OPERATIONS RESEARCH, COMPUTER PROGRAMMING AND NUMERICAL ANALYSIS

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ACHARYA NAGARJUNA UNIVERSITY

CENTRE FOR DISTANCE EDUCATION

NAGARJUNA NAGAR,

GUNTUR

ANDHRA PRADESH

LESSON - 1

AIMS AND SCOPE OF OPERATIONS RESEARCH

Objectives

The objectives of this lesson are to:

- Learn the factors which are responsible for development of Operations Research
- Know the difficulties that are faced by a decision maker
- Learn how Operations Research is evolved as a subject
- Learn the various definitions of Operations Research given by eminent scientists
- Able to select a suitable model and formulate it more accurately and skillfully
- Learn different types of models
- Able to differentiate the models
- Recognize the problems which belong to a particular type of model

Structure

1.0 Introduction

1.1 Evaluation of Operations Research

1.2 Nature & Definition of Operations Research

1.3 Model formulation

1.4 Scope of Operations Research

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1.0 INTRODUCTION

If one aspires to become a successful manager, an important talent one must develop is decision-making. As a student, you have already made a great many decisions like which courses to undergo or which type of interviews and jobs to hold.

Some of the difficulties that are normally faced by a decision-maker are:

- (i) complexity in business models;
- (ii) high cost of new technology, materials, labour; etc.
- (iii) more competition; and
- (iv) involvement of quick decisions to take.

All these factors, are real challenges for the present day decision makers. The decision-maker should use all of his/her ability based upon his previous experience and intuition. An effective decision depends on several factors like: economic, social and political. For example, a government decision on the location of a new factory would be based on the factors like: construction costs, labour costs, taxes, electricity costs and pollution control costs, transportation costs, and so on. Finding the location of a new medical/engineering college may be influenced by either central, state or local politics. Thus, understanding of the available scientific methods and utilization of these scientific methods will play vital role in taking decisions.

Operations Research (also known as a branch of quantitative methods or management science or decision science) is also a scientific approach that has evolved as an aid to the decision-maker in several types of organizations.

Thus Operations Research developed an increasing degree of importance in the theory as well as in practice of management. Some of the important factors which are responsible for this development are:

(i). Decision problems of present management are so complex. A systematic and scientifically based analysis can only provide better solutions.

(ii). Development of several types of quantitative models to solve the complex managerial problems.

(iii). Use of high-speed computers made it possible to minimize time and cost. So one can obtain the models to all real-life problems that arise in almost all organizations.

Operations Research is not a fixed formula. So, the problem is to be identified and defined. It should be analyzed and solved in a rational, logical, systematic and scientific manner based on the collected data, information. Operations Research is used only if quantitative models can be built upon. The models are to be modified time to time by the experience of the decision maker. The creative insight of the decision-maker is also important.

Self Assessment Question 1: *What are the difficulties that are faced by a decision maker ?*

1.1 EVALUATION OF OPERATIONS RESEARCH

Operations Research came into existence during world-war-II when the British and American military management called upon a group of scientists with diverse educational background in Physics, Biology, Statistics, Mathematics, Psychology etc., to develop and to apply a scientific approach to deal with strategic and technical problems of various military operations. The object of forming this group was to allocate scare resources in an effective manner in various military operations and to the activities within the operation. The name Operations Research came directly from the context "military operations".

Following the end of the war, the successive stories of military teams attracted the attention of industrial managers in U.K. and U.S.A. who were seeking solutions to their complex executive type problems. By 1948, it had taken held in U.K. and was in the process of achieving the same in U.S.A.

By the early 1950's, industries in U.S.A. had realized the importance of Operations Research in solving their management problems. During the year 1950, Operations Research achieved recognition as a subject worthy of academic study in the universities. Since then, the subject has been gaining more and more importance for the students of Economics, Management, Public Administration, Behavioural Science, Social Work, Mathematics, Commerce and Engineering.

'Operations Research Society of America' was formed in 1950. The International Federation of Operations Research Society was established in 1957. Later several international journals began to appear.

In India, Operations Research came into existence in 1949 with the opening of an operations research unit at the Regional Research Laboratory at Hyderabad and also in the Defence Science Laboratory at Delhi. For national planning and survey, an Operations Research unit was established in 1953 at the Indian Statistical Institute, Calcutta. In 1957, the Operations Research Society of India was formed. Prof. Mahalanobis first applied Operations Research in India by formulating second five-year plan with the help of Operations Research techniques. Planning Commission made the use of Operations Research techniques for planning the optimum size of the Caravelle fleet of Indian air lines. Some of the industries, namely, Hindustan Lever Ltd; Union Carbide, TELCO, Hindustan Steel, Imperial Chemical Industries, Tata Iron & Steel Company, Sarabhai Group, FCI etc. have engaged Operations Research teams. Kirlosker Company is using the assignment technique of Operations Research to maximize profit.

Textile Firms like, DCM., Binni's and Calico, etc., are using linear programming techniques. Among other Indian organizations using Operations Research are the Indian Railways, CSIR, Tata Institute of Fundamental Research, Indian Institute of Science, State Trading Corporation, etc. Present days, almost all the universities in India included Operations Research in their curriculum.

1.2 NATURE & DEFINITION OF OPERATIONS RESEARCH

1.2.1 Nature of Operations Research: As the name indicates, the subject, Operations Research involves research on (military) operations. This includes not only the approach but also the area of applications of the field. So, it is an approach to form the models and solve the related problems: how to coordinate and control the operations/activities within the organization under study.

If one wish to run an organization effectively, the problem that may arise frequently is the coordination among the conflicting goals of its various functional departments. To illustrate this, consider the problem of stocks of finished goods. Various departments of the organization may like to handle such problems differently. The marketing department has to transport a huge stock of a large variety of products to the customers keeping in view what product they want, and when they want. The finance department observes the stocks kept as capital tied up unproductively and tries strongly to reduce the stock. On the other side, the personnel department thinks for great advantage in labour relations whether there is a continuous level of production that leads to the steady employment. So different departments will act through their specialization. To optimize the whole system, the decision-maker must decide the best policy keeping in view all the factors and conflicting claims of various departments.

Operations Research provides not only a better solution but also helps to arrive at the optimal solutions to a problem that raised in an organization. The decisions taken by the decision-maker may not be acceptable to all the departments, but it should be an optimal strategy for the organization as a whole. In order to obtain such types of conclusions/solutions, the decision-maker must know different methods available in Operations Research.

1.2.2 Definition of operations Research: It is not possible to present uniformly acceptable definition of Operations Research. For definitions that are commonly used and widely accepted are given here.

(i) Operations Research concerned with scientifically deciding how best to design and operate man–machine systems, usually under conditions requiring the allocation of scarce resources.

- ***O.R. Society of America***

(ii) Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem.

- ***C.W. Churchman, R.L. Acokff and E.L. Arnoff***

(iii) Operations Research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control

- ***P.M. Morse and G.E. Kimball***

(iv) Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough going rationally in dealing with its decision problems.

- ***D.W. Miller and M.K. Starr***

(v) Operations Research is a scientific approach to problems solving for executive management.

- ***H.M. Wagner***

(vi) Operations Research is the art of giving bad answers to problems which otherwise have worse answers.

- ***T.L. Saaty***

(vii). Operations Research is a management activity pursued in two complementary ways-one half by the free and bold exercise of commonsense untrammeled by any routine, and the other half by the application of a repertoire of well established pre-created methods and techniques.

- Jaghit Singh

(viii). Operations Research can be associated with an art of winning the War without actually fighting it

- George B. Dantzig

(ix). Operations Research is a scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources. ***- H. A. Taha***

(x) 'The Operational Research quarterly' (a journal published on behalf of the U.K. Operational Research Society) carried out the following definition: "Operations Research is the application of the methods of science to complex problems arising to the direction and management of large systems of men, machines, materials and money in industries, business, government and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decision strategies and controls. The purpose is to help the management to determine the policies and actions scientifically".

1.2.3 Role of Computers in Operations Research: It is a fact that, computers have played an essential role in the development of Operations Research. Because computations in most of the Operations Research techniques are so complex and involved that these techniques would be of no practical use without computers. Many large scale applications of Operations Research techniques may take weeks, months and sometimes years even to yield the results manually where as they require only few minutes on the computer. So the computers have become a vital and integral part of Operations Research. Currently, Operations Research methodology and computer methodology are growing up in chorus. It may be obvious that in the near future the line dividing the two methodology are growing up simultaneously. It seems that in the near future, the two sciences will combine to form a more general and comprehensive science as the line dividing the two methodologies will disappear.

1.3 MODEL FORMULATION

A model may be considered to be accurate if it is an idealized substitute for the actual system under study. A model is said to be valid if it can provide a reliable prediction of the system's performance.

Most of our thinking and virtually all applications of Operations Research in business take place in the context of models.

A model is a general term denoting any idealized representation or abstraction of a real-life system or situation. The objective of the model is to identify all aspects of the situation and to identify significant factors and their inter-relationships. A model helps in decision-making as it provides a simplified description of complexities and uncertainties of a problem at hand in logical structure. A main advantage of modeling is it allows the decision-maker to check the behaviour of a system without interfering with on-going operations.

In general the first crucial step in any Operations Research model is the definition of the alternatives or the decision variables of the problem. Next, the decision variables are used to construct the objective function and the constraints of the model. With the three steps, we can generalize the Operations Research model as maximize or minimize objective function subject to constraints.

Verbal statements usually precede the formulation of a mathematical decision model. These statements prove a word picture or image of reality from which the mathematical model may evolve. The verbal description and the mathematical model should be closely related. Once a model is formulated, one can apply different scientific procedures that are suitable for the model.

Formulation of mathematical models requires functions and variables. These variables are to be identified from the real situation/system for which the model is to be formed. The variables must be related to the important factors in the system under consideration. A function is a mathematical expression that represents a relation among some of the variables identified. The effectiveness of the model depends on the selection of proper variables and related functions, by the decision–maker.

In formulating a mathematical model, one attempts to consider all components of the system that are relevant to the system's effectiveness. A large number of variables/parameters may involve and many of them may help us in analyzing the system. All the relevant important variables of the system are to be carefully translated into mathematical symbols or terms. As it is very difficult to include all the parameters in constructing the model, it is a common practice to consider as many variables as possible, which help us in analyzing the system significantly. In case, if some significant parameters are omitted, then the model may provide misleading results until they are detected and included. So the models formed are to be tested and updated by the decision-maker, depending on the changes that occur in the real system, over the time.

Now we present a simple example involving two variables:

1.3.1 Example: A furniture manufacturing company plans to make two products namely chairs and tables from its available resources, which consists of 400 board feet of wood and 450 man-hours. To make a chair it requires 5 board feet and 10 man-hours, and yields a profit of Rs. 45, while each table uses 20 board feet and 15 man-hours and has a profit of Rs. 80. The problem to be identified here is: in such a situation 'how many chairs and tables the company can make so that it gets maximum profit'.

Now we formulate the mathematical model related to the above situation.

Formulation: Let x_1 be the number of chairs to be produced and
x_2 be the number of tables to be produced.

First we formulate the constraint related to board. Since a chair requires 5 board feet, x_1 chairs require $5x_1$ board feet. Since a table requires 20 board feet, x_2 tables require $20x_2$ board feet. Therefore the total requirement is $5x_1 + 20x_2$. Since the available board is 400 feet we get that

$$5x_1 + 20x_2 \leq 400.$$

Now we formulate the constraint related to man-hours. Since a chair requires 10 man-hours, x_1 chairs require $10x_1$ man-hours. Since a table requires 15 man-hours, x_2 tables require $5x_2$ man-hours. So total man-hours required is $10x_1 + 15x_2$. Since the availability of man-hours is 450, we have that $10x_1 + 15x_2 \leq 450$.

Since the profit on one chair is Rs. 45, the profit on x_1 chairs is $45x_1$. Since the profit on one table is Rs.80, the profit on x_2 chairs is $80x_2$. Therefore the total profit is $45x_1 + 80x_2$. That is, profit = $45x_1 + 80x_2$. So the profit function 'f 'is given by $f = 45x_1 + 80x_2$. Since the profit is to be maximized, we write Max $f = 45x_1 + 80x_2$. Since the number of chairs, number of tables is non-negative, we have $x_1 \geq 0, x_2 \geq 0$.

Hence the mathematical model is

$$5x_1 + 20x_2 \leq 400,$$

$$10x_1 + 15x_2 \leq 450, \quad x_1 \geq 0, x_2 \geq 0, \quad \text{Max } f = 45x_1 + 80x_2.$$

1.3.2 Types of Models: Models and their manipulation are very important. In Operations Research / Management science. Decision makers must understand the operations and select appropriate courses of action for attaining organizational objectives. Simulation is the process of manipulating a model of reality instead of reality itself.

To represent an operational system, a model may be used. To explain and describe some characteristic of the system under study, a model is employed. Models give the basis for experimental investigation which yields information in a smaller amount time and at less cost where as direct manipulation of the system itself costs high and consumes more time. To manipulate reality is often impossible, for example complex industrial system. So we use models with ease.

Various basic types of models may be described as follows:

1. Physical Models

These models include all forms of diagrams, graphs and charts. Most of these models are designed to deal with specific problems. By listing significant factors and inter-relationship, we can convert it into a pictorial form. Physical models are more helpful for a decision maker in his analysis. For example, a bar chart can be used in presenting the previous production records of the company and it is also useful for forecasting. Such models are easy to build and observe. These models cannot be manipulated, but can be used for prediction. The two types of physical models are: Iconic models; and Analog models.

a) **Iconic models**: An icon is an image or likeness of a physical object or system. For example, a globe is an iconic model of the earth. An Iconic model, which is an abstract physical replica of a system, is usually based on a smaller scale than the original. The range of the problems araised in management areas for which these models can be applied is very narrow. But iconic models mainly used in those fields which are related to engineering and science. For example Flight simulators are used by Indian Airlines and Air India to train their pilots and the members of the crew. These Flight simulators are iconic models or replica of different types of aircrafts. The trainees feel as if they are piloting the real aircraft. Iconic models of several Japanese cars used to train trainee drivers.

b) **Analog models**: Analog models are closely related to iconic models. But, they are not replicas of the original systems. They are small physical systems that have similar characteristics and work like the original object system it represents. For example, children’s toys, model rail-roads, are fall in the category “Analog models”. The objectives of constructing these models is to understand the system. Analog models are less specific, less concrete but easier to prepare/manipulate than the iconic models.

2. Mathematical /Symbolic Models

The complexity of relationships in some systems cannot be represented physically, or the physical representation may be quite cumbersome and may take more time to construct/manipulate. So, the abstract model may be used with the aid of symbols (i.e., letters, numbers, etc). Mathematical models are more useful in several real life business/organizations/situations.

All mathematical models may contain three types of variables: Dependent variables, Decision variables and Uncontrollable variables. A **dependent variable** reflects the level of effectiveness of the system. A dependent variable is a variable that depends on a variable which already exists. The **decision variables** are those involving the choice factors. The decision variables are to be manipulated and controlled by the decision-maker. The **uncontrollable variables** are those variables which are not under the control of the decision-maker. The variables that are not dependent variables are called as independent variables.

Models are designed to represent the operations under study, by an idealized example of reality, in order to explain the essential relationships involved.

Self Assessment Question 2:

What is the type of the following 'models' ?

(i). Children toy: Cycle

(ii). A system of equations representing a Business problem.

(iii) Rocket simulations.

(iv) Histogram, pie-Chart, flow-chart.

1.4 SCOPE OF OPERATIONS RESEARCH

1.4.1 A Plan for Applying Operations Research

The application of Science and the Scientific method to operational problems faced by managers is described as Operations Research.

The aim of Operations Research is to provide the decision maker with an objective basis for evaluating the operations under his or her control. If a systematic plan is carried out, then this aim may be achieved with greater success if it is pursued in accordance with a systematic plan. Such a plan needs to provide a procedure that gives better conclusions because such a plan is a conceptual construct and useful in placing the area of analysis in proper perspective. The forth coming part of this chapter explains a systematic plan for the application of Operations Research.

1.4.2 Methodology of Operations Research

Different Operations Research techniques, are used to arrive at an optimal solution of the problem which is based on some optimality criteria. A given problem may be solved by using the following algorithm (in Operations Research) is as follows:

Step 1: Identify /Define the problem.
Step 2: Select a suitable mathematical model
Step 3: Get the solution of the model
Step 4: Examine the model and the solution. If necessary remodel the problem and go to step 3.
Step 5: Establish the controls
Step 6: Implement the solution

We now elaborate the above steps

Step 1: Identify/Define the Problem: This is the first crucial step as it sets the boundaries for all that follows. The formulation of the problem is lead by the identification of all significant interactions between the problem area and the other operations of an organization as a whole. Each problem may require different approaches to formulate it. Thus problem definition involves defining the scope of the problem under investigation.

Step 2: Select a suitable mathematical model: The next step after establishing the criteria of optimality, is to determine the specific mathematical relationships (or models) which exist among all (controllable and uncontrollable variables). They are formulated by an equation or a system of equations or inequalities. Here we are translating the definition of the problem into a Mathematical form consisting of relationships among variables identified.

Step 3: Get the Solution of the model: Depends on the type of the business model, we use different Operations Research techniques to get the actual solution. So the correctness of the model solution depends on the suitable well defined optimization algorithm that used.

Step 4: Examine the model and the solution: A model is a representation of a business problem under study, which in turn includes some specified aspects of reality. A model is a valid model if it provides a reliable prediction of the system's performance.

Step 5: Establish the controls: Control processes are necessary to ensure that the solution suggested by the model results in the predicted changes in performance. Beyond initial implementation, controls are necessary for maintenance of the solution.

Step 6: Implementing the solution: The decision-maker has not only to identify good decision alternatives but also to select alternatives that are capable of being implemented. This implies an assessment of the organizational climate for change and the decision-maker's abilities to move the organization. The behavioral aspects of change are exceedingly important to the successful implementation of results. In any case, the decision-makers who are in the best position to implement results must be aware of the objective, assumptions, commissions and limitations of the model.

Self Assessment Question 3: *Write an algorithm to solve a given problem by OR Technique ?*

1.4.3 Various Operations Research Techniques

The following classes of problems given below are solved by certain standard techniques or prototype models of Operations Research which can be helpful to a decision-maker. However, each one of these models of Operations Research involves detailed studies which is out of the scope of this book. So in our context we are merely listing these to arouse your interest.

Allocation models: Allocation models are concerned with the allocation of scarce resources in order to optimize the given objective function, subject to certain constraints. Procedures for solving such type of problems are collectively called mathematical programming problems. Mathematical programming problems are of two types. They are linear and non-linear in nature.

The difference between linear and non-linear programming problems is based on the nature of the objective function and / or constraints. For example, linear programming is used in the formulation and solution of those problems where the objective function is linear and constraints are in the form of linear inequalities. ‘Transportation and Assignment Problems’ are viewed as special cases of linear programming. A naval base can improve the fleet utilization by using transportation model which will determine the deployment pattern of various ships on different routes. Similarly, the assignment model can be used for assigning an optimal number of sales representatives to various sales territories in terms of sales potential and number of customers in each territory.

Inventory models: The differences in the timing or location of demand and supply originated the inventory models. This is applicable to the raw materials for a production process or finished goods stocked by a retailer, wholesaler or manufacturer. Inventory models are concerned with the determination of the optimal (called economic) order quantity and frequency considering such factors as demand per unit time, cost of placing orders, costs associated with goods held in inventory and cost of running short. For example, the manager of a warehouse has to constantly balance the cost of carrying inventory, the loss of sales on account of shortages and administrative cost of ordering the stocks.

Waiting line (Queuing) models: To wait in line (or queue) is a common experience for us. Waiting to get a cinema ticket, train ticket etc. in a long queue is unpleasant. These queuing systems needs modeling with the operating characteristics (for example, average length of waiting time of queue; average time spent in the queue by the customer, etc.). The objective of these models is to minimize the total costs of providing service, costs of obtaining service, and also to minimize the average time spent in queue.

Markovian models: These models are suitable in the situation where the status of the system, called its ‘states’, can be defined by some descriptive measure of numerical value and where the system moves from one state to another on a probability basis. These models, consists of a sequence of events in which the probability of occurrence for an event depends upon the immediately preceding event. Markovian models have been successfully applied in several cases such as: to analyse consumer buying patterns, to forecast bad debts, for planning personnel needs, to analyse equipment replacement, and so on.

Network models: A project is a human undertaking with a clear beginning, and a clear ending. These models enable us to study with complexities and interdependencies involved in large projects. Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) are examples of such models that are used for planning, scheduling and controlling complex projects that are networks of multiple activities.

Sequencing models: These models are dealt with the selection of an appropriate sequence of performing a series of jobs to be done on service facilities (such as machines) so as to optimize efficiency measures of performance of the system. Thus, sequencing is concerned with such a problem in which efficiency measures depends upon the sequence of performing a series of jobs.

Replacement models: These models deal with the study of the technique of formulating the appropriate replacement policy in situations where some items or machinery need replacement for some reason.

Simulation models: Simulation models are special class of mathematical models in management decision-making. Simulation is an experimental method that used to study behaviour over time. Instead of implementing a decision and observing the results in a real world situation, a model can be formed and tested. It is a “trail & error” method.

Decision theory: Decision theory assists the decision-maker in analyzing complex problems with several alternatives and consequences. The basic objective of decision theory is to provide more concrete information regarding these consequences, so that the decision-maker may be able to identify the better course of action. Mainly there are four different states of decision environment as given below:

States of Decisions	**Consequences**
Certainty	Deterministic
Risk	Probabilistic
Uncertainty	Unknown
Conflict	Influenced by an opponent

Decision-making under certainty: This certainty state exists when all the information required to make a decision is known and available. A linear programming problem, is categarised in "Decision making under certainty". So we call it as a deterministic model.

Decision-making under risk: The word "risk" refers to the situation where the probabilities of certain outcomes (consequences) are either known or can be estimated. These arise in many business situations Some examples are:
(i). Whether a particular product be introduced ?
(ii). Whether to build a new plant or expand the present one ?
(iii). How many saries should a textile factory produce in the next month ?

In each of the above situations we have the "decision-making under risk".

Decision-making under uncertainty: The uncertainty state refers to the state where the probabilities of certain outcomes are not known and difficult to estimate. It is similar to decision-making under risk with one major difference, i.e., in this type of situation, we have no knowledge of the probabilities of future events. In such cases, we can express our degree of optimism or convert the problem to risk with reasonable accuracy.

Decision-making under conflict: **"**Conflict" exists when the interests of two or more decision-makers are in a competitive situation. If decision maker X benefits from a course of action (or a strategy) he adopts, it is only because decision-maker Y taken a certain course of action. So in such a state of situation, the decision-makers are interested not only in what they do but also in what other firms. We know that the decision of one firm will affect the sales of the other firms.

1.4.4 Operations Research and Managerial Decision-Making

In their daily activities, Executives at all levels in business and industry have to face the problem of making decisions at every point. Operations Research presents the executive with quantitative basis for decision-making, and improves his ability to make long-range plans and to solve everyday problems of running a business and industry with greater efficiency, competence and confidence.

The study of Operations Research approach in decision-making is advantageous due to the following facts:

1 Better control: To every routine work, it is a difficult and costly affair to afford continuous executive supervision. To identify the problem areas, an analytical quantitative basis by the executive may be bestowed in an Operations Research approach. This in turn boosts the effectiveness of the executive in achieving the specific objective.

2 Better system: Operations Research approach is also initiated to analyse a particular problem of decision-making such as best location for factories, warehouses as well as the economical means of transportation, production scheduling, replacement of an old machine, etc.

3 Better decisions: It is essential that the best information and decision-making tools must be available to the executive, since an incorrect decision can have serious consequences,. An Operations Research approach may present the executive with techniques that will enable him to make decisions in a better way and with as much precision as possible. That means, Operations Research approach provides the executive an improved insight into how he makes his decisions so that he can improve his decision-making process.

Operations Research in Management

The industrial business problems analysed by Operations Research approach are categorized as follows:

1 Finance

(a) Analysis of Cashflow, long-range capital requirements, dividend politics, investments portfolios

(b) Credit policies, credit risks and delinquent account procedures

(c) Complaint procedures and Claims.

2 Management of Marketing

(a) Selection of Product, time the competitive actions

(b) Number of sales representatives, routing of sales representatives, frequency of calling on accounts, percentage of time to be spent on prospects against current customers

(c) Advertising media with respect to cost and time

(d) Physical distribution of goods

(e) Warehousing decisions

(f) To predict the behavior of consumer and examine it.

3 Management of Purchasing

(a) Rules for buying supplies under different prices

(b) Quantities determination and purchases timing

(c) Bidding policies

(d) Exploration strategies and exploitation of new material sources

(e) Policies replacement

4 Management of Production

(a) Balancing of Assembly line

(b) Aggregate production planning and smoothing

(c) Scheduling and sequencing the Production

(d) Production stabilisation and employment stabilisation, training, lay-offs and optimum product-mix

(e) Maintenance policies and preventive maintenance

(f) Crew sizes maintenance

(g) Scheduling of project and allocation of resources.

5 Management of Personnel

(a) Equitable salaries determination

(b) The age and skills mixing

(c) Recurring policies and jobs assignment

(d) Planning of manpower, scheduling and deployment

(e) Absenteeism management

(f) Negotiations of labor management.

6 Research and Development

(a) Areas of concentration of research determination and development

(b) Selection of project

(c) Alternative design and reliability

(d) Determination of time-cost trade-off and control of development projects.

1.4.5 Applications/Scope of Operations Research

Due to the organized development, Operations Research has been applied to many different areas of research for military, government and industry. Population explosion, unemployment, poverty and hunger are the crucial problems faced by the Third World countries. The basic problem in most of the developing countries is to remove them as soon as possible. So there is a great scope for economists, statisticians, administrators, politicians and the technicians working in a team to solve this problem by an Operations Research approach. Not only this, Operations Research is useful in the following various important fields.

1. In Agriculture. Because of shortage of food by the population explosion, every country is facing the problem of –

(i) optimum allocation of land to various crops in accordance with the climatic conditions; and

(ii) optimum distribution of water from various resources like cannals, wells, tanks etc. for irrigation purposes. Thus under the prescribed restrictions, there is a need of determining best policies.

2. In Finance. It has become very necessary for every government to have a careful planning for the economic development of her country in these modern times of economic crisis. Operations Research-techniques can be successfully applied:

(i) to maximize the per capita income with minimum resources;

(ii) to find out a better plan for the company;

(iii) to determine the best replacement policies, etc.

3. In Industry. If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using Operations Research techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist Operations Research techniques in business management. Thus Operations Research is useful to the Industry Director in deciding optimum allocation of various limited resources such as men, machines, material, money, time, etc. to arrive at the optimum decision.

4. In Marketing. A marketing Administrator (Manager) can decide, with the help of Operations Research techniques:

(i) where to distribute the products for sale so that the total cost of transportation etc. is minimum,

(ii) the minimum per unit sale price,

(iii) the size of the stock to meet the future demand,

(iv) how to select the best advertising media with respect to time, cost, etc.

(v) how, when, and what to purchase at the minimum possible cost ?

5. In personnel Management. A personnel manager can use Operations Research techniques:

(i) to appoint the most suitable persons on minimum salary,

(ii) to determine the best age of retirement for the employees,

(iii) to find out the number of persons to be appointed on full time basis when the workload is seasonal (but not continuous)

6. In Production Management. A production manager can use Operations Research techniques:

(i) to find out the number and size of the items to be produced;

(ii) in scheduling and sequencing the production run by proper allocation of machines;

(iii) in calculating the optimum product mix; and

(iv) to select, locate, and design the sites for the production plants.

7. In L. I. C. Operations Research is also applicable to enable the L. I. C officers to decide:

(i) what should be the premium rates for various modes of policies,

(ii) how best the profits could be distributed in the cases of with profit policies ? etc.

Finally, we can say : wherever there is a problem there is Operations Research. The applications of Operations Research cover the whole extent of many things.

1.5 SUMMARY

Operations Research is the applications of methods of mathematical science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of systems incorporating measurements of factors such as chance and risk with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help the management to determine its policy and actions scientifically.

In this lesson we discussed the origin, definition, scope and applications of Operations research. Methodology of Operations research is also provided. Some models of Operations research are presented.

1.6 TECHNICAL TERMS

Operations Research	:Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem.
Allocation models	:This class of models is concerned with the allocation of scarce resources so as to optimize the given objective function, subject to certain constraints
Physical Models	:These models include all forms of diagrams, graphs and charts.
Dependent variable	:A dependent variable is a variable that depends on a variable which already exists.
Decision variable	:The decision variables are those involving the choice factors.
Uncontrollable variables	:The uncontrollable variables are those variables which are not under the control of the decision-maker.
Inventory model	:Inventory models are concerned with the determination of the optimal (called economic) order quantity and frequency considering such factors as demand per unit time, cost of placing orders, costs associated with goods held in inventory and cost of running short.
Waiting line (Queuing) models	:These models attempt to predict the operating characteristics (e.g., average length of waiting time of queue; average time spent in the queue by the customer, etc.) of queuing systems.
Markovian models	:These models are applicable in the situation where the status of the system, called its ‘states’, can be defined by some descriptive measure of numerical value and where the system moves from one state to another on a probability basis.
Decision theory	:Decision theory assists the decision-maker in analyzing complex problems with numerous alternatives and possible consequences

1.7 ANSWERS TO SELF ASSESSMENT QUESTIONS

Ans 1: The difficulties that are faced by a decision maker are complexity in business models; high cost of new technology, materials, labour; etc. ; more competition; and involvement of quick decisions to take.

Ans 2: (i). Children toy: Cycle, is an iconic model

(ii). A system of equations representing a Business problem, is a mathematical model.

(iii) Rocket simulations are iconic models.

(iv) Histogram, pie-Chart and flow-chart are analog models.

Ans 3: An algorithm to solve a given problem by OR Technique is:

Step 1: Identify /Define the problem.

Step 2: Select a suitable mathematical model

Step 3: Get the solution of the model

Step 4: Examine the model and the solution. If necessary remodel the problem and go to step 3.

Step 5: Establish the controls

Step 6: Implement the solution

1.8 MODEL QUESTIONS

1.8.1 Model Question 1: Give the historical background of Operations Research ?

Ans: See 1.1

1.8.2 Model Question 2: What is Operations Research ?

Ans: See 1.2.2

1.8.2 Model Question 3: What are the essential characteristics of Operations Research ?

Ans: Decision making, Scientific Approach, Objective, Inter disciplinary team Approach, Digital Computer

1.8.4 Model Question 4: Discuss various classification schemes of models ?

Ans: See 1.4.3

1.8.5 Model Question 5: Describe briefly the different phases of Operations Research ?

Ans: See 1.4.2

1.8.6 Model Question 6: Discuss the significance and scope of Operations Research in modern management ?

Ans: See 1.4.4 & 1.4.5

1.8.7 Model Question 7: List at least two applications of Operations Research in each functional area of management.

Ans: See 1.4.4.

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Name of the Lesson Writer

Prof. Bhavanari Satyanarayana

LESSON - 2

FORMULATION OF LPP & GRAPHICAL METHOD

Objectives

The objectives of this lesson are to:

- Learn the fundamental definition of LPP
- Know the formulation of Mathematical Models
- Identify the variables in a given actual system under study
- Understand the algorithm of Graphical Method
- Learn to solve the LPP with two variables by Graphical Method

Structure

2.0 Introduction
2.1 Mathematical Formulation
2.2 Linear Programming
2.3 Algorithm for Graphical Method
2.4 Graphical Method – Solved Problems
2.5 Summary
2.6 Technical terms
2.7 Answers to Self Assessment Questions
2.8 Model Questions
2.9 Reference Books

2.0 INTRODUCTION

The usefulness of linear programming as a tool for optimal decision making and resource allocation is based on its applicability to many diversified decision problems. The effective use and application require, as a first step, the formulation of the model when the problem is presented. In this section we shall illustrate the formulation of linear programming problems in various different situations.

A linear programming problem can easily be solved by graphical method when it involves two decision variables. In this section two approaches Search and Iso-profit function will be discussed.

2.1 MATHEMATICAL FORMULATION

2.1.1 Model formulation: Verbal statements usually precede the formulation of a mathematical decision model. These statements prove a word picture or image of reality from which the mathematical model may evolve. The verbal description and the mathematical model should be closely related. Once a model is formulated, one can apply different scientific procedures that are suitable for the model.

Formulation of mathematical models requires functions and variables. These variables are to be identified from the real situation/system for which the model is to be formed. The variables must be related to the important factors in the system under consideration. A function is a mathematical expression that presents a relation between the some of the variables identified. The effectiveness of the model depends on the selection of proper variables and related functions, by the decision–maker.

In formulating a mathematical model, one attempts to consider all components of the system that are relevant to the system's effectiveness. A large number of variables/parameters may involve and many of them may help us in analyzing the system. All the relevant important variables of the system are to be carefully translated into mathematical symbols or terms. As it is very difficult to include all the parameters in constructing the model, it is a common practice to consider as many variables as possible, which help us in analyzing the system significantly. In case, if some significant parameters are omitted, then the model may provide misleading results until they are detected and included. So the models formed are to be tested and updated by the decision-maker, depending on the changes that occur in the real system, over the time.

A model may be considered to be accurate if it is an idealized substitute for the actual system under study. A model is said to be valid if it can provide a reliable prediction of the system's performance.

Now we present simple examples involving two variables.

2.1.2 Example: A furniture manufacturing company plans to make two products namely chairs and tables from its available resources, which consists of 400 board feet of wood and 450 man-hours. To make a chair it requires 5 board feet and 10 man-hours, and yields a profit of Rs. 45, while each table uses 20 board feet and 15 man-hours and has a profit of Rs. 80. The problem to be identified here is: in such a situation 'how many chairs and tables the company can make so that it gets maximum profit'.

Now we formulate the mathematical model related to the above situation.

Formulation: Let x_1 be the number of chairs to be produced and
x_2 be the number of tables to be produced.

First we formulate the constraint related to board. Since a chair requires 5 board feet, x_1 chairs require $5x_1$ board feet. Since a table requires 20 board feet, x_2 tables require $20x_2$ board feet. Therefore the total requirement is $5x_1 + 20x_2$. Since the available board is 400 feet we get that

$$5x_1 + 20x_2 \leq 400.$$

Now we formulate the constraint related to man-hours. Since a chair requires 10 man-hours, x_1 chairs require $10x_1$ man-hours. Since a table requires 15 man-hours, x_2 tables require $5x_2$ man-hours. So total man-hours required is $10x_1 + 15x_2$. Since the availability of man-hours is 450, we have that $10x_1 + 15x_2 \leq 450$.

Since the profit on one chair is Rs. 45, the profit on x_1 chairs is $45x_1$. Since the profit on one table is Rs.80, the profit on x_2 chairs is $80x_2$. Therefore the total profit is $45x_1 + 80x_2$. That is, profit = $45x_1 + 80x_2$. So the profit function 'f 'is given by $f = 45x_1 + 80x_2$. Since the profit is to be maximized, we write Max $f = 45x_1 + 80x_2$. Since the number of chairs, number of tables is non-negative, we have $x_1 \geq 0, x_2 \geq 0$.

Hence the mathematical model is

$5x_1 + 20x_2 \leq 400,$

$10x_1 + 15x_2 \leq 450,$

$x_1 \geq 0, x_2 \geq 0,$ Max $f = 45x_1 + 80x_2.$

2.1.3 Example: Formulate the mathematical model for the following:

(Diet Problem): Vitamin–A and Vitamin–B are found in food–1 and food–2. One unit of food–1 contains 5 units of vitamin–A and 2 units of vitamin–B. One unit of food–2 contains 6 units of vitamin–A and 3 units of vitamin–B. The minimum daily requirement of a person is 60 units of vitamin–A and 80 units of Vitamin–B. The cost per one unit of food–1 is Rs.5/- and one unit of food–2 is Rs.6/-. Assume that any excess units of vitamins are not harmful. Find the minimum cost of the mixture (of food–1 and food–2) which meets the daily minimum requirements of vitamins.

Formulation: Suppose x_1 = the number of units of food–1 in the mixture, x_2 = the number of units of food–2 in the mixture.

Now we formulate the constraint related to vitamin-A. Since each unit of food–1 contains 5 units of vitamin–A, we have that x_1 units of food–1 contains $5x_1$ units of vitamin–A. Since each unit of food–2 contains 6 units of vitamin–A, we have that x_2 units of food–2 contains $6x_2$ units of vitamin–A. Therefore the mixture contains $5x_1 + 6x_2$ units of vitamin-A. Since the minimum requirement of vitamin–A is 60 units, we have that $5x_1 + 6x_2 \geq 60$.

Now we formulate the constraint related to vitamin–B. Since each unit of food–1 contains 2 units of vitamin–B we have that x_1 units of food–1 contains $2x_1$ units of vitamin-B. Since each unit of food–2 contains 3 units of vitamin–B, we have that x_2 units of food–2 contains $3x_2$ units of vitamin–B. Therefore the mixture contains $2x_1 + 3x_2$ units of vitamin–B. Since the minimum requirement of vitamin–B is 80 units, we have that $2x_2 + 3x_2 \geq 80$.

Now we formulate the cost function. Given that the cost of one unit of food–1 is Rs. 5/- and one unit of food–2 is Rs. 6/-. Therefore x_1 units of food–1 costs Rs. $5x_1$, and x_2 units of food–2 costs Rs. $6x_2$. Therefore the cost of the mixture is given by Cost = $5x_1 + 6x_2$. If we write z for the cost function, then we have $z = 5x_1 + 6x_2$. Since cost is to be minimized, we write $\min z = 5x_1 + 6x_2$.

Since the number of units (x_1 or x_2) are always non-negative we have that $x_1 \geq 0$, $x_2 \geq 0$. Therefore the mathematical model is

$$5x_1 + 6x_2 \geq 60$$

$$2x_1 + 3x_2 \geq 80$$

$$x_1 \geq 0,\ x_2 \geq 0, \quad \min z = 5x_1 + 6x_2.$$

2.1.4 Example: A company sells two different products A and B. The company make a profit of Rs. 30 and Rs. 20 per unit of products A and B respectively. The two products are produced in a common production process. The production process has 25000 man-hours. To produce one unit of product-A, it takes two man-hours and to produce one unit of product–B, it takes three man-hours. The general manager feels that the maximum number of units of A that can be sold is 6000 and the maximum number of units of B that can be sold is 9000 units. Formulate the mathematical model.

Formulation: Suppose x_1 is the number of units of product-A produced, and x_2 is the number of units of product–B produced.

Now we formulate the constraint related to man-hours. Since one unit of product–A takes 2 man-hours, we have that x_1 units of product–A take $2x_1$ man-hours. Since one unit of product–B takes 3 man-hours, we have that x_2 units of product–B take 3x man-hours. Therefore x_1 units of product–A and x_2 units of product–B take $2x_1 + 3x_2$ man-hours. Since the availability is 25000 man–hours, we have that $2x_1 + 3x_2 \leq 25000$.

Since the maximum number of units of product–A that can be sold is 6000, we have that $x_1 \leq 6000$. Since the maximum number of units of product–B that can sold is 9000, we have that $x_2 \leq 9000$. Since the number of units is always non-negative, we have $x_1 \geq 0, x_2 \geq 0$.

Now we formulate the profit function. The company makes a profit of Rs. 30 on one unit of product–A and Rs.20 on one unit of product-B. Therefore the profit on x_1 units of A is 30x, and profit on x_2 units of B is $20x_2$. Therefore profit on products A and B is given by the function: $z = 30x_1 + 20x_2$ where z denotes the profit. Hence the mathematical model is

$$2x_1 + 3x_2 \leq 25000$$

$$x_1 \leq 6000$$

$$x_2 \leq 9000, \quad x_1 \geq 0, x_2 \geq 0, \; z = 30x_1 + 20x_2.$$

2.2 LINEAR PROGRAMMING

Linear Programming is an important branch of Operations Research. Certain problems arising in trade, commerce, industry, military operations etc., can be solved by making use of our knowledge related to the systems containing either linear equations or inequalities.

In these problems, we have to maximize or minimize a given linear function of two or more unknown quantities subject to the condition that the variables are non-negative and they satisfy the given system of linear equations or inequalities. These conditions to be satisfied are called linear **constraints**. The term linear implies that the mathematical relations given in the problem are linear relations (If all the variables in the given constraints are of the first degree, then we say that the constraints are linear). The function to be maximized or minimized is known as the **objective function**, and the method of maximizing or minimizing the function is called **Linear Programming**.

This linear programming was used in planning war operations during the second world war when it was necessary to economize the expenditure, minimize the loses and maximize the damage to the enemy. It is now widely used in planning the economic activities and formulation of national plans.

Let us first recall our knowledge of linear equations and inequalities, graphical methods of representing the linear equations and inequalities.

(i) An equation of the first degree in two or more variables is called a linear equation. If the sign ' = ' of equality in any equation is replaced by one of the signs of inequality " $>$, $<$, $\geq$., $\leq$ ", it becomes a linear inequality.

(ii) $ax + by + c = 0$ is the standard form of a linear equation in x and y where a and b are not simultaneously equal to ‘0’. It can be expressed in the form $y = mx + c$.

(iii) The equation $ax + by + c = 0$ or $y = mx + c$ can be graphically represented by a straight line. In the line $y = mx + c$, the number m is the slope of the line and the number c is the distance from the origin to the point where the line intersects the Y–axis. If $c = 0$ (in any of these equations), then the line passes through the origin. ax + by + c = 0 can be put in the other form as $y = -\frac{a}{b}x - \frac{c}{a}$. The slope is $-\frac{a}{b}$ and the y-intercept (the distance from the origin to the point of intersection of the lines ax + by +c = 0 and Y-axis.) is $-\frac{c}{a}$.

(iv) If m is positive, then the line slopes down from right to left. If m is negative, then the line slopes form left to right.

(v) The line ax + by + c = 0 divides the plane into three disjoint sets of points:

(a) the set of points which satisfy the equation and hence belong to the line;

(b) the set of points which satisfy the inequality ax + by + c < 0; and

(c) the set of all points which satisfy the inequality ax + by + c > 0.

[Note: The equation ax + by + c = 0 is represented by a line while the inequalities ax + by + c > 0, and ax + by + c < 0 represent regions. The inequalities ax + by + c ≥ 0 and ax + by + c ≤ 0 are represented by the respective regions ax + by + c > 0, ax + by + c < 0 together with the line ax + by + c = 0].

2.2.1 Note: Linear programming deals with the class of programming problems for which all relations among the variables are linear. The relation must be linear both in the constraints and in the function to be optimized.

A general linear-programming problem is to find a vector $(x_1, x_2, \ldots, x_n)$ which minimizes/maximizes the linear form/objective function

$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ subject to the linear constraints

$a_{1_1} x_1 + a_{1_2} x_2 + \ldots + a_{1_n} x_n = b_1$

$a_{2_1} x_1 + a_{2_2} x_2 + \ldots + a_{2_n} x_n = b_2$

… … … … … … … …

$a_{m_1} x_1 + a_{m_2} x_2 + \ldots + a_{m_n} x_n = b_m$

and $x_j \geq 0$ for $j = 1, 2, \ldots, n$. Here a_{i_j}, b_i and c_i are given constants and $m < n$.

A given Linear Programming problem can be solved in many ways. In this book we present algorithms for some important methods (graphical method, simplex method, two-phase method, revised simplex method, etc.,) of solving the given Linear programming problems.

2.2.2 Advantages of Linear Programming

Following are certain advantages of linear programming:

1. Linear programming helps in attaining the optimum use of productive factors. It also indicates how a decision-maker can employ his productive factors effectively by selecting and distributing these elements.

2. Linear programming techniques improves the quality of decisions. User of this technique become more objective and less subjective.

3. Linear programming gives possible and practical solutions since there might be other constraints operating outside of the problem which must be taken into account. Just because we can produce so many units does not mean that they can be sold. It allows modification of its mathematical solution for the sake of convenience to the decision-maker.

4. Highlighting of bottlenecks in the production processes is the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet demand while other remains idle for some of the time.

2.2.3 Limitations of Linear Programming:

Inspite of having wide field of applications, there are some limitations associated with it, which are given below:

1. Sometimes objective function and constraints are not linear. For example, generally, the constraints in real life situations concerning business and industrial problems are not linearly related to the variables.

2. There is no guarantee that it will give integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job. Rounding off the solution to the nearest integer will not yield an optimal solution. In such cases other method would be used.

3. Linear programming model does not like into consideration the effect of time and uncertainty. Thus it shall be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large-scale problems can not be solved with linear programming techniques even when assistance of computer is available. For it the main problem can be decomposed into several small problems and solved separately.

5. Parameters appear in the model are assumed to be constant but in real-life situation, they are frequently neither known nor constants.

6. It deals with only single objective, where as in real life situations we may come across more than one objective.

2.2.4 Application Areas of Linear Programming

Linear programming is the most widely used technique of decision making in business and industry and in various other fields. In this section, we will discuss a few of the broad application area of linear programming.

1. *Agricultural Applications*: These applications falls into two categories, farm economics and farm management. The former deals with agricultural economy of a nation or region, while the later is concerned with the problems of the individual farm.

The study of farm economics deals with interregional competition and optimum spartial allocation of crop production. Efficient production patterns were specified by linear programming model under regional land resources and national demands constraints.

Linear programming can be applied in agricultural planning, e.g., allocation of limited resources such as acreage, labour, water supply and working capital, etc. in such a way as to maximize net revenue.

2. *Military Applications*: Military applications include the problem of selecting an air weapon system against gurillas so as to keep them pinned down and at the same time minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bombs dropped on a set of targets and the problem of community defence against disaster, the solution of which yields the number of defence units that should be used in a given attack in order to provide the required level of protection at the lowest possible cost.

3. *Production Management*: Linear programming can be applied in production management for determining product-mix, product smoothing and assembly line-balancing.

4. *Marketing Management*: Linear programming helps in analyzing the effectiveness of advertising campaign and time based on the available advertising media. It also helps traveling sales force in finding the shortest route for his tour starting from his home city and visiting once each of the specified cities before returning to his home city.

5. *Man-power Management*: Linear programming allows personnel manager to analyse personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

6. *Physical Distribution*: Linear programming determines the most economic and effective manner of locating manufacturing plants and distribution centers for physical distribution.

Other applications of linear programming include in the area of administration, education, inventory control, fleet utilization, awarding contracts and capital budgeting etc.

2.3 ALGORITHM FOR GRAPHICAL METHOD

2.3.1 An Algorithm for solving a linear programming problem by Graphical Method : (This can be applied for only problems with two variables).

Step–I: Consider the given linear programming problem with two variables (if the given problem has more than two variables, then we cannot solve it by using this graphical method).

Step–II: Consider a given inequality. Suppose it is in the form $a_1x_1 + a_2x_2 \le b$ (or $a_1x_1 + a_2x_2 \ge b$). Then consider the relation $a_1x_1 + a_2x_2 = b$. Find two distinct points (k, l), (c, d) that lie on the straight line $a_1x_1 + a_2x_2 = b$. This can be find easily: If $x_1 = 0$, then $x_2 = \frac{b}{a_2}$. If $x_2 = 0$, then $x_1 = \frac{b}{a_1}$.

Therefore $(k, l) = (0, \frac{b}{a_2})$ and $(c, d) = (\frac{b}{a_1}, 0)$ are two points on the straight line $a_1x_1 + a_2x_2 = b$.

Step–III: Represent these two points (k, l), (c, d) on the graph which denotes X–Y-axis plane. Join these two points and extend this line to get the straight line which represents $a_1x_1 + a_2x_2 = b$.

Step–IV: $a_1x_1 + a_2x_2 = b$ divides the whole plane into two half planes, which are $a_1x_1 + a_2x_2 \leq b$ (one side) and $a_1x_1 + a_2x_2 \geq b$ (another side). Find the half plane that is related to the given inequality.

Step–V: Do step-II to step-IV for all the inequalities given in the problem. The intersection of the half-planes related to all the inequalities and
$x_1 \geq 0, x_2 \geq 0$, is called the **feasible region (** or **feasible solution space)**. Now find this feasible region.

Step–VI: The feasible region is a multisided figure with corner points A, B, C, …(say). Find the co-ordinates for all these corner points. These corner points are called as **extreme points**.

Step–VII: Find the values of the objective function at all these corner/extreme points.

Step–VIII: If the problem is a maximization (minimization) problem, then the maximum (minimum) value of z among the values of z at the corner/extreme points of the feasible region is the optimal value of z. If the optimal value exists at the corner/extreme point, say A(u, v), then we say that the solution $x_1 = u$ and $x_2 = v$ is an optimal feasible solution.

Step–IX : Write the conclusion (that include the optimum value of z, and the co-ordinates of the corner point at which the optimum value of z exists).

2.4 GRAPHICAL METHOD – SOLVED PROBLEMS

2.4.1 Problem: A grain merchant deals with two items: rice and wheat. He could invest only Rs. 15000, and had a room for atmost 80 bags. A bag of rice costs of Rs. 250, and a bag of wheat costs Rs.150. He gains Rs.15 on a bag of rice, and Rs.12 on a bag of wheat. Assuming that he sells all the quantity he purchases, find how he should invest the money in order to get the maximum profit ?

Solution: Let the number of bags of rice be x and the number of bags of wheat be y. Since both are non-negative, we have $x \geq 0$, $y \geq 0$.
He has space to store only 80 bags. So $x + y \leq 80$.

He could invest only Rs.15000. Cost of x bags of rice is Rs.250x; and cost of y bags of wheat is Rs.150y. Therefore $250x + 150y \leq 15000$. This inequality can be written as $5x + 3y \leq 300$.

He makes a profit of Rs.15 on one bag of rice, and Rs.12 on a bag of wheat; and so the total profit he makes is Rs. (15x + 12y). The profit is to be maximized. So we have Max $z = 15x + 12y$. So the mathematical model is

$$x + y \leq 80$$
$$5x + 3y \leq 300$$
$$x \geq 0,\ y \geq 0, \quad \text{Max } z = 15x + 12y.$$

Now, graph the inequalities and find out the solution set. (Since all quantities involved are positive it is enough if we have the 1st quadrant).

$x + y = 80$

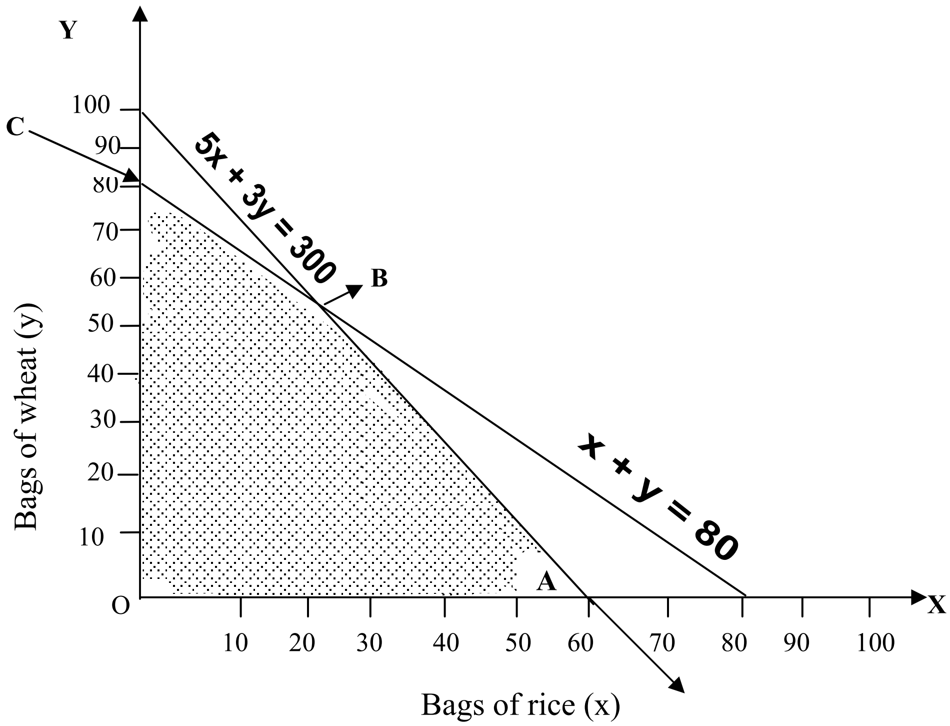
x	0	40	80
y	80	40	0

$5x + 3y = 300$

x	60	30	0
y	0	50	100

Some points on $x + y = 80$ are (80, 0); (40, 40); (0, 80), and some points on $5x + 3y = 300$ are (60, 0); (30, 50), (0, 100).

Mark these points and draw the graphs representing the equations.



The solution set/space is the closed polygonal region 0ABC.

Every point in this region satisfies the given constraints and is a feasible solution. So the region 0ABC is also called as the feasible region. From this region we have to find the point at which the value of the objective function

$z = 15x + 12y$ is maximum.

Now we have to find the values of z at the various vertices/extreme points of the polygonal region and decide the maximum.

Value of z at 0(0, 0) = 0 + 0 = 0.

Value of z at A(60, 0) = 15 x 60 + 0 = 900.

Value of z at B(30, 50) = (15 x 30 + 12 x 50) =1050.

Value of z at C(0, 80) = (15 x 0 + 12 x 80) = 960.

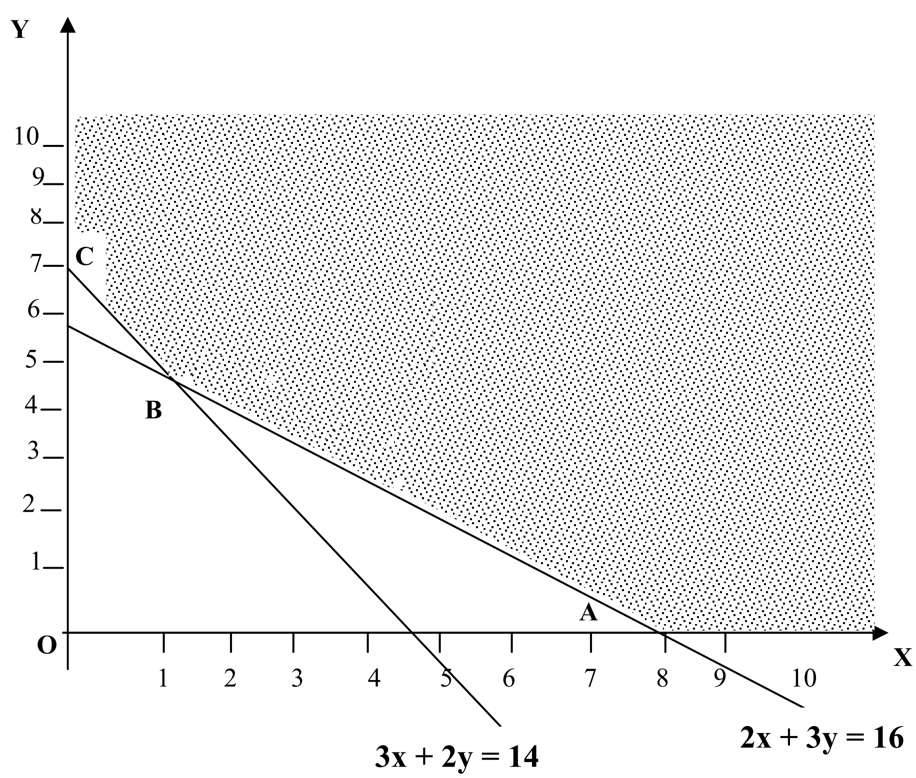
Maximum profit = Rs.1050 at x = 30, and y = 50.

Conclusion: If he invest Rs.7500 for 30 bags of rice, and Rs.7500 for 50 bags of wheat, then he gets a maximum profit of Rs. 1050.

2.4.2 Example: 1 Kg. of food–I contains 3 units of vitamin A and 2 units of vitamin C, while 1 Kg. of food-II contains 2 units of vitamin A and 3 units of vitamin C. They cost Rs.10 per Kg. and Rs.8 per Kg. respectively. A dietician wants to mix the two kinds and prepare a mixture so that it contains at least 14 units of vitamin A and 16 units of vitamin C. Find how the dietician should mix them so that the cost to him may be minimum.

Solution: Let the mixture contain x Kg of food-I and y Kg. of food–II. So $x \geq 0; y \geq 0$. x Kg. of food-I contains 3x units of vitamin A, and y Kg of food-II contains 2y units of vitamin A. Therefore the mixture contains $3x + 2y$ units of vitamin A. Therefore $3x + 2y \geq 14$.

Similarly the mixture contains $2x + 3y$ units of vitamin C. Therefore $2x + 3y \geq 16$.
The objective function is, say z, the cost of the mixture which is Rs.$(10x + 8y)$. This has to be minimized. So we have $\min z = 10x + 8y$.



$3x + 2y = 14$

x	0	4
y	7	1

$2x + 3y = 16$

x	8	5
y	0	2

The graphs of $3x + 2y \geq 14$ and $2x + 3y \geq 18$ are the half-planes not containing the origin. The shaded region in the graph represents the feasible region. Its vertices are A(8, 0), B(2, 4) and C(0, 7).

Value of z at A(8, 0) = Rs.80.

Value of z at B(2, 4) = Rs.52.

Value of z at C(0, 7) = Rs.56.

Therefore the minimum value of z exists at the point B(2, 4). So the dietician has to mix 2 Kg. of food-I, and 4Kg of food-II so that the cost is minimum.

2.4.4 Example: A furniture manufacturing company plans to make two products chairs and tables from its available resources, which consists of 400 board feet of wood and 450 man-hours. To make a chair it requires 5 board feed and 10 man-hours and yields a profit of Rs.45. Each table uses 20 board feet and 15 man-hours and has a profit of Rs.80. How many chairs and tables, the company can make so that it gets maximum profit. (i) Formulate the mathematical model; (ii) Solve this problem by graphical method.

Solution: (i) Already we converted the given problem into the mathematical model (Refer Example No. 2.1.2). The related mathematical model is

$$5x_1 + 20x_2 \leq 400$$

$$10x_1 + 15x_2 \leq 450$$

$$x_1 \geq 0, x_2 \geq 0, \ \max z = 45x_1 + 80x_2,$$

where x_1 = the number of chairs, and x_2 = the number of tables.

(ii). Since there are only two variables in the system, we can solve it by graphical method. To draw the graph for the constraints, we consider

$$5x_1 + 20x_2 = 400 \text{ and } 10x_1 + 15x_2 = 450.$$

We draw these two lines on the graph. Consider the line $5x_1 + 20x_2 = 400$ (say, line-1). If $x_1 = 0$, then $x_2 = 20$. Now $(x_1, x_2) = (0, 20)$ is a point on line-1. If $x_2 = 0$ then $x_1 = 80$. Therefore $(x_1, x_2) = (80, 0)$ is a point on line-1. Represent these two points on the graph and join them to get line-1 (L–1).

Consider line–2 (that is, $10x_1 + 15x_2 = 450$). If $x_1 = 0$ then $x_2 = 30$. If $x_2 = 0$ then $x_1 = 45$.

Therefore (0, 30) and (45, 0) are two points on line–2. If we join these two points, then we get L-2 (line-2). The dotted region in the graph represents the set of all points which satisfy the constraints $5x_1 + 20x_2 \le 400$, $10x_1 + 15x_2 \le 450$, $x_1 \ge 0$, $x_2 \ge 0$.

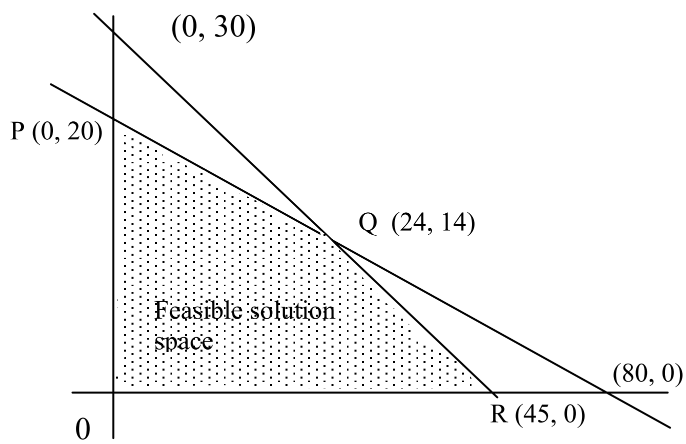
The point of intersection of the two lines $5x_1 + 20x_2 = 400$, and $10x_1 + 15x_2 = 450$ is Q(24,14).

A result in linear programming says that optimal solution exists at one of the extreme points of the feasible region (The feasible region means the set of all points which satisfy the constraints and non-negativity restrictions).

Now we consider the values of z at the extreme/corner points of this region.

The value of z at O(0,0) is 0. The value of z at P(0,20) is 1600.

The value of z at Q(24,14) is 2200. The value of z at R(45,0) is 2025.



The maximum value of z (that is 2,200) exists at the point Q(24, 14).

Conclusion: The company will get the maximum profit of Rs. 2200 if it manufacture $x_1 = 24$ chairs and $x_2 = 14$ tables.

2.4.5 Self Assessment Question 1: Solve the following L.P.P. by Graphical method $x_1 + 4x_2 \le 80$, $2x_1 + 3x_2 \le 90$, $x_1 \ge 0$, $x_2 \ge 0$, Max $z = 45x_1 + 80x_2$.

2.4.6 Self Assessment Question 2: Solve the following linear programming problem by using Graphical method.

Solve $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$, $x_1, x_2 \ge 0$, Max $z = 5x_1 + 3x_2$.

Basic Solutions

2.4.7 Result: Let $\{a_1, a_2, \ldots, a_n\}$ be a basis for a vector space V over F, $0 \neq b \in V$, $b = \alpha_1 a_1 + \ldots + \alpha_i a_i + \ldots + \alpha_n a_n$ and $\alpha_i \neq 0$ for some i, $\alpha_i \neq 0$. Then $B = \{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b\}$ is a basis for V.

(Proof is not necessary)

2.4.8 Definition: Given a system of m simultaneous linear equations in n unknowns, AX = b, m < n and m = rank of A. If any (m×m)-non-singular matrix B is chosen from A, and if all the (n–m) variables that are not associated with the columns of this matrix B are set equal to zero, then the solution to the resulting system of equations is called a **basic solution**. The m variables that are associated with the columns of B are called **basic variables**.

2.4.9 Note: (i) Let B be a (m×m)-matrix formed by m linearly independent columns of A. Then $X_B = B^{-1}b$ is a basic solution.

(ii) Since m columns are to be chosen from n columns, to form B, the number of possible basic solutions is ${}^n c_m = \dfrac{n!}{m!(n-m)!}$.

(iii) To obtain all ${}^n c_m$ solutions, every set of m columns of A would have to be linearly independent [follows from the definition "Basic solution").

2.4.10 Definitions: (i) A basic solution X_B to the system AX = b is said to be **degenerate** if one or more of the basic variables vanish.

(ii) A solution that satisfy the non-negativity restrictions is called a **feasible solution**.

(iii) Any feasible solution that optimizes the objective function is called an **optimal feasible solution**.

2.4.11 Self Assessment Question 3: Find all basic solutions for the system of simultaneous equations

$$2x_1 + 3x_2 + 4x_3 = 5$$
$$3x_1 + 4x_2 + 5x_3 = 6$$

2.4.12 Self Assessment Question 4: Given $x_1 + 4x_2 - x_3 = 3$, $5x_1 + 2x_2 + 3x_3 = 4$. Determine the basic solutions.

2.4.13 Note: The following conditions for a linear programming problem $AX = b, X \geq 0$ are equivalent: (i) All basic solutions exist; and (ii) Any set of m columns of A are linearly independent. (Follows from the definition of basic solution).

2.4.14 Lemma: Let $x_1, x_2, \ldots, x_m$ is a basic solution and $a_1, a_2, \ldots, a_m$ are corresponding column vectors in A [Equivalently, $a_1x_1 + \ldots + a_mx_m = b$ and a_i's are linearly independent). Then the following are equivalent: (i) $x_i = 0$ for some i (that is, the given basic solution is degenerate); and
(ii) $\{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots a_m, b\}$ is a linearly dependent set.

(Proof is not necessary)

2.4.15 Theorem: Let $X_B = B^{-1}b$ denotes a basic solution. Then the following conditions are equivalent: (i) X_B is non-degenerate; and (ii) b and every set of (m-1) columns from B forms a linearly independent set.

(Proof is not necessary)

2.4.16 Theorem: The following conditions are equivalent for a given linear programming problem $AX = b, X \geq 0$.
(i) All basic solutions exist and all are non-degenerate, and
(ii) Every set of m columns from the augmented matrix Ab = (A, b) is linearly independent.

(Proof is not necessary)

Euclidean Spaces

2.4.17 Note: (i) $E^n = \{ (a_1, a_2, \ldots, a_n) \;/\; a_i \in R, 1 \leq i \leq n\}$ where R = the set of all real numbers. Addition and scalar multiplication were defined as: $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$, $\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. This E^n is called the **n-dimensional Euclidean space**.
We define $(x_1, x_2, \ldots, x_n).\,(y_1, \ldots, y_n) = x_1y_1 + \ldots + x_ny_n$ and $|\,(x_1, x_2, \ldots, x_n)\,| = (x_1^2 + \ldots + x_n^2)^{1/2}$.
Distance between $(a_1, a_2, \ldots, a_n), (b_1, \ldots, b_n)$ is defined as $[\Sigma\, (a_i - b_i)^2]^{1/2}$.

2.4.18 Definitions: (i) Any subset of E^n is called a **point set**; (ii). A **hyper sphere** X in E^n with center at $a \in E^n$ and radius $\varepsilon > 0$ is defined as follows: $X = \{x \in E^n \;/\; |x - a| = \varepsilon\}$.

[Note that if n = 2 then $X = \{x \in E^2 \;/\; |x - a| = \varepsilon\} = \{x \in E^2 \;/\; (x_1 - a_1)^2 + (x_2 - a_2)^2 = \varepsilon^2\}$. Here $a = (a_1, a_2)$. It is clear that X is a circle. That is, if n = 2 then hypersphere is a circle. Similarly one can observe that if n = 3 then hypersphere is a sphere].

(iii) The **inside of a hypersphere** with center at a and radius ε is the set of points $X = \{x \in E^n / |x - a| < \varepsilon\}$. This is nothing but, the ε-neighborhood about the point $a \in E^2$.

(iv) A point $a \in E^n$ is an **interior point** of a set A if there exists an ε-neighborhood about a which is contained in A.

(v) A point $a \in E^n$ is a boundary point of the set A if every ε-neighborhood about 'a' contains points which are in the set A and also points which are not in A.

(vi) If A contains only its interior points, then A is called an **open** set. A set A is said to be **closed** if it contains all of its boundary points.

(vii) A is said to be bounded from below if there exists $r \in E^n$ such that $r \leq a$ for all $a \in A$ [that is, if $r = (r_1, r_2, \ldots, r_n)$ then $r_i \leq a_i$ $(1 \leq i \leq n)$ for all $a = (a_1, a_2, \ldots, a_n) \in A$]. Now it is clear that a set which is bounded from below has a lower limit on each component of every point in A.

2.4.19 Definitions: (i) In E^n, we define the **line** through the two points $x_1, x_2 \in E^n$ $(x_1 \neq x_2)$ to be $X = \{x \in E^n / x = \lambda x_2 + (1 - \lambda)\, x_1 \text{ where } \lambda \in R\}$.

(ii) The **line segment** joining $x_1, x_2 \in E^n$ is the set $X = \{x \;/\; x = \lambda x_2 + (1-\lambda)\, x_1, 0 \leq \lambda \leq 1\}$.

(iii) Let $c_i \in R$, $1 \leq i \leq n$ such that not all $c_i = 0$ and $z \in R$. The set $H = \{ (x_1, x_2, \ldots, x_n) \in E^n \;/\; c_1x_1 + c_2x_2 + \ldots + c_nx_n = z\}$ is called a **hyperplane** for the given values c_i and z.

[Note that in a linear programming problem, the set of all $x = (x_1, \ldots, x_n)$ yielding a given value of the objective function z is a hyperplane].

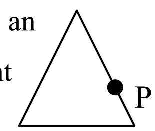
Convex Sets

2.4.20 Definitions: (i) A set X is **convex** if for any points x_1, x_2 in X, the line segment joining these points is also in X. (That is, $x_1, x_2 \in X, 0 \leq \lambda \leq 1 \Rightarrow \lambda x_2 + (1-\lambda)x_1 \in X$). By convention, a set containing only a single point is also a convex set.

(ii) $\lambda x_2 + (1-\lambda)x_1$ (where $0 \leq \lambda \leq 1$) is called a **convex combination** of x_1 and x_2.

(iii) A point x of a convex set X is said to be an **extreme point** if there do not exist $x_1, x_2 \in X$ ($x_1 \neq x_2$) such that $x = \lambda x_2 + (1-\lambda)x_1$ for some λ with $0 < \lambda < 1$.

2.4.21 Example: (i) In a triangle, the interior forms a convex set. Vertices are extreme points. Any point that lies on a side is a boundary point. So every boundary point need not be an extreme point. For example, in the figure on right side, p is a boundary point but not an extreme point .



(ii) Consider the circle $X = \{(x_1, x_2) \;/\; x_1^2 + x_2^2 \leq 1\}$ in R^2. It is a convex set. Every point on the circumference is both boundary and extreme point.

2.4.22 Definition: (i) A convex combination of a finite number of points $x_1, x_2, \ldots, x_m$ is defined as a point $x = \mu_1 x_1 + \ldots + \mu_m x_m$ where $\mu_i \geq 0$ for $1 \leq i \leq m$ and $\mu_1 + \mu_2 + \ldots + \mu_m = 1$.

(ii) The set of all convex combinations of a finite number of points is called the **convex polyhedron** spanned by these points.

(iii) A convex polyhedron spanned by (n+1) points (in E^n) which do not lie on a hyperplane is called a **simplex**.

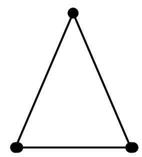
2.4.23 Result: Let X be a convex set and $x_i \in X$, $1 \leq i \leq m$, $\mu_i \geq 0$ for $1 \leq i \leq m$ and $\mu_1 + \ldots + \mu_m = 1$. Then $\mu_1 x_1 + \ldots + \mu_m x_m \in X$. [In the other words, if X is convex and $x_i \in X$, $1 \leq i \leq m$, then the convex polyhedron spanned by x_i, $1 \leq i \leq n$ is a subset of X].

(Proof is not necessary)

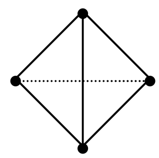
2.4.24 Definition: The **convex hull** of a set A is defined as the intersection of all convex sets containing A.

2.4.25 Example: (i) Consider $E^2 = R \times R$. By definition $\{(x_1, x_2) \in E^2 \;/\; c_1x_1 + c_2x_2 = k\}$ is a hyperplane in E^2. We know that this set is nothing but a straight line in the two dimensional Euclidean space $R^2 = E^2$.

(ii) By definition, a simplex in E^2 is a convex polyhedron / convex hull spanned by (n+1 = 2+1 = 3) points which do not lie on a hyperplane. Therefore in E^2, a simplex is the convex set generated by three points which do not lie on a straight line. Therefore a simplex in $R^2 = E^2$ is a triangle.



(iii) Consider E^3. In E^3, a hyperplane is a plane. A simplex in E^3 is the convex polyhedron spanned by four points 4, no three of them lies in a plane. Hence a simplex in E^3 is a prism (or tetrahedron).



(iv) A simplex in R is a line segment.

(v) Note that in $R^n = E^n$, a hyperplane is of dimension (n–1) and a simplex is of dimension n.

2.4.26 Theorem: Given any closed convex set X and a point y. Then either $y \in X$, or there exists a hyperplane H such that $y \in H$, and X is contained in one open half space produced by H.

(Proof is not necessary)

2.4.27 Note: In the proof of above theorem we used the following fact: Suppose $u \neq w$. Since $u, w \in X$, $(1-\lambda)w + \lambda u$ for $(0 \leq \lambda \leq 1)$ is in X. Now by selection of w, we have $|[(1-\lambda)w + \lambda u] - y| \geq |w - y|$

$\Rightarrow |(w - y) + \lambda(u - w)| \geq |w - y| \Rightarrow |(w - y) + \lambda(u - w)|^2 \geq |w - y|^2$

$\Rightarrow \{(w - y) + \lambda(u - w)\}^2 \geq (w - y)^2 \Rightarrow (w-y)^2 + 2\lambda(w-y)(u - w) + \lambda^2 (u - w)^2 \geq (w - y)^2$

$\Rightarrow 2\lambda(w - y)(u - w) + \lambda^2(u - w)^2 \geq 0 \Rightarrow 2(w - y)(u - v) + \lambda(u - w)^2 \geq 0.$

We know that $\lambda \geq 0$ and this is true for all $\lambda \geq 0$. Taking limit as $\lambda \to 0$ we get that $2(w - y)(u - w) \geq 0$

$\Rightarrow (w - y)(u - w) \geq 0.$

2.4.28 Definition: Given a boundary point w of a convex set A. Then CX = z is called a supporting hyperplane at w if Cw = z and if all the points of A lies in one closed half space produced by the hyperplane (that is, $Cu \geq z$ for all $u \in A$ or $Cu \leq z$ for all $u \in A$).

2.4.29 Lemma: Let X be a closed convex set, $y \notin X$ and $w \in X$ such that $\| y - w\| = \min \{\| y - u\| \ / \ u \in X\}$. Then w is a boundary point of X.

(Proof is not necessary)

2.4.30 Lemma: Let X be a closed convex set, $y \notin X$ and $w \in X$ such that $\| w - y\| = \min \{\| u - y\| \ / \ u \in X\}$. Then there exist a supporting hyperplane at w (to X).

(Proof is not necessary)

2.4.31 Theorem: If w is a boundary point of a closed convex set X, then there is atleast one supporting hyper plane at w.

(Proof is not necessary)

2.4.32 Result: Consider a linear programming problem AX = b, max z = CX. If a hyperplane CX = z corresponds to the optimal value of z, then no point on the hyperplane can be an interior point of the convex set X of all feasible solutions.

(Proof is not necessary)

2.4.33 Note: From the above note, we can conclude that if x_0 is an optimal solution for a linear programming problem, then x_0 must be a boundary point of the convex set of all feasible solutions.

2.4.34 Note: Any hyperplane contains no interior points (or interior of a hyperplane is empty).

Verification: Let H = {X / CX = z} be the given hyperplane. In a contrary way, suppose $x_0 \in H$ and x_0 is an interior point of H. Now there exists $\varepsilon > 0$ such that $S_\varepsilon(x_0) \subseteq H$. Write $x_1 = x_0 + \frac{\varepsilon}{2}.\frac{C}{|C|}$. Then $x_1 - x_0 = \frac{\varepsilon}{2}.\frac{C}{|C|} \Rightarrow |x_1 - x_0| = \frac{\varepsilon}{2} < \varepsilon \Rightarrow x_1 \in S_\varepsilon(x_0) \subseteq H$. Now $Cx_1 = C(x_0 + \frac{\varepsilon}{2}.\frac{C}{|C|}) = Cx_0 + \frac{\varepsilon}{2}.\frac{C^2}{|C|} = z + \frac{\varepsilon}{2}|C| > z \Rightarrow Cx_1 > z$, a contradiction (since $x_1 \in H$).

2.4.35 Lemma: A closed convex set X which is bounded from below has an extreme point.

(Proof is not necessary)

2.4.36 Theorem : A closed convex set X which is bounded from below has extreme points in every supporting hyperplane.

(Proof is not necessary)

2.4.37 Theorem : The set of all feasible solutions for a given L.P.P. $AX = b$, max $z = CX$, $X \geq 0$ is a convex set.

(Proof is not necessary)

2.4.38 Theorem: If a non-empty closed, bounded and convex set X has only a finite number of extreme points y_i, $1 \leq i \leq m$, then every point in X is a convex combination of y_i, $1 \leq i \leq m$.

(Proof is not necessary)

2.4.39 Note: From the above theorem, we have that every closed, bounded, convex set with a finite number of extreme points is a convex polyhedron.

2.4.40 Definitions: (i) Let x_1, x_2 be two distinct extreme points of the convex set X. The line segment joining x_1 and x_2 is called an **edge of the convex set** if it is the intersection of X with a supporting hyperplane.

(ii) If x^* is an extreme point, and if there exists another point $x \in X$ such that the line $\{ x^* + \lambda(x - x^*) \ / \ \lambda \geq 0 \}$ is in X and also it is in the intersection of X with a supporting hyperplane, then this line is said to be an edge of the convex set X which originates at x^* and extends to infinity.

(iii) Two distinct extreme points x_1, x_2 of the convex set X are called **adjacent** if the line segment joining x_1 and x_2 is an edge of the convex set.

2.4.41 Definitions: (i) A subset S of E^n is called a **cone** if $\{ \lambda x \ / \ x \in S \text{ and } \lambda \geq 0 \} \subseteq S$.

(ii) If $X \subseteq E^n$, then $C = \{\mu x \ / \ \mu \geq 0, x \in X\}$ is called the **cone generated** by X.

(iii) 0 is an element of every cone and it is called the **vertex** of the cone.

(iv) If C is a cone, then $\overline{C} = \{y \in E^n \ / \ -y \in C\}$ is called a **negative cone**.

(v) If C_1 and C_2 are two cones, then $C_1 + C_2 = \{u + v \ / \ u \in C_1, v \in C_2\}$.

(vi) A subset S of E^n is called a **convex cone** if it is a cone and a convex set.

2.4.42 Result: Let $\phi \neq S \subseteq E^n$. (i) S is a convex cone, if and only if (ii) $v_1, v_2 \in S, \ \mu \geq 0 \Rightarrow \mu.v_1 \in S$ and $v_1 + v_2 \in S$.

(Proof is not necessary)

2.4.43 Result : (i) Sum of two cones is a cone. (ii) Sum of a finite number of cones is a cone.

(Proof is not necessary)

2.4.44 Result : The cone generated by a convex set X is a convex cone.

(Proof is not necessary)

2.4.45 Definitions: (i) Let $x \in E^n$. The set $L = \{\mu x \ / \ \mu \geq 0\}$ is defined as an **half line** (note that half line is a cone generated by single point). In this case, we say that L is the **half line** generated by x.

(ii) The **dimension** of a cone C is defined as the maximum number of linearly independent vectors in C.

(iii) If L_i, $1 \leq i \leq n$ are half lines, then $C = \Sigma L_i = \{\Sigma x_i \ / \ x_i \in L_i, 1 \leq i \leq n\}$ is called a **convex polyhedral cone**.

2.4.46 Note: (i) Since each half line is convex, the sum of finite number of half lines is also convex. Hence convex polyhedral cone is convex.

(ii) If the half lines L_i's are generated by x_i, $1 \leq i \leq m$ then convex polyhedral is $\{\sum_{i=1}^{m} \mu_i x_i \ / \ \mu_i \geq 0\}$.

2.4.47 Theorem: The cone $C = \{\mu x \ / \ x \in X, \mu \geq 0\}$ generated by convex polyhedron X is a convex polyhedral cone.

(Proof is not necessary)

2.4.48 Problem: Is the union of two convex sets is a convex set ? Justify your answer

Solution: Union of two convex sets need not be convex.
(**Example**) Consider the set R of all real numbers. Write $X_1 = [0, 2]$ and $X_2 = [4, 6]$. We know that all the intervals in R are convex. Write $X = X_1 \cup X_2$. Now we wish to show that X is not convex. Now $1 \in X_1 \subseteq X$ and $5 \in X_2 \subseteq X$. Therefore $1, 5 \in X$. Take $\lambda = ½$. Then $3 = \frac{1+5}{2} = ½(1) + ½(5) = \lambda 1 + (1-\lambda)5$ is a convex combination of 1 and 5, but it is not an element of X. Hence X is not a convex set. Therefore we may conclude that union of two convex sets need not be a convex set.

2.4.49 Problem: Prove that a hyperplane is a closed set

Solution: Let H be a hyperplane. By the definition of the hyperplane $H = \{X \in R^n / CX = z\}$ where $C \in R^n$ and $z \in R$.

To show that H is closed, it is enough to show that every limit point of H is in H. Let Y be a limit point of H. Then there exists a sequence $\{X_n\}$ of elements from H such that $X_n \to Y$. Since $X_n \in H$, we have that $CX_n = z$. Consider $CY = C(\lim X_n) = \lim CX_n = \lim z = z$. Therefore $CY = z$, which implies $Y \in H$. Thus we proved that every limit point of H is in H. Hence H is a closed set.

2.5 SUMMARY

Linear Programming is one of the most important optimization (maximization/minimization) techniques developed in the field of Operations Research

We used graphical method for solving Linear Programming Problem when only two variables were considered.

In this lesson properties of Linear Programming Problem and the graphical method of solving a Linear Programming Problem are discussed.

2.6 TECHNICAL TERMS

Constraints	: An upper limit on the availability of a resource or a lower limit on necessary levels to achieve.
Corner (or Extra) points	: The points of feasible solution region formed by the constraints and the non-negativity restrictions of Linear Programming Problem.
Feasible solution region	: The region defined by the constraints and the non-negativity restrictions of the Linear Programming problem.
Linear programming	: A linear deterministic model used to solve the problem of allocating limited resources among competing activities.
Non-negativity restrictions	: The conditions that require the values of the decision variables to be either zero or more than zero
Objective functions	: An equation that specifies the dependent relationship between the decision objective and the decision variables.

2.7 ANSWERS TO SELF ASSESSMENT QUESTIONS

Answer to self Assessement Question 1(2.4.5): $x_1 = 24$, $x_2 = 14$ and max $z = 2200$.

Answer to Self Assessment Question 2 (2.4.6): $x_1 = 1.053$, $x_2 = 2.368$.

and Max $z = 5 \times 1.053 + 3 \times 2.368 = 12.37$.

Answer to Self Assessment Question (2.4.11):

The possible basic solutions are (i) $x_1 = -2$, $x_2 = 3$, (ii) $x_1 = -½$, $x_3 = 3/2$, and (iii) $x_2 = -1$, $x_3 = 2$.

Answer to Self Assessment Question (2.4.12): The set of all basic solutions are (i) $x_2 = 13/14$, $x_3 = 5/7$, (ii) $x_1 = 13/8$, $x_3 = -11/8$, and (iii) $x_1 = 5/9$, $x_2 = 11/18$).

2.8 MODEL QUESTIONS

2.8.1 Model Question 1. A firm manufactures two products, each of which must be processed through two departments, 1 and 2. The hourly requirements per unit for each product in each department, the weekly capacities in each department, selling price per unit, labour cost per unit, and raw material cost per unit are summerised as follows :

	Product A	Product B	Weakly capacity
Department 1	3	2	120
Department 2	4	6	260
Selling price per unit	Rs. 25	Rs. 30	
Labour cost per unit	Rs. 16	Rs. 20	
Raw material cost per unit	Rs. 4	Rs. 4	

The Problem is to determine the number of units to be produced of each product so as to maximize total contribution of profit. Formulate this as a linear Programming problem.

Ans: Maximize $z = 5x_1 + 6x_2$, subject to the constraints $3x_1 + 2x_2 \leq 120$; $4x_1 + 6x_2 \leq 260$; $x_1, x_2 \geq 0$.

2.8.2 Model Question 2. Vijay Electric Company produces two products P_1 and P_2. Products are produced and sold on a weekly basis. The weekly production cannot exceed 25 for product P_1 and 35 for product P_2 because of limited available facilities. The company employs total of 60 workers. Product P_1 requires 2 man-weeks of labour, while P_2 requires one man-week of labour. Profit margin on P_1 is Rs. 60 and on P_2 is Rs. 40. Formulate it as a linear programming problem to maximize the total profit and solve the problem by using Graphical method.

Ans: Maximize (total profit) $z = 60x_1 + 40x_2$ subject to the constraints $x_1 \leq 25$; $x_2 \leq 35$; $2x_1 + x_2 = 60$; $x_1, x_2 \geq 0$. Maximum profit (= 60 X 25 + 40 X 10) = Rs. 1900.

2.8.3 Model Question 3. The standard weight of a special purpose brick is 5 kg and contains two basic ingredients B_1 and B_2. B_1 costs Rs. 5 per kg and B_2 costs Rs. 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of B_1 and minimum of 2 kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.

Ans: Minimize (total cost) $z = 5x_1 + 8x_2$ subject to the constraints $x_1 \leq 4$; $x_2 \geq 2$; $x_1 + x_2 = 5$; $x_1 \geq 0$; $x_2 \geq 0$. $B_1 = 3$ kgs. $B_2 = 2$ kgs. Minimum cost = Rs. 31.

2.8.4 Model Question 4. The manager of an oil refinery must decide on the optimum mix of two possible blending processes of which the inputs and outputs per production run are as follows:

	Input (units)		Output (units)	
Process	Crude A	Crude B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amount available of crudes A and B is 200 units and 150 units respectively. Market requirements show that atleast 100 units of gasoline X and 80 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Solve the L.P. problem by graphical method.

Ans: Maximize (total profit) $z = 300x_1 + 400x_2$

subject to the constraints $5x_1 + 4x_2 \leq 200$; $3x_1 + 5x_2 \leq 150$; $5x_1 + 4x_2 \geq 100$; $8x_1 + 4x_2 \geq 80$; $x_1 \geq 0$, $x_2 \geq 0$

$x_1 = 400/13$ units under process 1 and $x_2 = 150/13$ units under process 2. Maximum profit is Rs. 1,80,000/13.

2.8.5 Model Question 5. A firm is engaged in breeding chicken. The chicken are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y, Z), it is necessary to buy two additional products, say A and B. One unit of product A contains 36 units of X , 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs Rs. 20 per unit and product B Rs. 40 per unit. Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphical method.

Minimise (total cost) $z = 20x_1 + 40x_2$

Subject to the constraints : $36x_1 + 6x_2 \geq 108$; $3x_1 + 12x_2 \geq 36$; $20x_1 + 10x_2 \geq 100$; $x_1 \geq 0$; $x_2 \geq 0$

4 units of product A and 2 units of product B. minimum cost = Rs. 160.

2.8.6 Model Question 6: Find all basic solutions for the system $x_1 + 2x_2 + x_3 = 4$, $2x_1 + x_2 + 5x_3 = 5$.

Solution : Here A is matrix is $B = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$. In this case $2x_2 + x_3 = 4$, $x_2 + 5x_3 = 5$.

If we solve this, then $x_2 = \frac{5}{3}$ and $x_3 = \frac{2}{3}$. Therefore $x_2 = \frac{5}{3}$, $x_3 = \frac{2}{3}$ is a basic feasible solution.

(ii) If $x_2 = 0$, then the basis matrix is $B = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$. In this case $x_1 + x_3 = 4$, $2x_1 + 5x_3 = 5$.

If we solve this, then $x_1 = 5$ and $x_3 = -1$. Therefore $x_1 = 5$, $x_3 = -1$ is a basic solution. (Note that this solution is not feasible, because $x_3 = -1 < 0$).

(iii) If $x_3 = 0$, then the basis matrix is $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. In this case $x_1 + 2x_2 = 4$.

$2x_1 + x_2 = 5$. If we solve this, then $x_1 = 2$, and $x_2 = 1$. Therefore $x_1 = 2$, $x_2 = 1$ is a basic feasible solution. Therefore (i) $(x_2, x_3) = (5/3, 2/3)$, (ii) $(x_1, x_3) = (5, -1)$, and (iii) $(x_1, x_2) = (2, 1)$ are only the collection of all basic solutions. Note that all these solutions are non-degenerate.

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Name of the Lesson Writer

Prof. Bhavanari Satyanarayana

LESSON - 3

SIMPLEX METHOD (3 Variables)
(APPLICATION OF LPP)

Objectives

The objectives of this lesson are to:

- Learn how to convert the given LPP into set of equalities
- Understand the fundamental theorem of LPP
- Know the concepts like: Slack variable, Surplus variable, feasible solution, basic feasible solution, unbounded solution
- Learn how to improve a given basic feasible solution
- Understand the algorithm of Simplex Method
- Develop the solving technique of Simplex Method

Structure

3.0 Introduction
3.1 Restatement of the given LPP
3.2 Fundamental Theorem of LPP
3.3 Basic Feasible Solutions
3.4 Unbounded Solutions
3.5 Optimality Criteria
3.6 Algorithm of Simplex Method
3.7 Summary
3.8 Technical terms
3.9 Answers to Self Assessment Questions
3.10 Model Questions
3.11 Reference Books

3.0 INTRODUCTION

Simplex method is a straight forward algebraic procedure which is a general method for solving any LP problem with two or more variables; where as the graphical method is applicable for problems with two variables. The algorithm for solving LP problems developed by G. Dantzig in 1947, is named as simplex method.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex of the feasible region) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated finitely many times. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps.

Rather than enumerating all the basic solutions (corner points or vertices of the feasible region) of the LP problem, the simplex method investigates only a “select few” of these solutions. This lesson not only describes the iterative nature of the method; but also provides the computational details of the simplex algorithm.

Next we see that unboundedness may point out the possibility of constructing a poor model. The most likely irregularities in such models are that one or more non redundant constraints have not been accounted for, and the parameters (constants) of some constraints may not have been estimated correctly.

3.1 RESTATEMENT OF THE GIVEN LPP

3.1.1 Restatement of the problem : Suppose the given L.P.P is as follows:

Max/Min $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the linear constraints/inequalities

$$a_{1_1} x_1 + a_{1_2} x_2 + \dots + a_{1_n} x_n \ \{\le, =, \ge\}\ b_1$$
$$a_{2_1} x_1 + a_{2_2} x_2 + \dots + a_{2_n} x_n \ \{\le, =, \ge\}\ b_2$$
$$\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots$$
$$a_{m_1} x_1 + a_{m_2} x_2 + \dots + a_{m_n} x_n \ \{\le, =, \ge\}\ b_m$$
$$\text{and } x_j \ge 0 \text{ for } j = 1, 2, \dots, n.$$

Here a_{i_j}, b_i and c_i are given constants and $m < n$.

(i) See all the b_i’s are non-negative (if one of the b_i’s is negative then multiply the corresponding inequality with (-1) on both sides).

(ii) Suppose an inequality contains the sign $\leq$ (with $b_i \geq 0$). For example, suppose the inequality is $a_1x_1 + a_2x_2 + \ldots + a_rx_r \leq b$. Then we introduce a new variable $x^1 \geq 0$ (where $x^1 = b - (a_1x_1 + a_2x_2 + \ldots + a_rx_r)$) and we convert the inequality into the equality $a_1x_1 + a_2x_2 + \ldots + a_rx_r + x^1 = b$. Here we call the variable x^1 as a **slack** variable.

(iii) Suppose an inequality contains the sign $\geq$ (with $b_i \geq 0$). For example, suppose the inequality is $a_1x_1 + a_2x_2 + \ldots + a_rx_r \geq b$. Then we introduce a new variable $x^1 \geq 0$ (where $x^1 = (a_1x_1 + \ldots + a_rx_r) - b$) and we convert the inequality into the equality $a_1x_1 + \ldots + a_rx_r - x^1 = b$. Here the variable x^1 is called a **surplus** variable.

(iv) Now the matrix A can be written as

$$A = \begin{bmatrix} a_{1_1} & a_{1_2} & \ldots & a_{1_r} & 1 & 0 & \ldots & 0 & \ldots & 0 \\ a_{2_1} & a_{2_2} & \ldots & a_{2_r} & 0 & 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i_1} & a_{i_2} & \ldots & a_{i_r} & 0 & 0 & \ldots & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m_1} & a_{m_2} & \ldots & a_{m_r} & 0 & 0 & \ldots & \ldots & \ldots & 0 \end{bmatrix}.$$

Note that in the matrix, 1's corresponds to the slack variables, -1's corresponds to the surplus variables.

3.1.2 Example: Convert $2x_1 + 3x_2 \leq 6$, $x_1 + 7x_2 \geq 4$, $x_1 + x_2 = 3$ into a set of equalities.

Solution: Since $2x_1 + 3x_2 \leq 6$ contains the sign " $\leq$ ", we introduce a slack variable $x_3 \geq 0$ and so we get the equality $2x_1 + 3x_2 + x_3 = 6$.

Since $x_1 + 7x_2 \geq 4$ contains the sign ' $\geq$' we use a surplus variable x_4 then the related equality is $x_1 + 7x_2 - x_4 = 4$.

Therefore the given system is $2x_1 + 3x_2 + x_3 = 6$, $x_1 + 7x_2 - x_4 = 4$, $x_1 + x_2 = 3$. Now the system (in the matrix form) is $AX = b$ where

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 7 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}.$$

3.2 FUNDAMENTAL THEOREM OF LPP

3.2.1 Fundamental Theorem of L.P.P.: Given a set of m simultaneous linear equations in n unknowns/variables, $n \geq m$, $AX = b$, with $r(A) = m$. If there is a feasible solution $X \geq 0$, then there exists a basic feasible solution.

Proof: Assume that there exists a feasible solution with p ($p \le n$) positive variables. Without loss of generality, we assume that the first p variables
$x_1, x_2, .., x_p$ are positive. Then this feasible solution can be written as

$$\sum_{j=1}^{p} x_j a_j = x_1 a_1 + \dots + x_p a_p = b \text{ (where } a_j \text{ is the } j^{th} \text{ column of A) ----- (i)}$$

and hence $x_j > 0$ for $1 \le j \le p$ and $x_j = 0$ for $p + 1 \le j \le n$. Now the vectors a_j, $1 \le j \le p$ may be linearly independent or linearly dependent.

Case (i): Suppose $\{ a_j \ / \ 1 \le j \le p \}$ is a linearly independent set. Since $r(A) = m$, the maximum number of columns in A which form a linearly independent set is m. Therefore $p \le m$.

Part–I: If $p = m$ then the given solution is a basic solution. Since $x_i > 0$ for $i = 1$ to p, this solution is non-degenerate. Hence if $p = m$ then the given solution is a non-degenerate basic feasible solution.

Part–II: Suppose $p < m$. Then there are $(m–p)$ columns of A which together with the p columns, form a basis for E^m. Without loss of generality we may assume that these $(m–p)$ columns are $a_{p+1}, a_{p+2}, \dots, a_m$. If we write $x_{p+1} = x_{p+2} = \dots = x_m = 0$, then we get a degenerate basic feasible solution.

Case (ii): Suppose $\{a_j \ / \ 1 \le j \le p \}$ is a linearly dependent set. Now we reduce the number of positive variables step by step until the columns associated with these positive variables are linearly independent. For this we follow the following steps.

Step–I: Since $\{a_j \ / \ 1 \le j \le p \}$ are 1inearly dependent , there exist

α_j, $1 \le j \le p$ (not all zero) such that $\sum_{j=1}^{p} \alpha_j a_j = 0$. Therefore $\alpha_k \ne 0$ for some k.. Without loss of

generality we assume that $\alpha_k > 0$. For any k with $\alpha_k \ne 0$, we have $a_k = - \sum_{\substack{j\ne k \\ j=1}}^{p} \frac{\alpha_j}{\alpha_k} a_j$ -------- (ii).

Take r such that $\frac{x_r}{\alpha_r} = \min \left\{ \frac{x_j}{\alpha_j} \ / \ \alpha_j > 0 \right\}$ ----- (iii).

By substituting (ii) in (i), we get $b = \sum_{\substack{j\ne r \\ j=1}}^{p} x_j a_j + x_r \left(- \sum_{\substack{j\ne r \\ j=1}}^{p} \frac{\alpha_j}{\alpha_r} a_j\right) = \sum_{\substack{j\ne r \\ j=1}}^{p} \left[x_j - x_r\left(\frac{\alpha_j}{\alpha_r}\right) \right] a_j$ ------ (iv).

In (iv), we have a solution and the number of non-zero variables in this solution is less than or equal to $(p - 1)$.

Step–II: Now we show that the solution that exists in (iv) is a feasible solution. To show this we have to show that $x_j - x_r(\frac{\alpha_j}{\alpha_r}) \geq 0$.

If $\alpha_j = 0$, then $x_j - x_r(\frac{\alpha_j}{\alpha_r}) = x_j - x_r(\frac{0}{\alpha_r}) = x_j > 0$.

If $\alpha_j < 0$, then $x_j - x_r(\frac{\alpha_j}{\alpha_r}) = x_j + x_r \frac{|\alpha_j|}{\alpha_r} \geq 0$.

If $\alpha_j > 0$, then $x_j - x_r(\frac{\alpha_j}{\alpha_r}) = \alpha_j\left[\frac{x_j}{\alpha_j} - \frac{x_r}{\alpha_r}\right] \geq 0$ (because $\frac{x_r}{\alpha_r} \leq \frac{x_j}{\alpha_j}$, by (iii)).

Hence $a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_p$ are columns corresponding to $(p-1)$ non-negative variables.

Step–III: If $a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_p$ are linearly independent, then by case-(i), a basic feasible solution exists. If $a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_p$ are not linearly independent, we employ the process in step-I (of case-ii) to reduce the number of variables until the corresponding columns become 1inearly independent. Finally, we get a basic feasible solution (by case (i)).

3.3 BASIC FEASIBLE SOLUTIONS

3.3.1 Some definitions and Notations:

(i) The j^{th} column of A is denoted by a_j for $1 \leq j \leq n$.

(ii) Form a matrix B with m linearly independent columns of A. Now the columns of B forms a basis for E^m. The columns of B will be denoted by $b_1, b_2, \ldots, b_m$. So $B = (b_1, b_2, \ldots, b_m)$.

(iii) Any column a_j of A can be written as a linear combination of $b_1, b_2, \ldots, b_m$. Therefore

$$a_j = y_{1_j} b_1 + y_{2_j} b_2 + \ldots + y_{m_j} b_m = \sum_{i=1}^{m} y_{i_j} b_i = (b_1, \ldots, b_m)\begin{bmatrix} y_{1_j} \\ y_{2_j} \\ \vdots \\ y_{m_j} \end{bmatrix} = By_j \quad \text{where} \quad y_j = \begin{bmatrix} y_{1_j} \\ y_{2_j} \\ \vdots \\ y_{m_j} \end{bmatrix}.$$

Hence $y_j = B^{-1}a_j$. Here, in y_{i_j} the subscript i refers to b_i and j refers to a_j.

(iv) The corresponding basic solution is $X_B = B^{-1}b$, where

$X_B = [x_{B_i}, \ldots, x_{B_m}]$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. If $x_{B_i} > 0$, then we say that b_i is in the basis at positive level.

and a degenerate solution is a basic solution to the system AX = B if one or more of the basic variables vanish.

(v) We write C_B: = $(C_{B_1}, C_{B_2}, \ldots, C_{B_m})$ where C_{B_i} is the coefficient of X_{B_i} in the objective function z. If X_B is a basic feasible solution, then the corresponding value of the objective function is $z = C_BX_B$. (We also write $z_B = C_BX_B$).

(vi) A new variable z_j is given by $z_j = y_{1_j} C_{B_1} + y_{2_j} C_{B_2} + \ldots + y_{m_j} C_{B_m} = \sum_{i=1}^{m} y_{i_j} C_{B_i} = C_B.Y_j$.

Now for each a_j in A, we have a ‘ z_j ’. If there is a change in the columns of B, then there may be a change in the value of z_j.

Now we understand the above notation with the help of the following example:

3.3.2 Example: Consider the L.P.P: $4x_1 + 2x_2 + x_3 + x_4 = 2$,

$x_1 + 2x_2 + 3x_3 - x_5 = 1$, $x_i \geq 0$, $1 \leq i \leq 5$, max $z = 2x_1 + x_2 + 3x_3$, x_4 is a slack variable and x_5 is a surplus variable .

(i). Here $a_1 = [4, 1]$, $a_2 = [2, 2]$, $a_3 = [1, 3]$, $a_4 = [1, 0]$, $a_5 = [0, -1]$, $b = [2, 1]$.

(ii). Let us write $B = [b_1, b_2]$ with $b_1 = a_3$ and $b_2 = a_1$. (Clearly b_1, b_2 are linearly independent columns in E^2).

(iii). $X_B = B^{-1}b = -\frac{1}{11}\begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/11 \\ 5/11 \end{bmatrix} \Rightarrow \begin{bmatrix} X_{B_1} \\ X_{B_2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2/11 \\ 5/11 \end{bmatrix}$ is a basic feasible solution (Note that here, $x_2 = x_4 = x_5 = 0$).

(iv). $C_B = [C_{B_1}, C_{B_2}] = [c_3, c_1] = [3, 2]$.

(v). Any vector a_j in A can be written as linear combination of b_j’s.

For example, $Y_2 = B^{-1}a_2 = -\frac{1}{11}\begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} y_{1_2} \\ y_{2_2} \end{bmatrix}$.

Therefore $a_2 = y_{1_2} b_1 + y_{2_2} b_2 = y_{1_2} a_3 + y_{2_2} a_1 = \frac{6}{11} a_3 + \frac{4}{11} a_1$.

(vi). $z_2 = C_B Y_2 = [3, 2]\begin{bmatrix} 6/11 \\ 4/11 \end{bmatrix} = \frac{26}{11}$ and $z = C_B X_B = [3, 2]\begin{bmatrix} 2/11 \\ 5/11 \end{bmatrix} = \frac{16}{11}$.

3.3.3 To reduce a given feasible solution into a basic feasible solution:

Suppose $x_1, x_2, \ldots, x_k$ is a feasible solution.

Case-(i): If the corresponding vectors $a_1, a_2, \ldots, a_k$ are 1inearly independent then the given solution is also basic.

Case-(ii): Suppose $a_1, a_2, \ldots, a_k$ are 1inearly dependent. Then there exist scalars α_i, $1 \le i \le k$. (not all zero) such that $\alpha_1 a_1 + \ldots + \alpha_k a_k = 0$.

(i) Consider $\frac{x_r}{\alpha_r} = \min \{\frac{x_i}{\alpha_i} \mid \alpha_i > 0\}$.

(ii) The new values of the variables are to be calculated by using the formula: $\hat{x}_j = x_j - \frac{x_r}{\alpha_r}.\alpha_j$.

(iii) Here we got a new solution in which $\hat{x}_r = 0$.

Therefore the number of the column vectors corresponding to new feasible solution is less than that of the given solution.

If these vectors related to the new solution are linearly independent then the new solution that we got is a basic feasible solution. Otherwise, we repeat the process.

3.3.4 Example: Consider the system $x_1a_1 + x_2a_2 + x_3a_3 = b$, $a_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $a_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $b = \begin{bmatrix} 11 \\ 14 \end{bmatrix}$.

Given a feasible solution $2a_1 + 3a_2 + a_3 = b$. Reduce this feasible solution to a basic feasible solution.

Solution: Given system is $x_1\begin{bmatrix}2\\3\end{bmatrix} + x_2\begin{bmatrix}1\\1\end{bmatrix} + x_3\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}11\\14\end{bmatrix}$. So the equations are

$$2x_1 + x_2 + 4x_3 = 11$$

$$3x_1 + x_2 + 5x_3 = 14.$$

Given feasible solution is $(x_1, x_2, x_3) = (2, 3, 1)$. Now consider $\{a_1, a_2, a_3\}$. Since this set is a linearly dependent set, the given solution is not basic. So we have to reduce the number of variables to find a new solution.

Since a_1, a_2, a_3 are linearly dependent, there exist scalars $\alpha_1, \alpha_2, \alpha_3$ (not all zero) such that $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$.

That is, $2\alpha_1 + \alpha_2 + 4\alpha_3 = 0$ ------- [1]

$3\alpha_1 + \alpha_2 + 5\alpha_3 = 0$ --------[2].

[1] and [2], gives $-\alpha_1 - \alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$.

Substituting $-\alpha_1 = +\alpha_3$ in [1], we get $+2\alpha_1 + \alpha_2 - 4\alpha_1 = 0 \Rightarrow \alpha_2 = 2\alpha_1$, $\alpha_1 = \frac{\alpha_2}{2} = -\alpha_3$.

Now take $\alpha_1 = 1$. Then $\alpha_2 = 2$ and $\alpha_3 = -1$.

Now $\frac{x_r}{\alpha_r} = \min\{\frac{x_i}{\alpha_i} / \alpha_i > 0\} = \min\{\frac{2}{1}, \frac{3}{2}\} = \frac{3}{2} = \frac{x_2}{\alpha_2}$. Therefore $r = 2$ and so the new value $\hat{x}_2$ of the variable x_2 will be equal to zero. The formula to calculate the new values of the variables is given by

$$\hat{x}_i = x_i - \frac{x_r}{\alpha_r}.\alpha_i .$$

Therefore $\hat{x}_1 = x_1 - \frac{x_2}{\alpha_2}.\alpha_1 = 2 - \frac{3}{2} \times 1 = \frac{1}{2}$, $\hat{x}_2 = x_2 - \frac{x_2}{\alpha_2}.\alpha_2 = 0$,

$\hat{x}_3 = x_3 - \frac{x_2}{\alpha_2}.\alpha_3 = 1 - \frac{3}{2}(-1) = \frac{5}{2}$. Hence the new solution is $\begin{bmatrix}\hat{x}_1\\\hat{x}_3\end{bmatrix} = \begin{bmatrix}1/2\\5/2\end{bmatrix} = \begin{bmatrix}x_1\\x_3\end{bmatrix}$.

Now $|(a_1 \ \ a_3)| = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0.$

Therefore $\{a_1, a_3\}$ is a linearly independent set. Hence the solution $x_1 = 1/2$, $x_3 = 5/2$ is a basic feasible solution.

Improving a basic feasible solution

3.3.5 Theorem: Given a basic feasible solution $X_B = B^{-1}b$ to a L.P.P. $AX = b$ with the value of the objective function $z = C_B X_B$. If for any column a_j in A but not in B, the condition $c_j - z_j > 0$ holds, and atleast one $y_{i_j} > 0$ for $i = 1, \dots, m$, then it is possible to obtain a new basic feasible solution $\hat{X}_B = \hat{B}^{-1} b$ (by replacing one of the columns of B by a_j, we get $\hat{B}$) and the new value of the objective function satisfies $\hat{z} \geq z$. Furthermore, if the given basic solution is not degenerate then $\hat{z} > z$.

(Proof is not necessary)

3.3.6 Theorem: If for some a_j in A but not in B, $z_j - c_j > 0$ and atleast one $y_{i_j} > 0$, then it is possible to obtain a new basic feasible solution by replacing one of the columns in B by a_j, and the new value of the objective function $\hat{z}$ satisfies $\hat{z} \leq z_B$. Moreover if the given basic solution is not degenerate, then $\hat{z} < z_B$.

(Proof is not necessary)

3.4 UNBOUNDED SOLUTIONS

3.4.1 Theorem: Given a basic feasible solution $X_B = B^{-1}b$ to a L.P.P., $AX = b$, $X \geq 0$. For this solution, there exists a column a_j in A not in B such that $z_j - c_j < 0$ and $y_{i_j} \leq 0$ for $1 \leq i \leq m$. Then there exists a feasible solution in which $(m + 1)$ variables can be different from zero, with the value of the objective function being arbitrarily large. In this case, the problem has an unbounded solution, if the objective function is to be maximized.

(Proof is not necessary)

3.4.2 Theorem: Let $X_B = B^{-1}b$ be a basic feasible solution. If for some a_j, $z_j - c_j > 0$ and $y_{i_j} \le 0$ $(1 \le i \le m)$, then there exist feasible solutions, in which (m+1) variables can be different from zero, with the value of the objective function being arbitrarily small. In such a case, the problem has an unbounded solution, if z is to be minimized.

(Proof is not necessary)

3.5 OPTIMALITY CRITERIA

3.5.1 Theorem: Given a basic feasible solution $X_B = B^{-1}b$ with $Z_0 = C_BX_B$ to the linear programming problem $AX = b$, $X \ge 0$, max $z = CX$ such that $z_j - c_j \ge 0$ for every column a_j in A (but not in B). Then z_0 is the maximum value of z subject to the constraints, and the basic feasible solution is an optimal basic feasible solution.

(Proof is not necessary)

3.5.2 Theorem: Given a basic feasible solution $X_B = B^{-1}b$ with $z_0 = C_BX_B$ to the L.P.P., $AX = b$, $X \ge 0$, min $z = CX$ such that $z_j - c_j \le 0$ for every column a_j in A. Then z_0 is the minimum value of z subject to the constraints, and the basic feasible solution is an optimal basic feasible solution.

(Proof is not necessary)

3.5.3 Note: In the process of solving a linear programming problem (by simplex method), the following information plays important role.

(i) If $z_j - c_j < 0$ for some j and $y_{i_j} \le 0$ for all i = 1, …, m, then by using the theorem 3.4.1, we can conclude that the given L.P.P., has an unbounded solution.

(ii) If $z_j - c_j \ge 0$ for all j, then by using the theorem 3.5.1, we can conclude that the present solution is an optimal basic feasible solution.

Alternative Optima

3.5.4 Theorem: If $X_1, X_2, .., X_k$ are k different optimal feasible solutions to a L.P.P., $AX = b$, $X \ge 0$. Max $z = CX$, then any convex combination of $X_1, X_2, …, X_k$ is also an optimal feasible solution.

(Proof is not necessary)

3.5.5 Note:(i) If $X_1, …, X_k$ are basic feasible optimal solutions, then any convex combination of X_i, $1 \le i \le k$ is an optimal feasible solution. This convex combination may not be basic. Therefore if two or more basic feasible optimal solutions exist then an infinite number of optimal feasible solutions exist (since the number of such convex combinations is infinite).

(ii) From the statement of the above theorem, we conclude that the set of variables yielding the optimal value of the objective function may not be unique.
We say that "There are alternative optima" if there exist more than one optimal solution.

3.5.6 Theorem: Suppose there exists an optimal basic solution with a column a_j in A but not in B such that $z_j - c_j = 0$ and $y_{i_j} \le 0$ for $1 \le i \le m$. Then there exists an optimal solution with (m+1) variables.

(Proof is not necessary)

Extreme points & basic feasible solutions:

3.5.7 Theorem: If X_B is a basic feasible solution, then it is an extreme point of the set of all feasible solutions.

(Proof is not necessary)

3.5.8 Theorem: Let X^* be an extreme point of the set of all feasible solutions for a L.P.P., AX = b, $X \ge 0$. Then X^* is a basic feasible solution. [Equivalently, the columns $a_1, a_2, \ldots, a_k$ of A associated with non-zero components of X^*, must be linearly independent].

(Proof is not necessary)

3.5.9 Theorem: If X is a feasible solution to a L.P.P., then the following are equivalent:
(i) X is basic; (ii) X represents an extreme point of the set of all feasible solutions.

(Proof is not necessary)

3.6 ALGORITHM OF SIMPLEX METHOD

3.6.1 Note: (**i**) Let X_B be a basic solution with B = $(b_1, \ldots, b_m)$. Then $a_j = \sum_{i=1}^{m} y_{i_j} b_i$ or $y_j = B^{-1}a_j$ ----- (i).
Suppose we replace b_r by a column a_k in A which is not in B. By (i),

$$a_k = \sum_{i=1}^{m} y_{i_k} b_i \ .$$

Therefore $y_{r_k} b_r = -\sum_{\substack{i=1 \\ i \ne r}}^{m} y_{i_k} b_i + a_k \Rightarrow b_r = -\sum_{\substack{i=1 \\ i \ne r}}^{m} \frac{y_{i_k}}{y_{r_k}} b_i + \frac{1}{y_{r_k}} a_k$ ----- (ii).

Now by substituting (ii) in (i), we get

$$a_j = \sum_{\substack{i=1\\ i\neq r}}^{m} y_{i_j} b_i + y_{r_j} b_r = \sum_{\substack{i=1\\ i\neq r}}^{m} y_{i_j} b_i + y_{r_j} \left[- \sum_{\substack{i=1\\ i\neq r}}^{m} \frac{y_{i_k}}{y_{r_k}} b_i + \frac{1}{y_{r_k}} a_k \right]$$

$$= \sum_{\substack{i=1\\ i\neq r}}^{m} \left[y_{i_j} - y_{r_j} \frac{y_{i_k}}{y_{r_k}} \right] b_i + y_{r_j} \frac{1}{y_{r_k}} a_k = \sum_{i=1}^{m} \hat{y}_{i_j} \hat{b}_i = \hat{B} \hat{y}_j \text{ ------(iii)},$$

where $\hat{B} = [\hat{b}_1, \ldots, \hat{b}_m]$ with $\hat{b}_1 = b_1, \ldots, \hat{b}_r = a_j, \ldots, \hat{b}_m = b_m$ and $\hat{b}_r = a_j = a_k$.

$$\hat{y}_{i_j} = y_{i_j} - y_{r_j} \cdot \frac{y_{i_k}}{y_{r_k}} \quad \text{if } i \neq r \text{ and } \hat{y}_{r_j} = \frac{y_{r_j}}{y_{r_k}} \quad \text{------------ (iv).}$$

Therefore $\hat{y}_j = \hat{B}^{-1} a_j$ is given by $\hat{y}_j = [\hat{y}_{1_j}, \ldots, \hat{y}_{m_j}]$ of (iii) and (iv). Now we know how to compute $\hat{y}_j$ in terms of the new basis and its elements.

(ii) <u>To compute</u> $\hat{z}_j - c_j$:

$$\hat{z}_j - c_j = \hat{c}_B \hat{y}_j - c_j = \sum_{i=1}^{m} \hat{c}_{B_i} \hat{y}_{i_j} - c_j = \sum_{\substack{i=1\\ i\neq r}}^{m} \hat{c}_{B_i} \hat{y}_{i_j} + \hat{c}_{B_r} \hat{y}_{r_j} - c_j$$

$$= \sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i} \left(y_{i_j} - y_{r_j} \cdot \frac{y_{i_k}}{y_{r_k}} \right) + \hat{c}_{B_r} \cdot \frac{y_{r_j}}{y_{r_k}} - c_j \quad \text{------- (v) (by above (iv), } c_{B_i} = \hat{c}_{B_i} \text{ for } i \neq r).$$

Since $c_{B_i} (y_{r_j} - y_{r_j} \cdot \frac{y_{r_k}}{y_{r_k}}) = 0$, we add this term on the right side of the (v). Then we get

$$\hat{z}_j - c_j = \sum_{i=1}^{m} c_{B_i} \left(y_{r_j} - y_{r_j} \cdot \frac{y_{i_k}}{y_{r_k}} \right) + \frac{y_{r_j}}{y_{r_k}} . c_k - c_j$$

$$= \left[\sum_{i=1}^{m} c_{B_i} y_{i_j} - \sum_{i=1}^{m} c_{B_i} \frac{y_{r_j}}{y_{r_k}} . y_{i_k} \right] + \frac{y_{r_j}}{y_{r_k}} . c_k - c_j$$

$$= z_j + \frac{y_{r_j}}{y_{r_k}} \left[c_k - \sum_{i=1}^{m} c_{B_i} y_{i_k} \right] - c_j = z_j + \frac{y_{r_j}}{y_{r_k}} [c_k - z_k] - c_j$$

$$= z_j - c_j - \frac{y_{r_j}}{y_{r_k}} (z_k - c_k).$$

Now we know how to compute $\hat{z}_j - c_j$ in terms of old values $(z_j - c_j)$ and $(z_k - c_k)$.

iii) ***Table for simplex method***:

C_B	Basis vectors	b/X_B	C_1	C_2	….	C_n	±M	…	±M
			a_1	a_2	….	a_n	q_1	…	q_s
C_{B_1}	b_1	x_{B_1}	y_{1_1}	y_{1_2}	…	y_{1_n}	$y_{1_{(n+1)}}$	…	$y_{1_{(n+s)}}$
C_{B_2}	b_2	x_{B_2}	y_{2_1}	y_{2_2}	…	y_{2_n}	$y_{2_{(n+1)}}$	…	$y_{2_{(n+s)}}$
…	…	…	…	…	…	…	…	…	…
C_{B_m}	b_m	x_{B_m}	y_{m_1}	y_{m_2}	…	y_{m_n}	$y_{m_{(n+2)}}$	…	$y_{m_{(n+s)}}$
		z	z_1-c_1	z_2-c_2	…	z_n-c_n			

3.6.2 Algorithm to find a solution for a given Linear Programming Problem with the Objective Function Max z = CX:

STEP -I : (i) The given problem is to maximize the objective function

$$z = c_1x_1 + c_2x_2 + \ldots + c_rx_r$$

subject to the given m linear inequalities / equalities of the form

$$\left.\begin{array}{l} \sum_{j=1}^{r} a_{i_j} x_j \ \{\le, =, \ge\} b_i \text{ for } 1 \le i \le m, \text{ and the non-negativity} \\ \text{restrictions } x_j \ge 0 \text{ for } 1 \le j \le r. \end{array}\right\} \text{----> (i)}$$

(ii) All b_i's should be non-negative. If necessary, we multiply the inequality by (- 1) to obtain $b_i \ge 0$.

(iii) If a constraint is in the form $\sum_{j=1}^{r} a_{kj} x_j \le b_k$ for some k , then add a new variable x^1 on L.H.S., so that $\sum_{j=1}^{r} a_{k_j} x_j + x^1 = b_k$ $(x^1 \ge 0)$. Here x^1 is called a *slack variable*.

If a constraint is in the form $\sum_{j=1}^{r} a_{k_j} x_j \ge b_k$ for some k, then subtract a new variable x^* on L.H.S., so that $(\sum_{j=1}^{r} a_{k_j} x_j) - x^* = b_k$ $(x^* \ge 0)$. Here x^* is called a *surplus variable*.

(iv). Each slack or surplus variable is assigned a price of zero (that is, the corresponding c_i is zero)

(v) Now the problem get the form Max $z = CX$, $AX = b$, $X \geq O$, $A = (a_1, a_2, \ldots a_n)$. Note that A is an $m \times n$ matrix.

STEP -II : i) Observe that the column corresponding to a slack variable is of the form e_i (that is 1 is i^{th} place and zero elsewhere). The column corresponding to a surplus variable is of the form $-e_i$.

ii) Examine the matrix A whether it contains the identity matrix I_m (interchange the columns if necessary).

iii) If A do not contain I_m, then we add sufficient number of artificial variables (to appropriate equations) in order to obtain the identity matrix.

iv) We assume that the price of an artificial variable is a very large negative value denoted by –M. We use the same value –M for all artificial variables.

v) Use the identity matrix I_m as the basis matrix B. Then the initial basic solution is given by $X_B = B^{-1}b = I_m^{-1}b = I_m b = b$.

Since each $b_i \geq 0$, this basic solution X_B is feasible.

STEP–III: i) Construct the initial table with $x_B = B^{-1}b = I_m^{-1}b = I_m b = b$, $y_j = B^{-1}a_j = I_m^{-1}a_j = I_m a_j = a_j$.

ii) $z = C_B X_B = C_B b$.

iii) $z_j - c_j = C_B y_j - c_j$ for $1 \leq j \leq n$.

STEP–IV: i) We apply the *Optimality Criterion*: If all $z_j - c_j \geq 0$ then the present basic feasible solution is optimal.

ii) If one or more $z_j - c_j < 0$ then we apply the Simplex Criterion–I:

If $z_k - c_k = \min \{z_j - c_j \ / \ z_j - c_j < 0\}$ then a_k enters the basis.

[If there is a tie for minimum value, any one of the tied vectors can be chosen to enter the basis. A simple rule to break the tie is to choose from among the tied vectors, the one with the smallest j-index].

iii) If $y_{i_k} \le 0$ for all i then there is an unbounded solution involving the vectors in the basis and a_k. In this case, we stop the process.

iv) If atleast one $y_{i_k} > 0$ (for $1 \le i \le m$) then a new basic feasible solution can be found having $\hat{z} \ge z$.

STEP - V: If atleast one $y_{i_k} > 0$ then we apply *Simplex Criterion* II:

If $\frac{x_{B_r}}{y_{r_k}} = \min_i \{\frac{x_{B_r}}{y_{i_k}} / y_{i_k} > 0\}$ then the vector in column r of the basis (that is, b_r) is removed (replaced by a_k).

[If there is a tie in selecting column r, here, then any one of the tied columns can be removed. A convenient way to break the tie is to choose the column with the largest y_{i_k}. If this do not break the tie, select from the last group of columns].

STEP VI: Compute the new table using the following formulae:

i) $\hat{y}_{i_j} = y_{i_j} - \frac{y_{i_k}}{y_{r_k}} . y_{r_j}$ for all j (here $1 \le i \le m+1$ and $i \ne r$); $\hat{y}_{i_j} = \frac{y_{r_j}}{y_{r_k}}$ for all j.

ii) $\hat{z}_j - c_j = (z_j - c_j) - (\frac{y_{r_j}}{y_{r_k}})(z_k - c_k)$;

iii) C_{B_r} is to be replaced by C_k (the column under C_B);

iv) b_r is to be replaced by a_k (the column under b).

Step VII: _Return to Step IV.

Step VIII: END

Note : It can be verified that $z_j - c_j = 0$ for all j where a_j is in the basis.

$z_j - c_j = c_B y_j - c_j = c_B e_j - c_j$ ($y_j = e_j$ if a_j is in the basis, because

$a_j = b_j = 0.b_1 + \dots + 1.b_j + \dots + 0.b_m = e_j B$

$\Rightarrow y_j = B^{-1} a_j = e_j) = c_j - c_j = 0.$

3.6.3 Example: Solve the following linear programming problem by using simplex method.

Solve $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10, \quad x_1, x_2 \ge 0, \quad \text{Max } z = 5x_1 + 3x_2.$

STEP1: (i) The given L.P.P. is

$3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10, \quad x_1, x_2 \ge 0, \quad \text{Max } z = 5x_1 + 3x_2.$

(ii) Here all b_i's are non-negative. Therefore there is no need to multiply (any equation) by (-1).

(iii) Since the given inequalities $3x_1 + 5x_2 \le 15$ and $5x_1 + 2x_2 \le 10$ contains the sign $\le$, we introduce two slack variables x_3 and x_4. Now the equations are

$3x_1 + 5x_2 + x_3 = 15,$ and $5x_1 + 2x_2 + x_4 = 10.$

(Since there is no given inequality with ' $\ge$ ' sign, there is no surplus variable in this system of equations).

(iv) Here $c_3 = 0$ and $c_4 = 0$. Therefore Max $Z = 5x_1 + 3x_2 + 0x_3 + 0x_4.$

(v) Now the given L.P.P. is in the following form

$3x_1 + 5x_2 + 1.x_3 + 0.x_4 = 15$

$5x_1 + 2x_2 + 0.x_3 + 1.x_4 = 10.$

Here $a_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$, and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Now the problem is in the form $AX = b$, $X \ge 0$, Max $z = CX$..

Step-II: i) The columns corresponding to x_3 and x_4 are $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ respectively.

ii) The matrix A contains I_2 (the identity matrix).

iii) Since A contains I_2, there is no necessity to add artificial variables.

iv) -----------

v) Now $B = I_2 = \begin{bmatrix}1\,0\\0\,1\end{bmatrix}$ and the initial basic feasible solution is

$X_B = b = \begin{bmatrix}1 & 5\\1 & 0\end{bmatrix}$.

Step-III :

i) .

C_B	Vectors in Basis	b/X_B	$C_1 = 5$	$C_2 = 3$	$C_3 = 0$	$C_4 = 0$
			a_1	a_2	a_3	a_4
$C_{B_1} = 0$	$b_1 = a_3$	$X_{B_1} = 15$	$y_{1_1} = 3$	$y_{1_2} = 5$	$y_{1_3} = 1$	$y_{1_4} = 0$
$C_{B_2} = 0$	$b_2 = a_4$	$X_{B_2} = 10$	$y_{2_1} = 5$	$y_{2_2} = 2$	$y_{2_3} = 0$	$y_{2_4} = 1$
Table – I		$z = C_BX_B = 0$	$z_1 - c_1 = -5$	$z_2 - c_2 = -3$	$z_3 - c_3 = 0$	$z_4 - c_4 = 0$

$z = C_B.b = C_{B_1} X_{b_1} + C_{B_2} X_{B_2} = (0)(15) + 0(10) = 0;$

$z_1 - c_1 = C_By_1 - c_1 = C_{B_1} y_{1_1} + C_{B_2} y_{2_1} - c_1 = 0 + 0 - 5 = -5;$

$z_2 - c_2 = C_By_2 - c_2 = C_{B_1} y_{1_2} + C_{B_2} y_{2_2} - c_2 = 0 - c_2 = -3;\quad z_3 - c_3 = 0 = z_4 - c_4.$

<u>STEP</u> - IV: i) At this stage, all the $z_j - c_j$ are not ≥ 0. So the present solution is not optimal.

ii) $z_k - c_k = \min \{z_j - c_j \,/\, z_j - c_j < 0\} = \min \{-5, -3\} = -5 = z_1 - c_1$.

So a_1 enters the basis.

iii) $y_{i_1} \geq 0$ for i = 1, 2. Therefore, at this stage, we do not know whether the problem has unbounded solution.

iv) Since $y_{i_1}, y_{i_2} \geq 0$, there exist a new basic feasible solution with $\hat{z} \geq z$..

Step – V: Here k = 1, $y_{1_1} = 3 > 0$ and $y_{2_1} = 5 > 0$.

$$\frac{x_{B_r}}{y_{r_k}} = \min_i \left\{ \frac{x_{B_r}}{y_{i_k}} \;/\; y_{i_k} > 0 \right\} = \min \left\{ \frac{x_{B_1}}{y_{r_k}}, \frac{x_{B_2}}{y_{2_k}} \right\} = \min \left\{ \frac{15}{3}, \frac{10}{5} \right\} = \frac{10}{5} = \frac{x_{B_2}}{y_{2_k}}.$$ Therefore r = 2.

Hence $b_r = b_2 = a_4$ is to leave the basis.

Step–VI: Now observe the table–I. a_1 is to enter and a_4 is to leave the basis.

<u>Calculations for Row–2</u>: Here i = 2 = r. The formula is $\hat{y}_{r_j} = \frac{y_{r_j}}{y_{r_k}}$ for all j.

$$\hat{y}_{2_0} = \frac{y_{2_0}}{y_{2_1}} = \frac{10}{5} = 2, \quad \hat{y}_{2_1} = \frac{y_{2_1}}{y_{2_1}} = \frac{5}{5} = 1, \quad \hat{y}_{2_2} = \frac{y_{2_2}}{y_{2_1}} = \frac{2}{5} = 0.4,$$

$$\hat{y}_{2_3} = \frac{y_{2_3}}{y_{2_1}} = \frac{0}{5} = 0, \quad \hat{y}_{2_4} = \frac{y_{2_4}}{y_{2_1}} = \frac{1}{5} = 0.2.$$

<u>Calculations for Row –1</u>: (Here i = 1 ≠ r). Formula: $\hat{y}_{1_j} = y_{1_j} - \frac{y_{i_k}}{y_{r_k}} y_{r_k}$.

$$\frac{y_{1_k}}{y_{r_k}} = \frac{y_{1_1}}{y_{2_1}} = \frac{3}{5} = 0.6$$ (since r = 2, k = 1). Now $\hat{y}_{1_j} = \hat{y}_{1_j} - 0.6\, y_{2_j}$.

$$\hat{y}_{1_0} = y_{1_0} - 0.6\, y_{2_0} = 15 - 0.6 \times 10 = 9, \quad \hat{y}_{1_1} = y_{1_1} - 0.6\, y_{2_1} = 3 - 0.6 \times 5 = 0,$$

$$\hat{y}_{1_2} = y_{1_2} - 0.6\, y_{2_2} = 5 - 0.6 \times 2 = 3.8, \quad \hat{y}_{1_3} = y_{1_3} - 0.6\, y_{2_3} = 1 - 0.6 \times 0 = 1,$$

$$\hat{y}_{1_4} = y_{1_4} - 0.6\, y_{2_4} = 0 - 0.6 \times 1 = -0.6.$$

To calculate $\hat{z}_j - c_j$: Formula $\hat{z}_j - c_j = (z_j - c_j) - y_{r_j} \cdot \frac{z_k - c_k}{y_{r_k}}$.

Since r = 2, k = 1 we have $\frac{z_k - c_k}{y_{r_k}} = \frac{z_1 - c_1}{y_{2_1}} = \frac{-5}{5} = -1$.

Therefore $\hat{z}_j - c_j = (z_j - c_j) - y_{r_j}(-1) = (z_j - c_j) + y_{2_j}$.

$\hat{z}_1 - c_1 = (z_1 - c_1) + y_{2_1} = -5 + 5 = 0$, $\hat{z}_2 - c_2 = (z_2 - c_2) + y_{2_2} = -3+2 = -1$, $\hat{z}_3 - c_3 = (z_3 - c_3) + y_{2_3}$

$= 0 + 0 = 0$, $\hat{z}_4 - c_4 = (z_4 - c_4) + y_{2_4} = 0 + 1 = 1$.

Now we have the following table.

c_B	Vectors in Basis	X_B	$c_1 = 5$	$c_2 = 3$	$c_3 = 0$	$c_4 = 0$
			a_1	a_2	a_3	a_4
0	a_3	9	0	3.8	1	-0.6
5	a_1	2	1	0.4	0	0.2
Table-II	$z_j - c_j$	z = 10	$z_1 - c_1 = 0$	$z_2 - c_2 = -1$	$z_3 - c_3 = 0$	$z_4 - c_4 = 1$

Step - VII: Now we have to go to Step IV.

(i) All $z_j - c_j$ are not ≥ 0. Therefore the present solution is not optimal.

(ii) $z_2 - c_2 < 0$ and this is only i with $z_i - c_i < 0$. Therefore a_2 enters the basis in the next step.

Now we go to Step – V to find the vector b_r to leave the basis.

$$\frac{x_{B_r}}{y_{r_k}} = \min_i \left\{ \frac{x_{B_i}}{y_{i_k}} \Big/ y_{i_k} > 0 \right\} = \min \left\{ \frac{9}{3.8}, \frac{2}{0.4} \right\} = \frac{9}{3.8} = \frac{x_{B_i}}{y_{1_2}}.$$

Therefore r = 1 and hence $b_r = b_1$ leaves the basis. Now the next table is given.

c_B	Vectors in Basis	X_B	5	3	0	2
			a_1	a_2	a_3	a_4
3	a_2	$\frac{9}{3.8} = 2.368$	0	1	$\frac{1}{3.8} = 0.2632$	$\frac{-0.6}{3.8} = -0.1579$
5	a_1	1.053	1	0	-0.1053	0.2632
Table-III	$z_j - c_j$	z = 12.37	$z_1 - c_1 = 0$	$Z_2 - c_2 = 0$	$z_3 - c_3 = 0.2632$	$z_4 - c_4 = 0.8423$

In this Table –3, $z_j - c_j \geq 0$ for all j. Therefore the present basic solution is optimal. This optimal solution exists when $B = \{a_1, a_2\}$, $x_1 = 1.053$, $x_2 = 2.368$.

Therefore Max $z = c_1 x_1 + c_2 x_2 = 5 \times 1.053 + 3 \times 2.368 = 12.37$.

A Minimization Problem

3.6.4. Problem:

Solve

$$x_1 + 3x_2 + 2x_3 + 5x_4 \leq 20$$

$$2x_1 + 16x_2 + x_3 + x_4 \geq 4$$

$$3x_1 - x_2 - 5x_3 + 10x_4 \leq -10$$

$$x_j \geq 0 \quad \text{for } 1 \leq j \leq 4$$

$$\text{Min } z = -2x_1 - x_2 - 4x_3 - 5x_4 .$$

Solution: To convert this problem into a maximization problem, we take Min z = -Max(- z) = - Max $\bar{z}$ where $\bar{z} = -z = 2x_1 + x_2 + 4x_3 + 5x_4$.

To get $b \geq 0$, we multiply the third equation by (-1). Then the system is

$$x_1 + 3x_2 + 2x_3 + 5x_4 \leq 20$$

$$2x_1 + 16x_2 + x_3 + x_4 \geq 4$$

$$-3x_1 + x_2 + 5x_3 - 10x_4 \geq 10.$$

$$\text{Max } \bar{z} = 2x_1 + x_2 + 4x_3 + 5x_4.$$

By adding slack and surplus variables, we get

$$x_1 + 3x_2 + 2x_3 + 5x_4 + x_5 = 20$$

$$2x_1 + 16x_2 + x_3 + x_4 - x_6 = 4$$

$$-3x_1 + x_2 + 5x_3 - 10x_4 - x_7 = 10$$

Observe the values of $z_j - c_j$'s in the above table-II:- Among them (-8) is minimum and this value corresponds to the column "a_2". So the vector "a_2" enters the basis.

Since $\dfrac{x_{B_r}}{y_{r_k}} = \min \left\{ \dfrac{x_{B_i}}{y_{i_k}} \, / \, y_{i_k} > 0 \right\} = \dfrac{4}{7}$, the vector "$a_4$" leaves the basis.

C_B	Vectors in basis	X_B	2	3	-5	0	0
			a_1	a_2	a_3	a_4	a_5
3	a_2	4/7	0	1	1/7	2/7	1/7
2	a_1	45/7	1	0	12/14	5/7	-1/7
Table- III		102/7	0	0	50/7	16/7	1/7

Observe Table- III. Here all $z_j - c_j \geq 0$. So the present solution is optimal. Therefore $x_1 = 45/7$ and $x_2 = 4/7$ is an optimal basis feasible solution and the maximum value of Z is $\frac{102}{7}$.

3.6.5 SELF ASSESSMENT QUESTION 1: Apply simplex procedure to solve the L.P.P: Maximize $z = 3x_1 + 4x_2$ subject to $5x_1 + 4x_2 \leq 200$; $3x_1 + 5x_2 \leq 150$; $5x_1 + 4x_2 \geq 100$; $8x_1 + 4x_2 \geq 80$, $x_1 \geq 0$, $x_2 \geq 0$.

3.6.6 SELF ASSESSMENT QUESTION 2 : Apply simplex procedure to solve the L.P.P: Maximize $z = 2x_1 + x_2$ subject to $4x_1 + 3x_2 \leq 12$, $4x_1 + x_2 \leq 8$, $4x_1 - x_2 \leq 8$, $x_1, x_2 \geq 0$.

3.6.7 SELF ASSESSMENT QUESTION 3 : Using simplex method find $x_1, x_2 \geq 0$ to Maximize $z = 3x_1+2x_2$ subject to the constraints $x_1 + x_2 \leq 4$, $x_1 - x_2 \leq 2$

3.6.8 SELF ASSESSMENT QUESTION 4: Solve the following problem by simplex procedure:- Maximize $z = x_1 + 2x_2$ subject to $5x_1 + 3x_2 \leq 15$, $2x_1 + 6x_2 \leq 24$, $x_1, x_2 \geq 0$.

A problem with unbounded solution

3.6.9 Problem:(An example of a problem with unbounded solution)

Solve $14x_1 + x_2 - 9x_3 + 3x_4 = 7$

$$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5$$

$$3x_1 - x_2 - x_3 \leq 0 \quad \text{where} \quad x_i \geq 0,\ 1 \leq i \leq 4$$

$$\text{Max } z = 107x_1 + x_2 + 2x_3.$$

Solution: (Major steps are mentioned here – Detailed computation left for exercise):

Add x_5, x_6 (slack variables) to 2nd and 3rd constraints. Divide the first constraint by 3.

Then the system in matrix form is

$$\begin{bmatrix} 14/3 & 1/3 & -3 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 5 \\ 0 \end{bmatrix}$$

C_B	Vectors in basis	X_B	107	1	2	0	0	0
			a_1	a_2	a_3	a_4	a_5	a_6
0	A_4	7/3	14/3	1/3	-3	1	0	0
0	A_5	5	16	½	-6	0	1	0
0	A_6	0	[3]	-1	-1	0	0	1
Table–I		0	-107	-1	-2	0	0	0

From Table – I it is clear that a_6 leaves the basis and a_1 enters in the next iteration.

C_B	Vectors in basis	X_B	107	1	2	0	0	0
			A_1	a_2	A_3	a_4	a_5	a_6
0	a_4	7/3	0	17/9	-4/9	1	0	-14/9
0	a_5	5	0	35/6	-2/3	0	1	-16/3
107	a_1	0	1	-1/3	-1/3	0	0	1/3
Table – II		0	0	-110/3	-113/3	0	0	

Observe Table – II. Here min $\{z_j - c_j < 0\} = z_3 - c_3 = -113/3$ and in the column under a_3 we have $y_{i_k} \leq 0$ for all i. Hence unbounded solution exists for this problem.

3.7 SUMMARY

This lesson provides the computational details of a Simplex iteration that include the rules for determining the entering and leaving variables as well as for stopping the computations when the optimum solution has been reached.

The simplex algorithm is an iterative (step-by-step) procedure for solving Linear Programming Problems. It consists of
(i). trial basic feasible solution for the given system of equations.
(ii). Testing whether it is an optimal solution.
(iii). Improving the first trail solution by a set of rules, and repeating the process till an optimal solution is obtained.

Example shows how unboundedness, both in the solution space and in the objective value can be recognized in the simplex tableau. An example is also provided.

3.8 TECHNICAL TERMS

Simplex method :An algorithm for solving LP problems

Algorithm :A formalized systematic finite procedure for solving problems.

Basic variables :A collection Variables whose columns form a linearly independent set are called basic variables.

Degenerate Solution :degenerate solution is a basic solution to the system AX = B if one or more of the basic variables vanish.

Slack Variable :A variable used to convert a less than or equal to constraint into an equality constraint by adding it to the left hand side of the constraint.

Surplus Variable :A variable used to convert a greater than or equal to constraint into an equality constraint by subtracting it from the left hand side of the constraint.

Unbounded Solution :A Linear Programming problem is said to have unbounded solution (i). if Max {z / z = cx, x is an element from feasible region} is infinite, in case of maximization problem

(or)

(ii). If Min {z / z = cx, x is an element from feasible region} is $-\infty$, in case of minimization problem.

3.9 ANSWERS TO SELF ASSESSMENT QUESTIONS

SELF ASSESSMENT QUESTION 1 (3.6.5): Apply simplex procedure to solve the L.P.P: Maximize $z = 3x_1 + 4x_2$ subject to $5x_1 + 4x_2 \leq 200$; $3x_1 + 5x_2 \leq 150$; $5x_1 + 4x_2 \geq 100$; $8x_1 + 4x_2 \geq 80$, $x_1 \geq 0$, $x_2 \geq 0$.

Solution: Given problem is a maximization problem.

Since all b_i's are ≥ 0, there is no need to multiply with (-1). Now add slack variables x_3, x_4 to the first two equations. Use surplus variables x_5 & x_6 in the inequalities with sign "≥". By adding artificial variables x_{a_1}, x_{a_2} where ever necessary, we get the following system.

$$\begin{aligned} 5x_1 + 4x_2 + x_3 &= 200 \\ 3x_1 + 5x_2 \quad + x_4 &= 150 \\ 5x_1 + 4x_2 \quad - x_5 \quad + xa_1 &= 100 \\ 8x_1 + 4x_2 \quad - x_6 \quad + xa_2 &= 80 \end{aligned}$$

Now the system in the matrix four is Ax = b where

$$A = \begin{bmatrix} 5 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 5 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 0 & 0 & -1 & 0 & 1 & 0 \\ 8 & 4 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ q_1 \\ q_2 \end{bmatrix}, \ b = \begin{bmatrix} 200 \\ 150 \\ 100 \\ 80 \end{bmatrix}.$$

Each surplus and slack variables have a price of zero, and price of artificial variables will be (-M).

C_B	Vectors in basis	X_B	3	4	0	0	0	0	-M	-M
			A_1	a_2	a_3	a_4	a_5	a_6	q_1	q_2
0	a_3	200	5	4	1	0	0	0	0	0
0	a_4	150	3	5	0	1	0	0	0	0
-M	q_1	100	5	4	0	0	-1	0	1	0
-M	q_2	80	8	4	0	0	0	-1	0	1
Table – I		-180M	-13M-3	-8M-4	0	0	M	M	0	0

Observe the table – I. Among $z_j - c_j$ values –13M-3 is minimum. So the vector a_1 enters the basis.

Since $\frac{x_{B_r}}{y_{r_k}} = \min\{\frac{x_{B_i}}{y_{i_k}} / y_{i_k} > 0\} = \frac{80}{8}$, the vector "$q_2$" leaves the basis.

C_B	Vectors in basis	X_B	3	4	0	0	0	0	-M	-M
			a_1	a_2	a_3	a_4	a_5	a_6	q_1	q_2
0	a_3	150	0	3/2	1	0	0	5/8	0	-5/8
0	a_4	120	0	7/2	0	1	0	3/8	0	-3/8
-M	q_1	50	0	3/2	0	0	-1	5/8	1	-5/8
3	a_1	10	1	½	0	0	0	-1/8	0	1/8
Table - II		-50M+30	0	(-3M-5)/2	0	0	M	$\frac{-5M-3}{8}$	M	(3/8) +M

Observe the table –II. Among $z_j - c_j$ values ((-3M-5)/2) is minimum. So the vector a_2 enters the basis.

Since $\frac{x_{B_r}}{y_{r_k}} = \min\{\frac{x_{B_i}}{y_{i_k}} / y_{i_k} > 0\} = \frac{10}{1/2}$, the vector q_1 leaves the basis.

C_B	Vectors in basis	X_B	3	4	0	0	0	0	-M
			a_1	a_2	a_3	a_4	a_5	a_6	q_1
0	a_3	120	-3	0	1	0	0	1	0
0	a_4	50	-7	0	0	1	0	5/4	0
-M	q_1	20	-3	0	0	0	-1	+1	1
4	a_2	20	2	1	0	0	0	-1/4	0
Table - III		-20M+80	3M+5	0	0	0	M	-M-1	0

Observe table –III. Among $z_j - c_j$ values (-M-1) is minimum. This value corresponds to the column a_6.

So "a_6" enters the basis. Since $\frac{x_{B_r}}{y_{r_k}} = \min_i\{\frac{x_{B_i}}{y_{i_k}} / y_{i_k} > 0\} = \frac{20}{1}$, the vector "$q_1$" leaves the basis.

C_B	Vectors in basis	X_B	3	4	0	0	0	0
			a_1	A_2	a_3	a_4	a_5	a_6
0	a_3	100	0	0	1	0	1	0
0	a_4	25	13/4	0	0	1	**5/4**	0
0	a_6	20	-3	0	0		-1	1
4	a_2	25	5/4	1	0	0	-1/4	0
Table – IV		100	2	0	0	0	-1	0

Observe Table – IV. Among $z_j - c_j$ values (-1) is minimum. So "a_5" enters the basis. Since $\frac{x_{B_r}}{y_{r_k}} = \min_i \{\frac{x_{B_i}}{y_{i_k}} \,/\, y_{i_k} > 0\}$, the vector "$a_4$" leaves the basis.

C_B	Vectors in basis	X_B	3	4	0	0	0	0
			a_1	a_2	a_3	a_4	a_5	a_6
0	a_3	80	13/5	0	1	-4/5	0	0
0	a_5	20	-13/5	0	0	4/5	1	0
0	a_6	40	-28/5	0	0	4/5	0	1
4	a_2	30	12/20	1	0	1/5	0	0
Table – V		120	-3/5	0	0	4/5	0	0

Observe Table - V. Among $z_j - c_j$ values (-3/5) is minimum. This value corresponds to the column "a_1". Therefore the vector "a_1" enters the basis.

Since $\frac{x_{B_r}}{y_{r_k}} = \min \{\frac{x_{B_i}}{y_{i_k}} \,/\, y_{i_k} > 0\} = \frac{80}{13/5}$, the vector a_3 leaves the basis.

C_B	Vectors in basis	X_B	3	4	0	0	0	0
			a_1	a_2	a_3	a_4	a_5	a_6
3	a_1	400/13	1	0	5/13	-4/13	0	0
0	a_5	100	0	0	1	0	1	0
0	a_6	2760/13	0	0	28/13	-12/13	0	1
4	a_2	150/13	0	1	-3/13	5/13	0	0
Table – VI		1800/13	0	0	3/13	8/13	0	0

Observe Table – VI. Since $z_j - c_j \geq 0$, the solution is optimal.

maximum value of Z is $\frac{1800}{13}$, $x_1 = 400/13$, $x_2 = 150/13$, $x_5 = 100$; $x_6 = 2760/13$

SELF ASSESSMENT QUESTION 2 (3.6.6.): Apply simplex procedure to solve the L.P.P. maximize $z = 2x_1 + x_2$ subject to $4x_1 + 3x_2 \le 12$, $4x_1 + x_2 \le 8$, $4x_1 - x_2 \le 8$, $x_1, x_2 \ge 0$.

Solution: Given problem is a maximization problem. Since all b_i's are ≥ 0 there is no need to multiply with (-1).

Since the inequalities contains the sign "≤", we add slack variables x_3, x_4 and x_5. Then the given system is,

$$4x_1 + 3x_2 + x_3 = 12$$

$$4x_1 + x_2 + x_4 = 8$$

$$4x_1 - x_2 + x_5 = 8$$

Now the system in matrix four is AX = b where

$$A = \begin{bmatrix} 4 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, b = \begin{bmatrix} 12 \\ 8 \\ 8 \end{bmatrix}.$$

Now the columns corresponding to x_3, x_4, x_5 forms an identity matrix. Also each slack variable is assigned a price of zero.

C_B	Vectors in basis	X_B	2	1	0	0	0
			a_1	a_2	a_3	a_4	a_5
0	a_3	12	4	3	1	0	0
0	a_4	8	[4]	1	0	1	0
0	a_5	8	4	-1	0	0	1
Table - I		0	-2	-1	0	0	0

Observe $z_j - c_j$ values in the table – I. $z_j - c_j = -2$; $z_2 - c_2 = -1$; $z_3 - c_3 = 0$; $z_4 - c_4 = 0$; $z_5 - c_5 = 0$. Since (-2) is minimum among the $z_j - c_j$ values, the vector "a_1" enters the basis.

Now $\frac{x_{B_r}}{y_{r_k}} = \min\{\frac{x_{B_i}}{y_{i_k}} \ / \ y_{i_k} > 0\} = \min\{\frac{12}{4}, \frac{8}{4}, \frac{8}{4}\}$. The minimum value "$\frac{8}{4}$" corresponds to the rows corresponding to the vectors a_4 & a_5. So there is a tie in selecting the vector to leave the basis. A convenient way to break this tie is to select from the last group of columns. So we select a_4 to sent out from the basis.

C_B	Vectors in basis	X_B	2	1	0	0	0
			a_1	a_2	a_3	a_4	a_5
0	a_3	4	0	[2]	1	-1	0
2	a_1	2	1	¼	0	¼	0
0	a_5	0	0	-2	0	-1	1
Table- II		4	0	-½	0	½	0

Observe table – II. Here $z_1 - c_1 = 0$; $z_2 - c_2 = -1/2$; $z_3 - c_3 = 0$; $z_4 - c_4 = ½$; $z_5 - c_5 = 0$.

Since $z_k - c_k = \min\{z_j - c_j \ / \ z_j - c_j < 0\} = -1/2$, the vector "$a_2$" enters the basis.

Since $\frac{x_{B_r}}{y_{r_k}} = \min\{\frac{x_{B_i}}{y_{i_k}} \ / \ y_{i_k} > 0\} = \frac{4}{2}$, the vector "$a_3$" leaves the basis.

C_B	Vectors in basis	X_B	2	1	0	0	0
			a_1	a_2	a_3	a_4	a_5
1	a_2	2	0	1	½	-1/2	0
2	a_1	3/2	1	0	-1/8	3/8	0
0	a_5	4	0	0	1	-2	1
Table – III		5	0	0	¼	¼	0

Observe Table – III. Here all $z_j - c_j \geq 0$. So the present solution is optimal. Therefore $x_1 = 3/2$; $x_2 = 2$; $x_5 = 4$ is an optimal basis feasible solution and the maximum value of z is 5.

SELF ASSESSMENT QUESTION 3 (3.6.7) : Using simplex method find $x_1, x_2 \geq 0$ to Maximize $z = 3x_1+2x_2$ subject to the constraints $x_1 + x_2 \leq 4$, $x_1 - x_2 \leq 2$

Solution: Given problem is a maximization problem.

Since all b_i's are "≥" 0, there is no need to multiply with (-1).

Since the equation contains "≤" sign, we add slack variables x_3 and x_4. Then the system is

$$x_1 + x_2 + x_3 \quad = 4$$

$$x_1 - x_2 \quad + x_4 = 2$$

Now the system in the matrix form is Ax = b, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Now the columns corresponding to x_3, x_4 forms the identity matrix. Now each slack variable is assigned a price of zero. That is, the corresponding prices c_3 and c_4 are equal to zero.

C_B	Vectors in basis	X_B	3	2	0	0
			A_1	A_2	a_3	a_4
0	a_3	4	1	1	1	0
0	a_4	2	[1]	-1	0	1
Table – I		0	-3	-2	0	0

Observe $z_j - c_j$ values. Here $z_1 - c_1 = -3$; $z_2 - c_2 = -2$; $z_3 - c_3 = 0$; $z_4 - c_4 = 0$.

Since (-3) is minimum among the $z_j - c_j$ values, the vector a_1 enters the basis.

Since $\frac{x_{B_r}}{y_{r_k}} = \min_i \{\frac{x_{B_i}}{y_{i_k}} / y_{i_k} > 0\} = \min \{4/1, 2/1\} = 2$, the vector "$a_4$" leaves the basis.

C_B	Vectors in basis	X_B	3	2	0	0
			a_1	A_2	a_3	a_4
0	a_3	2	0	[2]	1	-1
3	a_1	2	1	-1	0	1
Table – II		6	0	-5	0	0

Now in the above Table – II, $z_1 - c_1 = 0$; $z_2 - c_2 = -5$; $z_3 - c_3 = 0$; $z_4 - c_4 = 0$.

Since $z_k - c_k = \min \{z_j - c_j / z_j - c_j < 0\} = -5$, the vector a_2 enters the basis.

Since $\frac{x_{B_r}}{y_{r_k}} = \min_i \{\frac{x_{B_i}}{y_{i_k}} / y_{i_k} > 0\} = 2/2$, the vector "$a_3$" leaves the basis.

C_B	Vectors in basis	X_B	3	2	0	0
			a_1	A_2	a_3	a_4
2	a_2	1	0	1	½	-1/2
3	a_1	3	1	0	½	3/2
Table – III		11	0	0	2	7/2

In the Table – III, all $z_j - c_j \geq 0$. So the present solution is optimal.

Therefore $x_1 = 3$; $x_2 = 1$ is an optimal basic feasible solution and maximum value of z is 11.

3.10 MODEL QUESTIONS

3.10.1 Model Question 1 : Consider the L.P.P. max $z = 2x_1 + x_2 + 3x_3$, subject to the constraints $4x_1 + 2x_2 + x_3 + x_4 = 2$, $x_1 + 2x_2 + 3x_3 - x_5 = 1$, $x_i \geq 0$, $1 \leq i \leq 5$,

(i). Find a_1, a_2, a_3, a_4, a_5, b

(ii) Find X_B with $B = (a_3, a_1)$

(iii) Find c_B with $B = (a_3, a_1)$

(iv) Find y_2. Also write a_2 as a Linear combination of a_3, a_1.

(v) Find z_2 and $z = C_BX_B$

Answer:

(i). Here $a_1 = [4, 1]$, $a_2 = [2, 2]$, $a_3 = [1, 3]$, $a_4 = [1, 0]$, $a_5 = [0, -1]$, $b = [2, 1]$.

(ii). $X_B = B^{-1}b = \begin{bmatrix} 2/11 \\ 5/11 \end{bmatrix}$

(iii). $C_B = [C_{B_1}, C_{B_2}] = [c_3, c_1] = [3, 2]$.

(iv). $y_2 = B^{-1}a_2 = \begin{bmatrix} 6/11 \\ 4/11 \end{bmatrix}$.

$a_2 = \frac{6}{11} a_3 + \frac{4}{11} a_1$.

(v). $z_2 = C_BY_2 = \frac{26}{11}$ and $z = C_BX_B = \frac{16}{11}$.

3.10.2 Model Question 2 : Given system is $2x_1 + 7x_2 + 5x_3 + x_4 = 16$,

$3x_1 + 2x_2 + x_3 + 2x_4 = 10$.

Given feasible solution is $(x_1, x_2, x_3, x_4) = (1, 1, 1, 2)$. Find a basic feasible solution.

Answer: $(x_1, x_2) = \left(\frac{38}{17}, \frac{28}{17}\right)$ is a basic feasible solution.

3.10.3 Model Question 3 : Solve the following L.P.P. by simplex method $x_1 + 4x_2 \le 80$, $2x_1 + 3x_2 \le 90$, $x_1 \ge 0$, $x_2 \ge 0$, Max $z = 45x_1 + 80x_2$.

Ans: $x_1 = 24$, $x_2 = 14$ and max $z = 2200$.

3.10.4 Model Question 4 : Maximize $z = 2x_1 + 3x_2 - 5x_3$ subject to $x_1 + x_2 + x_3 = 7$, $2x_1 - 5x_2 + x_3 \ge 10$, $x_1, x_2, x_3 \ge 0$.

Ans: Max $z = 102/7$ and $x_1 = 45/7$, $x_2 = 4/7$.

3.10.5 Model Question 5 : Solve the following L.P.P. Max $z = 2x_1 + 3x_2 - 5x_3$ subject to $x_1 + x_2 + x_3 \le 7$, $2x_1 - 5x_2 + x_3 \ge 10$ where $x_1, x_2, x_3 \ge 0$.

Ans: $x_1 = 45/7$, $x_2 = 4/7$ and Max $z = 102/7$.

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Name of the Lesson Writer

Prof. Bhavanari Satyanarayana

LESSON - 4

TRANSPORTATION PROBLEM

Objectives

The objectives of this lesson are to:

- Observe a Transportation problem as a Linear Programming Problem
- Understand how to tabulate a Transportation problem
- Discuss few methods of finding initial basic feasible solution of a given Transportation problem
- Know how to solve a given Transportation problem

Structure

4.0 Introduction
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4.0 INTRODUCTION

All linear programming problems can be solved by the simplex method, however, certain class of linear programming problems, due to their specialized structure lend themselves to solutions by other methods which are computationally more efficient than the simplex method. Transportation models are such type of problems and deals with the transportation of a homogenous product manufactured at several plants (factories) to a number of different destinations (warehouses). The objective is to satisfy the destinations requirements within the plants capacity constraints at the minimum transportation cost.

There are different types of transportation models and the simplest of them that is more standard in the literature was first presented by F. L. Hitchcock (1941). It was further discussed by T. C. Koopmans (1949). The linear programming formulation of the transportation problem was stated by G. B. Dantzig (1951). Several extensions of this model and methods have been subsequently developed.

4.1 PRELIMINARY CONCEPTS

4.1.1 The meaning of "Transportation Problem": A product quantity a_i is available at origin i ($1 \le i \le m$) and b_j is the quantity required at destination j ($1 \le j \le n$). The cost of shipping one unit from origin i to destination j will be c_{i_j}. We assume that $\sum_{i=1}^{m} a_j = \sum_{j=1}^{n} b_j$ (that is, the quantity available is equal to the quantity required). (The transportation problems in which the quantity available is equal to the quantity required, are called balanced transportation problems). We wish to determine the shipping schedule that minimizes the total cost of shipping.

Suppose x_{i_j} is the quantity shipped from origin i to destination j. Then the problem is

$$\sum_{j=1}^{n} x_{i_j} = a_i, \quad a_i > 0, \text{ for } i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{i_j} = b_j, \quad b_j \ge 0, \text{ for } j = 1, 2, \ldots, n,$$

$$x_{i_j} \ge 0, \text{ for all } i \text{ and } j, \quad \min z = \sum_{i,j} c_{i_j} x_{i_j}.$$

If we convert these constraints into our standard form of a linear programming problem, then we have

$$
\begin{array}{llllllllll}
x_{1_1} + x_{1_2} + \ldots + x_{1_n} & & & & = a_1 \\
& x_{2_1} + x_{2_2} + \ldots . + x_{2_n} & & & = a_2 \\
& & \ldots \ \ldots \ \ldots \ \ldots \ \ldots & & \ldots \\
& & & x_{m_1} + x_{m_2} + \ldots + x_{m_n} & = a_m \\
x_{1_1} & + x_{2_1} + \ \ldots & + x_{m_1} & & = b_1 \\
\quad x_{1_2} & + x_{2_2} + \ldots & + x_{m_2} & & = b_2 \\
\quad \ldots \ \ldots & \ldots \ \ldots & \ldots \ \ldots & & \ldots \\
\qquad x_{1_n} & + x_{2_n} + \ldots & + x_{m_n} & & = b_n
\end{array}
$$

$x_{i_j} \geq 0$, $\min z = \sum_{ij} c_{i_j} x_{i_j}$.

Now we write this system in the matrix form.

$X = [x_{1_1}, \ x_{1_2}, \ldots, \ x_{1_n}, \ x_{2_1}, \ x_{2_2}, \ldots, \ x_{2_n}, \ \ldots, \ x_{m_1}, \ x_{m_2}, \ \ldots, \ x_{m_n}]$,

$b = [a_1, \ a_2, \ \ldots, \ a_m, \ b_1, \ b_2, \ \ldots, \ b_n]$,

$$
A = \left[\begin{array}{ccccc}
1_n & 0 & 0 & \ldots & 0 \\
0 & I_n & 0 & \ldots & 0 \\
0 & 0 & I_n & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1_n \\
I_n & I_n & I_n & \ldots & I_n
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} \text{m rows} \\
\left.\right\} \text{n- rows}
\end{array}
\quad \Bigg\} \ \text{m + n rows}
$$

$\underbrace{\qquad\qquad\qquad}_{\text{mn columns}}$

So, A is an $(m + n) \times (mn)$–matrix, and the system is given by $AX = b$, $X \geq 0$, $\min z = \sum_{ij} c_{i_j} x_{i_j}$.

4.1.2 Example: For two origins (with available quantities a_1, a_2) and four destinations (with requirement quantities b_1, b_2, b_3, b_4), the problem has the following matrix form:

$$AX = b \quad \text{where } A = \begin{bmatrix} 1_4 & 0 \\ 0 & 1_4 \\ I_4 & I_4 \end{bmatrix}, \quad b = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \text{ and } X = [x_{1_1}, x_{1_2}, x_{1_3}, x_{1_4}, x_{2_1}, x_{2_2}, x_{2_3}, x_{2_4}].$$

That is,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \\ x_{1_3} \\ x_{1_4} \\ x_{2_1} \\ x_{2_2} \\ x_{2_3} \\ x_{2_4} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

4.1.3. Note: In the above we have converted the transportation problem into the standard form of a linear programming problem and hence we can use simplex method. Since in the above form, the matrix A has a simple and special structure, it is possible to solve it by efficient computational procedures than the simplex method. One of such procedures is transportation problem solving method.

4.1.4 Note: In the above form $AX = b$ of the transportation problem, consider A.

(i) (The sum of first m rows of A) – (The sum of the last n-rows of A) = 0. Therefore $|A| = 0$, and hence $r(A) < m + n$.

(ii) Consider the matrix D formed by taking columns n, 2n, 3n, …,mn, 1, 2, …, (n - 1) and rows 1, 2, …, m + n – 1 (note that the last row of A being omitted). So the bottom left corner part of D is equal to zero.

Then $D = \begin{bmatrix} I_m & F \\ 0 & I_{n-1} \end{bmatrix}$ with $F = \begin{bmatrix} I_{n-1} \\ 0 \end{bmatrix}$. Also $|D| = 1$. Hence $r(A) = m + n - 1$.

(iii) Since $r(A) = m + n - 1$, we have that any basis matrix for the problem $AX = b$ should contain $(m + n - 1)$ linearly independent vectors. So the number of basic variables is not more than $(m + n - 1)$.

(iv) When allocation is completed, it is necessary to check that the number of allocations is equal to (m + n - 1) where m is the number of origins and n is the number of destinations. It may so happen that at the initial stage itself or at subsequent iterations the allocations are less than (m + n - 1). In such a case, we say that the solution is **degenerate**.

(v) The general form of a table for the transportation problem is given here.

	D_1	D_2	…	D_j	…	D_n	a_i
O_1	c_{1_1} x_{1_1}	c_{1_2} x_{1_2}	…	c_{1_j} x_{1_j}	…	c_{1_n} x_{1_n}	a_1
O_2	c_{2_1} x_{2_1}	c_{2_2} x_{2_2}	…	c_{2_j} x_{2_j}	…	c_{2_n} x_{2_n}	a_2
…	…	…	…	…	…	…	…
O_i	c_{i_1} x_{i_1}	c_{i_2} x_{i_2}	…	c_{i_j} x_{i_j}	…	c_{i_n} x_{i_n}	a_i
…	…	…	…	…	…	…	…
O_m	c_{m_1} x_{m_1}	c_{m_2} x_{m_2}	…	c_{m_j} x_{m_j}	…	c_{m_n} x_{m_n}	a_m
b_j	b_1	b_2	…	b_j	…	b_n	$\Sigma a_i = \Sigma b_j$

4.1.5 Example: A company has three factories F_1, F_2 and F_3 with production capacity 100, 250 and 150 units per week respectively. These units are to be shipped to four warehouses W_1, W_2, W_3 and W_4 with requirement of 90, 160, 200 and 50 units per week respectively. The transportation costs (in Rupees) per unit between factories and warehouses are given below:

Warehouses / Factories	W_1	W_2	W_3	W_4	Capacity (a_i)
F_1	30 x_{11}	25 x_{12}	40 x_{13}	20 x_{13}	100
F_2	29 x_{21}	26 x_{13}	35 x_{13}	40 x_{13}	250
F_3	31 x_{13}	33 1	37 x_{13}	30 x_{13}	150
Requirement (b_j)	90	160	200	50	500

The problem of the company is to distribute the available product to different warehouses in such a way so as to minimize the total transportation cost for all possible factory-warehouse shipping patterns.

Mathematical Formulation: Here x_{ij}'s represent the number of units of the product shipped from the i-th factory (i = 1, 2, 3) to j-th warehouse (j = 1, 2, 3, 4). Each x_{ij} is a non-negative integer. If in a solution the x_{ij} value is missing for a cell, means that no quantity is shipped from the factory-i to warehouse-j. The objective of the company is to find the number of units to be supplied from various factories to warehouses so that the total transportation cost is minimum. There are, in all, $3 \times 4 = 12$ possible shipping routes by which company may allocate shipments.

Now we can formulate the mathematical model in the linear programming form as follows:

$$\begin{aligned} \text{Min } Z = {} & 30x_{11} + 25x_{12} + 40x_{13} + 20x_{14} \\ & + 29x_{21} + 26x_{22} + 35x_{23} + 40x_{24} \\ & + 31x_{31} + 33x_{32} + 37x_{33} + 30x_{34} \end{aligned}$$

subject to the constraints

(i) Requirement constraints

$$x_{11} + x_{21} + x_{31} = 90$$
$$x_{12} + x_{22} + x_{32} = 160$$
$$x_{13} + x_{23} + x_{33} = 200$$
$$x_{14} + x_{24} + x_{34} = 50$$

(ii) Capacity constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 100$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 250$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 150$$

(iii) the total requirement = the total Capacity (that is,
90 + 160 + 200 + 50 = 100 + 250 + 150),

and $x_{ij} \geq 0$.

The above mathematical formulation of the transportation problem is nothing but a linear programming problem containing $3 \times 4 = 12$ decision variables x_{ij}'s and $3 + 4 = 7$ constraints.

4.1.6 Note: (i) For a Transportation Problem, to get a feasible solution, it is necessary that total capacity equals the total requirement

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(ii) In case, if capacity $\left(\sum a_i\right)$ is greater than requirement $\left(\sum b_j\right)$, then a dummy warehouse may be used to create desired equality. In cases capacity is less than requirement, then a dummy factory may be introduced. In such cases, the transportation cost in both the dummy cases is assumed to be zero.

(iii) If the total capacity equals to the total requirement, the problem is called as balanced transportation problem. Otherwise it is called as transportation problem.

4.1.7 Definitions: i) A directed path from the cell (i, j) to the cell (u, v) in the transportation table is defined to be an ordered set of cells {(i, j), (i, k), (q, k), …, (i, v)} or {(i, j), (s, j), (s, t), …, (u, v)} such that any two adjacent cells are in the same row or in the same column. Each cell occurs only once in the ordered sets. The cell (i, j) is called the initial cell of the path and the cell (u, w) is called the terminal cell.

ii) A directed branch is a line segment joining an ordered pair of cells which lie either in the same row or in the same column of the transportation table. The first cell of the ordered pair is called the initial point of the branch, and the second cell is called the end point of the branch.

iii) A directed loop is a directed path such that the first cell in the ordered set is the same as the last cell and the first branch is orthogonal to the last branch.

iv) A simple directed path from cell (i, j) to cell (u, v) is a directed path such that in any row or column of the transportation table, there are no more than two cells in the set of cells which defines the path.

v) A simple directed loop is a directed loop that has no more than two cells in any row or column of the transportation table.

vi) A set of cells in a transportation table is said to be connected if there exists a directed path involving only cells in the set, that joins any cell in the set to any other cell in the set.

vii) The cells of a transportation table corresponding to a given set of (m + n – 1) basis vectors will be referred to as basic cells .

viii) A tree is a connected set of cells without loops.

4.2 NORTH WEST CORNER METHOD

This method is one of the systematic and easier method for obtaining an initial basic feasible solution (allocation).

4.2.1 North-West Corner Method: (To find an initial basic feasible solution):

Suppose that a_i = quantity available at origin i,

b_j = quantity required at destination j.

Write x_{1_1} = min $\{a_1, b_1\}$. If $a_1 = b_1$, then write $x_{2_1} = 0$, and x_{2_2} = min $\{a_2, b_2\}$.

If $a_1 < b_1$ then $x_{1_1} = a_1$, and x_{2_1} = min $\{b_1 - a_1, a_2\}$.

If $a_1 > b_1$ then $x_{1_1} = b_1$. In this case requirements of destination–1satisfied.

Now x_{12} = min $\{a_1 - b_1, b_2\}$.

In this way, we continue to find the basic variables x_{i_j}. These cells do not form a loop. For all the non-basic variables we have $x_{i_j} = 0$. We do not enter the values of the non-basic variables in the table. In the table, we enter only the basic variable values and we circle these values.

4.2.2 North - West Corner Method (NWCM) – An Algorithm

Step 1: Select the north-west (upper left-hand) corner cell of the transportation table for a shipment and allocate as many units as possible equal to minimum between available capacity and requirement, i. e. min $\{a_1, b_1\}$.

Step 2: Adjust the capacity and requirement numbers for the next allocations.

Step 3: (a) If the capacity for the first row is exhausted, then movedown to the first cell in the second row and first column and go to step 2.

(b) If the requirement for the first column is exhausted, then move horizontally to the next cell in the second column and first row and go to step 2.

Step 4: If for any cell, capacity equals requirement, then the next allocation of magnitude zero can be made in cell either in the next row or column.

Step 5: Continue the procedure until the total available quantity is fully allocated to the cells as required.

4.3 THE STEPPING – STONE ALGORITHM

4.3.1 The Stepping –Stone Algorithm to solve a transportation problem:

Step–I: Enter the values a_i, b_j, c_{i_j} at their appropriate places of the transportation table. Find initial basic feasible solution with (m + n –1) variables x_{i_j}. Enter these values x_{i_j}'s in the appropriate cells and circle these values. This enable us to identify the basic cells easily.

Step-II: For the vectors not in the basis, we calculate $z_{i_j} - c_{i_j}$ by using the formula:

$z_{i_j} - c_{i_j} = (c_{i_r}^{B} - c_{u_r}^{B} + \ldots - c_{s_w}^{B} c_{s_j}^{B}) - c_{i_j}$

Where {(i, j), (i, r), (u, r), …, (s, w), (s, j), (i, j)} is the unique loop involving (i, j) and some of the basic cells.

Step – III: We need not compute $z_{i_j} - c_{i_j}$ for the basic cells because for these cells $z_{i_j} - c_{i_j} = 0$.

If all $z_{i_j} - c_{i_j} \leq 0$ then the present basic feasible solution is optimal. If not, we calculate

$$z_{s_t} - c_{s_t} = \max \{ z_{i_j} - c_{i_j} \; / \; z_{i_j} - c_{i_j} > 0 \}.$$

Then the vector p_{s_t} enters the basis and so x_{s_t} becomes positive in the next step.

Step–IV: The vector to leave the basis would be determined from

$$\frac{x_{q_r}^{B}}{y_{s_k}^{q_r}} = \min \left\{ \frac{x_{\alpha_\beta}^{B}}{y_{s_t}^{\alpha_\beta}} \; / \; y_{s_t}^{\alpha_\beta} > 0 \right\}.$$

Consider the loop continuing (s, t) and express $z_{s_t} - c_{s_t}$. If c_{α_β} has positive sign, then only $y_{s_t}^{\alpha_\beta} > 0$.

Otherwise $y_{s_t}^{\alpha_\beta} \leq 0$.

Note that for transportation problem, $y_{i_j} = \pm 1$ or 0. Therefore the above formula is same as

$x_{q_r}^{B} = \min \{ x_{\alpha_\beta}^{B} \; / \; y_{s_t}^{\alpha\beta} > 0 \}$. Then $p_{q_r}^{B}$ is removed from the basis.

New values are to be computed by using the following formulas:

(i) $\hat{x}_{s_t} = x_{q_r}^{B}$.

(ii) Consider the loop involving (s, t) and express $z_{s_t} - c_{s_t}$. If $c_{\alpha_\beta}^{B}$ has coefficient +1, then

$\hat{x}_{\alpha_\beta}^{B} = x_{\alpha_\beta}^{B} - x_{q_r}^{B}$. If $c_{\alpha_\beta}^{B}$ has coefficient (- 1), then $\hat{x}_{\alpha_\beta}^{B} = x_{\alpha_\beta}^{B} + x_{q_r}^{B}$.

(iii) If $c_{\alpha_\beta}^{B}$ is not in the loop involving (s, t) then, $\hat{x}_{\alpha_\beta}^{B} = x_{\alpha_\beta}^{B}$.

Now we calculate $z_{i_j} - c_{i_j}$ and then go to step-III.

Step V: Stop

Step VI: End

4.3.2 Example: Solve the following Transportation Problem by using Stepping Stone Algorithm.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	31
b_j	30	50	20	40	30	11	181

Table - IA

Solution: By North-West corner method we found the initial basic solution. The initial basic solution is given by the following table.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2 (30)	1 (20)	3 -2	3 -4	2 -1	5 -4	50
O_2	3 0	2 (30)	2 (10)	4 -4	3 -1	4 -2	40
O_3	3 2	5 -1	4 (10)	2 (40)	4 (10)	1 3	60
O_4	4 -1	2 0	2 0	1 -1	2 (20)	2 (11)	31
b_j	30	50	20	40	30	11	181

Table–IB

To find $z_{i_j} - c_{i_j}$, we consider the corresponding loop, and then use the formula:

$z_{i_j} - c_{i_j} = (c_{i_r}{}^B - c_{u_r}{}^B + \ldots + c_{s_j}{}^B) - c_{i_j}$.

$z_{1_1} - c_{1_1} = 0$ (since x_{1_1} is a basic variable), $z_{2_1} - c_{2_1} = (2 - 1 + 2) - 3 = 0$,

$z_{3_1} - c_{3_1} = (4 - 2 + 2 - 1 + 2) - 3 = 2$, $z_{4_1} - c_{4_1} = (2 - 4 + 4 - 2 + 2 - 1 + 2) - 4 = -1$,

$z_{1_2} - c_{1_2} = 0$, $z_{2_2} - c_{2_2} = 0$, $z_{3_2} - c_{3_2} = (4 - 2 + 2) - 5 = -1$,

$z_{4_2} - c_{4_2} = (2 - 4 + 4 - 2 + 2) - 2 = 0$, $z_{1_3} - c_{1_3} = (1 - 2 + 2) - 3 = -2$,

$z_{2_3} - c_{2_3} = 0$, $z_{4_3} - c_{4_3} = (2 - 4 + 4) - 2 = 0$.

At present, not all $z_j - c_j \leq 0$. So the present initial basic feasible solution is not optimal. Since

$z_{s_t} - c_{s_t} = \max \{ z_{i_j} - c_{i_j} \; / \; z_{i_j} - c_{i_j} > 0 \} = \max \{2, 3\} = 3 = z_{3_6} - c_{3_6}$,

the vector p_{3_6} enters the basis.

The loop involving (3, 6) is $z_{3_6} - c_{3_6} = c_{4_6} - c_{4_5} + c_{3_5} - c_{3_6}$. Here

c_{3_6} and c_{3_5} have coefficients + 1.

Therefore $y_{36}^{46} = +1$, $y_{36}^{35} = +1$. Since $\min \{ x_{4_6}, x_{3_5} \} = \min \{10, 11\} = 10 = x_{3_5}$, the vector p_{3_5} leaves the basis.

$\hat{x}_{3_6} = x_{3_5} = 10$. Since c_{4_6}, c_{3_5} have positive signs in the expression of $z_{3_6} - c_{3_6}$, we have

$\hat{x}_{4_6}^B = x_{4_6}^B - 10 = 11 - 10 = 1$, $\hat{x}_{3_5}^B = x_{35}^B - 10 = 10 - 10 = 0$.

Since c_{4_5} and c_{3_6} have negative signs in the expression, we have

$\hat{x}_{4_5} = x_{45}^B + 10 = 20 + 10 = 30$, $\hat{x}_{3_6} = x_{36}^B + 10 = 0 + 10 = 10$.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2, (30)	1, (20)	3, -2	3, -4	2, -4	5, -7	50
O_2	3, 0	2, (30)	2, (10)	4, -4	3, 4	4, -5	40
O_3	3, 2	5, -1	4, (10)	2, (40)	4, -3	1, (10)	60
O_4	4, 2	2, 3	2, 3	1, 2	2, (30)	2, (1)	31
b_j	30	50	20	40	30	11	181

Table-II

Observe that in the above table-II, not all $z_{i_j} - c_{i_j} \le 0$. So we have to construct the next table. To find the vectors to enter the basis, consider

$$z_{s_t} - c_{s_t} = \max \{ z_{i_j} - c_{i_j} \ / \ z_{i_j} - c_{i_j} > 0 \}$$

$$= \max \{2, 2, 3, 3, 2\} = 3 = z_{4_3} - c_{4_3} = z_{4_2} - z_{4_2} .$$

Here to break the tie we select $z_{4_3} - c_{4_3}$ at random. So p_{4_3} enters the basis.

Since $x^B_{q_r} = \min\{ x^B_{\alpha_\beta} \ / \ y^{\alpha\beta}_{s_t} > 0 \} = \min \{ x_{4_6}, \ x_{3_3} \} = \min \{1, 10\} = 1 = x_{4_6}$, The vector p_{4_6} leaves the basis.

(Here consider: $z_{4_3} - c_{4_3} = c_{4_6} - c_{3_6} + c_{3_3} - c_{4_3}$. $y_{s_t} = 1 > 0$ if the c_{s_t} have the positive sign. $y_{s_t} =$ 0 or –1 if c_{s_t} have negative sign. Here c_{4_6} , c_{3_3} have positive sign. So $y_{4_6} > 0$, $y_{3_3} > 0$).

$\hat{x}_{4_3} = x_{4_6} = 1.$

c_{4_6} and c_{3_3} have positive sign in the expression of $z_{4_3} - c_{4_3}$. So $\hat{x}_{4_6} = x_{4_6} - 1 = 0$, $\hat{x}_{3_3} = x_{3_3} - 1 = 9$.

c_{3_6} has negative sign in the expression of $z_{4_3} - c_{4_3}$.

$\hat{x}_{3_6} = x_{3_6} + 1 = 10 + 1 = 11$ (since c_{3_6} have negative sign in expression of

$z_{4_3} - c_{4_3}$). For the other basic variables, $\hat{x}_{i_j} = x_{i_j}$. Now we form

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	(30)	(20)	-2	-4	-1	-7	50
O_2	0	(30)	(10)	-4	-1	-5	40
O_3	2	-1	(9)	(40)	0	(11)	60
O_4	-1	0	(1)	-1	(30)	-3	31
b_j	30	50	20	40	30	11	181

Table-III

In the same way we get the following table IV & table V.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	(21)	(29)	-2	-2	-1	-5	50
O_2	0	(21)	(19)	-2	-1	-3	40
O_3	(9)	-3	-2	(40)	-2	(11)	60
O_4	-1	0	(1)	1	(30)	-1	31
b_j	30	50	20	40	30	11	181

Table-IV

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	(20)	(30)	-2	-2	0	-5	50
O_2	0	(20)	(20)	-2	0	-3	40
O_3	(10)	-3	-2	(39)	-1	(11)	60
O_4	-2	-1	-1	(1)	(30)	-2	31
b_j	30	50	20	40	30	11	181

Table-V

Observe table–V. All $z_{i_j} - c_{i_j}$ are non-positive. Hence the present basic solution is optimal. The solution is $x_{1_1} = 20$, $x_{1_2} = 30$, $x_{2_2} = 20$, $x_{2_3} = 20$, $x_{3_1} = 10$, $x_{3_4} = 39$, $x_{3_6} = 11$, $x_{4_4} = 1$, $x_{4_5} = 30$. Therefore

min z = (2 x 20) + (1 x 30) + (2 x 20) + (2 x 20) + (3 x 10) + (2 x 39) + (1 x 11) + (1 x 1) + (2 x 30) = 40 + 30 + 40 + 40 + 30 + 78 + 11 + 1 + 60 = 330.

4.3.3 SELF ASSESSMENT QUESTION (SAQ) 1:

Solve the following transportation problem using stepping–stone algorithm.

Destinations

Origins	D_1	D_2	D_3	D_4	D_5	Availability
O_1	3	2	7	5	2	27
O_2	3	5	6	0	7	12
O_3	2	1	1	3	2	10
Requirement	5	15	20	1	8	49

4.4 METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION

Determination of initial basic feasible solution – Different methods

Already we know the method entitled "North-West Corner Method" to find initial basic feasible solution for a given Transportation problem. In the following we present four different methods.

4.4.1 Matrix Minima Method: Determine the smallest cost in the entire table. Suppose this occurs in the cell (i, j) [If there is a tie, select the cell for which (i + j) is smallest].

Set $x_{i_j} = \min \{a_i, b_j\}$.

If $a_i < b_j$ then $x_{i_j} = a_i$ and then cross off the row-i.

If $b_j < a_i$ then $x_{ij} = b_j$ and then cross off the column-j.

If $a_i = b_i$ then $x_{ij} = a_i = b_j$ and then cross off either row-i or column-j, but not both. Continue in this way till we get the basic feasible solution.

4.4.2 Example : Find the initial basic feasible solutions for the following problem through matrix minima method.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	31
b_j	30	50	20	40	30	11	181

Solution: The solution obtained through the matrix minima method was given by the following table:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2 (0)	1 (50)	3	3	2	5	50
O_2	3 (20)	2	2 (20)	4	3	4	40
O_3	3 (10)	5	4	2 (9)	4 (30)	1 (11)	60
O_4	4	2	2	1 (31)	2	2	31
b_j	30	50	20	40	30	11	181

For this example the solution is given by
$x_{11} = 0$, $x_{12} = 50$, $x_{21} = 20$, $x_{23} = 20$, $x_{31} = 10$, $x_{34} = 9$,

$x_{35} = 30$, $x_{36} = 11$, $x_{44} = 31$.

4.4.3 Note: In the above table, we have that the number of basic variables is less than $(m + n - 1) = 4 + 6 - 1 = 9$. So we enter 0 in the cell (1, 1). The advantage is (after placing 0 in this (1, 1) cell), there is a directed path [for any given cell (i, j) of the table] that involves only the given cell and some of the basic cells.

4.4.4 Vogel's method: This technique has been suggested by Vogel. For each row-i, find the lowest c_{i_j} and the next lowest cost c_{i_t}. Compute $(c_{i_j} - c_{i_t})$. So for each row we get a number $(c_{i_j} - c_{i_t})$. Proceed in the same way for each column. Now we have $(n + m)$ numbers. Among these numbers, select the largest number. Suppose that the largest of these numbers was associated with the difference in column-j. Let cell (i, j) contain the lowest cost in column j. Then write $x_{i_j} = \min \{a_i, b_j\}$.

If $a_i < b_j$ then $x_{i_j} = a_i$. If $b_j < a_i$ then $x_{i_j} = b_j$. If $a_i = b_j$ then $x_{i_j} = a_i = b_j$.

Next find the differences for the remaining part of the table (this should be done at every step).
Now select the next largest $(c_{i_j} - c_{i_t})$ among the differences. Repeat the same process till all the row and column constraints are satisfied. In the process, if maximum difference (among the $(c_{i_j} - c_{i_t})$'s) is not unique then an arbitrary choice can be made (generally we select the cell with $i + j$ minimum).

4.4.5 Advantages of Vogel's method: This method is preferred over the other two methods because the initial basic feasible solution obtained is either optimal or very close to the optimal solution. Therefore, the amount of time require to calculate the optimum solution is greatly reduced.

4.4.6 Vogel's method – An algorithm:

Step 1: Calculate the penalty for each row and for each column of the transportation table. (The penalty for a given row and column is nothing but the difference between the smallest cost and the next smallest cost element in that particular row or column.)

Step 2: Select either the row or the column with the largest penalty. In this select row (column), choose the cell with the smallest cost and allocate the maximum possible quantity to this cell. Delete that row (column) in which capacity (requirement) is exhausted. (Whenever the largest penalty among rows and columns is not unique, make an arbitrary selection)

Step 3: Repeat steps 1 and 2 for the reduced table until the entire capacities are distributed to fill the requirement at different warehouses.

4.4.7 Example: Find the initial basic feasible solutions for the following problem through Vogel's method.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	31
b_j	30	50	20	40	30	11	181

Solution: The solution obtained through the Vogel's method was given by the following table:

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	1	2	3	4	5	6
O_1	2 (30)	1 (20)	3	3	2	5	50	0	1	-	-	-	-
O_2	3	2 (30)	2 (10)	4	3	4	40	0	0	0	1	1	0
O_3	3	5	4 (9)	2 (40)	4	1 (11)	60	1	1	1	3	0	1
O_4	4	2	2 (1)	1	2 (30)	2	31	0	0	0	0	0	0
b_j	30	50	20	40	30	11	181						
1	1	1	0	1	0	1	Differences						
2	-	1	0	1	0	1							
3	-	0	0	1	1	1							
4	-	0	0	-	1	1							
5	-	0	0	-	1	-							
6	-	0	0	-	-	-							

4.4.8 SELF ASSESSMENT QUESTION 2: Find the initial basic feasible solutions for the following problem through Vogel's method.

TABLE	I	II	III	IV	Availability
1	7	4	6	6	34
2	6	6	8	7	15
3	9	7	7	6	12
4	8	2	7	5	19
Requirement	21	25	17	17	

4.5 SUMMARY

In this lesson, Transportation Problem as a Linear Programming Problem has been introduced and also meaning of Transportation problem is given. Some terms viz. directed path, directed branch, directed loop, etc. are defined. To find an initial basic feasible solution various methods are discussed. Some of them are North West Corner method, Column minimum method, Row minima method, Matrix minima Method and Vogel's method. Finally problems are solved by using the above said methods. The stepping stone algorithm to solve a given transportation problem is introduced and solution to few problems are included. The examples considered in this chapter have more or less been confined to problems of distribution. The transportation models however have found applications in a variety of areas such as production scheduling, sales territory assignment, location of plants, and even in the hotel industry.

4.6 TECHNICAL TERMS

Transportation Problem :A product quantity a_i is available at origin i ($1 \le i \le m$) and b_j is the quantity required at destination j ($1 \le j \le n$). The cost of shipping one unit from origin i to destination j will be c_{ij}.

The Transportation Problem is to Transport various units available at origins to different destinations in such a way that the total transportation cost is a minimum.

Directed Path :A directed path from the cell (i, j) to the cell (u, v) in the transportation table is defined to be an ordered set of cells {(i, j), (i, k), (q, k), …, (i, v)} or {(i, j), (s, j), (s, t), …, (u, v)} such that any two adjacent cells are in the same row or in the same column. Each cell occurs only once in the ordered sets. The cell (i, j) is called the initial cell of the path and the cell (u, w) is called the terminal cell.

Directed Branch :A directed branch is a line segment joining an ordered pair of cells which lie either in the same row or in the same column of the transportation table

Simple Directed Path :A simple directed path from cell (i, j) to cell (u, v) is a directed path such that in any row or column of the transportation table, there are no more than two cells in the set of cells which defines the path.

Simple Directed Loop :A simple directed loop is a directed loop that has no more than two cells in any row or column of the transportation table.

North West Corner method :A method to find an initial basic feasible solution

The Stepping Stone algorithm :A method to solve a transportation problem

Column minimum method :A method to Determine an initial basic feasible solution

Row minima method :A method to Determine an initial basic feasible solution

Matrix minima method :A method to Determine an initial basic feasible solution

Vogel's method :A method to Determine an initial basic feasible solution

4.7 ANSWERS TO SELF ASSESSMENT QUESTIONS

Self Assessment Question 1 (4.3.3): Solve the following transportation problem using stepping–stone algorithm.

Destinations

Origins	D_1	D_2	D_3	D_4	D_5	Availability
O_1	3	2	7	5	2	27
O_2	3	5	6	0	7	12
O_3	2	1	1	3	2	10
Requirement	5	15	20	1	8	49

Solution:

Here we used North–West corner method to find initial basic feasible solution for the given Transportation Problem.

Table–I

Origin	D_1	D_2	D_3	D_4	D_5	a_i
O_1	[3] (5)	[2] (15)	[7] (7)	[5] 4	[2] 6	27
O_2	[3] -1	[5] -4	[6] (12)	[0] 8	[7] 0	12
O_3	[2] -5	[1] -5	[1] (1)	[3] (1)	[2] (8)	10
b_j	5	15	20	1	8	49

$z_{s_t} - c_{s_t} = \max \{ z_{i_j} - c_{i_j} \,/\, z_{i_j} - c_{i_j} > 0\} = \max \{4, 6, 8\} = 8 = z_{2_4} - c_{2_4}$. Therefore x_{2_4} enters the basis $z_{2_4} - c_{2_4} = (c_{2_3} - c_{3_3} + c_{3_4}) - c_{2_4}$. Now x_{q_r} leaves the basis if $x_{q_r} = \min \{ x^{B}_{\alpha_\beta} \,/\, y^{24}_{\alpha_\beta} > 0\} = \min \{ c_{2_3}, c_{3_4} \} = \min \{12, 1\} = 1 = x_{3_4}$. $\hat{x}_{2_3} = x_{2_3} - 1 = 12 – 1 = 11$, $\hat{x}_{3_3} = x_{3_3} + 1 = 1 + 1 = 2$; $\hat{x}_{3_4} = x_{3_4} - 1 = 1- 1 = 0$.

Table–II

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	[3] (5)	[2] (15)	[7] (7)	[5] -4	[2] 6	27
O_2	[3] -1	[5] -4	[6] (11)	[0] (1)	[7] 0	12
O_3	[2] -5	[1] -5	[1] (2)	[3] -8	[2] (8)	10
b_j	5	15	20	1	8	49

$z_{s_t} - c_{s_t} = \max \{ z_{i_j} - c_{i_j} \,/\, z_{i_j} - c_{i_j} > 0\} = \max \{6\} = z_{1_5} - c_{1_5}$. Therefore x_{1_5} enters the basis. $z_{1_5} - c_{1_5} = (c_{3_5} - c_{3_3} + c_{1_3}) - c_{1_5}$. Now x_{q_r} leaves the basis if $x_{q_r} = \min \{ x^{B}_{\alpha_\beta} / \; y^{15}_{\alpha_\beta} > 0\} = \min \{ x_{3_5}, x_{1_3} \} = \min \{8, 7\} = 7 = c_{1_3}$.

$\hat{x}_{3_5} = x_{3_5} - 7 = 8 - 7 = 1$; $\hat{x}_{3_3} = x_{3_3} + 7 = 2 + 7 = 9$; $\hat{x}_{1_3} = x_{1_3} - 7 = 7 - 7 = 0$.

Table – III

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	[3] (5)	[2] (15)	[7] -6	[5] -10	[2] (7)	27
O_2	[3] 5	[5] 2	[6] (11)	[0] (1)	[7] 0	12
O_3	[2] 1	[1] 1	[1] (9)	[3] -8	[2] (1)	10
b_j	5	15	20	1	8	49

$z_{s_t} - c_{s_t} = \max\{z_{i_j} - c_{i_j} \;/\; z_{i_j} - c_{i_j} > 0\} = \max\{5, 2, 7, 1\} = 5 = z_{2_1} - c_{2_1}$.

Therefore x_{2_1} enters the basis.

$z_{2_1} - c_{2_1} = (c_{2_3} - c_{3_3} + c_{3_5} - c_{1_5} + c_{1_1}) - c_{2_1}$. Now x_{q_r} leaves the basis if $x_{q_r} = \min\{x^B_{\alpha\beta} \;/\; y^{21}_{\alpha\beta} > 0\} = \min\{c_{2_3}, c_{3_5}, c_{1_1}\} = \min\{11, 1, 5\} = 1 = c_{3_5}$.

$\hat{x}_{2_3} = x_{2_3} - 1 = 11 - 1 = 10$; $\hat{x}_{3_3} = x_{3_3} + 1 = 9 + 1 = 10$;

$\hat{x}_{3_5} = x_{3_5} - 1 = 1 - 1 = 0$; $\hat{x}_{1_5} = x_{1_5} + 1 = 7 + 1 = 8$.

$\hat{x}_{1_1} = x_{1_1} - 1 = 5 - 1 = 4$.

Table – IV

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	[3] (4)	[2] (15)	[7] -1	[5] -5	[2] (8)	27
O_2	[3] (1)	[5] -3	[6] (10)	[0] (1)	[7] -5	12
O_3	[2] 0	[1] -4	[1] (10)	[3] -8	[2] 0	10
b_j	5	15	20	1	8	49

Therefore min z = (3 × 4) + (2 × 15) + (2 × 8) + (3 × 1) + (6 × 10) + (0 × 1) + (1 × 10) = 12 + 30 + 16 + 3 + 60 + 0 +10 = 131.

SAQ 2 (4.4.8): An initial feasible solution using Vogel's method was obtained and included in the following Table V.

TABLE – I	I	II	III	IV	Availability	Penalty
1	7	4	6	6	34	(2)
2	6	6	8	7	15	(0)
3	9	7	7	6	12	(1)
4	8	2	7	5	19	(3)
Requirement	21	25	17	17		
Penalty	(1)	(2)	(1)	(1)		

The largest of the penalties is (3) and this value corresponds to the fourth row. The least cost cell in this row is the cell (4,2). Allocate 19 to this cell and cross out the fourth row. After adjusting the availability and requirement. We compute penalties for the resultant table.

TABLE - II						
	7	4	6	6	34	(2)
	6	6	8	7	15	(0)
	9	7	7	6	12	(1)
	(crossed out)	19 [2]	(crossed out)	(crossed out)	19-19=0	
	21	25-19 =6	17	17		
	(1)	(2)	(1)	(1)		

In table-II, the largest of the penalties is 2, which corresponds to first row and second column. We arbitrarily select the first row. The least cost cell in this row is (1, 2). We allocate 6 to this cell and cross out the second column. Next we calculate the new penalties. The new penalties are given in table-III.

TABLE – III

7	6 (4)	6	6	34-6=28	(0)
6		8	7	15	(1)
9		7	6	12	(1)
	19 (2)			0	
21	6 –6=0	17	17		
(1)		(1)	(0)		

Observe table-III. As the highest penalty exist at the first column, at the third column, at the second row and at the third row, we arbitrarily select third row. In this row the least cost cell is the (3, 4). This cell receives 12, and we cross out the third row.

TABLE–IV

7	6 (4)	6	6	28	(0)
15 (6)				15-15=0	
9			12 (6)	12-12=0	
	19 (2)			0	
6	0	17	17–12 =5		
(1)		(1)	(0)		

Since only first row is left, the final allocations are made with the remaining values. The final table with complete allocations is given below:

TABLE – IV

6 (7)	6 (4)	17 (6)	5 (6)
15 (6)	6	8	7
9	7	7	12 (6)
8	19 (2)	7	5

Total shipping cost is 6(7) + 6(4) + 17(6) + 5(6) + 15(6) + 12(6) + 19(2) = 398.

4.8 MODEL QUESTIONS

4.8.1. Obtain an initial basic feasible solution to the following transportation problem:

Stores					
Warehouse	I	II	III	IV	Availability
A	7	3	5	5	34
B	5	5	7	6	15
C	8	6	6	5	12
D	6	1	6	4	19
Demand	21	25	17	17	80

Ans: The total transportation cost = 6 x 7 + 6 x 3 + 17 x 5 + 5 x 5 + 15 x 5 + 12 x 5 + 19 x 1 = Rs.324

4.8.2 : Determine an initial basic feasible solution to the following transportation problem using North-West Corner method.

Destination

		E	E	F	G	Supply
Source	A	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
	Demand	200	225	275	250	

Ans : $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{23} = 125$, $x_{33} = 150$, $x_{34} = 250$; total cost = Rs.12,200.

4.8.3. Determine an initial basic feasible solution to the above transportation problem using Vogel's method.

Ans: $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{23} = 125$, $x_{34} = 250$; total cost = Rs.12,200

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Name of the Lesson Writer

Prof. Bhavanari Satyanarayana

LESSON - 5

GAME THEORY

Objectives

The objectives of this lesson are to:

- Introduce the concepts: Pay-off matrix, Optimal strategy, pure strategy in a game
- Know the proof of fundamental theorem for two-person zero-sum game
- Appreciate relevance of a game and its related Linear Programming Problem
- Represent a game in the form of a matrix

Structure

5.0 Introduction
5.1 Preliminary concepts
5.2 Reduction of a game to Linear Programming Problem
5.3 The Principle of Dominance
5.4 Conversion of Linear Programming Problem to a game problem
5.5 Solved Problems
5.6 Summary
5.7 Technical terms
5.8 Answers to Self Assessment Questions
5.9 Model Questions
5.10 Reference Books

5.0 INTRODUCTION

Game theory is concerned with a type of decision problems characterized by conflict or competition among two or more competitors e.g., union leaders and management involved in collective bargaining, political negotiations, advertising and promotional decisions, etc.

The objective of game theory is to determine the rules of rational behavior in game (decisions under conditions of conflict or competition) situations where the outcome resulting from a decision made by one individual called a player depends not only on that individual' choice, but also on the course of action taken by other interested individuals. Thus, a player is not interested only in what he is doing as an individual but also in what his opponent is doing and vice-versa. A game is finite when each player has a finite number of moves and finite number of choices at each move.

Game theory provides a systematic quantitative method for analyzing competitive situations in which the competitors (or players) make use of logical processes and techniques in order to determine an optimal strategy for winning. Since many situations in business involve competition, game theory is of considerable theoretical interest.

Game theory models can be classified in a number of ways, depending on such factors as the number of players, sum of gains and losses and the number of strategies employed in the game. For example, if the number of players are two, we refer to the game as two-person game. Similarly, if the number of players are three or more, the game is refer to as n-person game.

The zero-sum property means that the payoff to one player and payoff to other player together sum to zero. This means that one player's gain is another's loss and vice-versa so that the sum of net gains is zero. If the sum of gains and losses does not equal zero, the game is a non zero-sum game.

The usual distinction in game theory is between two-person games and games involving three or more persons (n-person games which is still not fully developed. It is precisely this limitation that has restricted the application of game theory from many real life applications. Therefore, the discussion of this chapter will be limited to the presentation and analysis of two-person zero-sum games. The underlying assumptions, the rules of the game are given as follows:

1. The player act rationally and intelligently.
2. Each player has available to him a finite set of possible courses of action.
3. The players attempt to maximize gains and minimize losses.
4. All relevant information is known to each players.
5. The players make individual decisions without direct communication.
6. The players simultaneously select their respective courses of action.
7. The pay off is fixed and determined in advance.

Games of strategy deal with situations where there are conflicts of interest between two or more persons.

A game may be an indoor game such as chess or bridge. More generally, games involve conflict situations in economic, social, political, or military activities. Players can influence the final outcome, and hence that the outcome is not controlled purely by chance. By a **game** we shall mean a set of rules for playing. These rules may describe the moves, who makes moves, when they are made, what is information available to each of the players, what terminates a play of the game, etc.

After the termination of one play of the game, we imagine that there are certain pay-offs to each participant. (Here, these are usually through of as pay-offs in money). If the sum of the pay-offs to all participants at the end of the play is zero, then the game is called a zero-sum-game. Two-person games involve a conflict of interest between two persons (Here persons may refer to people, companies, countries, groups, etc,.) we restrict our attention to zero-sum two-person games.

5.1 PRELIMINARY CONCEPTS

5.1.1 Definition: One player writes an a sheet of paper what he will do under all possible circumstances at each move in a play of the game. This is called a **strategy**. We assume that the number of strategies of a player is finite. a_{i_j} = the pay off to player-1 if player–1 chooses strategy i and player-2 chooses strategy j. a_{i_j} may be positive, negative or zero. In a game, if player–1 has m strategies, player–2 has n strategies then $A = (a_{i_j})_{m\times n}$ is called the **pay-off matrix**. Consider a two-person zero-sum game. Suppose $A = (a_{i_j})$ is the pay-off matrix, where a_{i_j} is the pay-off to player–1 when player-1 and player-2 choose strategies i and j respectively. If player–1 choose strategy i, then he is sure of getting $\min_j a_{i_j}$ (no matter what player–2 does). It then seems wise for player–1 to choose the strategy which gives the maximum of these minimums. That is, $\max_i \min_j a_{i_j}$.

If player–2 chooses strategy-j, then he is sure that player–1 will not get more than $\max_{i} a_{i_j}$ (no matter what the player–1 does). Then player–2 attempts to minimize his maximum loss. So player–2 selects the strategy which gives minimum of these maxima. That is,

$$\min_{j} \max_{i} a_{i_j} .$$

If there is a_{r_k} such that $a_{r_k} = \max_{i} \min_{j} a_{i_j} = \min_{j} \max_{i} a_{i_j}$, then the game is said to have a **saddle point**. In this situation, we say that the **optimal strategy** for player–1 is r and for player-2 is k.

The strategies considered up to present, are called **pure strategies**.

5.1.2 Definition: Assume that u_i = probability of player–1 to select pure strategy-i, and ν_j = probability of player–2 to select pure strategy-j with $\sum_{i}^{m} u_i = 1$, $u_i \geq 0$, and $\sum_{j=1}^{m} v_j = 1$, $\nu_j \geq 0$. These strategies with probability vectors $U = (u_1, \ldots, u_m)$, $V = (\nu_1, \ldots, \nu_n)$ are called **mixed strategies.**

$U^1AV = \sum_{i,j} u_i \; a_{i_j} \nu_j$ is called the **expected winning** for player-1 when player–1 and player-2 used the mixed strategies U and V respectively. This expected winning is denoted by **E(U, V)**.

5.1.3 Definition: Player-1 wishes to find a mixed strategy U which maximizes his winnings. For any U he chooses, he is sure that his expected winnings will be atleast $\min_{V} E(U, V)$. Then he maximizes this expression over U so that his expected winnings are atleast

$$M_1{*} = \max_{U} \min_{V} E(U, V) .$$

If player-2 uses mixed strategy V, then he is sure that the expected winnings of player-1 is not more that $\operatorname{Max}_{U} E(U, V)$. Player–2 tries to select mixed strategy using

$$M_2^* = \min_{V} \max_{U} E(U, V).$$

If there exists mixed strategies U^*, V^* (for player–1, player-2 respectively) such that $E(U^*, V^*) = M_1^* = M_2^*$, then the game is said to have a **generalized saddle point**. In this case, we write $M^* = M_1^* = M_2^*$. M^* (the expected winnings of player-1) is called the **value of the game**.

5.1.4 Statement of fundamental theorem for two-person zero-sum game: There exist mixed strategies U^*, V^* for player-1 and player–2 such that $U^* \geq 0$, $V^* \geq 0$, $1.U^* = 1$, $1.V^* = 1$ and $E(U^*, V^*) = M_1^* = M_2^*$. (we get the proof for this theorem, in the sequel).

5.1.5 Problem: Check whether the game with the given pay-off matrix, has a saddle point. Here A is player-1 and B is player–2.

$$\begin{array}{cc} & B \\ & \begin{array}{cc} 1 & 2 \end{array} \\ A \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}$$

Solution: Here Both A and B have two strategies. $\min_j a_{1_j} = -1$, $\min_j a_{2_j} = -1$.

Therefore $\max_i \min_j a_{i_j} = \max\{-1, -1\} = -1$.

$\max_i a_{i_1} = 1$, $\max_i a_{i_2} = 1$. Therefore $\min_j \max_i a_{i_j} = \min\{1, 1\} = 1$.

Hence $\max_i \min_j a_{i_j} \neq \min_j \max_i a_{i_j}$. Therefore the game do not have a saddle point

5.1.6 Problem: Find the saddle point for the given two-person zero-sum game (if the saddle point exists).

Player-II

Player–I Strategies	a	b	c
A	5	10	4
B	4	5	2
C	4	8	3
D	6	7	5

Solution: Here a, b, c are strategies of player-II and A, B, C, D are strategies of player-I. max min a_{i_j} = max {row minima} = max {4, 2, 3, 5} = 5.

min max a_{i_j} = min{column maxima} = min {6, 10, 5} = 5.

Hence max min a_{i_j} = min max a_{i_j}. Therefore the game has a saddle point. The value of the game is 5.

Player-II

Player–I

Strategies	a	b	c	Row minimum
A	5	10	4	4
B	4	5	2	2
C	4	8	3	3
D	6	7	5	5
Column maximum	6	10	5	

Player-II

Player–I

Strategies	a	b
A	4	7
B	6	3
C	5	4

5.1.7 Problem: Consider the following pay-off matrix. What can you say about saddle point? What can you say about the value of the game if the saddle point do not exist?

Solution: In the given matrix, observe row-minimum and column maximum.

Min max a_{i_j} = 6 and

Max min a_{i_j} = 4

Hence saddle point do not exists for this game. The value of the game lies between 4 and 6.

Player-II

Player–I

Strategies	a	b	Row minimum
A	4	7	4
B	6	3	3
C	5	4	4
Column maximum	6	7	

5.1.8 Note: If the saddle point do not exists then the value of the game (that is M) satisfies max min a_{i_j} $\leq$ M $\leq$ min max a_{i_j}.

5.1.9 Self Assessment Question 1: Find the range and value of p and q which will render the entry (2,2) a saddle point for the game

Player B

	2	4	5
Player A	10	7	Q
	4	p	6

5.2 REDUCTION OF A GAME TO LINEAR PROGRAMMING PROBLEM

5.2.1 Note: (i) Suppose player-1 uses a mixed strategy $U \geq 0$, $1.U = 1$ and player-2 uses pure strategy-j, then expected winnings for player–1 are $\sum_{i=1}^{m} a_{i_j} u_i$ where $U = (u_1, \ldots, u_m)$. So player–1 can be sure that his expected winnings will be atleast M_1 if there exists a mixed strategy U such that.

$$\sum_{i=1}^{m} a_{i_j} u_i \geq M_1 \text{ for } 1 \leq j \leq n \text{ ------- (i)}$$

Player-1 wishes to maximize M_1. Without loss of generality, we assume that $M_1 > 0$. If we write $w_i = (u_i / M_1)$ then (i) becomes $\sum_{i=1}^{m} a_{i_j} w_i \geq 1$ for $1 \leq j \leq n$ or in the matrix form $A^1W \geq 1^1$ where $W = (w_1, \ldots, w_m)$, $w_i \geq 0$. Write $z = w_1 + \ldots + w_m = 1.w$.

Then $z = 1.w = w_1 + \ldots + w_m = \frac{u_1}{M_1} + \ldots + \frac{u_m}{M_1} = \frac{1}{M_1}(\Sigma u_i) = \frac{1}{M_1}.1 = \frac{1}{M_1}$.

Since M_1 is to be maximized, z is to be minimized. Hence the linear programming problem for player-1 is $A^1W \geq 1^1$, $W \geq 0$, $\min z = 1.W$.

The value of the game = $M_1 = (\frac{1}{\min z})$ and $U = W.M_1$.

(ii) Player–2 is trying to find a mixed strategy V which will give the smallest M_2 satisfying

$$\sum_{i=1}^{m} a_{i_j} v_j \leq M_2 \text{ for } 1 \leq i \leq m \text{ ------- > (ii).}$$

So player-2 wishes to minimize M_2. Without loss of generality we assume that $M_2 > 0$.

Write $x_j = \frac{v_j}{M_2}$. Clearly $x_j \geq 0$. Now (ii) be comes $\sum_{i=1}^{m} a_{i_j} x_j \leq 1$ for $1 \leq i \leq m$ or in matrix form $AX \leq 1^1$, $X \geq 0$ where $X = (x_1, \ldots, x_n)$.

Write $z^1 = 1.X = x_1 + x_2 + \dots + x_n$. Then $z^1 = x_1 + \dots + x_n = \frac{v_1}{M_2} + \dots + \frac{v_n}{M_2} = \frac{1}{M_2}(\Sigma v_i) = \frac{1}{M_2}.1 = \frac{1}{M_2}$. Since player-2 wishes to minimize M_2, z^1 is to be maximized. Hence the linear programming problem for player-2 is

$$AX \le 1^1,\ X \ge 0,\ \max z^1 = 1.X.$$

The value of the game = $M_2 = \left(\frac{1}{\max z^1}\right)$ and $V = XM_2$.

Now it is clear that **the linear programming problems obtained for player-1 and player-2 are dual linear programming problems**.

(iii) Note that the nature of the game unchanged if a constant added to each a_{i_j} (Hence we may assume $M_1 > 0$ and $M_2 > 0$). Write $U = (1, 0, \dots, 0)$, $V = (1, \dots, 0)$ (These are nothing but pure strategies). Then $X = \frac{V}{M_2}$, $W = \frac{U}{M_1}$ (as in above (i) and (ii)) are feasible solutions for the primal and dual linear programming problems. By a known theorem (in duality theory) {[statement of that theorem: For any feasible solutions X for primal, W for dual $CX \le dW$. So $\max z = \max CX \le dW$]. We can find optimal solution for primal. Since primal has feasible solution, we can find optimal solution for dual. (Since primal is the dual of the dual)}, we get optimal solutions $\hat{X}$, $\hat{W}$ for primal and dual respectively. In such a case, we have that $C\hat{X} = d\hat{W}$. This shows that $\min z = \max z^1$. Hence $M_1^* = \max M_1 = \min M_2 = M_2^*$. This says that there exist mixed strategies such that $M_1^* = M_2^*$. (Hence **we proved the fundamental theorem of two-person zero sum-game**).

5.2.2 Example: Solve the following game whose pay-off matrix is given by

$$\text{player-A}\ \begin{matrix} 1 \\ 2 \end{matrix} \overset{\text{player-B} \atop \begin{matrix} 1 & \ \ 2 \end{matrix}}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Solution: The corresponding linear programming problem for player–2 (that is, B) is $AX \le 1^1$, $X \ge 0$, max $Z^1 = 1.X$. Here $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $X = (x_1, x_2)$. Now we may solve it by graphical method or simplex method (left as an exercise).

5.2.3 Problem: Develop a linear programming model to solve the given two-person zero-sum game.

		Player-II	
Player -I	Strategies	a	b
	A	4	7
	B	6	3
	C	5	4

Solution: The pay-off matrix for player-1 is $\begin{bmatrix} 4 & 7 \\ 6 & 3 \\ 5 & 4 \end{bmatrix}$. Therefore the linear programming problem for player-2 is

$$4x_1 + 7x_2 \le 1$$
$$6x_1 + 3x_2 \le 1$$
$$5x_1 + 4x_2 \le 1$$
$$x_1 \ge 0,\ x_2 \ge 0, \quad \text{Max } z = x_1 + x_2.$$

The remaining part left as an exercise. The answer is $x_1 = 2/15$, $x_2 = 1/15$, max $z = 1/5$.

Now the value of the game is $M_2 = \dfrac{1}{\text{max } z} = 5$ and mixed strategy for player-2 is $V = (v_1, v_2)$ $= (x_1.M_2, x_2.M_2) = (\frac{2}{15}.5, \ \frac{1}{15}.5) = (\frac{2}{3}, \frac{1}{3})$.

5.2.4 Problem: Develop a linear programming problem to solve the following two-player zero-sum game.

		Player-II		
Player-I	Strategies	a	b	c
	A	5	10	4
	B	4	5	2
	C	4	8	3
	D	6	7	5

Hint: Observe that min max a_{i_j} = 5 and max min

$a_{i_j} = 5$. So the problem has saddle point. Value of the game = 5.

The linear programming problem for player-2 is

$5x_1 + 10x_2 + 4x_3 \leq 1$, $4x_1 + 5x_2 + 2x_3 \leq 1$, $4x_1 + 8x_2 + 3x_3 \leq 1$, $6x_1 + 7x_2 + 5x_3 \leq 1$, $x_1, x_2, x_3 \geq 0$, max $z = x_1 + x_2 + x_3$.

5.2.5 Self Assessment Question 1: What is a zero-sum two-person game ?

5.3 THE PRINCIPLE OF DIMINANCE

While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the queuel.

In a given pay-off matrix A, we say that the ith row dominates the kth row if $a_{i_j} \geq a_{k_j}$ for all j = 1, 2, …, n and $a_{i_j} > a_{k_j}$ for at least one j.

In such a situation player A will never use strategy corresponding to kth row, because he will gain less for choosing such a strategy.

Similarly, we say the pth column in the matrix dominates the qth column if $a_{i_p} \leq a_{i_q}$ for all i = 1, 2, …, m. and $a_{i_p} < a_{i_q}$ for at least one i.

In this case, the player B will loose more by choosing the strategy for the qth column than by choosing the strategy for the pth column. So, he will never use the strategy corresponding to the qth column. When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) form that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

5.3.1 Problem: Solve the game with the following pay-off matrix.

$$\text{Player A}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\overset{\text{Player B}}{\overset{\begin{matrix}\text{I} & \text{II} & \text{III} & \text{IV}\end{matrix}}{\begin{bmatrix} 4 & 2 & 3 & 6 \\ 3 & 4 & 7 & 5 \\ 6 & 3 & 5 & 4 \end{bmatrix}}}$$

Solution: First, consider the minimum of each row

Row	Minimum Value
1	2
2	3
3	3

Maximum of {2, 3, 3} = 3.

Next consider the maximum of each column

Column	Minimum Value
1	6
2	4
3	7
4	6

Minimum of {6, 4, 7, 6} = 4.

The following conditions holds:

Max{row minima} ≠ min{column maxima}.

Therefore, we see that there is no saddle point for the game under consideration.

Compare column II and III

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in Column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.

Now, we have the reduced game

$$\begin{array}{c} \\ \text{Player B} \\ \end{array}$$

$$\text{Player A}\;\begin{array}{c}1\\2\\3\end{array}\overset{\begin{array}{ccc}\text{I} & \text{II} & \text{IV}\end{array}}{\begin{bmatrix}4 & 2 & 6\\3 & 4 & 5\\6 & 3 & 4\end{bmatrix}}.$$

For this matrix again, there is no saddle points. Column II does not dominates column IV.
The choice is for player B. So player B will give up his strategy 4.

The game reduces to the following

$$\text{Player B}$$

$$\text{Player A}\;\begin{array}{c}1\\2\\3\end{array}\overset{\begin{array}{cc}\text{I} & \text{II}\end{array}}{\begin{bmatrix}4 & 2\\3 & 4\\6 & 3\end{bmatrix}}.$$

This matrix has no saddle pint.

The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following:

$$\begin{bmatrix}3 & 4\\6 & 3\end{bmatrix}.$$

Again there is no saddle point. We have a 2 × 2 matrix. Take this matrix as $\begin{bmatrix}a & b\\c & d\end{bmatrix}$.

5.4 CONVERSION OF LINEAR PROGRAMMING PROBLEM TO A GAME PROBLEM

5.4.1 Note (i): Suppose $DX \le d,\; X \ge 0,\; \max z = CX$ ----- (1)

is the given linear programming problem and

$$D^1W \ge C^1,\; W \ge 0,\; \min z^1 = d^1W \text{ ------- } (2)$$

is its dual. For any feasible solution X, W of (1) and (2) respectively, we know that $CX \le d^1W$ or $CX - d^1W \le 0$. Also when optimal solution exists max z = min z^1. If we can find feasible solution satisfying $CX - d^1W \ge 0$, then $CX = d^1W$ and hence X and W are optimal (by a known theorem in duality theory).

(ii) Introduce a variable t > 0, vectors U and V such that $X = \frac{U}{t}$, $W = \frac{V}{t}$. Then $DX \le d$

$$\Rightarrow DX - d \le 0 \Rightarrow D(\frac{U}{t}) - d \le 0 \Rightarrow DU - dt \le 0$$

$$\left.\begin{aligned} &\Rightarrow -DU + td \ge 0. \\ \text{Similarly } D^1W \ge C^1 &\Rightarrow D^1V - tC^1 \ge 0. \\ CX - d^1W \ge 0 &\Rightarrow CU - d^1V \ge 0 \end{aligned}\right\} \text{----------} (3)$$

(iii) Since t can be arbitrary so large, we impose an additional condition

$1.U + 1.V + t = 1$ ----------- (4)

(iv) Now consider the game for which the problem of player-1 is

$$-DU + td \ge M.(1^1), \quad D^1V - tc^1 \ge M.(1^1),$$

$$CU - d^1V \ge M.(1^1), \quad [V, U, t] \ge 0, \; 1.[V, U, t] = 1,$$

and M is to be maximized. Therefore in the matrix form, the problem is

$$\begin{bmatrix} 0 & -D & d \\ D^1 & 0 & -C^1 \\ d^1 & C & 0 \end{bmatrix} \begin{bmatrix} V \\ U \\ t \end{bmatrix} \ge M.(1^1).$$

(v) Since we thought the above matrix form is the problem for player-1, the transpose of this matrix is pay-off matrix for player-1. Hence the pay-off matrix of the game related to the given linear programming problem is $\begin{bmatrix} 0 & D^1 & d^1 \\ -D & 0 & C \\ d & -C^1 & 0 \end{bmatrix}$.

5.4.2 Note: (i) The above matrix is a skew–symmetric matrix. (ii) A game represented by a skew–symmetric pay-off matrix is called a **symmetric game**.

5.5 SOLVED PROBLEMS

5.5.1 Problem: Solve the game whose pay-off matrix is given here

$$\text{Player-A}\ \overset{\text{Player-B}}{\begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix}}$$

Solution Now we solve this problem by two different methods.

Direct method: Consider the given problem. Suppose A is player-I and B is player-II. Linear programming problem for Player-II is $AX^1 \leq 1^1$, where A is the matrix in the given problem, max $z = x_1 + x_2 + x_3$. The matrix form is $\begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(Now we give only outline of the solution). By adding slack variables x_4, x_5, x_6, we get

$x_1 + 7x_2 + 2x_3 + x_4 = 1$, $6x_1 + 2x_2 + 7x_3 + x_5 = 1$, $5x_1 + x_2 + 6x_3 + x_6 = 1$.

C_B	Vectors in basis	X_B	1	1	1	0	0	0
			a_1	a_2	a_3	a_4	a_5	a_6
0	a_4	1	1	7	2	1	0	0
0	a_5	1	[6]	2	7	0	1	0
0	a_6	1	5	1	6	0	0	1
Table – I		0	-1	-1	-1	0	0	0

C_B	Vectors in basis	X_B	1	1	1	0	0	0
			a_1	a_2	a_3	a_4	a_5	a_6
0	a_4	5/6	0	[20/3]	19/6	1	-1/6	0
1	a_1	1/6	1	1/3	7/6	0	1/6	0
0	a_6	1/6	0	-2/3	1/6	0	-5/6	1
Table – II		1/6	0	-2/3	1/6	0	1/6	0

C_B	Vectors in basis	x_B	1	1	1	0	0	0
			a_1	a_2	a_3	a_4	a_5	a_6
1	a_2	1/8	0	1	19/40	3/20	-1/40	0
1	a_1	1/8	1	0	121/6	-1/20	21/120	0
0	a_6	1/72	0	0	29/60	1/10	-23/60	1
Table – III		1/4	0	0	$\frac{2477}{120}$	$\frac{1}{10}$	$\frac{18}{120}$	0

In Table–III, all $z_j - c_j$'s are ≥ 0. The present solution is optimal. Max z $= \frac{1}{4}$, M = value of the game $= \frac{1}{\max z} = 4$. Then $x_1 = \frac{1}{8}$, $x_2 = \frac{1}{8}$. So mixed strategy for player-II is $(x_1.M, x_2.M, x_3.M)$ $= \{\frac{1}{8}.4, \frac{1}{8}.4, 0(4)\} = (\frac{1}{2}, \frac{1}{2}, 0)$. This strategy is the optimal strategy for player-II.

Other method:

Given pay-off matrix is
$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} B_1 \;\; B_2 \;\; B_3 \\ \begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix} \end{array}$$
. By the principle of dominance, we can neglect the B_3 for player–B (since elements of B_3 are $\geq$ the corresponding elements of B_1).

Then we get
$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} B_1 \;\; B_2 \\ \begin{pmatrix} 1 & 7 \\ 6 & 2 \\ 5 & 1 \end{pmatrix} \end{array}$$
. Since the elements of $A_3 \leq$ the corresponding elements of A_2, we can neglect A_3.

Then the remaining matrix is
$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} B_1 \;\; B_2 \\ \begin{pmatrix} 1 & 7 \\ 6 & 2 \end{pmatrix} \end{array}$$
. For this problem saddle point do not exists.

Let p be the probability of A_1 for player–I, q be the probability of B_1 for player–II. Then (1- p) is the probability of A_2 for player-I, and (1- q) be the probability of B_2 of player–II.

If player-II selects mixed strategy including B_1 and B_2, then with respect to player-I we have $1.q + 7(1 – q) = 6.q + 2.(1 - q) \Rightarrow q = ½$.

If player-I selects mixed strategy including A_1 and A_2, then with respect to player-II we have $1.p + 6(1 - p) = 7p + 2(1 - p) \Rightarrow p = 2/5, 1 – p = 3/5$.

Then the value of the game with respect to player-I (expected value of player-I) is $p[1.q + 7(1 - q)] + (1 - p)[6q + 2(1 - q)] = 4$. Similarly expected value with respect to player-II is $q[1.p + 6(1 - p)] + (1 - q)[(7p + 2(1 - p)] = 4$. So Value of the game = 4. The mixed optimal strategies for player-1 and player-2 are (2/5, 3/5, 0) and (1/2, 1/2, 0) respectively.

5.6 SUMMARY

In this lesson, the concept of Game theory and some preliminary concepts viz. strategy, pay-off matrix, saddle point etc. have been introduced. And also the fundamental theorem of two person zero-sum game is obtained. Reduction of a game to Linear Programming problem and conversion of LPP to a game problem are discussed. Finally, problems are solved by using the above said techniques.

5.7 TECHNICAL TERMS

Game Theory :Game theory is concerned with a type of decision problems characterized by conflict or competition among two or more competitors e.g., union leaders and management involved in collective bargaining, political negotiations, advertising and promotional decisions, etc.

Finite Game : A game is finite when each player has a finite number of moves and finite number of choices at each move

Two-person game :if the number of players are two, we refer to the game as two-person game.

n-person game :if the number of players are three or more, the game is refer to as n-person game.

The zero-sum property : If the sum of the pay-offs to all participants at the end of the play is zero, then the game is called a zero-sum-game.

Strategy : One player writes an a sheet of paper what he will do under all possible circumstances at each move in a play of the game. This is called a strategy.

Pay-off matrix : In a game, if player–1 has m strategies, player–2 has n strategies then $A = (a_{i_j})_{m\times n}$ is called the pay-off matrix.

Saddle point : If there is a_{r_k} such that $a_{r_k} = \max_i \min_j a_{i_j} = \min_j \max_i a_{i_j}$, then the game is said to have a saddle point.

Optimal Strategy : If there is a_{r_k} such that $a_{r_k} = \max_i \min_j a_{i_j} = \min_j \max_i a_{i_j}$, then the game is said to have a saddle point. In this situation, we say that the optimal strategy for player–1 is r and for player-2 is k.

Mixed Strategy : Assume that u_i = probability of player–1 to select pure strategy-i, and ν_j = probability of player–2 to select pure strategy-j with $\sum_i^m u_i = 1$, $u_i \geq 0$, and $\sum_{j=1}^m v_j = 1$, $\nu_j \geq 0$. These strategies with probability vectors $U = (u_1, \ldots, u_m)$, $V = (\nu_1, \ldots, \nu_n)$ are called mixed strategies.

Generalised Saddle point : If there exists mixed strategies U^*, V^* (for player–1, player-2 respectively) such that $E(U^*, V^*) = M_1^* = M_2^*$, then the game is said to have a generalized saddle point.

Value of the game : we write $M^* = M_1^* = M_2^*$. M^* (the expected winnings of player-1) is called the value of the game.

The fundamental theorem of two person zero sum game : There exist mixed strategies U^*, V^* for player-1 and player–2 such that $U^* \geq 0$, $V^* \geq 0$, $1.U^* = 1$, $1.V^* = 1$ and $E(U^*, V^*) = M_1^* = M_2^*$. (we get the proof for this theorem, in the sequel).

Symmetric game : A game represented by a skew–symmetric pay-off matrix is called a symmetric game.

5.8 ANSWERS TO SELF ASSESSMENT QUESTIONS

5.8.1 Answer to SAQ 5.1.9: First ignoring the values of p and q determine the maximin and minimax values of the payoff matrix as below:

		Player B			
		B_1	B_2	B_3	Row Min
	A_1	2	4	5	2
Player A	A_2	10	7	q	7
	A_3	4	p	6	4
Column Max.		10	7	6	

Since the entry (2, 2) is a saddle point, maximin value $\underline{v} = 7$, minimax value $\overline{v} = 7$.

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$. Hence the range of p and q will be $p \leq 7$, $q \geq 7$.

5.8.2 Answer to SAQ 5.2.5: A game with only two players (say, Player A and Player B) is called a 'two person zero-sum game' if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero.

5.9 MODEL QUESTIONS

5.9.1. What are the assumptions made in the theory of games ?

Ans: See the Section 5.0.

5.9.2. Develop a linear programming model to solve the given two-person zero-sum game.

Ans: Problem 5.2.3

	Player-II		
Player -I	Strategies	a	b
	A	4	7
	B	6	3
	C	5	4

5.9.3 Solve the game whose pay-off matrix is given here

$$\text{Player-A}\overset{\text{Player B}}{\begin{pmatrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{pmatrix}}$$

Ans: Problem 5.5.1.

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Name of the Lesson Writer

Prof. Bhavanari Satyanarayana