c - \frac{1}{n} < x_n < c$.

Clearly $c - x_n < \frac{1}{n}$ and so $\lim (c - x_n) = 0$, So that $\lim (x_n) = c$

Since each x_n is irrational, $g(x_n) = x_n + 3$.

$\Rightarrow \lim g(x_n) = \text{liom}(x_n + 3) = \lim (x_n) + 3 = c + 3$

$\Rightarrow 2c = c + 3 \Rightarrow c = 3.$

Thus if g is continuous at c and c is rational thus c must be 3.

If c is irrational, choose again (x_n) to be a sequence of rational numbers such that $\lim (x_n)=0$.

If g is continuous at c, $\lim g(x_n) = g(c)$

$\Rightarrow 2c = c + 3 \Rightarrow c = 3$, this is impossible, since c is assumed to be irrational number.

Thus g is not continuous at any number, rational or irrational different from 3. When c = 3, g (c) = 6. So $|g(x) - g(3)| = |2x - 6|$ or $|x + 3 - 6|$

$= 2|x - 3|$ or $|x - 3|$ according as x is rational or irrational.

In either case, $|g(x) - g(3)| \le 2|x - 3|$. If $\in > 0$, $|g(x) - g(3)| < \in$ if $|x - 3| < \in/2$

Thus g is continuous at 3 and at no other value.

15.4.11 SAQs

(a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous at c and f(c) > 0 then show that there is $\delta > 0$ such that f(x) > 0 whenever $x \in V_\delta(c)$.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $Z(f) = \{x \in \mathbb{R} \,/\, f(x) = 0\}$, (x_n) is a sequence in Z(f) and $\lim (x_n) = x$, then show that $x \in Z(f)$.

(c) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le k|x - y|$ for all x, y and some k > 0. Show that f is continuous on $\mathbb{R}$.

(d) Define $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x \ne 2$. Does there exist $L \in \mathbb{R}$ such that if we set f(2)=L, f becomes a continuous function?

(e) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and f(r) = r if r is rational. Show that f(x) = x if $x \in \mathbb{R}$.

(f) Let $c \notin A$, c be a cluster point of A, $f : A \to \mathbb{R}$, $\lim\limits_{x \to C} f(x) = \ell$.

Define $F : A \cup \{c\} \to \mathbb{R}$ by F(x) = f(x) if $x \in A$, $F(c) = \ell$. Show that F is continuous at c.

15.5. Combinations of continuous functions:

Analogons to limits of functions and sequences we consider continuity of the sum, difference, product and multiple by a number as well.

We fix $A \subseteq \mathbb{R}$, $b \in \mathbb{R}$, $c \in A$, $f, g : A \to \mathbb{R}$.

15.5.1 Theorem: If f and g are continuous at c, then (a) f + g is continuous at c (b) f–g is continuous at c (c) b f is continuous at c for $b \in \mathbb{R}$.

Proof: If c is not a cluster point of A, (a), (b), (c) are trivially valid.

If c is a cluster point of A, then $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$.

There by 14.4.5, $\lim_{x\to c} (f + g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c) = (f + g)(c)$, hence (a) holds.

$\lim_{x\to c} (f - g)(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = f(c) - g(c) = (f - g)(c)$, hence (b) holds.

$\lim_{x\to c} (bf)(x) = \lim_{x\to c} b \,.\, f(x) = b \lim_{x\to c} f(x) = b.f(c) = (bf)(c)$, hence (c) holds.

SAQ 15.5.2 Prove theorem 15.5.1 directly from the definition.

15.5.3 Theorem: If f and g are defined on A, $c \in A$ and f, g are continuous at c then

(a) f g is continuous at c.

(b) If $g(x) \neq 0$ for $x \in A$ then $\frac{f}{g}$ is continuous at c.

Proof: If c is not a cluster point of A then the conclusion holds trivially.

Assume that c is a cluster point of A. From the hypothesis $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$.

By theorem 14 $\lim_{x\to c} (f g)(x) = \lim_{x\to c} f(x) \lim_{x\to c} g(x) = f(c)\, g(c) = (fg)(c)$ hence (a) holds.

When $g(x) \neq 0$ for $x \in A$, by theorem 14.

$$\lim_{x\to c} \left(\frac{f}{q}\right)(x) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)} = \left(\frac{f}{g}\right)(c) \text{, hence (b) holds.}$$

15.5.4 Corollary: Let $A \subset \mathbb{R}$, $c \in A$, $f_i : A \to \mathbb{R}$ be continuous at C for $1 \leq I \leq n$. Then (a) $f_1 + f_2 + \ldots + f_n$ is continuous at c. (b) $f_1 . f_2 \ldots f_n$ is continuous at c.

Proof: Use induction.

15.5.5 Corollary: (a) If f is a polynominal then f is continuous on $\mathbb{R}$.

(b) if f and g are polynomials and z (g) = $\{ \alpha_1, \alpha_2 \ldots. \alpha_n\}$ is the set of roots of $g(\alpha) = 0$, then $\frac{f}{g}$ is continuous an $\mathbb{R} - Z$ (g).

Proof: (a) Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ when $a_n \neq 0$ and $n \geq 0$ then $\otimes$ $f = a_0 + a_1 f_1 + a_2 f_1^2 + \ldots. + a_n f_1^n$ where $f_1(x) = x$ for all x.

Since the function f_1 is continuous on $\mathbb{R}$, by 15.5.2 $f_1, f_1^2, \ldots f_1^n$ is continuous.

(b) f and g are continuous on $\mathbb{R} - z(g) = \{x / g(x) \neq 0\}$ $g(x) \neq 0$ on $\mathbb{R} - Z(g)$. So by 15.5.3, $\frac{f}{g}$ is continuous on $\mathbb{R} - Z(g)$.

15.5.6 The sine function 'sin' is continuous on $\mathbb{R}$ we use the following facts : For all n, y, z in $\mathbb{R}$

(1) $|\sin z| \leq 1$ (ii) $|\cos z| \leq 1$ and (iii) $\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$

If $c \in \mathbb{R}$ then we have $|\sin x - \sin c| = \left| 2 \cos\left(\frac{x+c}{2}\right) \sin\left(\frac{x-c}{2}\right)\right|$

$$= 2 \left|\sin\left(\frac{x-c}{2}\right)\right| . \left|\cos\left(\frac{x+c}{2}\right)\right|$$

$$\leq 2 \frac{|x-c|}{2} . 1 = |x - c|.$$

For given $\in > 0$, we choose $\delta = \in$. If $|x - c| < \delta$ then $|\sin x - \sin c| \leq |x-c| < \delta = \in$ $\Rightarrow$ sin is continuous at c. Since $c \in \mathbb{R}$ is arbitary, it follows that sin is continuous on $\mathbb{R}$.

15.5.7 Theorem: Let $A \subseteq \mathbb{R}$, and $f : A \to \mathbb{R}$.

(a) If f is continuous at a point $c \in A$, then |f| is continuous at c.

(b) If f is continuous on A, then |f| is continuous on A.

Proof: (a) If f is continuous at c, given $\in > 0$ there is $\delta > 0$ such that $|f(x) - f(c)| < \in$ if $x \in A$ and $|x - c| < \delta$ since $||a| - |b|| \leq |a - b|$ for any real numbers a,b. We get $|1f(k)| - |f(c)|| \leq |f(x) - f(c)| < \in$ if $x \in A$ and $|x - c| < \delta$.

Since $\in > 0$ is arbitrary, it follows that |f| is continuous at c.

(b) If f is continuous on A, f is continuous at every c. $\in$.A.

$\Rightarrow$ |f| is continuous at every $c \in A$

$\Rightarrow$ |f| is continuous on A.

15.5.8 Corollary: Let f, g : $A \to \mathbb{R}$, $c \in A$ and f, g be continuous at c. Then H(x) = max {f(x), g(x) } for $x \in A$ and h(x) = min {f(x), g(x) } for $x \in A$. Then H and h are continuous at c.

Proof: For any a, b in $\mathbb{R}$.

max (a, b) = $\dfrac{(a+b)+|a-b|}{2}$ and min (a,b) = $\dfrac{(a+b)-|a-b|}{2}$

Hence for $x \in A$, H(x) = $\dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$

$$= \frac{(f+g)+|f-g|}{2}(x)$$

Since f and g are continuous at c, f + g, f – g hence |f – g|, and hence $\dfrac{(f+g)+|f-g|}{2}$ are continuous at c. So H is continuous at c. Similarly h is continuous at c.

15.5.9 Theorem: Let f : $A \to \mathbb{R}$ be continuous at $c \in A$, $f(A) \subseteq B$ and g : $B \to \mathbb{R}$ be continuous at f(c), Then the composite function h : $A \to \mathbb{R}$ defined by h(x) = (gof) (x) = g(f(x)) for $x \in A$ is continuous at c.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h = gof"'] & F(A) \subseteq B \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

Proof: Since $f(A) \subseteq B$, $f(c) \in B$. Let $\in > 0$.

Since g is continuous at f(c), there is $\beta > 0$ such that $Y \in B \cap V_\beta(f(c))$

$\Rightarrow |g(y) - g(f(c))| < \in$.

Since f is continuous at c, corresponding to $\beta > 0$, there is $\delta > 0$ such that $x \in A \cap V_\delta(c)$

$\Rightarrow |f(x) - f(c)| < \beta$

$\Rightarrow f(x) \in V_\beta(f(c)) \Rightarrow |g(f(x)) - g(f(c))| < \in$

$\Rightarrow |h(x) - h(c)| < \in$

Since β corresponds to $\in$ and $\delta > 0$ depends on β δ corresponds to $\in$. Since $\in > 0$ is arbitrary, it follows that h is continuous at c.

15.5.10 Corollary: If $f : A \to \mathbb{R}$ is continuous, $g : B \to \mathbb{R}$ is continuous and $f(A) \subseteq B$ then the composite function $h = g \circ f$ is continuous on A.

Proof: If $c \in A$, f is continuous at c and g is continuous at f(c), hence by theorem 15.5.9 h is continuous at c. This holds for every $c \in A$, so h is continuous on A.

15.5.11. Examples:

(i) If $f: A \to \mathbb{R}$, $f(x) \geq 0$ for $x \in A$ and c is a cluster point of A then $\lim\limits_{x\to c} f(x) = \ell \Rightarrow \lim\limits_{x\to c} \left(\sqrt{f}\right)(x) = \sqrt{\ell}$. Consequently it follows that if $f : A \to \mathbb{R}$ is continuous at $c \in A$, then $\sqrt{f}$ is continuous at c.

We can prove this result from 15.5.8 as follows. We make use of the fact that the function $g(x)=\sqrt{x}$ is continuous on $\mathbb{R}$. We know that if $f:A \to (0,\infty)$ is continuous at c and $g : (0, \infty) \to (0, \infty)$ is continuous at f(c) then gof is continuous.

Since $(gof)(x) = g(f(x)) = \sqrt{f(x)}$ it follows that $\sqrt{f}$ is continuous at c.

(ii) The function $g : [0,\infty)\to \mathbb{R}$ defined by

$g(x) = \sin \frac{1}{x}$ if $x \neq 0$ and $g(0) = 0$ is proved to be discontinuous at 0. we show that g is continuous on $(0,\infty)$.

$$\begin{tikzcd}
(0,\infty) \arrow[r, "f"] \arrow[rr, bend right, "g=hof"'] & \mathbb{R} \arrow[r, "h"] & \mathbb{R}
\end{tikzcd}$$

The function $f(x) = \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$.

The function h(x) = sin x is continuous on $\mathbb{R}$. Also f(0,∞) $\subseteq \mathbb{R}$. Hence h of is continuous on (0,∞). For x ∈ (0,∞), (h o f) (x) = h (f(x)) = sin $\frac{1}{x}$ = g(x). ∴so g is continuous on (0,∞)

(iii)Define F: $\mathbb{R} \to \mathbb{R}$ by f(x) = x + 1 and g: $\mathbb{R} \to \mathbb{R}$ by g(1) = 0 and g(x) = 2 if x ≠ 1.
f(x) = 1 if x = 0 so (gof) (0) = g(f(0)) = g(1) = 0.
If x ≠ 0, (gof) (x) = g(f(x)) = g(x+1) = 2
So (gof) (x) = 0 if x = 0 and 2 if x ≠ 0.
⇒ $\lim\limits_{x\to 0}$ (gof) (x) = 2 $\lim\limits_{x\to 0}$ (gof) (x) = (gof) (0) Hence gof is continuous at 0.

15.5.12 Examples: We call a function F: $\mathbb{R} \to \mathbb{R}$ additive if f(x+y) = f(x) + f(y) for all x,y in $\mathbb{R}$. We proved in 14.5.14 that if x if rational, then f(x) = x.f(1) and that if $\lim\limits_{x\to 0}$ f(x) = L then L = 0

(a) We show that if f is continuous at some x_0 then f is continuous on $\mathbb{R}$.
If f is continuous at x_0, given ∈ > 0, there is δ >0 such that $|f(x) - f(x_0)| < \in$ if $|x - x_0| < \delta$
⇒ $|f(x- x_0)| = |f(x) - f(x_0)| < \in$ if $|x - x_0| < \delta$
⇒ $|f(y)| < \in$ if $|y| < \delta$. ⇒ f is continuous at 0, since f(0) = 0.
Let x ∈ $\mathbb{R}$ and lim (x_n) = x, then lim $(x_n - x)$ = 0. Since f is continuous at 0, then lim $f(x_n - x)$ = 0.
⇒ lim $(f(x_n) - f(x))$ = 0 ⇒ lim $f(x_n)$ = f (x)
Hence f is continuous at 0.
(b) We show that if f is continuous on $\mathbb{R}$ and f is additive then f(x) = x. f(1) for all x∈$\mathbb{R}$.
We know that f(x) = x f(1) if x is rational.

If x is any irrational number, we choose a sequence (x_n) of rational numbers such that lim (x_n) = x . Since f is continuous at x, it follows that
f(x) = lim ($f(x_n)$) = lim x_n f(1) lim (x_n) = f(1).x.

15.5.13 SAQ Determine the points of continuity of

(i) $f(x) = \dfrac{x^2+2x+1}{x^2+1}$ ($x \in \mathbb{R}$) (ii) $g(x) = \cos\sqrt{1+x^2}$ ($x \in \mathbb{R}$)

15.5.14 SAQ Show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is discontinuous while |f| is continuous at every point of $\mathbb{R}$.

15.5.15 SAQ Show that if f and g are continuous on $\mathbb{R}$ and f(r) = g(r) for every rational number 'r' then f = g.

15.5.16 SAQ Show that if f is continuous on $\mathbb{R}$ and $f\left(\dfrac{m}{2^n}\right) = 0$ for every integer m and natural number n then f = 0.

15.6. Continuous functions on an interval :

In this section we concentrate on the properties of the range of a continuous function on an interval. We derive the famous boundedness theorem, the maximum minimum theorem the intermediate value theorem and finally end up with the presentation of intervals theorem. We recall the following definitions.

15.6.1 Definition: $f: A \to \mathbb{R}$ is said to be bounded on A if there is $M \in \mathbb{R}$ such that $0 \le |f(x)| \le M$ for all $x \in A$.

15.6.2. Remark: (1) $f: A \to \mathbb{R}$ is unbounded on A if for no M in $\mathbb{R}$ satisfies $|f(x)| \le M$ for all $x \in A$.

This happens if and only if for any M > 0 there is at least one x depending on M so that $|f(x)| > M$.

(2) A continuous function is not necessarily bounded. For example if A is any unbounded interval say $[0,\infty)$ and f(x) = x for $x \in [0,\infty)$, then f is not bounded on $[0, \infty)$ because if M > 0, $M + 1 \in [0, \infty)$ and f (M + 1) = M + 1 > M.

(3) The function $f(x) = \dfrac{1}{x}$ is continuous on (0,1] but not bounded.

(4) The function $f(x) = \frac{1}{x}$ if x > 0, f (0) = 0 is defined on [0, 1] but unbounded. Also f is not continuous at 0.

(5) Boundedness of f on A is graphically explained in an equivalent way as follows.
$f: A \to \mathbb{R}$ is bounded iff the graph of f lies in the horizontal strip bounded by the horizontal line $y = \pm M$ for some M > 0.

15.6.3. Boundednesss Theorem: Let I = [a, b] be a closed and bounded interval and $f: I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Proof: The proof is by contradiction. Suppose $f : I \to \mathbb{R}$ is continuous but not bounded on I. if $n \in N$ then there is $x_n \in I$ such that $|f(x_n)| \geq n$. The sequence (x_n) is choosen from I which is a bounded set, so (x_n) is bounded. By Bolzano – weierstrass theorem there is a convergent sub sequence (x_{n_k})

Let $x^1 = \lim x_{n_k}$. Since $a \leq x_{n_k} \leq b$ for every n_k, it follows that $a \leq = \lim x_{n_k} \leq b$

Hence $a \leq x^1 \leq b$. Since f is continuous at $x^1 \in$ I., it follows that $\lim f(x_{n_k}) = f(x^1)$

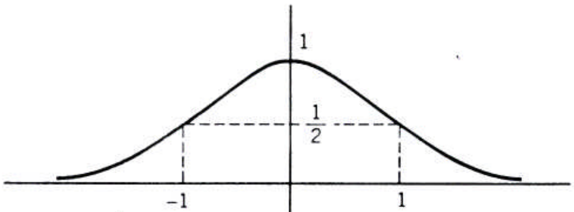
$\Rightarrow \left\{f\left(x_{n_k}\right)\right\}$ must be bounded. But this is a contradiction. Since $\left|f\left(x_{n_k}\right)\right| > n_k > k$ for $k \in \mathbb{N}$.

Since $|f(x_{n_k})| \geq k$, it follows that $|f(x^1)| = \lim |f(x_{n_k})| \geq k$ for every $k \in$ N. This is impossible since $k \in \mathbb{N}$, $|f(x^1)| \leq k$. Therefore f is bounded on I.

Remark: The boundedness of the interval I and the inclusion of the end points as members in I are essential as is evident from the examples 15.6.2(2) and (3).

Example; $f(x) = \frac{1}{x^2+1}$ for $x \in \mathbb{R}$. clearly f is continuous on $\mathbb{R}$. it is easy to see that if $I_1 = (-1, 1)$ the $f(I_1) = (½ , 1]$, which is not open interval.

Also if $I_2 = [0, \infty)$ then $f(I_2) = (0,1]$ which is not a closed interval.



Graph of $f(x) = 1/(x^2+1)$ $(x \in \mathbb{R})$

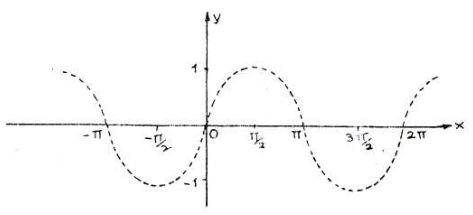
15.6.4 Maximum – Minimum Theorem:

Definition: Let $A \subseteq \mathbb{R}$ and let $F: A \to \mathbb{R}$. We say that f has an absolute maximum on A if there is a point $x^0 \in A$ such that $f(x^0) \geq f(x)$ for all x in A.

We say that f has an absolute minimum on A. if there is a point $x_0 \in A$ such that $f(x_0) \leq f(x)$ for all $x \in A$. We say that x^0 is an absolute maximum point for f on A and that x_0 is an absolute minimum point for f on A.

15.6.5. Remarks: (i) The geometrical interpretation is that at an absolute maximum x^0 of the graph of f lies below the horizontal line $y = f(x^0)$

(ii) A function may attain absolute maximum or minimum at several values of x.
For example we consider the sine function and the parabola $y = x^2$ defined on $\mathbb{R}$.



y = Sin x

y = sin x absolute maximum = 1, attained at $\frac{\Pi}{2}, \frac{5\Pi}{2}, \ldots.$

Absolute minimum = -1, attained at $\frac{3\Pi}{2}, \frac{7\Pi}{2} \ldots.$

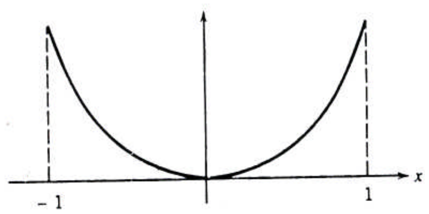


Figure 5.3.2 The function $g(x) = x^2$ $(|x| \leq 1)$.

$y = x^2$, absolute maximum in [-1,1] attained at -1,1.

Warning: Theorem 15.6.4 guarantees that if $f: [a, b] \to \mathbb{R}$ is continuous then $f([a, b]) = \{ f(x) / a \leq x \leq b\}$ is bounded and there exists c, d in [a, b] such that $f(c) \leq f(x) \leq f(d)$ for all x in [a, b]. There is no guarantee that $c < d$ or $a = c$ and $b = d$.
However $f(c) \leq f(a) \leq f(d)$ and $f(c) \leq f(b) \leq f(d)$

We will prove soon that for every y such that $f(c) \leq y \leq f(d)$ there is x in [a, b] such that $f(x) = y$.

Thus $[f(c), f(d)] \subseteq f([a,b]) \subseteq [f(c), f(d)]$ so that $f([a,b]) = [f(c), f(d)]$.

There is no guarantee that f(a) = inf f[a,b] or f(b) = sup f[a,b].

(iii) We note that a continuous function on a set A does not necessarily have an absolute maximum or an absolute minimum on the set. For example $f(x) = \frac{1}{x}$ has neither an absolute maximum nor an absolute minimum on the set $A = (0, \infty)$ There can be no absolute maximum for f on A, since f is not bounded on A and there is no point at which f attains the value = inf $\{ f(x) / x \in A\}$

The same function has neither absolute maximum nor absolute minimum when it is restricted to the set (0, 1) while it has both an absolute maximum at x = 1 and an absolute minimum at x = 2 when it is restricted to the set [1, 2].

In addition $f(x) = \frac{1}{x}$ has an absolute maximum at x = 1 but no absolute minimum when restricted to the set $[1, \infty)$.

15.6.6 Maximum – Minimum Theorem:

Let I = [a, b] be a closed bounded interval and $f: I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

Proof: Consider the range of f, $f(I) = \{ f(x) / x \in I\}$.

By the boundedness theorem, f (I) is a bounded subset of $\mathbb{R}$, hence f has supremum and infimum.

Let $s^* = \sup f(I)$ and $s_* = \inf f(I)$. We claim that there are points x^* and x_* in I such that $s^* = f(x^*)$ and $s_* = f(x_*)$ Since $s^* = \sup f(I)$, if $n \in N$ $s^* - \frac{1}{n}$ is not an upper bound of the set f(I). Consequently there is a number $x_n \in I$ such that $s^* - \frac{1}{n} < f(x_n) \leq s^* \; \forall \; n \in \mathbb{N}$.

Since I is bounded, the sequence (x_n) is bounded. Therefore, by Bolzano – wirestress theorem, there is a sub sequence (x_{n_r}) of (x_n) that converges to some number x^*. since

$a \le (x_{n_r}) \le b$ for every n_r

$a \le x^* \le b$ if $x^* \in$ I. Since f is continuous at x^* and $\lim x_{n_r} = x^*$, $\lim f(x_{n_r}) = f(x^*)$

Since $s^* - \frac{1}{n_r} < f(x_{n_r}) \le s^*$ for all $r \in \mathbb{N}$

We conclude from sequeze theorem for sequences that $\lim\limits_{n_r} f(x_{n_r}) = s^*$. Therefore we have $f(x^*) = \lim (f(x_{n_r})) = s^* = \sup f(I)$. Hence x^* is an absolute maximum point of f in I.

The proof for absolute minimum is similar.

15.6.7 Location of roots:

It is of importance to find the points of intersection of the graph of a continuous function f with the x – axis. Any such x is a solution of $f(x) = 0$. Thus the question before us is "when does a function f become zero?". We prove here a fundamental theorem called the intermediate value (also called location of roots) theorem. Our proof is by he method of bisection in Numerical Analysis.

15.6.9 Location of roots theorem:

Let $I = [a, b]$ and let $f: I \to \mathbb{R}$ be continuous on I. if $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exist a member $c \in (a, b)$ such that $f(c) = 0$.

Proof: Assume that $f(a) < 0 < f(b)$

We will generate a sequence of intervals by successive bisections.

Let $I_1 = [a_1, b_1]$ where $a_1 = a_1$, $b_1 = b$ and let P_1 be the mid point, $P_1 = (½)(a_1+b_1)$. If $f(P_1)=0$ we take $c = P_1$ and we are done. If $f(P_1) \neq 0$ then either $f(P_1) > 0$ or $f(P_1) < 0$

If $f(P_1) > 0$, $f(a_1) < 0 < f(P_1)$. We set $a_2 = a_1$, $b_2 = P_1$, while if $f(P_1) < 0$, $f(P_1) < 0 < f(b_1)$ so we set $a_2 = P_1$, $b_2 = b_1$. In wither case, the interval $I_2 = [a_2, b_2]$ satisfies

(i) $I_2 \subseteq I_1$ and (ii) $f(a_2) < 0 < f(b_2)$. Also $b_2 - a_2 = (½)(b_1 - a_1)$.

We continue the bisection process. Suppose that the intervals $I_1, I_2 \dots\dots\dots I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k)>0$ and we set $P_k = (½)(a_k + b_k)$

If f $(P_k) = 0$ we take $c = P_k$ and we are done. If f ($P_k) > 0$ the f $(a_k) < 0 <$ f (P_k). we set $a_{k+1} = a_k$, $b_{k+1} = b_k$.

While if $f(P_k) < 0$ we set $a_{k+1} = p_k$, $b_{k+1} = b_k$.

we write $I_{k+1} = [a_{k+1}\ b_{k+1}]$ then $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0$, f $(b_{k+1}) > 0$. Also

$(b_{k+1} - a_{k+1}) = ½ (b_k - a_k)$.

If this process terminates by locating a point p_n such that f $(P_n) = 0$ then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n = [a_n , b_n]$ such that for any $n \in \mathbb{N}$, we have f $(a_n) < 0 <$ f (b_n) and $b_n - a_n = \frac{b-a}{2^{n-1}}$. If follows from the nested interval property that there is a point $c \in I_n$ for all $n \in \mathbb{N}$.

Since $a_n \le c \le b_n$ for all $n \in \mathbb{N}$, we have $0 \le c - a_n \le b_n - a_n = \frac{b-a}{2^{n-1}}$ and

$0 \le b_n - c \le b_n - a_n = \frac{b-a}{2^{n-1}}$

By squeeze theorem it follows that lim $a_n = c =$ lim b_n

Since f is continuous at c we have lim f $(a_n) = f(c) =$ lim f (b_n)

Since f $(a_n) < 0$ for all $n \in \mathbb{N}$, f (c) = lim f($a_n) \le 0$

Since f $(b_n) \ge 0$ for all $n \in \mathbb{N}$, f (c) = lim $f(b_n) \ge 0$.

Thus we conclude that f (c) = 0. Consequently c is a root of f in [a, b].

Since f (a) < 0 = f(c), $a \neq c$. similarly $c \neq b$ so $c \in (a, b)$.

So far we have considered the domain of a continuous function to be a closed and bounded interval. There are other types of intervals not necessarily closed nor bounded. And interval may be unbounded, where one of the end points is $+ \infty$ or $- \infty$ an open interval or half open and bounded where one of the end points is excluded from the interval. Let us remember here that a sub set I of $\mathbb{R}$ with at least two elements is an interval whenever $\alpha \in I$, $\beta \in I$ and $\alpha < \beta \Rightarrow [\alpha , \beta] \subseteq I$.

15.6.9 Bolzano's Intermediate value Theorem: Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. if a, $b \in I$ and if $k \in \mathbb{R}$ satisfied f (a) < k < f (b), them there exist a points $c \in I$ between a and b such that f (c) = k.

Proof: Here I is an interval, not necessarily closed and bounded. We may assume that a < b and let g(x) = f(x)- k, then g(a) < 0 < g(b) Location of roots theorem is applied to [a, b] and g, there is a point c with a < c < b such that g (c) = 0 ⇒ f(c) – k = 0 ⇒ f (c) = k.

If b < a we apply the above technique to h (x) on [b, a] where h (x) = k – f(x). Clearly h (b) < 0 < h(a)

⇒ There is a point c with b < c < a such that h (c) = 0 ⇒ k – f (c) = 0 ⇒ f (c) = k.

15.6.10 Corollary: Let I = [a, b] be a closed, bounded interval and let f: I $\to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying inf f (I) ≤ k ≤ sup f(I); then there is a $c \in I$ such that f (c) = k.

Proof: From the Maximum – Minimum theorem there are points c^* and c_* in I such that inf f(I) = $f(c_*) \le k \le f(c^*)$ = sup f(I). Then by Bolzano's Theorem there is $c \in I$ such that f(c) = k.

15.6.11 Application: Suppose f: [0,1] → [0,1] is continuous. Then there is a x in [0,1] such that f(x) = x.

write g(x) = x – f(x) g is continuous on [0,1] since 0 ≤ f(0) ≤ 1 and o ≤ f(1) ≤ 1,

g(0) = 0 ~ f(0) ≤ 0 and g(1) = – f (1) ≥ 0. Thus g(0) ≤ 0 ≤ g(1). If g(o) = 0, f (0) = 0.

If g(1) = 0, f(1) = 0

If g(0) < 0 < g(1), by location of roots theorem, there is a c in (0,1) such that g(c) = 0.

For this c, f(c) =c.

15.6.12 Application of Bisection Theorem:

The equation $f(x) = xe^x - 2 = 0$ has a root c in [0,1], because f is continuous on this internal and f(0) = -2 < 0 and f(1) = e – 2 > 0. We construct the following table, where the sign of $f(P_n)$ determines the internal at the next step. The far right column is an upper bound on the error when P_n is used to appropriate the root c, because we have $|P_n - c| \le \frac{1}{2}\ (b_n - a_n) = \frac{1}{2^n}$.

We will find an approximation P_n with error less the 10^{-2}

n	**a_n**	**b_n**	**P_n**	**$f(P_n)$**	**½ $(b_n$-$a_n)$**
1	0	1	0.5	-1.176	0.5
2	0.5	1	0.75	-0.412	0.25
3	0.75	1	0.875	+0.099	0.125
4	0.75	0.875	0.8125	-0.169	0.0625
5	0.8125	0.875	0.84375	-0.0382	0.03125
6	0.84375	0.875	0.859375	+0.0296	0.015620
7	0.84375	0.859375	0.8515625	-	0.0078125

We have stopped at n = 7, obtaining c = P_7 = 0.8515625 with error less than 0.0078125. Thus is the first step in which the error is less than 10^{-2}. The decimal places values of P_7 past the second place cannot be taken seriously, but we can conclude that 0.843 < c < 0.860.

Let us examine how the change of sign takes place

Internal	Sign position
$[P_2\ P_3]$	$f(P_2) < 0 < f(P_3)$
$[P_3\ P_4]$	$f(P_3) > 0 > f(P_4)$
$[P_5\ P_6]$	$f(P_5) < 0 < f(P_6)$
$[P_6\ P_7]$	$f(P_6) < 0 < f(P_7)$

15.6.13 Theorem: Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous. Then the set $f(I) = \{f(x)/x \in I\}$ is a closed bounded interval.

Proof: Let m = inf f(I) and M = sup f(I), then from the Maximum – minimum Theorem, m and M belong to f(I). Moreover we have $f(I) \subseteq [m.M]$. If k is any element of [m, M] then by 15.6.10 there is a point $c \in I$ such that f(c) = k. Hence $k \in f(I)$ and we conclude that $[m, M] \subseteq f(I)$ $\therefore$ f(I) = [m, M] which is closed and bounded interval.

15.6.14. Preservation of Intervals Theorem:

Let I be an interval and let $f: I \to \mathbb{R}$ be continuous. Then the set f(I) is an interval.

Proof: Here I could be any interval, not necessarily of the form [a, b]. To show that f (I) is an interval, it is enough to prove that either f(I) is a singletion set or $\alpha \in f(I)$, $\beta \in f(I)$, $\alpha < \beta \Rightarrow [\alpha, \beta] \subseteq f(I)$.

If f (I) has only one element say k then f(I) = {k} = [k, k].
Assume that f(I) has at least two elements.
Let $\alpha, \beta \in f(I)$ with $\alpha < \beta$ then there exists a, b in I such that $\alpha == f(a)$ and $\beta = f(b)$.

Further it follows from 15.6.9 that if $k \in [\alpha, \beta]$ then there is $c \in I$ such that $f(c) = k \in f(I)$. There fore $[\alpha, \beta] \subseteq f(I)$, showing f(I) is an interval.

15.6.15 Example: Let I = [a, b], $f: I \to \mathbb{R}$ be continuous on I. If for every $x \in I$ there is a $y \in I$ such that $|f(y)| \le \frac{1}{2}|f(x)|$; show that there is c in I such that f(c) = 0.

Solution: Choose $x_0 \in I$. There is $(x_n) \subset I$ such that $|f(x_n)| \le \frac{1}{2^n}|f(x_0)|$ for every n. Since $a \le x_n \le b$ for every n, (x_n) is bounded. By Bolzano – Weierstross theorem there is a convergent sub sequence (x_{n_k}) of (x_n). If $x = \lim x_{n_k}$, $a \le r_{n_k} \le b$ for every $n_k \Rightarrow a \le x \le b$.
Since f is continuous and $\lim x_{n_k} = x$, $\lim f(x_{n_k}) = f(x)$. Hence $|f(x)| = \lim |f(x_{n_k})|$.

Since $|f(x_{n_k})| \le \lim \frac{1}{2^{n_k}}|f(x_0)| \le \frac{1}{2^k}|f(x_0)|$ for every k.

$|f(x)| = \lim |f(x_{n_k})| \le \lim \frac{1}{2^k}|f(x_0)| = 0.$

Since $0 \le |f(x)| \le 0$. it follows that f(x) = 0.

15.6.16 Short answer questions:

a) If $f: [a, b] \to (0, \infty)$ is continuous, show that there is $\alpha > 0$ such that $f(x) \ge \alpha$. $\forall x \in [a,b]$

b) Let I = [a, b], $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$ be continuous on I. show that the set $E = \{x \in I / f(x) = g(x)\}$ has the property that $(x_n) \subseteq E$ and $x_n \to x_0 \Rightarrow x_0 \in E$.

c) Show that the polynomial $P(x) = x^4 + 7x^3 - 9$ has at least tow real roots. Locate them.

d) Show that a polynomial with real coefficients and odd degree has a real root.

e) Let $f: [a, b] \to \mathbb{R}$ be continuous $f(a) < 0 < f(b)$ and $W = \{ x \in [a, b] / f(x) < 0\}$. If $w = \sup W$, show that $f(w) = 0$.

f) Examine which open (closed) intervals are mapped by $f(x) = x^2$ ($x \in \mathbb{R}$) onto open (closed) intervals.

g) If $f: [0,1] \to \mathbb{R}$ is continuous and has only rational values, show that f must be a constant function.

15.7 Uniform continuity

15.7.1 Example: we know that the function g defined on $(0, \infty)$ by $g(x) = \dfrac{1}{x}$ is continuous, $\lim\limits_{x\to 0} g(x) = \infty$ and $\lim\limits_{x\to 0} g(x) = 0$.

If $u \in (0,\infty)$, $g(x) - g(u) = \dfrac{1}{x} - \dfrac{1}{u} = \dfrac{u-x}{ux}$ If $|x - u| < \dfrac{u}{2}$, $u - \dfrac{u}{2} < x < u + \dfrac{u}{2}$

$\Rightarrow \dfrac{u}{2} < x < \dfrac{3u}{2} \Rightarrow \dfrac{2}{3u} < \dfrac{1}{x} < \dfrac{2}{u}$. $\Rightarrow |g(x) - g(u)| \le \dfrac{|x-u|}{u}\,\dfrac{2}{u} = \dfrac{2}{u^2}|x - u|$

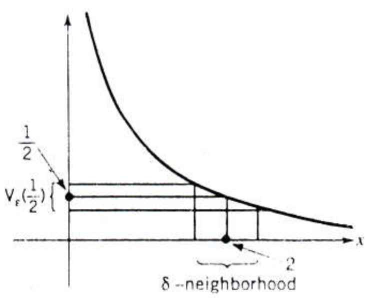
if $\in > o$, $\dfrac{2}{u^2}|x-u| < \in$ if $|x-u| < \dfrac{u^2}{2}$ Thus if $|x-u| < \min\left\{\dfrac{u}{2}, \text{and } \dfrac{\in u^2}{2}\right\}$, $|g(x)-g(u)| < \in$

We choose $\delta(\in) = \min\left\{\dfrac{u}{2}, \dfrac{\in u^2}{2}\right\}$. Thus when $u = \dfrac{1}{2}$, $\delta\left(\dfrac{1}{2}\right) = \min\left\{\dfrac{1}{u}, \dfrac{\in}{8}\right\}$

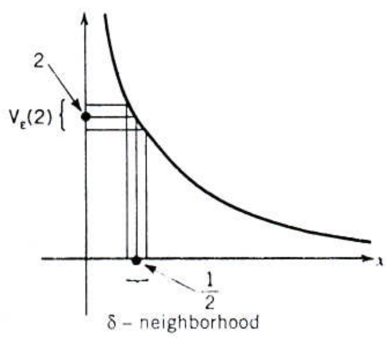
while when $u = 2$, $\delta(2) = \min\{1, 2\in\}$ We observe that the δ changes with $\in$, u.

Infact it is impossible to find $\delta > 0$ which does not depend upon u but depends only on $\in$.

This is clear from the figures.



$g(x) = 1/x \quad (x > 0)$.



$g(x) = 1/x \quad (x > 0)$.

15.7.2 Definition: Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. We say that f is uniformly continuous on A if for each $\in > 0$ there is a $\delta(\in) > 0$ such that if $x, y \in A$ are numbers satisfying $|x - y| < \delta(\in)$ then $|f(x) - f(y)| < \in$

15.7.3. Example: If $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for all x in $\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

For given $\in > 0$, choose $\delta = \in$ such that for $x_1, x_2 \in \mathbb{R}$ and $|x_1 - x_2| < \delta$,

$|f(x_1) - f(x_2)| = |x_1 - x_2| < \delta = \in$.

Since $x_1, x_2 \in \mathbb{R}$ are arbitrary, it follows that f is uniformly continuous on $\mathbb{R}$.

15.7.4 Theorem: It $f : A \to \mathbb{R}$ is uniformly continuous on A then it is continuous on A.

Proof: Let $\in > 0$ be given.

f is uniformly continuous on A $\Rightarrow$ there is $\delta > 0$ such that $x_1, x_2 \in A$ and $|x_1 - x_2| < \delta$, $|f(x_2) - f(x_1)| < \in$. Let $a \in A$. If $x \in A$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \in \Rightarrow$ f is continuous at $a \in A$.

since $a \in A$ is arbitrary. f is continuous on A.

Note: As is evident from 15.7.1, the converse of the above theorem is not true. i.e. if f is continuous on A then f need not be uniformly continuous on A.

15.7.5. We consider yet another example: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in A$. f is continuous but not uniformly continuous on $\mathbb{R}$.

That f is continuous on $\mathbb{R}$ is proved in 15.4.10 (d). If f were uniformly continuous on $\mathbb{R}$ and $\in > 0$ there must exist $\delta > 0$ such that if $x_1, x_2 \in \mathbb{R}$, $|x_1 - x_2| < \delta$ then $|f(x_1 - f(x_2)| < \in$

Let $x_1 > 0$ and $x_2 = x_1 + \frac{\delta}{2} \in \mathbb{R}$. Then $|x_1 - x_2| = \frac{\delta}{2} < \delta$.

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| = |x_1 - x_2|\,|x_1 + x_2| \\
&= \frac{\delta}{2}(2x_1 + \frac{\delta}{2}) = \delta x_1 + \frac{\delta^2}{4}
\end{aligned}
$$

$\Rightarrow \delta x_1 + \frac{\delta^2}{4} < \in \;\Rightarrow \delta x_1 < \in$. This must happen for all $x_1 > 0$. since $\delta > 0$, by Archimedean property there is $n \in \mathbb{N}$ such that $n\,\delta > \in$.

Therefore take $x_1 = n$ and $x_2 = x_1 + \frac{\delta}{2}$, $|f(x_1) - f(x_2)| > \in$

This implies that no $\delta > 0$ exists such that $x_1, x_2 \in A$ $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \in$.
Hence f is not uniformly continuous on $\mathbb{R}$.

15.7.6 Non-uniform continuity criteria:

Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. Then the following statements are equivalent.

(i) f is not uniformly continuous on A.

(ii) There exist $\in_0 > 0$ such that for every $\delta > 0$, there are points x_δ, u_δ in A such $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \in_0$

(iii) There is $\in_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim (x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \in_0$ for all $n \in \mathbb{N}$.

Example: We consider the function of $g(x) = \frac{1}{x}$ with domain $A = \{x/x \in \mathbb{R}$ and $x > 0\}$.

Let $x_n = \frac{1}{n}$ and $x_n = \frac{1}{2n}$, then $\lim (x_n - y_n) = \lim (\frac{1}{n} - \frac{1}{2n}) = \lim (\frac{1}{2n}) = 0$ and

$|g(x_n) - g(y_n)| = |n - 2n| = n$ for all $n \in \mathbb{N}$.
Hence $\lim g(x_n)$ does not exist.
$\Rightarrow$ g is not uniformly continuous an A.

15.7.7 Uniform continuity Theorem: Let I be a closed and bounded interval and let $f: I \to \mathbb{R}$ be continuous an I. Then f is uniformly continuous on I.

Proof: Suppose f is not uniformly continuous on I.

$\Rightarrow$ There is $\in_0 > 0$ and the sequences (x_n) and (y_n) in I such that $|x_n - y_n| < \frac{1}{n}$ and

$|f(x_n) - f(y_n)| \geq \in_0$ for all $n \in N$.
Since I is bounded, the sequence (x_n) is bounded. By Bolzam welirtross There is a sub sequence (x_{n_k}) of (x_n) that converges to an element $z \in I$.

Since $|x_{n_k} - y_{n_k}| \le \frac{1}{n_k} \le \frac{1}{k}$ and $\lim \frac{1}{k} = 0$, $\lim (x_{n_k} - y_{n_k}) = 0$. Since $\lim x_{n_k} = z$, $\lim u_{n_k} = \lim (u_{n_k} - x_{n_k} + x_{n_k}) = \lim (u_{n_k} - x_{n_k}) + \lim (x_{n_k}) = 0 + z = z \Rightarrow (u_{n_k})$ also converges to z. Since f is continuous at z, thus both the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ must converge to f(z).

But this is not possible since $|f(x_n) - f(x_n)| \ge \in_0$ for all $n \in N$. $\Rightarrow$ f is not continuous at the point $z \in I$. which is a contradiction. Hence f is uniformly continuous on I.

15.7.8. Theorem: If f and g are uniformly continuous on a sub set A of $\mathbb{R}$. then f + g. f – g and c f are uniformly continuous on A where c is any real number.

Proof: Let $\in > 0$ since f, g are uniformly continuous on $\mathbb{R}$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$|f(x) - f(y)| < \frac{\in}{2}$ if x, y belong to A and $|x - y| < \delta_1$ and $|g(x) - g(y)| < \frac{\in}{2}$ if x, y belong to A and $|x - y| < \delta_2$.

If $\delta(\in) = \min\{\delta_1, \delta_2\}$ then both the above inequalities hold good and hence if $x \in A$, $y \in A$ and $|x - y| < \delta$,

$|(f + g)(x) - (f + g)(y)| = |f(x) + g(x) - f(y) - g(y)|$

$= |f(x) - f(y) + g(x) - g(y)| \le |f(x) - f(y)| + |g(x) - g(y)| < \frac{\in}{2} + \frac{\in}{2} = \in.$

Hence f + g is uniformly continuous on A.

Uniform continuity of f – g can be proved similarly.

If c = 0, c f = 0 and the zero function is clearly uniformly continuous.

If $c \ne 0$ and $\in > 0$ there is $\delta > 0$ such that for $x \in A$ $y \in A$ and $|x - y| < \delta$,

$|f(x) - f(y)| < \frac{\in}{|c|} = \in.$

Hence c f is uniformly continuous.

15.7.9 Theorem: If f and g are uniformly continuous on A and are both bounded on A then f . g is uniformly continuous on A.

Proof: Since f and g are both bounded on A, there is M > o such that |f(x)| < M and |g(x)| < M for $x \in A$.

Also for any x, y in A, |(fg)(x) – (fg) (y)| = |f(x). g(x) – f(y). g(y)|

= |f(x) (g(x) – g(y)) + g(y) (f(x) – f(y))|

$\leq$ |f(x| |g(x) – g(y)| + |g(y)| |f(x) – f(y)| $\rightarrow$ (1)

Now if $\in > 0$ there exist $\delta_1, \delta_2 > 0$ such that $|f(x) - f(y)| < \frac{\in}{2M}$ if $x, y \in A$ and $|x-y| < \delta_1$

and $|g(x) - g(y)| < \frac{\in}{2M}$ if $x, y \in A$ and $|x-y| < \delta_2$

Let $\delta(\in) = \min\{\delta_1, \delta_2\}$.

From (i) for x, y in A with $|x - y| < \delta(\in)$.

$|(fg)(x) - (fg)(y)| < M.\frac{\in}{2M} + M.\frac{\in}{2M} = \in$.

Hence f g is uniformly continuous on A.

Remark: If f is uniformly continuous on a set A, there is no guarantee that f is bounded on A. The simplest example is the identity function f(x) = x on $\mathbb{R}$ However if A = [a, b] and f is uniformly continuous on A then f is continuous an [a, b] hence bounded.

15.7.11. Examples:

1. $f(x) = \frac{1}{x}$ is uniformly continuous on $[a,\infty)$ where a > 0.

   If $x \geq a$, $y \geq a$, $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{|x||y|} = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2}$.

   So if $\in > o$, $|f(x) - f(y)| < \in$ if $\frac{|x-y|}{a^2} < \in$, $|x - y| < \in a^2$.

   Since this holds for all x, y in $[a,\infty)$ with $|x - y| < \in a^2$, $|f(x) - f(y)| < \in$.

   Hence f is uniformly continuous on $[a, \infty)$.

2. $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ if a > o but not an $(0, \infty)$.

   Let $x \geq a$, $y \geq a$ and $\in > 0$.

$|f(x) - f(y)| = |\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{1}{x} - \frac{1}{y}|\ |\frac{1}{x} + \frac{1}{y}|$

$$= \frac{|x-y|}{xy}\left(\frac{1}{x} + \frac{1}{y}\right) \le \frac{2}{a}\ \frac{|x-y|}{a^2} < \in \quad \text{if } |x-y| < \frac{a^3}{2} \in.$$

Hence f is uniformly continuous on $[a, \infty)$ if $a > 0$.

If $\delta > 0$ and $n > \frac{1}{2^\delta}$, $\left|\frac{1}{n} - \frac{1}{2n}\right| = \frac{1}{2n} < \delta$. $\left|f\left(\frac{1}{n}\right)\right| = |n^2 - 4n^2| = 3n^2 > 6$ whenever

$0 < \in < 1$. So f is not uniformly continuous on $(0, \infty)$

3. $f(x) = \frac{1}{1+x^2}$ is uniformly continuous on $\mathbb{R}$.

Let x, y be real numbers and $\in > 0$.

$$|f(x) - f(y) = \frac{|x^2 - y^2|}{(1+x^2)(1+y^2)} = \frac{|x-y||x+y|}{(1+x^2)(1+y^2)}$$

$$\le |x-y| \frac{|x|}{1+x^2} . \frac{|y|}{1+y^2} \le |x-y| < \in \quad \text{if } |x-y| < \in$$

Hence f is uniformly continuous on $\mathbb{R}$.

15.8 Solution to SAQ's

15.4.11 (a) Let $\in = f(c)$. There is $\delta > 0$ such that $|f(x) - f(c) < \in$ if $|x - c| < \delta$.

$\Rightarrow f(c) - f(c) < f(x) < f(c) + f(c)$ if $x \in V_\delta(c) \Rightarrow f(x) > 0$ if $x \in V_\delta(c)$.

(b) Let (x_n) be a sequence in $Z(f) \Rightarrow f(x_n) = 0$

Since $\lim (x_n) = x$ and f is continuous at x, then $\lim f(x_n) = f(x)$.

since $f(x_n) = 0$ for every n, it follows that $f(x) = \lim f(x_n) = 0 \Rightarrow x \in z(f)$.

(c) Since $|f(x) - f(y)| < k|x - y|$ for all x, y, if $\in > o$ and $|x - y| < \frac{\in}{k}$,

$|f(x) - f(y)| < k . \frac{\in}{k} = \in$. Hence f is continuous at every y in $\mathbb{R}$. Follows that

f is continuous on $\mathbb{R}$.

(d) $F(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x+2)\,(x-2)}{x-2} = x+2$ for $x \neq 2$.

2 is a cluster point of $\mathbb{R} - \{2\}$.

Also $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}(x+2) = 4$.

Hence if we define $F(2) = 4$ and $F(x) = f(x)$ for $x \neq 2$ the function F is continuous at 2.

(e) Given $f(r) = r$ if r is rational and f is continuous on $\mathbb{R}$.

If c is a rational, choose (r_n) such that r_n is rational and $c - \dfrac{1}{n} < r_n < c$. Then $\lim r_n = c$. By continuity of f at c it follows that $f(c) = \lim f(r_n) = \lim r_n = c$. Hence $f(x) = x$ if $x \in \mathbb{R}$.

(f) We are given $f: A \to \mathbb{R}$, $c \notin A$, c is a cluster point of A and $\lim\limits_{x\to c} f(x) = \ell$.

Then the function $F: A \cup \{c\} \to \mathbb{R}$ is defined by $F(x) = f(x)$ if $x \in A$.

$= \ell$ if $x = c$.

We prove continuity of F on $A \cup \{c\}$ If $a \in A$ then since f is continuous at a, given $\in>0$ there is $\delta > 0$ such that $|f(x) - f(a)| < \in$ if $x \in A$ and $|x-a| < \delta$. This is true for every $\in > 0$ so F is continuous at a. Since $\lim\limits_{x\to c} f(x) = \ell$,

given $\in > 0$ there is $\delta > 0$ such that $0 < |x-c| < \delta$ and $x \in A$

$\Rightarrow |f(x) - \ell| < \in$.

If $x \in A \cup \{c\}$ $|F(x) - F(l)| = \begin{cases} 0 \text{ if } x = c \\ |f(x) - \ell| \text{ if } x \neq c \end{cases}$

So that if $|x - c| < \delta$ and $x \in A$ then

$|F(x) - F(\ell)| \leq |f(x) - \ell| < \in$

$\Rightarrow$ F is continuous at c.

15.5.2: Let $\in > 0$.

Since f is continuous at c, there is $\delta_1 > 0$ such that $x \in A$, $|x - c| < \delta_1$, then

$|f(x) - f(c)| < \dfrac{\in}{2}$.

Since g is continuous at c, there is $\delta_2 > 0$ such that $x \in A$, $|x - c| < \delta_2$, then $|g(x)-g(c)| < \frac{\in}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and $|x - c| < \delta$, $x \in A$. Then

$|(f + g)(x) - (f + g)(c)| = |f(x) - f(c) + g(x) - g(c)|$

$\leq |f(x) - f(c)| + |g(x) - g(c)| < \frac{\in}{2} + \frac{\in}{2} = \in$.

Hence f + g is continuous at c. Similarly f – g is continuous at c. We now prove that b f is continuous at c.

Let $\in > 0$ since f is continuous at c corresponding to $\frac{\in}{1+|b|}$ there is b > 0 such that

if $x \in A$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \frac{\in}{|b|+1}$

Let $x \in A$ and $|x - c| < \delta$.

$|(bf(x) - (bf)(c)| = |b(f(x) - f(c)| = |b|\,|f(x) - f(c)| < |b| \frac{\in}{|b|+1} < \in$.

Hence bf is continuous at c.

15.5.13

(i) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 1}$ $(x \in \mathbb{R})$

Let $f_1(x) = 1$, $f_2(x) = x$ then f_1, f_2 are continuous on $\mathbb{R}$.

$f(x) = \dfrac{(f_1 + f_2)^2(x)}{(f_1 + f_2^2)(x)}$

Since f_1, f_2 are continuous on $\mathbb{R}$, $f + f_2$ $(f_1 + f_2)^2$, f_2^2, $f_1 + f_2^2$ are continuous on $\mathbb{R}$.

Also $(f_1 + f_2^2)(x) \neq 0$ for all $x \in \mathbb{R}$ and hence f is continuous on $\mathbb{R}$.

(ii) $g(x) = \cos\sqrt{1+x^2}$ $(x \in \mathbb{R})$.

$f_1(x) = 1$ is continuous on $\mathbb{R}$.

$f_2(x) = x$ is continuous on $\mathbb{R}$.

So $f_1 + f_2^2$ is continuous on $\mathbb{R}$. Also $(f_1 + f_2^2)(x) \geq 0\ \forall\ x \in \mathbb{R}$

So $h(x) = \sqrt{(f_1 + f_2^2)}\,(x)$ is continuous on $\mathbb{R}$.

Also k(y) = cos y is continuous on $\mathbb{R}$.

Hence g(x) = cos $\sqrt{1+x^2}$ = (koh) (x) is continuous on $\mathbb{R}$.

15.5.14: Suppose c is rational and $x_n = c+\frac{\sqrt{2}}{n}$. Then x_n is irrational and $|x_n - c|=\frac{\sqrt{2}}{n}$

$\Rightarrow \lim |x_n - c| = 0. \Rightarrow \lim x_n = c$. But $\lim f(x_n) = -1$ while f(c) = 1. So $\lim f(x_n) \neq f(c)$.

Hence f is not continuous at c.

Suppose c is irrational. For each $n \in \mathbb{N}$, choose a rational number y_n such that

$c < y_n < c + \frac{1}{n}$. Then $\lim y_n = c$. $f(y_n) = 1$ for every $n \Rightarrow \lim f(y_n) = 1$, while f(c) = -1

So $\lim f(y_n) \neq f(c)$. Hence f is not continuous at c.

This shows that f is not continuous at any point.

However |f(x)| = 1 for all x.

So |f| is continuous on $\mathbb{R}$.

15.5.15: Let x be any irrational number.

For $n \in \mathbb{N}$ choose a rational number x_n such that $|x - x_n| < \frac{1}{n}$. Then $\lim x_n = x$ and $f(x_n) = g(x_n)$ for every $n \in \mathbb{N}$. Since f is continuous at x and g is continuous at x, f(x) = $\lim f(x_n) = \lim g(x_n)$ = g(n). Hence f(x) = g(x) for all x in $\mathbb{R}$.

15.5.16: Let f be continuous on $\mathbb{R}$ and $f\left(\frac{m}{2^n}\right) = 0$ for $m \in Z$ and $n \in \mathbb{N}$.

Then f(x) = 0 for every $x \in \mathbb{R}$.

Clearly f(0) = 0

Assume that x > 0. There is $m \in \mathbb{N}$ such that $0 < \frac{x}{m} < 1$. There is $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \frac{x}{m} < \frac{1}{2^{n-1}} \Rightarrow \frac{m_0}{2^n} < x < \frac{m_0+1}{2^n}$. This is true for every $m > m_0$. Thus for every $m > m_0$, $m \in N$ there is $x_m \in N$ such that $\frac{m}{2^{xm}} < x < \frac{m+1}{2^{xm}} \Rightarrow 0 \leq x - \frac{m}{2^{xm}} < \frac{1}{2^{xm}}$.

Hence $\lim \frac{m}{2^{xm}} = x$. Since f is continuous, $f(x) = \lim f\left(\frac{m}{2^{xm}}\right) = 0$. The proof is similar when x < 0. Hence $f(x) = 0 \ \forall \ x \in \mathbb{R}$.

SAQ 15.6.16

a) Since f: [a, b] $\to$ (0, ∞) is continuous, f([a,b]) is a closed and bounded interval [α, β] where α = inf {f(x) / a $\le$ x $\le$ b} and β = sup {f(x) / a $\le$ x $\le$ b}. The infimum is also attained so that there is $x_0 \in$ [a, b] such that $f(x_0) = \alpha$. Since $f(x_0) \in (0, \infty)$, $\alpha = f(x_0) > 0$. Since $f(x_0)$ = inf {f(x) / a $\le$ x $\le$ b}, it follows that $f(x) > f(x_0) = \alpha > 0$ for x $\in$ [a,b].

b) Let h(x) = f(x) – g(x) for x $\in$ I. h is continuous on I and

E = Z(h) = {x $\in$ I / h(x) = 0}. $(x_n) \subset E \subset I$ and $x_n \to x_0$.

$\Rightarrow a \le x_0 \le b$, since $a \le x_n \le b$ for every n, so $x_0 \in$ I. Since h is continuous on E and $\lim x_n = x_0$, $h(x_0) = \lim h(x_n) = 0$ since for every $n \in \mathbb{N}$, $h(x_n) = f(x_n) - g(x_n) = 0$, hence $x_0 \in$ E.

c) $P(x) = x^4 + 7x^3 - 9 = x^3 (x + 7) - 9 > 0$ if x < 0 and x + 7 < 0 then $x^3 (x + 7) > 9$ or x > 0, x + 7 > 0 and $x^3 (x + 7) > 9$

If x < -7, x^3 and x + 7 are both < 0. So try x = -8, P (-8) = $8^3 - 9 > 0$, P (-7) = -9<0

So one root lies in (-8, -7)

P(1) < 0, P(2) > 0 $\Rightarrow$ one root his in (1,2).

d) Let $P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n-1,\, x}{}^{2n-1-,}$ where $a_{2n-1} > 0$.

Then $\frac{P(x)}{x^{2n-1}} = \frac{a_0}{x^{2n-1}} + \frac{a_1}{x^{2n}} + \ldots + a_{2n-1}$.

$\Rightarrow \lim\limits_{x\to\infty} \frac{P(x)}{x^{2n-1}} = a_{2n-1} > 0$ Similarly, $\lim\limits_{x\to-\infty} P(x) = -\infty$

Hence there exist $\delta_1 > 0 > \delta_2$ such that $P(x) > 1 > -1 > P(y)$ if $x > \delta_1 > 0 > \delta_2 \ge y$.

Since P is continuous on $[\delta_2, \delta_1]$ and $P(\delta_1) > 0 > P(\delta_2)$ there is $c \in [\delta_2, \delta_1]$ such that P(c) = 0. The proof is similar when $a_{2n-1} < 0$.

e) x $\in$ W $\Rightarrow$ f(x) < 0. If n $\in$ N. w - $\frac{1}{n}$ is not an upper bound, hence there is $x_n \in$ W such that $w - \frac{1}{n} \le x_n \le W \Rightarrow f(x_n) < 0$ and $0 < w - x_n < \frac{1}{n}$ for every $n \in \mathbb{N}$

$\Rightarrow \lim x_n = w \Rightarrow \lim f(x_n) = f(w)$. Since $f(x_n) < 0$ for every n, $f(w) \leq 0$. If $f(w) < 0$ then $f(b) > 0 > f(w)$. So there is $x_0 \in [w, b]$ such that $f(x_0) = 0$

$\Rightarrow x_0 \in W \Rightarrow x_0 \leq w$ but $x_0 > w$. This is a contradiction. Hence $f(w) = 0$.

(f) Let $f(x) = x^2$

If $a > b > 0$ and $f(x) = x^2$ then $f((b,a)) = (b^2, a^2)$ and $f([a,b]) = [b^2, a^2]$

If $a< b<0$ then $f((a,b)) = (b^2, a^2)$ and $f([a,b]) = [b^2, a^2]$

If $a<0<b$ then $f((a,b)) = (0,c)$ and $f([a,b]) = [a,c]$ where $c = \max \{a^2, b^2\}$.

(g) If $f : [0,1] \to \mathbb{R}$ is continuous then the range of f namely $f([0,1])$ is a closed and bounded interval. An interval contains both rationals and irrationals if it is not a singleton. Since the hypothesis is that $f(x)$ is rational for all x, it follows that $f([0,1])$ is a singleton. In this case $f(x) = K$ for all $x \in [0,1]$, K being a rational number.

15.7.10

Since f is uniformly continuous on A. Corresponding to $\in = \frac{1}{2}$ there is $\delta > 0$ such that if $x \in A$, $y \in A$ and $|x - y| < \delta$. $|f(x) - f(y)| < \frac{1}{2}$.

$\Rightarrow | |f(x) - |f(y)| | < \frac{1}{2}$ if $x \in A$, $y \in A$ and $|x - y| < \delta$. If f is not bounded in A, then for every $n \in \mathbb{N}$,. there is $x_n \in A$ such that $|f(x_n)| > \text{man}\{n, f(x_{n-1}) + 1\}$ ………… (1)

Since $(x_n) \subseteq A$ and A is bounded there is a convergent subsequence (x_{n_k}) of (x_n). Write $x_{n_k} = y_k$. Since (y_k) converges, (y_k) is a Cauchy sequence. So corresponding to the δ obtained above there is a $K \in \mathbb{N}$ such that $|y_w - y_s| < \delta$ if $w > s \geq K$.

$\Rightarrow | |f(y_u) | - |f(y_s) | < \frac{1}{2}$ for $u > s \geq K$. From (1), $|f (x_n) - f(x_m) | > 1$ if $n > m$.

Hence $1 < | |f(y_u| | - |f (y_s) | | < \frac{1}{2}$. This is a contradiction. Hence f is bounded on A.

15.9 Summary:

The definition of a continuous function, some Algebraic properties of continuous functions and the structure of the range of a continuous function on a closed and bounded interval are studied in detail. Some important examples such as Dirichlet function, Thomae function are discussed. Discontinuity of a function and uniform continuity are also studied.

15.10 Technical Terms

Continuity, Discontinuities, Uniform continuity, Intermediate value property

15.11 Exercise 15.11 A

1. Let $a < b < c$. Suppose that f is continuous on [a,b], that g is continuous on [b,c] and that $f(b) = g(b)$. Define h on [a,c] by $h(x) = f(x)$ for $x \in [a,b]$ and $h(x) = g(x)$ for $x \in (b,c]$ prove that h is continuous on [a,c].

2. If $x \in \mathbb{R}$, we define [x] to be the greatest integer $n \in Z$ such that $n \le x$. the function $x \to [x]$ is called the greatest integer function. Determine the points of continuity of the following functions.

   (a) $g(x) = x\,[x]$ (b) $h(x) = [\sin x]$

   (c) $k(x)$ $\left[\frac{1}{x}\right]$, $(x \neq 0)$

3. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\in > 0$, there exists a nbd V_δ (c) of c such that if $x, y \in A \cap V_\delta$ 9c) then $|f(x) - f(y)| < \in$.

4. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > o$. show that there exists a nbd V_δ (c) of c such that of if $x \in V_\delta$ (c) then $f(x) > 0$.

5. Let $A \subseteq B \subseteq \mathbb{R}$ .let $f: B \to \mathbb{R}$ and let g be the restriction of f to A (ie, $g(x) = f(x)$ for $x \in A$).

   (a) If f is continuous at $c \in A$, show that g is continuous at c.

   (b) Show by an example that if g is continuous at c, it need not follow that f is continuous at c.

6. Let A = (0, ∞) and f: A → $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ n \text{ if } x \text{ is rational and } x = \dfrac{m}{n} \text{ in its simplest form} \end{cases}$$

Prove that f is unbounded in every open internal. Also prove that f is discontinuous at every point of A.

(**Hint:** Given a < b, show that there are infinity may n ∈ N such that n is the denominator of some rational number in (a, b).)

Exercise 15.11.B

1. Determine the points of continuity of the following functions.

   (a) $f(x) = \sqrt{x + \sqrt{x}}$ $(x \geq 0)$

   (b) $g(x) = \dfrac{\sqrt{1 + |\sin x|}}{x}$ $(x \neq 0)$

2. Show that f: A → $\mathbb{R}$ is continuous on A ⊆ $\mathbb{R}$ and if n ∈ $\mathbb{N}$ then the function defined by $f^n(x) = (f(x))^n$ for x ∈ A is continuous on A.

3. Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that (a) the sum f + g is continuous at c (ii) the product f g is continuous at c.

4. Determine the points of continuity of the function f(x) = x-[x], x ∈ R.

5. Let g be defined on $\mathbb{R}$ by g(1) = 0 and g(x) = 2 if x ≠ 1; and let f(x) = x + 1 for al x ∈ $\mathbb{R}$. Show that $\lim\limits_{x \to 0}$ (gof) (x) ≠ (gof) (c). why does this contradict Theorem 15.5.9 ?

6. Let f, g be defined on $\mathbb{R}$ and let c ∈ $\mathbb{R}$. Suppose that $\lim\limits_{x \to c} f = b$ and that g is continuous at b. show that $\lim\limits_{x \to c}$ (gof) (x) = g(b).

7. Given an example of a discontinuous function f on [0, 1] such that f^2 is continuous on [0, 1].

8. Let f: $\mathbb{R}$ → $\mathbb{R}$ be continuous on $\mathbb{R}$ and let P = {x ∈ $\mathbb{R}$ / f(x) > 0} If c ∈ P, show that there exists a nbd V_δ (c) ⊆ P.

9. If f and g are continuous on $\mathbb{R}$, let g = {x ∈ $\mathbb{R}$ / f(x) ≥ g(x)} If $(s_n) \subseteq S$ and lim (s_n) = s, show that s ∈ S.

10. Let g: $\mathbb{R} \to \mathbb{R}$ satisfy g(x + y) = g(x) . g(y) for all x, y in $\mathbb{R}$.

    (a) Show that g(0) = 0 or 1 and g(0) = 0 ⇒ g(x) = 0 for all x. Assume that g(0) ≠ 0.

    (b) Show that $g(-x) = g\left(\frac{1}{x}\right)$ if x ≠ 0

    (c) Show that $g(n) = (g(1))^n$ if n ∈ Z

    (d) Show that $g\left(\frac{m}{n}\right) = (g(1))^{m/n}$ if $\frac{m}{n}$ is rational

    (e) Show that if g is continuous at 0 then g is continuous at every c.

    (f) Show that if g(a) = 0 for some a ∈ $\mathbb{R}$ show that g(x) = 0 for all x.

11. Let f, g: $\mathbb{R} \to \mathbb{R}$ be continuous at a point c, and let h(x) = sup {f(x), g(x)} for x ∈ $\mathbb{R}$ show that h(x) = ½ (f(x) + g(x)) + ½ |f(x) – g(x)| for all x ∈ $\mathbb{R}$. Use this to show that h is continuous at c.

Exercise 15.11.c

1. Let f be continuous on [0, 1] to $\mathbb{R}$ nad such that f(0) = f(1). Prove that there exists a point c in [0, ½] such that f(c) = f(c + ½).
   [Hint: Consider g(x) = f(x) – f(x + ½)].

2. Show that the equation x = cos x has a solution in [0, ½].
   Use the Bisection method and a calculate to find an approximate solution of this equation, with error less then 10^{-3}.

3. Show that two function $f(x) = 2 \log x + \sqrt{x} - 2$ has root in [1, 2]. Use the Bisection method and find the root with error less than 10^{-2}

4. Show that the function f(x) = (x-1) (x-2) (x-3) (x-4) (x-5) has five roots in [0, 7]. If the Bisection method is applied on this interval, how many of the roots are located in this interval.

5. Let f: $\mathbb{R} \to \mathbb{R}$ be continuous

   (a) If $\lim_{x\to\infty} f(x) = 0$, show that f is bounded in [0, ∞).

(b) If $\lim_{x\to-\infty} f(x) = 0$, show that f is bounded in $(-\infty, 0]$.

(c) If (a) and (b) both hold show that f is bounded on $\mathbb{R}$.

(d) Show that when (c) holds f attains maximum and minimum.

(e) If $f(x) = \frac{1}{x^2+1}$, show that f satisfies above conditions.

Find inf f(x) and sup f(x)? Does f attain infimum on $\mathbb{R}$? Does f attain supremum on $\mathbb{R}$?

6. Show that the function $f(x) = \frac{1}{x^2+1}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

7. Use the non-uniform criterion to show that the following functions are not uniformly continuous on the given sets.

(a) $f(x) = x^2$, $A = [0, \infty)$

(b) $g(x) = \sin\left(\frac{1}{x}\right)$, $B = (0, \infty)$.

8. If f(x) = x and g(x) = sin x, show that f and g are uniformly continuous on $\mathbb{R}$, but their product f g is not uniformly continuous on $\mathbb{R}$.

9. Prove that if f and g are each uniformly continuous on $\mathbb{R}$ then the composite function fog is uniformly continuous on $\mathbb{R}$.

10. If f is uniformly continuous on $A \subseteq \mathbb{R}$ and $|f(x) \geq K > 0$ for all $x \in A$, show that $\frac{1}{f}$ is uniformly continuous on A.

Exercise 15.11.D.

(a) Using the formula $\cos A - \cos B = 2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{B-A}{2}\right)$ show that if $0 \leq x < y \leq ½$, $\cos y < \cos x$.

(b) Using the result that if f(c) < 0 and f is continuous at x_0 then there is $\delta > 0$ such that $f(x) < 0$ for $x \in V_\delta$ (c) prove that if $x_0^2 < \cos x_0$ for some $x_0 \in (0, \prod/2)$ then $x^2 < \cos x$ in some nbd $V_\delta(x_0) \leq (0, \frac{\Pi}{2})$.

(c) If $h(x) = \min\{x^2, \cos x\}$ $0 \leq x \leq \prod/2$, show that h is continuous on $[0, \frac{\Pi}{2}]$.

(d) If h attains minimum at x_0 show that $0 < x_0 \le \frac{\Pi}{2}$

Show that if $h(x_0) = \min\{h(x) / 0 \le x \le \frac{\pi}{2}\}$ and $h(x_0) = \cos x_0$ then $h(x) = \cos x$ in some nbd $V_\delta(x_0)$ in $(0, \frac{\pi}{2})$.

(e) Hence show that $x_0^2 \not< \cos x_0$

(f) Prove similarly that $x_0^2 \not> \cos x_0$.

(g) Finally conclude that if $h(x_0) = \min\{h(x) / 0 \le x \le \frac{\pi}{2}\}$ then $x_0^2 = \cos x_0$.

(h) By applying location of roots theorem to the function $f(x) = x^2 - \cos x$ on $[0, \frac{\pi}{2}]$. Prove that there is $x_0 \in (0, \frac{\pi}{2})$ so that $x_0^2 = \cos x_0$.

15.12 Model Examination Questions:

1. Let $f: A \to \mathbb{R}$ and $c \in A$. then show that f is continuous at c iff for every $\in > o$ there is a $\delta > 0$ such that for all $x \in A \cap V_\delta(c)$, $f(x) \in V_\in(f(c))$.
2. Show that the greatest integer function defined on $\mathbb{R}$ by $g(x) = [x] = n$ where n is an integer and $n \le x < n + 1$ is discontinuous at every integer values of x and continuous otherwise.
3. Let $g: [0, \infty) \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ if $x > 0$ and $g(0) = K$ where $K \in \mathbb{R}$.

   Then show that g is not continuous at 0.
4. Define $f: \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 2x \text{ if } x \text{ is rational} \\ x+3 \text{ if } x \text{ is irrational} \end{cases}$

   Find all x at which g is continuous.
5. Define $f(x) = \frac{x^2 - 4}{x - 2}$ for $x \neq 2$. Does there exist $L \in \mathbb{R}$ such that if we let $f(2) = L$, f becomes a continuous function?

6. If f and g are defined on A, $c \in A$ and f, g are continuous at c then show that (a) f g is continuous at c. (b) If $g(x) \neq 0$ for $x \in A$ then $\frac{f}{g}$ is continuous at c.

7. Let $f: A \to \mathbb{R}$ be continuous at $c \in A$. $f(A) \subseteq B$ and $g: B \to \mathbb{R}$ be continuous at f(c). Then show that the composite function $h: A \to \mathbb{R}$ defined by $g(x) = (gof)(x) = g(f(x))$ for $x \in A$ is continuous at c.

8. The function $g: [0, \infty) \to \mathbb{R}$ defined by $g(x) = \sin \frac{1}{x}$ if $x \neq 0$ and $g(0) = 0$ is proved to be discontinuous at 0. show that g is continuous on $(0, \infty)$.

9. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x+1$ and $g: \mathbb{R} \to \mathbb{R}$ by $g(1) = 0$ and $g(x) = 2$ if $x \neq 1$. Then show that gof is continuous at $x = 0$.

10. Determine the points of continuity of

(i) $f(x) = \dfrac{x^2 + 2x + 1}{x^2 + 1}$ $(x \in \mathbb{R})$ (ii) $g(x) \cos \sqrt{1 + x^2}$ $(x \in \mathbb{R})$

11. Show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} -1 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases},$$

is discontinuous while |f| is continuous at every point of $\mathbb{R}$.

12. Let I = [a, b] be a closed and bounded interval and $f: I \to \mathbb{R}$ be continuous on I. Then show that f is bounded on I.

13. State and prove Bolzano's Intermediate value theorem.

14. Suppose $f: [0, 1] \to [0, 1]$ is continuous. Then show that there is a x in [0, 1] such that $f(x) = x$.

15. Let I be an interval and let $f: I \to \mathbb{R}$ be continuous. Then show that the set f(I) is an interval.

16. Show that the polynomial $P(x) = x^4 + 7x^3 - 9$ has at least two real roots. Locate them.

17. Examine which open intervals are mapped by $f(x) = x^2$ $(x \in \mathbb{R})$ onto open intervals.

18. If $f: A \to \mathbb{R}$ is uniformly continuous on A, then show that it is continuous on A. Is the converse true? Justify your answer.

19. Show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$ where $a > 0$.

20. Let I be a closed and bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then show f is uniformly continuous on I.

Answers

15.11.A.2 (a) Continuous at x if x is not a non zero integer

(b) Continuous at x if $\sin x \notin \{0, 1\}$

(c) Continuous at x if $x \notin z$.

15.11.B. (1) $[0, \infty)$ (2) $\mathbb{R} - \{0\}$.

(4) {x} for x is not an integer}

(7) $f(x) = 1$ if $x \in Q$, $f(x) = -1$ if $x \notin \mathbb{R}$.

11.11.c. 5(e) inf $f(x) = 0$ sup $f(x) = 1$, both are not attained.

15.13 Model Practical Problem with solution:

Show that $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist but the function defined by $g(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $g(0) = 0$ is continuous at 0.

To show that $\lim_{x\to 0} f(x)$ does not exist and g is continuous at 0.

Definitions: (1) Let $E \subseteq \mathbb{R}$; c a cluster point of E $f: E \to \mathbb{R}$ and $l \in \mathbb{R}$.

We say that $f(x) \to l$ as $x \to c$, in symbols $\lim_{x\to c} f(x) = \ell$ if for every $\in > 0$ there corresponds $\delta_\in > 0$ such that if $x \in E$ and $o < |x - c| < \delta_\in$ there $|f(x) - \ell| < \in$.

(2) Let $E \subseteq \mathbb{R}$, $c \in E$. we say that $g: E \to \mathbb{R}$ is continuous at c if for every $\in > 0$ there corresponds $\delta_\in > 0$ such that if $x \in E$ and $|x\text{-}c| < \delta_\in$ hen $|f(x) - f(c)| < \in$.

Result to be used: (i) If $E \subseteq \mathbb{R}$, c limit point of E and $f: E \to \mathbb{R}$ $\lim_{x\to c} f(x) = l$ implies that for every $f(x_n) = \ell$.

(ii) **Archimedis Principle**: If $x \in \mathbb{R}\ \exists\ n_x \in N \ni n_x > x$.

Stepwise division of he solution to show that $\lim\limits_{x\to 0} f(x)$ does not exist.

(i) $\lim\limits_{n\to\infty} \frac{1}{n\pi} = \lim\limits_{n\to\infty} \frac{1}{(4n+1)\frac{\pi}{2}} = 0$

(ii) $\lim\limits_{n\to\infty} f\left(\frac{1}{n\pi}\right) \neq \lim\limits_{n\to\infty} f\left(\frac{1}{(4n+1)\frac{\pi}{2}}\right)$

Solution: (i) Let $\in > o$, $N_\in$ be any positive integer $\ni N_\in > \frac{1}{\pi \in}$.

Such $n \in \mathbb{N}$ and $n \geq N_\in$ then $\frac{1}{n} \leq \frac{1}{N_\in} < \Pi \in \Rightarrow \frac{1}{n\Pi} < \in$.

Since $o < \frac{1}{n\pi} < \in$ whence $n \geq N_\in$ $\lim \frac{1}{n\pi} = 0$.

(ii) Let $\in > 0$, $N_\in^1$ be any positive integer $\ni N_\in^1 > \frac{1}{2\pi \in}$.

If $n \in \mathbb{N}$ and $n \geq N_\in^1$ then $n > \frac{1}{2\pi \in}$

$\Rightarrow 4n + 1 > 4x > \frac{2}{\pi \in}$

$\Rightarrow (4n+1)\frac{\pi}{2} > \frac{1}{\in}$

$\Rightarrow \frac{1}{(4n+1)\frac{\pi}{2}} < \in$.

Since $0 < \frac{1}{(4n+1)\frac{\pi}{2}} < \in$ if $n \in \mathbb{N}$ and $n \geq N_\in^1$, $\lim\limits_{n} \frac{1}{(4n+1)\frac{\pi}{2}} = 0$.

Since $f\left(\frac{1}{n\pi}\right) = \sin n\pi = 0\ \forall\ n \in \mathbb{N}$,

$$|f(\frac{1}{n\pi}) - 0| = 0 < \in\ \forall \in > 0 \text{ and } n \in \mathbb{N}.$$

Hence $\lim\limits_{n} f\left(\frac{1}{n\pi}\right) = 0$.

Since $f\left(\dfrac{1}{(4n+1)\frac{\pi}{2}}\right)$ = sin (4n+1) $\dfrac{\Pi}{2}$ = 1 $\forall$ n $\in$ $\mathbb{N}$,

$\left| f\left(\dfrac{1}{(4n+1)\frac{\pi}{2}}\right) - 1\right| = 0 < \in$ for every $\in$ > o and $\forall$ n $\in$ $\mathbb{N}$

Hence $\lim\limits_{n} f\left(\dfrac{1}{(4n+1)\frac{\pi}{2}}\right) = 1$.

By result (1) $\lim\limits_{x \to 0} f(x)$ does not exist.

(ii) If x $\in$ $\mathbb{R}$ $|g(x)| = |x \sin \dfrac{1}{x}| \leq |x|$

If $\in$ > 0 then $\forall$ x $\in$ x $|g(x)| \leq |x| < \in$ whenever $|x| < \in$.

Hence g is continuous at 0.

K. SAMBASIVA RAO

Sir Isaac Newton 1643-1727

Isaac Newton was the greatest English mathematician of his generation. He laid the foundation for differential and integral calculus. His work on optics and gravitation make him one of the greatest scientists the world has known.

LESSON – 16

DIFFERENTIATION – I

16.1 Objective: The student is introduced to the notion of derivative, its dependence of continuity, linearity properties, the chain rule for the derivative, connection with maxima and minima, mean value theorems the intermediate value property for the derivative and finally a variety of applications through illustrations.

16.2 Structure:

This lesson contains the following components.

16.3 Introduction
16.4 Definition and elementary properties.
16.5 Caratheodory theorem and chain rule.
16.6 Mean value theorems
16.7 Examples and applications
16.8 Solutions to short answer questions (SAQs)
16.9 Summary
16.10 Technical terms
16.11 Exercises & Answers
16.12 Model examination questions
16.13 Model practical problem with solution.

16.3 Introduction :

A curve was generally described as locus of points satisfying some geometric condition and a taugent line was usually obtained through some geometric construction. We consider the "secants" i.e., chords through a point c in the domain of a function f which is usually an interval (containing c).

The slope of a sceant at c through a point (x,f(x)) where $x \in I$ is

$$\phi(x) = \frac{f(x) - f(c)}{x - c} \ (x \neq c, x \in I) \qquad \text{------ (1)}$$

If this secant converges to some $\ell \in \mathbb{R}$ as $x \to c$ we say that f has tangent at c, namely the line L through (c, f(c)) with slope l. Thus a tangent to a curve at a point is the "limiting position" of the secants through P. This intutive geometric consideration leads one to the definition of the derivative of a function. It was Pierre de Fermat who found a connection between the problem of finding the maxima and minima of a curve and the deviative. Sir Issac Newton discovered in the late 1660's a relation between the tangent lines to a curve and the velocity of a moving particle. The most significant observation made by Newton and Leibnitz independently was that the areas under curves could be calculated by reversing the differentiation process. This ultimately lead to the most coherent theory, now known as the differentiate and intyral calculus.

16.4 Definition and elementary properties:

16.4.1 Definition: Let $I \subseteq \mathbb{R}$ be an interval, $c \in I$ and $f : I \to \mathbb{R}$. We say that a real number L is a derivative of f at c if for every $\in > 0$ there is $\delta > 0$ such that if $x \in I$ and $0 < |x - c| < \delta$

$$\Rightarrow \left| \frac{f(x) - f(c)}{x - c} - L \right| < \in$$

In this case we say that f is differentiable at c and write $f^1(c)$ for L.

In other words we say that f is differentiable at c if the function $\phi : I - \{c\} \to \mathbb{R}$ defined by converges to a limit which we denote by $f'(c)$

If f: I $\rightarrow \mathbb{R}$ is differentiable at every point of I then we say that f is differentiable on I.

In this case the function defined on I that maps x to $f'(x)$ is called the derivative of f and is denoted by f'.

Notation: We also use Df and $\frac{df}{dx}$ for the derivative of f.

Thus (D f) (c) = $\left(\frac{df}{dx}\right)_{x=c}$ = f' (c)

6.4.2 (a) Example: If f: $\mathbb{R} \rightarrow \mathbb{R}$ is defined by f(x) = x^2 then for any c $\in \mathbb{R}$, and x $\neq$ 0

$$\frac{f(c+x)-f(c)}{x} = \frac{(x+c)^2-c^2}{x} = \frac{x(x+2c)}{x} = x + 2c$$

Hence $\lim\limits_{x\rightarrow 0} \frac{f(c+x)-f(c)}{x} = 2c$

So f is differentiable at every c $\in \mathbb{R}$ and the derivative $f' = \mathbb{R} \rightarrow \mathbb{R}$ is defined by f' (x) = 2x (x $\in \mathbb{R}$)

16.4.2(b) Example: If K $\in \mathbb{R}$ and f(x) = k for all x in $\mathbb{R}$ the constant function f is differentiable and f' (x) = 0 for x $\in \mathbb{R}$

If c $\in \mathbb{R}$ and x $\neq$ c

$$\frac{f(x)-f(c)}{x-c} = \frac{k-k}{x-c} = 0$$

$$\Rightarrow \lim_{x\rightarrow c} \frac{f(x)-f(c)}{x-c} = 0$$

Hence the constant function f is differentiable on $\mathbb{R}$ and f' (x) = 0 for x in $\mathbb{R}$.

Examples:

16.4.2 (c) h(x) = $\sqrt{x}$ for x > 0

if c > 0 $\frac{h(x)-h(c)}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c}$

and 0 < x $\neq$ c $= \frac{\left(\sqrt{x}-\sqrt{c}\right)}{\left(\sqrt{x}-\sqrt{c}\right)\left(\sqrt{x}+\sqrt{c}\right)}$

$$= \frac{1}{\sqrt{x}+\sqrt{c}}$$

Since $\lim\limits_{x\to c} \sqrt{x} = \sqrt{c}$,

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \frac{1}{2\sqrt{c}} \ (c > 0)$$

Hence h is differentiable at c and $h'(c) = \dfrac{1}{2\sqrt{c}}$

16.4.2(d) Example

$h(x) = \dfrac{1}{\sqrt{x}}$ (x > 0)

for c > 0 and x ≠ c, x > 0

$$\frac{h(x)-h(c)}{x-c} = \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{c}}}{x-c} = \frac{-\left(\sqrt{x}-\sqrt{c}\right)}{\sqrt{x}\sqrt{c}\ \ (x-c)}$$

As above $\lim\limits_{x\to c} \dfrac{h(x)-h(c)}{x-c} = \lim\limits_{x\to c} \dfrac{-1}{\sqrt{x}\sqrt{c}} \dfrac{\left(\sqrt{x}-\sqrt{c}\right)}{(x-c)}$

$$= \frac{-1}{\sqrt{c}} \lim_{x\to c} \frac{1}{\sqrt{x}} \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}}$$

$$= \frac{-1}{\sqrt{c}} \cdot \frac{1}{\sqrt{c}} \cdot \frac{1}{2\sqrt{c}}$$

$$= \frac{-1}{2} \frac{1}{c\sqrt{c}}$$

16.4.3 Theorem

If I is an internal and f: I $\to \mathbb{R}$ is differentiable at c $\in$ I then f is continuous at c.

Proof: For x $\in$ I, x ≠ c write $\phi(x) = \dfrac{f(x)-f(c)}{x-c}$

Then ϕ is defined on I – {c}, c is a cluster point of I – {c}

$\phi(x)(x-c) = f(x) - f(c)$

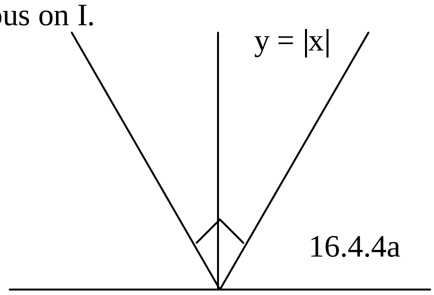
and $\lim\limits_{x\to c} \phi(x) = \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$

Also $\lim\limits_{x\to c} (x-c) = 0$

Hence $\lim\limits_{x\to c} f(x) - f(c) = \lim\limits_{x\to c} \phi(x)(x-c) = \lim\limits_{x\to c} \phi(x) \lim\limits_{x\to c} (x-c) = f'(c) \cdot 0 = 0$

Hence $\lim_{x \to c} f(x) = f(c)$ So f is continuous at c.

16.4.4 Corollary: If $f : I \to \mathbb{R}$ is differentiable on I then f is continuous on I.



graph of f(x) = |x|

Example: If f is continuous at c is not necessarily true that f is differentiable at c. Define $f(x) = |x|$ for $x \in \mathbb{R}$

Continuity of f at 0:

If $\in > 0$, $|f(x) - f(0)| = |x| < \in$ if $|x| < \delta(\in)$ where $\delta(\in) = \in$

Non differentiability at 0:

For $x \neq 0$ $\phi(x) = \dfrac{f(x) - f(0)}{x - 0} = \dfrac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

$\Rightarrow \lim_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0} = 1$ and $\lim_{x \to 0-} \dfrac{f(x) - f(0)}{x - 0} = -1$

Hence f is not differentiable at 0.

16.4.4. Example: Define $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

f is defined on $\mathbb{R}$ and for $x \neq 0$ $\dfrac{f(x) - f(0)}{x - 0} = \sin \dfrac{1}{x}$

We know that the function $\sin \dfrac{1}{x}$ $(x \neq 0)$ is not convergent at 0. Hence f is not differentiable at c. However for $x \in \mathbb{R}$ $|f(x) - f(0)| = |f(x)| \leq |x|$ so that if $\in > 0$ and $\delta(\in) = \in$, $|f(x) - f(0)| \leq |x|$ so that if $\in > 0$ and $\delta(\in) = \in$, $|f(x) - f(0)| \leq |x| < \in$ whenever $|x - 0| < \delta$. Hence f is continuous at 0.

16.4.5 Theorem: If $f: I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at c then f + g and f–g are differentiable at c and (i) $(f + g)'(c) = f'(c) + g'(c)$ and (ii) $(f - g)'(c) = f'(c) - g'(c)$

Proof: We first prove (i). Proof of (ii) is similar. By the differentiability of f and of g at c, We have $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c)$ and $\lim_{x \to c} \dfrac{g(x) - g(c)}{x - c} = g'(c)$

For $x \neq c, x \in I$ $\dfrac{(f+g)(x) - (f+g)(c)}{x - c} = \dfrac{f(x) - f(c)}{x - c} + \dfrac{g(x) - g(c)}{x - c}$

Hence $\lim\limits_{x \to c} \dfrac{(f+g)(x) - (f+g)(c)}{x - c} = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} + \lim\limits_{x \to c} \dfrac{g(x) - g(c)}{x - c}$

$= f'(c) + g'(c)$. (14.5.1(i))

Hence f + g is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$

16.4.6 Corollary:

If $n \in \mathbb{N}$ and $f_i: I \to \mathbb{R}$ is differentiable at $c \in I$ and $1 \leq i \leq n$ then $f_1 + \ldots . f_n$ is differentiable at c and $(f_1 = \ldots f_n)'(c) = f_1'(c) = \ldots f_n'(c)$

Proof: The proof is by induction on n. The statement is true when n = 2.

Assume that the statement is true for n let $f_1, \ldots . f_{n+1}$ be any (n + 1) functions on I each of which is differentiable at c. Write $g = f_2 + \ldots .. + f_{n+1}$ since f_i is differentiable at c for $2 \leq i \leq n+1$ by induction hypothesis g is differentiable at c and $g'(c) = (f_2 + \ldots .. + f_{n+1})'(c) = f_2'(c) + \ldots . + f_{n+1}'(c)$ Since f_1 and g are differentiable at c, by 16.4.5

$f_1 + g = f_1 + \ldots .. + f_{n+1}$, is differentiable at c and $(f_1 + \ldots .. + f_{n+1})'(c) = (f_1 + g)'(c) = f_1'(c) + g(c) = f_1'(c) + f_2'(c) + \ldots .. + f_{n+1}'(c)$

As the statement is valid for n+1 functions whenever it is valid for n functions the statement is valid for all n.

16.4.7 Corollary:

If $f_i : I \to \mathbb{R}$ is differentiable on I for $1 \leq i \leq n$ then $f_1 + \ldots \ldots . + f_n$ is differentiable on I and $(f_1 + \ldots .. + f_n)' = f_1' + \ldots . + f_n'$.

Proof: By the above corollary, differentiability of $f_1 + \ldots .. + f_n$ at every c in I follows from the differentiability of each f_i at c . Hence $f_1 + \ldots .. f_n$ is differentiable on I. Morever if $x \in I$ $(f_1 + \ldots .. f_n)'(x) = f_1'(x) + \ldots . + f_n'(x) = (f_1' + \ldots . + f_n')' =(x)$

Since this holds good for every $x \in I$, follows that $(f_1' + \ldots . + f_n')' = f_1' + \ldots . + f_n'$.

16.4.8 Theorem:

If f: I $\to$ $\mathbb{R}$ is differentiable at c $\in$ I and k $\in$ $\mathbb{R}$ then kf is differentiable at c and $(kf)'(c) = k\, f'(c)$

Proof: Write $\phi(x) = \dfrac{f(x)-f(c)}{x-c}$ for x $\in$ c.

Then $\lim\limits_{x\to c} \phi(x) = f'(c)$

$\Rightarrow \lim\limits_{x\to c} k\,\phi(x) = k\, f'(c)$

$\Rightarrow \lim\limits_{x\to c} \dfrac{k\,f(x)-k\,f(c)}{x-c} = k\, f'(c)$

$\Rightarrow$ (k f) is differentiable at c and $(k\,f)'(c) = k\, f'(c)$

16.4.9 Theorem:

If f: I $\to$ $\mathbb{R}$ and g: I $\to$ $\mathbb{R}$ differentiable at c $\in$ I then f g is differentiable at c and $(f\,g)'(c) = f(c)\, g'(c) + f'(c)'\, g(c)$

Proof: For x $\neq$ c in I

$$\frac{(f\;g)(x)-(f\;g)(c)}{x-c} = \frac{f(x)\,(g(x)-g(c))}{x-c} + \frac{g(c)\,(f(x)-f(c))}{x-c}$$

since f, g are differentiable at c,

$\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$ and $\lim\limits_{x\to c} \dfrac{g(x)-g(c)}{x-c} = g'(c)$

Since f is differentiable at c, f is continuous at c so

$\lim\limits_{x\to c} f(x) = f(c)$

Hence $\lim\limits_{x\to c} \dfrac{(f\;g)(x)-(f\;g)(c)}{x-c} = f(c)\, g^1(c) + g(c)\, f^1(c)$ (by theorem 14.5.2)

Hence f g is differentiable at c and $(f\,g)'(c) = f(c)\, g'(c) + g(c)\, f'(c)$.

16.4.10 Corollary:

If $n \in \mathbb{N}$ and $f_i : I \to \mathbb{R}$ is differentiable on I for $1 \le i \le n$ then $f_1 f_2 \ldots f_n$ is differentiable at c and $(f_1 \ldots f_n)'(c) = \sum_{i=1}^{n} \{f_1(c) f_2(c) \ldots f_i'(c) \ldots f_n(c)\}$

16.4.11 Proof: see exercise 8

Corollary: If $n \in \mathbb{N}$ and $f_i : I \to \mathbb{R}$ is differentiable on I for $1 \le i \le n$ then $f_1 f_2 \ldots f_n$ is differentiable on I.

Proof: See exercise 9

16.4.12 Corollary:

If $f: I \to \mathbb{R}$ is differentiable at $c \in I$ and $n \in \mathbb{N}$, f^n is differentiable at c and $(f^n)'(c) = n f^{n-1}(c) f'(c)$

Proof: Put $f_1 = f_2 = \ldots f_n = f$ in corollary 16.4.10

16.4.13 Theorem:

If $f: I \to \mathbb{R}$ is differentiable at $c \in I$ and $f(c) \neq 0$ then $\frac{1}{f}$ defined by $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$ is defined in $V_\delta(c) \cap I$ for some $\delta > 0$ and $\frac{1}{f}$ is differentiable at c and

$$\left(\frac{1}{f}\right)'(c) = \frac{-f'(c)}{(f(c))^2}$$

Proof: Since f is differentiable at c f is continuous at c since $f(c) \neq 0$ there is $\delta > 0$ such that if $x \in V_\delta(c) \cap I$, then $f(x) \neq 0$

So $\frac{1}{f}$ is defined in $V_\delta(c) \cap I$. If $x \in V_\delta(c) \cap I$ and $x \neq c$

$$\frac{\left(\frac{1}{f}\right)(x) - \left(\frac{1}{f}\right)(c)}{x - c} = \frac{f(c) - f(x)}{f(x)\, f(c)\, (x - c)}$$

$$= -\frac{f(x) - f(c)}{x - c} \cdot \frac{1}{f(x)\, f(c)}$$ Since f is continuous at c, $\lim_{x \to c} f(x) = f(c)$

Since f(c) ≠ 0 $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{f(c)}$ Also $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$

Hence $\lim_{x\to c} \frac{\left(\frac{1}{f}\right)(x) - \left(\frac{1}{f}\right)(c)}{x-c} = \lim_{x\to c} \frac{-\left(f(x)-f(c)\right)}{x-c} \frac{1}{f(x)} \quad \frac{1}{f(c)} = - \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.

$\lim_{x\to c} \frac{1}{f(x)} \quad \frac{1}{f(c)} = - \frac{f`(c)}{(f(c))^2}$

16.4.14 Theorem:

Suppose f: I → ℝ and g: I → ℝ are differentiable at c ∈ I and g(c) ≠ 0. Then the quotient function q = $\frac{f}{g}$, defined in a nbd of c is differentiable at c and

$$q'(c) = \left(\frac{f}{g}\right)'(c) = \frac{f'(c)\, g(c) - f(c)\, g'(c)}{(g(c))^2}$$

Proof: Since g(c) ≠ 0 there is a nbd V_δ (c) such that g(x) ≠ o if x ∈ V_δ (c) ∩ I.

So that q(x) = $\frac{f(x)}{g(x)}$ is meaningful for x ∈ V_δ (c) ∩ I.

For x ∈ V_δ (c) ∩ I,

$$\frac{q(x)-q(c)}{x-c} = \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)}{x-c}$$

$$= \frac{f(x)\, g(c) - f(c)\, g(x)}{g(x)\, g(c)\, (x-c)}$$

$$= \frac{f(x)\,(g(c)-g(x)) + g(x)\,(f(x)-f(c))}{g(x)\, g(c)\, (x-c)}$$

$$= \frac{f(x)}{g(x)\, g(c)} \cdot \frac{g(c)-g(x)}{x-c} + \frac{1}{g(c)} \cdot \frac{f(x)-f(c)}{x-c}$$

since $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$ and $\lim_{x\to c} \frac{g(x)-g(c)}{x-c} = g'(c)$ it follows

that $\lim_{x\to c} \frac{q(x)-q(c)}{x-c} = \frac{f(c)}{g(c)^2} \cdot (-g'(x)) + \frac{1}{g(c)} f'(c)$

$$= \frac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$$

Hence $q = \frac{f}{g}$ is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$$

16.4.15 Let s(x) = sin x and c(x) = cos x

Assume that $s'(x) = \cos x$ and $c' = -\sin x$

We show that $D \tan x = (\sec x)^2$

$$\left.\begin{array}{l} D \tan x = (\sec x)^2 \\ D \operatorname{Sec} x = (\sec x)(\tan x) \end{array}\right\} \text{ for } x \neq (2k+1)\frac{\pi}{2},\ n \in Z$$

and

$$\left.\begin{array}{l} D \cot x = -(\operatorname{cosec} x)^2 \\ D \operatorname{cosec} x = -(\operatorname{cosec} x) \cot x \end{array}\right\} \text{ for } x \neq k\pi,\ k \in Z$$

By the quotient rule $D \tan x = D\,\frac{\sin x}{\cos x} = \frac{(D \sin x)\cos x - (D \cos x)\sin x}{(\cos x)^2}$

whenever $\cos x \neq 0$

$\Rightarrow D \tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = (\sec x)^2$ if $x \neq (2n+1)\frac{\pi}{2}$

$D \operatorname{Sec} x = D\left(\frac{1}{\cos x}\right)$ $(x \neq \frac{\pi}{2}(2n+1),\ n \in Z)$

$$= \frac{0 - 1(-\sin x)}{(\cos x)^2}$$

$= \sec x \tan x$ for $x \neq \frac{\pi}{2}(2n+1)$

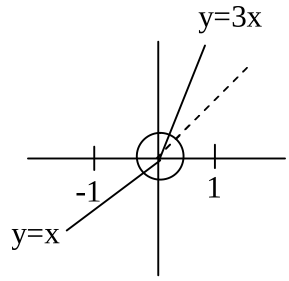
Similarly we can prove that

$$\left.\begin{array}{l} D \cot x = -(\operatorname{cosec} x)^2 \\ \text{and } D \operatorname{cosec} x = -(\operatorname{cosec} x) \cot x. \end{array}\right\} \text{ for } x \neq n\pi,\ n \in Z.$$

16.4.16

Draw the graph of g(x) = 2x + |x| and show that g is differentiable except at 0

Find $g'(x)$ for $x \neq 0$



$$g(x) = \begin{cases} 3x \text{ if } x > 0 \\ x \text{ if } x < 0 \\ 0 \text{ if } x = 0 \end{cases}$$

observe that at 0 the turn is not "smooth".

For x > 0 $\dfrac{g(x)-g(0)}{x-0} = 3$ $\quad \lim\limits_{x\to 0+} \dfrac{g(x)-g(0)}{x-0} = 3$

For x < 0 $\dfrac{g(x)-g(0)}{x-0} = 1$ $\quad \lim\limits_{x\to 0-} \dfrac{g(x)-g(0)}{x-0} = 1$

So g is not differentiable at 0

It is easy to show that g is differentiable at other points and $g'(x) = 3$ if x > 0 and

1 if x < 0.

16.4.16 (b)

If r > 0 is a rational number and g: $\mathbb{R} \to \mathbb{R}$ is defined by

$g(x) = x^r \sin \dfrac{1}{x}$ if x > 0 and

g(0) = 0

determine r for which g is differentiable at 0.

For x ≠ o $\dfrac{g(x)-g(0)}{x-0} = x^{r-1} \sin \dfrac{1}{x} = 0$

If r > 1 $\lim\limits_{x\to 0} x^{r-1} = 0$ so $\lim\limits_{x\to 0} x^{r-1} \sin \dfrac{1}{x} = 0 \Rightarrow$ g is differentiable at o and $g'(0) = 0$ if r > 1

If r = 1 $\lim\limits_{x\to 0} x^{r-1} \sin \dfrac{1}{x}$ does not exist

$\Rightarrow$ g is not differentiable at o

If 0 < r < 1, $\lim\limits_{x\to 0} x^{r-1} = \infty$

So when $x_n = \dfrac{1}{(4n+1)\frac{\pi}{2}}$. Sin $\dfrac{1}{x} = 1$ and hence

$$\frac{g(x_n)-g(0)}{x_n-0} = \left(\frac{2}{\pi}\right)^{r-1} \left(\frac{1}{(4n+1)}\right)^{r-1} = \left(\frac{\pi}{2}\right)^{1-r} (4n+1)^{1-r}$$

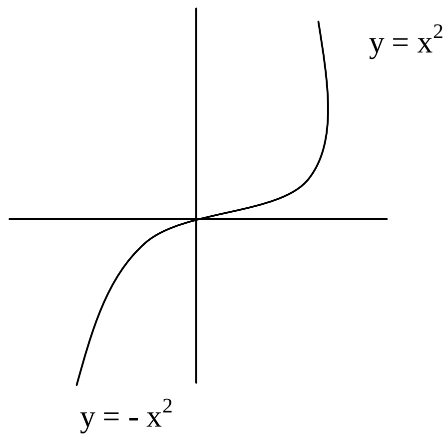
$$\Rightarrow \lim_{x_n\to 0} \frac{g(x_n)-g(0)}{x_n-0} = \lim_{n\to\infty} \left(\frac{\pi}{2}\right)^{(1-r)} (4n+1)^{1-r} = \infty$$

If $y_n = \frac{1}{n\pi}$, $\frac{g(y_n)-g(0)}{y_n-0} = \left(\frac{1}{n\pi}\right)^{r-1}$. Sin $n\pi = 0$

$$\Rightarrow \lim_{y_n\to 0} \frac{g(y_n)-g(0)}{y_n-0} = 0$$

Thus $\frac{g(x)-g(0)}{x-0}$ does not converge at the cluster point 0

Hence g is not differentiable at 0.



16.4.16 (c)

Draw the graph of h defined by

$$h(x) = \begin{cases} x^2 \text{ if } x \geq 0 \\ -x^2 \text{ if } x < 0 \end{cases}$$

The seemingly problematic point is 0

At 0: For $x > 0$ $\quad \frac{h(x)-h(0)}{x-0} = x$

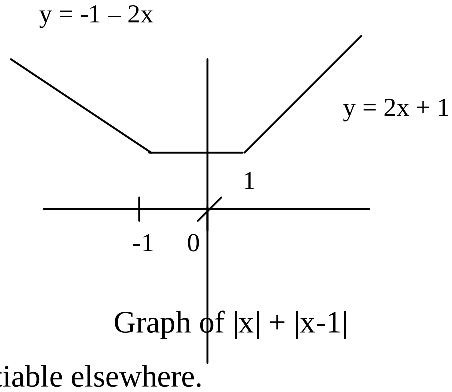
For $x < 0$ $\quad \frac{h(x)-h(0)}{x-0} = -x$

So for $x \neq 0$ $\quad \left|\frac{h(x)-h(0)}{x-0}\right| = |x|$

$\lim_{x\to 0} |x| = 0$ Hence $\lim_{x\to 0} \left|\frac{h(x)-h(0)}{x-0}\right| = 0 \Rightarrow h$ is differentiable at 0.

It is easy to verify that $h'(x) = \begin{cases} 2x \text{ if } x \geq 0 \\ -2x \text{ if } x < 0 \end{cases}$

Hence h is differentiable on $\mathbb{R}$ and $h'(x) = 2|x|$.



Graph of |x| + |x-1|

16.4.16 (d)

Discuss the differentiability of $f(x) = |x| + |x + 1|$

If $x \ge 0$, $f(x) = 2x + 1$.

If $-1 \le x \le 0$ $f(x) = -x + x + 1 = 1$ If $x < -1$ $f(x) = -x - 1 - x = -1-2x$

Observe the graph of f. The turns at -1 and 0 are not "smooth"

We show that f is not differentiable at these points and is differentiable elsewhere.

Non differentiability at 0: For $x > 0$, $\dfrac{f(x)-f(0)}{x-0} = \dfrac{2x+1-1}{x} = 1$

If $-1 < x < 0$ $\dfrac{f(x)-f(0)}{x-0} = \dfrac{1-1}{x} = 0$

For $x \ni -1 < x < \infty$ and $x \ne 0$

$$\frac{f(x)-f(0)}{x-0} = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x < 0 \end{cases}$$

so that $\lim\limits_{x\to 0+} \dfrac{f(x)-f(0)}{x-0} = 1$

and $\lim\limits_{x\to 0-} \dfrac{f(x)-f(0)}{x-0} = 0$

Hence $\dfrac{f(x)-f(0)}{x-0}$ does not converge at the cluster point 0. Hence f is not differentiable at 0.

Non differentiability at – 1:

If $-1 < x < 0$ $\dfrac{f(x)-f(-1)}{x-(-1)} = \dfrac{1-1}{x+1} = 0$ If $x < -1$ $\dfrac{f(x)-f(-1)}{x-(-1)} = \dfrac{-1-2x-1}{x+1} = -2$

As above we can show that

$$\lim_{x\to -1+} \frac{f(x)-f(-1)}{x+1} = 0 \text{ and } \lim_{x\to -1-} \frac{f(x)-f(-1)}{x+1} = -2$$

Hence f is not differentiable at – 1.

Differentiability of f at other points and that $f'(x) = \begin{cases} 2 \text{ if } x > 0 \\ 0 \text{ if } -1 < x < 0 \\ -2 \text{ if } x < -1 \end{cases}$ can be proved easily

16.5 Caratheodory Theorem:

Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there is a function ϕ defined on I such that

(i) ϕ is continuous at c and

(ii) $\phi(x)(x - c) = f(x) - f(c)$ for $x \in I$

In this case $\phi(c) = f^1(c)$.

Proof: Suppose f is differentiable at c. Write $\phi(x) = \frac{f(x) - f(c)}{x - c}$ if $x \in I - \{c\}$ and $\phi(c) = f^1(c)$

Then $\lim_{x \to c} \phi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) = \phi(c) \Rightarrow \phi$ is continuous at c

It is clear from the definition of ϕ that $\phi(x)(x - c) = f(x) - f(c)$ for $x \in I$

Conversely suppose there is a function ϕ on I satisfying (i) and (ii).

Then by the continuity of ϕ at c we get $\phi(c) = \lim_{x \to c} \phi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

Hence f is differentiable at c and $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \phi(c)$

Hence f is differentiable at c and $f'(c) = \phi(c)$.

16.5.2 Illustration:

$f(x) = x^3$ ($x \in \mathbb{R}$) for any $c \in \mathbb{R}$ $f(x) - f(c) = x^3 - c^3 = x^2 + xc + c^2$

$f(c) = x^3 - c^3 = (x-c)(x^2 + xc + c^2)$

The function $\phi: \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2+xc+c^2$ is continuous at c and

$f(x) - f(c) = (x-c)\phi(x)$ ($x \in \mathbb{R}$) Hence f is differentiable at c and $f'(c) = \phi(c) = 3c^2$

16.5.3 Example:

Given $f: \mathbb{R} \to \mathbb{R}$, $c \in \mathbb{R}$ and f is differentiable at c and $f(c) = 0$

Show that $|f|$ is differentiable at c if and only if $f'(c) = 0$

Solution:

Since f is differentiable at c and f(c) = 0 there is a $\phi: \mathbb{R} \to \mathbb{R}$ such that

(i) ϕ is continuous at c

(ii) $\phi(x)(x-c) = f(x) = f(x) - f(c)$ for $x \in \mathbb{R}$ and $(\because f(c) = 0)$

(iii) $\phi(c) = f'(c)$

|f| is differentiable at c if and only if there is $\psi: \mathbb{R} \to \mathbb{R}$ such that

(i) ψ is continuous at c

(ii) $\psi(x)(x-c)$, $|f(x)| - |f(c)| = |f(x)| =$ $(x \in \mathbb{R})$

and (iii) $\psi(c) = |f'(c)|$.

Thus if f is differentiable at c, from (ii) and we must have

$\psi(x)(x-c) = |f(x)| = |\phi(x)|\,|x-c|$

i.e. $\psi(x) = |\phi(x)| \dfrac{|x-c|}{(x-c)}$

since ψ is continuous at c we have from (iii)

$$|f'(c)| = \psi(c) = \lim_{x\to c} |\phi(x)| \frac{|x-c|}{(x-c)} = \lim_{x\to c+} |\phi(x)| \; (x > c \Rightarrow x - c > 0) = \lim_{x\to c} |\phi(x)| = |\phi(c)|$$

also $|f'(c)| = \psi(c) = \lim\limits_{x\to c-} -|\phi(x)|$ $(x < c \Rightarrow x-c<0) = -\lim\limits_{x\to c} |\phi(x)| = -|\phi(c)|$

$\Rightarrow |f'(c)| = -|f(c)| \Rightarrow |f'(c)| = 0 \Rightarrow f'(c) = 0$ conversely if $f'(c) = 0$

Write $\psi(x) = \dfrac{|f(x)|}{x-c}$ if $x \neq c$ and $\psi(c) = 0$. Then $\lim\limits_{x\to c} \psi(x) = \lim\limits_{x\to c} \dfrac{|f(x)|}{x-c}$

$$= \lim_{x\to c} \frac{|f(x)-f(c)|}{x-c} = \lim_{x\to c} \left|\frac{f(x)-f(c)}{x-c}\right| \frac{|x-c|}{x-c}$$

since $\dfrac{|x-c|}{x-c} = \pm 1$ and $\lim\limits_{x\to c} \left|\dfrac{f(x)-f(c)}{x-c}\right| = |f'(c)| = 0$

$\lim\limits_{x\to c} \psi(x) = 0 = \psi(c)$

This proves continuity of ψ at c. Also $\psi(c) = 0 = -f'(c) = |f'(c)|$. Hence ψ satisfies the conditions (i) (ii) (iii) stated above Hence |f| is differentiable at c.

16.5.4 Chain rule:

Let I, J be intervals in $\mathbb{R}$, f: I $\rightarrow \mathbb{R}$ and g: I $\rightarrow \mathbb{R}$ be functions such that f(J) $\subseteq$ I and c $\in$ J. if g is differentiable at f(c) and f is differentiable at c then the composite function gof is differentiable at c and

$$(gof)'(c) = g'(f(c))\, f'(c)$$

Proof: We apply Caratheodory's theorem. Write h = gof and d = f(c). since f is differentiable at c, there is a function ϕ defined on J such tat

(i) ϕ is continuous at c,

(ii) $\phi(x)(x-c) = f(x) - f(c)$ and for x $\in$ J and

(iii) $\phi(c) = f'(c)$

since g is differentiable at d = f(c), there is a function ψ defined on I such that

(iv) ψ is continuous at d,

(v) $\psi(y)(y-d) = g(y) - g(d)$ for y $\in$ I and

(vi) $\psi(d) = g'(d)$

write $\theta(x) = \psi(f(x)) . \phi(x)$ for x $\in$ J Since ψ is continuous at d = f(c) and ϕ is continuous at c. θ is continuous at c.

For x $\in$ J, y = f(x) $\in$ f(J) $\subseteq$ I so $\theta(x)(x-c) = \psi(f(x))\,\phi(x)(x-c) = \psi(f(x))\, f(x) - f(c) = \psi(y)(y-d) = g(y) - g(d) = g(f(x)) - g(f(c)) = h(x) - h(c)$.

Again by carathiodory theorem h is differentiable at c and

$h'(c) = \theta(c) = \psi\, f(c) . \phi(c) = g'(f(c)\, f'(c)$.

Notation: If g is differentiable on I, f is differentiable on J and f(I) $\subseteq$ J then by the chain rule, for x $\in$ I $(gof)'(x) = (g' of)(x).\ f'(x)$. Thus $(gof)' = (g' of) .\ f'$.

Using the D-notation in place of "/" for derivative, we have D (gof) = ((Dg) of). (Df).

16.5.5 Applications of chain rule:

16.5.5(a) If f: I $\rightarrow \mathbb{R}$ is differentiable on I there for n $\in \mathbb{N}$ f^n is differentiable and for x$\in$I. $(f^n)'(x) = n f^{n-1}(x)\, f'(x)$

Reason: write $g(y) = y^n$ for $y \in \mathbb{R}$ and $h = gof$ Then $h(x) = g(f(x)) = (f(x))^n$. By chain rule $h'(x) = g'(f(x))\, f'(x) = n(f(x))^{n-1} f'(x)$

16.5.5 (b) If $f: I \to \mathbb{R}$ is differentiable and $f(x) \neq 0 \neq f'(x)$ for all $x \in I$ then $\frac{1}{f}$ is differentiable on I and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2} \text{ for } x \in I$$

Reason: Write $h(y) = \frac{1}{y}$ for $y \in \mathbb{R}, y \neq 0$

Then h is differentiable on $\mathbb{R} - \{0\}$ and $h'(y) = \frac{-1}{y^2}$ for $y \neq 0$

Hence by the chain rule hof is differentiable on I and $(hof)'(x) = h'(f(x))\, f'(x)$

$$\Rightarrow \left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{f^2(x)}$$

16.5.5(c)

Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ f is differentiable on $\mathbb{R} - \{0\}$ and $f'(x) = \left(\cos\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$ for $x \neq 0$ Write $g(x) = \frac{1}{x}$ $(x \neq 0)$ and $S(x) = \sin x$ $(x \in \mathbb{R})$ g is differentiable at all $x \neq 0$ and $g'(x) = \frac{-1}{x^2}$ for $x \neq 0$ and S is differentiable at all x and $S'(x) \cos x$. also $f(x) = \sin\frac{1}{x} = S(g\,x) = (Sog)(x)$ By chain rule f is differentiable for all $x \neq 0$ and $f'(x) = S'(g(x))\, g'(x)$

$$= \left(\cos\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{-1}{x^2}\cos\frac{1}{x}.$$

16.5.5 (d)

Using (c) and product rule show that the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

is differentiable on $\mathbb{R}$ and $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ $(x \neq 0)$ and $f'(0) = 0$

We have $D\left(\sin \frac{1}{x}\right) = \frac{-1}{x^2} \cos \frac{1}{x}$ for $x \neq 0$ from (c) above and $D(x^2) = 2x$ for all x.

So by product rule, $D\left(x^2 \sin \frac{1}{x}\right) = (D(x^2)) \sin \frac{1}{x} + x^2 D\left(\sin \frac{1}{x}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ $(x\neq 0)$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$$

$\Rightarrow$ (Df) (0) = 0.

16.5.5 (e)

$g(x) = x^2 \sin \frac{1}{x^2}$ $(x \neq 0)$ and $g(0) = 0$

$$\frac{g(x) - g(0)}{x - 0} = x \sin \frac{1}{x^2} \quad (x \neq 0)$$

$$\Rightarrow \left|\frac{g(x) - g(0)}{x - 0}\right| = \left|x \sin \frac{1}{x^2}\right| = |x| \left|\sin \frac{1}{x^2}\right| \leq |x|$$

Let $\in > 0$. If $0 < |x| < \in$, $\left|\frac{g(x) - g(0)}{x - 0} - 0\right| \leq |x| < \in$ Hence $\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = 0$

So g is differentiable at 0 and $g'(0) = 0$. If $x \neq 0$ the functions t^2, $\frac{1}{t^2}$ and sin t are differentiable at x so by the product and chain rules, g is differentiable at c.

Further $g'(x) = x^2 \cos \frac{1}{x^2} \left(-\frac{2}{x^3}\right) + 2x \sin \frac{1}{x^2}$

$$= 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$$

When $n \in \mathbb{N}$ and $x_n = \frac{1}{\sqrt{(2n+1)\,\Pi}}$, $\frac{1}{x_n^2} = (2n+1)\Pi$ so that $\sin \frac{1}{x_n^2} = 0$ and $\cos \frac{1}{x_n^2} = -1$

And hence $g'(x_n) = 2\sqrt{2n+1}\ \Pi$ Since $\lim (g'(x_n)) = \infty$ / g' is unbounded.

16.6 Mean value Theorems

The mean value theorem relates the values of a function to values of its derivative. This theorem permits one to draw conclusions about the nature of a function from the information about its derivative. Information about relative maxima and minima from information about vanishing of the derivative as well, is drawn using this theorem.

Geometrically the mean value theorem says that every chord of the curve y = f(x) has a parallel tangent line. Because of its many useful consequences the mean value theorem may be called THE FUNDAMENTAL THEOREM OF DIFFERENTIAL CALCULUS.

Let I be an interval and c an interior point of I.

Definitions: f: I $\to$ $\mathbb{R}$ is said to have a relative maximum at c $\in$ I if there is a nbd V = V_δ(c) of c such that f(x) $\le$ f(c) if x $\in$ V $\cap$ I.
f is said to have a relative minimum at c $\in$ I if there is a nbd V = V_δ (c) of c such that f(c) $\le$ f(x) if x $\in$ V $\cap$ I. we say that f has a relative extremum at c $\in$ I if f has either a relative minimum at c.

Recall that f has absolute maximum (minimum) at c if f(x) $\le$ f(c) (resply f(x) $\ge$ f(c)) for all x $\in$ I. f is said to be monotonically increasing in I if f(x) $\le$ f(y) whenever x < y and x $\in$ I, y $\in$ I f is said to be strictly monotonically increasing in I if f(x) < f(y) whenever x<y and x $\in$ I, y $\in$ I monotonically decreasing and strictly monotonically decreasing are defined analogously. We usually drop “monotonically” and say f is increasing / decreasing instead of monotonically increasing / monotonically decreasing.

Interior extension theorem:

Let c be an interior point of I at which f: I $\to$ $\mathbb{R}$ has relative extremum. If f is differentiable at c then f' (c) = 0
Proof: We may assume that f has relative maximum at c. In case f has relative minimum at c, -f has relative maximum at c. Since c is an interior point there is $\delta_1 > 0$ such that (c - δ_1, c + δ_1) $\subseteq$ I. Since f has relative maximum there is $\delta_2 > 0$ such that (c - δ_2, c + δ_2) $\subseteq$ I and f(x) $\le$ f(c) for x $\in$ (c - δ_2, c + δ) if f' (c) > 0 there is $\delta_3 > 0$ such that (c - δ_3, c + δ_3) $\subseteq$ I and $\left|\dfrac{f(x)-f(c)}{x-c} - f`(c)\right| < \dfrac{f'(c)}{2}$ if x $\in$ (c - δ_3, c + δ_3) and x $\ne$ c

$\Rightarrow \dfrac{f(x) - f(c)}{x - c} > \dfrac{f'(c)}{2} > 0$ if $x \in (c - \delta_3, c + \delta_3)$ and $x \neq c \rightarrow (1)$

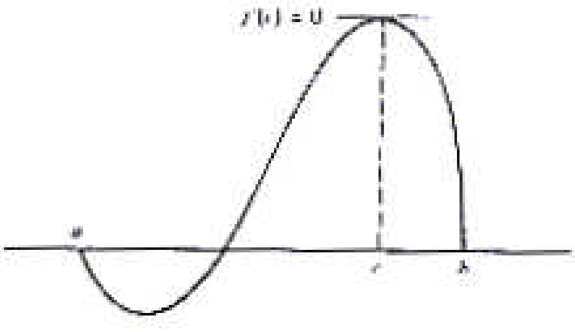
If $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, $(c - \delta, c + \delta) \subseteq (c - \delta_i, c + \delta_i) \subseteq I$ for I = 1,2,3 and so $f(x) \leq f(c)$ for $x \in (c - \delta, c + \delta)$. If $c < x < c + \delta$, $x - c > 0$ and $f(x) - f(c) \leq 0$ so that $\dfrac{f(x) - f(c)}{x - c} \leq 0$, contradicting (1). Hence $f'(c) \not> 0$ i.e $f'(c) \leq 0$. Similarly by considering $(c-\delta, c)$ we can show that $f'(c) \geq 0$. Hence $f'(c) = 0$.

16.6.3 Corollary: If $f: I \rightarrow \mathbb{R}$ has relative extremum at an interior point $c \in I$ then
(i) either f is not differentiable at c or (ii) f is differentiable at c and $f'(c) = 0$

16.6.3 Remark: Theorem 16.6.2 does not guarantee differentiability of the function at a relative extremum but merely assures that the derivative, if exists, at a relative extremum must vanish. For example it is clear that the absolute value function $f(x) = |x|$ has relative, infact absolute minimum extremum at zero but f is not differentiable at 0.

16.6.4 Rolle's Theorem: Let I = [a, b] and suppose that f is continuous on I, differentiable on (a,b) and that f(a) = f(b) = 0. Then thee exists at least one point c in (a,b) such that $f'(c) = 0$.

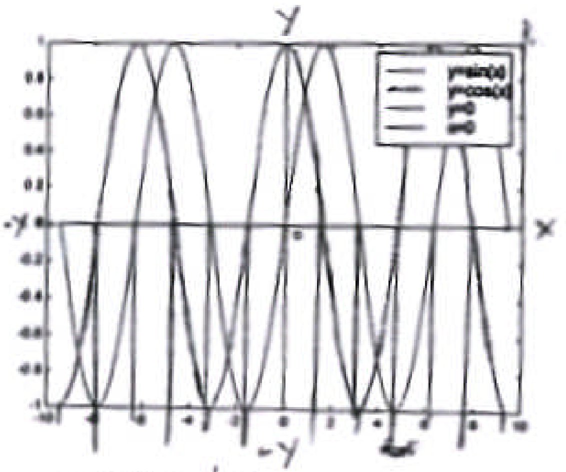
Proof: If $f(x) = 0$ for all $x \in I$ then $f'(x) = 0$ for all $x \in (a,b)$.
So we may assume that $f(x_0) \neq 0$ for some x_0. We may further assume that $f(x_0) > 0$ as other wise we may consider – f. Since f is continuous on [a,b] f is bounded and attains maximum there is c in [a,b] such that $f(c) \geq f(x)$ for all $x \in [a,b]$. Since $f(c) \geq f(x_0) > 0$ and $f(a) = f(b) = 0$ $a \neq c \neq b$. i.e. $c \in (a,b)$. Since f has absolute maximum at c, f has relative maximum at c. Hence by the interior extension theorem $f'(c)$, if exists, must be zero. It is assumed that f is differentiable in (a,b) so f is differentiable at c. Thus $f'(c) = 0$



Remark: Rolle's theorem guarantees the existence of root of $f'(x) = 0$ between two consecutive roots of $f(x) = 0$, when f is differentiable.

16.6.5 Application: Rolles' theorem can be used for the location of roots of a function.

16.6.5 Example (i): The roots of sin and cos functions interlace each other: i.e. between any two roots of sin x there is a root of cos x and vice versa. This follows from Rolle's theorem and the fact that D sin x = cos x and D cos x = -sin x. If $\sin \alpha = \sin \beta = 0$ and $\alpha < \beta$ then by Rolle's theorem there exists r in (α, β) such that cos r = D sin r = 0. The other part follows from the fact that D cos x = - sin x.



• Roots of f′(x)
• Roots of f(x) 16.6.5(i)

16.6.5 (ii) Example: If f is a polynomial of degree n > 1 with real coefficients and α, β are real numbers such that $\alpha < \beta$ and $f(\alpha) = f(\beta) = 0$ i.e. if α, β are roots of the polynomial f then there is a root r in (α, β) such that $f'(r) = 0$. In particular suppose $c_0, c_1, \ldots. c_n$ are real numbers such that $\frac{c_0}{1} + \frac{c_1}{2} + \ldots\ldots\ldots + \frac{c_n}{n+1} = 0$

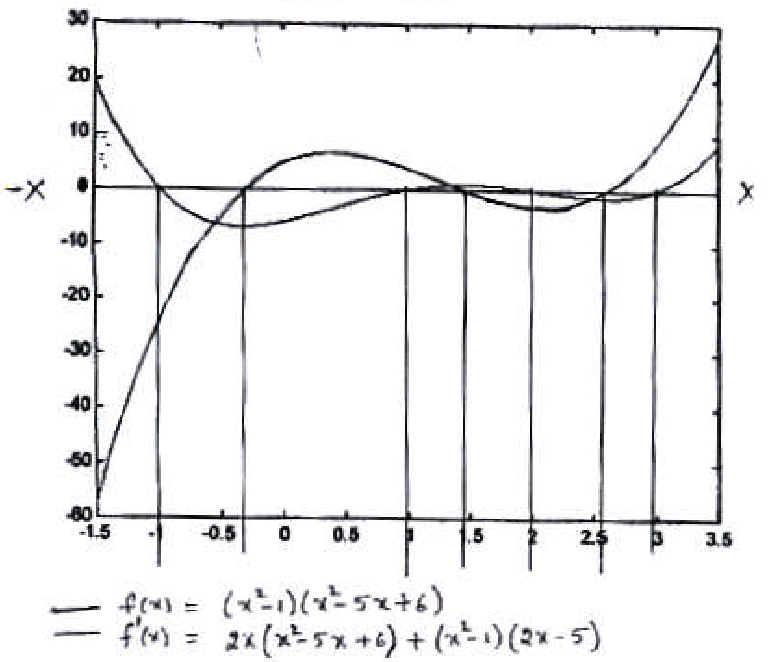
Then the polynomial $f(x) = c_0 + c_1x + \ldots\ldots\ldots.. + c_n x^n$ has a root in (0,1)

To show this we consider $g(x) = c_0 x + c_1 \frac{x^2}{2} + \ldots\ldots + c_n \frac{x^{n+1}}{n+1}$

Since g(0) = g(1) = 0 by Rolle's theorem there is $\alpha \in (0,1)$ such that $g'(\alpha) = 0$

But $g^1(x) = f(x)$. hence f has a root in (0,1)

16.6.5 (iii)



16.5.5 (iii) We consider yet another example. et $f(x) = (x^2 - 1)(x^2 - 5x + 6)$. Then $f^1(x) = 2x(x^2 - 5x + 6) + (x^2 - 1)(2x - 5)$ he graphs of f and f' are given in Figure 16.6.5(iii) It is evident from this figure that each one of the roots of $f'(x) = 0$ lies in consecutive roots of f(x) = 0.

16.6.6 The mean value theorem (Lagrange's Mean value theorem):

Suppose f is continuous on a closed interval I = [a,b] and that f has a derivative in the open interval (a,b). Then there exists atleast one point c in (a,b) such that

$f(b) - f(a) = f'(c)(b - a)$

Proof: Define $g : I \to \mathbb{R}$ by $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$ g is continuous in [a,b].

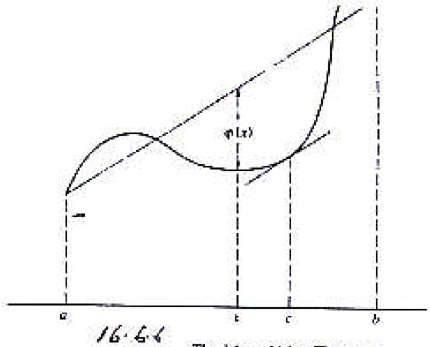
g is differentiable in (a,b) and $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$ and $g(a) = 0 = g(b)$

So by Rolle's theorem $g'(c) = 0$ for some $c \in (a,b)$

Since $g'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

We get $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

so that $f(b) - f(a) = (b - a)\, f'(c)$



16.6.6 The Mean Value Theorem

Geometrical Interpretation:

The theorem can be interpreted geometrically as it guarantees the existence of a tangent to the curve y = f(x) which is parallel to the chord jointing the end points (a, f(a)), (b, f(b)).

16.6.7 Theorem: Suppose that f is continuous on I = [a,b], and differentiable on (a,b) and that $f'(x) = 0$ for all x in (a,b). Then f is constant on I.

Proof: We show that f(x) = f(a) for all x such that $a \le x \le b$

Suppose $a < x \le b$. Then f is continuous on [a,x] and differentiable on (a,b). So by the Mean value theorem there exists $c \in (a,x)$ such that

$f(x) - f(a) = (x - a)\, f'(c) = (x - a) \,.\, 0 = 0$

Hence f(x) = f(a). This is true for all x such that $a < x \le b$

Hence f(x) = f(a) for $x \in I$.

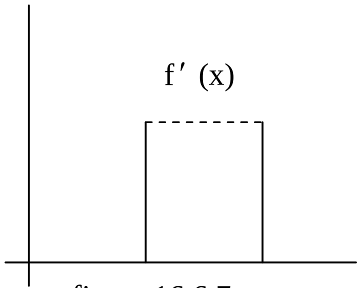


figure 16.6.7

16.6.8 Corollary: Suppose f and g are continuous on I = [a,b] and differentiable on (a,b) and $f'(x) = g'(x)$ for all $x \in (a,b)$ Then there is a constant k such that

$f(x) = g(x) + k$ for $x \in [a,b]$

Proof: The function $h : I \to \mathbb{R}$ defined by h(x) = f(x) – g(x) is continuous on [a,b], differentiable on (a,b) and $h'(x) = f'(x_1 - g'(x) = 0$ for all $x \in (a,b)$. hence by the above theorem there is a constant k such that h(x) = k for all $x \in [a,b]$

So f(x) = g(x) + k for all $x \in [a,b]$.

Hence $f'(x) \geq 0$ on I. Proof of (ii) is similar or we may apply (1) for – f.

Monotonicity: Let us recall that f: $I \to \mathbb{R}$ is monotonically (strictly) increasing on I if $f(x) \leq f(y)$ (resply $f(x) < f(y)$) whenever $x \in I$, $y \in I$ and $x < y$. f is said to be monotonically (strictly) decreasing on I if $f(y) \leq f(x)$ (resply $f(y) < f(x)$) whenever $x \in I$, $y \in I$ and $x < y$.

16.6.9 Theorem: Let $f : I \to \mathbb{R}$ be differentiable on I. Then

(i) f is increasing on I if and only if $f'(x) \geq 0$ for $x \in I$

(ii) f is decreasing on I if and only if $f'(x) \leq 0$ for $x \in I$.

Proof: (i) Suppose $f'(x) \geq 0$ for $x \in I$. If $x \in I$, $y \in I$ and $x < y$ then f is continuous in [x,y] and differentiable in (x,y). So by the mean value theorem there exists $z \in (x,y)$ such that $f(y) - f(x) = (y - x)\, f'(z)$. Since $y - x > 0$ and $f^1(z) \geq 0$ $(y - x)\, f'(z) \geq 0$ so $f(y) \geq f(x)$. This holds for every $x < y$ in I. Hence f is monotonically increasing on I.

If $x \in I$ and $x > c$, $x - c > 0$ and $f(x) \geq f(c) \Rightarrow \dfrac{f(x) - f(c)}{x - c} \geq 0$

If $x < c \Rightarrow x - c < 0$ and $f(x) \leq f(c) \Rightarrow \dfrac{f(x) - f(c)}{x - c} \geq 0$

Hence for $x \neq c$ and $x \in I$ $\quad \dfrac{f(x) - f(c)}{x - c} \geq 0$

Hence $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = f'(c) \geq 0$. This is true for every c in I.

Hence $f'(x) \geq 0$ on I. Proof of (ii) is similar or we may apply (i) for f.

Remark: Call $f : I \to \mathbb{R}$, increasing at a print $c \in I$ if there is a neighborhood $(c-\delta, c+\delta) \leq I$ of c such that f is increasing on $(c - \delta, c + \delta)$ We may define strictly increasing at a point, (strictly) decreasing at a point and monotonic at a point in a similar way. The analogue of monotonicity for differentiable functions (Theorem 16.6.9 does not hold good for monotonicity at a point. More precisely, it is not necessarily true if $f'(c) > 0$ then f is strictly increasing at c.

16.6.10 Example: Let g(x) = $\begin{cases} x + 2x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

if $x \neq 0$ $\quad \frac{g(x) - g(0)}{x - 0} = 1 + 2x \sin \frac{1}{x}$ Since $\lim\limits_{x \to c} x \sin \frac{1}{x} = 0$,

$g'(0) = \lim\limits_{x \to 0} \frac{g(x) - g(0)}{x - 0} = 1$ Hence $g'(0) > 0$

We show that g is not monotonically increasing in any nbd of 0.

Let $\delta > 0$ choose $n \in \mathbb{N}$ such that $0 < \frac{1}{n\pi} < \delta$.

Then $g\left(\frac{1}{n\pi}\right) = \frac{1}{n\pi} + 0 = \frac{1}{n\pi} > 0$ (since $\sin n\pi = 0$) $g(-\frac{1}{n\pi}) = -\frac{1}{n\pi} < 0$ $(-\delta < -\frac{1}{n\pi} < 0)$

Hence $g(\frac{1}{n\pi}) > 0 > g(-\frac{1}{n\pi})$ Hence g is not increasing in any nbd of 0.

First derivative test for extrema:

Let f be continuous on [a,b], c be any point of [a,b], f: $[a,b] \to \mathbb{R}$ be such that f is differentiable in (a,c) and in (c,b). Then

(i) If there is a neighborhood $V = (c - \delta, c + \delta)$ of c such that $V \subseteq I$ and $f'(x) \geq 0$ if $c - \delta < x < c$ and $f'(x) \leq 0$ if $c < x < c + \delta$ then f has relative maximum at C.

(ii) If there is a neighborhood V. $(c - \delta, c + \delta)$ of c such that $V \subseteq I$ and $f'(x) \leq 0$ if $c - \delta < x < c$ and $f'(x) \geq 0$ if $c < x < c + \delta$ then f has relative minimum at c.

Proof: We prove (i) Proof (ii) is similar.

Assume that there is a nbd $V = (c - \delta, c + \delta)$ of c such that (i) holds. If $c - \delta < x < c$ then f is continuous in [x,c] and differentiable in (x,c). So by the mean value theorem there is $\alpha \in (x,c)$ such that $f(x) - f(c) = (x - c) f'(\alpha)$

Since $x < \alpha$, $x - c < 0$ and by (i) $f'(\alpha) \geq 0$ so $(x - c) f'(\alpha) \leq 0$, hence $f(x) - f(c) \leq 0$.

Thus if $c - \delta < x < c$, $f(x) \leq f(c)$ ………………………………………….. (A)

If $c < x < c + \delta$ we apply the mean value theorem to f on [c,x].

There exists β in (c,x) such that $f(x) - f(c) = (x - c)\, f'(\beta)$

Since $x > c$, $x - c > 0$ and $f'(\beta) \le 0$ so $f(x) - f(c) \le 0$

Thus if $c < x < c + \delta$, $f(x) \le f(c)$ ………………………………………… (B)

From (A) and (B) $f(x) \le f(c)$ if $c - \delta < x < c + \delta$

Hence f has relative maximum at c.

In lesson 15 on continuous functions we proved that if f is continuous on [a,b] then f is bounded, assumes its maximum and minimum values in the interval [a,b] and possesses intermediate value property i.e. if $a \le x \le b$, $a < y \le b$ and $f(x) < \lambda < f(y)$ there is a c in between x and y such that $f(c)=\lambda$.

We now extend this mean value property to derivatives as well. The hypothesis in the intermediate value theorem is that f be continuous on [a,b]. However a derivative is not necessarily continuous as can be seen from the following.

16.6.12 Example: Let $f(x) = x^2 \sin \frac{1}{x}$ if $x \ne 0$ $\quad f(0) = 0$

Solution: f is differentiable at $x \ne 0$ and by the product and chain rules $f'(x)=2x \sin \frac{1}{x} - \cos \frac{1}{x}$

As $\frac{f(x)-f(0)}{x} = x \sin \frac{1}{x}$ for $x \ne 0$ $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = 0$

Clearly $f'(x)$ does not converge at 0 as $\lim \cos \frac{1}{x_n} = 1$ when $x_n = \frac{1}{2n\pi}$ and

$\lim \cos \frac{1}{x_n} = 0$ when $x_n = \frac{1}{(2n+1)\frac{\pi}{2}}$.

Thus it is not necessarily true that a derivative is continuous.

Before proving intermediate value property for the derivative of a function we first prove a lemma.

16.6.13 Lemma: Let I be an interval in $\mathbb{R}$, $c \in I$ and $f : I \to \mathbb{R}$ be differentiable at c. Then

(a) If $f'(c) > 0$ there is $\delta > 0$ such that $f(x) > f(c)$ if $c < x < c + \delta$ and $x \in I$.

(b) If $f'(c) < 0$ there is $\delta > 0$ such that $f(x) > f(c)$ if $c - \delta < x < c$ and $x \in I$.

Proof: Suppose $f'(c) > 0$ then for $\in = f'(c)$ there is $\delta > 0$ such that if

$x \in (c - \delta, c + \delta) \cap I$ and $x \neq c$

$$\left|\frac{f(x) - f(c)}{x - c} - f`(c)\right| < f'(c)$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} > f'(c) - f'(c) = 0 \text{ if } c - \delta < x < c + \delta \text{ and } x \in I.$$

if $c < x < c + \delta$ and $x \in I$ $0 < x - c < \delta$ so $f(x) - f(c) = (x - c)\dfrac{f(x) - f(c)}{x - c} > 0$

$\Rightarrow f(x) > f(c)$ if $c < x < c + \delta$ and $x \in I$

This proves (a)

If $f'(c) < 0$ we take $\in = - f'(c) > 0$. As above there exists $\delta > 0$ such that

$\dfrac{f(x) - f(c)}{x - c} < 0$ if $c - \delta < x < c + \delta$, $x \in I$ and $x \neq c$

If $c - \delta < x < c$ and $x \in I$ then $x - c < 0$ so that

$$f(x) - f(c) = (x - c)\frac{f(x) - f(c)}{x - c} > 0$$

$\Rightarrow f(x) > f(c)$ if $c - \delta < x < c$, and $x \in I$.

16.6.14 Darboux theorem:

If f is differentiable on [a,b] and k is a number between $f'(a)$ and $f'(b)$ then there is at least one c in (a,b) such that $f'(c) = k$.

Proof: We may assume that $f'(a) < k < f'(b)$ as the proof is similar when $f'(b)<k<f'(a)$. Define $g' \ [a,b] \to \mathbb{R}$ by $g(x) = kx - f(x)$. Since f is differentiable in [a,b] so is g. and $g'(x) = k - f'(x)$. Since $f'(a) < k < f'(b)$

$g'(a) = k - f'(a) > 0 > k - f'(b) = g'(b)$

Hence there is a $\delta > 0$ such that $g(x) > g(a)$ if $a < x < a + \delta$ and $x \in [a,b]$.

Since g is continuous on [a,b], g is bounded and attains maximum in [a,b]; i.e. there is $x_0 \in [a,b]$ such that $g(x) \le g(x_0)$ for all x in [a,b].

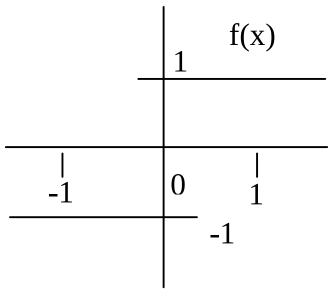
Since $g(x) > g(a)$ if $x \in [a,b]$ and $a < x < a + \delta$, it follows that $x_0 \neq 0$. ------------ (1)

Again since $g^1(b) = k - f^1(b) < 0$, there is $\delta > 0$ such that

if $x \in [a,b]$ and $b - \delta < x < b$ $g(x) > g(b)$ As above it follows that $x_0 \neq b$.

Hence $a < x_0 < b$. Since g has maximum at x_0 and is differentiable at x_0, $g'(x_0) = 0$

Since $g'(x_0) = 0$, $f'(x_0) = k$. The theorem is thus proved.



16.6.15 Example:

The function $g : [-1, 1] \to [-1, 1]$ defined by

$$g(x) = \begin{cases} -1 \text{ if } -1 \le x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } 0 < x \le 1 \end{cases}$$

Cannot be the derivative of any function; i.e. there is no f define on [-1, 1] such that $f'(x) = g(x)$. Because if such a function f exists then f' must have the intermediate value property. However g does not have this property.

16.6.16 Applications of Mean value Theorem:

(i) The mean value theorem can be applied for approximate calculations and finding error estimates.

Suppose we have to find an approximation for $\sqrt{105}$

Write $f(x) = \sqrt{x}$. $(x > 0)$. Then $f^1(x) = \dfrac{1}{2\sqrt{x}}$

We know that $\sqrt{100} < \sqrt{105} < \sqrt{121}$ ie $10 < \sqrt{105} < 11$

But 10 and 11 are "far away" from $\sqrt{105}$

Write a = 10 and b = 105 and apply the mean value theorem for f on [a, b]. There exists c between 100 and $\sqrt{105}$ Such that $\sqrt{105} - \sqrt{100} = 5.\ f^1(c) = \dfrac{5}{2\sqrt{c}}$

Thus $10 < \sqrt{c} < \sqrt{105} < 11$

$$\Rightarrow \frac{1}{11} < \frac{1}{\sqrt{c}} < \frac{1}{10}$$

$$\Rightarrow \frac{5}{2(11)} < \frac{5}{2\sqrt{c}} < \frac{5}{2(10)}$$

$$\Rightarrow \frac{5}{22} < \sqrt{105} - \sqrt{100} < \frac{5}{20} = \frac{1}{4}.$$

$$\Rightarrow 10 + \frac{5}{22} < \sqrt{105} < 10 + \frac{1}{4}.$$

$$\Rightarrow 10\,.\,2272 < \sqrt{105} < 10.25$$

Thus starting with (10, 11) we are able to find a smaller interval that includes $\sqrt{105}$ - namely (10.2272, 10.2500)

we further improve this interval. since $\sqrt{105} < 10.2500$ and $10 < \sqrt{c} < \sqrt{105}$

we get $\sqrt{c} < 10.2500$

$$\Rightarrow \frac{5}{2\sqrt{c}} > \frac{5}{2(10.25)} = \frac{5}{20.50} = \frac{5x2}{41} > .2439$$

Thus $.2439 < \frac{5}{2\sqrt{c}} = \sqrt{105} - 10 \Rightarrow 10.2439 < \sqrt{105} < 10.25$

Clearly the interval (10.2439, 10.25) is contained in (10.2272, 10.25)

(ii) Apply the Mean value theorem to prove that $e^x > 1 + x$ if $x \in \mathbb{R}$

Write $f(x) = e^x$ for $x \in \mathbb{R}$. Then f is differentiable and $f'(x) = e^x$.

If x > 0 f is continuous on [0, x] and is differentiable; so by the mean value theorem there exists c in (0, x) such that

$$\frac{f(x)-f(0)}{x-0} = f'(c)$$

$$\Rightarrow \frac{e^x - e^0}{x} = e^c$$

since $f'(x) = e^x > 0$ f is monotonically increasing

Hence $\frac{e^x - e^0}{x} = e^c > e^0 = 1 \Rightarrow e^x > e^0 + x = 1 + x$ if $x > 0$

If $x < 0$ $\frac{e^x - e^0}{x} = e^c$ for some c in (x, 0)

$\Rightarrow \frac{e^x - 1}{x} = e^c = \frac{1}{e^{-c}} < 1$

$\Rightarrow e^x - 1 > x$ (since $x < 0$)

$\Rightarrow e^x > 1 + x$ if $x < 0$.

Clearly $e^0 = 1$ so $e^x > 1 + x$ for all x

(iii) Show by using the mean value theorem that

$\frac{x-1}{x} < \ln x < x - 1$ for $x > 1$

Solution: If $x > 1$, $f(x) = \ln x$ is continuous in [1, x] and differentiable in (1, x).

So, by the Mean value theorem there is $c \in (1, x)$ such that

$\frac{f(x) - f(1)}{x-1} = f'(c) = \frac{1}{c}. \Rightarrow \frac{\ln x - \ln 1}{x-1} = \frac{1}{c}$

Since $1 < c < x$, $\frac{1}{x} < \frac{1}{c} < 1 \Rightarrow \frac{1}{x} < \frac{\ln x}{x-1} < 1 \Rightarrow \frac{x-1}{x} < \text{lin } x < x\text{-}1$

(iv) Apply mean value theorem and prove that if $a > 0$, $b > 0$ and $0 < \alpha < 1$ then $a^{\alpha} b^{1-\alpha} \le \alpha a + (1-\alpha) b$.

write $g(x) = \alpha x - x^{\alpha}$ for $x \ge 0$. Then $g'(x) = \alpha - \alpha x^{\alpha-1} = \alpha (1 - x^{\alpha-1})$

If $0 < x<1$, $\frac{1}{x} > 1$ so $\left(\frac{1}{x}\right)^{1-\alpha} > 1 \Rightarrow x^{\alpha-1} < 1 \Rightarrow g'(x) = \alpha (1 - x^{\alpha-1}) < 0$ if $0 < x<1$.

If $x > 1$, $0 < \frac{1}{x} < 1 \Rightarrow 0 < \left(\frac{1}{x}\right)^{1-\alpha} < 1 \Rightarrow x^{1-\alpha} > 1 \Rightarrow x^{\alpha-1} < 1 \Rightarrow 1 - x^{\alpha-1} > 0$ if $x>1$

So $g'(x) > 0$ if $x > 1$

Now if $x > 1$ by the Mean value theorem there exists c in (1, x) such that

$\frac{g(x) - g(1)}{x-1} = g'(c) > 0$

Since $x - 1 > 0$, $g(x) - g(1) = \frac{g(x) - g(1)}{x-1}(x\text{-}1) > 0$

$\Rightarrow$ g(x) > g(1) = α -1 if x > 1 $\Rightarrow \alpha x - x^{\alpha} > \alpha - 1$ if x > 1 $\Rightarrow x^{\alpha} < \alpha x + (1-\alpha)$

If a > b > 0, $\left(\frac{a}{b}\right)^{\alpha} < \alpha \cdot \frac{a}{b} + (1-\alpha) \Rightarrow a^{\alpha} b^{1-\alpha} < \alpha a + (1-\alpha)$ b.

(v) If a > 0, b > 0 and 0 < α < 1 show that $a^{\alpha} b^{1-\alpha} \leq a\,\alpha + b\,(1-\alpha)$

write $g(x) = \alpha x - x^{\alpha}$ for $x \geq 0$. g is differentiable in $(0, \infty)$ and

$g'(x) = \alpha (1 - x^{\alpha-1})$ if 0 < x

since $0 < \alpha < 1$, $g'(x) < 0$ if 0 < x < 1 and $g'(x) > 0$ if x > 1

If 0 < x < y by the Mean value theorem there exists $c \in (x, y)$ such that

$$\frac{g(y)-g(x)}{y-x} = g'(c) \text{ so that } g(y) - g(x) = (y-x)\, g^{1}(c) \qquad (1)$$

In particular if 0 < x < 1 there exists c such that x < c < 1 and g(1) – g(x) = (1-x) $g'(c) < 0$ so that g(1) < g(x). Hence g(x) > α - 1 if 0 < x < 1.

If x > 1 by (1) there is c in (1, x) such that g(x) – g(1) = (x-1) $g'(c)$

Since c > 1, $g'(c) > 0$ and x-1 > 0 so that g(x) > g(1) = α-1

Thus for all x > 0, $x \neq 1$ $\alpha x - x^{\alpha} > \alpha - 1$ i.e $x^{\alpha} < \alpha x + 1 - \alpha$ if x > 0 and $x \neq 1$

clearly x = 1 $\Rightarrow x^{\alpha} = \alpha x + 1 - \alpha = 1$. If a > 0 b > 0 and $a \neq b$ put $x = \frac{a}{b}$

Then $\left(\frac{a}{b}\right)^{\alpha} < \frac{\alpha a}{b} + (1-\alpha) \Rightarrow a^{\alpha} b^{1-\alpha} < \alpha a + (1-\alpha)$ b.

if a = b clearly $a^{\alpha} b^{1-\alpha} = a\,\alpha + (1-\alpha)b$

(vi) **Bernoulli's inequality:**

If $\alpha > 1$ and $x > -1$ $\quad (1+x)^{\alpha} > 1 + \alpha x$

Proof: When α is a natural number, $\alpha > 1$ $(1+x)^{\alpha+1} = (1+x)^{\alpha} (1+x)$

So if we assume that $(1+x)^{\alpha} > 1 + x\,\alpha$ we get $(1+x)^{\alpha+1} > (1 + x\,\alpha)(1+x)$

$= 1 + (\alpha+1) x + x^2 \alpha > 1 + x(\alpha+1)$

However $(1+x)^1 = 1 + x.1$ and $(1+x)^2 = 1 + x^2 + 2x > 1+2x$.

So the inequality holds when α = 2 and it is true for α+1 whenever it holds for α.

So by induction $(1+x)^{\alpha} \geq (1+x\alpha)$ for all natural numbers $\alpha > 1$

In the general case we write $h(x) = (1+x)^{\alpha}$. Then $h'(x) = \alpha(1+x)^{\alpha-1}$

If $x > 0$ we apply the Mean value theorem to h on [0, x]. There exists c in (0, x) such that $h(x) - h(0) = (x-0)\, h'(c) = x\alpha\, (1+c)^{\alpha-1} > x\,\alpha \qquad \because\ \alpha-1 > 0\ \&\ 1+c > 1$

Hence $(1+x)^{\alpha} - 1 > x\alpha$. This implies that $(1+x)^{\alpha} > 1 + x\alpha$ if $x > 0$

If $-1 < x < 0$ apply the Mean value theorem to h on [x,0]. There exists c in [x, 0] such that $(0-x)\, h'(c) = h(0) - h(x)$. $\Rightarrow h(x) - h(0) = x\, h'(c)$

$\Rightarrow (1+x)^{\alpha} - 1 = x\, \alpha(1+c)^{\alpha-1} > x\,\alpha \ \Rightarrow (1+x)^{\alpha} > 1 + x\alpha$

16.6.17 Examples:

(i) $f(x) = x + \dfrac{1}{x}\ (x \neq 0)$

$f'(x) = 1 - \dfrac{1}{x^2} = 0$ if $x = \pm 1$.

$f'(x) > 0$ iff $1 > \dfrac{1}{x^2}$ i.e $|x| > 1$

and $f'(x) < 0$ iff $|x| < 1$ so f is increasing (strictly) in $(-\infty, -1) \cup (1, \infty)$

and strictly decreasing in (-1, 1)

$f(1) = 2$ and $f(-1) = -2$

also $f(-x) = -f(x)$ so f is an add function

(ii) $g(x) = 3x - 4x^2$

$= -(4x^2 - 3x)$

$= -\left(2x - \dfrac{3}{4}\right)^2 + \dfrac{9}{16}$

$g'(x) = 3 - 8x = 0$ if $x = \dfrac{8}{3}$

$g''(x) = -8 < 0$

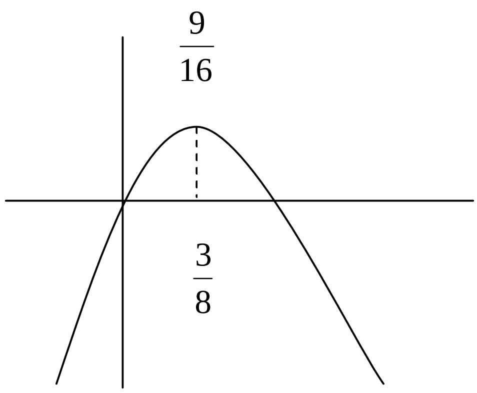
so g has maximum at $x = \dfrac{3}{8}$

maximum value is $g\left(\dfrac{3}{8}\right) = \dfrac{9}{16}$

$g'(x) < 0$ if $x > \dfrac{3}{8}$

g is strictly increasing in $(\dfrac{3}{8}, \infty)$

g is strictly decreasing in $(-\infty, \dfrac{3}{8})$



(iii) $K(x) = x^4 + 2x^2 - 4$

$K'(x) = 4x^3 + 4x$

$= 4x(x^2 + 1) = 4x(x^2 + 1)$

$= 0$ if $x = 0$

$> o$ if $x > 0$

< 0 if $x < 0$

Hence the curve $y = K(x)$ increases in $(0, \infty)$ and decreases in $(-\infty, 0)$

Also $K(x) = (x^2 + 1)^2 - 5 \geq 1 - 5 = -4$ for all x so

K(x) has minimum at x= 0.

(iv) Show that the function $f(x) = x^5 + 4x + 3$ is continuous, strictly increasing and the inverse g is differentiable at c = 1 and show that $g'(8) = \frac{1}{9}$

Solution: $f'(x) = 5x^4 + 4 > 0$ for all $x \in \mathbb{R}$

So f is strictly increasing on $\mathbb{R}$

Hence f is bijective

If $g = f'$ then for $x \in \mathbb{R} \Rightarrow g(f(x)) = x$

$f'(x) = 5x^4 + 4$ and

$g'(f(x)\ f'(x) = 1$ g is differentiable on $\mathbb{R}$.

so $g'(f(x)) = \dfrac{1}{5x^4 + 4} = \dfrac{1}{f`(x)}$

When x = 1, f(x) = 8 so $g'(8) = \dfrac{1}{5+4} = \dfrac{1}{9}$

16.6.18 Short Answer Question:

a) If f: $\mathbb{R} \to \mathbb{R}$ is differentiable at c show that $\lim_{n} (n(f(c + \frac{1}{n}) - f(c))) = f'(c)$

b) Show that there exist functions f: $\mathbb{R} \to \mathbb{R}$ such that $\lim_{n} (n(f(c + \frac{1}{n}) - f(c)))$ exists in $\mathbb{R}$ but f is not differentiable at c.

c) Discuss the differentiability of $f(x) = x^{\frac{1}{3}} (x \in \mathbb{R})$ at 0.

d) Discuss the differentiability at 0 of $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

e) If f is differentiable on $\mathbb{R}$ show that |f| is differentiable at all $x \in \mathbb{R}$ for which $f(x) \neq 0$. Evaluate $|f|'(x)$ whenever it exists.

f) Differentiate (i) $(\sin (x^k))^m$ $m \in \mathbb{N}$ and $k \in \mathbb{N}$ $(x \in \mathbb{R})$ (ii) $\tan (x^2)$ $\left(|x| < \frac{\pi}{2}\right)$

g) Assume that $L : (0, \infty) \to \mathbb{R}$ is such that $L'(x) = \frac{1}{x}$. Calculate the derivatives of

(i) $L(2x + 3)$ (ii) $(L(x^2))^3$ (iii) $L(a x)$ $(a > 0, x > 0)$ and (iv) $L(L(x))$ $(x > 0, L(x) > 0)$

(h) Prove straddle lemma:

Let $f: I \to \mathbb{R}$ be differentiable at $c \in I$. Given $\in > 0$ show that there exists $\delta(\in)>0$ such that if $c - \delta(c) < x \leq C \leq \upsilon < C + \delta(\in)$

$|f(\upsilon) - f(u) - (\upsilon\text{-}u)\, f'(c)| < \in (\upsilon - u)$.

i) Apply the Mean value theorem to show that $-x \leq \sin x \leq x$ for all $x \geq 0$ and $-|x| \leq \sin x \leq |x|$ for all x.

j) Show that $|\sin x - \sin y| \leq |x - y|$

k) Let $f: [a, b] \to \mathbb{R}$ be continuous on [a, b] and differentiable on (a, b)

If $\lim_{x\to a} f'(x) = A$ show that f is differentiable at a and $f'(a) = A$

l) Find the maxima and minima of x^2-3x+5

and the intervals in which f increases and decreases.

m) Let $h(x) = x^3-3x-4$. Find the maxima and minima for h and the intervals in which h is increasing and decreasing.

16.7 Solutions to SAQ's

(a) If f: $\mathbb{R} \to \mathbb{R}$ is differentiable at c then lim $\left(n\left(f\left(c+\frac{1}{n}\right) - f(c)\right)\right) = f'(c)$

Solution: Given $\in > 0$ there is $\delta > 0$ such that if $0 < |x-c| < \delta$, $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \in$

If $n \in \mathbb{N}$ and $n > \frac{1}{\delta}$, $\frac{1}{n} < \delta$ so putting $x = c + \frac{1}{n}$

we get $\left|\frac{f(c+\frac{1}{n}) - f(c)}{1/n} - f'(c)\right| < \in$

Thus if $n > \frac{1}{\delta}$ and $n \in \mathbb{N}$, $\left| n\left(f\left(c + \frac{1}{n}\right) - f(c)\right) - f'(c)\right| < \in$

Hence lim $\left\{n\left(f\left(c + \frac{1}{n}\right) - f(c)\right)\right\} = f^1(c)$

Solution: (b) Define $f(x) = x \sin \frac{\pi}{x}$ if $x \neq 0$; $f(0) = 0$

If $n \in \mathbb{N}$ $n\left\{f\left(0+\frac{1}{n}\right) - f(0)\right\} = n\left\{\frac{1}{n} \sin n\pi\right\} = \sin n\pi = 0$. So lim $n\left(f\left(\frac{1}{n}\right) - f(0)\right) = 0$

But, if $x \neq 0$, $\frac{f(x)-f(0)}{x-0} = \sin \frac{1}{x}$ and $\sin \frac{1}{x}$ does not converge at the point 0.

Hence f is not differentiable at 0.

Solution (c)

$f(x) = x^{1/3}$ ($x \in \mathbb{R}$) is not differentiable at 0.

$$\frac{f(x)-f(0)}{x-0} = \frac{x^{1/3}}{x} = \frac{1}{x^{2/3}}$$

lim $\frac{1}{n} = 0$ but since $f\left(\frac{1}{n}\right) = n^{2/3}$, lim $f\left(\frac{1}{n}\right) = +\infty$ Hence f is not differentiable at 0.

Solution (d):

$f(x) = x^2$ if x is rational and 0 if x is irrational f is differentiable at 0 and $f'(0) = 0$

for $x \neq 0$ $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{x^2 - 0}{x - 0} = x$ if x is rational

$= \dfrac{0 - 0}{x - 0} = 0$ if x is irrational

Hence $\left|\dfrac{f(x) - f(0)}{x - 0}\right| \leq |x|$

If $\in > 0$, $|x| < \in$ if $0 < |x-0| = |x| < \in$

Hence $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$

So f is differentiable at o and $f'(0) = 0$.

SAQ (e)

If f is differentiable on $\mathbb{R}$ then $|f|$ is differentiable at all $x \in \mathbb{R}$ for which $f(x) \neq 0$.

At such x, $|f|'(x)$ is given by

$$|f|'(x) = (\operatorname{sgn} f(x))\, f'(x) = \begin{cases} f'(x) \text{ if } f(x) > 0 \\ -f'(x) \text{ if } f(x) < 0 \end{cases}$$

The absolute value function $g(x) = |x|$ is differentiable at all $x \neq 0$ and

$$g'(x) = \left.\begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \end{cases}\right\} = \operatorname{sgn} x$$

if $h(x) = |f(x)|$ then $h = gof$. Since f is differentiable on $\mathbb{R}$ and g on $\mathbb{R} - \{0\}$, h is differentiable at all x for which $f(x) \neq 0$ and for such x,

$|f|'(x) = L' = g'(f(x))\, f'(x)$

$= (\operatorname{sgn} f(x))\, f'(x)$

In particular let $f(x) = x^2 - 1$ for $x \in \mathbb{R}$.

$$|f(x)| = |x^2 - 1| = \begin{cases} x^2 - 1 \text{ if } |x| > 1 \\ 1 - x^2 \text{ if } |x| < 1 \end{cases}$$

$f'(x) = 2x$

$$\text{So } |f|'(x) = (\operatorname{sgn} f(x));\ f'(x) = \begin{cases} 2x \text{ if } |x| > 1 \\ -2x \text{ if } |x| < 1 \end{cases}$$

SAQ (f)

(i) $(\sin x^k)^m \in \mathbb{N}, k \in \mathbb{N}$

Let $h(x) = \sin(x^k)$. $h'(x) = K(\sin x^k)\, x^{k-1}$

Then $(h^m)' = m\, h^{(m-1)}(x)\, h'(x)$

$= m(\sin x^k)^{m-1}\, k.\ \sin x^k \,.\, x^{k-1}$

$= k\, m\, (\sin x^k)^m \,.\, x^{k-1}$

$\Rightarrow (h^m)'(x) = k\, m \,.\, x^{k-1}, (\sin x^k)^m$

(ii) $\tan(x^2) \qquad (|x| < \frac{\pi}{2})$

Let $h(x) = \tan(x^2) \quad |x| \ (< \frac{\pi}{2})$

$\Rightarrow h'(x) = \sec^2 x^2, 2x$

$= 2x \sec^2 x^2.$

SAQ (g)

Assume that $L\,(0, \infty) \to \mathbb{R}$ is such that $L'(x) = \frac{1}{x}$ for $x > 0$. Calculate the derivative of

(i) $f(x) = L(2x+3)$ for $x > 0$

By chain rule $f'(x) = L'(2x+3) \,.\, 2 = \dfrac{2}{2x+3}$

(ii) $g(x) = (L(x^2))^3$ for $x > 0$

$g'(x) = 3(L(x^2))^2\, L'(x^2) \,.\, 2x$

$= 3(L(x^2)^2 \,.\, \dfrac{2x}{x^2} = \dfrac{6}{x}(L(x^2))^2$

(iii) $h(x) = L(a\,x)$ for $a > 0 \quad x > 0$

$h'(x) = L'(a\,x) \,.\, a = \dfrac{a}{ax} = \dfrac{1}{x}.$

(iv) $k(x) = L(L(x))$ when $L(x) > 0, x > 0$

$= L'(L(x)) \,.\, L'(x)$

$= \dfrac{1}{L(x)} \,.\, \dfrac{1}{x} = \dfrac{1}{x\,L(x)}.$

SAQ (h)

Straddle lemma

Let f: $I \to \mathbb{R}$ be differentiable at $c \in I$ Given $\in > 0$ there exists $\delta(\in) > 0$ such that if $C - \delta(\in) < u \le c \le \upsilon < c + \delta(\in)$, $|f(\upsilon) - f(u) - (\upsilon - u) f^1(c)| \le \in (\upsilon - u)$

Proof: Since f is differentiable at c, given $\in > 0$ there exists $\delta(\in) > 0$ such that if $0 < |x-c| \le \delta(\in)$

$$\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \in$$

Hence if $c - \delta(\in) < u < c$

$$\left|\frac{f(u)-f(c)}{u-c} - f'(c)\right| < \in$$

$\Rightarrow |f(u) - f(c) - f'(c)(u-c)| < \in (c-u)$.

if $c < \upsilon < c + \delta(\in)$

$|f(\upsilon) - f(c) - f^1(c)(v-u)| < \in (v-c)$

$\Rightarrow |f(v) - f(u) - (v-u)\, f'(c)|$

$= |f(v) - f(c) - f'(c) . (v-c) + f(c) - f(u) - f'(c)(u-c)|$

$\le |f(v) - f(c) - f'(c)(v-c)| + |f(u) - f(c) - f'(c)(u-c)|$

$< \in (c-u + v-c) = \in (v-u)$

SAQ (i)

Apply the Mean value theorem to show that

$-x \le \sin x \le x$ for all $x \ge 0$ and $-|x| \le \sin x \le |x|$ for all x.

Solution:

Define $g(x) = \sin x$. Then $g'(x) = \cos x$

If $x > 0$ by the Mean value theorem applied to g on [0, x]

We get c in (0, x) so that $\frac{g(x)-g(0)}{x-0} = g'(c) \Rightarrow \frac{\sin x}{x} = \cos c \Rightarrow -1 \le \frac{\sin x}{x} \le 1$

$\Rightarrow -x \le \sin x \le x$ if $x > 0$ If $x = 0$ equality occurs and the above inequality holds if $x \ge 0$

If $x < 0$, $-x > 0$ so that $-(-x) \le -\sin(-x) \le (-x)$

$\Rightarrow x \le -\sin x \le -x \Rightarrow x \le \sin x \le -x$

Thus for all $x \in \mathbb{R}$ $\quad -|x| \le \sin x \le |x|$

SAQ (j)

Show that $|\sin x - \sin y| \le |x-y|$

Write $f(x) = \sin x$ $\quad$ Then $f'(x) = \cos x$

If $x < y$ $\quad$ f is continuous in [x, y] and differentiable in (x, y) so by the Mean value theorem there is z in (x, y) such that

$$\frac{f(x)-f(y)}{x-y} = f'(z) \Rightarrow \frac{\sin x - \sin y}{x-y} = \cos z \Rightarrow \frac{|\sin x - \sin y|}{|x-y|} = |\cos z| \le 1$$

$\Rightarrow |\sin x - \sin y| \le |x - y|$

SAQ (K)

Let $f: [a, b] \to \mathbb{R}$ be continuous on [a, b] and differentiable on (a, b). If $\lim\limits_{x\to a} f'(x) = A$, then f is differentiable at a and $f'(a) = A$.

Solution: Since $\lim\limits_{x\to a} f'(x) = A$, given $\in > 0$ there is δ such that

$0 < \delta < b-a$ and if $a \le x < a + \delta$ then $|f'(x) - A| < \in$

If $a < y < a + \delta$, we apply the Mean value theorem to f on [a, b], then there is x_0 such that

$a < x_0 < y$ and $\dfrac{f(y)-f(a)}{y-a} = f'(x_0)$

$$\Rightarrow \left|\frac{f(y)-f(a)}{y-a} - A\right| = |f'(x_0) - A| < \in$$

This is true for every $\in > 0$ so $\lim\limits_{y\to a} \dfrac{f(y)-f(a)}{y-a} = A \Rightarrow f'(a) = A$

SAQ (l)

$f(x) = x^2-3x+5 = (x - \frac{3}{2})^2 + \frac{11}{4}$ $\quad f'(x) = 2(x - \frac{3}{2})^2 = 0 \Leftrightarrow x = \frac{3}{2}$

$f'(x) = 2 > 0$ so f has minimum at $x = \frac{3}{2}$

$f'(x) > 0$ if $x > \frac{3}{2}$ and $f'(x) < 0$ if $x < \frac{3}{2}$

Hence f is increasing in $\left(\frac{3}{2}, \infty\right)$ and decreasing in $\left(-\infty, \frac{3}{2}\right)$

Remark: If $a \neq 0$ f has extremum value at $\frac{-b}{2a}$ $f(x) = ax^2 + bx + c$.

If $a > 0$ the extremum is minimum and If $a < 0$ the extremum is maximum

In either case the value is $f(a) = \frac{4ac - b^2}{4a}$

SAQ (m)

$h(x) = x^3 - 3x - 4$

$h'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x-1)(x+1)$

$h'(x) = 0$ if $x = -1$ or $x = 1$

$h'(x) < 0$ if $x^2 < 1$ ie $|x| < 1$

$h'(x) > 0$ if $x^2 > 1$ ie $|x| > 1$

Hence h is increasing (strictly) in $(-\infty, -1) \cup (1, \infty)$ and h is strictly decreasing in $(-1, 1)$ $h(-1) = -1 + 3 - 4 = -2$ $h(1) = 1 - 3 - 4 = -6$ $h(x) = x^3 \left(1 - \frac{3}{x} - \frac{4}{x^3}\right) \to -\infty$ as $x \to -\infty$

$h(x) \to \infty$ as $x \to +\infty$.

16.9 Summary

After introducing the notion of derivative at a point a few interesting consequences of differentiability of a function are proved and then Caratheodory theorem and the chain rule are established. These results are followed by the Mean value theorems and their applications to extremum problems. Finally number of applications and examples are provided to enable the student to gain experience in finding solutions for problems on differentiation.

16.10 Technical terms:

Differentiable, derivative, Intermediate value, maximum, minimum and extremum

16.11 Exercises

Use the definition to find the derivative of the following function

(i) $f(x) = \begin{cases} x^3 \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$

(ii) $h(x) = \begin{cases} \sqrt{x} \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$

(iii) $g(x) = \begin{cases} \frac{1}{x} \text{ if } x > 0 \\ \frac{1}{x^2} \text{ if } x < 0 \\ 0 \text{ if } x = 0 \end{cases}$

(iv) $h(x) = |x|^3 \, (x \in \mathbb{R})$

2. Is $f(x) = \frac{1}{x^5}$ $(x \neq 0)$, $f(0) = 0$ differentiable at 0?

3. $f(x) = \begin{cases} x^2 \text{ if x is rational} \\ 0 \text{ if x is irrtional} \end{cases}$

show that f is differentiable at zero. Find $f'(0)$

4. Which of the following are true? Justify.

(a) If f is differentiable at $c \in \mathbb{R}$, so is |f|

(b) If |f| is differentiable at $c \in \mathbb{R}$, so is f.

5. Differentiate and simplify

(a) $g(x) = \sqrt{5 - 2x + x^2}$

(b) $k(x) = \tan(x^2)$ $\quad |x| < \frac{\pi}{2}$

6. If f(x) = f(-x) for $x \in \mathbb{R}$, f is called an even function if f(x) = -f(-x) for $x \in \mathbb{R}$, f is called odd function. (a) Show that if f(x) = 0 if x is rational and f(x) = -x if x is irrational then f is even function.

(b) if g(x) = 0 if x is rational and g(x) = x if x is irrational then g is odd function

(c) If f is odd and differentiable, show that f' is an even function

(d) If f is even and differentiable show that f' is an odd function.

(e) Give examples of even functions and odd function which are not differentiable.

7. Let $h(x) = x^3+2x+1$, $x \in \mathbb{R}$. Assume that h is invertible show that h^{-1} is differentiable and find $((h^{-1}(y))^`$ when y = 0, and y = z

8. Prove corollary 16.4.10

9. Prove corollary 16.4.11

Answers to exercise:

1. (i) $f'(x) = \begin{cases} 3x^2 \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$

(ii) $h'(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, \text{ if } x > 0 \\ 0 \text{ if } x \le 0 \end{cases}$

(iii) $g'(x) = \begin{cases} \dfrac{-1}{x^2} \text{ if } x > 0 \\ \dfrac{-2}{x^3} \text{ if } x < 0 \end{cases}$

(iv) $h'(x) = \begin{cases} 3x^2 \; x \ge 0 \\ -3x^2 \; x < 0 \end{cases}$

and $g'(0)$ doesn't exist

2. f is not differentiable at x = 0

4. a. True

b. False eg: $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ -x \text{ if } x \text{ is irrational} \end{cases}$

then $|f| = x \quad \forall x \in \mathbb{R}$ is differentiable, but f is not differentiable

5. a. $\dfrac{x-1}{\sqrt{5-2x+x^2}}$, b. $2x \, \text{Sec}^2 (x^2)$

If $f(x) = \sin x \quad \left(x \in \left[\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right)$ show that f is monotonically increasing and one – one.

Let $f(x) = \begin{cases} x^n & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ where n is a positive integer

Show that f is differentiable when $x \neq 0$. Is f differentiable at 0? For what values of $n \in \mathbb{N}$ is f' differentiable?

Find the derivative of

(a) $\sqrt{5 - 2x + x^2}$ (b) $(\sin x^k)^m$ ($m \in \mathbb{N}$, $k \in \mathbb{N}$)

Find the points in $\mathbb{R}$ at which the following are differentiable and find the derivative at these points

(a) $|x + 1| + |x-1|$

(b) $2x + |x|$

(c) $x\,|x|$

(d) $|\sin x|$

If $f(x) = f(-x)$ for all $x \in \mathbb{R}$ and f is differentiable on $\mathbb{R}$ show that $f'(x) = -f'(-x)$

If $h(x) = x^3+2x+1$ show that h is one-one $\mathbb{R}$.

Find $(h^{-1})(x)$ when $x \in \{0, 1, -1\}$.

16.12 Model Examination Questions

1. State and prove chain rule for differentiation
2. Show that if f is differentiable in (a, b) then f is continuous in (a, b). Is the converse true? Justify.
3. State and prove Rolles' theorem
4. State and prove the mean value theorem
5. State and prove Caretheodory theorem for derivatives
6. State and prove Darboux theorem for derivatives

7. Show that if f is differentiable (a, b) and has maximum at $c \in (a, b)$ then $f'(c)=0$

8. If $f(x) = x^3 - x$ in [-1, 1] find all x in (-1, 1) such that $f'(x) = 0$

9. If r> o is a rational number and g: $\mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) = x^r \sin \frac{1}{x} \text{ if } x \neq 0 \text{ and } g(0) = 0$$

determine r for which g is differentiable at 0.

10. Discuss the differentiability of $f(x) = |x| + |x+1|$

11. Apply the mean value theorem to prove $e^x > 1 + x$ if $x \in \mathbb{R}$

16.13 Model Practical problem with solution

$$\text{Let } g(x) = \begin{cases} -1 \text{ if } -1 \le x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } 0 < x \le 1 \end{cases}$$

Show that g is not the derivative of any function on [-1,1]

Aim: To show that there does not exist a differentiable function f on [-1, 1] such that $f'(x) = g(x)$ for $-1 \le x \le 1$.

Definition: If $I \subseteq \mathbb{R}$ is an interval, $a \in I$, $f: I \to \mathbb{R}$

We say that f is differentiable at c if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists in } \mathbb{R}.$$

In this case we call the limit, the derivative of f at a and denote by $f'(a)$

If f is differentiable at every point of I we say that f is differentiable on I.

Result: Darboux theorem: If f is differentiable on [a, b] and K is any real number between $f'(a)$ and $f'(b)$ then there exists c in (a, b) such that $f'(x) = c$

Solution: Suppose there is a differentiable function f on [-1,1] such that $f'(x) = g(x)$ for all x in [-1, 1]. Then by Darboux theorem for every k between $-I = g(-1) = f'(1)$ and $I = g(1) = f'(1)$ there must exist c in (-1, 1) such that $g(c) = f'(c) = k$.

But this does not hold good as Range $g = \{-1, 0, 1\}$.

K. SAMBASIVA RAO

Lesson - 17

DIFFERENTIATION II - L'HOSPITAL'S RULES AND TAYLOR'S THEOREM

17.1 Objective:

In this lesson the student will be introduced to some useful techniques for solving problems on differential calculus based on "Indeterminate forms". Taylor's theorem and applications will also be placed before the student to enable to employ these techniques wherever necessary.

17.2 Structure:

17.3 Introduction

17.4 L'Hospital's Rules

17.5 Proofs for L'Hospital's Rules

17.6 Higher Order Derivatives

17.7 Solutions to SAQ's

17.8 Summary

17.9 Technical Terms

17.10 Exercises

17.11 Answers

17.12 Model Examination Questions

17.13 Model Practical Question with Solution

17.3 Introduction:

The (Lagrange's) mean value theorem relates the slope of the chord connecting the end points on the graph to the slope of the tangent at a point on the curve corresponding to an interior point of the domain. Because of various important consequences and applications this fundamental theorem has a special status in Differential Calculus. An important extension of this illustrious theorem is Cauchy's mean value theorem. Evaluation of the limit of a quotient is hitherto restricted to the situation where the denominator converges to a nonzero (finite) limit and the cases where $\lim\limits_{x} f(x) = \lim\limits_{x} g(x) = 0$. $\lim\limits_{x} g(x) = \infty$ were not discussed earlier, $\dfrac{f}{g}$ is said to be an indeterminate form when $\lim\limits_{x} f(x) = \lim\limits_{x} g(x) = 0$. The symbolism $\dfrac{0}{0}$ is used to refer to this situation. Differentiation plays significant role in situations of this type. The limit theorem, known as L'Hospital's rule (H silent and is pronced Lopila's Rule) first appeared in "L' Analyse dis infinement patits" written by Marquis Guillame Francors L'Hospital, though it was first discovered by John Bernouli from whom L'Hospital learnt the new differential calculus. The initial theorem was refined and extended and the various results are collectively referred to as L'Hospital's Rules. Other indeterminate forms are represented by the symbols $\frac{\infty}{\infty}$, $0, \infty$, 0^0, 1^∞, ∞^0 and $\infty - \infty$. We focus our attention on the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. The other indeterminate forms are usually reduced to one of the above two, by algebraic manipulations, taking logarithms and exponential.

A very useful technique in Analysis of real functions is the approximation of a function by polynomials. Brook Tayolor is the inventor of this technique that involves higher derivatives of the function. However a detailed discussion on the remainder which is the basis for estimation of the error was initiated by lagrange after a long gap.

17.4 L'Hospital's Rules:

17.4.1 Theorem: Let f, g be defined on [a, b], $f(a) = g(a) = 0$ and $g(x) \neq 0$ for $a < x < b$. If f and g are differentiable at a and $g'(a) \neq 0$ then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

Proof: If $a < x < b$,

$$\frac{f(x)}{g(x)} = \frac{f(x)-0}{g(x)-0} = \frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \frac{f(x)-f(a)}{x-a} \cdot \frac{x-a}{g(x)-g(a)}$$

Since $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ and $g'(a) \neq 0$ so that

$$\frac{1}{g'(a)} = \lim_{x \to a} \frac{x-a}{g(x)-g(a)}$$

We get $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and $= \frac{f'(a)}{g'(a)}$

17.4.2 Example: The result fails to hold if one of f(x) and g(a) is nonzers as is evident from the following example.

Let $f(x) = x + 4$ and $g(x) = 2x + 3$ for $x \in R$.

$\lim_{x \to 0} f(x) = 4$, $\lim_{x \to 0} g(x) = 3$ so $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{4}{3}$

$f'(x)=1$, $g'(x)=2$ for all x so that $\dfrac{f'(0)}{g'(0)}=\dfrac{1}{2}$

17.4.3 Cauchy Mean Value Theorem:

Let f and g be continuous on $[a,b]$ and $g'(x)\neq 0$ for all x in (a, b). Then there exists c in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

Proof: If $g(a)=g(b)$ by Roller's theorem there is some c in (a,b) such that $g'(c)=0$ contrary to the hypotheses. Hence $g(b)\neq g(a)$.

Define $\phi(x)=A(g(x)-g(a))-(f(x)-f(a))$ for $a\leq x\leq b$ where A is chosen so that $\phi(a)=\phi(b)=0$. This condition implies that $A=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

Clearly ϕ is differentiable in (a, b) and continuous in [a, b] and $\phi(a)=\phi(b)=0$.

Hence by Rolle's Theorem there is c in (a, b) such that $\phi'(c)=0$.

$$\Rightarrow \quad A \quad g'(c)-f'(c)=0$$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Geometric interpretation: The function $H:[a,b]\to R^2$ defined by $H(t)=(f(t),g(t))$ represents a curve in the plane. The above conclusion means that there is a point $H(c)=(f(c),g(c))$ on the curve so that the slope of the tangent at this point is equal to the slope of the chord jouring H(a) and H(b).

Remark: When $g(x)=x$ Cauchy mean value theorem reduces to the mean value theorem.

17.4.4: Rule I :

Let $-\infty\leq a<c<b\leq\infty$ and f, g be differentiable in (a, b) and let $g'(x)\neq 0$ for all $x\in(a,b)$. Suppose that $\lim\limits_{x\to a+} f(x)=\lim\limits_{x\to a+} g(x)=0$. Then

(a) If $\lim\limits_{x\to a+}\dfrac{f'(x)}{g'(x)}=L\in R$ then $\lim\limits_{x\to a+}\dfrac{f(x)}{g(x)}=L$

(b) If $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$ then $\lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = L$

Δ similar conclusions are valid when the left limits are taken at b and limits are taken at c.

17.4.5 Rule II:

Let $-\infty \le a < c < b \le \infty$ and f, g be differentiable in (a, b) and let $g'(x) \ne 0$ for all $x \in (a,b)$. Suppose that $\lim\limits_{x \to a+} g(x) = \pm\infty$. Then

(a) If $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = L \in R, \lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = L$

(b) If $\lim\limits_{x \to a+} \dfrac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\} \lim\limits_{x \to a+} \dfrac{f(x)}{g(x)} = L$

similar conclusions are valid when the left limits are taken at b and limits are taken at c.

Indeterminate forms such as $\infty - \infty$, $0.\infty$, 1^∞, 0^∞. 0^0, ∞^0 can be reduced to the above cases by algebraic multiplications and use of logarithmic and exponential functions.

Proofs of Rule I and Rule II are provided in Appendix.

17.4.6 Examples:

(i) Find $\lim\limits_{x \to 0+} \dfrac{\sin x}{\sqrt{x}}$.

$f(x) = \sin x, g(x) = \sqrt{x}$ are defined for x in $(0, \infty)$

$$\lim_{x \to 0+} \sin x = \lim_{x \to 0+} \sqrt{x} = 0$$

$f'(x) = \cos x, g'(x) = \dfrac{1}{2\sqrt{x}}$ for $x \in (0, \infty)$

$$\lim_{x \to 0+} \frac{f'(x)}{g'(x)} = \frac{\cos x}{\frac{1}{2\sqrt{x}}} = \frac{\sqrt{x}\cos x}{2}$$

$\lim\limits_{x \to 0+} \sqrt{x} = 0$ and $\lim\limits_{x \to 0} \cos x = 1$.

Hence $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{x}} = \lim\limits_{x \to 0} 2\sqrt{x} \cos x = 0$.

(ii) Find $\lim\limits_{x\to 0+}\dfrac{1-\cos x}{\sqrt{x}}$:

Cos x and $\sqrt{x}$ are defined for $x\in(0,\infty)$

$\lim\limits_{x\to 0+} 1-\cos x=0$, $\lim\limits_{x\to 0+}\sqrt{x}=0$

$(1-\cos x)' = \sin x, (\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$

$\lim\limits_{x\to 0+}\dfrac{\sin x}{\frac{1}{2\sqrt{x}}}=\lim\limits_{x\to 0+} 2\sqrt{x}\sin x=0$

Another Method:

for $x>0$, $\dfrac{1-\cos x}{\sqrt{x}}=-\left(\dfrac{\cos x-\cos 0}{x}\right)\sqrt{x}$.

$\Rightarrow \lim\limits_{x\to 0+}\dfrac{1-\cos x}{\sqrt{x}}=\lim\limits_{x\to 0+}-\left(\sqrt{x}\right)\dfrac{\cos x-\cos 0}{x}$

$=\lim\limits_{x\to 0+}\sqrt{x}\ .\ \lim\limits_{x\to 0+}\dfrac{\cos x-\cos 0}{x}$

= 0.0 = 0.

(iii) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

Method 1 : The function $f(x)=e^x$. So

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0)=e^0=1$$

Method 2: Let $f(x)=e^x-1$ and $g(x)=x$. Then $f'(x)=e^x, g'(x)=1$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0} g(x)=0 \text{ and } \lim_{x\to 0}\frac{f'(x)}{g'(x)}=\lim_{x\to 0}\frac{e^x}{1}=e^0=1 .$$

Hence $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$.

(iv) $\lim_{x\to 0}\frac{e^x-1-x}{x^2}=\frac{1}{2}$.

Let $f(x)=e^x-1-x, g(x)=x^2$.

$\lim_{x\to 0} f(x)=\lim_{x\to 0} g(x)=0$ $\quad f'(x)=e^x-1, g'(x)=2x$ and $\lim_{x\to 0} f'(x)=\lim_{x\to 0} g'(x)=0$.

$f''(x)=e^x,\ \ g''(x)=2$.

by (iii) above, $\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\lim_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$

Hence $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f''(x)}{g''(x)}=\frac{1}{2}$.

17.4.7 Examples:

(a) $\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}=\lim_{x\to 0+}\frac{\cos x}{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}}=\lim_{x\to 0+}2\sqrt{x}\cos x=0$

Here $f(x)=\sin x$ and $g(x)=\sqrt{x}$ are defined $(0,\infty)$ $\lim_{x\to 0+} f(x)=\lim_{x\to 0+} g(x)=0$

$g'(x)=\frac{1}{\sqrt{2}\sqrt{x}}\neq 0$ in $(0,\infty)$ Hence Rule I applies.

(b) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$

Here $f(x)=1-\cos x$ and $g(x)=x^2$ are derived on $(0,\infty),(-\infty,0)$ $\lim_{x\to 0} f(x)=\lim_{x\to 0} g(x)=0,$ $g'(x)\neq 0$ in $(0,\infty)$ and $(-\infty,0)$

Hence Rule I applies. We have to consider left limit and right limit at 0.

$\lim_{x\to 0+}\frac{f(x)}{g(x)}=\lim_{x\to 0+}\frac{f'(x)}{g'(x)}=\lim_{x\to 0+}\frac{\sin x}{2x}$ and $\lim_{x\to 0-}\frac{f(x)}{g(x)}=\lim_{x\to 0-}\frac{f'(x)}{g'(x)}=\lim_{x\to 0-}\frac{\sin x}{2x}$

The same process is to be adopted to find $\lim_{x\to 0}\frac{\sin x}{x}$.

Observe that we cannot consider directly the limit as $x\to 0$ as $g'(0)=0$.

(c) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=\lim\limits_{x\to 0}\dfrac{e^x}{1}=1$.

Here $f(x)=e^x-1, g(x)=x$, $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} g(x)=0$, $g'(x)=1$ for all x. Hence Rule I applies.

Observe that application of Rule I requires that the derivative of e^x is e^x. However this derivative is obtained at 0 by considering the limit on l.h.s.

(d) $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}=\lim\limits_{x\to 1}\dfrac{1/x}{1}=1$

Here $f(x)=\ln x, g(x)=x-1$. The functions are defined on $(0,\infty)$ $\lim\limits_{x\leftarrow 1} f(x)=\lim\limits_{x\leftarrow 1} g(x)=0$

$g'(x)=1$ for all $x\in(0,\infty)$ and $f'(x)=\dfrac{1}{x}$.

Hence Rule I applies.

Here also $\lim\limits_{x\leftarrow 1}\dfrac{\ln x}{x-1}=\lim\limits_{x\to 1}\dfrac{\ln x-\ln 1}{x-1}=(\ln x)'$ at $x=1$ which is obtained directly.

(e) $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$.

Here $f(x)=\ln x, g(x)=x$. are defined on $(0,\infty)$ and $f'(x)=\dfrac{1}{x}, g'(x)=1$. By Rule II

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0\ .$$

(f) $\lim\limits_{x\to\infty} e^{-x}.x^2$.

$f(x)=x^2, g(x)=e^x. f'(x)=2x, g'(x)=e^x$

By Rule II $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}=\lim\limits_{x\to\infty}\dfrac{2x}{e^x}$.

We apply Rule II again to $2x$ and e^x and get $\lim\limits_{x\to\infty}\dfrac{2x}{e^x}=\lim\limits_{x\to\infty}\dfrac{2}{e^x}=0$.

(g) $\lim\limits_{x\to 0+}\dfrac{\ln\sin x}{\ln x}$.

$f(x) = \ln \sin x, g(x) = \ln x$ on $\left(0, \frac{\pi}{2}\right)$.

$f'(x) = \frac{\cos x}{\sin x} \quad g'(x) = \frac{1}{x}$. Rule II is applicable.

$$\lim_{x \to 0+} \frac{\ln \sin x}{\ln x} = \lim_{x \to 0+} \frac{(\cos x/(\sin x)}{1/x} = \lim_{x \to 0+} \cos x . \frac{x}{\sin x}$$

$$= \lim_{x \to 0+} \cos x . \lim_{x \to 0+} \frac{x}{\sin x} = 1.$$

17.4.8 Other indeterminate forms.

Many indeterminate forms such as $\infty - \infty$, 0.0, 1^{∞}, 0^0, ∞^0 can be reduced to one of the L'Hospital's Rules stated earlier by use of algebraic manuplations and the use of logarithmic and exponential functions. Instead of formulating these variations as theorems, we illustrate the pertinent techniques by means of examples.

Examples:

(a) Let $I = \left(0, \frac{\pi}{2}\right)$ and consider $\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ which has the indeterminate form $\infty - \infty$

$$\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0+} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0+} \frac{\cos x - 1}{x \cos x + \sin x} = \lim_{x \to 0+} \frac{-\sin x}{2\cos x - x \sin x}$$

$$= \frac{0}{2} = 0$$

(b) Let $I = (0, \infty)$ and consider $\lim_{x \to 0+} x \ln x$ which has the indeterminate form $0.(-\infty)$ we have

$$\lim_{x \to 0+} (x \ln x) = \lim_{x \to 0+} \frac{\ln x}{1/x} = \lim_{x \to 0+} \frac{1/x}{-1/x^2} = \lim_{x \to 0+} (-x) = 0.$$

(c) $\lim_{x \to 0+} (x^x)$ $(x > 0)$. This is of the form 0^0, We have $x^x = e^{x \ln x}$

Since the function $y \to e^y$ is continuous at 0.

Hence $\lim_{x \to 0+} x^x = \lim_{x \to 0+} \left(e^{x \ln x}\right) = e^0 = 1$.

(d) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x, (x>1)$. This is of the form 1^∞. Write $\left(1+\frac{1}{x}\right)^x = e^{x\ln\left((1+\frac{1}{x}\right)}$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{-1}\left(-x^{-2}\right)}{-x^{-2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1$$

Since $y \to e^y$ is continuous at 1, we have $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$.

(e) $\lim_{x\to 0+}\left(1+\frac{1}{x}\right)^x (x>0)$. This is of the form ∞^0.

$$\lim_{x\to 0+} x\ln\left(1+\tfrac{1}{x}\right) = \lim_{x\to 0+}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to 0+}\frac{1}{1+\frac{1}{x}} = 0$$

Thus we have $\lim_{x\to 0+}\left(1+\frac{1}{x}\right)^x = e^0 = 1$.

17.4.9 SAQ: Let f be differentiable on $(0,\infty)$ and suppose that $\lim_{x\to\infty} f(x) + f'(x) = L$.

Show that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.

17.4.10 SAQ: Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

and $g(x) = \sin x$ if $x \in R$. Show that $\lim_{x\to 0}\frac{f(x)}{g(x)} = 0$. Explain why L'Hospital rule is not applicable.

17.5. Proofs for L'Hospital's Rules:

17.5.1 Proof for L' Hospital's Rule I :-

Proof: If $a<\alpha<\beta<b$ and $g(\beta) = g(\alpha)$, by the Mean Value Theorem.

There is $c \in (\alpha,\beta)$ such that $g'(c) = \frac{g(\beta)-g(\alpha)}{\beta-\alpha} = 0$ which is a contradiction. So $g(\beta) \neq g(\alpha)$

whenever $\alpha \neq \beta$.

By the Cauchy Mean Value Theorem applied to f, g on $[\alpha, \beta]$ there is $u \in (\alpha, \beta)$ such that

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}=\frac{f'(u)}{g'(u)} \qquad \rightarrow \quad (2)$$

(a): suppose $L=\lim_{x\to a+}\frac{f'(x)}{g'(x)} \in R$. If $\in > 0$ there is $\delta > 0$ such that $a < a+\delta < b$ and if $x \in (a,c)$

where $c = a+\delta$, $L-\frac{\in}{2}<\frac{f'(x)}{g'(x)}<L+\frac{\in}{2}$. $\qquad \rightarrow \quad (3)$

Thus any u that corresponds to $[\alpha, \beta]$ satisfying (2) satisfies (3),

$$L-\frac{\in}{2}<\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}<L+\frac{\in}{2}$$

Letting $\alpha \to a+0$ we get $L-\in < L-\frac{\in}{2} \leq \frac{f(\beta)}{g(\beta)} \leq L+\frac{\in}{2} < L+\in$. if $a < \beta \leq c$.

$\Rightarrow \left|\frac{f(\beta)}{g(\beta)}-L\right| < \in$ if β belongs to (a, c]

Hence $\lim_{\beta\to a+}\frac{f(\beta)}{g(\beta)}=L=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$

(b) If $L=+\infty$, for every $M>0$ there corresponds $c \in (a,b)$

such that if $a<x<c$ $\quad \frac{f'(x)}{g'(x)}>M$.

suppose $a<\beta<c$. For every α in (a,β) there exists u in (α,β) satisfying

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}=\frac{f'(u)}{g'(u)}$$

$\Rightarrow$ for ever α in (a,β), $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)}>M$.

Since $\lim_{\alpha\to a+} f(\alpha)=\lim_{\alpha\to a+} g(\alpha)=0$, we by letting $\alpha \to a+ \frac{f(\beta)}{g(\beta)} \geq M$.

This holds for every $\beta \in (a,c)$. Hence $\lim_{x\to a+} \frac{f(x)}{g(x)} = +\infty = L$. The proof of the following L'Hospital's rule for left limits is similar to the above result, hence omitted.

17.5.2 Theorem:

Let $-\infty \le a < b \le \infty$, f, g be differentiable on (a, b) and $g'(x) \ne 0$ for all x in (a, b). Suppose further that $\lim_{x\to b-0} f(x) = 0 = \lim_{x\to b-0} g(x) = 0$.

Then if $\lim_{x\to b-0} \frac{f'(x)}{g'(x)} = L$ exists in R or $\lim_{x\to b-0} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$. Then $\lim_{x\to b-0} \frac{f(x)}{g(x)} = L$.

The following corollary is an immediate consequence of L' Hospitals rules for left and right limits.

Corollary: Let $-\infty \le a < c < b \le \infty$; f and g be differentiable on (a, b) and $g'(x) \ne 0$ for all x in (a, b). Suppose further that $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$. Then if $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$ where $L \in R$ or $L \in \{-\infty, \infty\}$

then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$.

Proof for L' Hospital's Rule II:

17.5.3 : If $a < \alpha < \beta < b$ there is $u \in (\alpha, \beta)$ such that $\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(t)}{g'(t)}$.

(a) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = l \in R$ and $\in > 0$ there exists $c \in (a,b)$ such that

$$\left|\frac{f'(x)}{g'(x)} - l\right| < \in \text{ if } a < x \le c . \Rightarrow \quad l - \in < \frac{f'(x)}{g'(x)} < l + \in \text{ if } a < x \le c .$$

$$\Rightarrow \quad l - \in < \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} < l + \in \text{ if } a < \alpha < \beta \le c \quad \rightarrow \qquad (1)$$

Fix β in (a, c). since $\lim_{x\to a+} g(x) = +\infty$ there is c_1.

such that $a < c_1 < \beta$ and $g(\alpha) > 1 + |g(\beta)|$ for $\alpha \in (a, c_1)$

$\Rightarrow \quad g(\alpha) > 0$ and $g(\alpha) > g(\beta)$ if $\alpha \in (a, c_1)$.

$$\Rightarrow \quad \frac{g(\alpha)-g(\beta)}{g(\alpha)} > 0.$$

$$\Rightarrow \quad (l-\in)\frac{g(\alpha)-g(\beta)}{g(\alpha)} < \frac{f(\alpha)-f(\beta)}{g(\alpha)} < (l+\in)\frac{g(\alpha)-g(\beta)}{g(\alpha)}$$

$$\Rightarrow \quad (l-\in)\left(1-\frac{g(\beta)}{g(\alpha)}\right) < \frac{f(\alpha)}{g(\alpha)} - \frac{f(\beta)}{g(\alpha)} < (l+\in)\left(1-\frac{g(\beta)}{g(\alpha)}\right)$$

Since $\lim_{x\to a+} g(x) = +\infty \qquad \lim_{\alpha\to a+} \frac{g(\beta)}{g(\alpha)} = 0$.

Hence for any δ in (0, 1) there exists d in (a, c_1) such that

$0 < \frac{g(\beta)}{g(\alpha)} < 1$ and $\frac{|f(\beta)|}{g(\alpha)} < \delta$ if $\alpha \in (a,d)$.

Hence $(l-\in)(1-\delta)-\delta < \frac{f(\alpha)}{g(\alpha)} < (l+\in)+\delta$.

If we take the above $\delta \to 0 < \delta < \min\left\{1, \in, \frac{\in}{1+|l|}\right\}$ then we get

$l - 2\in \le \frac{f(\alpha)}{g(\alpha)} \le l + 2\in$ if $a < \alpha < d$.

i.e. $\left|\frac{f(\alpha)}{g(\alpha)} - l\right| < 2\in$ if $a < \alpha < d$

Hence $\lim_{\alpha\to a+} \frac{f(\alpha)}{g(\alpha)} = l$.

(b) If $l = +\infty$ and $M > 1$ there exists $c \in (a,b)$ such that $\frac{f'(u)}{g'(u)} > M$ for all $u \in (a,c)$.

As above if $a < \alpha < \beta < c$, we get $\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} > M$

Since $g(x) \to \infty$ as $x \to a+$ as above we fix β in (a, c) and find c_1 in (a, β) such that

$g(\alpha) > 2g(\beta)$ and $g(\alpha) > 0$ so that $0 < \dfrac{g(\beta)}{g(\alpha)} < \dfrac{1}{2}$ if $a < c < c_1$

This fields $\dfrac{f(\alpha) - f(\beta)}{g(\alpha)} > M\left(1 - \dfrac{g(\beta)}{g(\alpha)}\right) > \dfrac{M}{2}$.

So that $\dfrac{f(\alpha)}{g(\alpha)} > \dfrac{1}{2}M + \dfrac{f(c)}{g(c)} > \dfrac{1}{2}(M-1)$ for α in (a, c_1).

This holds for every $M > 1$.

$$\text{Hence } \lim_{\alpha \to a+} \frac{f(\alpha)}{g(\alpha)} = +\infty$$

The proof similar when $l = -\infty$.

Remark: L'Hospitals rule II holds for the left limit at b.

Corollary: Let $-\infty \le a < c < b \le \infty$ and f and g be differentiable in (a, b) and $g'(x) \ne 0$ for all x in (a, b).

Further suppose that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = l \in R \text{ or } l \in \{-\infty, \infty\}.$$

If $\lim_{x \to} g(x) = +\infty$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = l .$$

17.6 Higher Derivatives:

Let I be an interval and $f : I \to R$ be differentiable on I. For every x in I we write $f'(x)$ for the derivative of f at x. Thus $f' : I \to R$ defines a function on I. If f' is differentiable at $x \in I$ we call the derivative of f' at x, the second derivative of f at x and denote this by $f''(x)$. If f' has derivative on I, we call the function $f'' : I \to R$, the second derivative function. We thus inductively define $f^{(n)}(x)$ at $x \in I$ as the derivative of $f^{(n-1)}$ at x. In order that $f^{(n)}$ exists at x it is necessary that $f^{(n-1)}$ is defined in a neighborhood of I. The sequence $\{f^{(n)}(x) / n \ge 0\}$ where

$f^{(0)}(x) = f(x)$ and $f^{(n)}(x) = \left(f^{(n-1)}\right)^1(x)$ for $n \geq 1$ is called the sequence of successive derivatives at x.

These successive derivatives play an important role in dermining the behaviour of the function f at x.

17.6.1 Taylor's Theorem:

Let $n \in N$, $I = [a,b], f : I \to R$ be such that f and its derivatives $f', f'', ... f^{(n)}$ are continuous on I and $f^{(n+1)}$ exists on (a, b). If $x_0 \in I$, then for any x in I there is a point c between x_0 and x such that $f(x) = f(x_0) + \frac{(x-x_0)}{1!} f^1(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$.

Proof: We apply Rolle's theorem to the function

$$G(t) = F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0) \text{ where}$$

$$F(t) = f(x) - f(t) - (x-t) f'(t) - ... - \frac{(x-t)^n}{n!} f^{(n)}(t). \text{ for } t \in J$$

J, being be the closed interval with end prints x_0 and x.

Clearly $G(x_0) = G(x) = 0$ and $F'(t) = \frac{(x-t)^n}{n!} f^{(n+1)}(t)$

By Rolle's Theorem, there exists c in J with $x_0 \neq c \neq x$ such that $G'(c) = 0$.

$$\Rightarrow F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0) = 0$$

$$\Rightarrow F(x_0) = -\frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}.F'(c) \qquad = \frac{1}{n+1}.\frac{(x-x_0)^{n+1}}{(x-c)^n}.\frac{(x-c)^n}{n!}.f^{(n+1)}(c)$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

$$\Rightarrow f(x) = f(x_0) + (x-x_0) f'(x_0) + ... + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

17.6.2 Lagrange's form of the Remainder:

We write $P_n(x) = f(x_0) + (x - x_0) f'(x_0) + ... + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0)$

P_n is called the n-th degree Taylor Polynomial.

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{(n+1)}$$

is called the Lagrange (or derivative) form of the remainder.

We then get $f(x) = P_n(x) + R_n(x)$.

17.6.3 Remark:

The Remainder $R_{n+1}(x)$ after n terms in the Taylor's theorem involves x, n and c that depends on x. The theorem guarantees the existence of one c satisfying the condition of the theorem but this c is not necessarily unique. As such $R_n(x)$ is not a function of x. However it is possible in some cases that for every $\in > 0$ there is $N_\in \in N$ such that for any fixed x and every n depending on $x, \in$ and $n \geq N_\in$ what ever the c be $|R_n(x)| < \in$.

In this case it is customary to say that $R_n(x) \to 0$ as $n \to \infty$.

17.6.4

If $f(x) = e^x$ show that the remainder term in Taylor's theorem for each x_0 "Converges to Zero" (see remark at the end).

Let $f(x) = e^x$. then for $n \in N$ $\quad f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$.

Hence by Taylor's theorem for $x \in R$, $x > 0$ and $n \in N$ there is c_{n+1} such that

$0 < c_{n+1} < x$ and

$$e^x = f(x) = f(0) + xf^{(1)}(0) + ... + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c_{n+1})$$

$$= 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + f^{(n+1)}(c_{n+1}).$$

So that the remainder after n terms is

$$a_{n+1} = \frac{x^{n+1} e^{c_{n+1}}}{(n+1)!}..$$

$\frac{a_{n+1}}{a_n} = \frac{x}{n+1} \cdot \frac{e^{c_{n+1}}}{e^{c_n}}$. Here $e^{c_{n+1}} < e^x$ and $e^{c_n} > 1$.

$\Rightarrow \quad \frac{a_{n+1}}{a_n} < \frac{xe^x}{n+1}$ since x is independent of n and $\lim \frac{1}{n+1} = 0$ it follows $\lim a_n = 0$.

APPLICATIONS OF TAYLOR'S THEOREM:

17.6.5 Use Taylor's Theorem with n=2 to approximate $\sqrt[3]{1+x}$ for $x > -1$.

Write $f(x) = (1+x)^{\frac{1}{3}} \quad (x > -1)$

$$f'(x) = \frac{1}{3}(1+x)^{\frac{-2}{3}}, \quad f'(0) = \frac{1}{3}.$$

$$f''(x) = \frac{1}{3}\left(\frac{-2}{3}\right)(1+x)^{\frac{-5}{3}} \quad f''(0) = \frac{-2}{9}.$$

By Taylor's theorem with n = 2 and $x_0 = 0$ we get

$f(x) = 1 + \frac{1}{3}x - \frac{2}{3}.\frac{1}{2}.\frac{1}{3}.x^2 + \frac{1}{3!}f'''(c)x^3$ for some c between 0 and x.

$$f'''(c) = \frac{10}{27}(1+c)^{-8/3}.$$

Thus $\left| f(x) - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2\right) \right| \delta < \frac{1}{3!}.\frac{10}{27}|1+c|^{\frac{-8}{3}}$. When $x > 0$, $c > 0$ so $0 < (1+c)^{\frac{-8}{3}} < 1$.

$$\Rightarrow \quad \left| f(x) - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2\right) \right| < \frac{5}{81}.$$

For example when $x = 0.3$

$f(0.3) = 1 + \frac{1}{3} \cdot \frac{3}{10} - \frac{1}{9} \cdot \frac{9}{100} = 1.09$ and $(1+c)^{-8/3} < \frac{5}{81}\left(\frac{3}{10}\right)^3 = \frac{1}{600} < 0.17X10^{-2}$.

$\Rightarrow \quad \left|\sqrt[3]{1.3} - 1.09\right| < .5X10^{-2}$

(b) Approximate e with an error less than 10^{-5}.

Write $g(x) = e^x$. then $g^{(k)}(x) = e^x$ for all x and $k \in N$.

so that $g^{(k)}(0) = 1$ for all $k \in N$. The Taylor Polynomial $P_n(x)$ for $g(x) = e^x$ is given by

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

The Remainder for $x = 1$ is given by

$R_n(1) = \frac{e^c}{(n+1)!}$ for some $c \in (0,1)$.

Since $e^c < e < 3$ we have to find $n \in N$ such that $R_n(1) < \frac{3}{(n+1)!} < 10^{-5}$

i.e. $(n+1)! > 3X10^5$

By Direct computation, $8! = 45,360$ and $9! = 3,62,880$

so that $8! < 3$ x $10^5 < 9!$

Thus we get $n+1=9$ and hence $n = 8$.

Thus the required approximation is

$e \approx P_8(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\ldots + \frac{1}{8!} \quad = 2.71828$, the error being less than 10^{-5}.

(e) Show that $1 - \frac{x^2}{2} \le \cos x$ if for all $x \in R$

Let $f(x) = \cos x, \quad x_0 = 0$. Taylor's Theorem with n = 2 gives $\cos x = 1 - \frac{x^2}{2} + R_2(x)$ where

$$R_2(x) = \frac{f'''(c)}{3!} x^3 = \sin c . \frac{x^3}{6}.$$

Since If $0 \le x \le \pi,\ 0 < c < \pi \Rightarrow c > 0$ and $\frac{x^3}{6} \ge 0 \Rightarrow R_2(x) \ge 0$.

If $-\pi \le x \le 0$ then $-\pi < c < 0 \Rightarrow \sin c \le 0$ and $\frac{x^3}{6} < 0 \Rightarrow R_2(x) \ge 0$.

So $1 - \frac{x^2}{2} \le \cos x$ for all x in $[-\pi, \pi]$.

If $|x| \ge \pi$ then $1 - \frac{x^2}{2} \le -3 < -1 \le \cos x$.

Hence $\cos x \ge 1 - \frac{x^2}{2}$ for all $x \in R$.

17.6.6 SAQ: Show that $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$. Using these in equalities find approximations for $\sqrt{1.2}$ and $\sqrt{2}$.

17.6.7 SAQ: For any $k \in N$ and $x > 0$ show that

$$x - \frac{x^2}{2} + \frac{x^3}{3} \text{............} - \frac{x^{2k}}{2k} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} - ... - \frac{x^{2k}}{2k} + \frac{x^{2k+1}}{2k+1}$$

17.6.8 SAQ: If $0 \le x \le 1$ and $n \in N$ show that

$$\left| \ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3} - ... + (-1)^{n-1}\frac{x^n}{n} \right| < \frac{x^{n+1}}{n+1}$$

Use this to approximate $\ln(1.5)$ with an error less than

(a) .01 (b) .001

Let us recall that if I is an interval in R and $c \in I$, $f : I \to R$ has relative maximum (minimum) at c if there is a nighborhood $V_\delta(c) \subseteq I$ such that $f(x) \le f(c)$ $(f(x) \ge f(c))$ if $x \in I$ and that f has relative extremum at c if f has either a relative maximum or a relative minimum at c. Applying Taylors theorem we now derive sufficient conditions for relative maximum and relative minimum.

17.6.9 Theorem:

Let I be an interval, x_0 be a point of I, $n \ge 2$ and suppose that the derivatives $f', f'', ... f^{(n)}$

exist and are continuous in a neighborhood of x_0 and $f'(x_0) = f''(x_0) = ...f^{(n-1)}(x_0) = 0,$ but $f^{(n)}(x_0) \neq 0$.

(i) If n is even and $f^{(n)}(x_0) > 0,$, f has a relative maximum at x_0.

(ii) If n is even and $f^{(n)}(x_0) < 0$, f has a relative minimum at x_0.

(iii) If n is odd then f has neither relative maximum nor relative minimum at x_0.

Proof: Write $P_{n-1}(x) = f(x_0) + (x - x_0) f'(x_0) + ... + \dfrac{(x - x_0)^{n-1}}{(n-1)!} f^{(n-1)}(x_0)$ and

$R_{n-1}(x) = \dfrac{(x - x_0)^n}{n!} f^{(n)}(c)$ where c is in the open interval with end points x_0 and x.

If $x \in I$, by Taylor's Theorem we get c between x_0 and x such that $f(x) = P_{n-1}(x) + R_{n-1}(x)$ where R_{n-1} corresponds to this c.

Since $f^{(n)}$ is continuous in $V_\delta(x_0)$ and $f^{(n)}(x_0) \neq 0$ there is neighborhood U centred at x_0 such that $U \subseteq V_\delta(x_0)$ and if $x \in U$. $f^{(n)}(x)$ and $f^{(n)}(x_0)$ have the same sign. If $x \in U$, as c lies in x_0 and x, c belongs to U so that $f^{(n)}(c)$ and $f^{(n)}(x_0)$ will have the same sign.

(a) Suppose n is even:

Case (i): If $f^{(n)}(x_0) > 0$ then for $x \in U$ and c satisfying the condition of Taylor's theorem, $f^{(n)}(c) > 0$ and $(x - x_0)^n \geq 0$. so that $R_{n-1}(x) \geq 0$. Hence $f(x) \geq f(x_0)$ for $x \in U$. Therefore f has relative minimum at x_0.

Case (ii): If $f^{(n)}(x_0) < 0$ then as above $f^{(n)}(c) < 0$ and $(x - x_0)^n \geq 0$. So that $R_{n-1}(x) \leq 0$ and hence f has relative maximum at x_0.

(b) Suppose n is odd. then $(x - x_0)^n > 0$ if $x > 0$ and $(x - x_0)^n < 0$ if $x < x_0$. Consequently $R_{n-1}(x)$ and $R_{n-1}(y)$ have opposite signs if $x < x_0 < y$ in U. Thus f cannot have relative maximum or relative minimum at x_0.

17.6.10 Example:

Determine whether $x = 0$ is a point of relative extremum of

(a) $\sin x - x$ (b) $x^3 + 2$

(a) Let $f(x) = \sin x - x$.

$f'(x) = \cos x - 1$ $f'(0) = 0$

$f''(x) = -\sin x - 1$ $f''(0) \neq 0$.

Since $f''(0) < 0$ f has relative maximum at 0.

(b) Let $f(x) = x^3 + 2$

$f'(x) = 3x^2 + 2$ $f'(0) = 2$.

If f has relative extremum at 0, $f'(0)$ must be zero.

Hence f does not have relative extremum at 0.

17.6.11 Leibnitzs Rule for the n[th] derivative of the Product:

If f and g are differentiable in (a, b) then for $n \in N$ and $x \in I$

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x) \qquad \rightarrow \qquad (1)$$

Write P_n for (1), a_k for $f^{(k)}(x)$ and b_k for $g^{(k)}(x)$.

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$= \binom{1}{1} f^{(1)}(x) g^{(0)}(x) + \binom{1}{0} f^{(0)}(x) g^{(1)}(x)$, hence P_1 is true assume that P_n holds.

Differentiating l.h.s. and r.h.s of (1).

$$(fg)^{(n+1)}(n) = \sum_{k=0}^{n} \binom{n}{k} \left\{ f^{(n-k+1)}(x) g^{(k)}(x) + f^{(n-k)}(x) g^{(k+1)}(x) \right\}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left\{ a_{n-k+1} b_k + a_{n-k} b_{k+1} \right\}$$

$$= \binom{n}{0} a_{n+1} b_0 + \left\{ \binom{n}{0} + \binom{n}{1} \right\} a_n b_1 + \ldots\ldots\ldots\ldots + \left\{ \binom{n}{n-1} + \binom{n}{n} \right\} a_1 b_n + \binom{n}{n} a_0 b_{n+1}$$

$$= a_{n+1}b_0 + \sum_{k=0}^{n-1}\left\{\binom{n}{k} + \binom{n}{k+1}\right\}a_{n-k}b_{k+1} + a_0 b_{n+1}$$

$$= a_{n+1}b_0 + \sum_{k=0}^{n-1}\left\{\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}\right\}a_{n-k}b_{k+1} + a_0 b_{n+1}$$

$$= a_{n+1}b_0 + \sum_{k=0}^{n-1}\frac{n!(n+1)}{(k+1)!(n-k)!}a_{n-1}b_{k+1} + a_0 b_{n+1}$$

$$= a_{n+1}b_0 + \sum_{k=0}^{n-1}\frac{(n+1)!}{(k+1)!(n-k)!}a_{(n+1)-(k+1)}b_{(k+1)} + a_0 b_{n+1}$$

$$= \sum_{k=0}^{n-1}\binom{n+1}{k}a_{n-k}b_{k+1}$$

$$= \sum_{k=0}^{n+}\binom{n+1}{k}f^{(n-k)}(x)g^{k+1}(x)$$

Hence P_{n+1} is true. Thus P_n holds for all $n \in N$.

17.7 Solutions to SAQ's:

17.4.9 SAQ:

Let f be differentiable on $(0,\infty)$ and suppose that $\lim_{x\to\infty}\left(f(x)+f'(x)\right)=L$.

Show that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.

Solution: Write $g(x) = f(x)e^x$ and $h(x) = e^x$. g and h are differentiable on $(0,\infty)$.

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} e^x = \infty$$

$$g'(x) = (f(x)+f'(x))e^x, \quad h'(x) = e^x$$

So $\lim_{x\to\infty}\frac{g'(x)}{h'(x)} = \lim_{x\to\infty}\{f(x)+f'(x)\} = L$.

Hence by L'Hospital's Rule II $\lim_{x\to\infty}\frac{g(x)}{h(x)}=L$

$\Rightarrow \lim_{x\to\infty} f(x)=L \Rightarrow \lim_{x\to\infty} f'(x)=0$.

17.4.10 SAQ: Let $f(x)=x^2$ for x rational

= 0 for x irrational,

and $g(x)=\sin x$ for $x\in R$. Show that $\lim_{x\to 0}\frac{f(x)}{g(x)}=0$. Explain why L' Hospital rule cannot be used.

Solution:

$$f(x)=\begin{cases} x^2 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational}\end{cases}$$

So, $\left|\frac{f(x)-f(0)}{x-0}\right|\le |x|$ $\Rightarrow \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$ $\Rightarrow f'(0)=0$.

Suppose $x\neq 0$ and $y\neq x$.

x	y	$\frac{f(y)-f(x)}{y-x}$
rational	rational	$\frac{y^2-x^2}{y-x}=y+x$
rational	irrational	$\frac{-x^2}{y-x}$
irrational	rational	$\frac{y^2}{y-x}$
irrational	irrational	0

So $\lim_{y\to x}\frac{f(y)-f(x)}{y-x}$ doesn't exist when $x\neq 0$.

So $f'(x)$ doesn't exist in $R-\{0\}$.

So L'Hospital's rule is not applicable for any g.

However when $g(x) = \sin x, \quad g'(x) = \cos x$

$$\frac{f(x)}{g(x)} = \begin{cases} \dfrac{x^2}{\sin x} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

$$0 \le \left|\frac{f(x)}{g(x)}\right| \le \left|\frac{x^2}{\sin x}\right| \le |x| \left|\frac{x}{\sin x}\right| = \frac{|x|}{\left|\dfrac{\sin x}{x}\right|}$$

$\lim\limits_{x\to 0} \dfrac{Sinx}{x} = 1$, so $\lim\limits_{x\to 0} \dfrac{|x^2|}{|Sinx|} = 0$ and $f^{'}(0) = 0$, $g^{'}(0) = 1$.

Hence $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 0 = \dfrac{f'(0)}{g'(0)}$.

17.6.6 SAQ:

Show that $1 + \frac{1}{2}x - \frac{1}{2}x^3 \le \sqrt{1+x} \le 1 + \frac{1}{2}x$ if $x > 0$

using these inequalities find approximations for $\sqrt{1.2}$ and $\sqrt{2}$.

Solution : Let $f(x) = \sqrt{1+x}$ for $x > 0$

$$f'(x) = \frac{1}{2}(1+x)^{\frac{-1}{2}} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{1}{2}.\frac{-1}{2}.(1+x)^{\frac{-1}{2}} \qquad f''(0) = \frac{-1}{4}$$

$$f'''(x) = \frac{1}{2}.\frac{-1}{2}.\frac{-3}{2}(1+x)^{\frac{-5}{2}} = \frac{3}{8}(1+x)^{\frac{-5}{2}}$$

If $x > 0$ by Taylor's theorem there exist c_1 and c_2 in $(0, x)$ such that

$$f(x) = \sqrt{1+x} = f(0) + xf'(0) + \frac{x^2}{2!} f''(c_1) \text{ and also}$$

$$f(x)=\sqrt{1+x}=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(c_2).$$

Thus $\sqrt{1+x}=1+\frac{1}{2}x+\frac{x^2}{2}.\left(\frac{-1}{4}\right)(1+c_1)^{-3/2} \quad =1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{x^3}{3!}.\frac{3}{8}.(1+c_2)^{-5/2}$

Since $\frac{x^2}{8}(1+c_1)^{-3/2}>0, \quad \sqrt{1+x}<1+\frac{1}{2}x$

Since $\frac{x^3}{16}(1+c_2)^{-5/2}>0 \quad \sqrt{1+x}>1+\frac{1}{2}x-\frac{1}{8}x^2$

Thus we get if $x>0$,

$$1+\frac{1}{2}x-\frac{1}{8}x^2<\sqrt{1+x}<1+\frac{1}{2}x$$

Since $\sqrt{1.2}=\sqrt{1+0.2}$ so we have $1+\frac{1}{2}.\frac{2}{10}-\frac{1}{8}.\frac{4}{100}<\sqrt{1.2}<1+\frac{1}{2}.\frac{2}{10}$.

on simplification we get $1.095<\sqrt{1.2}<1.1$

Similarly $1+\frac{1}{2}-\frac{1}{8}<\sqrt{2}<1+\frac{1}{2} \Rightarrow 1.375<\sqrt{2}<1.5$

17.6.7 SAQ:

For any $k\in N$ and $x>0$ show that

$$x-\frac{x^2}{2}+\frac{x^3}{3}-...........-\frac{x^{2k}}{2k}<\ln(1+x)<x-\frac{x^2}{2}+\frac{x^3}{3}-.........-\frac{x^{2k}}{2k}+\frac{x^{2k+1}}{2k+1}$$

Solution: For $x>0$ $(ln(1+x))'=\frac{1}{1+x}$.

Hence $(\ln(1+x))^{(k+1)}=\left(\frac{1}{1+x}\right)^{(k)}=(-1)^k(1+x)^{-(k+1)}k!$.

$\Rightarrow$ The Taylor's Polynomial is given by

$$P_n(x)=x-\frac{x^2}{2}+........+(-1)^{n-1}\frac{x^n}{n}$$

and the remainder is given by

$$R_n(x) = (-1)^n c^{n+1}.\frac{x^{n+1}}{n+1} \text{ for some } c \in (0, x)$$

Thus if $x > 0$ and $n = 2k$, $R_n(x) = R_{2k}(x) > 0$

and if $n = 2k+1$, $R_n(x) = R_{2k+1}(x) < 0$

Hence the required inequality follows.

17.6.8 :

If $0 \le x \le 1$ and $n \in N$ show that

$$\left| \ln(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3} - .... + (-1)^{n-1}\frac{x^n}{n} \right| < \frac{x^{n+1}}{n+1} \qquad \rightarrow \quad (1)$$

use this to approximate $\ln 1.5$ with an error less than

(a) .01 (b) .001

Solution: Let $f(x) = \ln(1+x)$ for $0 \le x \le 1$. Then for $n \in N$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n-1}$ so that $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

By Taylor's theorem

$$f(x) = P_n(x) + R_n(x)$$

Where $P_n(x) = f(0) + xf'(0) + ... + \frac{x^n}{n!} f^{(n)}(0)$

and $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ for a suitable $c \in (0, x)$

So $P_n(x) = 0 + x - \frac{x^2}{2} + ... + (-1)^{n-2}\frac{x^{n-1}}{(n-1)!} + (-1)^{n-1}\frac{x^n}{n!}$

and $R_n(x) = \frac{(-1)^{n-1}}{n+1}(1+x)^{-n-1} x^{n+1}$

$$\Rightarrow \quad |R_n(x)| < \frac{x^n}{n+1}$$

$$\Rightarrow \left| f(x) - P_n(x) \right| < \frac{x^n}{n+1}$$

To find approximation for $\ln(1.5)$ we use the inequality (1)

Since $\left|\ln(1+x) - P_n(x)\right| < \frac{x^n}{n+1}$ and the maximum limit for error in (a) is $.01 = \frac{1}{100}$, we find n so that $\frac{x^n}{n+1} < \frac{1}{100}$ when $x = \frac{1}{2}$ and see direct computation that $\frac{1}{2^5.5} = \frac{1}{160} < \frac{1}{100} < \frac{1}{2^4.4}$.

Thus when n = 4

$$\left|\ln(1.5) - P_4\left(\frac{1}{2}\right)\right| < \frac{1}{160} < \frac{1}{100}.$$

Hence $\frac{1}{2} - \frac{1}{2^2.2} + \frac{1}{2^3.3} - \frac{1}{2^4.4} = 0.40$ is the r required approximation for $\ln(1.5)$ with error .01.

The maximum limit for the error in (b) is $.001 = \frac{1}{1000}$.

As above we can show that

$$\left|\ln(1.5) - P_5\left(\frac{1}{2}\right)\right| < .001$$ so that

$P_5\left(\frac{1}{2}\right) = 0.405$ is the required approximation in this case.

17.8 Summary:

This lesson covers several important results in differential calculus including Cauchy Mean Value theorem, L'Hospital's rules Taylor's theorem sufficient conditions for relative extrema and so on and a good number of applications such as finding n th roots and polynomial approximations involving higher order derivatives with error estimations.

17.9 Technical Terms:

Indeterminate forms - Higher order derivatives - Local extrema error bounds.

17.10 Exercises:

1. Suppose that f and g are continous on [a, b], differentiable on (a, b) that $g(x) \neq 0$ for $x \in [a, b]$.

Let $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$. If B = 0 and $\lim_{x\to c}\frac{f(x)}{g(x)}$ exists in R show that A = 0.

2. In addition to the hypothesis in excercise 1 above

Let $g(x) > 0$ for $x \in [a,b]$, $x \neq c$, show that if B = 0

(1) $\lim_{x\to c}\frac{f(x)}{g(x)} = \infty$ if $A > 0$ and

(2) $\lim_{x\to c}\frac{f(x)}{g(x)} = -\infty$ if $A < 0$.

3. Evaluate the following limits where the domain of the quotient is as indicated

(a) $\lim_{x\to 0+}\frac{\ln(x+1)}{\sin x}$ $\left(0, \frac{\pi}{2}\right)$

(b) $\lim_{x\to 0+}\frac{\tan x}{x}$ $\left(0, \frac{\pi}{2}\right)$

(c) $\lim_{x\to 0+}\frac{\ln\cos x}{x}$ $\left(0, \frac{\pi}{2}\right)$

(d) $\lim_{x\to 0+}\frac{\tan x - x}{x^3}$ $\left(0, \frac{\pi}{2}\right)$

(e) $\lim_{x\to 0}\frac{Arc\tan x}{x}$ $(-\infty, \infty)$

(f) $\lim_{x\to 0}\frac{1}{x(\ln x)^2}$ $(0,1)$

(g) $\lim_{x\to 0+} x^3\ln x$ $(0,\infty)$

(h) $\lim_{x\to\infty}\frac{x^3}{e^x}$ $(0,\infty)$

(i) $\lim_{x\to\infty}\frac{\ln x}{x^2}$ $(0,\infty)$

(j) $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$ $(0,\infty)$

(k) $\lim_{x\to 0}\log\sin x$ $(0,\pi)$

(l) $\lim_{x\to\infty}\frac{x+\ln x}{x\ln x}$ $(0,\infty)$

4. Evaluate the follwing limits

(a) $\lim_{x\to 0+} x^{2x}$ $(0,\infty)$

(b) $\lim_{x\to 0}\left(1+\frac{3}{x}\right)^x$ $(0,\infty)$

(c) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ $(0,\infty)$

(d) $\lim_{x\to 0+}\left(\frac{1}{x}+\frac{1}{Arc\tan x}\right)$ $(0,\infty)$

(e) $\lim_{x\to\infty} x^{\frac{1}{x}}$ $(0,\infty)$

(f) $\lim_{x\to 0+}(\sin x)^x$ $(0,\pi)$

(g) $\lim_{x\to 0+} x^{\sin x}$ $(0,\infty)$

(h) $\lim_{x\to\frac{\pi}{2}-}(\sec x-\tan x)$ $\left(0,\frac{\pi}{2}\right)$

5. Let $f(x)=\sin ax.(a\neq 0)$. Find $f^{(n)}(x)$ for $n\in N$ and $x\in R$.

6. If $\left|f(x)-f(y)\right|\le (x-y)^2$ for all x,y in R show that f is constant function.

7. If $g(x)=\sin x$ show that the remainder term in Taylor's theorem converges to zero as $n\to\infty$ for each fixed x_0 and x.

8. Show that $\left|\sin x-\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)\right|<\frac{1}{5040}$ if $|x|\le 1$.

9. Determine whether or not x = 0 is a point of relative extremum of

(a) $\sin x+\frac{x^3}{6}$ (b) $\cos x-1+\frac{x^2}{2}$

10. If $x>0$ show that $x-\frac{x^2}{2}<\ln(1+x)<\frac{x^2}{2(1+x)}$.

11. Show that the function f defined by

$f(x)=1-x$ if $0\le x\le 1$

$=x-1$ if $1\le x\le 2$

has minimum at x = 1. Is f differentiable at x = 1?

12. Show that $\cos x<1-\frac{x^2}{2}+\frac{x^4}{24}$ for $x\in R$.

13. If $f(x)=\left|x^3\right|$ show that f possesses the first three derivatives at all $x\neq 0$ and $f'''(0)$ does not exist while $f'(0)=f''(0)=0$

(Hint: If $x\ge 0$ $f'(x)=3x^2$, $f''(x)=6x$

If $x<0$ $f'(x)=-3x^2$, $f''(x)=-6x$)

17.11 Answers:

	a	b	c	d	e	f	g	h	i	j	k	l
3	1	1	0	$\frac{1}{3}$	1	∞	0	0	0	0	0	0
4	1	1	e^3	0	1	1	1	0				

5 $f^{(2n)}(x) = (-1)^n a^{2n} \sin ax$

$f^{(2n+1)}(x) = (-1)^n a^{2n+1} \cos ax$

6 (a) No (b) relative minimum at 0.

17.12 Model Examination Questions:

1. State and prove Rolle's Theorem.
2. State and prove Cauchy Mean Value Theorem.
3. Verify Lagranges Mean Value Theorem for $f(x) = \cos ax$ in $\left[0, \frac{\pi}{2}\right]$
4. Find $\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}}$.
5. Show that $\lim_{x \to 0+} \left(1 + \frac{1}{x}\right)^x = 1$
6. Show that $x - \frac{x^3}{6} < \sin x$ if $x \in R$.
7. Show that if f is differentiable in (a, b) and $f'(x) \geq 0$ for all $x \in (a,b)$ then $f(x) \geq f(y)$ when $x > y$.
8. If $g(x) = |x|^3$ show that $g'''(x)$ does not exist when $x = 0$.
9. Derive Leibnitz's rule for the n th derivative of the product

$$(fg)^n(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x).$$

10. Show that $1+\frac{1}{2}x-\frac{x^2}{8}<\sqrt{1+x}<1+\frac{1}{2}x$ if $x>0$.

17.13 Model Practical Questions with Solution:

Question: Let $f(x)=2x^4+x^4\sin\frac{1}{x}$ if $x\neq 0$ and $f(0)=0$. Show that f has relative minimum at 0 but that the derivative assumes both positive and negative values in every neighborhood of 0.

1. AIM:

(i) f has absolute minimum at o, i.e. $f(x)\geq f(0)$ for all $x\in R$

(ii) If $\delta>0$ there exist x, y in $v_\delta(0)$ such that $f'(x)<0<f'(y)$.

2. Definitions:

(a) **Absolute minimum** : If $E\subset R$, $f:E\to R$ is said to have absolute minimum at $a\in R$ if $f(x)\geq f(a)$ for all $x\in E$.

(b) **Derivative at** $x\in E\subseteq R$ **:** If $F\subseteq R$, $a\in E$ and a is a limit point of E, $A\subseteq R$ is said to be a derivative of f at a if $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=A$.

(c) **Neighborhood of a point** : If $a\in R,\delta>0,V_\delta(a)=\{x\in R/|x-a|<\delta\}$ is called a neighborhood of a.

3. **Results to be used in the solution:**

(i) $|\sin\theta|\leq 1$ for all $\theta\in R$.

(ii) Archimedis Principle.

4. Solution: (i) : If $x\in R$, $x^4\geq 0$ and from $3(i)$, $-1\leq\sin\frac{1}{x}\leq 1$

$$\Rightarrow -x^4\leq x^4\sin\frac{1}{x}\leq x^4$$

$$\Rightarrow 0\leq x^4=2x^4-x^4\leq 2x^4+x^4\sin\frac{1}{x}\leq 3x^4$$

$\Rightarrow$ $f(x)\geq f(0)$ for all x.

(ii) $f'(x) = 8x^3 + 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$

$= x^2\left(8x + 4x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$

So $f'(x) > 0$ or $f'(x) < 0$ accordinaling as $\frac{f'(x)}{x^2} = 8x + 4x\sin\frac{1}{x} - \cos\frac{1}{x} > 0$ or < 0.

If $n \in N$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$

So if n even and $x = \frac{1}{n\pi}$

$\frac{f'(x)}{x^2} = \frac{8}{n\pi} - 1 < 0$ if $\frac{8}{n\pi} < 1$ i.e. if $n > \frac{8}{\pi}$ and if *m* is odd and $x = \frac{1}{m\pi}$.

$\frac{f'(x)}{x^2} = \frac{8}{m\pi} - (-1) = \frac{8}{m\pi} + 1 > 0$

If $\delta > 0$ we choose an even inter n and an odd integer $mn > \frac{8}{\pi}, \frac{1}{n\pi} < \delta$ and $\frac{8}{m\pi}, < \delta$.

This is possible by the Archimedis Principle.

- K. Sambasiva Rao

Lesson - 18

RIEMANN INTEGRATION - I

18.1 Objective:

To introduce the notion of Riemann Integral for an arbitrary function $f:[a,b]\to R$, provides a good number of examples and establish that Riemann integrability of f implies boundedness of f and Riemann integral obeys linearity conditions and order property.

18.2 Structure:

18.3 Introduction

18.4 Definition and Examples

18.5 Linearity and Other Properties

18.6 Examples

18.7 Solutions to SAQ's

18.8 Summary

18.9 Technical Terms

18.10 Exercises

18.11 Model Examination Questions

18.12 Model Practical Problem with Solutions

18.3 Introduction:

The integral, mostly known as antiderivative is very much used in finding Lengths of Curves, Areas of plane regions and volumes etcetra. Taking inspiration from the idea that the area bounded by a plane curve may be approximated by finding the area of the rectangular regions on the bases obtained by partitioning the domain into successive nonoverlapping intervals, the Riemann sums are defined without imposing boundedness of the function defining the curve. For each partition of the domain [a, b] into nonoverlapping intervals we assign tags in the subintervals and find the sum of the areas of the areas of the rectangles formed by the intervals $[x_{i-1},x_i]$ and having height $f(t_i)$, t_i being the tag in $[x_{i-1},x_i]$. This sum is the Riemann sum. If these Riemann sums approach a "limit" when the max length of the sub intervals of P is "reduced" f is said to be Riemann integrable. This definition has an advantage over other definitions including the one that uses upper and lower sums as we need not assume that f is bounded.

In this lesson we present the definition of Riemann integral, show its uniqueness, derive boundedness of a Riemann Integrable function and provide a good number of examples.

18.4 Definitions and Notation:

Let I be a closed and bounded interval in $\mathbb{R}, I=[a,b]$. By partition of [a, b] we mean a finite collection of points $P:\{a=x_0,x_1,\text{..........}x_n=b\}$ where $x_0<x_1<\text{..........}<x_n$. For

$1 \le i \le n, [x_{i-1}, x_i]$ is called the ith subinterval of P. We also write $P = \{[x_{i-1}, x_i]\}^n{}_{i=1}$. We call $\Delta x_i = x_i - x_{i-1}$, the length of the ith subinterval $[x_{i-1}, x_i]$ and write

$\|P\|$ = maximum $\{(x_i - x_{i-1}) / 1 \le i \le n\}$ is called the norm (also mesh) of R.

A tagged partition of [a, b] is a partition of [a, b] with a collection of points $\{t_i\}$ where t_i lies in the ith subinterval of the partition i.e. $t_i \in [x_{i-1}, x_i]$. If $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ is a partition of [a, b] and $x_{i-1} \le t_i \le x_i$ for $i = 1, 2, \ldots\ldots\ldots\ldots n$ then $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition. We define norm $\dot{P}$ = norm P and write $\|\dot{P}\| = \|P\|$.

If $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is a tagged partition of [a, b] and $f : [a, b] \to \mathbb{R}$ we define the Riemann sum of f corresponding to $\dot{P}$ to be the number $S(f, \dot{P}) = \sum_{i=1}^{n} f(t_i) \Delta x_i$; where $\Delta x_i = x_i - x_{i-1}$.

If $f(x) > 0$ when $a \le x \le b$ then for every i $A_i = (x_i - x_{i-1}) f(t_i)$ is the area of the rectangle with base $[x_{i-1}, x_i]$ and height $f(t_i)$.

Further $S(f, \dot{P}) = \sum_{i=1}^{n} A_i$ is the sum of the areas of the rectangles formed by the heights $f(t_i)$. and the bases $[x_{i-1}, x_i]$.

18.4.1 Definition:

$f : [a, b] \to R$ is said to be Riemann integrable if there exists a number A such that for every $\in > 0$ there corresponds $\delta_\in > 0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \delta_\in$

$$\left| S(f, \dot{P}) - A \right| < \in .$$

Any number A satisfying the above conditon is called Riemann integeral of A and we write

$\lim_{\|P\| \to 0} S(f, \dot{P}) = A = \int_a^b f dx$. $\int_a^b f dx$ is also denoted by $\int_a^b f$.

The set of all Riemann integrable functions over [a, b] is denoted by R[a, b].

Note: When we write $\lim_{\|P\|\to 0} S(f, \dot{P}) = A$, it does not really mean that l.h.s. is a limit as we have not so far defined limit except for functions defined on a subset of $\mathbb{R}$ at a limit point and for a sequence.

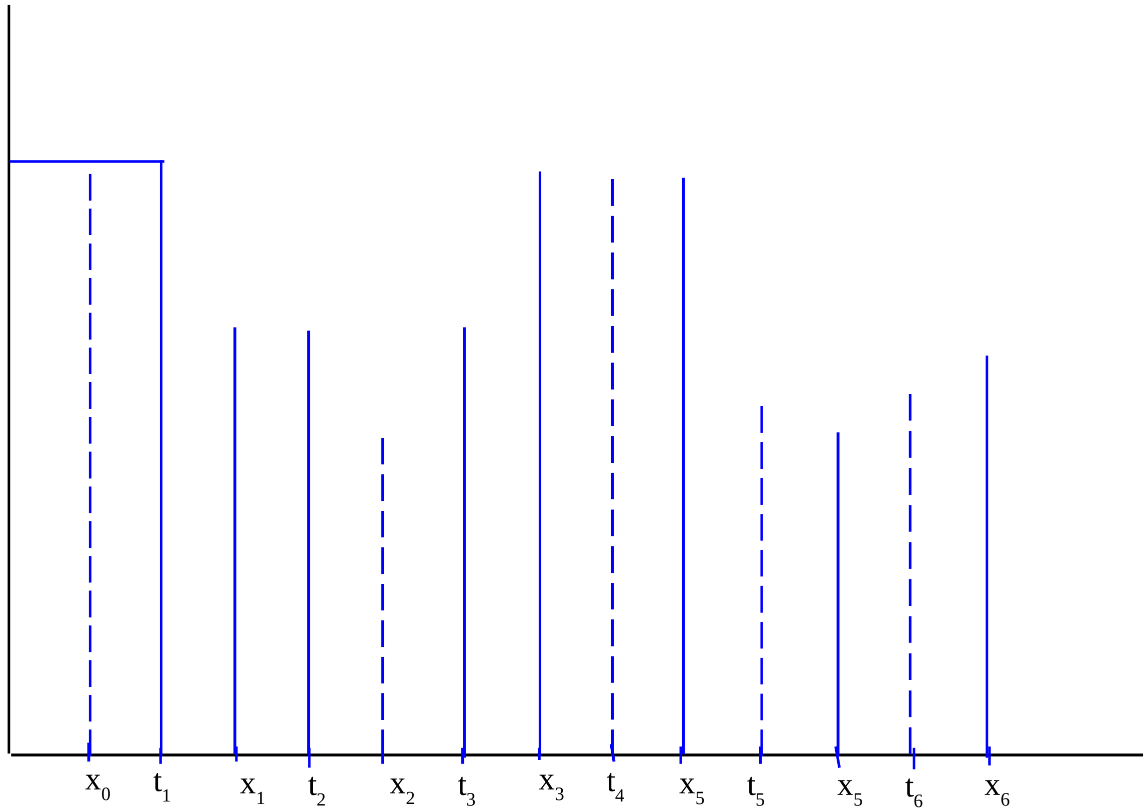


Fig. 18.2.1

18.4.2 Examples:

(i) Calculate the norm of $P:\{0,1,1.5,2,3,4,4\}$

$I_1:[0,1]$, $I_2:[1,1.5]$ $I_3:[1.5,2]$ $I_4:[2,3.4]$ and $I_5:[3.4,4]$.

If $l([a,b]) = b-a$ then $l(I_1)=1$, $l(I_2)=.5=l(I_3)$, $l(I_4)=1.4$ and $l(I_5)=.6$.

So $\|P\| = \max\{l(I_1),l(I_2),l(I_3),l(I_4),l(I_5)\} = \max\{1,0.5,0.5,1.4,0.6\} = 1.4$

(ii) Find $\|Q\|$ where $Q=\{0,.5,2.5,3.5,4\}$

$I_1=[0,.5]$, $I_2=[.5,2.5]$, $I_3=[2.5,3.5]$, $I_4=[3.5,4]$, $l(I_1)=.5=l(I_4)$, $l(I_2)=2$,

$l(I_3)=1$, $\|Q\|=2$.

(iii) Find $S(f,\dot{P})$ where $f:[0,4]\to\mathbb{R}$ is given by $f(x)=x^2$ and $P=\{[x_{i-1},x_i],t_i\}_{i=1}^{5}$, t_i is the left end point of $[x_{i-1}, x_i]$ and P is the partition in (i) above

clearly $t_1=0, t_2=1, t_3=1.5, t_4=2, t_5=3.4$

$$f(t_1)=0\, f(t_2)=1\, f(t_3)=2.25\, f(t_4)=4\, f(t_5)=11.56$$

So $S(f,P)=0(x_1-x_0)+1(x_2-x_1)+2.25(x_3-x_2)+4(x_4-x_3)+11.56(x_5-x_4)$

$$=.5+1.125+5.6+6.936=14.161$$

(iv) Calculate $S(f,\dot{P})$ when $f:[0,4]\to\mathbb{R}$ is given by $f(x)=x^2$, P is as in (i) and the tags are right end points.

$t_1=1$, $t_2=1.5$, $t_3=2$, $t_4=3.4$, $t_5=4$, $f(t_1)=1$, $f(t_2)=2.25$, $f(t_3)=4$, $f(t_4)=11.56$, $f(t_5)=16$, $S(f,\dot{P})=1.1+(2.25)(.5)+(4)(.5)+(11.56)(1.4)+(16)(0.6)=29.909$

18.4.3 : Every constant function f is Riemann integrable on [a, b] and $\int_a^b f dx = k(b-a)$

Where $f(x)=k$ for all x in [a, b].

Let $P=\{([x_{i-1},x_i],t_i)\}_{i=1}^{2}$ be any tagged partition

$$S(f,P)=\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1}) \quad = k.\sum_{i=1}^{n}(x_i-x_{i-1}) \quad = k(b-a).$$

Hence If $\in>0$ and $\delta(\in)=\in$ and $\dot{P}$ is any tagged partition with $\|\dot{P}\|<\in$

$$|S(f,P)-k(b-a)|=0<\in. \text{ Hence } f\in R[a,b] \text{ and } \int_a^b f dx = k(b-a)$$

18.4.4 SAQ: If $f\in R[a,b]$ and if $\{P_n\}$ is any sequence of tagged partitions of [a, b] such that

$$\lim_n \|\dot{P}_n\| = 0 \quad \text{then } \lim_n S(f,P_n)=\int_a^b f dx\ .$$

18.4.5 SAQ:

Let $g(x) = \begin{cases} 0 \text{ if } 0 \le x \le 1 \text{ and } x \text{ is rational} \\ \frac{1}{x} \text{ if } 0 \le x \le 1 \text{ and } x \text{ is irrational} \end{cases}$

Show that there is a sequence $\{\dot{P}_n\}$ of tagged partitions of [0, 1] such that $\lim_n S(g, \dot{P}_n) = 0$.

Show also that $g \notin R[0,1]$.

18.4.6 SAQ: Suppose $f:[a,b] \to \mathbb{R}$ is bounded, $\dot{P}_n$ and $\dot{Q}_n$ are tagged partitions such that $\lim\left(\left\|\dot{P}_n\right\|\right) = \lim\left(\left\|\dot{Q}_n\right\|\right) = 0$ but $S(f, \dot{P}_n) \neq \lim S(f, \dot{Q}_n)$. Show that $f \notin R[a,b]$.

18.4.7 Theorem: If $f \in R[a,b]$ then the value of the integral is unique.

Proof: Suppose L and L' are real numbers such that for every $\in > 0$ there is $\delta_1 > 0$ such that for every tagged partition $\dot{P}$ of [a, b] with $\left\|\dot{P}\right\| < \delta_1$, $\left|S(f, \dot{P}_1) - L\right| < \frac{\in}{2}$ and also $\delta_2 > 0$ such that for every tagged partition $\dot{Q}$ with $\left\|\dot{Q}\right\| < \delta_2$ $\left|S(f, \dot{Q}) - L'\right| < \frac{\in}{2}$.

Let δ = minimum $\{\delta_1, \delta_2\}$ and $\dot{R}$ be any tagged partition with $\left\|\dot{R}\right\| < \delta$. Then

$$\left|S(f, \dot{R}) - L\right| < \frac{\in}{2} \text{ and } \left|S(f, \dot{R}) - L'\right| < \frac{\in}{2}.$$

$$\Rightarrow \; 0 \le \left|L - L'\right| \le (L - S(f, \dot{R})) + \left|S(f, \dot{R}) - L'\right| < \frac{\in}{2}.$$

This holds for every $\in > 0$ so $L = L'$.

Boundedness of $f \in R[a,b]$:

18.4.8 Theorem: If $f \in R[a,b]$ then f is bounded.

Proof: Let $\in = 1$. There exists $\delta > 0$ such that if $\dot{P}$ is any tagged partition with $\left\|\dot{P}\right\| < \delta$

$$\left|S(f,\dot{P})-\int_a^b fdx\right|<1$$

$\Rightarrow \left|S(f,\dot{P})\right|<1+\left|\int_a^b fdx\right| = M_1$ say. Fix $m\in\mathbb{N}$ such that $m>\dfrac{b-a}{\delta}$ and $x\in[a,b]$.

Let $\dot{P}$ be the tagged partition of [a, b] obtained by dividing [a, b] into subintervals of equal length $\dfrac{b-a}{m}$. Let $P=\{a=x_0<x_1<..........<x_m=b\}$ and let $t_j=x_j$ if $x\in[x_{j-1},x_j]$ and $t_i=x$ if $x\in[x_{i-1},x_i]$ and $P=[[x_{i-1},x_i],t_i]_{i=1}^m$

$$S(f,\dot{P}) = f(x)(x_i-x_{i-1})+\sum_{j\neq i} f(x_j)(x_j-x_{j-1}) = \left\{f(x)+\sum_{j\neq i} f(x_j)\right\}\left(\frac{b-a}{m}\right)$$

Hence $M_1>\left|S(f,\dot{P})\right|\geq\left|f(x)+\sum_{j\neq i} f(x_j)\right|\left(\dfrac{b-a}{m}\right)$

$$\Rightarrow \frac{M_1 m}{b-a}>|f(x)|-\left|\sum_{j\neq i} f(x_j)\right| \quad \{Since\,|a+b|\geq|a|-|b|\}$$

$$\Rightarrow |f(x)|<\frac{M_1 m}{b-a}+\left|\sum_{j\neq i} f(x_j)\right|\leq\frac{mM_1}{b-a}+\sum_{j=1}^{m}|f(x_j)|=M \text{ say}$$

Since $x\in[a,b]$ is arbitrary and R.H.S. is independent of x, it follows that f is bounded.

18.4.9 Short Answer Questions:

a) Show that $f:[a,b]\to R$ is Riemann integrable if and only if there exists $L\in\mathbb{R}$ such that for every $\in>0$ there corresponds $\delta_\in>0$ such that if $\dot{P}$ is any tagged partition with $\|\dot{P}\|\leq\delta_\in$ then $\left|S(f,\dot{P})-L\right|\leq\in$.

b) Show that the Dirichlet function $f:[0,1]\to R$ defined by

$$f(x)=\begin{cases}1 \text{ if } 0\le x\le 1 \text{ and } x \text{ is rational}\\ 0 \text{ if } 0\le x\le 1 \text{ and } x \text{ is irrational}\end{cases}$$

doesn't belong to $R[0,1]$.

c) If $f:[a,b]\to\mathbb{R}$ is defined by $f(x)=0$ except for a finite number of points $x_1,x_1,.........x_n$ in [a, b] show that $f\in R[a,b]$ and $\int_a^b fdx=0$.

c) Let $\dot{P}=\{(I_j,t_j)\}_{j=1}^{n}$ be a tagged partition of [a,b] and $a\le c_1<c_2\le b$. If U_1 is the union of all subintervals I_j such that $c_1\le t_j\le c_2$ show that $U_1\subseteq\left[c-\|P\|,\quad c+\|P\|\right]$

18.5 Linearity Properties of the Integral:

18.5.1 Theorem: Suppose that $f\in R[a,b]$ and $g\in R[a,b]$ then $f+g\in R[a,b]$ and $\int_a^b(f+g)dx=\int_a^b fdx+\int_a^b gdx$

Proof: Let $\in>0$. Since $f\in R[a,b]$ there exists $\delta_1>0$ such that for every partition P of [a, b] with $\|P\|<\delta_1$

$$\left|S(f,P)-\int_a^b fdx\right|<\frac{\in}{2} \qquad\to\qquad (1)$$

Since $g\in R[a,b]$, there exists $\delta_2>0$ such that for every tagged partition of $\dot{P}$ of [a, b] with $\left\|\dot{P}\right\|<\delta_2$

$$\left|S(g,P)-\int_a^b gdx\right|<\frac{\in}{2} \qquad\to\qquad (2)$$

Let $\delta_{\in} = \min\{\delta_1, \delta_2\}$. $\delta_{\in} > 0$. If $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \delta$ then $\|\dot{P}\| < \delta_1$ as well as $\|\dot{P}\| < \delta_2$ so (1) and (2) both hold.

Let $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$. then $S(f, \dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

$S(g, \dot{P}) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$ and $S(f+g, \dot{P}) = \sum_{i=1}^{n}[f(t_i) + g(t_i)](x_i - x_{i-1})$

$$= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) + \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$$

$$= S(f, \dot{P}) + S(g, \dot{P})$$

Hence $= \left| S(f+g, \dot{P}) - \int_a^b f dx - \int_a^b g dx \right|$

$$= \left| S(f, \dot{P}) + S(g, \dot{P}) - \int_a^b f dx - \int_a^b g dx \right| \leq \left| S(f, \dot{P}) - \int_a^b f dx \right| + \left| S(g, \dot{P}) - \int_a^b g dx \right| < \frac{\in}{2} + \frac{\in}{2} = \in$$

Since this holds for every tagged partition with $\|\dot{P}\| < \delta$ it follows that $f + g \in R[a,b]$ and

$$\int_a^b (f+g)dx = \int_a^b f dx + \int_a^b g dx.$$

18.5.2. Corollary : If $n \in \mathbb{N}$, $f_i \in R[a,b]$ for $1 \leq i \leq n$ then $f_1 + ... + f_2 \in R[a,b]$ and $\int_a^b (f_1 + ... + f_n)dx = \int_a^b f_1 dx + ... + \int_a^b f_n dx$

Proof: We prove this result by the principle of Mathematical induction. When $n = 1$ $l.h.s = \int_a^b f_1 dx = r.h.s.$ Assume for $n-1 \in \mathbb{N}$ and let $f_1, ...f_n$ be n functions. Such that $f_i \in R[a,b]$ for $1 \leq i \leq n$. Write $f = f_2 + ... + f_n$. By induction hypothesis $f \in R[a,b]$ and

$\int_a^b fdx = \int_a^b f_2 dx + ... + \int_a^b f_n dx$. By 18.5.1 $f_1 + f \in R[a,b]$ and $\int_a^b (f_1 + f)dx = \int_a^b f_1 dx + \int_a^b fdx$.

Since $f_1 + f = f_1 + f_2 + ... + f_n$ it follows that $f_1 + f_2 + ... + f_n \in R[a,b]$ and

$$\int_a^b (f_1 + ... + f_n)dx = \int_a^b f_1 dx + \int_a^b fdx \quad = \int_a^b f_1 dx + \int_a^b f_2 dx + ... + \int_a^b f_n dx.$$

18.5.3 Theorem:

If $f \in R[a,b]$ and $K \in \mathbb{R}$, $Kf \in \mathbb{R}[a,b]$ and $\int_a^b (Kf)dx = K\int_a^b fdx$.

Proof: If $k = 0$, $Kf = 0$ and hence lhs = rhs = 0.

Assume that $K \neq 0$. Since $f \in R[a,b]$, given $\in > 0$ there exists $\delta > 0$ such that for every tagged partition $\dot{P}$ of [a, b] with $\left\|\dot{P}\right\| < \delta$,

$$\left| S(f, \dot{P}) - \int_a^b fdx \right| < \frac{\in}{|K|}.$$ If $\left\|\dot{P}\right\| < \delta$ and $\dot{P} = \left\{\left([x_{i-1}, x_i], t_i\right)\right\}_{i=1}^n$,

$S(f, P) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$ and hence $S(Kf, \dot{P}) = \sum_{i=1}^n Kf(t_i)(x_i - x_{i-1})$

$$= K\sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = KS(f, \dot{P})$$

Hence $\left| S(Kf, \dot{P}) - K\int_a^b fdx \right| = \left| KS(f, \dot{P}) - K\int_a^b fdx \right| = |K|\left| S(f, \dot{P}) - \int_a^b fdx \right| < |K|\frac{\in}{|K|} = \in$

Hence $Kf \in R[a,b]$ and $\int_a^b Kfdx = K\int_a^b fdx$.

18.5.4 Theorem: If $f \in R[a,b], g \in R[a,b]$ and $f(x) \leq g(x)$ for all x in [a, b] then $\int_a^b fdx \leq \int_a^b gdx$.

Proof: Since f and g are integrable $g-f$ is integrable. Since $f(x) \le g(x)$ for all $x \in [a,b]$ $(g-f)(x) \ge 0$ for all $x \in [a,b]$.

Since $\int_a^b (g-f)dx = \int_a^b gdx - \int_a^b fdx$. It is enough to show that if $g \in R[a,b]$ and $g(x) \ge 0$ for all $x \in [a,b]$ then $\int_a^b gdx \ge 0$. To this end we show that $\int_a^b gdx > -\in$ for every $\in > 0$.

If $\in > 0$ there is $\delta > 0$ such that for any tagged partition $\dot{P}$ with $\left\|\dot{P}\right\| < \delta$,

$$\left|S(g,\dot{P}) - \int_a^b gdx\right| < \in \text{ i.e } S(g,\dot{P}) - \in < \int_a^b gdx < S(g,\dot{P}) + \in .$$

If $\dot{P} = \left\{\left([x_{i-1}, x_i], t_i\right)\right\}_{i=1}^m$ $\quad S(g,\dot{P}) = \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) \ge 0$

$\Rightarrow \; -\in < \int_a^b gdx$. This is true for every $\in > 0$ so $\int_a^b gdx \ge 0$.

18.5.5 Corollary : If f is bounded, $f \in R[a,b]$ and $m \le f(x) \le M$ for all $x \in [a,b]$

Then $m(b-a) \le \int_a^b fdx \le M(b-a)$.

The constant functions $f_1(x) = m$ and $f_2(x) = M$ for $x \in [a,b]$ are integrable and $f_1(x) \le f(x) \le f_2(x)$ for all x.

So $\int_a^b f_1 dx \le \int_a^b fdx \le \int_a^b f_2 dx$ (by theorem 18.5.4)

$\Rightarrow \; m(b-a) \le \int_a^b fdx \le M(b-a)$.

18.6 Riemann Integrability Criteria:

We now state a few criteria for Riemann integrability of a function $f:[a,b] \to \mathbb{R}$. The first two criteria will be proved in lesson 19. The third criterion will be accepted as a fact without proof as even though the proof is well within our reach it is more technical.

Cauchy Criterion: A function $f:[a,b]\to\mathbb{R}$ belongs to R[a, b] if and only if for every $\in>0$ there exists $\eta_{\in}>0$ such that if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of [a, b] with $\|\dot{P}\|<\eta_{\in}$ and $\|\dot{Q}\|<\eta_{\in}$, then $\left|S(f,\dot{P})-S(f,\dot{Q})\right|<\in$.

The Squeeze Theorem: A function $f:[a,b]\to\mathbb{R}$ belongs to R[a, b] if and only if for every $\in>0$ there exists $\alpha_{\in}$ and $\varpi_{\in}$ in R[a, b] with $\alpha_{\in}(x)\le f(x)\le\varpi_{\in}(x)$ for all x in [a, b] and

$$\int_a^b(\varpi_{\in}-\alpha_{\in})dx<\in.$$

For Bounded Functions: Let $f:[a,b]\to R$ be bounded. Then the following are equivalent

a) $f\in R[a,b]$

b) For every $\in>0$ there exists a partition $P_{\in}$ such that if $\dot{P}_1$ and $\dot{P}_2$ are any tagged partitions having the same subintervals as $P_{\in}$ then

$$\left|S(f,\dot{P}_1)-S(f,\dot{P}_2)\right|<\in.$$

(c) For every $\in>0$ there exists a partition

$P_{\in}=\{a=x_0<x_1<.........<x_n=b\}$ such that

If $m_i=\inf\{f(x)/<x_{i-1}\le x\le x_i\}$ and

$M_i=\sup\{f(x)/x_{i-1}\le x\le x_i\}$ then

$$O(f,P_{\in})=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})<\in.$$

18.6.1 Example: Show directly from definition that the following function $f\in R([0,3])$ and find $\int_d^3 fdx$.

$$f(x) = \begin{cases} 2 & \text{if } 0 \le x \le 1 \\ 3 & \text{if } 1 < x \le 2 \end{cases}$$

Solution: Let $P = \{0 = x_0 < \dots\dots\dots < x_m \le 1 < x_{m+1} < \dots\dots < x_n = 3\}$ by any partition, $t_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$.

$$\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}, \ \dot{P}_1 = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{m}, \ \dot{P}_2 = \{([x_{i-1}, x_i], t_i)\}_{i=m+1}^{n}$$

Clearly $S(f, \dot{P}) = S(f, \dot{P}_1) + S(f, \dot{P}_2)$.

Also $f(t_i) = 2$ if $i \le m$

= 3 if $i > m + 2$

$f(t_{m+1}) = 2$ if $x_m \le t_{m+1} < 1 < x_{m+1}$

= 3 if $x_m \le 1 < t_{m+1} < x_{m+1}$

$$S(f, \dot{P}_1) = \sum_{i=1}^{m} 2(x_i - x_{i-1}) = 2(x_n - x_0) \quad \rightarrow \quad (1)$$

$$S(f, \dot{P}_2) = f(t_{m+1})(x_{m+1} - x_m) + \sum_{i=m+2}^{m} 3(x_i - x_{i-1})$$

$$= f(t_{m+1})(x_{m+1} - x_m) + 3(3 - x_{m+1}).$$

Now there are two possibilities :

either $x_m \le t_{m+1} \le 1$ or $x_m \le 1 < t_{m+1} \le x_{m+1}$

Case (i) : If $x_m \le t_{m+1} \le 1 < x_{m+1} f(t_{m+1}) = 2$. In this case $S(f, \dot{P}_2) = 2(x_{m+1} - x_m) + 3(3 - x_{m+1})$

$= 9 - 2x_m - x_{m+1} \quad \rightarrow \quad (2)$

Case (ii) : If $x_m \le 1 < t_{m+1} < x_{m+1}$, $f(t_{m+1}) = 3$ so that $S(f, \dot{P}_2) = 3(x_{m+1} - x_m) + 3(3 - x_{m+1})$

$= 9 - x_{m+1}$ in case (i) and $S(f, \dot{P}) = 2(x_m - 0) + 3(3 - x_m) = 9 - x_m$ in case (ii)

Thus

$$S(f,P)=\begin{cases}8+(1-x_{m+1})\text{ in case (i)}\\ 8+(1-x_m)\text{ in case (ii)}\end{cases}$$

$$\Rightarrow |S(f,P)-8|=\begin{cases}|1-x_{m+1}|\text{ in case (i)}\\ |1-x_m|\text{ in case (ii)}\end{cases}$$

Since $x_m \le 1 < x_{m+1}$ it follows that $\left|S(f,\dot{P})-8\right| \le (x_{m+1}-x_m)$

If $\in > 0$ and $\left|\dot{P}\right| < \in$ it follows that $x_{m+1}-x_m < \in \left|S(f,\dot{P})-8\right| < \in$ whenever $\left\|\dot{P}\right\| < \in$.

Hence $f \in R[0,3]$ and $\int_0^3 f dx = 8$.

18.6.2 Example: Show that $\int_0^1 x dx = \frac{1}{2}$.

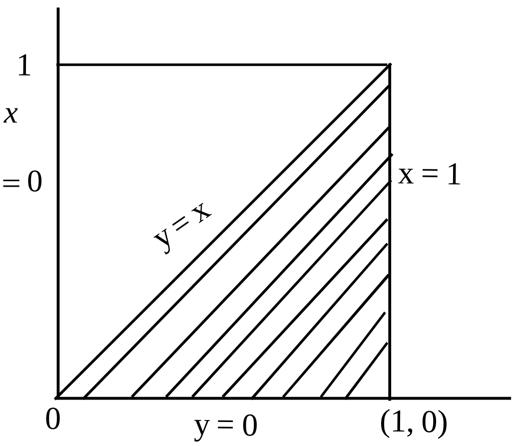


Fig. 18.6.2

From the figure the region bounded by the function $h(x)=x$ over [0, 1] is the triangular region bounded by the lines $y=0$ $y=x$ and $x=1$. As the area of this region is $\frac{1}{2}$.

For any tagged partition $\dot{P}=\left\{\left([x_{i-1},x_i],t_i\right)\right\}_{i=1}^n$, $S(h,\dot{P})=\sum_{i=1}^n t_i(x_i-x_{i-1})$.

Let $\dot{P}_0$ the parition with tag $q_i=\frac{x_i+x_{i-1}}{2}$.

Thus $\quad S(h,\dot{P}_0)=\frac{1}{2}\sum_{i=1}^{n}(x_i+x_{i-1})(x_i-x_{i-1}) \;=\frac{1}{2}\sum_{i=1}^{n}\; x_i^{\,2}-x_{i-1}^{2} \;=\frac{1}{2}(1^2-0^2)=\frac{1}{2}$

$\Rightarrow \quad \left|S(h,\dot{P})-\frac{1}{2}\right| \;=\; |S(h,P)-S(h,P_0)| \;=\; \left|\sum_{i=1}^{n}(t_i-q_i)(x_i-x_{i-1})\right| \le \sum_{i=1}^{n}|t_i-q_i|(x_i-x_{i-1})$.

$\le \sum_{i=1}^{n}|x_i-x_{i-1}|(x_i-x_{i-1})$ since $x_{i-1}\le t_i\le x_i$ and $x_{i-1}\le q_i\le x_i$

$\le \sum_{i=1}^{n}|x_i-x_{i-1}|(x_i-x_{i-1}) \le \|\dot{P}\|\sum_{i=1}^{n}(x_i-x_{i-1})=\|\dot{P}\|$

Thus if $\in>0$ and $\|\dot{P}\|<\in$ we get $\left|S(h,\dot{P})-\frac{1}{2}\right|<\in$.

Hence $h\in R[0,1]$ and $\int_0^1 hdx=\frac{1}{2}$.

18.6.3 Example:

Let $a\le x^*\le b;\;\; f:[a,b]\to R$ be defined by $f(x^*)=1$ and $f(x)=0$ if $x\ne x^*$.

Show that $f:[a,b]\to R$ and $\int_a^b fdx=0$

Let $P=a=x_0<x_1<..........<x_n=b$ any partition of [a, b]

$t_i\in[x_{i-1},x_i] \qquad x_{j-1}\le x^*\le x_j$ and $\qquad \dot{P}=\left\{\left(\left[x_{i-1},x_i\right],t_i\right)\right\}_{i=1}^{n}$

Then $\quad f(t_i)=0$ if $i\ne j,$

$$f(t_j)=\begin{cases}0 \text{ if } t_j\ne x^*\\ 1 \text{ if } t_j=x^*\end{cases}$$

Hence $S(f,\dot{P})=\begin{cases}0 \text{ if } t_j\ne x^*\\ x_j-x_{j-1} \text{ if } t_j=x^*\end{cases}$

so that $0 \le S(f,\dot{P}) \le x_j - x_{j-1} \le \|\dot{P}\|$. It follows that $\left|S(f,\dot{P})\right| < \in$

Hence $f \in R[a,b]$ and $\int_a^b f dx = 0$.

18.6.4 Example:

Define $f:[0,1] \to R$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\} \\ 0, \text{otherwise} \end{cases}$$

Show that $f \in R[0,1]$ and $\int_0^1 f dx = 0$.

Let $\dot{P}$ be any tagged partition of [0, 1]. $\dot{P}_0$ be the part of $\dot{P}$, where the tags are different from $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$ and $\dot{P}_1$ be the part with tags $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. We also assume that these tags are end points of the sub intervals so that each tag occurs twice-once as left end point of $[x_{i-1}, x_i]$ and also as right end point of $[x_{i-1}, x_i]$. Since $f(t_i) = 0$ if $t_i \notin \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$, $S(f,\dot{P}_0) = 0$. So

$$S(f,\dot{P}) = S(f,\dot{P}_0) + S(f,\dot{P}_1) = S(f,\dot{P}_1)$$

Also $S(f,\overset{1}{P_1}) \le \sum_{i=1}^{4} 1\left(\frac{i}{5} - x_{i-1}\right) + \left(x_i - \frac{i}{5}\right) \le 2\sum_{i=1}^{4}\left(x_i - x_{i-1}\right) \le 8\|\dot{P}\|$

So if $\in > 0$, let $\delta = \frac{\in}{8}$, for any tagged partition having $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ as tags at end points of the intervals we get $S(P,f) \le 8\|\dot{P}\| < \in$.

Hence $f \in R[0,1]$ and $\int_0^1 f dx = 0$.

18.6.5 Example:

Define $G:[0,1]\to R$ by $G(x)=x$ if $x=\dfrac{1}{n}$ for some $n\in\mathbb{N}$ and 0 otherwise.

Show that G : [0 ,1] and $\int_0^1 G dx = 0$

Solution: If $\in > 0$ the set $E_\in = \left\{ \dfrac{1}{n} / n \in \mathbb{N}, \dfrac{1}{n} \geq \in \right\}$ is finite and $E_\in = \left\{ 1, \dfrac{1}{2} \ldots\ldots\ldots \dfrac{1}{N} \right\}$ where $N \leq \dfrac{1}{\in} < N+1$, Let $\delta = \dfrac{\in}{2N}$ and $\dot{P}$ be any tagged partition of [0, 1] with $\left\|\dot{P}\right\| < \delta$.

Let $\dot{P}_1$ be that part of $\dot{P}$ where the tags belong to $E_\in$ and $\dot{P}_2$ be the remaining part of $\dot{P}$ so that if a tag t corresponds to an interval in $\dot{P}_2$, $G(t)=0$.

Hence $S(G,\dot{P}) = S(G,\dot{P}_1)\cdot$

For every $\left([x_{i-1}, x_i], t_i\right)$ in P_1 $0 < G(t_i) \leq 1$ and $x_i - x_{i-1} < \delta$

Hence $S(G,\dot{P}) = S(G,\dot{P}_1)$

$$\leq \sum_{[x_{i-1},x_i]\in P_1} \delta < 2N\delta = \in$$

$$S(G,P) = S(G,\dot{P}_1) = \sum G(t_i)(x_i - x_{i-1}) \text{ (where } t_i \in E_\in \text{)}$$

$$\leq \sum (x_i - x_{i-1}) \text{ where } t_i \in E_\in \ (\because E(t_i) = 1.)$$

In this summation it is possible that t_i is the tag for two adjecent sub intervals. As the length of each sub interval $\leq \delta$, the above summation can atmost be 2δ X the number of tags in $E_\in$. As the maximum number of tages in $E_\in$ is N the above summation is $2\delta \ N$.

Thus for every $\in > 0$ there is $\delta > 0$ such that for every $\dot{P}$ with $\left\|\dot{P}\right\| < \delta$ $S(G,\dot{P}) < \in \cdot$

Hence $G \in R[0,1]$ and $\int_0^1 G dx = 0$.

18.6.6 Example:

Consider the Thomae function H defined on [0, 1] by

$$H(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x = 0 \\ \frac{1}{x} \text{ if } 0 \neq x \text{ is rational and } x = \frac{m}{n} \text{ in its simplest form.} \end{cases}$$

We already proved in lesson 15 that H is continuous at every irrational number and discontinuous at every rational number. We prove here that $H \in R[0,1]$ and $\int_0^1 Hdx = 0$.

Let $\in > 0$ and $E_\in = \{x / 0 \leq x \leq 1$ and $H(x) \geq \frac{\in}{2}\}$. If $x \neq 0$ and $H(x) > \frac{\in}{2}$ then $x = \frac{2}{\in}$. Hence the number of denominators n is finite and for each n the number of $m \ni 0 < m < n$ and $g < d(m,n) = 1$ is finite, $E_\in$ is a finite set.

Let N be the number of elements in $E_\in$ and $\delta = \frac{\in}{4N}$.

If $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \delta$ we show that $S(H, \dot{P}) < \in$.

Let $\dot{P}$ be any such partition and $\dot{P}_1$ be the subset of P_1 having tags in $E_\in$ and $\dot{P}_2$ be the remaining part of $\dot{P}$, having tags outside $E_\in$.

If [0, 1] is any interval in $\dot{P}_1$ with tag t and t is an end point then t is an end point of the adjacent interval as well. Thus the maximum number of intervals with tags in $E_\in$ is 2N. Each such interval has length less than δ. So the total length of the intervals with tags in $E_\in$ is atmost $2N\delta = 2N\frac{\in}{4N} = \frac{\in}{2}$.

Since $0 \leq H(t) \leq 1$ for every tag t

$$S(H, \dot{P}_1) = \sum_{\dot{P}_1} \{H(t_i)(x_i - x_{i-1}) / t_i \in [x_{i-1} - x_i]\} \leq \sum_{[x_{i-1}, x_i] \in \dot{P}_1} (x_{i-1}, x_i) < \frac{\in}{2}.$$

If [a, b] is an interval in $\dot{P}_2$ with tag t, then $0 \le H(t) < \frac{\in}{2}$

Hence $S(H, \dot{P}) = S(H, \dot{P}_1) + S(H, \dot{P}_2) < \frac{\in}{2} + \frac{\in}{2} = \in$.

Hence $H \in R[0,1]$ and $\int_0^1 H dx = 0$.

18.6.7 Example:

Let $f:[a,b] \to R$, $g:[a,b] \to R$ be such that $f(x) = g(x)$ except at a point $c \in [a,b]$

Then $f \in R[a,b]$ if and only if $g \in R[a,b]$. In this case $\int_a^b f dx = \int_a^b g dx$.

Solution: Let $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be any tagged partition of [a, b] and $c \in [x_{i-1}, x_i]$. Then for $j \ne i$ $f(t_j) = g(t_i)$ and $f(t_i) = g(t_i)$ if $t_i \ne c$ while $f(t_i) \ne g(t_i)$ if $t_i = c$.

Hence $\left|S(f, \dot{P}) - S(g, \dot{P})\right| = \left|f(t_i) - g(t_i)\right|(x_i - x_{i-1}) \le \left|f(c) - g(c)\right| \left\|\dot{P}\right\| \quad \to \quad (1)$

Now suppose $f \in R[a,b]$ and $\in > 0$. There exists $\delta_1 > 0$ such that if

$$\left\|\dot{P}\right\| < \delta_1, \quad \left|s(f, \dot{P}) - \int_a^b f dx\right| < \frac{\in}{2} \quad \to \quad (2)$$

If $\left\|\dot{P}\right\| < \frac{\in}{2(\left|f(c) - g(c)\right|)} \left|S(f, \dot{P}) - S(g, \dot{P})\right| < \frac{\in}{2}$.

Hence if $\delta = \min\left\{\delta_1, \frac{\in}{2(\left|f(c) - g(c)\right|)}\right\}$ and $\left\|\dot{P}\right\| < \min\{\delta_1 \delta_2\}$

$$\left|S(g, \dot{P}) - \int_a^b f dx\right| \le \left|S(g, \dot{P}) - S(f, \dot{P})\right| + \left|S(f, \dot{P}) - \int_a^b f dx\right| < \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence $g \in R[a,b]$ and $\int_a^b g dx = \int_a^b f dx$.

By symmetry if $g \in R[a,b]$, $f \in R[a,b]$ and $\int_a^b fdx = \int_a^b gdx$.

18.7 Solutions to SAQ:

18.4.4 SAQ: Let $\in > 0$. Then there exists $\delta > 0$ such that $\left|\int_a^b fdx - S(f,\dot{P}_n)\right| < \in$ if $\left\|\dot{P}_n\right\| < \delta$

Since $\lim_n \left\|\dot{P}_n\right\| = 0$ there exists $N \in \mathbb{N}$ such that $\left\|\dot{P}_n\right\| < \delta$ if $n \geq N$.

Hence if $n \geq N$ $\left|\int_a^b fdx - S(f,\dot{P}_n)\right| < \in$.

$\Rightarrow \lim_n S(f,\dot{P}_n) = \int_a^b fdx$.

18.4.5 SAQ: $g(x) = 0$ if x is rational, $0 \leq x \leq 1$ and $\frac{1}{x}$ if x is irrational $0 \leq x \leq 1$.

For each $n \in \mathbb{N}$ let $P_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < ...... < \frac{n}{n} = 1\right\}$ and $t_i = \frac{i}{n}.t_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ and $g(t_i) = 0$.

Then $S(g\,\dot{P}_n) = 0$ where t_i is the tag for $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ clearly $\left\|\dot{P}_n\right\| = \frac{1}{n}$ so $\lim \left\|\dot{P}_n\right\| = 0$ and $\lim_n S(g,\dot{P}_n) = 0$.

18.4.6 SAQ:

Suppose $f \in R[a,b]$. Given $\in > 0$ there exists $\delta > 0$ such that for every tagged partition $\dot{P}$ with $\left\|\dot{P}\right\| < \delta$, $\left|S(f,\dot{P}) - \int_a^b f\right| < \frac{\in}{2}$.

Since $\lim \left\|\dot{P}_n\right\| = \lim \left\|\dot{Q}_n\right\| = 0$ for the above δ. there corresponds $N \in \mathbb{N}$ such that

$\left\|\dot{P}_n\right\| < \delta$ and $\left\|\dot{Q}_n\right\| < \delta$ if $n \geq N$.

$\Rightarrow \left|S(f, \dot{P}_n) - \int_a^b f\right| < \frac{\in}{2}$ and $\left|S(f, \dot{Q}_n) - \int_a^b f\right| < \frac{\in}{2}$ for $n \geq N$.

$\Rightarrow \left|S(f, \dot{P}_n) - S(f, \dot{Q}_n)\right| < \in$ for $n \geq N$.

$\Rightarrow \lim_n \left\{S(f, \dot{P}_n) - S(f, \dot{Q}_n)\right\} = 0$.

This contradicts the hypothesis that $\lim_n S(f, \dot{P}_n) \neq \lim_n S(f, \dot{Q}_n)$.

18.4.9 (a) SAQ:

$f \in R[a,b] \Rightarrow$ there exists $A \in \mathbb{R}$ such that for every $\in > 0$ there corresponds $\delta > 0$ such that for any taggled partition $\dot{P}$ of [a, b] with $\left\|\dot{P}\right\| < \delta$

$$\left|S(f, \dot{P}) - A\right| < \in.$$

$\Rightarrow$ If $\left\|\dot{P}\right\| \leq \frac{\delta}{2}$, $\left|S(f, \dot{P}) - A\right| < \in$,

Hence $\left\|\dot{P}\right\| \leq \frac{\delta}{2} \Rightarrow \left|S(f, \dot{P}) - A\right| \leq \in$

conversely if A is such that for every $\in > 0$ there corresponds $\delta > 0$ such that for any tagged partition $\dot{P}$ of [a, b] with If $\left\|\dot{P}\right\| \leq \delta$, $\left|S(f, \dot{P}) - A\right| \leq \in$ there for every $\dot{P}$ with $\left\|\dot{P}\right\| < \delta$ (This implies $\left\|\dot{P}\right\| \leq \delta$) $\left|S(f, \dot{P}) - A\right| < 2\in$

Since $\in > 0$ is arbitrary it follows that $f \in R[a,b]$.

18.4.9 (b) SAQ:

$$f(x) = \begin{cases} 1 \text{ if } 0 \le x \le 1 \text{ and } x \text{ is rational} \\ 0 \quad \text{otherwise} \end{cases}$$

If $\dot{P}$ is a tagged partition of [0, 1] with all tags rationals, $S(f, \dot{P}) = 1$

If all the tags of $\dot{P}$ are irrational then $S(f, \dot{P}) = 0$.

Hence there is no single A such that for $\in = \frac{1}{2}$ there corresponds $\delta > 0$ satisfying

$$\left\|\dot{P}\right\| < \delta \Rightarrow \left|S(f, \dot{P}) - A\right| < \frac{1}{2}.$$

Hence $f \notin R[0,1]$

18.4.9 (c) SAQ:

Let $\dot{P} = \left\{\left(\left[y_{i-1}, y_i\right], t_i\right)\right\}_{i=1}^{n}$ be any tagged partition of [a, b].

Then $\left|S(f, \dot{P})\right| = \left|\sum_{i=1}^{n} f(t_i)(y_i - y_{i-1})\right| \le \sum_{i=1}^{n} \left|f(t_i)\right|(y_i - y_{i-1}) \le \sum_{i=1}^{n} \left|f(t_i)\right| \left\|\dot{P}\right\|$

For every $x \in [a,b]$, $f(x) = \begin{cases} 0 \\ \text{or} \\ c_i \text{ for some } c_i \end{cases}$

Hence $\left|f(x)\right| \le \left|c_1\right| + \ldots\ldots. + \left|c_r\right|$.

Hence $\left|S(f, \dot{P})\right| \le \left(\sum_{i=1}^{r} \left|c_i\right|\right) \left\|\dot{P}\right\|$.

If $\in > 0$ and $\delta = \dfrac{\in}{\sum_{i=1}^{n} \left|c_i\right|}$ then if $\left\|\dot{P}\right\| < \delta$

$\left|S(f, \dot{P})\right| < \in$.

Hence $f \in R[a,b]$ and $\int_a^b f dx = 0$.

18.4.9 (d) SAQ:

Let $\dot{P} = \{(I_j, t_j)\}_{j=1}^{n}$ be a tagged partition of [a, b] and $c_1 < c_2$.

If $u \in I_j$ and $c_1 < t_j < c_2$ show that $c_1 - \|\dot{P}\| \le u \le c_2 + \|\dot{P}\|$.

Let $I_j = [x_{j-1}, x_j]$. Then $x_{j-1} \le u \le x_j$, $c_1 \le t_j < c_2$ and $x_{j-1} \le t_j < x_j$

Since $x_j - x_{j-1} \le \|\dot{P}\|$, $x_j \le x_{j-1} + \|\dot{P}\|$

$\Rightarrow \quad c_1 \le t_j \le x_j \le x_{j-1} + \|\dot{P}\|$

$\Rightarrow \quad c_1 - \|\dot{P}\| \le x_{j-1} \le u \quad \to \quad (1)$

$\Rightarrow \quad c_1 - \|\dot{P}\| \le u$.

Also $x_j - \|\dot{P}\| \le x_{j-1} \le t_j \le c_2$

$\Rightarrow \quad u \le x_j < c_2 + \|\dot{P}\| \quad \to \quad (2)$

From (1) and (2) if follows that $u \in \left[c_1 - \|\dot{P}\|, c_2 + \|\dot{P}\|\right]$.

18.8 Summary:

The notion of Riemann Integral is introduced via tagged partitions. Uniqueness of the Riemann Integral, Boundedness of a Riemann integrable function, linearity properties of the Riemann integral are established. A good number of examples with solutions are included.

18.9 Technical Terms:

Tagged partition - norm of a partition - Riemann integral.

18.10 Exercises:

1. If $I = [0,4]$ calculate the norms of the following partitions

(a) $P_1 = \{0,1,2,4\}$ (b) $P_2 = \{0,2,3,4\}$

2. If $f(x) = x^2$, $0 \le x \le 4$ calculate the Riemann sums for

(a) $P_1 = \{0,1,2,4\}$ with tags at the left end points of the intervals.

(b) $P_1 = \{0,1,2,4\}$ with tags at the left end points of the intervals.

(c) $P_2 = \{0,2,3,4\}$ with tags at the left end point of intervals.

(d) $P_2 = \{0,2,3,4\}$ with tags at the right end points of the intervals.

3. Let $f(x) = 2$ if $0 \le x < 1$ and $f(x) = 1$ if $1 \le x < 2$

Show that $f \in R[0,2]$ and $\int_0^2 f dx = 3$.

4. Let $g(x) = 2$ if $0 \le x < 1$, $f(1) = 1$ and $f(x) = 3$ if $1 < x \le 2$

Show that $g \in R[0,1]$ and $\int_0^2 g dx = 5$.

5. Use mathematical induction to prove that if $n \in \mathbb{N}$, $f_i \in R[a,b]$ and $k_i \in \mathbb{R}$ for $1 \le i \le n$ then $k_1 f_1 + \ldots + k_n f_n \in R[a,b]$ and $\int_a^b (k_1 f_1 + \ldots + k_n f_n) dx = k_1 \int_a^b f_1 dx + \ldots + k_n \int_a^b f_n dx$.

6. Give examples of f and g defined on [a, b] such that $g \in R[a,b]$, $f(x) \le g(x)$ if $x \in [a,b]$ but $f \notin R[a,b]$.

7. Show that if $f:[a,b]\to\mathbb{R}$ satisfies $f(x)=0$ except at a finite number of points then $f\in R[a,b]$ and $\int_a^b fdx=0$.

8. If $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are such that $f(x)=g(x)$ except a finite number of points in [a, b]. Show that $f\in R[a,b]$ if and only if $g\in R[a,b]$ and in this case $\int_a^b fdx=\int_a^b gdx$.

9. Let $0\le a<b$, $f(x)=x^2$ if $a\le x\le b$,

$\dot{P}:\{([x_{i-1}\le q_i\le x_i],q_i)\}_{i=1}^{n}$ where $q_i=\sqrt{x_i^2+x_ix_{i-1}+x_{i-1}^2}$ prove the following

(i) $0\le x_{i-1}\le q_i\le x_i$

(ii) $f(q_i)(x_i-x_{i-1})=\dfrac{x_i^3-x_{i-1}^3}{3}$

(iii) $S(f,\dot{P})=\dfrac{b^3-a^3}{3}$

(iv) Show that $f\in R[a,b]$ and $\int_a^b fdx=\dfrac{1}{3}(b^3-a^3)$.

10. Let $f\in R[a,b]$ and $a<c<b$. Show that the function $g:[a+c,b+c]\to\mathbb{R}$ defined by $g(x)=f(x-c)$ belongs to $R[a+c,b+c]$. Also prove that $\int_a^b fdx=\int_{a+c}^{b+c} gdx$.

11. If $f\in R[a,b]$ and $|f(x)|\le M$ for $x\in[a,b]$ show that $\left|\int_a^b f(x)dx\right|\le M(b-a)$.

12. Let $\dot{P}$ be tagged partition of [o, 3].

(a) Show that the union U_1 of all subintervals in $\dot{P}$ with tages in [0,1] satisfies

$$\left[0,1-\|\dot{P}\|\right]\subseteq U_1\subseteq\left[0,1-\|\dot{P}\|\right].$$

(b) Show that the union U_2 of all subintervals in $\dot{P}$ with tags in [1,2] satisfies

$$\left[1+\left\|\dot{P}\right\|, 2-\left\|\dot{P}\right\|\right] \subseteq U_2 \subseteq \left[1-\left\|\dot{P}\right\|, 2+\left\|\dot{P}\right\|\right]$$

13. Let $\dot{P} = \left\{\left(I_j, t_j\right)\right\}_{i=1}^{n}$ be a tagged partition of [a, b] and $a \le c_1 < c_2 \le b$.

14. If $t_j \in \left[c_1+\left\|\dot{P}\right\|, c_2+\left\|\dot{P}\right\|\right] \cap I_j$, show that $t_j \in [c_1, c_2]$

Let $f(x)=2$ if $0 \le x < 1$ and $f(x)=1$ if $1 \le x < 2$ show that $f \in R[0,2]$ and $\int_0^2 f dx = 3$.

Let $g(x)=2$ if $0 \le x < 1$, $g(1)=3$ and $g(x)=1$ if $1 < x \le 2$ show that $g \in R[0,2]$ and $\int_0^2 g dx = 3$.

15. Let $0 \le a > b$; $f(x)=x^2$; $a = x_0 < x_1 < .... < x_n = b$; and $t_i = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$. Show that

(a) $x_{i-1} \le t_i \le x_i$

(b) If $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ $S(f, \dot{P}) = \frac{b^3 - a^3}{3}$.

(c) Using an argument similar to Example 18.6.2. show that $\int_a^1 x^2 dx = \frac{b^3 - a^3}{3}$.

18.9 Model Examination Questions:

1. Define Riemann integral and show that the integral of a function $f:[a,b] \to R$ is unique.

2. Show that if $f \in R[a,b]$, f is bounded.

3. If $f(\frac{1}{2}) = 1$ and $f(x) = 0$ if $0 \le x \le 1$ and $x \ne \frac{1}{2}$ show that $f \in R[0,1]$ and that $\int_0^1 f dx = 0$.

4. Show that if $f \in R[a,b]$ and $g \in R[a,b]$ then $f+g \in R[a,b]$ and $\int_a^b (f+g)dx = \int_a^b fdx + \int_a^b gdx$.

5. Show that if $f(x)=0$ if x is rational and $f(x)=1$ is x is irrational then f is not Riemann integrable on [0,1]

18.10 Model Practical Problem with Solution:

$$\text{Let } f(x) = \begin{cases} 0 \text{ if x is irrational} \\ 1 \text{ if x is rational} \end{cases}$$

Show that $f \in R[0,1]$ and $\int_0^1 fdx = 0$.

Definition: $f:[a,b] \to R$ is said to be **Riemann Integrable** if there exists real number A with the following property:

For every $\in > 0$, there corresponds $\eta > 0$ such that whenever $\dot{P}$ is any tagged partition with $\|\dot{P}\| < \eta$

$|S(f,\dot{P}) - A| < \in$. The number A is called the **Riemann integral** of f over [a, b] and is denoted by

$A = \int_a^b fdx$ or $A = \int_a^b f$.

A **tagged partition** is a division $\dot{P}$ of [a, b] into subintervals $[x_{i-1}, x_i]$ with “tags” t_i chosen in $[x_{i-1}, x_i]$ and is written $\dot{P} = \{([x_{i-1}, x_i]), t_i\}_{i=1}^n$ $x_{i-1} \le t_i x_i$ where $a = x_0 < x_1 < .... < x_n = b$.

$$\|\dot{P}\| = \max(x_i - x_{i-1}).\ 1 \le i \le n,\ S(f,\dot{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

Solution: Let $\dot{P} = \{0 = x_0 < x_1 < ....x_n = 1\}$ be any tagged partition with any tag $t_i \in [x_{i-1}, x_i]$.

$$S(f,\dot{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = f(t_n)(x_n - x_{n-1}) \text{ (since } f(t) = 0 \text{ if } t < 1)$$

$= x_n - x_{n-1}$ if $t_n = 1$ and 0 if $t_n < 1$.

Hence if $\in > 0$ and $\left\|\dot{P}\right\| < \in$ then $\left|S(f, \dot{P})\right| = S(f, \dot{P}) \leq x_n - x_{n-1} < \left\|\dot{P}\right\| < \in$.

Thus $\forall \in > 0$ we choose $\delta_{\in} = \in$. This $\delta_{\in}$ satisfies the requirement in the definition of Riemann integral so that $f \in R[0,1]$ and $\int_0^1 f dx = 0$.

- C. Sandhya

Lesson - 19

Riemann Integrability:

19.1 Objective:

We present some criteri of Riemann integrability of a function $f:[a,b]\to R$ issuing these criteria we identify some classes of integrable functions we also study Riemann integrabili of a function when the domain is split into subintervals.

19.2 Structure:

19.3 Introduction

19.4 Conditions for Integrability

19.5 Step Functions

19.6 Other Classes of Integrable Functions

19.7 Additivity Theorem

19.8 Answers to SAQ's

19.9 Summary

19.10 Technical Terms

19.11 Excercises

19.12 Model Practical Problems with Solutions

19.3 Introduction:

This lesson is devoted for the study of conditions for Riemann integrability of a function. We study in detail the cauchy criterion and the squeeze theorem which propose conditions for integrability of a function in terms of tagged partitions. At times these techniques may be lengthy to adopt. As such we also mention other equilent conditions without proof. We establish Integrability of step functions, continuoues functions and monotone functions. We then proceed to establish additivity theorems for integration. Which deal with splitting or clubbing intervals. Some useful theorems concerning the integrability of the product, substitution theorem and so on are also presented here. Other important properties are included in short answer questions examples and exercises.

19.4 Conditions of Intigrability:

19.4.1 Theorem (Cauchy Criterian for Riemann Integrability):

$f:[a,b]\to R$ belongs to R[a, b] iff for every $\in>0$ there corresponds $\eta_\in>0$ such that if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of [a, b] with $\|\dot{P}\|<\eta_\in$ and $\|\dot{Q}\|<\eta_\in$ then

$\left|S(f,\dot{P})-S(f,\dot{Q})\right|<\in \quad \rightarrow \quad (1)$

Proof : $(\Rightarrow)$ Suppose $f \in R[a,b]$. Then by definition for every $\in > 0$ there corresponds $\eta_{\in} > 0$ such that for every tagged partition $\dot{P}$ of [a, b] with $\|\dot{P}\| < \eta_{\in}$ $\left|S(f,\dot{P})-\int_a^b f dx\right| < \frac{\in}{2}$.

If $\dot{P}$ and $\dot{Q}$ are tagged partitions of [a, b] with norm $< \eta_{\in}$

$$\left|S(f,\dot{P})-S(f,\dot{Q})\right|$$

$$= \left|S(f,\dot{P})-L+L-S(f,\dot{Q})\right|, \text{ where } L=\int_a^b f dx$$

$$\leq \left|S(f,\dot{P})-L\right|+\left|L-S(f,\dot{Q})\right|$$

$$< \frac{\in}{2}+\frac{\in}{2}=\in.$$

Converse: $(\Leftarrow)$ suppose that for every $\in > 0$ there corresponds $\eta_{\in} > 0$ such that for all tagged partitions $\dot{P}$ and $\dot{Q}$ of [a, b] with norms $< \eta_{\in}$, $\left|S(f,\dot{P})-S(f,\dot{Q})\right|<\in \quad \rightarrow \quad (2)$

Takng $\in=\frac{1}{n}$ where $n \in N$ we choose $\eta_n > 0$ such that (2) holds whenever $\dot{P}$, $\dot{Q}$ are tagged partitions of [a, b] with norm $< \eta_{\in}$.

Let $\delta_n = \min\{\eta_1,.........\eta_n\}$. Then $\delta_{n+1} < \eta_i$ for $1 \leq i \leq n+1$ hence $\delta_{n+1} \leq \delta_n$ for all n.

Choose tagged partition $\dot{P}$ of [a, b] such that $\|\dot{P}\| < \delta_n$. If $\in > 0$ and N is a natural number such that $N > \frac{1}{\in}$ then for $m > n \geq N$, $\delta_m \leq \delta_n \leq \delta_N$ so that $\|\dot{P}_m\| < \delta_m < \delta_N$ and $\|\dot{P}_n\| < \delta_n \leq \delta_N$

Hence by (2) with $\in = 1/N$

$$\left| S(f, \dot{P}_m) - S(f, \dot{P}_n) \right| < \frac{1}{N} < \in .$$

Therefore $(S(f, \dot{P}_n))$ is a cauchy sequence in R.

Let $A = \lim_n S(f, \dot{P}_n)$.

We show that for $\in > 0$ there is $\eta_{\in} > 0$ such that for all tagged partitions of [a, b] with $\|\dot{P}\| < \eta_{\in}$ $\left| S(f, \dot{P}) - A \right| < \in$.

So that $f \in R[a, b]$ and $\int_a^b f dx = A$.

Let $\in > 0$: Since $\lim_n S(f, \dot{P}_n) = A$, there is $N \in N$ such that

$$\left| S(f, \dot{P}) - A \right| < \frac{\in}{2} \text{ for } n \geq N \qquad \rightarrow \qquad (3)$$

We may assume that $N > \frac{2}{\in}$.

Let $\eta_{\in} = \delta_N$. If $\dot{P}$ is any tagged partition of [a, b] with $\|\dot{P}\| < \eta_{\in}$ there $\|\dot{P}\| < \delta_N$ and $\|\dot{P}\| < \delta_N$

so by (2), $\left| S(f, \dot{P}) - S(f, \dot{P}_N) \right| < \frac{1}{N} < \frac{\in}{2}$ $\qquad \rightarrow \qquad$ (4)

If $\|\dot{P}\| < \eta_{\in}$, $\left| S(f, \dot{P}) - A \right|$

$$= \left| \left\{ (S(f, \dot{P}) - S(f, \dot{P}_N) \right\} + \left\{ S(f, \dot{P}_N) - A \right\} \right|$$

$$\leq \left| (S(f, \dot{P}) - S(f, \dot{P}_N) \right| + \left| S(f, \dot{P}_N) - A \right|$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in .$$

19.4.2 Qqueeze Theorem:

Let $f:[a,b]\to R$. Then $f \in R[a,b]$ if and only if for every $\in > 0$ time exist $\alpha_\in$ and $\varpi_\in(x)$ in such that

(i) $\alpha_\in(x) \le f(x) \le \varpi_\in(x)$ for $x \in [a,b]$

(ii) $\alpha_\in \in R[a,b]$, $\varpi_\in \in R[a,b]$ and

(iii) $\int_a^b (\varpi_\in - \alpha_\in)dx = 0$.

Proof: If $f \in R[a,b]$ and $\in > 0$ choose $\alpha_\in = \varpi_\in = f$. Then $\alpha_\in$, $\varpi_\in$ satisfy (i), (ii) and (iii) conversely suppose that for every $\in > 0$ there exist functions $\alpha_\in$, $\beta_\in$ satisfying (i) (ii) and (iii).

Since $\alpha_\in \in R[a,b]$ and $\in > 0$ there exists $\delta_1 > 0$

Such that for every tagged partition $\dot{P}$ of [a, b] with $\|\dot{P}\| < \delta_1$,

$$\left| S(\alpha_\in, \dot{P}) - \int_a^b \alpha_\in dx \right| < \frac{\in}{3} \qquad \rightarrow \qquad (1)$$

Similarly $\delta_2 > 0$ such that for every tagged partition $\dot{Q}$ of [a, b] with $\|\dot{Q}\| < \delta_2$

$$\left| S(\varpi_\in, \dot{Q}) - \int_a^b \varpi_\in dx \right| < \frac{\in}{3} \qquad \rightarrow \qquad (2)$$

From (+)

(1) implies $\int_a^b \alpha_\in dx - \frac{\in}{3} < S(\alpha_\in, \dot{P}) < \int_a^b \alpha_\in dx + \frac{\in}{3}$ if $\|\dot{P}\| < \delta_1$

(2) implies $\int_a^b \varpi_\in dx - \frac{\in}{3} < S(\varpi_\in, \dot{Q}) < \int_a^b \varpi_\in dx + \frac{\in}{3}$ if $\|\dot{Q}\| < \delta_2$

Let $\delta_\in = \min\{\delta_1, \delta_2\}$ and $\dot{P}$ be any tagged partition of [a, b] with $\|\dot{P}\| < \delta_\in$.

Since $\alpha_\in \leq f(x) \leq \varpi_\in(x)$ for $x \in [a,b]$

$$S(\alpha_\in, \dot{P}) \leq S(f, \dot{P}) \leq S(\varpi_\in, \dot{P})\cdot$$

$$\Rightarrow \qquad \int_a^b \alpha_\in dx - \frac{\in}{3} < S(\alpha_\in, \dot{P}) \leq S(f, \dot{P}) \leq S(\varpi_\in, \dot{P}) < \int_a^b \varpi_\in dx + \frac{\in}{3}$$

$$\Rightarrow \qquad \int_a^b \alpha_\in dx - \frac{\in}{3} < S(f, \dot{P}) < \int_a^b \varpi_\in dx + \frac{\in}{3} \qquad \rightarrow \qquad (3)$$

III^ly if $\dot{Q}$ is any tagged partition with $\|\dot{Q}\| < \delta_\in$.

$$\int_a^b \alpha_\in dx - \frac{\in}{3} < S(f, \dot{Q}) < \int_a^b \varpi_\in dx + \frac{\in}{3} \qquad \rightarrow \qquad (4)$$

(3)+(4)

$$\Rightarrow \qquad \left| S(f, \dot{P}) - S(f, \dot{P}) \right| < \int_a^b (\varpi_\in - \alpha_\in) dx + \frac{2\in}{3}.$$

$$< \frac{\in}{3} + \frac{2\in}{3} = \in.$$

Thus for every $\in > 0$ there is $\delta_\in > 0$ such that for every $\dot{P}$ and $\dot{Q}$ with $\|\dot{P}\| < \delta_\in$ and $\|\dot{Q}\| < \delta_\in$

$$\left| S(f, \dot{P}) - S(f, \dot{Q}) \right| < \in \ .$$

By cauchy criterion $f \in R[a,b]$.

19.4.3 SAQ: Apply Cauchy criterion for the function

$g:[0,3] \to R$ defined in 18.6 by

$$g(x) = \begin{cases} 2 \text{ if } 0 \leq x \leq 1 \\ \\ 3 \text{ if } 1 < x < 3 \end{cases}$$

(b) SAQ: Apply Cauchy criterion to show that the dirichlet function is not Riemann integrable.

(c) SAQ: Show that $f:[a,b]\to R$ is not Riemann integrable if and only if there exists $\in > 0$ such that for every $n\in N$ there exist tagged partitions $\dot{P}_n$ and $\dot{Q}_n$ such that $\lim\|\dot{P}_n\| = \lim\|\dot{Q}_n\| = 0$ but for all $n\in N$, $\left|S(f,\dot{P}_n) - S(f,\dot{Q}_n)\right| \geq \in_0$

(d) Let $H(x) = \begin{cases} x+1 & \text{if } x\in[0,1] \quad x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$

(e) Let Let $H(x) = \begin{cases} k & \text{if } x = \frac{1}{k},\ k\in N \text{ and} \\ 0 & \text{if } 0\leq x\leq 1 \text{ and } x\neq \frac{1}{k} \text{ for any } k\in N \end{cases}$

Show that $H\notin R[0,1]$.

19.4.4 Criteria for Integrability of a bounded function:

We state without proof some criteria for integrability of a bounded function $f:[a,b]\to R$.

Theorem: Let $f:[a,b]\to R$ be a bounded function then the following are equivalent.

(a) $f\in R[a,b]$.

(b) For every $\in > 0$ there corresponds a partition $P_\in$ of [a, b] such that for every tagged partitions $\dot{P}_1$ and $\dot{P}_2$ with the same subintervals as $P_\in$

$$\left|S(f,\dot{P}_1) - S(f,\dot{P}_2)\right| < \in.$$

(c) For every $\in > 0$ there corresponds a partition $P_\in = \{I_j\}_{j=1}^{n}$

Such that if $I_j = [x_{j-1}, x_j]$, $\sup\{f(x)/x\in[x_{j-1},x_j]\} = M_j$ and if

$\inf\{f(x)/x\in[x_{j-1},x_j]\}=m_j$ then the oscillatory sum $0(f,P_\in)=\sum_{j=1}^{n}(M_j-m_j)(x_{j-1},x_j)<\in$.

19.5 Integrability of Step Functions:

19.5.1 Definition:

$f:[a,b]\to R$ is called an elementary step function if there is an interval $J\leq[a,b]$ such that

$$f(x)=\begin{cases}1 \text{ if } x\in J\\ 0 \text{ if } x\in[a,b]\setminus J\end{cases}$$

f is denoted by ϕ_J .

Note: J may include both the end points or one and only one of them or none. The case $J=\{c\}$ is also not ruled out.

Definition: $f:[a,b]\to R$ is called a step function if f has a finite number of distinct values, each value being assumed on a finite number of intervals.

19.5.2 Examples:

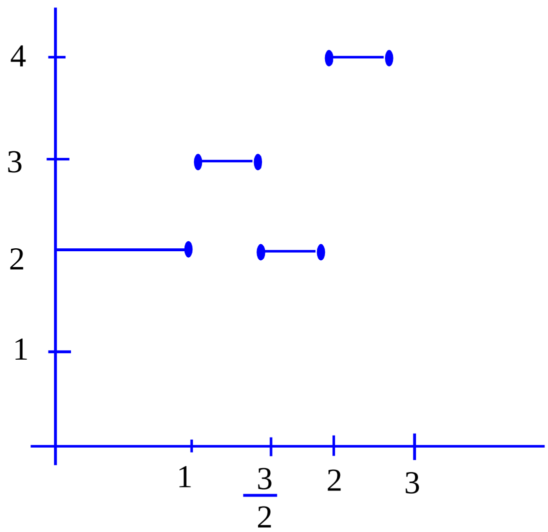
(i) Define f on [0, 3] by $f(x)=2$ if $0\leq x\leq 1$

$=3$ if $1<x\leq\frac{3}{2}$

$=2$ if $\frac{3}{2}<x\leq 3$

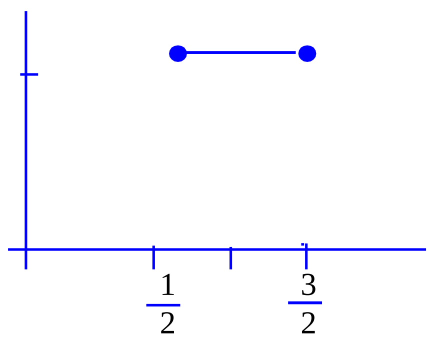
$=4$ if $2<x\leq 3$.

f is a step function.



(ii) The function defined on [0, 2] by

$$f(x)=\begin{cases}0 \text{ if } 0\leq x<\frac{1}{2} \text{ or } \frac{3}{2}\leq x\leq 2\\ 1 \text{ if } \frac{1}{2}\leq x<\frac{3}{2}\end{cases}$$



is a step function but not an elementary step function.

Note: Where as a step function has a finete number of values true. That a function that assumes asfinite number of values on [a. b] is necessarily a step function. For example refer f on [0, 1] by

$$f(x) = \begin{cases} 0 \text{ if } x \in \left\{ \frac{1}{n} / n \in N \right\} \\ 1 \text{ otherwise} \end{cases}$$

clearly $f(x)$ =either 0 or 1 but neither of the sets $f^{-1}\{0\} = \left\{ \frac{1}{n} / n \in N \right\}$ and its complement in [0, 1] is an internal or a finite union of intervals.

19.5.3 SAQ:

Every step function is a linear combination of elementary step functions.

19.5.4 SAQ:

If f is the step function defined on [a, b] by $f(x) = 1$ if $x = c$ where $a \le c \le b$ (i fixed)

= 0 otherwise

Then $f \in R[a,b]$ and $\int_a^b f dx = 0$.

19.5.5 Lomma:

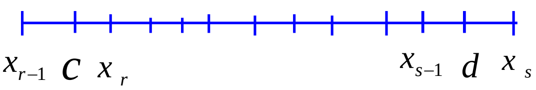
Let $[c,d] \subseteq [a,b]$, $f:[a,b] \to R$ be defined by

$$f(x) = \begin{cases} \alpha \text{ if } c \le x \le d \\ = 0 \text{ otherwise} \end{cases}$$

Where $\alpha \neq 0$.

Then $f \in R[a,b]$ and $\int_a^b f dx = \alpha(d-c)$.

Proof: Let $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ be any tagged partition of [a, b]



$x_{r-1} < c \le x_r < x_{s-1} \le d < x_s$

Clearly $S(f, \dot{P}) = f(t_r)(x_r - x_{r-1}) + \alpha(x_s - x_{s-1}) + f(t_s)(x_s - x_{s-1})$ we have the following possibilities for $f(t_r)$ and $f(t_s)$

$$f(t_r) = \begin{cases} 0 & \text{if } x_{r-1} \le t_r < c \\ \alpha & \text{if } c \le t_r \le x_r \end{cases}$$

$$f(t_s) = \begin{cases} \alpha & \text{if } x_{s-1} \le t_s < d \\ 0 & \text{if } d \le t_s \le x_s \end{cases}$$

Hence

$$S(f,\dot{P}) = \begin{cases} \text{(i) } \alpha \ (x_s - x_r) \text{ if } f(t_r)=0, \ f(t_s)=\alpha \\ \text{(ii) } \alpha \ (x_{s-1} - x_r) \text{ if } f(t_r)=0 \quad f(t_s)=0 \\ \text{(iii) } \alpha \ (x_s - x_{r-1}) \text{ if } f(t_r)=\alpha = f(t_s) \\ \text{(iii) } \alpha \ (x_{s-1} - x_{r-1}) \text{ if } f(t_r)=\alpha, f(t_s)=0 \end{cases}$$

Hence

$$S(f,\dot{P}) = \begin{cases} \text{(i) } \alpha \ (d-c)+\alpha(x_s - d - x_r + c) \\ \text{(ii) } \alpha(d-c)+\alpha(x_{s-1} - d + c - x_r) \\ \text{(iii) } \alpha(d-c)+\alpha(x_s - d + c - x_{r-1}) \\ \text{(iii) } \alpha(d-c)+\alpha(x_{s-1} - d + c - x_{r-1}) \end{cases}$$

$$\Rightarrow \quad \left|S(f,\dot{P}) - \alpha(d-c)\right| \le |\alpha|(x_s - x_{s-1}) + (x_r - x_{r-1}) \le 2|\alpha|\left\|\dot{P}\right\|$$

If $\left|S(f,\dot{P}) - \alpha(d-c)\right| < \in$ if $\left\|\dot{P}\right\| < \dfrac{\in}{2|\alpha|}$.

This holds for every $\in > 0$ so $f \in R[a,b]$ and $\int_a^b f dx = \alpha(d-c)$

19.5.6 Lemma: If J is a subinterval of [a, b] and has end points c,d where $c<d$ then $Q_J \in R[a,b]$ and $\int_a^b Q_J dx = d-c$.

Proof: The possibilities for J are [c,d], [c, d), (c,d] and (c, d) in 19.5.5 we proved the result when J=[c,d], write $J_0 = (c,d)$.

Let $P = a = x_0 < x_1 < .... < x_n = b$, $t_i \in [x_{i-1}, x_i]$ and $\dot{P}$ the corresponding Tagged Partition.

Then $\left| S(\phi_{J,}\ \dot{P}) - S(\phi_{J_0}\ \dot{P} \right|$

$$= \left| \sum_{i=1}^{n} \left(Q_J(t_i) - Q_{J_0}(t_i)\right)\left(x_i - x_{i-1}\right) \right|$$

$$\leq \sum_{i=1}^{n} \left| Q_J(t_i) - Q_{J_0}(t_i) \right| \left(x_i - x_{i-1}\right)$$

For any x, $Q_j(x) - Q_{j_0}(x) = \begin{cases} 0 & \text{if } x \neq c \\ 1 & \text{if } x = c \end{cases}$

Hence $\left| S(\phi_{J,}\ \dot{P}) - S(\phi_{J_0}\ \dot{P} \right| \leq \left\| \dot{P} \right\|$.

If $Q_j \in R[a,b]$ then for every $\in > 0$ there is $\delta > 0$ such that $< \delta < \frac{\in}{2}$ and for every tagged partition $\dot{P}$ of [a, b] with $\left\|\dot{P}\right\| < \delta$

$$\left| S(\phi_{J,}\ \dot{P}) - \int_a^b \phi_J\, dx \right| < \frac{\in}{2}.$$

For such $\dot{P}$, $\left| S(\phi_{J_0,}\ \dot{P}) - \int_a^b \phi_J\, dx \right|$.

$$\leq \left| S(f, \phi_{J_0}) - S(f, \phi_J) \right|$$

$$+ \left| S(f, \phi_J) - \int_a^b \phi_J \, dx \right|$$

$$< \left\| \dot{P} \right\| + \frac{\in}{2} + \frac{\in}{2} = \in$$

Hence $Q_{j_0} \in R[a,b]$ and $\int_a^b Q_{J_0} dx = \int_a^b Q_J dx = d - c$.

19.5.7 Theorem:

If $\phi : [a,b] \to R$ is a step function the $\phi \in R[a,b]$.

Proof: Since ϕ is a step function, ϕ can be expressed as linear combination of elementary step functions (SAQ 19.5.3).

Write $\phi = K_1 \phi_{J_1} + ... + K_m \phi_{J_m}$, where J_i has end points say $c_i \leq d_i$ for $1 \leq i \leq m$.

Then $\phi_{J_i} \in R[a,b]$ and $\int_a^b \phi_{J_i} dx = \phi_i - c_i$

$$\phi \in R[a,b] \text{ and } \int_a^b \phi \, dx = \sum_{i=1}^{m} k_i \int_a^b \phi_{J_i} dx = \sum_{i=1}^{m} k_i (d_i - c_i)$$

19.5.8. SAQ:

If $f : [a,b] \to R$ and $\dot{P}$ is a tagged partition of [a, b] then show that there is a step function ϕ on [a, b] such that $\int_a^b \phi = S(f, \dot{P})$.

19.6 Other Classes of Integrable functions:

19.6.1 Theorem:

If $f : [a,b] \to R$ is continuous then $f \in R[a,b]$.

Proof: Let $\in > 0$. Since f is uniformly continuous continuous on [a, b], there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\in}{b-a}$ if $|x - y| < \delta$ and x, y belong to [a, b]. Divide [a, b] into n equal parts to form a partition $P:\{a = x_0 < x_1 < ....x_n = b\}$ so that $x_i - x_{i-1} = \dfrac{b-a}{n} < \delta \;\forall\, i$. Since f is continous on $[x_{i-1}, x_i]$ f is bounded and attains bounds. So there exist u_i and v_i in $[x_{i-1}, x_i]$ such that $f(u_i) \le f(t) \le f(v_i)$ if $x_{i-1} \le t \le x_i$

Let $\alpha_\in(t) = f(u_i)$ and $\alpha_\in(b) = f(v_i)$ if $x_{i-1} \le t \le x_i$ and $\alpha_\in(b) = f(u_n)$ and $\varpi_\in(b) = f(v_n)$

$\alpha_\in$ and $\varpi_\in$ being step functions, are Riemann integrable and for every t in [a, b], $\alpha_\in(t) \le f(t) \le \varpi_\in(t)$.

Further $\displaystyle\int_a^b (\varpi_\in - \alpha_\in)dt = \sum_{i=1}^{n} (f(v_i) - f(u_i)(x_i - x_{i-1})$

$$< \frac{\in}{b-a}(b-a) = \in .$$

Hence by the squeeze theorem $f \in R[a,b]$.

19.6.2 SAQ:

Prove theorem 19.6.1 using (c) of the criteria of integrability for bounded functions.

19.6.3 Theorem:

If f is monotonic in [a, b] then $f \in R[a,b]$

Proof: if $f(a) = f(b)$ for all $x \in [a,b]$, hence $f \in R[a,b]$. Assume that $f(a) \ne f(b)$ and f is monotonically is [a, b].

Let $\in > 0$, $\delta = \dfrac{\in}{f(b) - f(a)}$ and $P = \{a = x_0 < x_1 < .... < x_n = b\}$ be any partition with $\|\dot{P}\| < \delta$.

For $1 \le i \le n$ define $\alpha_\in(t) = f(x_{i-1})$ and $\varpi_\in(t) = f(x_i)$ if $x_{i-1} \le t < x_i$ for $1 \le i < n$ and $\alpha_\in(b) = f(b)$ and $= \text{lub}\left|f^2(x) - f^2(y)\right| x, y.$

Being step functions $\alpha_\in$ and $\varpi_\in$ are Riemann integrable on [a, b] and $\alpha_\in(t) \le f(t) \le \varpi_\in(t)$ if $a \le t \le b$.

Further $\displaystyle\int_a^b \varpi_\in - \alpha_\in)dt = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) < \frac{\in}{f(b) - f(a)}(f(x_n) - f(x_0)) = \in$.

Hence by the squeeze theorem $f \in R[a,b]$.

19.6.4 SAQ: Prove theorem 19.6.2 using (c) of the criteria of integrability for bounded functions.

19.6.5 Theorem: If $f \in R[a,b]$ then $f^2 \in R[a,b]$/

Proof: Since $f \in R[a,b]$, f is bounded Let $u > 0$ be such that $|f(x)| \le M$ for all x in [a, b].

If x, y belong to [a, b], $\left|f^2(x) - f^2(y)\right| = \left|f(x) - f(y)\right|\left|f(x) + f(y)\right|$

$$\le \left|f(x) - f(y)\right|(\left|f(x)\right| + \left|f(y)\right|$$

$$\le 2k\left|f(x) - f(y)\right|$$

Since $f \in R[a,b]$, given $\in > 0$ there is a partition

$P_{\in} = \{a = x_0 < x_1 < \ldots\ldots\ldots < x_n = b\}$ such that

$$0(f, P_{\in}) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \frac{\in}{2k}.$$

Where $O(f, P_{\in}) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \in$

and $m_i^{'} = \text{glb}\{f^2(x) / x \in [x_{i-1}, x_i]\}$

$M_i^{'}$ - $m_i^{'} = \text{lub}\{\left|f^2(x) - f^2(y)\right| x, y$ belong to $[x_{i-1}, x_i]\}$

$\le 2k\,\text{lub}\{\left|f(x) - f(y)\right| x, y$ belong to $[x_{i-1}, x_i]\}$

$= 2k(M_i - m_i)$

Hence $O(f^2, P_{\in}) = \sum_{i=1}^{n}(M_i^{'} - m_i^{'})(x_i - x_{i-1})$

$$\le 2k\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$$

$$= 2kO(f, P_{\in}) < \in.$$

This is true for every $\in > 0$ so $f^2 \in R[a,b]$.

19.6.6 Theorem: If $f \in R[a,b]$ and $g \in R[a,b]$ then $fg \in R[a,b]$

Proof: $f \in R[a,b]$ and $g \in R[a,b]$

$\Rightarrow (f+g) \in R[a,b]$ and $f-g \in R[a,b]$

$\Rightarrow (f+g)^2 \in R[a,b]$ and $(f-g)^2 \in R[a,b]$

$$\Rightarrow \frac{(f+g)^2+(f-g)^2}{4} \in R[a,b]$$

$\Rightarrow fg \in R[a,b]$.

19.6.7 Theorem: If $f \in R[a,b]$, $|f| \in R[a,b]$ and $\left|\int_a^b f dx\right| \le \int_a^b |f| dx$.

Proof ; Since $f \in R[a,b]$, there is a partition $P_\in = \{a = x_0 < x_1 < ....< x_n = b\}$ of [a, b] such that

$$O(f, P_\in) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \in$$

Where $M_i = \text{lub}\,\{f(x) / x_{i-1} \le x \le x_i\}$ and

$m_i = \text{glb}\,\{f(x) / x_{i-1} \le x \le x_i\}$

Then $M_i - m_i = \text{lub}\{|f(x) - f(y)| / x_{i-1} \le x, y \le x_i\}$

Let $M_i^{'} = \text{lub}\,\{|f(x)| / x_{i-1} \le x \le x_i\}$ and

$m_i^{'} = \text{glb}\,\{|f(x)| / x_{i-1} \le x \le x_i\}$

Then $M_i^{'} - m_i^{'} = \text{lub}\{||f(x)| - |f(y)|| / x_{i-1} \le x, y \le x_i\}$

Since $||f(x)| - |f(y)|| \le |f(x) - f(y)|$

We have $M_i^{'} - m_i^{'} \le M_i - m_i$ for $1 \le i \le n$ so

$$O(f^1, P_\in) = \sum_{i=1}^{n} (M_i^{'} - m_i^{'})(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$= \in .$

This is true for every $\in > 0$. so $|f| \in R[a,b]$.

Since $-|f|(x) \leq f(x) \leq |f|(x) \forall x \in [a,b]$

We now have $-\int_a^b |f|(x) \leq \int_a^b fdx \leq \int_a^b |f|dx$ by 18.

Hence $\left|\int_a^b fdx\right| \leq \int_a^b |f|dx$.

19.6.8 Short answer Questions:

(a) If $f:[a,b] \to R$ is continuous, $f(x) \geq 0$ for all x in [a, b] and $\int_a^b fdx = 0$ show that $f(x) = 0$ for all x in [a, b].

(b) Give an example of a function $f:[a,b] \to R$ such that $f \in R[c,b]$ if $a < c \leq b$ and $f \notin R[a,b]$.

(c) If f is bounded and the restriction f_c of f to [c, b] is Riemann integrable for every c in [a, b] show that $f \in R[a,b]$ and $\lim_{c \to a+} \int_c^b f_c dx = \int_c^b f dx$.

(d) If f is continuous on [a, b] show that there exists c in [a, b] such that $(b-a)f(c) = \int_a^b fdx$.

(e) If f and g are continous on [a, b] and $g(x) > 0$ for all x in [a, b] show that there exists c in [a, b] such that $\int_a^b fgdx = f(c)\int_a^b gdx$.

(f) If f and g are continous on [a, b] and if $\int_a^b fdx = \int_a^b gdx$ show that there exists c in [a, b] such that $f(c) = g(c)$.

(g) If $f:[a,b]\to R$ is bounded and continous except at a finite number of point in [a, b] then show that $f \in R[a,b]$.

19.7 Additivity Theorems:

19.7.1 Additivity Theorem: Let $f:[a,b]\to R$ and $c\in(a,b)$, and f, the restriction of f to [a, c] and f_2 be the restriction of f to [c, b]. Then $f \in R[a,b]$ if and only if $f_1 \in R[a,c]$ and $f_2 \in R[c,b]$.

In this case $\int_c^b fdx = \int_a^c fdx = \int_c^b fdx$.

Proof: $\Leftarrow$ suppose $f_1 \in R[a,c]$, $f_2 \in R[c,b]$ and write $\int_a^c f_1 dx = L_1$ and $\int_c^b f_2 dx = L_2$.

Since f_1, f_2 are integrable, f_1, f_2 are bounded, hence there exists M > 0 such that $|f(x)| \le M$ for $x\in[a,b]$.

Since $f_1 \in R[a,c]$ and $f_2 \in R[c,b]$, given $\in > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for every tagged partitions $\dot{P_1}$ of [a, c] and $\dot{P_2}$ of [c, b] with $\|\dot{P}_1\| < \delta_1$ and $\|\dot{P}_2\| < \delta_2$

$$\left|S(f_1,\dot{P}_1)-L_1\right| < \frac{\in}{3} \text{ and } \left|S(f_2,\dot{P}_2)-L_2\right| < \frac{\in}{3} \qquad \to (1)$$

Let $\delta_\in = \delta = \min\left\{\delta_1, \delta_2, \frac{\in}{6M}\right\}$ and $\dot{P}$ a tagged partition of [a, b] with $\|\dot{P}\| < \delta$

and $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ so that

$$a = x_0 \le t_1 \le x_1 \le t_{2x} \le \text{..........} \le x_n = b$$

Then for some k where $1 \le k \le n$, $t_{x-1} \le c < x_k$.

Let $\dot{P_1} : \left\{a = x_o \le t_1 \le x_1 \le \text{..........} . \le x_{k-2} \le t_{k-1} \le x_{k-1} \le t_k^{'} = c\right\}$ and

$$\dot{P}_2 : \left\{c = t_k^{''} < x_k \le t_{k+1} \le \text{.........} \quad \text{.....} \le k_n = b\right\}$$

$\dot{P}_1$ is a tagged partition of [a, c] with $\|\dot{P}_1\| < \delta \le \delta$,

$\dot{P}_2$ is a tagged partition of [c, b] with $\|\dot{P}_2\| < \delta \le \delta$,

$$S(f, \dot{P}_0) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$S(f_1, \dot{P}_1) = \sum_{i=1}^{k-1} f(t_i)(x_i - x_{i-1}) + f(c)(c - x_{k-1})$$

$$S(f_2, \dot{P}_2) = f(c)(x_k - c) + \sum_{i=k+1}^{n} f(t_i)(x_i - x_{i-1})$$

$$\Rightarrow S(f, \dot{P}) - S(f_1, \dot{P}_1) - S(f_2, \dot{P}_2)$$

$$= f(t_k)(x_k - x_{k-1}) - f(c)(x_k - c) - f(c)(c - x_{k-1})$$

$$= \left(f(t_i) - f(c)\right)(x_k - x_{k-1}).$$

$$\Rightarrow \left|S(f, \dot{P}) - S(f_1, \dot{P}_1) - S(f_2, \dot{P}_2)\right|$$

$$= \left|f(t_k) - f(c)\right|(x_k - x_{k-1})$$

$$\le \left(\left|f(t_k)\right| + \left|f(c)\right|\right)\|\dot{P}\|$$

$$< 2M.S < \frac{\in}{3} \qquad \rightarrow \qquad (2)$$

Hence $\left|S(f, \dot{P}) - L_1 - L_2\right|$

$$\le \left|S(f, \dot{P}) - S(f_1, \dot{P}_1) - S(f_2, \dot{P}_2)\right| + \left|S(f_1, \dot{P}_1) - L_1\right| + \left|S(f_2, \dot{P}_2) - L_2\right|$$

$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} = \in$ (from (1) and (2))

Hence $f \in R[a,b]$ and

$$\int_a^b fdx = L_1 + L_2 = \int_a^c fdx + \int_c^b fdx$$

($\Rightarrow$) Conversely suppose $f \in R[a,b]$. Then for every $\in > 0$ there corresponds $\eta_\in = \eta > 0$ such that if $\dot{P}$ and $\dot{Q}$ are tagged partitions of [a, b] with $\|\dot{P}\| < \eta$ and $\|\dot{Q}\| < \eta$ then $|S(f,\dot{P}) - S(f,\dot{Q})| < \in$.

Let $\dot{P}_1$, $\dot{P}_2$ be tagged partitions of [a, c] with $\|\dot{R}\| < \eta$. Write $\dot{P}$ for the tagged partition of [a, b] obtained by clubbing $\dot{P}_1$ and $\dot{R}_1$ and $\dot{Q}$ be the tagged partition of [a, b] obtained by clubbing $\dot{P}_2$ and $\dot{R}_1$.

Then $\|\dot{P}\| < \eta$ and $\|\dot{Q}\| < \eta$

$S(f,\dot{P}) = S(f_1,\dot{P}_1) + S(f_2,\dot{R}_2)$ and

$S(f,\dot{Q}) = S(f_1,\dot{P}_2) + S(f_2,\dot{R}_1)$

$$\Rightarrow \left|S(f_1,\dot{P}_1) - S(f_1,\dot{P}_2)\right| = \left|S(f,\dot{P}) - S(f,\dot{Q})\right| < \in$$

Hence $f_1 \in R[a,c]$. Similarly $f_2 \in R[c,b]$

Hence by the first part,

$$\int_a^b fdx = \int_a^c f_1 dx + \int_c^b f_2 dx .$$

19.7.2 Corellary: If $f \in R[a,b]$ and [c, d] $\leq$ [a, b] then the restriction of f to [c, d] is in R[c, d].

Proof: Since $f \in R[a,b]$ and $c \in [a,b]$ the restriction f_1 of f to [c, b] is in R[c, b]. Since $d \in [c,b]$ the restriction f_2 of f_1 to [c, d] is in R[c, d]. But f_2 is the restriction of f to [c, d]. Hence the restriction of f to [c, d] is in R[c, d].

19.7.3 Corollary: If $f \in R[a,b]$ and $a = c_0 < c_1 < .... < c_m = b$ then the restriction of f to each of the subintervals is in $R[c_{i-1}, c_i]$ and $\int_a^b fdx = \int_{c_o}^{c_1} fdx + \int_{c_1}^{c_2} fdx + ... + \int_{c_{n-1}}^{c_n} fdx$

Proof: We prove this by induction on n. When n=1 the corollary hold by 19.7.2. Assume that the statement is valid for n - 1 Let $a = c_0 < c_1 < .... < c_m = b$. Then by corollary - $f \in R[a,b]$ and R[c, b] and $\int_a^b fdx = \int_a^{c_1} fdx + \int_{c_1}^{c_n} fdx$.

Applying induction hyposis for the prints $c_1, c_2, ...\ ..c_n = b$, $f \in R[a,b]$ $2 \le i \le n$.

We get $\int_{c_1}^{b} fdx = \int_{c_1}^{c_2} fdx + ...\ .. + \int_{c_{n-1}}^{c_n = b} fdx$

Hence $\int_a^b fdx = \int_{c_0}^{c_1} fdx + ...\ .. + \int_{c_n - 1}^{b} fdx$.

19.7.4 Defination: Suppose $f \in R[a,b]$ and $a < \alpha < \beta < b$.

We write $\int_\beta^\alpha fdx = \int_\alpha^\beta fdx$ and $\int_\beta^\alpha fdx = 0$

19.7.5 Theorem: If $f \in R[a,b]$ and α, β, γ are any numbers in [a, b]

$$\int_\alpha^\beta fdx = \int_\alpha^\gamma fdx + \int_\gamma^\beta fdx \qquad \rightarrow \qquad (2)$$

in the sense that if two the three in legals in the above equality hold the theory also holds and equality occurs.

Proof: (1) holds if $\int_\alpha^\beta fdx + \int_\alpha^\gamma fdx + \int_\gamma^\alpha fdx = 0$

Write $L(\alpha,\beta,\gamma)$ for $\int_{\alpha}^{\beta} fdx + \int_{\beta}^{\gamma} fdx + \int_{\gamma}^{\alpha} fdx = 0$

clearly $L(\alpha,\beta,\gamma) = L(\alpha,\gamma,\beta) = L(\gamma,\beta,\alpha) = -L(\alpha,\beta,\gamma)$

It is clear that if f is into two of $R[\alpha,\beta]$, $R[\beta,\gamma]$, $R[\alpha,\beta]$ then f belongs to the third class and

Then $\alpha < \beta < \gamma \quad L(\alpha,\beta,\gamma) = -L(\beta,\alpha,\gamma)$.

Moreover $L(\alpha,\beta,\gamma) = 0$ since

$$\int_{\alpha}^{\beta} fdx + \int_{\beta}^{\gamma} fdx = +\int_{\alpha}^{\gamma} fdx = -\int_{\gamma}^{\alpha} fdx .$$

The proof is similar in the remaining cases.

19.8 Answers to Short Answer Questions:

SAQ: Apply cauchy criterian for the function $g:[0,3] \to R$ defined in example by $g(x) = 2$ if $0 \le x \le 1$ and 3 if $1 < x \le 3$. For $\in > 0$ find $\eta_{\in} > 0$ such that if $\|\dot{P}\| < \eta_{\in}$ and $\|\dot{Q}\| < \eta_{\in}$ then $\left|S(g,\dot{P}) - S(g,\dot{Q})\right| < \in$.

Solution: Let $\dot{P}$ be a tagged partition of [0, 3] with $\|\dot{P}\| < \delta$ where $\delta > 0$; $\dot{P}_1$ be the subset of $\dot{P}$ having its tags in [0, 1] and let $\dot{P}_2$ be the subset of $\dot{P}$ with tags in (1, 3].

Then $S(g,\dot{P}) = S(g,\dot{P}_1) + S(g,\dot{P}_2) \qquad \to \qquad (1)$

If U_1 is the union of all subintervals in $\dot{P}_1$ there by 18.........

$$\left[0, 1 - \|\dot{P}\|\right] \subseteq U_1 \subseteq \left[0, 1 + \|\dot{P}\|\right].$$

Also if U_2 is the union of all subintervals in $\dot{P}_2$ then by 18..........

$$\left[1+\left\|\dot{P}\right\|, 2-\left\|\dot{P}\right\|\right] \subseteq .U_2 \subseteq \left[1-\left\|\dot{P}\right\|, 2+\left\|\dot{P}\right\|\right]$$

consequently $2(1-\delta) \subseteq S(g, \dot{P}_1) \leq 2(1+\delta)$ and

$$3(2-\delta) \subseteq S(g, \dot{P}_2) \leq 3(2+\delta)$$

Adding these in equalities and using (1) we get

$$8-5\delta \leq S(g, \dot{P}) \leq 8+5\delta \qquad \rightarrow \qquad (2)$$

If $\dot{Q}$ is any tagged partition with $\left\|\dot{Q}\right\| < \delta$,

$$8-5\delta \leq S(g, \dot{Q}) \leq 8+5\delta \qquad \rightarrow \qquad (3)$$

From (2) and (3) we get

$$\left|S(g, \dot{P}) - S(g, \dot{Q})\right| < 10\delta .$$

Hence If $\in > 0$ $\left|S(g, \dot{P}) - S(g, \dot{Q})\right| < \in$ if $\delta < \dfrac{\in}{10}$

and $\left\|\dot{P}\right\| < \delta$, $\left\|\dot{Q}\right\| < \delta$. We thus choose $\eta_{\in} = \dfrac{\in}{10}$.

SAQ 19.4.3 (b) : The Dorichlet function g is defined on [0, 1]

by $g(x) = 1$ if $x \in [0,1]$ and x is rational while

$g(x) = 0$ if $x \in [0,1]$ and x is irrational.

This function was already shown to be not Riemann integrable.

We deduce this form cachy criterion as follows:

If $\dot{P}$ is any partition of [0, 1] $P = \{0 = x_0 < x_1 < .... < x_n = 1\}$ let $t_i \in [x_{i-1}, x_i]$ be arational number and $s_i \in [x_{i-1}, x_i]$ be an irrational number.

Let $\dot{P_1} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ and $\dot{P_2} = \{([x_{i-1}, x_i], s_i)\}_{i=1}^{n}$

$$S(g,\dot{P}_1)=\sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = 1$$

and $S(g,\dot{P}_2)=\sum_{i=1}^{n} g(s_i)(x_i - x_{i-1}) = 0$.

Hence $\left|S(g,\dot{P}_1)-S(g,\dot{P}_2)\right| = 1$.

As P is arbitary $\left\|\dot{P}_1\right\| = \left\|\dot{P}_2\right\| = \left\|\dot{P}\right\|$ is also arbitray.

Thus cauchy critierion fails here when $0 < \in < 1$.

SAQ 19.4.3 (c): $f:[a,b]\to R$ is not Riemann integrable if and only if there exists $\in_0 > 0$ such that for every $n \in N$ there correspond tagged partitions $\dot{P}_n$ and $\dot{Q}_n$ with $\left\|\dot{P}_n\right\| < \frac{1}{n}$, $\left\|\dot{Q}_n\right\| < \frac{1}{n}$ and

$$\left|S(f,\dot{P}_n)-S(f,\dot{Q}_n)\right| \geq \in_0$$

Solution: ($\Rightarrow$): Suppose $f \notin R[a,b]$. Then by cauchy criterion, there exists $\in_0 > 0$ such that for $\delta = \frac{1}{n}$ where $n \in N$, there corresponds $\dot{P}_n$ and $\dot{Q}_n$ with $\left\|\dot{P}_n\right\| < \frac{1}{n}$, $\left\|\dot{Q}_n\right\| < \frac{1}{n}$ and

$$\left|S(f,\dot{P}_n)-S(f,\dot{Q}_n)\right| \geq \in_0.$$

($\Leftarrow$) : Conversely suppose this condition holds. For any $\delta > 0$ we can choose $N \in N$ such that $\frac{1}{N} < \delta$. The corresponding tagged partitions $\dot{P}_N$ and $\dot{Q}_N$ with $\left\|\dot{P}_N\right\| < \delta$, $\left\|\dot{Q}_N\right\| < \delta$ and

$$\left|S(f,\dot{P}_N)-S(f,\dot{Q}_N)\right| \geq \in_0.$$

Hence $f \notin R[a,b]$.

S.A.Q. 19.4.3(d) :

Let $H(x) = \begin{cases} x+1 \text{ if } x \in [0,1], x \text{ rational} \\ 0 \text{ if } x \in [0,1], x \text{ irrational} \end{cases}$

For $n \in N$ Let $\dot{P_n} = \left\{\left(\left[\frac{J-1}{n}, \frac{J}{n}\right], t_j\right)\right\}_{j=1}^{n}$ and

$$\dot{Q_n} = \left\{\left(\left[\frac{J-1}{n}, \frac{J}{n}\right], s_j\right)\right\}_{j=1}^{n}$$

Where $t_j = \frac{j}{n}$ and $s_j \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$, s_j irrational $\left\|\dot{P_n}\right\| = \left\|\dot{Q}_n\right\| = \frac{1}{n} \forall n$ and

$$S(H, \dot{P}_n) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}+1\right) = \frac{3}{2} + \frac{1}{2^n} \text{ and } S(H, \dot{Q}_n) = 0.$$

Hence $\left|S(H, \dot{P}_n) - S(H, \dot{Q}_n)\right| > 2 \forall n \in N$.

Hence $H \notin R[0,1]$.

S.A.Q. 19.4.3(e) :

Let $H(x) = \begin{cases} k \text{ if } x = \frac{1}{k}, k \in N \\ 0 \text{ otherwise in } [0,1] \end{cases}$

To show that $H \notin R[0,1]$:

Method 1 : H is not bounded hence $H \notin R[0,1]$.

Method 2 : Let $\dot{P_n} = \left\{a < \frac{1}{n} < \frac{2}{n} < ....< \frac{n}{n} = 1\right\}$.

Let $0 < a < \frac{1}{n}$ and a be irrational number.

Let $t_i = \frac{1}{n}$, $t_i = a + \frac{j-1}{n}$ for $1 < i \le n$.

Then t_1 is rational while $t_2, t_3, \ldots\ldots\ldots\ldots t_n$ are irrational numbers and $\frac{i-1}{n} < t_i \le \frac{i}{n}$ for every n.

with these tages to P_n, we have $S(H, \dot{P}_n) = n\frac{1}{n} = 1 \forall n$

If $\dot{Q}_n$ is the partition P_n with tags $s_j \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$, s_j being irrational, $S(H, \dot{Q}_n) = 0$.

We thus have $\left\|\dot{P}_n\right\| = \left\|\dot{Q}_n\right\| = \left\|P_n\right\| = \frac{1}{n} \to 0$ as $n \to \infty$.

While $S(H, \dot{P}_n) - S(H, \dot{Q}_n) = 1$. for every n

Hence $H \notin R[0,1]$.

S.A.Q. 19.5.3 : Every step function is a linear combination of elementary step functions.

Proof: Let $f : [a,b] \to R$ be a step function, with values $c_1, \ldots\ldots\ldots c_n$ and for each i, $1 \le i \le n$

Let E_i be the subset of [a, b] such that $x \in E_i \Leftrightarrow f(x) = c_i$.

In otherwords $E_i = f^{-1}(\{c_i\})$. By the defintion of a step function each E_i is a finite union of dispoint intervals.

Let $E_i = J_{i_1} \cup J_{i_2}, \ldots\ldots J_{i_{n_i}}$

Then $f(x) = \sum_{i=1}^{n} \sum_{j=1}^{ni} c_1 Q_{J_{i_g}}$

Illustration : Let f we defined on [0, 4] by

$f(x) = 1$ if $0 \le x \le 1$ or $3 < x < \frac{7}{2}$

$= 2$ if $1 < x \le \frac{3}{2}$ or $\frac{7}{2} \le x \le 4$

$= -1$ if $\frac{3}{2} < x \leq 2$ or $\frac{9}{4} < x \leq 3$

$= 0 \qquad 2 < x \leq \frac{9}{4}$

See fig 19.5.3 for graph of the above function.

$$E_1 = \{x / f(x) = 1\} = [0,1] \cup \left(3, \frac{7}{2}\right)$$

$$E_2 = \{x / f(x) = 2\} = \left(1, \frac{3}{2}\right] \cup \left[\frac{7}{2}, 4\right]$$

$$E_3 = \{x / f(x) = -1\} = \left(\frac{3}{2}, 2\right] \cup \left[\frac{9}{4}, 3\right]$$

$$E_4 = \{x / f(x) = 0\} = \left(2, \frac{9}{4}\right]$$

So $f = \phi_{[0,1]} + \phi_{\left[\frac{3}{2},\frac{7}{2}\right]} + 2\left(\phi_{\left(1,\frac{3}{2}\right]} + \phi_{\left(\frac{7}{4},4\right]}\right) - \phi_{\left(\frac{3}{2},2\right]} - \phi_{\left(\frac{9}{4},\frac{3}{2}\right]}$

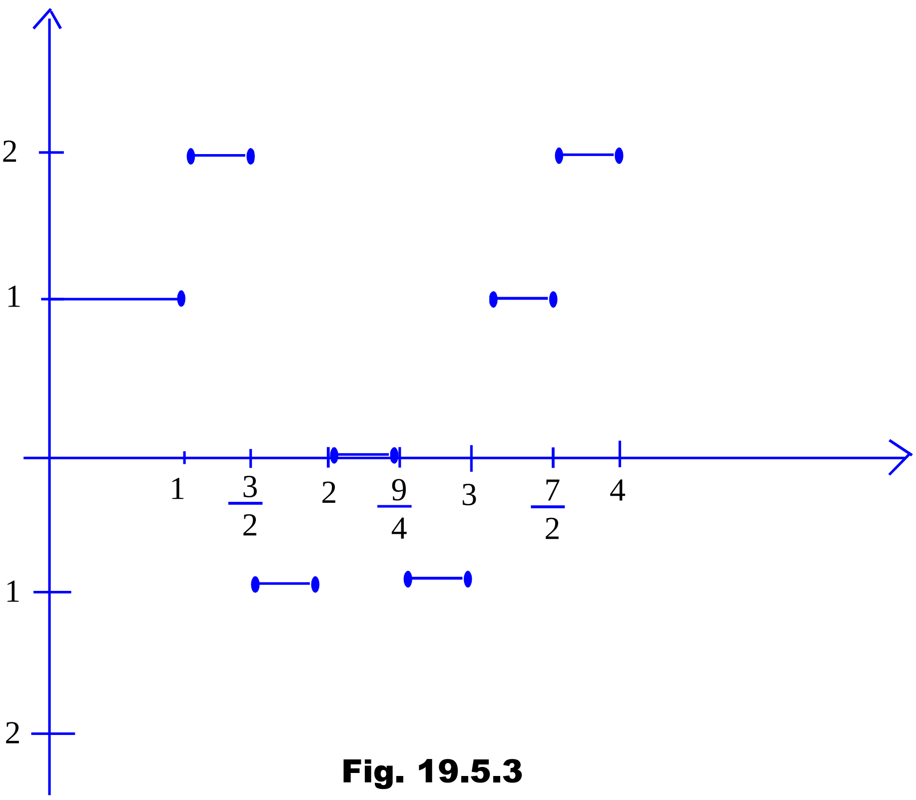


Fig. 19.5.3

19.5.4 : If $a \le c \le b$ and $f(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \ne c \end{cases}$

Hence $f \in R[a,b]$ and $\int_a^b f dx = 0$.

Solution: Let $P = \{a = x_0 < x_1 < ....x_{i-1} \le c < x_i \le .... < x_n = b\}$ by any partition of [a, b] and $x_{i-1} \le c < x_i$.

Let $t_j \in \lfloor x_{j-1}, x_j \rfloor$ be any tag for $1 \le j \le n$.

Then $f(t_j) = 0$ if $j \in \{i-1, i\}$. There are two possibilities namely $c = x_{i-1}$ or $c \ne x_{i-1}$.

Where $c = x_{i-1}$ the possibilities for t_{i-1} and t_i are as follows.

(1) $t_{i-1} = c = t_i$ $(\Rightarrow f(t_{i-1}) = f(t_i) = 1)$

(2) $t_{i-1} = c \ne t_i$ $(\Rightarrow f(t_{i-1}) = 1, f(t_i) = 0)$

(3) $t_{i-1} \ne c = t_i$ $(\Rightarrow f(t_{i-1}) = 0, f(t_i) = 1)$

When $c \ne x_{i-1}$ two possibilities arise

(4) $c = t_i (\Rightarrow f(t_i) = 1)$

(5) $x_i - x_{i-2}$

In cases (4) and (5) $f(t_{i-1}) = 0$

Thus $S(f, \dot{P}) = f(t_{i-1})(x_{i-1} - x_{i-2}) + f(t_i(x_i - x_{i-1})$

$= x_i - x_{i-2}$ in case (1)

$= x_i - \bar{1}x_{i-2}$ in case (2)

$= x_i - x_{i-1}$ in cases (3) and (4)

$= 0$ in case (5)

In all these cases $\left|S(f,\dot{P})\right| = \left|S(f,\dot{P})-0\right|$

$$= S(f,\dot{P}) \leq 2\left\|\dot{P}\right\|$$

Hence if $\in > 0$ for any tagged partition $\dot{P}$ with $\left\|\dot{P}\right\| < \frac{\in}{2}$

$$\left|S(f,\dot{P})-0\right| < \in.$$

Hence $f \in R[a,b]$ and $\int_a^b f dx = 0$.

Note: From this result if follows that Reimann integrability of a function and the value of the integral reimann inchanged by changing the value of the function at a point.

19.5.8 : If $\dot{P}$ is a tagged partition of [a, b] and $f:[a,b] \to R$ then there is a step function ϕ on [a, b] with $\int_a^b \phi = S(f,P)$.

Proof: Let $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ so that

$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq .... \leq t_n \leq x_n \leq b$$

Then $S(f,\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$

Let $J_i = [x_{i-1}, x_i]$. Then $\int_a^b \phi_{J_i} dx = (x_{i-1}, x_i)$.

write $\phi = \sum_{i=1}^{n} f(t_i)\phi_{J_i}$

Then ϕ is a step function and

$$\int_a^b \phi = \sum_{i=1}^{n} f(t_i)\int_a^b \phi_{J_i} = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \int_a^b \phi = \sum_{i=1}^{n} f(t_i)\int_a^b \phi_{J_i} = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = S(f,\dot{P})$$

19.6.2 SAQ : (Second proof of 19.6.1.)

Theorem : If f is continuous in [a, b] then $f \in R[a,b]$.

Proof: Since f is continuous in [a, b] f is uniformly continuous in [a, b]. So given $\in > 0$ there exists $\delta > 0$ such that for all x, y in [a, b] with $|x-y| < \delta$, $|f(x) - f(y)| < \dfrac{\in}{2(b-a)}$.

Let $n \in \mathbb{N}$ be such that $n > \dfrac{b-a}{\delta}$ and $P_{\in} = \{a = x_0 < x_1 < \dots\dots\dots < x_n = b\}$ where $x_i - x_{i-1} = \dfrac{b-a}{n} < \delta$.

If $1 \le i \le n$ let $M_i = \text{lub}\{f(x) / x_{i-1} \le x \le x_i\}$ and $m_i = \text{glb}\{f(x) / x_{i-1} \le x \le x_i\}$

Then $M_i - m_i = \text{lub}\{|f(x) - f(y)| / x_{i-1} \le x, y \le x_i\}$

$$\le \frac{\in}{2(b-a)}.$$

Hence $0(f, P_{\in}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$

$$\le \frac{\in}{2(b-a)}(b-a).$$

$$= \frac{\in}{2} < \in$$

This is true for every $\in > 0$. Hence $f \in R[a,b]$.

19.6.4 SAQ : Second Proof of 19.6.3.

Theorem: If f is monotonic in [a, b] then $f \in R[a,b]$.

Proof: We may assume that f is monotronically increasing in [a, b]. If $f(a) = f(b)$ then f is a constant function, hence $f \in R[a,b]$. Assume that $f(a) < f(b)$.

If $\in > 0$ let $n \in \mathbb{N}$ and $n > \dfrac{(b-a)(f(b) - f(a))}{\in}$ and $P_{\in} = \{a = x_0 < x_1 < \dots\dots\dots < x_n = b\}$ where $x_i - x_{i-1} = \dfrac{b-a}{n}$.

Since f is monotonically increasing for each i, $1 \le i \le n$ and $x_{i-1} \le x \le x$

$f(x_{i-1}) \le f(x) \le f(x_i)$ so that

$$M_i = \text{lub}\{f(x)/x_{i-1} \le x \le x_i\} = f(x_i) \text{ and}$$

$$m_i = \text{glb}\{f(x)/x_{i-1} \le x \le x_i\} = f(x_{i-1}).$$

Hence $0(f, P_{\in}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$.

$$= \frac{b-a}{n} \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}$$

$$= \frac{(b-a)(f(b) - f(a)}{n}$$

$$= \in.$$

This is true for every $\in > 0$. Hence $f \in R[a,b]$.

SAQ : 19.6.8(a): If f is continuous on [a, b] $f(x) \ge 0$ for all x in [a, b] and $\int_a^b f dx = 0$ then $f(x) = 0$ for all x in [a, b].

Solution: If $f(c) > 0$ for some c in (a, b), there exists $\delta > 0$ such that $a < c - \delta < c + \delta < b$ and $f(x) > 0$ if $x \in [c - \delta, c + \delta]$ since f is continuous $\exists$ $u \in [c - \delta, c + \delta]$ such that $f(x) \ge f(u)$ for all x in $[c - \delta, c + \delta]$. Hence

$$0 = \int_a^b f dx \ge \int_{c-\delta}^{c+\delta} f dx \ge f(u) 2\delta > 0.$$

This is a contradiction. Hence $f(x) = 0$ for all x in (a, b).

since f is continous at a and b it follows that $f(a) = f(b) = 0$.

19.6.8(b) SAQ:

Let $f(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$

f is unbounded hence $f \notin R[0,1]$.

If $0 < a < 1$, f is continous on [a, 1], hence $f \in R[a,1]$

If f is bounded on [a, b] and the restriction of f to [c, b] is Riemann integrable on [c, b] for $c \in (a,b]$ then $f \in R[a,b]$ and $\int_c^b f_c(x)dx \to \int_c^b f(x)dx$, where $f_c : f/_{[c,b]}$

$\exists\ M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$.

Let $\in > 0$ and $a < c < b$ be such that $0 < (c-a) < \frac{\in}{4M}$ and f_c be the restriction of f to [c, b]. Then there exist functions α_c and ϖ_c such that $\alpha_c(x) \le f_c(x) \le \varpi_c(x)$ if $x \in [c,b]$ and $\int_c^b (\varpi_c - \alpha_c)dx < \frac{\in}{2}$.

(Squeeze Theorem 19.4.2)

Define $\alpha(x) = \begin{cases} \alpha_c(x) & \text{if } c \le x \le b \\ -M & \text{if } a \le x \le c \end{cases}$

and $\varpi(x) = \begin{cases} \varpi_c(x) & \text{if } c \le x \le b \\ M & \text{if } a \le x < c \end{cases}$

α l, ware step functions such that $\alpha(x) \le f(x) \le \varpi(x)$ if $x \in [a,b]$ and

$$\int_a^b (\varpi - \alpha)dx = \int_a^c (\varpi - \alpha)dx + \int_c^b (\varpi_c - \alpha_c)dx$$

$$< 2M(c-a) + \frac{\in}{2}$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in.$$

This is true for every $\in > 0$ so $f \in R[a,b]$.

19.6.8 (c) SAQ :

Since f is bounded there is a real number $M > 0$ such that $|f(x)| < M$ if $a \le x \le b$. Let $a < c < b$.

Define $\alpha(x) = \begin{cases} -M & \text{if } a \le x \le c \\ f(x) & \text{if } c \le x \le b \end{cases}$

and $\varpi(x) = \begin{cases} M & \text{if } a \le x \le c \\ f(x) & \text{if } c \le x \le b \end{cases}$

$$\alpha_1(x) = \begin{cases} -M & \text{if } a \le x < c \\ 0 & \text{if } c \le x \le b \end{cases}$$

$$\alpha_2(x) = \begin{cases} 0 & \text{if } a \le x < c \\ f(x) & \text{if } c \le x \le b \end{cases}$$

Then $\alpha = \alpha_1 + \alpha_2$, $\alpha(x) \le f(x) \le \varpi(x)$ for $x \in [a,b]$.

$\alpha_1 \in R[a,b]$, (see note after 19.5.4 solution)

$\alpha_2 \in R[a,b]$ (hypothesis).

Hence $\alpha \in R[a,b]$. Similarly $\varpi \in R[a,b]$

Morever $\varpi - \alpha = 2M$ if $a \le x < c$

$= 0$ if $c \le x < b$

and $\int_a^b (\varpi - \alpha)dx = 2M(c-a)$

Hence if $\in > 0$ $\int_a^b (\varpi - \alpha)dx < \in$ if $a < c < a + \frac{\in}{2M}$

Hence by squeeze theorem $g \in R[a,b]$.

Morever $|f(c) - g(a)| = \left| \int_c^b f - \int_c^b f \right|$

$$= \left| \int_a^c f \right|$$

$$\leq \int_a^c |f|.$$

$$< M(c-a) \text{ if } a < c < a + \frac{\in}{M}$$

Hence $\lim_{c \to a+} g(c) = g(a)$.

19.6.8 (d) SAQ:

If f is continuous on [a, b] there exists $c \in [a,b]$ such that

$$(b-a)f(c) = \int_a^b f dx.$$

Solution: Let $f(x_0) = \inf\left\{f(x)/a \leq x \leq b\right\}$ and $f(x_1) = \sup\left\{f(x)/a \leq x \leq b\right\}$

Then $f(x_0) \leq f(x) \leq f(x_1)$ if $a \leq x \leq b$.

$$\Rightarrow \int_a^b f(x_0)dx \leq \int_a^b f(x)dx \leq \int_a^b f(x_1)dx.$$

$$\Rightarrow f(x_0) \leq \frac{\int_a^b f(x)dx}{b-a} \leq f(x_1)$$

By the intermediate value theorem there exists $c \in [a,b]$

Such that $f(c) = \dfrac{\int_a^b f(x)dx}{b-a}$

19.6.8 (e) SAQ:

Let $g(x) > 0$ for all x in [a, b], f, g be continous on [a, b]

To show that there exists c in [a, b] such that

$$\int_a^b f(x)dx = f(c)\int_a^b gdx,$$

Let $f(x_0) = \inf\{f(x)/a \le x \le b\}$ and $f(x_1) = \sup\{f(x)/a \le x \le b\}$

Then $f(x_0) \le f(x) \le f(x_1)$ for all x in [a, b]

$\Rightarrow$ $f(x_0)g(x) \le f(x)g(x) \le f(x_1)g(x)$ for all x in [a, b] (Since $g(x) > 0$)

$$\Rightarrow \quad f(x_0)\int_a^b gdx \le \int_a^b fgdx \le f(x_1)\int_a^b gdx.$$

$$\Rightarrow \quad f(x_0) \le \frac{\int_a^b fgdx}{\int_a^b gdx} \le f(x_1) \quad \left(\int_a^b gdx > 0\right)$$

Hence there exists c in [a, b] such that

$$\frac{\int_a^b fgdx}{\int_a^b gdx} = f(c).$$

19.6.8 (f) SAQ:

If f and g are continous on [a, b] and $\int_a^b fdx = \int_a^b gdx$ then for some $c \in [a,b]$, $f(c) = g(c)$.

Let $h(x) = f(x) - g(x)$. Then h is continiuous on [a, b] and $\int_a^b hdx = 0$.

($\because \int_a^b hdx = \int_a^b fdx - \int_a^b gdx = 0$). If for some x_1, $h(x_1) < 0$ and for some $\frac{x}{2}$, $h(x_2) > 0$, by the intermediation value property there is some c between x_1, x_2 such that $h(c) = 0$

Suppose $h(x) > 0$ for all $x \in [a,b]$ then there is a $x_0 \in [a,b]$ such that $h(x) \ge h(x_0) > 0$ for all $x \in [a,b]$.

$\Rightarrow \quad 0 = \int_a^b h(x)dx \geq \int_a^b h(x_0)dx = h(x_0)(b-a) > 0$ (by 19.6.8(d))

This is a contradection. Hence it is not true that $h(x) > 0$ for all $x \in [a,b]$. Similarly it is not true that $h(x) < 0$ for all $x \in [a,b]$. Then for some x, $h(x) > 0$ and some y, $h(y) < 0$. In this case the conclusion holds, as proved above.

19.6.8 (g) SAQ:

If $f : [a,b] \to \mathbb{R}$ is bounded and is continous except at a finite number of points in [a, b] then $f \in R[a,b]$.

Proof: Let $E = \{c_1, ...c_n\}$ be the finite subset of [a, b] where $a \leq c_1 < c_2 < .... < c_n \leq b$ and f is discontinuous at each c_i and continuous on $[a,b] - \exists$. Let $M = \text{lub}\{|f(x)| / x \in [a,b]\}$. If $\in > 0$ choose $u_i, \mho_i (1 \leq i \leq n)$ such that $a \leq u_1 \leq c_1 \leq \mho_1 \leq u_2 \leq c_2 \leq \mho_2 \leq .... \leq u_n \leq c_n \leq \mho_n \leq b$ and for each i, $\mho_1 - u_i < \dfrac{\in}{4nM}$.

In each of the intervals $[a,u_1], [\mho_1, u_2] ....[\mho_{n-1}, u_n]$ and $[\mho_n, b]$

f is continuous, so f is Riemann integrable in each of these intervals. So there exist partitions $P_1, P_2, ...P_n, P_{n+1}$ of these intervals such that $O(f, P_i) > \dfrac{\in}{2(n+1)}$ for $1 \leq i \leq n+1$.

If s and t belong to $[\mho_1 - u_i]$, $|f(s) - f(t)| \leq |f(s) /+ f(t)| \leq 2M$

Hence if $M_i = \text{lub}\{f(s) / s \in [u_i, \mho_i]\}$ and $m_i = g\,\text{lb}\{f(t) / t \in [u_i, \mho_i]\}$

$M_i - m_i = \text{lub}\{|f(s) - f(t)| / s, t$ belong to $[u_i, \mho_i]\} \leq 2M$.

Hence $\sum_{i=1}^{n} (M_i - m_i)[\mho_i - u_i]\} \leq 2M . \dfrac{\in n}{4Mn} = \dfrac{\in}{2}$.

If $P_\in$ is the partition of [a, b] obtained by taking $P_1, ...P_{n+1}$ together

$$O(f, P_\in) = \sum_{i=1}^{n+1} 0(f, P_i) + \sum_{i=1}^{n} (M_i - m_i)[\mho_i - u_i]$$

$< \dfrac{\in (n+1)}{2(n+1} + \dfrac{\in}{2} = \in$. Hence $f \in R[a,b]$.

Summary: In this lesson Cauchy criterion for Riemann integrability, seneeze theorem are established and integrability of continuous functions and monotone functions are deduced from the squeeze theorem by using the integrability of step function. Another criterion for integrability involving supreme and in fima is stated without proof and a method of obtaining integrability of continuous functions and monotone functions, which does not require integrability of step functions is deduced. Additivity theorems are also established.

Technical terms :

Elementary step function - functional

Exercises:

1. Let $\alpha(x) = -x$, $\varpi(x) = x$ and f be the Direct function defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \text{ and } x \text{ is rational and} \\ 1 & \text{if } x \in [0,1] \text{ and } x \text{ is irrational} \end{cases}$$ show that

(a) $\alpha(x) = f(x) \le \varpi(x)$ for all $x \in [0,1]$.

(b) $\alpha \in R[0,1]$ and $\varpi \in R[0,1]$

(c) If $n \in \mathbb{N}$, $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{n-1}{n}\right\} t_i = \frac{1}{1}$ and s_i any irrational number in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ $(1 \le i \le n)$

$\dot{P_n} = \{P_n; t_i, t_n\}$ and $\dot{Q_n} = \{P_{n_1}, s_i, \ldots\ldots. s_n\}$

find $S(f, \dot{P}_n)$ and $S(f, \dot{Q}_n)$

(d) Does $f \in R[0,1]$?

2. Let $f \in R[a,b]$, $g(x) = f(x)$ if $a < x \le b$. Show that $g \in R[a,b]$ and $\int_a^b f dx = \int_a^b g dx$.

3. Let $f \in R[c,d]$ $[c,d] < [a,b]$, $g(x) = M$ if $a \le x < c$ and $g(x) = M^1$ if $d < x \le b$ while of $g(x) = f(x)$ if $c \le x < d$. Show that $g \in R[a,b]$.

4. Show that if $S(f, \dot{P})$ is a Riemann sumthen the x is a step function ϕ such that $\int_a^b \phi dx = S(f, \dot{P})$.

Hint for $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$, define $\phi(t) = f(t)$ in $[x_{i-1}, x_i]$ and $\phi(b) = f(b)$.

5. Let $f(x) = \begin{cases} 0 \text{ if } 0 \le x \le \frac{1}{2} \text{ or } \frac{1}{2} < x \le 1 \\ 1 \text{ if } x = \frac{1}{2} \end{cases}$

show that $f \in R[0,1]$ and $\int_0^1 f dx = 0$.

6. Let $f:[a,b] \to \mathbb{R}$; $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ be any tagged partition. Let ϕ be the step function given by $\phi(t) = f(t_i)$ if $t \in [x_{i-1}, x_i]$ show that $\int_a^b \phi(t)dt = S(f, \dot{P})$.

7. Show that if $g(x) = \sin\frac{1}{x}$ when $x \ne 0$ and $g(0) = 1$, then $g \in R[0,1]$, [Hint : Find the discontinoites of g)

8. Suppose $f \in R[-a,0]$. (a) if f is even $f(x) = g(x)$ for all x.

$\dot{P}_1 = \{0 \le t_1 \le x_1 \le t_2 \le x_2 \le \dots\dots\dots \le t_n \le x_n - a\}$ and

$\dot{P}_2 = \{-a \le -t_n \le -x_{n-1} \le \dots\dots\dots\dots \le x_1 \le t_1 \le 0\}$ show that

$S(f, \dot{P}_1) = S(f, \dot{P}_2)$.

(b) Show that if f is even $\int_a^a fdx < \int_0^a fdx$.

9. Suppose $f \in R[-a,a]$ and f is ....

Show that $\int_a^a fdx = 0$.

10. Let $f \in R[c,d]$, $[c,d] \le [a,b]$ and

$g(x) = \begin{cases} f(x) \text{ if } c \le x \le d \\ -M \text{ if } a < x < c \\ M \text{ if } d < x < b \end{cases}$ show that $g \in R[a,b]$

11. Let $f:[a,b]\to\mathbb{R}$, $\dot{P}=\left\{\left([x_{i-1},x_i],t_i\right)\right\}_{i=1}^{n}$ and $\dot{Q}_n=\left\{\left([x_{i-1},x_i],s_i=t_i\right)\right\}_{i=1}^{n}$ where $x_{i-1},x_i=\dfrac{b-a}{n}\forall i$. Assume that f is monotonically increasing on [a, b).

(a) Show that $f(a)(b-a)\le S(f,\dot{P}_n)\le S(f,\dot{Q}_n)\le f(b)(b-a)$

Hint: $f(a)\le f(x_i)\le f(b)\forall i$.

(b) $0\le S(f,\dot{Q}_n)-S(f,\dot{P}_n)\le f(b)-f(a)\dfrac{(b-a)}{n}$

Hint: $\sum_{i=1}^{n}\left\{f(x_i)-f(x_{i-1})\right\}=f(b)-f(a)$.

12. If f is continous on [-a, a] show that $\int_b^a f(x^2)dx=2\int_0^a f(x^2)$.

Hint: If $g(x)=f(x^2), g(x)=g(-x)$.

13. If f is continous on [-1, 1] show that $\int_0^{\pi/2} f(\cos x)dx=\int_0^{\pi} f(\sin x)dx=\dfrac{l}{2}\int_0^{\pi} f(\sin x)dx$

Hint: Examine certain Riemann sums.

14. Let $f:[a,b]\to\mathbb{R}$ be continuous and $f(x)\ge 0$ for all x write $M_n=\left\{\int_a^b f^n dx\right\}^{1/n}$ for $n\in\mathbb{N}$ and

$M=\sup\left\{f(x)/0\le x\le 1\right\}$. Show that $\lim M_n=M$.

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19.12 Model Practical Problems with solutions:

Let $f(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ -1 & \text{if } x \text{ is irrational}\end{cases}$

If $n\in\mathbb{N}$ and $P_n=\left\{0,\dfrac{1}{n},\ldots\ldots\ldots\ldots\dfrac{n}{n}=1\right\}$.

Let $\dot{P}_n = \left\{ \left(\left[\frac{i-1}{n}, \frac{i}{n} \right], t_i \right) \right\}_{i=1}^{n}$, $\dot{Q}_n = \left\{ \left(\left[\frac{i-1}{n}, \frac{i}{n} \right], s_i \right) \right\}_{i=1}^{n}$ where $t_i \in \left[\frac{i-1}{n}, \frac{i}{n} \right]$,

$s_i \in \left\{ \frac{i-1}{n}, \frac{i}{n} \right\}$, t_i is rational and s_i is irrational.

Show that $S(f, \dot{P}_n) = 1$ and $S(f, \dot{Q}_n) = -1$.

Deduce then $f \notin R[0,1]$. Show also that $|f| \in R[0,1]$.

Definition: $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable if $\exists A \in \mathbb{R}$ with the following property.

For every $\in > 0 \exists \eta > 0$ such that for every tagged partition $\dot{P}$ of [a, b] with $\left\| \dot{P} \right\| < \eta$,

$\left| S(f, \dot{P}) - A \right| < \in$.

Tagged partition $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ where $a = x_0 \le t_i \le x_1 \le t_2 \le x_2 \le .... \le x_{n-1} \le t_n \le x_n = b$

$\left\| \dot{P} \right\| = \max \{ (x_i - x_{i-1}) / 1 \le i \le n \}$.

The number A is called the Riemann integral of f over [a, b] and is denoted by $A = \int_a^b f dx$ or $A = \int_a^b f$.

If f is Riemann integrable over [a, b] we write $f \in R[a,b]$.

Result used: $f \in R[a,b]$ iff for every $\in > 0$ there corresponds $\eta > 0$ such that for every tagged partitions $\dot{P}$ and $\dot{Q}$ of [a, b] with $\left\| \dot{P} \right\| < \eta$ and $\left\| \dot{Q} \right\| < \eta$, $\left| S(f, \dot{P}) - S(f, \dot{Q}) \right| < \in$.

Solution: For $n \in \mathbb{N}$ and $\dot{P}_n$ and $\dot{Q}_n$ as given as in the problem .

$$S(f, \dot{P}_n) \sum_{i-1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) = 1$$ and

$$| f | \in R[0,1]$$

Hence $\left|S(f,\dot{P}_n)-S(f,\dot{Q}_n)\right|=|1-(-1)|=2$.

Moreover $\left\|\dot{P}_n\right\|=\frac{1}{2}=\left\|\dot{Q}_n\right\|$ $\quad \because \frac{i}{n}-\frac{i-1}{n}=\frac{1}{n}\ \forall\, i$

So for every $\delta>0$ we may choose $n\in\mathbb{N} \ni \frac{1}{n}<\delta$. The corresponding $\dot{P}_n$, $\dot{Q}_n$ satisfy $\left\|\dot{P}_n\right\|=\left\|\dot{Q}_n\right\|=\frac{1}{n}<\delta$ but $\left|S(f,\dot{P}_n)-S(f,\dot{Q}_n)\right|=2>\in$. Hence $f\notin R[0,1]$. However $|f(x)|=1$ for all $x\in[0,1]$ so $|f|\in R[0,1]$.

LESSON 20

THE FUNDAMENTAL THEOREMS OF CALCULUS

20.1 OBJECTIVE: In this lesson we propose to establish that integration and differentiation are inverse processes for contain classes of functions. Accordingly we prove two theorems known as fundamental theorems of calculus. We also prove the substitution theorem that simplifies the calculation of definite integrals.

20.1 Structure:

20.3 Introduction
20.4 Fundamental theorem of calculus (First form)
20.5 Fundamental theorem of calculus (Second form)
20.6 Some other theorems on Riemann integral
20.7 Short Answer questions
20.8 Solutions to SAQ's
20.9 Summary
20.10 Technical terms
20.11 Exercises
20.12 Model Examination Questions
20.13 Model practical problem with solution

20.3 Introduction:

It is customary to introduce to the beginners, integral as antiderivative, there by meaning that integration is the inverse process of differentiation. Yes! When we integrate a function and then differentiate the resultant we arrive at the same function again. Likewise when differentiate a function and then integrate the resultant we ultimately get back the original function. However this holds goal for certain classes of functions only. This lesson is devoted to a systematic study of these aspects in the form of Fundamental theorems of calculus. The first type fundamental theorems is concerned about integrating a derivative and that second type of fundamental theorem of calculus discusses differentiating a primitive.

20.4.1 The fundamental theorem of calculus (First form):

Definition: Let F, f be real functions defined on [a,b]. F is called an antiderivative or a primitive of f on [a, b] if $F^1(x) = f(x)$ for $a \leq x \leq b$.

Example: (i) sin x is a primitive of cos x and – cos x is a primitive of sin x.

(ii) $\frac{1}{2} x^2 + 3$ and $\frac{x^2+5}{2}$ are antidervatives of x

20.4.2 The Fundamental theorem (First form)

Suppose there is a finite set E in [a, b] and functions f and ϕ form [a, b] into $\mathbb{R}$ such that

(a) ϕ is continuous on [a, b]

(b) $\phi^1(x) = f(x)$ for every $x \in [a, b] \setminus E$ and

(c) $f \in R[a, b]$.

Then $\int_a^b f dx = \phi(b) - \phi(a)$

Proof: Let $\in > 0$ be given since $f \in R[a, b]$, there exists $\delta_\in > 0$ such that if P is any tagged partition with $\| P \| < \delta_\in$ $|S(f, P) - \int_a^b f dx| < \in$. We first assume that

$E = \{a, b\}$ Let $P = \{ ([x_{j-1}, x_j], u_j) \}_{j=1}^n$

Where each u_j satisfies the condition in the mean valve theorem

$\phi(x_j) - \phi(x_{j-1}) = \phi'(u_j)(x_j - x_{j-1})$ and $u_j \in (x_{j-1}, x_j)$. Then

$$\begin{aligned}
\phi(b) - \phi(a) &= \sum_{j=1}^{n} (\phi(x_j) - \phi(x_{j-1})) \\
&= \sum_{j=1}^{n} (\phi(u_j)(x_j - x_{j-1}) \\
&= \sum_{j=1}^{n} f(u_j)(x_j - x_{j-1}) \\
&= s(f, P)
\end{aligned}$$

Hence $| \phi(b) - \phi(a) - \int_a^b f dx | < \in$.

This is true for every $\in > 0$ so that $\int_a^b fdx = \phi(b) - \phi(a)$

The case when E has more points is considered in SAQ 20.4.

20.4.3 Example:

a) If $F(x) = ½ x^2 \; \forall x \in [a, b] \leq \mathbb{R}$

then $F^1(x) = x \; \forall x \in [a, b] \leq \mathbb{R}$

Further $F^1(x)$ is continuation, so it is in R [a, b]

Hence by the fundamental theorem, with E = ϕ, implies that

$$\int_a^b fdx = F(b) - F(a)$$

$$= ½ b^2 - ½ a^2 = ½ (b^2 - a^2)$$

b) If $G(x) = \tan^{-1} x$ for $x \in [a, b] \leq R$ then

i) $G^1(x) = \frac{1}{1+x^2} \; \forall x \in [a, b] \leq R$

ii) $G^1(x)$ is continuation in [a, b], so that $G \in R[a, b]$

Hence by the fundamental theorem (with E = ϕ) implies that

$$\int_a^b \frac{1}{1+x^2} dx = G(b) - G(a)$$

$$= \tan^{-1}b - \tan^{-1} a.$$

c) If $A(x) = |x| \; \forall x \in [-10, 10]$

$A^1(x) = +1$ if $x \in (0, 10]$; $A^1(x) = 0$ if $x = 0$

Since the signum function, sgn be defined by sgn (x) = +1 for x > 0

= 0 for x = 0

= -1 for x < 0.

We conclude

$A^1(x) = \text{sgn}(x) \; \forall x \in [-10, 10] \setminus \{0\}$.

Since the singum function is a step function, it belongs to R [-10, 10].

Therefore by fundamental theorem (with E = {0}) implies that

$$\int_{-10}^{10} \text{sgn}(x)\,dx = A(10) - A(-10)$$
$$= 10 - 10 = 0$$

d) If $H(x) = 2\sqrt{x}$ for $x \in [0, b] \leq R$ H is continuation on [0, b] and $H^1(x) = \frac{1}{\sqrt{x}}$ for $x \in (0, b]$

which is not bounded on $(0, b] \leq \mathbb{R}$

Hence it does not belong to R [0, b].

20.5 Fundamental theorem (2nd form)

20.5.1 Definition: If $f \in R[a, b]$, then the function defined by

$F(z) = \int_a^z f$ for $z \in [a, b]$ is called indefinite integral of f with base point 'a'.

20.5.2 Theorem: the indefinite integral F defined as in, is continuation on $[a, b] \leq \mathbb{R}$ infact, if $|f(x)| \leq M$, $\forall x \in [a, b]$, then $|F(z) - F(w)| \leq M (z - w| \; \forall z, w \in [a, b]$.

Proof: If the additive theorem,

If $z, w \in [a, b]$ & $w \leq z$, then

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f$$

$$= F(w) + \int_w^z f.$$

$$\Rightarrow \int_a^z f = F(z) - F(w) \quad \rightarrow (1)$$

$\therefore \underset{z \to w}{\text{Lt}} (F(z) - f(w)) = 0.$

$\Rightarrow \underset{z \to w}{\text{Lt}} F(z) = F(w).$

$\therefore$ The indefinite integral F is continuation on [a, b]

If $|f(x)| \leq M \; \forall x \in [a, b]$

$\Rightarrow -M \leq f(x) \leq M \; \forall x \in [a, b]$

If $f \in R[a, b]$, then for $w \leq z \in [a, b]$,

$f \in R\ [w, z\}$

$\Rightarrow -M\ (z\text{-}w) \le \int_w^z f \le M\ (z - w)$

$\Rightarrow \left|\int_w^z f\right| \le M\ (z - w)$

Hence form (1),

$|F(z) - F(w)| \le M\ (z\text{-}w)$

20.5.4 Fundamental theorem of calculus (second form)

Let $f \in R\ [a, b]$ and let f be continuous at a point $c \in [a, b]$. Then the indefinite integral as in is differentiable at c and $F^1(c) = f(c)$

Proof: Given that f is continuous at point $c \in [a, b]$

$\Rightarrow$ f is continuous at $c \in [a, b)$

$\Rightarrow$ for $\in > 0, \exists\ n_{\in} > 0\ \exists$ if

$c \le x < c + n_{\in}$, then

$f(c) - \in < f(x) < f(c) + \in \rightarrow (1)$

If we consider $h \in (0, n_{\in})$ then for $a, c, c + h \in [a, b)$, by the additive theorem,

if $f \in R\ [a, b] \Rightarrow f \in R\ [a, c]$,

$f \in R\ [c, c + h]$

$\&\ f \in R\ [a, c + h]$

$$F\ (c + h) = \int_a^{c+h} f = \int_a^c f + \int_c^{c+h} f$$

$$= F(c) + \int_c^{c+h} f$$

$$\therefore\ F\ (c+h) - F(c) = \int_c^{c+h} f$$

$(1) \Rightarrow f(c) - \in < f(x) < f(c) + \in$

$$\Rightarrow \int_c^{c+h}(f(c)-\in) < \int_c^{c+h} f(x) < \int_c^{c+h}(f(c)+\in)$$

$$\Rightarrow h(f(c) - \in) < \int_c^{c+h} f < h(f(c) + \in)$$

$$\Rightarrow f(c) - \in < \frac{1}{h}\int_c^{c+h} f < f(c) + \in$$

$$\Rightarrow f(c) - \in < \frac{1}{h} [F(c+h) - F(c) \} < f(c) + \in$$

$$\Rightarrow - \in < \frac{F(c+h)-F(c)}{h} - f(c) < \in$$

$$\Rightarrow \left| \frac{F(c+h)-F(c)}{h} - f(c) \right| \le \in$$

$$\Rightarrow \lim_{h\to 0} \frac{F(c+h)-F(c)}{h} = f(c)$$

$$\Rightarrow F^1 (c) = f(c)$$

Hence the indefinite integral is right differentiable at $c \in [a, b)$

$|||^{ly}$ we can prove for $c \in (a, b]$, the indefinite integral is left differentiable.

Hence the result.

20.5.5 Theorem: If f is continuation on [a, b]. Then the indefinite integral F defined in $\otimes$, is differentiable on [a, b] and $F^1(x) = f(x)\ \forall\ x \in [a, b]$

Proof: If f is continuous on [a, b], $f \in R$ [a, b].

By the above theorem 20.5.4 the result follows.

Aho F is continuous on $[\alpha, \beta]$

Hence by fundamental theorem,

$$\int_\alpha^\beta (f \circ \phi)\,\phi^1 = \int_\alpha^\beta f(\phi(x)\,\phi^1(x)\,dx$$

$$= \int_\alpha^\beta \psi^1(x)\,dx$$

$$= \psi(\beta) - \psi(\alpha)$$

$= F(\phi(\beta)) - F(\phi(\alpha))$

$= F(b) - F(a)$ When $b = \phi(\beta)$

$a = \phi(\alpha)$

$$= \int_a^b F^1(x)\,dx$$

$$= \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx$$

20.5.6 Example:

a) Consider the integral $\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$

Put $\phi(t) = \sqrt{t}$ for $t \in [1, 4]$

$\Rightarrow \phi^1(t) = \frac{1}{2\sqrt{t}}$ is continuous on [1, 4]

Let f(x) = 2 sin x then integral is of the form.

$(f \circ \phi)\,\phi^1$ and by the above theorem

$$\int_1^2 2\sin x\,dx = [-2\cos x]_1^2 = 2(\cos 1 - \cos 2)$$

20.5.7 Examples:

(a) Let f(x) = sgn x on [-1, 1] then $f \in R[-1, 1]$

If $x \geq 0$ $\int_{-1}^{x} f(t)\,dt = \int_{-1}^{0} -1\,dx + \int_{0}^{x} 1\,dx = x - 1$

If $x < 0$ $\int_{-1}^{x} f(t)\,dt = \int_{-1}^{x} -1\,dx = -x - 1$

Hence the indefinite integral F(x) = |x| - 1. (with base point -1).

However |x| is not differentiable at o, hence F is not differentiable at 0. Hence f is not an antiderivative of f on [-1, 1].

(b) Let h denote the Thomae function defined by

h(x) = 0 if x is irrational and $x \in [0, 1]$ and

$= \frac{1}{n}$ when x is rational, $x \in [0, 1]$ and $x = \frac{m}{n}$ in simplest form (when x = 0 we take $h(0) = 1$)

then $H(x) = \int_0^x h(t)\,dt = 0$ for $x \in [0, 1]$

Hence $H^1(x) = 0$ for every $x \in [0, 1]$

Since $H^1(x) \neq h(x)$ for when $x \in [0, 1]$ and x is rational,

H is not antiderivative of h on [0, 1].

20.6 Some other theorems on Riemann integral substitution theorem:

Let $J = [\alpha, \beta]$ and let $\phi : J \to \mathbb{R}$ have a continuation derivative on J. if $f : I \to \mathbb{R}$ is continuation on an internal I = [a, b] containing $\phi(J)$, then

$$\int_\alpha^\beta f(\phi(t)) \cdot \phi^1(t)\,dt = \int_{\phi(\alpha)}^{\phi(\beta)} f(x)\,dx$$

Proof: Since f is continuation function on the closed bounded interval [a, b], then there exists a function F defined by

$$F(w) = \int_a^w f(x)\,dx$$

Which is differentiable on [a, b] and $F^1(w) = f(w)\ \forall\ w \in [a, b]$

Let $\psi(w) = F(\phi(w))$ for $a \le w \le b$.

By chain rule,

$$\psi^1(w) = F^1(\phi(w))\,\phi^1(w)$$
$$= (f \circ \phi)(w) \cdot \phi^1(w)$$
$$= [(f \circ \phi)\,\phi^1](w) \text{ for } a < \alpha \le w \le \beta < b$$

The continuity of ϕ' implies the continuity of ϕ. Also the continuity of f and ϕ implies the continuity of the composite function $f o \phi$ and consequently, the product is continuation on $[\alpha, \beta]$.

20.6.2 Integration by parts: Let F and G be differentiable on [a, b], $f = F^1$ and $g = G^1$ belong to R [a, b].

Then $\int_a^b f\ G\ dx = F(b)\ G(b) - F(a)\ G(a) - \int_a^b F\ g\ dx$

Proof: Since F and G are differentiable on [a, b]

and $(F\ G)^1\ (x) = F(x)\ G^1\ (x) + F^1\ (x)\ G(x)$

$= F(x)\ g(x) + f(x)\ G(x)$ if $x \in [a, b]$

Since F and G are continuous and $f = F^1$, $g = G^1$ are Riemann integrable, $f\ G = F^1\ G$ and $Fg = FG^1$ are integrable on [a, b]. Therefore by the Fundamental theorem,

$$F\ (b)\ G(b) - F(a)\ G(a) = \int_a^b (FG)^1\ dx$$

$$= \int_a^b (fG)\ dx + \int_a^b Fg\ dx$$

Hence $\int_a^b (fG)\ dx = F_a\ (b)\ G(b) - F(a)\ G(a) - \int_a^b Fg\ dx$

20.6.3 Taylor's theorem with the Remainder

Suppose $f, f^1, \ldots.. f^{(n)}, f^{(n+1)}$ exist in [a, b] and $f^{(n+1)} \in R\ [a, b]$.

Then $f(b) = f(a) + (b\text{-}a)\ f^1(a) \ldots\ldots.. + (b\text{-}a)^n\ \dfrac{f^{(n)}(a)}{n!} + R_n$

Where $R_n = \dfrac{1}{n!} \int_a^b f^{(n+1)}(t)\ (b-t)^n\ dt$

Proof: The functions $F(t) = f^{(n)}\ (t)$ and $G(t) = \dfrac{(b-t)^n}{n!}$ are differentiable in [a, b) $F^1(t) = f^{(n+1)}\ (t)$ and $G^1\ (t) = - \dfrac{(b-t)^{n-1}}{(n-1)!}$ belong to R [a, b]. Then by the theorem on integration by p arts,

$$R_n = \int_a^b F^1 G\ dt = F(b)\ G(b) - F(a)\ G(a) - \int_a^b FG^1\ dt$$

$$\Rightarrow R_n = f^{(n)}(b)\ \frac{(b-a)^n}{n!} - f^n\ (a)\ \frac{(b-a)^n}{(n)!}$$

$$+ \frac{1}{(n-1)!} \int_a^b f^{(n)}(t)\ (b\text{-}t)^{n-1}\ dt$$

$$= \frac{-f^{(n)}(a)}{n!}\ (b-a)^n + \frac{1}{(n-1)!} \int_c^b f^{(n)}(t)\,(b-t)^{n-1} dt$$

Repeating this process with the integral on the right successively we ultimately get

$$R_n = \frac{-f^{(n)}(a)}{n!}\ (b\text{-}a)^n - \frac{f^{(n-1)}(a)}{(n-1)!}\ (b\text{-}a)^{n-1} \ldots\ldots\ldots - f(a) + f(b)$$

$$\Rightarrow f(b) = f(a) + (b\text{-}a)\ f^1(a) + \ldots\ldots \frac{(b-a)^n}{n!}\ f^{(n)}(a) + R_n$$

20.7 Short answer Questions

(a) Extend the proof the first fundamental theorem (first form) to the case of an arbitrary finite set

(b) If $n \in N$ let $F(x) = \frac{x^{n+1}}{n+1}$ for $x \in [a, b]$. Apply the Fundamental theorem of calculus (first form) to deduced that $\int_a^b x^n\ dx = \frac{b^{n+1} - a^{n+1}}{n+1}$

(c) If $g(x) = x$ for $|x| \geq 1$ and $-x$ for $|x| < 1$ and if $G(x) = \frac{1}{2}\ |x^2 - 1|$ Show that G is differentiable and $G^1(x) = g(x)$ except at $x = \pm 1$. Apply Fundamental theorem of calculus (first form) to deduce that $\int_{-2}^{3} g(x)\ dx = G(3) - G(-2) = \frac{5}{2}$.

(d) Let $f: [a, b] \to \mathbb{R}$ and $c \in \mathbb{R}$.

(i) If $\phi : [a, b] \to \mathbb{R}$ is an antiderivative of f on [a, b] show that $\phi_c(x)=\phi(x)+c$ is also an antiderivative of f on [a, b]

(ii) Conversely show that if ϕ_1 and ϕ_2 are antiderivative of f on [a, b] then there is $c \in \mathbb{R}$ such $\phi_1(x) = \phi_2(x) + c$ for all $x \in [a, b]$

(e) If $f \in R\,[a, b]$ and $c \in [a, b]$ show that he function F_c defined by $F_c(z) = \int_c^z f(t)\,dt$

for $z \in [a, b]$ called the indefinite integral of f with base point c satisfies

$F_a(z) - F_c(z) = F_a(c)$ for all z.

(f) Is that Fundamental theorem of calculus (first form) applicable to derive $\int_0^1 H(x)\,dx = 0$ where H is the thomae function?

20.4.6 Let $F: [0, \infty] \to \mathbb{R}$ be defined by $F(x) = (n-1)\,x - \dfrac{(n-1)n}{2}$ if $n-1 \le x < n$ and $n \in N$. show that F is continuous on $[0, \infty)$.

Evaluate $F^1(x)$ at points where F is differentiable and Apply the Fundamental theorem of calculus (first form) to evaluate $\int_a^b [x]\,dx$ where $[x]$ is the largest integer $\le x$.

20.4.7 Let $f: [a, b] \to \mathbb{R}$ be continuous on [a. b] $\sigma : [c, d] \to [a, b]$ be differentiable. If $G(x) = \int_a^{\sigma(x)} f(t)\,dt$ show that $G^1(x) = f(\sigma(x)\,\sigma^1(x)$.

(i) Cauchy Bunyakovsky – Schwartz in equality

If f, g belong to R [a, b] show that $\left|\int_a^b fg\,dx\right|^2 \le \int_a^b f^2\,dx \,.\, \int_a^b g^2\,dx\,.$

20.8 Solutions to short answer Questions

20.7. a Extend the proof of the Fundamental theorem of calculus (first form) to the case of an arbitrary finite set.

Proof: Suppose $E = \{a = c_0 < c_1 < \ldots < c_m = b\}$ be the set of points of [a, b] where either F is not differentiable or F is differentiable but $F^1(x) \neq f(x)$. Then for $1 \le i \le m$ $f \in R\,[c_{i-1}, c_i]$ and $\int_{c_{i-1}}^{c_i} f\,dx = F(c_i) - F(c_{i-1})$.

Hence $\int_a^b f\ dx = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f\ dx = \sum_{i=1}^{m} F(c_i) - F(c_{i-1}) = F(b) - F(a)$

Continuing in this process, we get

$|a_n r^n| \geq B$ for all $n \geq w$

consider $|a_n z^n| = |a_n z^n| \dfrac{r^n}{r^n}$

$$= |a_n r^n| \frac{|z|^n}{r^n}$$

$$= |a_n r^n| \left(\frac{|z|}{r}\right)^n$$

$$\geq B\left(\frac{|z|}{r}\right)^n \text{ for all } n \geq N.$$

Since $\dfrac{|z|}{r} > 1$, the series $\sum_{n=0}^{\infty}\left(\dfrac{|z|}{r}\right)^n$ diverges

Hence $\sum_{n=0}^{\infty} a_n z^n$ diverges when $|z| > \alpha$

$\therefore R \leq \alpha$ — (2)

from (1) and (2) we get

$$R = \alpha = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Hence $R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$ if it exists

Hence the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ is a power series with radius of convergence R

then $R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right|$ if it exists.

20.7.b Verify that the Fundamental theorem of calculus (first form) can be applied to the function $F(x) = \dfrac{x^{n+1}}{n+1}$ for x in [a, b] and $n \in N$ to deduce that $\int_a^b x^n\ dx = \dfrac{b^{n+1} - a^{n+1}}{n+1}$.

Verification: The function F is differentiable on [a, b], f(x) = $F^1(x) = x^n$ for all $x \in [a, b]$ (we may take E = ϕ) and $f \in R$ [a, b].

Hence the Fundamental theorem of calculus (first form) is applicable. By this theorem

$$\int_a^b x^n \, dx = \int_a^b f(x)\, dx = F(b) - F(a) = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

20.7.c Let G(x) = ½ $|x^2-1|$. Find x where G is not differentiable. Show that $G^1(x) = g(x)$ whenever G is differentiable where $g(x) = \begin{cases} x \text{ if } |x| \ge 1 \\ -x \text{ if } |x| < 1 \end{cases}$

Show that $\int_{-2}^{3} g(x)\, dx = G(3) - G(-2) = \frac{5}{2}$.

Solution: If |x| > 1 $G^1(x) = \lim\limits_{h\to 0} \frac{G(x+h) - G(x)}{h} = x$

If |x| < 1 $G^1(x) = \lim\limits_{h\to 0} \frac{G(x+h) - G(x)}{h} = -x$

$\lim\limits_{h\to 0+} \frac{G(1+h) - G(1)}{h} = \frac{1}{2h} \{(1+h)^2 - 1\} = 1$

Write $\lim\limits_{h\to 0+} \frac{G(1-h) - G(1)}{h} = \frac{1}{2h}\{(1-h)^2 - 1\} = -1$

Hence G is not differentiable at 1. similarly G is not differentiable at -1. Hence $G^1(x)$ = g(x) except at x = $\pm$ 1

Where G is not differentiable. The function g is Riemann integrable in [-2, 3]. Hence by the Fundamental theorem of calculus (first form)

$$\int_{-2}^{3} g(x)\, dx = G(3) - G(-2) = 5.$$

20.7.d Let f: [a, b] $\to \mathbb{R}$, $c \in \mathbb{R}$.

(i) If ϕ : [a, b] $\to \mathbb{R}$ is an antiderivative of f on [a, b] and $\phi_c(x) = \phi(x) + c$ for $x \in [a,b]$ show that ϕ_c is also an antiderivative of f.

(ii) conversely if ϕ_1, ϕ_2 are antiderivatives of f on [a, b] show that there is a $c \in \mathbb{R}$ such that $\phi_1(x) = \phi_2(x) + c$ for $x \in [a, b]$.

20.7.e (i) since ϕ is differentiable on [a, b] so is ϕ_c. Also $\phi_c^1(x) = \phi^1(x) = f(x)$ for $x \in [a, b]$

Hence ϕ_c is an antiderivative of f on [a, b].

(ii) Since $\phi_1^1(x) = \phi_2^1(x) = f(x)$ for al x in [a, b], $(\phi_1 - \phi_2)^1(x) = \phi_1^1(x)\ \phi_2^1(x) = 0$ for all x in [a, b]. Hence there exists $c \in \mathbb{R}$ such that $(\phi_1 - \phi_2)(x) = c$ for all $x \in [a, b]$. This implies that $\phi_1(x) = \phi_2(x) + c$ for $x \in [a, b]$.

20.7.f If $f \in R[a, b]$ and $c \in [a, b]$, the function $F_c(z) = \int_c^z f(t)\,dt$ for $z \in [a, b]$ is called the indefinite integral of f with base point c. Find a relation between F_a and F_c.

By definition $F_a(z) = \int_c^z f(t)\,dt$ for $z \in [a, b]$

$$= \int_a^c f(t)\,dt + \int_c^z f(t)\,dt$$

$$= \int_a^c f(t)\,dt + F_c(z).$$

Hence $F_a(z) - F_c(z) = F_a(c)$ for al $z \in [a, b]$.

Problem:

1. Find Radius of convergence of the series $\Sigma \frac{z^n}{n!}$

Sol: Consider the power series $\Sigma \frac{z^n}{n!}$

Here $a_n = \frac{1}{n!}$

Consider the Radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$R = \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right|$$

$$= \lim_{n\to\infty}|n+1|$$

$$R = \infty$$

Hence the series converges for all z.

20.7.g The Thomae function H : [0, `] $\to \mathbb{R}$ defined by H(x) = 0 if x is irrational and $x \in [0, 1]$ and H(x) = $\frac{1}{n}$ if x is rational in [0, 1] and x = $\frac{m}{n}$ in the simplest form $x \neq 0$ while H(0) = 1 then H is continuous at every irrational number and discontinuous at every rational number, H is not differentiable at any point of [a, b]. Also we know that $\int_0^1 H(x)\,dx = 0 = H(1) - H(0)$. This conclusion can not be derived from the fundamental theorem of calculus (first form) since there is no h: [0, 1] $\to \mathbb{R}$ such that $H^1(x) = h(x)$ for all but possibly many x in [0, 1].

Introduction:

In this topic we have to discuss about definition and some properties on power series.

Power series:

The series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the power series about the point 'a'.

Radius of convergence:

Let $\sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series we define $\frac{1}{R}$ = lim sup $|a_n|^{1/n}$. The number R is called radius of convergence of the power series.

Result: If $\sum_{n=0}^{\infty} a_n (z-a)^n$ is a power series with radius of convergence R thus

$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if it exists

Proof: Without loss of generality we prove the result for the power series $\sum_{n=0}^{\infty} a_n\, z^n$

Suppose $\alpha = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists.

20.7.h Let f(x) be defined for $x \in [0, \infty]$ by $F(x) = (n-1)\,x - (n-1)\,\frac{n}{2}$ for $x \in [n-1, n]$ and $n \in N$.

Show that F is continuous on $[0, \infty)$. Evaluate $F^1(x)$ where this derivative exists. Use this result to evaluate $\int_a^b [x]\,dx$ where [x] is the integer such that $[x] \le x < [x] + 1$.

Since $F(x) = (n-1)\{x - \frac{n}{2}\}$, for $n-1 \le x < n$, F is differentiable in (n-1, n) for $n \in N$ and $F^1(x) = (n-1)$ if $n-1 < x < n$.

Also $\lim_{h \to 0+} \frac{F(n+h) - F(n)}{h} = n$ while

$$\lim_{h \to 0-} \frac{F(n-h) - F(n)}{h} = n-1 \text{ for } n \in N$$

Thus F is not differentiable at n for $n \in N$. Thus many interval [a, b] where $a < b$ the set of points where F is not differentiable is finite. Hence the Fundamental theorem of calculus (first form) is applicable with f(x) = [x] and $\int_a^b [x]\,dx = F(b) - F(a)$.

For example $\int_1^3 [x]\,dx = F(3) - F(1) = 3$.

$$\int_{1/2}^{3/4} [x]\,dx = F(\tfrac{3}{4}) - F(\tfrac{1}{2}) = 0.$$

20.7.i Let $f : [a, b] \to \mathbb{R}$ be continuous, $\upsilon : [c, d] \to [a, b]$ be differentiable and let

$$G(x) = \int_a^{\upsilon(x)} f(t)\,dt$$

Show that $G^1(x) = f(\upsilon(x)\,\upsilon^1(x)$ for $\in [c, d]$.

Let $H(x) = \int_a^x f(t)\,dt$ since f is continuous, H is differentiable and $H^1(x) = f(x)$ for $x \in [a, b]$

Since $G(x) = H(\upsilon(x))$ and H & υ are differentiable, it follows by chain rule that

$$G^1(x) = H^1(\upsilon(x))\,\upsilon^1(x)$$
$$= f(\upsilon(x))\,\upsilon^1(x).$$

20.7.j Cauchy – Bunyakovsky – Schwarz in equality:

Let f,g $\in$ R [a, b]. Then $\left|\int_a^b fg\right|^2 \le \left(\int_a^b |fg|\right)^2 \le \int_a^b f^2 \int_a^b g^2$

As the first inequality is clear we prove the second inequality.

For every t $\in \mathbb{R}$ $(tf - g)^2 (x) > 0$ if x [a, b]

$\Rightarrow \int_a^b (t\,|\,f\,| - |\,g\,|)^2\, dx \ge 0.$

$\Rightarrow t^2 \int_a^b f^2 - 2t \int_a^b |\,fg\,| + \int_a^b g^2 \ge 0.\ \forall\, t \in \mathbb{R} \ \rightarrow (1)$

If $\int_a^b f^2 = 0,\ 0 \le \int_a^b |\,fg\,| \le \frac{1}{2t}\int_a^b g^2$

This holds for every t > 0. Letting $t \to +\infty$ we get $\int_a^b |\,fg\,| = 0.$

If $\int_a^b f^2 > 0$ put $t = \frac{\int_a^b |\,fg\,|}{\int_a^b f^2}$, $A = \int_a^b f^2$, $B = \int_a^b |\,fg\,|$ and $c = \int_a^b g^2$ in (1) we get $c - \frac{B^2}{A} \ge 0$

So that A C $\ge B^2$

This implies that $(\int_a^b |\,fg\,|)^2 \le (\int_a^b f^2) \le (\int_a^b g^2)$

20.9 Summary:

After completing this lesson the student will be able to appreciate the idea that integration and differentiation or inverse processes for a large class of functions which is established via the fundamental theorems of calculus.

20.10 Technical Terms: Integration by parts, indefinite integral, Anti derivative.

20.11 Exercises:

1. Let $B(x) = \begin{cases} -\frac{1}{2}x^2 \text{ if } x < 0 \\ \frac{1}{2}x^2 \text{ if } x \ge 0 \end{cases}$

   Show that $B^1(x) = |x|$ for all x.

   Using Fundamental theorem if calculus first form

   Show that $\int_a^b |\,x\,|\,dx = B(b) - B(a).$

2. Let $f \in R\,[a, b]$ and define $F(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$.

(a) Show that $\int_a^x f(t)\,dt = F(x) - F(c)$

(b) Show that $\int_x^{bx} f(t)\,dt = F(b) - F(x)$

(c) Show that $\int_x^{\sin x} f(t)\,dt = F(\sin x) - F(x)$ if $a \le \sin x \le b$ for $x \in [a, b]$

3. Show that $F^1(x) = 2x\,(1 + x^6)^{-1}$ if $F(x) = \int_0^{x^2} (1+t^3)^{-1}\,dt$

4. Show that $F^{-1}(x) = \sqrt{(1+x^2)} - 2x\sqrt{(1+x^4)}$ if $F(x) = \int_{x^2}^{x} \sqrt{1+t^2}\,dt$

(Hint: apply SAQ ………)

5. Let $f(x) = \begin{cases} x \text{ if } 0 \le x < 1 \text{ or } 2 \le x \le 3 \\ 1 \text{ if } 1 \le x < 2 \end{cases}$

Show that $F(x) = \int_0^x f(t)\,dt \begin{cases} \dfrac{x^2}{2} \text{ if } 0 \le x \le 1 \\ x - \dfrac{1}{2} \text{ if } 1 \le x \le 2 \\ \dfrac{x^2 + 2x - 5}{2} \text{ if } 2 \le x \le 3 \end{cases}$

Show that f is differentiable except at 1 and 2.

Show that F is not differentiable at 1 and 2.

6. If f: $\mathbb{R} \to \mathbb{R}$ is continuous and $c > 0$ show that

$g(x) = \int_{x-c}^{x+c} f(t)\,dt$ is differentiable in $\mathbb{R}$ and

show that $g^1(x) = f(x + c) - f(x - c)$.

7. If f: $[0, 1] \to \mathbb{R}$ is continuous and $\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt$ for all $x \in [0, 1]$ then show that $f(x) = 0$ for all $x \in [0, 1]$

(Hint $\int_0^x f(t)\,dt = \frac{1}{2} \int_0^1 f(t)\,dt$ differentiate)

8. Use the substitution theorem to evaluate the following integrals

(a) $\int_0^1 t\sqrt{1+t^2}\,dt$ $(1 + t^2 = u)$ (Ans: $\frac{1}{3}(2^{3/2}-1)$)

(b) $\int_0^2 t^2(1+t^3)^{-1/2}\,dt$ $(1 + t^3 - u)$ (Ans: $\frac{4}{3}$)

(c) $\int_1^4 \frac{\sqrt{1+\sqrt{t}}}{\sqrt{t}}\,dt$ $(\sqrt{t} = u - 1)$ (Ans: $\frac{4}{3}(3^{3/2} - 2^{3/2})$)

(d) $\int_1^4 \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$ $(\sqrt{t} = u)$ (Ans: 2(sin2 – sin1)

20.12 Model Examination Questions

1. State and prove the fundamental theorem of calculus (first form)

2. Show that if $f \in R\,[a, b]$ then the indefinite integral $F(z) = \int_a^z f(t)\,dt$ is continuous.

3. State and prove the fundamental theorem of calculus (second form)

4. State and obtain the formula for integration by parts.

5. Find $F^1(x)$ when $F(x) = \int_0^{x^2} \frac{dt}{1+t^3}$

6. If $\int_0^x f(t)\,dt = \int_x^1 f(t)\,dt$ for all $x \in [0, 1]$ and $f : [0, 1] \to \mathbb{R}$ is continuous, show that $f(x) = 0$ for all $x \in [0, 1]$.

20. Model Practical problem with solution

Let $f(x) = \begin{cases} 0 \text{ if } 0 \le x < 1 \\ 1 \text{ if } 1\ x \le 2 \end{cases}$

(a) Show that $f \in \mathbb{R}\ [0,2]$

(b) If $f(x) = \int_0^x f(t)\,dt$ show that F is continuous on [0, 2]

(c) Find all x for which f is differentiable at x.

Definitions:

(1) we say that $f \in R\,[a, b]$ if $\exists\, A \in \mathbb{R}$ such that for every $\in > 0$ there corresponds $\delta_{\in} > 0$ with the property that if $P = \{ ([x_{i-1}, x_i], t_i) \}_{i=1}^{n}$ is any tagged partition of $[a, b]$ i. e $t_i \in [x_{i-1}, x_i]\ \forall\ i,\ 1 \le i \le n$ with $\| p \| = \max (x_i - x_{i-1}) < \delta_{\in}$,

$$\left| \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1}) - A \right| < \in$$

This A is called the integral of f over [a, b] and is denoted by $\int_a^b f$

(2) we say that $f: [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$ and has derivative $f'(x)$ if $\in \mathbb{R}$ and $\lim_{y \to x} \dfrac{f(y) - f(x)}{y - x} = f^1(x)$.

Results used:

(1) If $f \in R\,[a, b]$ and $f(x) = g(x)$ for $x \in [a, b],\ x \ne x_0$

Where $x_0 \in [a, b]$ and $f(x_0) \ne g(x_0)$ then $g \in R\,[a, b]$ and $\int_a^b f\ dx = \int_a^b g\ dx$

(2) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b] \setminus E$ where E is a finite subset of [a, b] then $f \in R\,[a, b]$

(3) If $f \in R\,[a, b]$ then the function F defined on [a, b] by $F(x) = \int_a^x f(t)\,dt$ is continuous on [a, b]. If f is continuous at $x_0 \in [a, b]$ there f is differentiable at $x_0 \in [a, b]$ there F is differentiable at x_0 and $F^1(x_0)$ and $F^1(x_0) = f(x_0)$

Solution: Let g(x) = if $0 \le x \le 1$ Then $g \in R\,[0,1]$ Since g = f except at x = 1, $f \in R\,[0,1]$ and $\int_0^1 f(t)\,dt = \int_0^1 g(t)\,dt = 0$. (Result <u>1</u>)

Hence F(1) = 0. Also if $0 \le x < 1$, $F(x) = \int_0^2 f\ dt = 0$.

In [0, 2] f is continuous except at x = 1. So $f \in R\,[0,2]$ (Result <u>2</u>)

Also for $1 \le x \le 2$ $F(x) = \int_0^x f(t)\,dt \quad = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt$

$$= F(1) + \int_1^x f(t)\,dt = \int_1^x f(t)\,dt$$

$$= \int_1^x 1\,dt = x - 1.$$

Since $f \in R\,[0, 2]$, F is continuous on [0, 2] (Result 3)

Since f is continuous in [0, 2] except at 1, D is differentiable at x for $x \neq 1$

$$\lim_{y\to 1+} \frac{F(y) - F(x)}{y - 1} = \lim_{y\to 1+} \frac{y - 1}{y - 1} = 1$$

$$\lim_{y\to 1-} \frac{F(y) - F(x)}{y - 1} = \frac{0}{y - 1} = 0$$

Hence F is not differentiable at 1.

Hence F is differentiable on $[0, 2] - \{1\}$.

C. SANDHYA

Richard Dedekind (1831-1916)

Dedekind's major contribution was a redefinition of irrational numbers in terms of Dedekind cuts. He introduced the notion of an ideal which is fundamental to ring theory.

LESSON 10A :

REAL NUMBER SYSTEM

10.A:1 Objective: The aim of this lesson is to provide the student with the necessary preliminaries on set theory and introduce the Real number system as a 'complete ordered field'. The most commonly used properties - The Archimedian principle and density of Q in $\mathbb{R}$ are also presented for ready use of the student.

10.A.2 Structure

10.A.3 Introduction

10.A.4 Sets and functions

10.A.5 The Real number system

10.A.6 Absolute value

10.A.7 Bounded sets

10.A.8 Bounded functions

10.A.9 Intervals

10.A.10 Nested intervals

10.A.11 Some elementary inequalities

10.A.12 Solutions to SAQ's

10.A.13 Exercises

10.A.14 Model examination questions

10.A.15 Model practical problem with solution

10.A.3 Introduction

In this lesson we present the background needed for real analysis. We begin with the assumption that we “know” what a “set” is and make no attempt to define set.

After a brief introduction to the elements of set theory we introduce the principle of mathematical induction in various forms without proof. We then proceed to the definition of finite, infinite, denumerable and uncountable sets.

The order properties of $\mathbb{R}$ lead to the notion of bounded and unbounded sets. The supremum and infimum of a bounded set are defined and some important properties are discussed in this lesson. The notions of interval, Absolute valve, and nested intervals are also studied. We do not make any attempt to provide rigorous definitions to the trigonometric functions, exponential functions and logarithmic functions but make use of their properties wherever required.

10.A.4. Sets and Functions:

We assume familiarity with sets and elements and the relation “belongs to” denoted by $\in$. If a is an element and S is a set only one of the two – “a belongs to S”, “a does not belong to S” holds. In the first case we write $a \in S$ and in the second case we write $a \notin S$. The word set is synonymous with the words “class” “family” and “collection”. The familiar notations { ……… } and $\{x| P(x)\}$ are used to represent a set.

→ Let A and B be sets. We say that A is equal to B, in symbols $A = B$ if $x\in \Leftrightarrow x\in B$.

→ We say that A is a subset of B, in symbols $A \subseteq B$ or $B \supseteq A$ if $x \in B$ whenever $x \in A$. if $A \subseteq B$ and $A \neq B$ we write a $\underset{\neq}{\subset}$ B.

→ We assume that there is one and only one set denoted by ϕ which does not have any elements.

→ The union $A \cup B$ of sets A and B is defined by $A \cup B = \{ x \mid x \in A \text{ or } x \in B\}$

→ The intersection $A \cap B$ is defined by $A \cap B = \{ x \mid x \in A \text{ and } x \in B\}$

→ and the complement of B in A is defined by $A \setminus B = \{ x \mid x \in A \text{ and } x \notin B\}$

→ Let Δ be any set and for every $\alpha \in \Delta$, A_α is a set. The collection of sets $\{ A_\alpha \mid \alpha \in \Delta \}$is said to be "indexed" by the "index set Δ"

→ The union of the family of sets $\{ A_\alpha \mid \alpha \in \Delta \}$ and the intersection are defined by The union of $\{ A_\alpha \mid \alpha \in \Delta \} = \cup_{\alpha\in\Delta} A_\alpha = \{ x \mid x \in A_\alpha$ for some $\alpha \in \Delta\}$ and the intersection of $\{ A_\alpha \mid \alpha \in \Delta \} = \cap_{\alpha\in\Delta} A_\alpha = \{ x \mid x \in A_\alpha$ for every $\alpha\in \Delta\}$

→ If a and b are elements the ordered pair with "first component" a and "second component" b is (a,b)

→ (a,b) = (c,d)if and only if a = c, b = d. Thus (a,b) $\neq$ (b,a) if a $\neq$ b.

→ The Cartician product A x B of sets A and B is given by A x B = $\{ (a,b) \mid a \in A$ and $b \in B\}$. Any subset of A x B is called a relation.

→ A relation $F \subseteq A \times B$ is a function (also called a mapping) from A into B, in symbols $F: A \to B$ if for every $a \in A$ there is a unique $b \in B$ such that $(a,b) \in F$. This b corresponding to a such that $(a,b) \in F$ is called the image of a in F and is denoted by b = F (a). A is called the domain of F and B the codomain of F.

→ If $C \subseteq A$ $F (C) = \{f (a) \mid a \in C\}$. F (A) is called the range of F. If $b \in B$ $F^{-1} (b) = \{a \mid a \in A$ and $b = F (a)\}$ and if $B_1 \subseteq B$, $F^{-1} (B_1) = \{a \mid F (a) \in B_1\}$

→ $F: A \to B$ is said to be one-one (also called injection) if $x \in A$, $y\in A$ and $F (x) = F (y) \Rightarrow x = y$.

→ F is called onto (also called surjection) if B = F (A); i.e. for every $y \in B$ there is $x \in A$ such that F (x) = y

→ F is called one-one and on to (bijection) if F is one – one and onto, i e. for every $y \in A$ there is unique $x \in A$ such that F (x) = y.

→ We say that A is equipotent to B, in symbols $A \sim B$ if there is a bijection from A onto B.

10.A.4.1. **SAQ:** If $F : A \to B$ and $G: B \to C$ the composite $GoF: A \to C$ defined by (GoF) (x) = G (F(x)) is a function.

10.A.4.2 SAQ: If $F: A \to B$ is a bijection, $F^{-1}: B \to A$ defined by $F^{-1} = \{(y,x)\ y \in B, x \in A$ and $F (x) = y\}$ is a bijection.

10.A.4.3 SAQ: The identity function $I: A \to A$ defined by I (x) = x is a bijection.

10.A.4.4. SAQ: If F: A $\to$ B and G: B$\to$ C are bijections then show that the composite GoF : A $\to$ C is a bijection.

The restriction of a function F: A $\to$ B to a subset C $\subseteq$ A, denoted by F_{1_c} is the set F_1=F$\cap$(C x B), F_{1_c} : C $\to$ B defined by F_1 (x) = F (x) if x $\in$ C is a function and is denoted by F_1 = F/C

We now assume the existence of the set N of natural numbers whose elements are denoted by 1,2,3,….n….., N = {1,2,3,….n…..} and the existence of a relation called order relation on N satisfying 1 < 2 < 3 < … n < … we also assume the existence of arithmetic operations addition (+) and multiplication (.). We will discuss about these properties further very soon. We now state the famous well ordering principle and the principle of Mathematical induction .

An element n $\in$ A $\subseteq$ N is called the first (least) element of A if m $\in$ N and m < n $\Rightarrow$ m $\notin$ A.

Well ordering principle: Every nonempty subset of N has first element.

Principles of mathematical induction: Let S be a subset of N that possesses the two properties:

(1) 1 $\in$ S (2) If k $\in$ N and k $\in$ S, then k+1 $\in$ S. Then S = N.

10.A.4.5 SAQ: Prove the following by the principle of Mathematical induction.

(i) For each n $\in$ N, 1 +2 + …. + n = $\frac{n(n+1)}{2}$

(ii) For each n $\in$ N, $1^2 + 2^2 + \ldots\ldots\ldots + n^2 = \frac{1}{6}(n)(n+1)(2n+1)$

10.A.4.6 Principle of Mathematical Induction – Second version:

Let P(n) be a statement for n $\in$ N. If for some n_o $\in$ N, P(n_o) is true and if P (k+1) is true whenever k $\in$ N, K $\geq n_o$ and P(k) is true, then P(n) is true for all n $\in$ N and n $\geq n_o$.

10.A.4.7. Principles of strong (complete) induction:

Let $S \subseteq N$. If $1 \in S$ and $n + 1 \in S$ whenever $\{1,2,\dots n\} \subseteq S$ then $S = N$.

10.A.4.8. Finite and infinite sets: If $n \in N$ we write $J_n = \{1,2,\dots.n\}$.

A set A is said to be finite if either $A = \phi$ or else there is $n \in N$ such that $A \sim J_n$. A is said to be infinite if A is not finite. A is said to be countabley infinite or denumerable if $A \sim N$; i.e. there is a bijection from A onto N.

If A is countabley infinite and $x : N \rightarrow A$ is a bijection, it is customary to write $x(n)=x_n$ for $n \in N$ and represent A by $A = \{x_1, x_2, \dots.. x_n \dots.\}$ This represention of A is called an enumeration of A and is also written $A = \{x_n \mid n \in N\}$

If $n \in N$ x_n is called the n-th term of the enumeration. If $A = \phi$, A is said to have zero elements. If $A \sim J_n$ A is said to have n elements. If A is finite or denumerable to A, then A is said to be countable. i.e. If $A = \phi$ or $A \sim J_n$ for some n or $A \sim N$ then A is countable. If A is not countable, Then A is said to be uncountable and have uncountable many elements.

We use the following theorems wherever necessary. We state the theorems without proofs as our main interest is in analysis and not in set theory.

Uniqueness theorem: If S is a finite set then there is a unique $n \in N$ such that S has n elements.

10.A4.9. Theorem: The set N of natural numbers is infinite.

10.A.4.10. Theorem: The set Q of rational numbers is denumerable.

Theorem: If for $n \in N$ A_n is countable, $\bigcup\limits_{n \in 1=1} A_n$ is countable

10.A.4.11 Theorem: Suppose S and T are sets and that $T \subseteq S$.

(a) If S is a countable set, then T is a countable set.

(b) If T is an uncountable set then S is an uncountable set.

10.A.5. We now discuss some essential properties associated with the real number system. As already mentioned the natural number system N, the sets Z of integers, Q of rational numbers and the set $\mathbb{R}$ of real numbers are governed by the inclusions N $\underset{\neq}{\subset} Z \underset{\neq}{\subset} Q \underset{\neq}{\subset} \mathbb{R}$. These sets are equipped with two algebraic operations – addition, denoted by + and multiplication, denoted by. Addition assigns to each ordered pair (a,b) of real numbers, their sum, denoted by a + b and multiplication assigns a .b (also denoted by a b) to (a, b). Each of the sets N, Z, Q and $\mathbb{R}$ is "closed" under '+' as well as for "." in the sense that the sum a + b and the product a b of any pair (a, b) from each of these sets lie in the same set.

We now list of he fundamental laws of addition and multiplication.

10.A.5.1 (A) LAWS OF ADDITION:

A_1 :	If a, b are in $\mathbb{R}$, a + b = b + a	commutative law
A_2 :	If a, b, c are in $\mathbb{R}$ (a + b)+c = a + (b + c)	associative law
A_3 :	There is an element 0 (called zero) in $\mathbb{R}$. Such that a + 0 = a for every a in $\mathbb{R}$	existence of zero element
A_4 :	For every a in $\mathbb{R}$ there is an element u in $\mathbb{R}$ such that a + u = 0. Any such u is denoted by –a and is called inverse of a	existence of additive inverse Existence of additive inverse.

(M) Laws of Multiplication:

M_1 : If a, b are in $\mathbb{R}$ ab = ba (commutative law)

M_2 : If a, b, c are in $\mathbb{R}$ (ab) c = a (bc) associative law.

M_3 : There as an element of called unity in $\mathbb{R}$ such that 1 ≠ 0 and a1 = 1a = a for all a in $\mathbb{R}$. existence of unity

M_4: If a ∈ $\mathbb{R}$ and a ≠ 0 there a v in $\mathbb{R}$ such at av = va = 1. This v is called inverse of a and is denoted by a^{-1} or $\frac{1}{a}$ (existence of inverse for nonzero element)

(D) Distributive law: If a,b,c belong to $\mathbb{R}$ a (b+c) = ab+ac

With these axioms (R, +, .) becomes a "field".

Remark: The elements 0,1, -a and $\frac{1}{a}$ are unique.

Reason: If z_1 and z_2 satisfy A_3 then $Z_2 = Z_2 + Z_1 = Z_1$. If e_1 and e_2 satisfy M_3 then $e_1 = e_1 e_2 = e_2$

(O) Order properties of $\mathbb{R}$: There is a nonempty subset $\mathbb{R}^+$ of $\mathbb{R}$ called the set of positive real numbers. We write a > 0 (The symbol > is said greater than) if $a \in \mathbb{R}^+$.

$\mathbb{R}^+$ satisfies the following properties.

(i) $a > 0, b > 0 \Rightarrow a + b > 0$

(ii) $a > 0, b > 0 \Rightarrow ab > 0$

(ii) If $a \in \mathbb{R}$, exactly one of the following three holds:

(a) $a > 0$ (b) $a = 0$ (c) $-a > 0$

The third property is called the Trichotomy property. With these axioms $\mathbb{R}$ becomes ordered field. If a > 0 or a = 0 i.e. $a \geq 0$, we say a is nonnegative. If $-a > 0$ we say a is negative. If $-a > 0$ or a = 0 we say a is nonpositive. When a > 0 we also say that a is strictly positive. "Strictly negative" is also used in a similar way. We say that a > b (in $\mathbb{R}$) if $a - b > 0$. And $a \geq b$ if a > b or a = b. If a > b we also write b < a and say b is less than a. The symbol $\leq$ is used to specify < or =.

The Trichotomy law yields that if $a \in \mathbb{R}$ and $b \in \mathbb{R}$ exactly one of the following three **hold:** (a) a < b (b) a = b (c) b < a

Uniqueness of inverse:

If b_1, b_2 satisfy $a + b_1 = a + b_2 = 0$ then $b_1 = 0 + b_1 = (a + b_2) + b_1 = a + (b_2 + b_1) = a+(b_1+b_2) = (a + b_1) + b_2 = 0 + b_2 = b_2$.

Finally if u_1, u_2 satisfy $a\,u_1 = a\,u_2 = 1$ then $u_1 = 1u_1 = (au_2)\,u_1 = a\,({}_2\,u_1) = a(u_1\,u_2) = (a\underline{u}_1)\,u_2 = 1u_2 = u_2$

The existence of $-a$ corresponding to $a \in \mathbb{R}$, satisfying $a + (-a) = 0$ and of $\frac{1}{a}$ corresponding to $0 \neq a \in \mathbb{R}$ satisfying $a.\frac{1}{a} = 1$ allow us to define binary operations called subtraction on $\mathbb{R}$ and division on $\mathbb{R} \setminus \{0\}$ defined by a – b = a + (-b) and $\frac{a}{b}$ = $a(b^{-1}) = a\left(\frac{1}{b}\right)$ when $b \neq 0$. If a and b are real (rational) then so is a – b. While this does not hold for natural numbers. If $a \in \mathbb{R}$ we define $a^0 = 1$ if $a \neq 0$ and $a^n = (a^{n-1})\,a$ if $n \in \mathbb{N}$. a^n is called the n^{th} power of a. If n is a negative integer then $-n \in \mathbb{N}$ and we define $a^n = (a^{-1})^{-n}$ when $a \neq 0$.

10A 5.2 Theorem: There is no rational number r such that $r^2 = 2$.

Proof: Any nonzero rational number can be uniquely written as $r = \frac{m}{n}$ where $m \in Z$, $n \in \mathbb{N}$ and g.c.d. (m,n) = 1.

If $r = \frac{m}{n}$, $m \in Z$, $n \in \mathbb{N}$ and g.c.d. (m, n) = 1 and $r^2 = 2$ then $m^2 = 2n^2 \Rightarrow m^2$ is even $\Rightarrow$ m is even (why?) If m = 2k where $k \in \mathbb{N}$, $m^2 = 4k^2 = 2n^2 \Rightarrow n^2 = 2k^2 \Rightarrow$ n is even (as above) Since m and n are both even g.c.d. $(m,n) \geq 2$. This is a contraditction. Hence $r^2 \neq 2$ for any rational number r.

10.A.6 Absolute value:

10. A.6.1 Definition: If $a \in \mathbb{R}$, we define the absolute value of a, denoted by |a|, as follows:

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

Some elementary properties of the absolute value:

10.A.6.2 If $a \in \mathbb{R}$, $b \in \mathbb{R}$ the following hold good.

(a) $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$

(b) $|-a| = |a|$

(c) $|ab| = |a|\,|b|$, in particular $|a|^2 = |-a|^2 = a^2$

(d) $|\frac{a}{b}| = \frac{|a|}{|b|}$ if $b \neq 0$

(e) $-|a| \leq a \leq |a|$

(f) $|b| \leq a$ if and only if $-a \leq b \leq b$.

The proofs of the above statements are simple, hence we leave them as exercises.

10.A.6.3 Triangle inequality: If $a \in \mathbb{R}$, $b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$

Proof: If a and b have the same sign, i.e. either both a and b are positive or both of them are negative then a + b, a and b have the same sign so that $|a + b| = \pm (a + b) = (a + b)$ or $(-a - b) = |a| + |b|$

If $a < 0 < b$ then either $a + b < 0$ or $a + b \geq 0$. If $a + b < 0$, $|a+b| = -a - b = |a| - b \leq |a| + b = |a| + |b|$. If $a + b > 0$ $|a + b| = a + b = a + |b| \leq |a| + |b|$. If one of a,b say $a = 0$ then l.h.s. = r.h.s. = $|b|$

10.A.6.4 Corollary: If $a \in \mathbb{R}$ and $b \in \mathbb{R}$ then

(a) $||a| - |b|| \leq |a - b|$ and (b) $|a - b| \leq |a| + |b|$

Proof: (a) $|a| = |(a - b) + b| \leq |a - b| + |b| \Rightarrow |a| - |b| \leq |a - b|$

By symmetry $|b| - |a| \leq |b - a|$. Since $|a - b| = |b - a|$ and $||a| - |b|| = \pm (|a| - |b|)$

We get $||a| - |b|| \leq |a - b|$.

(ii) $|a - b| = |a + (-b)| \leq |a| + |b| = |a| + |b|$

10.A.6.5 Corollary: If $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $1 \leq i \leq n$ then

$|a_1 + \ldots\ldots + a_n| \leq |a_1| + \ldots\ldots\ldots + |a_n|$.

Proof: To prove this inequality we use the principle of mathematical induction.

When n = 1, then lhs = $|a_1|$ = r.h.s.

If n = 2, lhs = $|a + a_2| \leq |a_1| + + a_2)$ = rhs (by 0.6.3)

Assume that for $n \in \mathbb{N}$, $a_1, \ldots\ldots a_n \in \mathbb{R} \Rightarrow |a_1 + \ldots\ldots + a_n| \leq |a_1| + \ldots\ldots + |a_n|$

Let $a_1, \ldots\ldots a_{n+1}$ be real numbers. Then

$|a_1 + a_2 + ….. + a_{n+1}| \quad = |a_1 + (a_2 +……+ a_{n+1})|$

$\leq |a_1| + |a_2 + ….+a_{n+1}|$ by 0.6.3

$\leq |a_1| + |a_2| +….+ |a_{n+1}$

By induction the statement holds for all $n \in \mathbb{N}$

10.A.6.6. Examples:

(a) Determine that set $A = \{ x \mid x \in \mathbb{R}$ and $|2x+3| < 7\}$

$x \in A \Leftrightarrow |2x + 3| < 7 \Leftrightarrow - 7 < 2x + 3 < 7 \Leftrightarrow -5 < x < 2$

Hence $A = \{x \mid x \in \mathbb{R}$ and $-5 < x < 2\}$

(b) Determine the set $B = \{ x/x \in \mathbb{R}$ and $|x -1| < |x|\}$

Method: (i) If $x \geq 1$, $x – 1 \geq 0$ so $|x – 1| = x -1$ and $|x| = x$

Hence $x \in B \Leftrightarrow x – 1 < x$. Thus all $x \geq 1$ belong to B.

If $x \leq 0$ $x – 1 \leq -1 < 0$ and hence $|x – 1 | = 1 –x$ and $|x| = -x$ so that $|x| = -x < 1-x = |x| - 1$

Hence $x \leq 0 \Rightarrow x \notin B$.

Finally if $0 < x < 1$, $x – 1 < 0$ so that $|x – 1| = 1-x < x = |x| \Leftrightarrow 1 < 2x \Leftrightarrow ½ < x < 1$.

Thus $B = \{x\in R \mid ½ < x\}$.

Method (ii): Since $|x – 1| \geq 0$ and $|x| \geq 0$

$|x–1| \leq |x| \quad \Leftrightarrow |x–1|^2 < |x|^2 \Leftrightarrow (x–1)^2 < x^2 \Leftrightarrow x^2–2x + 1 < x^2 \Leftrightarrow 1 < 2x \Leftrightarrow x > ½$

Hence $B = \{ x \in \mathbb{R} \mid ½ < x\}$

(c) If a and b are real numbers then $a \leq b$ if and only if $a < b + \in$ for every $\in > 0$.

$a \leq b \quad \Rightarrow a – b > 0$

$\Rightarrow a – b < \in$ for every $\in > 0$

$\Rightarrow a < b + \in$ for every $\in > 0$.

$a > b \quad \Rightarrow a – b > 0$

$$\Rightarrow \frac{a-b}{2} > o$$

$$\Rightarrow b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a$$

In this case, when $\in = \frac{a-b}{2}$, b + $\in$ < a

Thus a $\leq$ b if and only if a < b + $\in$ for every $\in$ > 0

10.A.6.7 The Real line:

It is customary and convenient to represent the real number system on a straight line. We first a line, point 0 on the line, call it the origin. For any points x, x^1 on the line, x on the "right side" of 0 and x^1 on the left side we call the "ray" 0 x the positive half ray and the ray o x^1, the negative half ray. The line x^1 0 x is called a directed line. We first a point A, on he right half ray call the length $0A_1$ the unit length. The points A_2, A_3 … A_n … are fixed on the half ray 0x represents the natural number n. like wise we fix A_{-1} A_{-2}, A_{-3}, … on the negative half ray so that $0A_1 = 0A_{-1} = A_{-i-1}\ A_{-1}$. $\forall$ i These points represent the negative integers. If m $\in$ N, n $\in$ N and P lies on the right half ray such that n(op) = m(oA_1) = $0A_n$ then the point P represents the positive rational number $\frac{m}{n}$. It can be proved that for given m, n such P is uniquely fixed. Thus points represent positive rational numbers. Analogously the negative rational numbers are represented by points.

However the set Q of rational numbers is countable while any line has uncountably many points. The "gaps" represent irrational numbers. As the line has uncountably many points and the rationals are represented by countably many points the points left out are again uncountable. Each such point represents an irrational number and every irrational number corresponds to one and only one such point. From 0.5.2 there is a unique point in the "gap" which represents $\sqrt{2}$. Other irrational numbers that are frequently used are "e" and "π".

It is interesting to establish that the points on the real line representing the rationals and irrationals are not two "separate" subsets of $\mathbb{R}$ and are interlaced among them there by substantiating the fact that between any two real numbers there are infinitely many irrational numbers which is nothing but the denseness of rationals and irrationals in $\mathbb{R}$.

10.A.7. Bounded Sets:

10.A.7.1. Definitions: Anonempty subset A of $\mathbb{R}$ is said to be

(a) bounded above if there is $x \in \mathbb{R}$ such that $a \leq x$ for every $x \in A$

(b) bounded fellow if there is $y \in \mathbb{R}$ such that $y \leq a$ for every $a \in A$

(c) Bounded if A is bounded above and bounded fellow.

(d) $x \in \mathbb{R}$ is called an upper bound of A if $a \leq x$ for every $a \in A$.

(e) $y \in \mathbb{R}$ is called a lower bound of A if $y \leq a$ for every $a \in A$.

(f) $\mu \in \mathbb{R}$ is called a supremum (also called least upper bound) of A if

(i) μ is an upper bound of A and

(ii) $\mu \leq x$ if x is any upper bound of A; equivalently

(iii) if $x < \mu$ then x is not an upper found of A.

μ is denoted by $\mu = \sup A = \sup \{a \mid a \in A\}$ and also by μ = l.u.b.A. = lub $\{ a \mid a \in A\}$

(g) $\sigma \in \mathbb{R}$ is called an infimum (also called greatest lower found)of A if

(i) σ is a lower bound of A and

(ii) $y \leq \sigma$ if y is any lower bound of A; equivalently

(iii) if $\sigma < y$ then y is not lower found of A.

σ is denoted by $\sigma = \inf A = \inf \{a \mid a \in A\}$

and also by σ = g.l.b.A. = glb $\{a \mid a \in A\}$

10.A.7.2. completeness property of $\mathbb{R}$:

Every nonempty subset S of $\mathbb{R}$ which is bounded above has supremum. This property is also called supremum (l.u.b.) property of $\mathbb{R}$. We assume this without proof which is beyond the scope of this book. With this complete ness property, $\mathbb{R}$ becomes a "complete ordered field". The following elementary properties of bounds are stated as short answer questions.

10.A.7.3. SAQ: Let A and B be nonempty subsets of $\mathbb{R}$.

(i) If α and β are lub's of A then $\alpha = \beta$.

(ii) If α and β are glb's of B then $\alpha = \beta$.

(iii) If $-A = \{-a \mid a \in A\}$ then α is an upper bound of A if and only if $-\alpha$ is a lower bound of $-A$; hence lub $(-A)$ = glb (A)

(iv) If $a \in \mathbb{R}$, lub $(A + a)$ = (lub A) + a where $A + a = \{x + a \mid x \in A\}$

(v) If $a \in \mathbb{R}$ glb $(A + a)$ = (glb A) + a

(vi) Let B be bounded below and A be the set of all lower bounds of A. Then $a \in A, b \in B \Rightarrow a \le b$.

(vii) If A and B are subsets of $\mathbb{R}$ such that $a \in A, b \in B \Rightarrow a \le b$ then sup A $\le$ inf B.

(viii) $\phi \ne A \subseteq \mathbb{R}$ is bounded if and only if there is $M > 0$ such that $|a| \le M$ for every $a \in A$.

(ix) If M = sup A and m = inf A then $M - m = \sup\{\,|x - y : 1|\ x \in A, y \in A\}$

10.A.7.4. Theorem: The following are equivalent.

(a) : Every nonempty subset of $\mathbb{R}$ which is bounded above has l.u.b.

(b) : Every nonempty subset of $\mathbb{R}$ which is bounded below has g.l.b.

Proof: $(a \Rightarrow b)$: Let B be a nonempty subset of $\mathbb{R}$ which is bounded below and A be the set of lower bounds of B. Then A is nonempty. $a \in A$ and $b \in B \Rightarrow a \le b$.
$\Rightarrow$ A is bounded above $\Rightarrow$ A, has l.u.b say α (by (a)) $\Rightarrow \alpha \le b\ \forall\ b \in B$. $\Rightarrow \alpha$ is a lower bound of B. also if a is lower bound of B, $\alpha \le a$. $\Rightarrow \alpha$ = inf B. $\Rightarrow$ B has glb.

$(b \Rightarrow a)$: Let A be nonempty subset of $\mathbb{R}$ which is bounded above then the set B of all upper bounds of A is nonempty and satisfies.
$a \in A, b \in B \Rightarrow a \le b$. $\Rightarrow$ B is bounded below $\Rightarrow$ B has lub say β (by b) $\Rightarrow b \le \beta\ \forall\ b \in B$.
$\Rightarrow \beta$ is an upper bound of B. $b \le \beta \Rightarrow \beta$ =sup A. $\Rightarrow$ A has l.u.b.

10.A.7.5 The Archimedean Property: If $x \in \mathbb{R}$, there is a rational number n_x such that $n_x > x$.

Proof: We prove this by contradiction. Suppose the assertion were false. Then if $n \in \mathbb{N}$, $n \le x$.

$\Rightarrow \mathbb{N}$ is bounded above and x is an upper bound of $\mathbb{N}$.

$\Rightarrow \mathbb{N}$ has l.u.b (by completeness property of $\mathbb{R}$) say u.

$\Rightarrow$ u – 1, being less than u, is not an upper bound of $\mathbb{N}$.

$\Rightarrow u - 1 < m$ for some $m \in \mathbb{N}$.

$\Rightarrow u < m + 1$ and $m + 1 \in \mathbb{N}$.

This contradicts the assumption that u is an upper bound of $\mathbb{N}$. Thus our assumption that there is no n in $\mathbb{N}$ with n < x is false and hence there exists $n_x \in \mathbb{N}$ with $n_x > x$.

Collary: If $S = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ then inf S = 0

Proof: If $n \in \mathbb{N}$, $\frac{1}{n} > 0$. So 0 is a lower bound of S.

Since S is bounded below, by the completeness property of $\mathbb{R}$, S has g.l.b., say w. clearly $w \geq 0$. If $\in > 0$, by the Archimedean property, there is $n_{\in} \in \mathbb{N}$ such that $n_{\in} > \frac{1}{\in}$. This implies that $0 < \frac{1}{n_{\in}} < \in$. Hence $\in$ is not a lower bound S. This is true for every $\in > 0$. So g.l.b. S = 0.

10.A7.6 SAQ: **i)** If t > 0 there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < t$

ii) If y > 0 there exists $n \in \mathbb{N}$ such that $n - 1 \leq y < n$.

10.A7.7 Existence of $\sqrt{2}$ in $\mathbb{R}$

There is a positive real number r such that $m^2 = 2$.

Proof: Let $S = \{s \in \mathbb{R} \mid s \geq 0 \text{ and } s^2 < 2\}$. Since $1 = 1^2 < 2$, $1 \in S$. Moreover $s \in S \Rightarrow s \geq 0$ and $s^2 < 2 < 4 = 2^2$ so that $0 \leq s < 2$. Thus S is a nonempty subset of $\mathbb{R}$ which is bounded above. By the completeness property, S has l.u.b. Let x = sup S. Clearly x > 1. We prove that $x^2 = 2$ by showing that neither $x^2 < 2$ nor $x^2 > 2$. First assume that $x^2 < 2$.

We show that for some $n \in \mathbb{N}$, $(x + \frac{1}{n})^2 < 2$. Since $x + \frac{1}{n} \in S$ contrary to the assumption that x is an upper bound of S.

Since $x^2 < 2$, $2 - x^2 > 0$ since $x > 0$, $\frac{2-x^2}{2x+1} > 0$.

By the Archimedean property there is $n \in \mathbb{N}$ such that $n > \frac{2x+1}{2-x^2}$

For this n, $\frac{1}{n} < \frac{2-x^2}{2x+1} \Rightarrow \frac{(2x+1)}{n} < 2 - x^2 \Rightarrow x^2 + \frac{2}{n}x + \frac{1}{n} < 2$

$\Rightarrow x^2 + 2.\frac{1}{n}.x + \frac{1}{n} < 2$

$\Rightarrow \left(x + \frac{1}{n}\right)^2 < 2 \Rightarrow x + \frac{1}{n} \in S$ which is required to be shown. Hence $x^2 < 2$ is not possible.

Now assume that $x^2 > 2$. we show that for some $m \in N$, $x - \frac{1}{m}$ is an upper bound of S contrary to the assumption that x is sup S.

Since $x^2 > 2$, $\frac{x^2 - 2}{2x} > 0$. By the Archimedean property there exists $m \in N$ such that $\frac{x^2 - 2}{2x} > \frac{1}{m}$

Since $2x^2 > x^2 > x^2 - 2$, $x > \frac{x^2 - 2}{2x} > \frac{1}{m}$ so $x - \frac{1}{m} > 0$. Also $\left(x - \frac{1}{m}\right)^2 > 2 > s^2$ for every $s \in S$. Since s and $x - \frac{1}{m}$ are positive we get $x - \frac{1}{m} > s$ for every $s \in S$. This implies that $x - \frac{1}{m}$ is an upper bound of S.

Note: The number $\sqrt{2}$ is irrational. A "series representation" for $\sqrt{2}$ is given by

$$\sqrt{2} = 1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{3}{10^6} + \frac{5}{10^7} + \frac{6}{10^8} + -----$$

which is represented as an infinite decimal: $\sqrt{2} \sim 1.4\,1\,4\,2\,1\,3\,5\,6$ ………..

The series representation for π is

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \ldots\ldots\ldots$$

$\sim 3.\ 1\ 4\ 1\ 5\ 9\ 2\ 6 \ldots\ldots..$

These representations are “non terminating”.

The irrational number e has series representation given by $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots.$

which is non terminating. It is known that $2 < e < 3$.

10A.7.8 Density of rational numbers in $\mathbb{R}$:

If x and y are any real numbers with $x < y$ then there exists a rational number r with $x < r < y$.

Proof: (i) Assume that $x > 0$. since $y - x > 0$ there is $n \in N$ such that

$n - 1 < \frac{1}{y-x} < n$ so that $0 < \frac{1}{n} < y - x \Rightarrow 1 < n\,(y - x) \Rightarrow n\,x + 1 < n\,y$.

Again since $n\,x>0$. there is $m \in N$ with $m-1 \le n\,x<m. \Rightarrow n\,x<m<n\,y \Rightarrow x<\frac{m}{n}<y$.

we take $r = \frac{m}{n}$

(ii) If $x = 0$ since $0 < y$, $0 < \frac{y}{2} < y$. By (i) there is a rational number r such that $\frac{y}{2} < r < y$. This r satisfies $0 = x < \frac{y}{2} < r < y$ so that $x < r < y$.

(iii) If $x < 0 < y$, by (ii) there is a rational number r such that $0 < r < y$. This r satisfies $x < r < y$.

(iv) If $x < y < 0$ then $0 < -y < -x$. By (i) there is rational number r_1 such that $-y < r_1 < -x$.

If $r = -r_1$, r is a rational number satisfying $a < r < y$.

10.A.8.1 Bounded functions:

Any function $f : A \to \mathbb{R}$ is called a real valued function defined on A. if f is a real valued function defined on A, f is said to be bounded above or bounded below according as the range $f\,(A) = \{ f(x) \mid x \in A \}$ is bounded above or bounded below. If f is bounded above and bounded below then f is said to be bounded.

We write lub f, lub $\{f(x) \mid x \in A\}$ or $\underset{x\in A}{\text{lub}}\, f(x)$ for lub $f(A)$. We also sup for lub.

Similarly we write $\underset{A}{\text{glb}}\, f$, glb $\{f(x) \mid x\in A\}$ or $\underset{x\in A}{\text{glb}}\, f(x)$ for glb $f(A)$ and also use inf for glb. As a consequence of SAQ 07.3 (viii) we have the following.

10A.8.2 Theorem: $f\colon A \to \mathbb{R}$ is bound if and only if there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in A$.

10.A.8.3. Theorem: If $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ are such that

(i) for every $x \in A$ and $y \in A$, $f(x) \le g(y)$ then f is bounded above, g is bounded below and $\sup\limits_{x\in A} f(x) \le \inf\limits_{y\in A} g(y)$

(ii) for every $x \in A$ $f(x) \le g(x)$ then $\sup\limits_{x\in A} f(x) \le \sup\limits_{x\in A} g(x)$

Proof: (i) is a consequence of 0.73 (vii) To prove (ii) Let $\sup\limits_{x\in A} g(x) = \alpha$. Then if $x \in A$, $f(x) \le g(x) \le \alpha$ so that $f(x) \le \alpha$. Hence α is an upper bound of $\{f(x) \mid x \in A\}$ Hence $\sup \{f(x) \mid x \in A\} \le \alpha$.

10.A9. Intervals:

Extended real number system: We adjoin the two symbols $-\infty$ and ∞ to $\mathbb{R}$ and call the set $\mathbb{R} \cup \{-\infty, \infty\}$, The extended real number system we write $-\infty < x < \infty$ for all $x \in \mathbb{R}$
$x + \infty = \infty$, $x - \infty = -\infty$ for all $x \in \mathbb{R}$ and $x \cdot \infty = -\infty$ if $x > 0$ and $-\infty$ if $x < 0$

Definition: If $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a < b$ The open interval $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$
The closed interval $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$
The half open (half closed) intervals determined by a, b are
$(a,b] = \{x \in \mathbb{R} \mid a < x \le b\}$ $[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$
For the above intervals a is called the left end point and b is called the right end point.
The length of an interval with end points a,b where $a < b$ is defined to be $b - a$.
If $a \in \mathbb{R}$ we write
$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$

These are called infinite half open intervals.

$[a, \infty) = \{ x \in \mathbb{R} \mid x \geq a\}$ and

$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$ are called infinite half closed intervals

We also write $\mathbb{R} = (-\infty, \infty)$

Bounded intervals:

An interval which is a bounded set is called a bounded interval. Thus ϕ, [a,b], [a,b), (a,b), (a,b] are bounded intervals.

An interval with $\pm \infty$ as an end point is an unbounded interval.

Example: $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$

10.A.9.2 Characterisation Theorem for intervals:

If S is a subset of $\mathbb{R}$ that contains at least two elements is an interval If $x \in S$, $y \in S$ and $x < y$ then $[x, y] \subseteq S$ (∗) then S is an interval.

Proof: There are four possibilities for S.

(i) S is bounded (ii) S is bounded above but not below. (iii) S is bounded below but not above and (iv) S is not bounded above and not bounded below.

(i) Suppose S is bounded, a = inf S and b = sups.

In this case we show that S is an interval with left end point a and right end point b. Clearly $x \in S \Rightarrow a \leq x \leq b \Rightarrow S \leq [a, b]$. We show that $(a, b) \leq S$. Let $a < c < b$. Then by the infimum property. There is $x \in S$ such that $a < x < c$ and by the supremum property there is y in S such that $c < y < b$. By (*) $[x,y] \subseteq S$. Since $c \in [x, y]$, $c \in S$. This shows that $(a,b) \subseteq S \subseteq [a, b]$. If $a \in S$, $b \in S$, $[a,b] = S$. If $a \in S$, $b \notin S$ $[a,b] = S$. If $a \notin S$, $b \in S$ $(a, b] = S$. and if $a \notin S$, $b \notin S$, $(a,b) = S$

(ii) Suppose S is bounded above but unbounded below.

We show that $S = (-\infty, b)$ or $(-\infty, b]$ where b = supS. Clearly $(-\infty\ b) \subseteq S$ because if $x < b$ by the supremum property x is not an upper bound of S so that there is $d \in S$ such that $x<d$

Since S is unbounded below x is not a lower bound for S so that there is $c \in S$ such that $c<x$. Since $c \in S$, $d \in S$, by (*) $[c,d] \subseteq S$ hence $x \in S$. Also $S \subseteq (-\infty, b]$ because $x \in S \Rightarrow x \leq b$. Thus $(-\infty, b) \leq S \subseteq (-\infty, b)$. If $b \in S$, $S = (-\infty, b]$ and if $b \notin S$, $S = (-\infty, b)$.

(iii) Suppose S is bounded below and unbounded above. We show that $S = (a, \infty)$ or $[a, \infty)$ where $a = \inf S$. Clearly $x \in S \Rightarrow a \leq x$, hence $S \subseteq [a, \infty)$

If $x \in (a, \infty)$, $a < x$, so x is not a lower bound of S. Hence there is $c \in S$ such that $c < x$. Since S is not bounded above there is $d \in S$ such that $x < d$. Since $c \in S$, $d \in S$ and $c < d$, $[c, d] \subseteq S$ by (*) This implies that $x \in S$ whenever $x \in (a, \infty)$. Thus $(a, \infty) \subseteq S \subseteq [a, \infty)$

If $a \in S$ $S = [a, \infty)$ and otherwise $S = (a, \infty)$

(iv) The proof for $S = (-\infty, \infty)$ when S is not bounded above and not bounded below is similar.

10.A.10 Nested Intervals:

10.A.10.1 We say that a sequence of intervals $\{I_n\}$ is NESTED

If $I_n \supseteq I_{n+1}$ for very $n \in \mathbb{N}$, i.e. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\ldots\ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots\ldots\ldots\ldots\ldots$

10.A.10.2 Theorem: If $\{I_n\}$ is a sequence of closed and bounded intervals which are nested then there exists $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for every n.

Proof: Let $I_n = [a_n, b_n]$. Since $I_n \subseteq I_1$ $a_n \in I_1$ and $b_n \in I_1$ for all $n \in \mathbb{N}$ so that

$a_1 \leq a_n \leq b_n \leq b_1$ if $n \in \mathbb{N}$. If $k \in \mathbb{N}$, then $I_{n+k} \leq I_n$ so we get as above

$a_1 \leq a_n \leq a_{n+k} \leq b_{n+k} \leq b_n \leq b_1$ if $n \in \mathbb{N}$.

The set $\{a_n | n \in \mathbb{N}\}$ is bounded above. Let $\xi = \sup \{a_n \mid n \in \mathbb{N}\}$

Clearly $a_n \leq \xi$ for $n \in \mathbb{N}$. We show that every b_n is an upper bound of the set

$\{a_k \mid k \in \mathbb{N}\}$ so that $\xi \leq b_n$ for every n.

If $k < n$ $I_k \supseteq I_n \Rightarrow a_k \leq a_n \leq b_n \leq b_k \Rightarrow a_k \leq b_n$

If $n \leq k$ $I_n \supseteq I_k \Rightarrow a_n \leq a_k \leq b_k \leq b_n \Rightarrow a_k \leq b_n$

In either case $a_k \le b_n$ for every k. This is what we are required to prove since $a_n \le \xi \le b_n$ for all n, $\xi \in [a_n, b_n] = I_n$ for all $n \in \mathbb{N}$. This completes the proof.

10.A.10.3 Theorem: If $\{I_n \mid n \in \mathbb{N}\}$ is a nested sequence of closed and bounded intervals, $I_n = [a_n, b_n]$ such that

$\inf \{b_n - a_n \mid n \in \mathbb{N}\} = 0$ then the ξ that belongs to every I_n is unique.

Proof: Suppose $\xi \in I_n$ for all $n \in \mathbb{N}$ and $n \in I_n$ for all $n \in \mathbb{N}$.

We show that $\xi = n$ by proving that $\xi < n$ $\eta < \xi$ is not possible. Suppose $\xi < \eta$.

Then $a_n \le \xi < m \le b_n$ for all $n \in \mathbb{N}$.

$\Rightarrow 0 < \eta - \xi \le b_n - a_n$ for all $n \in \mathbb{N}$.

$\Rightarrow \eta - \xi$ is a lower bound of $\{b_n - a_n \mid n \in \mathbb{N}\}$

$\Rightarrow \eta - \xi \le 0 = \inf \{b_n - a_n \mid n \in \mathbb{N}\}$

This contradicts the assumption that $\xi < \eta$.

So $\xi < \eta$ is not possible. Similarly $\eta < \xi$ is not possible.

10.A.10.4 Theorem: The set $\mathbb{R}$ of real numbers is not countable.

Proof: Our proof is based on the fact that a subset of a countable set is countable. Thus if $\mathbb{R}$ were countable then the set I = [0,1] would be countable. We can then enumerate the set as say $I = \{x_1, x_2, \dots x_n \dots.\}$. First, choose a closed subinterval I_1 of I such that $x_1 \notin I_1$. Then choose a closed subinterval I_2 of I_1 such that $x_2 \notin I_2$. Inductively, assuming that I_1, I_2, ….. I_n and $x_r \notin I_r$ for $1 \le r \le n$ choose a closed and bounded sub interval I_{n+1} of I_n such that $x_{n+1} \notin I_{n+1}$. Thus $\{I_n\}$ is a nested sequence of closed and bounded intervals. So there is a $\xi \in I$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$. this implies that $\xi = x_m$ for some m but $x_m \notin I_m$. This is a contradiction. Hence I is not countable. This implies $\mathbb{R}$ is not countable.

10.A.10.5 Some elementary inequalities:

If $0 < b < a$, $n \in \mathbb{N}$, $n > 1$ then $n\,a^{n-1}(a-b) > a^n - b^n > n\,b^{n-1}(a-b)$

If 0 M a and $n \in \mathbb{N}$ there is one and only one $x > 0$ Such that $x^n = a$. this x is given by

$x = \text{l.u.b.}\ \{y \in \mathbb{R} \mid y > 0,\ y^n \text{ M } a\}$

This unique $x > 0$ such that $x^n = a$ where $a > 0$ is called the n^{th} root of a and is denoted by

$x = a^{\frac{1}{n}} = \sqrt[n]{a}$

If $a > 0$, $\dfrac{m}{n} = \dfrac{p}{q}$ where m and p are integer and $n, q \in \mathbb{N}$ then $(a^m)^{\frac{1}{n}} = (a^p)^{\frac{1}{n}}$

This is well defined as stated above and is also denoted by $a^{\frac{m}{n}}$ or $\sqrt[n]{a^m}$.

If $a_1 \ldots. a_n$ are positive real numbers then AM $(a_1, \ldots. a_n) \geq$ GM $(a_1, \ldots. a_n)$:

$\dfrac{a_1 + \ldots\ldots + a_n}{n} \geq (a_1 + \ldots. + a_n)^{\frac{1}{n}}$

If $a > 1$ and $x \in \mathbb{R}$

l.u.b. $\{a^r / r \in Q, r < x\}$ = g.l.b $\{a^r \mid r \in Q, r > x\}$

This equal value is called a to the power of x and is denoted by a^x.

Remark: The above two are equal when $x \in Q$.

Bernouli's inequality: $(1 + x)^n \geq 1 + n\,x$ if $x > -1$ and $n \in N$.

If $a > 0$, $b > 0$ and $a \neq 1$ there is unique x such that $a^x = b$.

This number x such that $a^x = b$ is called the logarithm of b to the base a and is written

$x = \log_a b$.

$\text{Log}_{10}\ b$ is denoted by Log b.

There is a unique real number $e > 0$ such that

$\dfrac{e^x - 1}{x} > 1 > \dfrac{e^{-x} - 1}{-x}$ for all $x \in \mathbb{R}$, $x > 0$.

This e is called the Napier base. We use the fact that $e^x > x$ if $x > 0$

Logarithms with the base e are called natural logarithms we write $l_n\, x$ for $\log_e x$.

10.A. 12 solutions to SAQs

10.A.4.1. $(x, y) \in GoF$, $(x, z) \in GoF$

$\Rightarrow y = G(F(x)$ and $z = G(F(x) \Rightarrow x = z$.

$x \in A \Rightarrow (x, F(x)) \in F \Rightarrow F(x) \in B \Rightarrow (F(x), G(F(x)) \in G \Rightarrow G(F(x) \in C$.

10.A.4.2. $y_1, \in B$, $x_1, x_2 \in A$; $(y_1\ x_1) \in F^{-1}$, $(y_1\ x_2) \in F^{-1} \Rightarrow (x_1\ y_1) \in F$, $(x_2, y_1) \in F$

$\Rightarrow y_1 = F(x_1) = F(x_2) \Rightarrow x_1 = x_2$ ($\because$ F us $1 - 1$)

Also $y_1 \in B \Rightarrow$ for some $x_1 \in A$, $(x_1\ y_1) \in F$

$\Rightarrow (y_1, x_1) \in F^{-1}$

since $\forall\ y_1 \in B$ there is a unique $x_1 \in A$. such that $(y_1\ x_1) \in F^{-1}$

This shows that $F^{-1}: B \rightarrow A$ is a function. F^{-1} is one-one since $F^{-1}(y_1) = F^{-1}(y_2) = x_1$

$\Rightarrow (y_1, x_1) \in F^{-1}$, $(y_2, x_1) \in F^{-1} \Rightarrow (x_1\ y_1) \in F$ and $(x_1\ y_2) \in F \Rightarrow y_1 = y_2$ since F is a function.

F^{-1} is on to since $x \in A$ and $F(x) = y$, $(y,x) \in F^{-1}$

$\Rightarrow y = F^{-1}(x)$

10.A.4.3 $I = \{(x, x) / x \in A\} = \{(x, y) \mid x \in A, y \in A, x = y\}$ $x \in A$ & $(x, y) \in I \Rightarrow y = x$. Hence I is a function. $(y, x) = (z, x) \in I \Rightarrow y = z$ so I is one- one. If $x \in A$ $(x, x) \in I \Rightarrow$ I is onto

10.A.4.4 By definition $F = \{x, y) \mid x \in A, y \in B$ and $y = F(x)\}$

$G = \{y, z) \mid y \in B, z \in C$ and $z = G(y)\}$

and $GoF = \{x, z) \mid x \in A, z \in C$ and $y = G(F(x)\}$

Then $x \in A \Rightarrow (x, G(F(x))) \in GoF$

Moreover $(x, z) \in GoF$, $(x, z^1) \in GoF$

$\Rightarrow z = G(F(x))$ and $z^1 = G(F(x)) \Rightarrow z = z^1$

Thus z in C is uniquely fixed so that $(x, z) \in GoF$. Hence $GoF : A \rightarrow C$ is a function

$(GoF)(x) = (GoF)(y) \Rightarrow G(F(x) = G(F(y))$

$\Rightarrow F(x) = F(y)$ since G is one – one $\Rightarrow x = y$ since F is one – one

So GoF is one.

If z $\in$ c, there is y $\in$ B such that z = G(y) and there is x $\in$ A such that y = F(x) so z = G (F(x). Hence GOF is onto.

10.A.4.5 (i): When n = 1, L.H.S = 1 = RHS. Suppose the statement is true for n = k. i e

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$

$$\sum_{j=1}^{k+1} j = (\sum_{j=1}^{k} j) + (k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)\,(\frac{k}{2}+1) = \frac{(k+1)\,(k+2)}{2}$$

Since the statement is true for k+1 whenever it is true for k and also for 1. The statement is valid for all n.

(ii) When n = 1, $\sum_{k=1}^{n} k^2 = 1 = \frac{n(n+1)(2n+1)}{6} = \frac{1.2.3}{6} = 1.$

The result holds for n = 1.

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\,\{\frac{(n)\,(2n+1)}{6} + (n+1)\}$$

$$= (n+1)\,\{\ \frac{n(2n+1)+6n+6}{6} = \frac{(n+1)\,(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)\,(n+2)\,(2n+3)}{6}$$

By the principle of mathematical induction the statement holds for all n. this completes the proof.

10.A.7.3:

(i) α = lub A and β upper bound $\Rightarrow \alpha \le \beta$. β = lub A and α upper bound $\Rightarrow \beta \le \alpha$.

$\Rightarrow \alpha \le \beta \le \alpha \Rightarrow \alpha = \beta$

(ii) similar to (i)

(iii) If a $\in$ A a $\le \alpha \Leftrightarrow -\alpha \le -a$

Hence $-\alpha$ is a lower bound of –A if α is an upper bound of –A.

If β = glb A, β is a lower bound of A. So - β is and upper bound of - A.

Further if r is an upper bound of $-A$, $-a \le r\ \forall\ a \in A \Rightarrow -r \le a\ \ \forall\ a \in A$.

$\Rightarrow -r$ is a lower bound of $A \Rightarrow -r \le \beta \Rightarrow -\beta \le r$. Hence $-\beta =$ lub $-A$

(iv) β is an upper bound of A.

$\Leftrightarrow x \le \beta\ \forall\ x \in A \Leftrightarrow x + a \le \beta + + a\ \forall\ x \in A \Leftrightarrow \beta + a$ is a n upper bound of AS+a

Hence (lub A) + a is an upper bound of A + a $\longrightarrow$ (a)

If α is an upper bound of A+a, $x \le \alpha + a\ \forall\ x \in A \Rightarrow x - a \le \alpha\ \forall\ x \in A \Rightarrow x - a \le$ lub A

$\Rightarrow x \le$ (lub A) + a $\longrightarrow$ (b)

Hence by (a) and (b) lub (A + a) = (lub A) +a

(v) similar to (iv)

(vi) Let B be bounded below and A be the set of all lower bounds of B. Then $a \in A$, $b \in B$.

$\Rightarrow a \le b$. If $a \in A$, a is a lower bound of B. $\Rightarrow a \le b\ \forall, b \in B \Rightarrow a \le b$. This is true for every $a \in A$. Hence $a \in A, b \in B \Rightarrow a \le b$.

(vii) Let A and B be nonempty subjects of $\mathbb{R}$ such that $a \in A, b \in B \Rightarrow a \le b$.

Then A is bounded above and every b in B is an upper bound of A. If $b \in B$, b is an upper bound of A. Hence $\alpha \le b$. Thus α is a lower bound of B. Again by the completeness axiom B has glb in $\mathbb{R}$ and $\alpha \le$ glb. Thus sup A $\le$ inf B.

(viii) Suppose A is nonempty, $A \subseteq \mathbb{R}$ and A is bounded. Then there exist $m_o \in \mathbb{R}$ and $M_0 \in \mathbb{R}$ such that $m_0 \le a \le M_0$ for all $a \in A$.

Let $M = + |m_0| + |M_0|$ Then $-M = -|m_0| + |M_0| \le |m_0| \le m_0 \le M_0 \le |M_0| \le |M_0| + |m_0| = M \Rightarrow -M \le a \le M$ if $a \in A \Rightarrow |a| \le M$ if $a \in A$.

conversely if there is $M > 0$ such that $|a| \le M$ for all $a \in A$, $-M \le a \le M$ for all $a \in A$.

$\Rightarrow$A is bounded above with an upper bound M and bounded below with lower bound $-M$.

Hence A is bounded.

(ix) Let $\phi \ne A \subseteq \mathbb{R}$, M = lub A and m = glb A.

$\Rightarrow m \le x \le M$ and $m \le y \le M$ if $x \in A$ and $y \in B \Rightarrow +m - M \le x - y \le M - m$ if $x \in A$ and $y \in B$. $\Rightarrow m - M \le y - x \le M - m$ if $x \in A$ and $y \in B$. $\Rightarrow |x - y| \le M - m$ if $x \in A$ and $y \in B$

If $0 < \in < M - m$ there is $x_1 \in A$ such that $M - \frac{\in}{2} < x_1$ and there is $y_1 \in A$ such that $m + \frac{\in}{2} > y_1$ Since $x_1 \in [m + \frac{\in}{2}, M - \frac{\in}{2}]$ and $y_1 \in [m+\frac{\in}{2}, M - \frac{\in}{2}]$ $|x_1-y_1| \le M-m - \in$. Hence $M - m - \in$ is not an upper bound of the set $\{ |x - y| / x \in A, y \in A\}$

Hence lub $\{ |x-y| / x \in A, y \in A\} = M - m$.

10.A.7.6:

(i) Solution: For every $t > 0$. $\frac{1}{t} > 0$ and $t. \frac{1}{t} = 1$ By Archimedean property $\therefore$ there is $n \in N$ Such that $n > \frac{1}{t} \Rightarrow 0 < \frac{1}{n} < t$

(ii) Solution: Given $y > 0$. Lt $S = \{m : m \in N, m > y\}$By Archimedean property for $y>0$ there exists $m \in N$ such that $m > y$.

Therefore $s \ne \phi$ and S is a subset of N. By Well ordering principle, S has a least member Let $n \in N$ be the least member of S. $n \in S \Rightarrow n > y$.

If $n = 1$ then $1 > y$ and $y > 0 \Rightarrow 0 < y < 1$ i.e. $n - 1 < y < n$

If $n \ne 1$ then $n - 1 \notin S$ and $n - 1 \in N$. $N - 1 \notin S \Rightarrow n - 1 \not> y \Rightarrow n - 1 \le y \Rightarrow n - 1 \le y < n$

10.A.13 Exercises:

1. $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$
2. If $A \subset x$ and $B \subset x$ show that

   (a) $(x \setminus A) \cap (x \setminus B) = x \setminus (A \cup B)$ and

   (b) $(x \setminus A) \cup (x \setminus B) = x \setminus (A \cap B)$

   These are called De Morgan's laws

3. Prove the distributive laws

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. The symmetric difference of sets A and B is defined by $A \Delta B = (A - B) \cup B - A$.

   $A \Delta B = (A \setminus B) \cup (B \setminus A)$ show that

   (i) $A \Delta B = (A \cup B) \setminus (A \cap B)$

   (ii) $A \Delta A = \phi$.

   (iii) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$ and

   (iv) $A \Delta B = B \Delta A$

5. If for each $n \in N$ $A_n = \{ (n + 1) k \mid k \in N\}$show that $A_1 \cap A_2 = A_5$

6. If f: $\mathbb{R} \setminus \{0\} \to \mathbb{R}$ is defined by

   $f(x) = \dfrac{1}{x^2}$, show that $f[-1,1] \setminus \{0\} = [1,a] = f((0,1])$.

7. Let $g(x) = x^2$ and $f(x) = x + 2$ for $x \in R$.

   show that gof = h and if E = [0,1] then h(E) = [4, 9]

8. If f: $A \to B$, $E \subseteq A$, $F \subseteq A$, $C \subseteq B$ and $H \subseteq B$ then show that

   (a) $f(E \cap F) \subseteq f(E) \cap f(F)$

   (b) $f(E \setminus F) \subseteq f(E) \setminus f(F)$

   (c) $f(E \cup F) \subseteq f(E) \cup f(F)$

   (d) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$

   (e) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

9. Let $f(x) = x^2$ $(x \in \mathbb{R})$

   $E = [-1,0]$, $F = [0,1]$ show that $f(E) = f(F) = [0,1]$ and $f(E \cap F) \neq [0,1]$

10. (a) Show that $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ is one-one form $\mathbb{R}$ into $\mathbb{R}$

    (b) Show that $f(\mathbb{R}) = (-1,1)$

11. Using the principle of mathematical induction prove the following

    (a) $\dfrac{1}{1.2} + \dfrac{1}{2.3} + ... + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$ for $n \in N$

(b) $1^3 + 2^3 + \ldots.. + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n \in N$

(c) $3 = 11 + \ldots. + (8n\text{-}5) = 4n^2 - n$ for $n \in N$

(d) $1^2 + 3^2 + \ldots + (2n\text{-}1)^2 = \frac{4n^3 - n}{3}$ for $n \in N$

(e) 3 divides $4n^3$- n for $n \in N$

(f) 6 divides $n^3 + 5n$ for $n \in N$

(g) 5^{2n}-1 is divisible by 8 for $n \in N$

10. Using principle of Mathematical induction prove the following

(a) $2^n > 3$ for all $n \in N$

(b) $2^n < n!$ for all $n \in N, n \geq 4$

(c) $2n - 3 \leq 2^{n-2}$ for all $n \geq 5, n \in N$

11. If S is any set and P(S) stands for the power set of S consisting of all subsets of S then show that P(S) has 2^n elements whenever S has n elements

(Hint: Use principle of Mathematical induction.)

12. If S and T are denumerable then show that $S \cup T$ is denumerable.

13. Using Mathematical induction prove Bernouli's inequality: If x>-1 then $(1+x)^n > 1+nx$

for all $n \in N$

14. If $0 < c < 1$ show that for $n \in N$ $c^n \geq c^{n+1} < c$.

15. If $c > 1$ show that $c^n \geq c$ for $n \in N$ and $c^{n+1} > c$.

16. Show that if $c > 1$ and $m \in N$, $n \in N$ show that $c^m > c^n$ whenever $m > n$

17. If $0 < c < 1$ and $m \in N$, $n \in N$ show that $c^m < c^n$ whenever $m > n$

18. Show that $\{x \in \mathbb{R} \mid |2x + 3| < 7\} = (-5,2)$

19. Show that $\{x \in \mathbb{R} \mid |x-1| < |x|\} = (½ ,\alpha)$

20. Show that if $(x, y) \leq (a, b)$ then $y - x < b - a$.

21. Show that $|x - a| < \in$ if and only if $a - \in < x < a + \in$

22. Show that $\{x \mid x \in \mathbb{R}, |x+1| + |x - 2| = 7\} = \{13,4\}$

23. Show that $|x-1| > |x+1|$ if and only if $x < 0$.

24. Show that $|x| + |x=1| < 2$ iff $-\frac{3}{2} < x < \frac{1}{2}$

25. Show that $4 < |x + 2| + | x - 1 < 5$ if and only if

$$x \in (-3, -\frac{5}{2}) \cup (\frac{3}{2}, 2)$$

26. Show that $V_r(a) \cap V_s(a) = V_t(a)$ where $t = \min\{r,s\}$

27. If $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a \neq b$, show that

for $0 < c < \frac{|b-a|}{2}$, $V_c(a) \cap V_c(b) = \phi$

28. If $a \in \mathbb{R}$, $b \in \mathbb{R}$ let $\max\{a, b\} = \begin{cases} b \text{ if } a \le b \\ a \text{ if } b \le a. \end{cases}$ and $\min\{a, b\} = \begin{cases} a \text{ if } a \le b \\ b \text{ if } b \le a. \end{cases}$

(a) Show that $\max\{a,b\} = \frac{a+b+|a-b|}{2}$

(b) Show that $\min\{a,b\} = \frac{a+b-|a-b|}{2}$

(c) Show that min { max (a, b}, max (b, c), max {c, a}}

= mid {a b, c is the 'middle number' among a, b, c; which lies between the remaining numbers.

29. Show that $\sup\{1 - \frac{1}{n} \mid n \in N\} = 1$.

30. Show that $\inf\{\frac{1}{n} - \frac{1}{m} \mid n \in N, m \in N\} = -1$.

31. Let x be a nonempty set x $f: x \to \mathbb{R}$, $g: x \to \mathbb{R}$ be bounded

Show that $\inf_{x \in x} (f(x) + g(x) \ge \inf_{x \in x} f(x) + \inf_{x \in x} g(x)$

32. If $x \in \mathbb{R}$ show that there is a unique $n \in \mathbb{Z}$, such that $n - 1 \le x < n$.

33. If $y > 0$. show that there is $n \in N$ such $\frac{1}{2^n} < y$.

34. Let $I_n=[a_n, b_n]$. If $I_n \supseteq I_{n+1}$ for all $n \in N$ show that $a_1 \le a_2 \le \ldots \le b_n \le b_{n-1} \le \ldots \le b_1$

35. If $I_n = [0, \frac{1}{n}]$ show that $\bigcap_{n \in N} I_n = \{0\}$

36. If $J_n = [0, \frac{1}{n}]$ show that $\bigcap_{n \in N} J_n = \phi$

37. If $k_n = [n, \infty)$ show that $k_1 \supseteq k_2 \supseteq \ldots \supseteq k_n \supseteq \ldots\ldots\ldots$ and $\bigcap_{n \in N} k_n = \phi$

10.A.14. Model Examination Questions

1. (a) Show that if $a \in \mathbb{R}$ and $b \in \mathbb{R}$ $| |a| - |b|| \le | a - b|$

   (b) Show that between any two real umbers there is a rational number

2. (a). Show that l.u.b. $\{ \frac{1}{m} + \frac{1}{n} | m \in N \text{ and } n \in N \} = 2$

   (b) Show that there is no rational number x such that $x^2 = 3$

3. (a). Show that for all positive integers n, n (n + 1 (n +2) is a multiple of 6

   (b) Deter mine the set $A = \{ x \in \mathbb{R} \setminus |x - 1| \le |x| \}$

4. Show that if $0 < b < a$ $n \in N$ and $n > 1$ then $n\, a^{n-1} (b - a) < b^n - a^n < n\, b^{n-1} (b - a)$

5. Show that if A and B are bounded above sup (A + B) = sup A + sup B

6. (a) Show that max $\{ a,b\} = \dfrac{a + b + | a - b |}{2}$

   (b) Show that $2^n \le n!$ for $n \in N$, $n \ge 4$.

10.A.15 Model practical problem with solution

Let A be the set of all rational numbers between 0 and 1 and B be the set of all irrational numbers between 0 and 1. If C = A +B = { a + b / a ∈ A, b ∈ B}show that C is bounded, sup C = 2 and inf C = 0.

Aim: (i) C is bounded (ii) sup C = 2 and (iii) inf C = 0

Definitions:

(1) $E \subseteq \mathbb{R}$ is said to be bounded if there exist α, β in $\mathbb{R}$ such that $\alpha \le x \le \beta$ for all x in E.

(2) A number l is called least upper bound or supremum of a set E, in symbols l = lub E = sup E if

   (a) l is an upper bound of E, i.e. $x \le l$ $\forall$ $x \in E$ and

   (b) If l^1 is any upper bound of E then $l \le l^1$.

(3) A number m is called greatest lower bound or infimum of E, in symbols m = glb E = inf E if (a) m is a lower bound of E, i.e. $m \le y$ $\forall$ $y \in E$ and (b) if m^1 is any lower bound of E then $m^1 \le m$.

A. Result used: If a and b are real numbers, $a < b$ there is a rational number x and an irrational number y such that $a < x < b$ and $a < y < b$.

Solution:

(i) To show that C is bounded:

$z \in C \Leftrightarrow \exists\, x \in A, y \in B \ni z = x + y$.

$x \in A \Leftrightarrow 0 \le x \le 1$ $y \in B \Leftrightarrow 0 < y < 1$.

Hence $0 < x + y < 1$

Hence $0 < z < 1$.

This holds $\forall\, z \in C$. Hence C is bounded.

(ii) sup C = z

From (i) z is an upper bound of C.

1 is not an upper bound of C because $1 < 1 + y^1\ \forall\ y^1 \in B$ and $1 + y^1 \in C$ since $1 \in A$ and $y^1 \in B$. Since 1 is not an upper bound of C any $m < 1$ is not an upper bound of C.

If $1 < m < 2$, $0 < m - 1 < 1$. By (A) there is an irrational number y such that $m - 1 < y < 1$. Then $z = 1 + y \in A$ and $m < Z$.

Hence if $m < 2$, m is not an upper bound of C. Thus sup C = 2.

(iii) To show that inf C = 0: From (i) 0 is a lower bound of C.

If $0 < m < 1$, by (A) there is an irrational number y^1 such that $0 < y^1 < m$. So m is not a lower bound of C. $\rightarrow$ (c)

Also 1 is not a lower bound of C because every member y^1 of B satisfies:

$y^1 = 0 + y^1 \in A + B = C$ and $0 < y^1 < 1$.

Hence any number $m > 1$ is also not a lower bound of C. Thus any $m > 0$ is not a lower bound of C. Then inf C = 0.

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